Some aspects of analysis on circles, disks, and annuli

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Abstract

These informal notes deal with some spaces of power series and Laurent series over fields with absolute value functions.

Contents

Part I
Preliminaries

1 Vector spaces and algebras

Let k be a field, let V be a vector space over k, and let X be a nonempty set. The space of all V-valued functions on X will be denoted $c(X, V)$. This is a vector space over k with respect to pointwise addition and scalar multiplication of functions.

The *support* of a V-valued function f on X is defined to be the set

$$\operatorname{supp} f = \{x \in X : f(x) \neq 0\}. \tag{1.1}$$

Let $c_{00}(X, V)$ be the set of V-valued functions f on X whose support has only finitely many elements. It is easy to see that this is a linear subspace of $c(X, V)$. Of course, $c_{00}(X, V)$ is the same as $c(X, V)$ when X has only finitely many elements.

If $f \in c_{00}(X, V)$, then the sum

$$\sum_{x \in X} f(x) \tag{1.2}$$

can be defined as an element of V by reducing to the sum over any nonempty finite subset of X that contains the support of f. The mapping from f to the sum (1.2) is linear as a mapping from $c_{00}(X, V)$ into V.

In particular, we can take $V = k$, considered as a one-dimensional vector space over itself. If $y \in X$, then let $\delta_y(x)$ be the k-valued function defined on X by

$$\begin{aligned} \delta_y(x) &= 1 \quad \text{when } y = x \\ &= 0 \quad \text{when } y \neq x, \end{aligned} \tag{1.3}$$

where 0 and 1 are the additive and multiplicative identity elements in k. Thus $\delta_y \in c_{00}(X, k)$ for every $y \in X$, and the collection of δ_y with $y \in X$ is a basis for $c_{00}(X, k)$, as a vector space over k.

To say that $\mathcal{A}$ is an (associative) *algebra* over k means that $\mathcal{A}$ is a vector space over k equipped with a binary operation of multiplication which is bilinear over k and satisfies the associative law. If multiplication on $\mathcal{A}$ also satisfies the commutative law, then $\mathcal{A}$ is said to be a *commutative algebra* over k.

Let $\mathcal{A}$ be an algebra over k, and suppose that $\mathcal{A}$ has a multiplicative identity element e, so that

$$e\,a = a\,e \tag{1.4}$$

for every $a \in \mathcal{A}$. Of course, e is unique when it exists, and $\mathcal{A} = \{0\}$ when $e = 0$. If $e \neq 0$, then $a \in \mathcal{A}$ is said to be *invertible* in $\mathcal{A}$ when there is a $b \in \mathcal{A}$ such that

$$a\,b = b\,a = e. \tag{1.5}$$

In this case, b is unique, and is denoted a^{-1}. If a_1, a_2 are invertible elements of $\mathcal{A}$, then $a_1\,a_2$ is invertible in $\mathcal{A}$ as well, with inverse $a_2^{-1}\,a_1^{-1}$.

If X is a nonempty set again, then the space $c(X, k)$ of k-valued functions on X is a commutative algebra over k, with respect to pointwise multiplication of functions. Let $\mathbf{1}_X$ be the k-valued function on X equal to the multiplicative identity element 1 in k at every $x \in X$. This is the multiplicative identity element in $c(X, k)$.

If V and W are vector spaces over k, then the space $\mathcal{L}(V, W)$ of all linear mappings from V into W is a vector space over k with respect to pointwise addition and multiplication. The space $\mathcal{L}(V) = \mathcal{L}(V, V)$ of all linear mappings from V into itself is an algebra over k, with composition of mappings as multiplication. The identity mapping $I = I_V$ on V is the multiplicative identity element in $\mathcal{L}(V)$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebras over k, then an (algebra) *homomorphism* from $\mathcal{A}_1$ into $\mathcal{A}_2$ is a linear mapping that preserves multiplication, as usual.

2 Polynomials and power series

Let k be a field, and let T be an indeterminate. As in [2, 5], we normally try to use upper-case letters like T for indeterminates, and lower-case letters for elements of k, or an algebra over k. A *formal polynomial* in T with coefficients in k can be expressed as

$$f(T) = \sum_{j=0}^{n} f_j\,T^j, \tag{2.1}$$

where n is a nonnegative integer and $f_j \in k$ for $j = 0, 1, \ldots, n$. Similarly, a *formal power series* in T with coefficients in k can be expressed as

$$f(T) = \sum_{j=0}^{\infty} f_j\,T^j, \tag{2.2}$$

where $f_j \in k$ for each nonnegative integer j. Thus a formal polynomial may be considered as a formal power series for which all but finitely many coefficients are equal to 0.

The spaces of formal polynomials and power series in T with coefficients in k are denoted $k[T]$ and $k[[T]]$, respectively. Let $\mathbf{Z}_+$ be the set of positive integers, so that $\mathbf{Z}_+ \cup \{0\}$ is the set of nonnegative integers. To be more precise,

$k[[T]]$ can be identified with the space $c(\mathbf{Z}_+ \cup \{0\}, k)$ of all k-valued functions on the set of nonnegative integers, by identifying $f(T) \in k[[T]]$ with $j \mapsto f_j$ as a k-valued function on $\mathbf{Z}_+ \cup \{0\}$. Using this identification, $k[T]$ corresponds to the space $c_{00}(\mathbf{Z}_+ \cup \{0\}, k)$ of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$ with finite support.

We can also consider $k[[T]]$ as a vector space over k with respect to termwise addition and scalar multiplication, which corresponds to pointwise addition and scalar multiplication of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$. Note that $k[T]$ is a linear subspace of $k[[T]]$, just as $c_{00}(\mathbf{Z}_+ \cup \{0\}, k)$ is a linear subspace of $c(\mathbf{Z}_+ \cup \{0\}, k)$.

Let (2.2) and

$$g(T) = \sum_{l=0}^{\infty} g_l\, T^l \tag{2.3}$$

be formal power series in T with coefficients in k. Their product

$$f(T)\, g(T) = h(T) = \sum_{n=0}^{\infty} h_n\, T^n \tag{2.4}$$

is defined as a formal power series in T with coefficients in k by putting

$$h_n = \sum_{j=0}^{n} f_j\, g_{n-j} \tag{2.5}$$

for each nonnegative integer n. Of course, this corresponds exactly to multiplying $f(T)$ and $g(T)$ formally, and collecting terms with the same power of T. It is well known and not difficult to check that $k[[T]]$ is a commutative algebra over k with respect to this definition of multiplication. It is easy to see that $k[T]$ is a subalgebra of $k[[T]]$, because $h_n = 0$ for all but finitely many $n \geq 0$ when $f_j = 0$ for all but finitely many $j \geq 0$ and $g_l = 0$ for all but finitely many $l \geq 0$.

We can also identify each element of k with the "constant polynomial" such that the coefficient of T^j is the given element of k when $j = 0$, and is equal to 0 when $j \geq 1$. It is easy to see that k corresponds to a subalgebra of $k[T]$ in this way. The constant polynomial corresponding to the multiplicative identity element 1 in k is the multiplicative identity element in $k[[T]]$.

If $a(T) \in k[[T]]$, then

$$\sum_{l=0}^{\infty} a(T)^l\, T^l \tag{2.6}$$

can be defined as an element of $k[[T]]$, as follows. If $l \in \mathbf{Z}_+$, then $a(T)^l$ is defined as an element of $k[[T]]$ using multiplication on $k[[T]]$, as before. If $l = 0$, then $a(T)^l$ is interpreted as being the constant polynomial corresponding to 1, as usual. Thus

$$\sum_{l=0}^{n} a(T)^l\, T^l \tag{2.7}$$

is defined as an element of $k[[T]]$ for every nonnegative integer n. If j is any nonnegative integer, then the coefficient of T^j in $a(T)^l\, T^l$ is equal to 0 when

$l > j$. This implies that the coefficient of T^j in (2.7) does not depend on n when $n \ge j$. In order to define (2.6) as a formal power series in T, we take the coefficient of T^j in (2.6) to be the same as the coefficient of T^j in (2.7) when $n \ge j$.

Observe that

$$(1 - a(T)\,T) \sum_{l=0}^{n} a(T)^l\,T^l = 1 - a(T)^{n+1}\,T^{n+1} \tag{2.8}$$

for each nonnegative integer n, by a standard computation. Using this, one can check that

$$(1 - a(T)\,T) \sum_{l=0}^{\infty} a(T)^l\,T^l = 1, \tag{2.9}$$

because the coefficient of T^j in the left side of (2.9) is the same as the coefficient of j in the left side of (2.8) when $n \ge j$. This shows that $1 - a(T)\,T$ has a mutliplicative inverse in $k[[T]]$, namely

$$(1 - a(T)\,T)^{-1} = \sum_{l=0}^{\infty} a(T)^l\,T^l. \tag{2.10}$$

If $f(T) \in k[[T]]$ is as in (2.2) and $f_0 \ne 0$, then it is easy to see that $f(T)$ has a multiplicative inverse in $k[[T]]$, by reducing to the case where $f_0 = 1$ and applying the remarks in the preceding paragraph. Conversely, if $f(T)$ has a multiplicative inverse in $k[[T]]$, then $f_0 \ne 0$. This follows from the fact that

$$f(T) \mapsto f_0 \tag{2.11}$$

is an algebra homomorphism from $k[[T]]$ onto k.

Let $\mathcal{A}$ be an algebra over k, and suppose that $\mathcal{A}$ has a multiplicative identity element e. If $a \in \mathcal{A}$ and $f(T) \in k[T]$ is as in (2.1), then $f(a)$ can be defined as an element of $\mathcal{A}$ by

$$f(a) = \sum_{j=0}^{n} f_j\,a^j, \tag{2.12}$$

where a^j is interpreted as being equal to e when $j = 0$, as usual. One can verify that the mapping $f(T) \mapsto f(a)$ defines an algebra homomorphism from $k[T]$ into $\mathcal{A}$.

3 Laurent polynomials and series

Let k be a field again, and let T be an indeterminate. A *formal Laurent polynomial* in T with coefficients in k can be expressed as

$$f(T) = \sum_{j=-n}^{n} f_j\,T^j, \tag{3.1}$$

where n is a nonnegative integer, and $f_j \in k$ for each integer j with $|j| \le n$. A *formal Laurent series* in T with coefficients in k can be expressed as

$$f(T) = \sum_{j=-\infty}^{\infty} f_j \, T^j, \tag{3.2}$$

where $f_j \in k$ for each integer j. Thus a formal Laurent polynomial may be considered as a formal Laurent series for which all but finitely many coefficients are equal to 0. Formal polynomials and power series in T may be considered as formal Laurent polynomials and series, respectively, for which the coefficients of T^j are equal to 0 when $j < 0$.

The space of formal Laurent series in T with coefficients in k can be identified with the space $c(\mathbf{Z}, k)$ of all k-valued functions on the set $\mathbf{Z}$ of integers, by identifying (3.2) with $j \mapsto f_j$ as a k-valued function on $\mathbf{Z}$. Similarly, the space of formal Laurent polynomials in T with coefficients in k can be identified with the space $c_{00}(\mathbf{Z}, k)$ of all k-valued functions on $\mathbf{Z}$ with finite support. The space of formal Laurent series in T with coefficients in k is a vector space over k with respect to termwise addition and scalar multiplication, which corresponds to pointwise addition and scalar multiplication of k-valued functions on $\mathbf{Z}$. The space of formal Laurent polynomials in T with coefficients in k is a linear subspace of the space of formal Laurent series in T with coefficients in k, just as $c_{00}(\mathbf{Z}, k)$ is a linear subspace of $c(\mathbf{Z}, k)$.

Put

$$c^+(\mathbf{Z}, k) = \{f \in c(\mathbf{Z}, k) : \operatorname{supp} f \subseteq \mathbf{Z}_+ \cup \{0\}\}, \tag{3.3}$$

which is a linear subspace of $c(\mathbf{Z}, k)$. We can identify $c(\mathbf{Z}_+ \cup \{0\}, k)$ with (3.3), by identifying a k-valued function on $\mathbf{Z}_+ \cup \{0\}$ with the element of (3.3) whose restriction to $\mathbf{Z}_+ \cup \{0\}$ is the given function. This corresponds to identifying formal power series in T with coefficients in k with formal Laurent series in T with coefficients in k for which the coefficients of T^j are equal to 0 when $j < 0$, as before. In particular, $k[[T]]$ corresponds to a linear subspace of the space of formal Laurent series in T with coefficients in k.

Similarly, put

$$c_{00}^+(\mathbf{Z}, k) = \{f \in c_{00}(\mathbf{Z}, k) : \operatorname{supp} f \subseteq \mathbf{Z}_+ \cup \{0\}\} = c_{00}(\mathbf{Z}, k) \cap c^+(\mathbf{Z}, k), \tag{3.4}$$

which is a linear subspace of $c_{00}(\mathbf{Z}, k)$. Using the identification of $c(\mathbf{Z}_+ \cup \{0\}, k)$ with (3.3) mentioned in the preceding paragraph, we get an identification of $c_{00}(\mathbf{Z}_+ \cup \{0\}, k)$ with (3.4). This corresponds to identifying formal polynomials in T with coefficients in k with formal Laurent polynomials in T with coefficients in k for which the coefficients of T^j are equal to 0 when $j < 0$. Thus $k[T]$ corresponds to a linear subspace of the space of formal Laurent polynomials in T with coefficients in k, which is the intersection of $k[[T]]$ with the space of formal Laurent polynomials in T with coefficients in k.

Let $f(T)$ be a formal Laurent series in T with coefficients in k, as in (3.2). If $f_j = 0$ for all but finitely many $j < 0$, then we may express $f(T)$ as

$$f(T) = \sum_{j >> -\infty} f_j \, T^j, \tag{3.5}$$

as in [2]. The space of formal Laurent series in T with coefficients in k of this type is denoted $k((T))$. This is a linear subspace of the space of formal Laurent series in T with coefficients in k, which corresponds to the linear subspace of $c(\mathbf{Z}, k)$ consisting of k-valued functions on $\mathbf{Z}$ that are equal to 0 at all but finitely many negative integers. Note that $k[[T]]$ and the space of formal Laurent polynomials in T with coefficients in k are linear subspaces of $k((T))$.

Let (3.5) and

$$g(T) = \sum_{l>>-\infty} g_l\, T^l \tag{3.6}$$

be elements of $k((T))$. Their product

$$f(T)\, g(T) = h(T) = \sum_{n=-\infty}^{\infty} h_n\, T^n \tag{3.7}$$

is defined as a formal Laurent series in T with coefficients in k by putting

$$h_n = \sum_{j=-\infty}^{\infty} f_j\, g_{n-j} \tag{3.8}$$

for each $n \in \mathbf{Z}$. More precisely, one can check that all but finitely many terms in the sum on the right side of (3.8) are equal to 0 in this situation, so that the sum defines an element of k. One can also verify that $h_n = 0$ for all but finitely many $n < 0$, so that (3.7) is an element of $k((T))$ too. Similarly, if $f(T)$ and $g(T)$ are formal Laurent polynomials in T, then (3.7) is a formal Laurent polynomial in T.

As before, one can check that $k((T))$ is a commutative algebra over k with respect to this definition of multiplication. The space of formal Laurent polynomials in T with coefficients in k is a subalgebra of $k((T))$. The product (3.7) reduces to the one defined in the previous section when $f(T)$ and $g(T)$ are formal power series in T, so that $k[[T]]$ corresponds to a subalgebra of $k((T))$ as well. Each element of k can be identified with the "constant" formal Laurent polynomial for which the coefficient of T^j is equal to the given element of k when $j = 0$, and is equal to 0 when $j \neq 0$. This identifies k with a subalgebra of $k((T))$, which corresponds to the subalgebra of $k[T]$ mentioned in the previous section.

In particular, the constant formal Laurent series corresponding to the multiplicative identity element 1 in k is the multiplicative identity element in $k((T))$. It is easy to see that every nonzero element $f(T)$ of $k((T))$ has a multiplicative inverse in $k((T))$, using the description of multiplicative inverses in $k[[T]]$ in the previous section. Thus $k((T))$ is a field, which contains a copy of k.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e, and let a be an invertible element of $\mathcal{A}$. If $f(T)$ is a formal Laurent polynomial in T with coefficients in k, as in (3.1), then

$$f(a) = \sum_{j=-n}^{n} f_j\, a^j \tag{3.9}$$

defines an element of $\mathcal{A}$. Of course, this reduces to (2.12) when $f(T) \in k[T]$. One can check that the mapping $f(T) \mapsto f(a)$ defines a homomorphism from the algebra of formal Laurent polynomials in T with coefficients in k into $\mathcal{A}$, as before.

4 Some inequalities

Let X be a nonempty set, and let f be a nonnegative real-valued function on X with finite support. Put

$$\|f\|_r = \Big(\sum_{x\in X} f(x)^r\Big)^{1/r} \tag{4.1}$$

for every positive real number r, and

$$\|f\|_\infty = \max_{x\in X} f(x). \tag{4.2}$$

Observe that

$$\|f\|_\infty \le \|f\|_r \tag{4.3}$$

for every $0 < r < \infty$. If $0 < r_1 \le r_2 < \infty$, then

$$\|f\|_{r_2}^{r_2} = \sum_{x\in X} f(x)^{r_2} \le \|f\|_\infty^{r_2 - r_1} \sum_{x\in X} f(x)^{r_1} = \|f\|_\infty^{r_2-r_1}\,\|f\|_{r_1}^{r_1}. \tag{4.4}$$

This implies that

$$\|f\|_{r_2} \le \|f\|_\infty^{1-(r_1/r_2)}\,\|f\|_{r_1}^{r_1/r_2} \le \|f\|_{r_1}, \tag{4.5}$$

using (4.3) in the second step. If $0 < r < \infty$, then it is easy to see that

$$\|f\|_r \le (\#\operatorname{supp} f)^{1/r}\,\|f\|_\infty, \tag{4.6}$$

where $\#\operatorname{supp} f$ is the number of elements of the support of f. It follows that

$$\lim_{r\to\infty} \|f\|_r = \|f\|_\infty, \tag{4.7}$$

using (4.2) and (4.6).

If a and b are nonnegative real numbers, then

$$\max(a,b) \le (a^r + b^r)^{1/r} \le 2^{1/r}\,\max(a,b). \tag{4.8}$$

This corresponds to (4.3) and (4.6), where the support of f has at most two elements. Hence

$$\lim_{r\to\infty} (a^r + b^r)^{1/r} = \max(a,b), \tag{4.9}$$

as in (4.7). Similarly, if $0 < r_1 \le r_2 < \infty$, then

$$(a^{r_2} + b^{r_2})^{1/r_2} \le (a^{r_1} + b^{r_1})^{1/r_1}, \tag{4.10}$$

by (4.5). This implies that

$$(a+b)^r \le a^r + b^r \tag{4.11}$$

when $0 < r \le 1$, by taking $r_1 = r$ and $r_2 = 1$ in (4.10).

Let f, g be nonnegative real-valued functions on X with finite support. If $0 < r \le 1$, then

$$\|f+g\|_r^r = \sum_{x \in X} (f(x)+g(x))^r \le \sum_{x \in X} f(x)^r + \sum_{x \in X} g(x)^r = \|f\|_r^r + \|g\|_r^r. \tag{4.12}$$

This uses (4.11) in the second step. If $1 \le r \le \infty$, then it is well known that

$$\|f+g\|_r \le \|f\|_r + \|g\|_r. \tag{4.13}$$

This is *Minkowski's inequality* for finite sums. Of course, (4.12) and (4.13) are the same when $r = 1$, in which case the inequalities are clearly equalities. It is also easy to verify (4.13) directly when $r = \infty$.

5 q-Metrics and q-semimetrics

Let X be a set, and let q be a positive real number. A nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ is said to be a *q-semimetric* on X if it satisfies the following three conditions. First,

$$d(x, x) = 0 \quad \text{for every } x \in X. \tag{5.1}$$

Second,

$$d(x, y) = d(y, x) \quad \text{for every } x, y \in X. \tag{5.2}$$

Third,

$$d(x, z)^q \le d(x, y)^q + d(y, z)^q \quad \text{for every } x, y, z \in X. \tag{5.3}$$

If we also have that

$$d(x, y) > 0 \quad \text{for every } x, y \in X \text{ with } x \ne y, \tag{5.4}$$

then $d(x, y)$ is said to be a *q-metric* on X. A q-metric or q-semimetric with $q = 1$ is also known simply as a *metric* or *semimetric*, as appropriate.

A nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ is said to be a *semi-ultrametric* on X if it satisfies (5.1), (5.2), and

$$d(x, z) \le \max(d(x, y), d(y, z)) \quad \text{for every } x, y, z \in X. \tag{5.5}$$

If (5.4) also holds, then $d(x, y)$ is said to be an *ultrametric* on X.

Observe that (5.3) is equivalent to saying that

$$d(x, z) \le (d(x, y)^q + d(y, z)^q)^{1/q} \quad \text{for every } x, y \in X. \tag{5.6}$$

The right side of of the inequality in (5.5) is the same as the limit of the right side of the inequality in (5.6) as $q \to \infty$, as in (4.9). Thus we may consider ultrametrics and semi-ultrametrics as q-metrics and q-semimetrics with $q = \infty$, respectively.

The right side of the inequality in (5.6) decreases monotonically in q, as in (4.10). This includes the right side of the inequality in (5.5) as the analogue of the right side of the inequality in (5.6) with $q = \infty$, by the first inequality in (4.8). If $0 < q_1 \le q_2 \le \infty$ and $d(x, y)$ is a q_2-semimetric on X, then it follows that $d(x, y)$ is a q_1-semimetric on X as well. Of course, this implies the analogous statement for q-metrics.

The *discrete metric* is defined on X by putting $d(x, y)$ equal to 1 when $x \ne y$, and to 0 when $x = y$, as usual. It is easy to see that the discrete metric is an ultrametric on X.

Let $d(x, y)$ be a nonnegative real-valued function defined for $x, y \in X$ again, and let a be a positive real number. It is easy to see that $d(x, y)$ is a q-semimetric on X for some $q > 0$ if and only if

$$d(x, y)^a \tag{5.7}$$

is a (q/a)-semimetric on X. Similarly, $d(x, y)$ is a q-metric on X if and only if (5.7) is a (q/a)-metric on X. This includes the $q = \infty$ case, with q/a interpreted as being ∞ too.

Let $d(x, y)$ be a q-semimetric on a set X for some $q > 0$. The *open ball* in X centered at a point $x \in X$ and with radius $r > 0$ is defined as usual by

$$B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}. \tag{5.8}$$

Similarly, the *closed ball* in X centered at $x \in X$ with radius $r \ge 0$ is defined by

$$\overline{B}(x, r) = \overline{B}_d(x, r) = \{y \in X : d(x, y) \le r\}. \tag{5.9}$$

If a is a positive real number, then

$$B_{d^a}(x, r^a) = B_d(x, r) \tag{5.10}$$

for every $x \in X$ and $r > 0$, and

$$\overline{B}_{d^a}(x, r^a) = \overline{B}_d(x, r) \tag{5.11}$$

for every $x \in X$ and $r \ge 0$.

The topology determined on X by $d(\cdot, \cdot)$ is defined as usual by saying that $U \subseteq X$ is an *open set* if for every $x \in U$ there is an $r > 0$ such that $B_d(x, r) \subseteq U$. It is easy to see that an open set with respect to $d(\cdot, \cdot)$ is the same as an open set with respect to (5.7) for any $a > 0$, because of (5.11). In particular, one can use this to reduce to the case of ordinary semimetrics, by taking $a = q$ when $q < 1$. One can check that this collection of open sets defines a topology on X, for which open balls are open sets, and closed balls are closed sets. If $d(\cdot, \cdot)$ is a q-metric on X, then X is Hausdorff with respect to this topology.

If $d(\cdot, \cdot)$ is a semi-ultrametric on X, then one can check that open balls are closed sets, and that closed balls in X of positive radius are open sets.

One can define uniform continuity of mappings with respect to q-semimetrics and q-metrics in the same way as for ordinary semimetrics and metrics, and with possibly different q's for the domain and range. Note that uniform continuity is preserved when the q-semimetrics or q-metrics on the domain or range are replaced with positive powers of themselves, as in (5.7).

Similarly, Cauchy sequences and completeness can be defined with respect to a q-metric, in the same way as for an ordinary metric. These are also preserved when the q-metric is replaced with a positive power of itself, as in (5.7).

6 q-Absolute value functions

Let k be a field, and let q be a positive real number. A nonnegative real-valued function $|\cdot|$ defined on k is said to be a *q-absolute value function* if it satisfies the following three conditions. First,

$$|x| = 0 \quad \text{if and only if} \quad x = 0. \tag{6.1}$$

Second,

$$|x\,y| = |x|\,|y| \quad \text{for every } x, y \in k. \tag{6.2}$$

Third,

$$|x + y|^q \le |x|^q + |y|^q \quad \text{for every } x, y \in k. \tag{6.3}$$

If these conditions hold with $q = 1$, then $|\cdot|$ is said to be an *absolute value function* on k. The standard absolute value functions on the fields $\mathbf{R}$ and $\mathbf{C}$ of real and complex numbers are absolute value functions in this sense.

If a nonnegative real-valued function $|\cdot|$ on a field satisfies (6.1) and (6.2), then it is easy to see that $|1| = 1$, where the first 1 is the multiplicative identity element in k, and the second 1 is the multiplicative identity element in $\mathbf{R}$. This uses the fact that $1^2 = 1$ in k, so that $0 < |1| = |1|^2$. Similarly, if $x \in k$ satisfies $x^n = 1$ for some positive integer n, then $|x| = 1$, because $|x|^n = |x^n| = 1$. In particular, $|-1| = 1$, where -1 is the additive inverse of 1 in k, because $(-1)^2 = 1$.

If $|\cdot|$ is a q-absolute value function on k for some $q > 0$, then

$$d(x, y) = |x - y| \tag{6.4}$$

defines a q-metric on k. This uses the fact that $|-1| = 1$ to get that (6.4) is symmetric in x and y.

A nonnegative real-valued function $|\cdot|$ defined on a field k is said to be an *ultrametric absolute value function* on k if it satisfies (6.1), (6.2), and

$$|x + y| \le \max(|x|, |y|) \quad \text{for every } x, y \in k. \tag{6.5}$$

In this case, (6.4) defines an ultrametric on k. The *trivial absolute value function* is defined on any field k by putting $|0| = 0$ and $|x| = 1$ for every $x \in k$ with $x \neq 0$.

This is an ultrametric absolute value function on k, for which the corresponding ultrametric is the discrete metric.

Note that (6.3) is equivalent to asking that

$$|x+y| \le (|x|^q + |y|^q)^{1/q} \quad \text{for every } x, y \in k. \tag{6.6}$$

The right side of this inequality tends to the right side of the inequality in (6.5) as $q \to \infty$, as in (4.9). An ultrametric absolute value function on k may also be considered as a q-absolute value function with $q = \infty$. If $0 < q_1 \le q_2 \le \infty$ and $|\cdot|$ is a q_2-absolute value function on k, then one can check that $|\cdot|$ is a q_1-absolute value function on k, using (4.8) and (4.10).

If p is a prime number, then the *p-adic absolute value* $|x|_p$ of a rational number x is defined as follows. Of course, we put $|0|_p = 0$. Otherwise, if $x \ne 0$, then x can be expressed as $p^j\,(a/b)$, where a, b, and j are integers, $a, b \ne 0$, and neither a nor b is an integer multiple of p, and we put

$$|x|_p = p^{-j}. \tag{6.7}$$

One can check that this defines an ultrametric absolute value function on the field $\mathbf{Q}$ of rational numbers. The corresponding ultrametric

$$d_p(x, y) = |x - y|_p \tag{6.8}$$

is known as the *p-adic metric* on $\mathbf{Q}$.

Let $|\cdot|$ be a q-absolute value function on a field k for some $q > 0$. If k is not already complete with respect to the associated q-metric (6.4), then one can pass to a completion, using standard arguments. The completion is also a field with a q-absolute value function, which contains k as a dense subfield. The completion is unique up to suitable isomorphic equivalence.

If p is a prime number, then the field $\mathbf{Q}_p$ of *p-adic numbers* is obtained by completing $\mathbf{Q}$ with respect to the p-adic metric, as in the preceding paragraph. The natural extensions of the p-adic absolute value and metric to $\mathbf{Q}_p$ are denoted in the same way as on $\mathbf{Q}$.

Let $|\cdot|$ be a nonnegative real-valued function on a field k again, and let a be a positive real number. One can check that $|\cdot|$ is a q-absolute value function on k for some $q > 0$ if and only if

$$|\cdot|^a \tag{6.9}$$

is a (q/a)-absolute value function. In this case, the (q/a)-metric associated to (6.9) is the same as the ath power of the q-metric (6.4) associated to $|\cdot|$. In particular, these (q/a) and q-metrics determine the same topology on k, as in the previous section.

Let $|\cdot|_1$ and $|\cdot|_2$ be q_1 and q_2-absolute value functions on k, for some $q_1, q_2 > 0$. If there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{6.10}$$

for every $x \in k$, then $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* on k. This implies that the topologies determined on k by the associated q_1 and q_2-metrics are the same, as in the preceding paragraph. Conversely, it is well known that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k when the topologies determined on k by the associated q_1 and q_2-metrics are the same. Note that one can reduce to the case where $q_1 = q_2 = 1$, by replacing $|\cdot|_1$ with $|\cdot|_1^{q_1}$ when $q_1 \le 1$, and similarly for $|\cdot|_2$.

7 q-Norms and q-seminorms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *q-seminorm* on V with respect to $|\cdot|$ on k for some positive real number q if it satisfies the following two conditions. First,

$$N(t\,v) = |t|\,N(v) \quad \text{for every } t \in k \text{ and } v \in V. \tag{7.1}$$

Second,

$$N(v+w)^q \le N(v)^q + N(w)^q \quad \text{for every } v, w \in V. \tag{7.2}$$

Note that (7.1) implies that $N(0) = 0$, by taking $t = 0$. If N also satisfies

$$N(v) > 0 \quad \text{for every } v \in V \text{ with } v \ne 0, \tag{7.3}$$

then N is said to be a *q-norm* on V with respect to $|\cdot|$ on k. As usual, we may simply use the terms *norm* and *seminorm* when $q = 1$.

If a nonnegative real-valued function N on V satisfies (7.1) and

$$N(v+w) \le \max(N(v), N(w)) \quad \text{for every } v, w \in V, \tag{7.4}$$

then N is said to be a *semi-ultranorm* on V with respect to $|\cdot|$ on k. If N also satisfies (7.3), then N is said to be an *ultranorm* on V with respect to $|\cdot|$ on k.

As before, (7.2) is equivalent to saying that

$$N(v+w) \le (N(v)^q + N(w)^q)^{1/q} \quad \text{for every } v, w \in V. \tag{7.5}$$

An ultranorm or semi-ultranorm on V may be considered as a q-norm or q-seminorm, as appropriate, with $q = \infty$, because the right side of the inequality in (7.5) tends to the right side of the inequality in (7.4) as $q \to \infty$, by (4.9). If $0 < q_1 \le q_2 \le \infty$ and N is a q_2-norm or q_2-seminorm on V with respect to $|\cdot|$ on k, then N is a q_1-norm or q_1-seminorm on V too, as appropriate, by (4.8) and (4.10).

If N is a q-norm or q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$, then

$$d(v,w) = d_N(v,w) = N(v-w) \tag{7.6}$$

defines a q-metric or q-semimetric on V, as appropriate. Note that $|\cdot|$ may be considered as a q_k-norm on k with respect to itself, and where k is considered as a one-dimensional vector space over itself.

Let a be a positive real number, and remember that $|\cdot|^a$ defines a (q_k/a)-absolute value function on k, as in the previous section. One can check that a nonnegative real-valued function N on V is a q-norm or q-seminorm with respect to $|\cdot|$ on k if and only if

$$N(v)^a \tag{7.7}$$

is a (q/a)-norm or (q/a)-seminorm on V with respect to $|\cdot|^a$ on k, as appropriate. Of course,

$$d_{N^a}(v,w) = N(v-w)^a = d_N(v,w)^a \tag{7.8}$$

for every $v, w \in V$ in this situation.

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k. In this case, the *trivial ultranorm* is defined on V by putting $N(0) = 0$ and $N(v) = 1$ for every $v \in V$ with $v \neq 0$. It is easy to see that this defines an ultranorm on V, for which the associated ultrametric as in (7.6) is the discrete metric on V.

Let $|\cdot|$ be any q_k-absolute value function on k again, and suppose that N is a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$. If $N(v) > 0$ for some $v \in V$, then one can check that $|\cdot|$ is a q-absolute value function on k.

8 Absolute values on $k((T))$

Let k be a field, let T be an indeterminate, and remember that $k((T))$ denotes the space of formal Laurent series in T with coefficients in k such that the coefficient of T^j is equal to 0 for all but finitely many $j < 0$, as in Section 3. If

$$f(T) = \sum_{j >> -\infty} f_j\, T^j \tag{8.1}$$

is a nonzero element of $k((T))$, then let $j_0(f(T))$ be the unique integer such that $f_{j_0(f(T))} \neq 0$ and $f_j = 0$ when $j < j_0(f(T))$. Otherwise, let us take $j_0(f(T)) = \infty$ when $f(T) = 0$. If

$$g(T) = \sum_{j >> -\infty} g_j\, T^j \tag{8.2}$$

is another element of $k((T))$, then it is easy to see that

$$j_0(f(T) + g(T)) \geq \min(j_0(f(T)), j_0(g(T))) \tag{8.3}$$

and

$$j_0(f(T)\, g(T)) = j_0(f(T)) + j_0(g(T)), \tag{8.4}$$

with suitable interpretations when any of these terms is ∞.

Let r be a positive real number with $r \leq 1$. If $f(T) \in k((T))$, then put

$$|f(T)|_r = r^{-j_0(f(T))} \tag{8.5}$$

when $f(T) \neq 0$, and $|0|_r = 0$. One can check that $|\cdot|_r$ defines an ultrametric absolute value function on the field $k((T))$, using (8.3) and (8.4). If $r = 1$, then $|\cdot|_r$ is the same as the trivial absolute value function on $k((T))$. If a is a positive real number, then $0 < r^a \leq 1$, and

$$|f(T)|_{r^a} = |f(T)|_r^a \tag{8.6}$$

for every $f(T) \in k((T))$.

Let us suppose from now on in this section that $r < 1$. If $l \in \mathbf{Z}$, then $f(T) \in k((T))$ satisfies

$$|f(T)|_r \leq r^{-l} \tag{8.7}$$

if and only if $j_0(f(T)) \geq l$, which is the same as saying that

$$f_j = 0 \quad \text{when } j < l. \tag{8.8}$$

Let $T^l\, k[[T]]$ be the collection of formal Laurent series in T with coefficients in k that can be expressed as T^l times a formal power series in T with coefficients in k. This is the same as the collection of $f(T) \in k((T))$ that satisfy (8.7), or equivalently (8.8).

A subset of $k((T))$ is bounded with respect to $|\cdot|_r$ when there is a finite upper bound for $|f(T)|_r$ for elements $f(T)$ of the set. This is the same as saying that the set is contained in $T^{l_0}\, k[[T]]$ for some $l_0 \in \mathbf{Z}$, by the remarks in the preceding paragraph. A sequence of elements of $k((T))$ is said to be bounded with respect to $|\cdot|_r$ when the set of terms of the sequence is bounded with respect to $|\cdot|_r$. If a sequence of elements of $k((T))$ converges to an element of $k((T))$ with respect to the ultrametric associated to $|\cdot|_r$, then that sequence is bounded with respect to $|\cdot|_r$, by standard arguments. Similarly, Cauchy sequences in $k((T))$ with respect to the ultrametric associated to $|\cdot|_r$ are bounded with respect to $|\cdot|_r$.

As before, $f(T), g(T) \in k((T))$ satisfy

$$|f(T) - g(T)|_r \leq r^{-l} \tag{8.9}$$

for some $l \in \mathbf{Z}$ if and only if

$$f_j = g_j \quad \text{when } j < l. \tag{8.10}$$

Given a sequence of elements of $k((T))$, we get for each $j \in \mathbf{Z}$ a corresponding sequence of coefficients of T^j in k. Let us say that the sequence is termwise eventually constant if for each $j \in \mathbf{Z}$, the corresponding sequence of coefficients of T^j in k is eventually constant. Similarly, let us say that the sequence converges termwise to $f(T) \in k((T))$ if for each $j \in \mathbf{Z}$, the terms of the corresponding sequence of coefficients of T^j in k are eventually equal to the coefficient f_j or T^j in $f(T)$.

If a sequence in $k((T))$ is a Cauchy sequence with respect to the ultrametric associated to $|\cdot|_r$, then the sequence is termwise eventually constant. If a sequence in $k((T))$ converges to $f(T) \in k((T))$ with respect to the ultrametric associated to $|\cdot|_r$, then the sequence converges termwise to $f(T)$. In both cases, the converse holds when the sequence is bounded with respect to $|\cdot|_r$.

If a sequence of elements of $k((T))$ is termwise eventually constant, then the eventual constant values of the coefficients of T^j for each $j \in \mathbf{Z}$ can be used to define a formal Laurent series $f(T)$ in T with coefficients in k. If the sequence in $k((T))$ is also bounded with respect to $|\cdot|_r$, then $f(T) \in k((T))$. It follows that $k((T))$ is complete with respect to the ultrametric associated to $|\cdot|_r$.

9 q-Banach spaces and infinite series

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k with a q-norm N with respect to $|\cdot|$ on k for some $q > 0$. If V is complete with respect to the q-metric associated to N, then V is said to be a *q-Banach space* with respect to N. If $q = 1$, then we may simply say that V is a *Banach space*. If V is not complete, then one can pass to a completion to get a q-Banach space, using standard arguments. The completion is unique up to suitable isometric linear equivalence.

As usual, an infinite series

$$\sum_{j=1}^{\infty} v_j \tag{9.1}$$

with terms in V is said to *converge* with respect to N if the corresponding sequence of partial sums

$$\sum_{j=1}^{n} v_j \tag{9.2}$$

converges to an element of V with respect to the q-metric associated to N. In this case, the value of the sum (9.1) is defined to be the limit of the partial sums (9.2). If (9.1) converges and $t \in k$, then $\sum_{j=1}^{\infty} t\, v_j$ converges in V too, with

$$\sum_{j=1}^{\infty} t\, v_j = t \sum_{j=1}^{\infty} v_j, \tag{9.3}$$

by standard arguments. Similarly, if $\sum_{j=1}^{\infty} w_j$ is another convergent series with terms in V, then $\sum_{j=1}^{\infty}(v_j + w_j)$ converges in V as well, with

$$\sum_{j=1}^{\infty}(v_j + w_j) = \sum_{j=1}^{\infty} v_j + \sum_{j=1}^{\infty} w_j. \tag{9.4}$$

The sequence (9.2) of partial sums is a Cauchy sequence in V with respect to the q-metric associated to N if and only if for every $\epsilon > 0$ there is a positive integer L such that

$$N\Big(\sum_{j=l}^{n} v_j\Big) < \epsilon \tag{9.5}$$

for every $n \geq l \geq L$. In particular, this implies that

$$\lim_{j \to \infty} N(v_j) = 0, \tag{9.6}$$

by taking $l = n$ in (9.5). Of course, if (9.1) converges in V, then the sequence of partial sums (9.2) is a Cauchy sequence with respect to the q-metric associated to N, because convergent sequences are Cauchy sequences.

Suppose for the moment that $q < \infty$. An infinite series (9.1) with terms in V is said to converge *q-absolutely* with respect to N if

$$\sum_{j=1}^{\infty} N(v_j)^q \tag{9.7}$$

converges as an infinite series of nonnegative real numbers. We may refer to this as *absolute convergence* with respect to N when $q = 1$. Observe that

$$N\Big(\sum_{j=l}^{n} v_j\Big)^q \le \sum_{j=l}^{n} N(v_j)^q \tag{9.8}$$

for every $n \ge l \ge 1$, by the q-norm version of the triangle inequality. If (9.1) converges q-absolutely with respect to N, then one can use (9.8) to verify that the Cauchy condition (9.5) holds. If V is a q-Banach space with respect to N, then it follows that (9.1) converges in V. One can also check that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big)^q \le \sum_{j=1}^{\infty} N(v_j)^q \tag{9.9}$$

under these conditions, using (9.8).

Suppose now that $q = \infty$, so that N is an ultranorm on V. Thus

$$N\Big(\sum_{j=l}^{n} v_j\Big) \le \max_{l \le j \le n} N(v_j) \tag{9.10}$$

for every $n \ge l \ge 1$, by the ultranorm version of the triangle inequality. If (9.6) holds, then it is easy to see that the Cauchy condition (9.5) holds too, using (9.10). This implies that (9.1) converges in V when V is a Banach space with respect to N. In this situation, one can verify that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \max_{j \ge 1} N(v_j), \tag{9.11}$$

using (9.10). Note that the maximum on the right side of (9.11) is automatically attained, because of (9.6). This is trivial when $v_j = 0$ for every $j \ge 1$, and otherwise the maximum can be reduced to finitely many terms.

10 Bounded linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k. Also let N_V, N_W be q_V, q_W-seminorms on V,

W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. A linear mapping T from V into W is said to be *bounded* with respect to N_V, N_W if there is a nonnegative real number C such that

$$N_W(T(v)) \le C\, N_V(v) \tag{10.1}$$

for every $v \in V$. This implies that

$$N_W(T(u) - T(v)) = N_W(T(u - v)) \le C\, N_V(u - v) \tag{10.2}$$

for every $u, v \in V$, so that T is uniformly continuous with respect to the q_V, q_W-semimetrics associated to N_V, N_W on V, W, respectively.

Let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from V into W with respect to N_V, N_W. If $T \in \mathcal{BL}(V, W)$, then put

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \ge 0 : (10.1) \text{ holds}\}, \tag{10.3}$$

where more precisely the infimum is taken over all nonnegative real numbers C for which (10.1) holds. It is easy to see that (10.1) holds with $C = \|T\|_{op}$, so that the infimum is automatically attained. One can check that $\mathcal{BL}(V, W)$ is a vector space over k with respect to pointwise addition and scalar multiplication, and that (10.3) is a q_W-seminorm on $\mathcal{BL}(V, W)$ with respect to $|\cdot|$ on k.

Let Z be another vector space over k, and let N_Z be a q_Z-seminorm on Z with respect to $|\cdot|$ on k, for some $q_Z > 0$. If T_1 is a bounded linear mapping from V into W with respect to N_V, N_W, and T_2 is a bounded linear mapping from W into Z with respect to N_W, N_Z, then their composition $T_2 \circ T_1$ is a bounded linear mapping from V into Z with respect to N_V, N_Z. More precisely, if $v \in V$, then

$$\begin{aligned} N_Z((T_2 \circ T_1)(v)) = N_Z(T_2(T_1(v))) &\le \|T_2\|_{op,WZ}\, N_W(T_1(v)) \\ &\le \|T_1\|_{op,VW}\, \|T_2\|_{op,WZ}\, N_V(v). \end{aligned} \tag{10.4}$$

This implies that $T_2 \circ T_1$ is bounded, with

$$\|T_2 \circ T_1\|_{op,VZ} \le \|T_1\|_{op,VW}\, \|T_2\|_{op,WZ}. \tag{10.5}$$

Let us suppose from now on in this section that N_W is a q_W-norm on W. This implies that (10.3) is a q_W-norm on $\mathcal{BL}(V, W)$. If W is also complete with respect to the q_W-metric associated to N_W, then one can check that $\mathcal{BL}(V, W)$ is complete with respect to the q_W-metric associated to (10.3), using standard arguments.

Suppose that N_V is a q_V-norm on V, and let V_0 be a linear subspace of V that is dense in V with respect to the q_V-metric associated to N_V. Note that the restriction of N_V to V_0 defines a q_V-norm on V_0 with respect to $|\cdot|$ on k. Let T_0 be a bounded linear mapping from V_0 into W, with respect to the restriction of N_V to V_0, and N_W on W. In particular, T_0 is uniformly continuous with respect to the restriction of the q_V-metric associated to N_V to V_0, and the q_W-metric associated to N_W on W. If W is complete with respect to the q_W-metric associated to N_W, then there is a unique extension of T_0 to a uniformly continuous mapping from V into W. This follows from a well-known extension theorem for uniformly continuous mappings between metric spaces, which is easily extended to q-metric spaces. In this situation, the extension is a bounded linear mapping from V into W, with the same operator q_W-norm as T_0 has on V_0.

11 Submultiplicative q-seminorms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k. Also let N be a q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. If

$$N(x\,y) \le N(x)\,N(y) \tag{11.1}$$

for every $x, y \in \mathcal{A}$, then N is said to be *submultiplicative* on $\mathcal{A}$. Similarly, N is said to be *multiplicative* on $\mathcal{A}$ if

$$N(x\,y) = N(x)\,N(y) \tag{11.2}$$

for every $x, y \in \mathcal{A}$.

Suppose that $\mathcal{A}$ has a multiplicative identity element e. If N is submultiplicative on $\mathcal{A}$, then $N(x) \le N(e)\,N(x)$ for every $x \in \mathcal{A}$. If $N(x) > 0$ for some $x \in \mathcal{A}$, then it follows that

$$N(e) \ge 1. \tag{11.3}$$

Similarly, if N is multiplicative on $\mathcal{A}$, then $N(x) = N(e)\,N(x)$ for every $x \in \mathcal{A}$. If $N(x) > 0$ for some $x \in \mathcal{A}$, then we get that

$$N(e) = 1. \tag{11.4}$$

Suppose that N is a submultiplicative q-norm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. If $\mathcal{A}$ is complete with respect to the q-metric associated to N, then $\mathcal{A}$ is said to be a *q-Banach algebra* with respect to N. As usual, one may simply say that $\mathcal{A}$ is a *Banach algebra* when $q = 1$. If $\mathcal{A}$ is not complete, then one can pass to a completion, by standard arguments, which is unique up to a suitable isometric isomorphic equivalence. Sometimes the requirement that $\mathcal{A}$ have a multiplicative identity element e with $N(e) = 1$ is included in the definition of a q-Banach algebra.

Let V be a vector space over k, and let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k. Consider the space $\mathcal{BL}(V) = \mathcal{BL}(V,V)$ of bounded linear mappings from V into itself, using N_V on the domain and range. This is an algebra over k, with composition of mappings as multiplication. The corresponding operator q_V-seminorm $\|\cdot\|_{op} = \|\cdot\|_{op,VV}$ defined in the previous section is submultiplicative on $\mathcal{BL}(V)$, as in (10.5). The identity mapping $I = I_V$ on V is bounded with respect to N_V, with

$$\|I\|_{op} = 1 \tag{11.5}$$

when $N(v) > 0$ for some $v \in V$, and $\|I\|_{op} = 0$ otherwise. Of course, I is the multiplicative identity element in $\mathcal{BL}(V)$. If N_V is a q_V-norm on V, and V is complete with respect to the q_V-metric associated to N_V, then $\|\cdot\|_{op}$ is a q_V-norm on $\mathcal{BL}(V)$, and $\mathcal{BL}(V)$ is complete with respect to the associated q_V-metric, as in the previous section. This means that $\mathcal{BL}(V)$ is a q_V-Banach algebra with respect to $\|\cdot\|_{op}$, at least if $V \neq \{0\}$, so that (11.5) holds, if that is included in the definition of a q-Banach algebra.

12 Multiplication operators

Let k be a field, and let $\mathcal{A}$ be an algebra over k. If $a \in \mathcal{A}$, then put

$$M_a(x) = a\,x \tag{12.1}$$

for every $x \in \mathcal{A}$. Thus M_a is a linear mapping from $\mathcal{A}$ into itself, where $\mathcal{A}$ is considered simply as a vector space over k. If $b \in \mathcal{A}$ too, then

$$(M_a \circ M_b)(x) = M_a(M_b(x)) = M_a(b\,x) = a\,b\,x = M_{a\,b}(x) \tag{12.2}$$

for every $x \in \mathcal{A}$. This means that

$$M_a \circ M_b = M_{a\,b} \tag{12.3}$$

as linear mappings from $\mathcal{A}$ into itself. It is easy to see that

$$a \mapsto M_a \tag{12.4}$$

is linear as a mapping from $\mathcal{A}$ into the algebra $\mathcal{L}(\mathcal{A})$ of linear mappings from $\mathcal{A}$ into itself, where $\mathcal{A}$ is considered as a vector space over k again. More precisely, (12.4) is an algebra homomorphism from $\mathcal{A}$ into $\mathcal{L}(\mathcal{A})$, because of (12.3). If $\mathcal{A}$ has a multiplicative identity element e, then M_e is the identity operator on $\mathcal{A}$. In this case, (12.4) is injective, because

$$M_a(e) = a\,e = a \tag{12.5}$$

for every $a \in \mathcal{A}$.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a submultiplicative q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. Let $a \in \mathcal{A}$ be given, and observe that

$$N(M_a(x)) = N(a\,x) \le N(a)\,N(x) \tag{12.6}$$

for every $x \in \mathcal{A}$. This means that M_a is bounded as a linear mapping from $\mathcal{A}$ into itself, with respect to N on the domain and range, and that

$$\|M_a\|_{op} \le N(a). \tag{12.7}$$

It follows that (12.4) is bounded as a linear mapping from $\mathcal{A}$ into $\mathcal{BL}(\mathcal{A})$, with respect to N on $\mathcal{A}$ and the corresponding operator q-seminorm $\|\cdot\|_{op}$ on $\mathcal{BL}(\mathcal{A})$.

More precisely, the corresponding operator q-seminorm of (12.4) is less than or equal to 1, by (12.7).

If $\mathcal{A}$ has a multiplicative identity element e, then

$$N(a) = N(M_a(e)) \le \|M_a\|_{op}\, N(e) \tag{12.8}$$

for every $a \in \mathcal{A}$. If $N(e) = 1$, then we get that

$$\|M_a\|_{op} = N(a) \tag{12.9}$$

for every $a \in \mathcal{A}$.

13 Supremum q-seminorms

Let X be a nonempty set, and let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$. Also let V be a vector space over k, and let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k for some $q_V > 0$. A V-valued function f on X is said to be *bounded* with respect to N_V if $N_V(f(x))$ is bounded as a nonnegative real-valued function on X. Let $\ell^\infty(X, V)$ be the space of V-valued functions on X that are bounded with respect to N_V, and put

$$\|f\|_\infty = \|f\|_{\ell^\infty(X,V)} = \sup_{x \in X} N_V(f(x)) \tag{13.1}$$

for every such function. One can check that $\ell^\infty(X, V)$ is a linear subspace of the space $c(X, V)$ of all V-valued functions on X, and that (13.1) defines a q_V-seminorm on $\ell^\infty(X, V)$ with respect to $|\cdot|$ on k.

Suppose for the moment that N_V is a q_V-norm on V, which implies that (13.1) is a q_V-norm on $\ell^\infty(X, V)$. If V is complete with respect to the q_V-metric associated to N_V, then $\ell^\infty(X, V)$ is complete with respect to the q_V norm associated to (13.1), by standard arguments.

Let $\mathcal{A}$ be an algebra over k, and consider the space $c(X, \mathcal{A})$ of all $\mathcal{A}$-valued functions on X. This is an algebra over k too, with respect to pointwise multiplication of functions. If $\mathcal{A}$ has a multiplicative identity element e, then the constant function on X equal to e at every point is the multiplicative identity element in $c(X, \mathcal{A})$.

Let $N_{\mathcal{A}}$ be a submultiplicative $q_{\mathcal{A}}$-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q_{\mathcal{A}} > 0$. Using $N_{\mathcal{A}}$, we can define the space $\ell^\infty(X, \mathcal{A})$ of $\mathcal{A}$-valued functions on X that are bounded with respect to $N_{\mathcal{A}}$, as before. It is easy to see that $\ell^\infty(X, \mathcal{A})$ is a subalgebra of $c(X, \mathcal{A})$, and that the corresponding supremum $q_{\mathcal{A}}$-seminorm $\|f\|_\infty$ is submultiplicative on $\ell^\infty(X, \mathcal{A})$.

Now let X be a nonempty topological space, and let V be a vector space over k with a q_V-seminorm N_V with respect to $|\cdot|$ on k for some $q_V > 0$ again. The q_V-semimetric on V associated to N_V determines a topology on V, as usual. Let $C(X, V)$ be the space of V-valued functions on X that are continuous with respect to this topology on V. This is a linear subspace of the space $c(X, V)$ of all V-valued functions on X, by standard arguments. Of course, if X is equipped with the discrete topology, then every function on X is continuous, so that $C(X,V)$ is the same as $c(X,V)$.

Let E be a nonempty compact subset of X. If $f \in C(X,V)$, then $f(E)$ is a compact subset of V. This implies that $f(E)$ is bounded with respect to N_V, so that $N_V(f(x))$ is bounded as a nonnegative real-valued function on E. Put

$$\|f\|_{sup,E} = \sup_{x \in E} N_V(f(x)). \tag{13.2}$$

One can check that this defines a q_V-seminorm on $C(X,V)$ with respect to $|\cdot|$ on k, which is the *supremum q_V-seminorm* associated to E and N_V.

Let

$$C_b(X,V) = C(X,V) \cap \ell^\infty(X,V) \tag{13.3}$$

be the space of V-valued functions on X that are both bounded and continuous with respect to N_V on V. This is a linear subspace of $C(X,V)$ and $\ell^\infty(X,V)$. If $f \in C_b(X,V)$, then we may also use $\|f\|_{sup}$ for the supremum q_V-seminorm (13.1). Note that $C_b(X,V)$ is a closed set in $\ell^\infty(X,V)$ with respect to the supremum q_V-semimetric, which is the q_V-semimetric associated to the supremum q_V-seminorm, by standard arguments. If X is compact, then every continuous V-valued function on X is bounded. If N_V is a q_V-norm on V, and V is complete with respect to the associated q_V-metric, then $\ell^\infty(X,V)$ is complete with respect to the supremum q_V-metric, as before. In this case, it follows that $C_b(X,V)$ is complete with respect to the restriction of the supremum q_V-metric to $C_b(X,V)$.

Let $\mathcal{A}$ be an algebra over k with a submultiplicative $q_{\mathcal{A}}$-seminorm $N_{\mathcal{A}}$ with respect to $|\cdot|$ on k for some $q_{\mathcal{A}} > 0$ again. Observe that $C(X,\mathcal{A})$ is a subalgebra of $c(X,\mathcal{A})$, by standard arguments. If E is a nonempty compact subset of X, then (13.2) is submultiplicative on $C(X,\mathcal{A})$. Of course, $C_b(X,\mathcal{A})$ is a subalgebra of both $C(X,\mathcal{A})$ and $\ell^\infty(X,\mathcal{A})$.

14 More on absolute value functions

Let k be a field, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. If $x \in k$ and $n \in \mathbf{Z}_+$, then let $n \cdot x$ be the sum of n x's in k. If there are positive integers n such that $|n \cdot 1|$ can be arbitrarily large, then $|\cdot|$ is said to be *archimedean* on k. Here 1 refers to the multiplicative identity element in k. Otherwise, $|\cdot|$ is said to be *non-archimedean* if there is a finite upper bound for $|n \cdot 1|$, $n \in \mathbf{Z}_+$. If $|\cdot|$ is an ultrametric absolute value function on k, then it is easy to see that

$$|n \cdot 1| \le 1 \tag{14.1}$$

for every $n \in \mathbf{Z}_+$, so that $|\cdot|$ is non-archimedean on k. Conversely, if $|\cdot|$ is non-archimedean on k, then it is well known that $|\cdot|$ is an ultrametric absolute value function on k.

Let $|\cdot|$ be a q-absolute value function on the field $\mathbf{Q}$ of rational numbers for some $q > 0$. A famous theorem of Ostrowski says that $|\cdot|$ is either equivalent to the standard absolute value function on $\mathbf{Q}$, or $|\cdot|$ is the trivial absolute value function on $\mathbf{Q}$, or $|\cdot|$ is equivalent to the p-adic absolute value function on $\mathbf{Q}$ for some prime number p.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and suppose that k is complete with respect to the associated q_k-metric. If $|\cdot|$ is archimedean on k, then another famous theorem of Ostrowski says that k is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, in such a way that $|\cdot|$ corresponds to a q_k-absolute value function on $\mathbf{R}$ or $\mathbf{C}$ that is equivalent to the standard absolute value function.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again. Thus

$$\{|x| : x \in k \setminus \{0\}\} \tag{14.2}$$

is a subgroup of the group $\mathbf{R}_+$ of positive real numbers with respect to multiplication. If 1 is not a limit point of (14.2) with respect to the standard Euclidean metric on $\mathbf{R}$, then $|\cdot|$ is said to be *discrete* on k. In this case, one can show that (14.2) is the same as the set of integer powers of a positive real number r, with $r = 1$ when $|\cdot|$ is the trivial absolute value function on k. Otherwise, if $|\cdot|$ is not discrete on k, then (14.2) is dense in $\mathbf{R}_+$ with respect to the standard Euclidean metric.

Suppose that $|\cdot|$ is archimedean on k, and let us check that $|\cdot|$ is not discrete on k. The archimedean property implies that k has characteristic 0, since otherwise there are only finitely many elements of k of the form $n \cdot 1$, with $n \in \mathbf{Z}_+$. Hence there is a natural embedding of $\mathbf{Q}$ into k. Using this embedding, we get a q_k-absolute value function on $\mathbf{Q}$, which is archimedean on $\mathbf{Q}$. The first theorem of Ostrowski mentioned earlier implies that this q_k-absolute value function on $\mathbf{Q}$ is equivalent to the standard absolute value function on $\mathbf{Q}$. In particular, this q_k-absolute value function on $\mathbf{Q}$ is not discrete on $\mathbf{Q}$. It follows that $|\cdot|$ is not discrete on k, as desired. Equivalently, if $|\cdot|$ is discrete on k, then $|\cdot|$ is non-archimedean on k, which implies that $|\cdot|$ is an ultrametric absolute value function on k, as before.

Part II
Sums and norms

15 ℓ^r q-Seminorms

Let X be a nonempty set, and let f be a nonnegative real-valued function on X. The sum

$$\sum_{x \in X} f(x) \tag{15.1}$$

can be defined as a nonnegative extended real number as the supremum of the sums

$$\sum_{x \in A} f(x) \tag{15.2}$$

over all nonempty finite subsets A of X. If (15.1) is finite, then f is said to be *summable* on X. It is easy to see that

$$\sum_{x \in X} t\, f(x) = t \sum_{x \in X} f(x) \tag{15.3}$$

for every positive real number t, where $t\,(+\infty)$ is interpreted as $+\infty$, as usual. Similarly, if g is another nonnegative real-valued function on X, then one can check that

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x). \tag{15.4}$$

Of course, the right side of this equation is interpreted as being $+\infty$ when either of the sums is $+\infty$. It is sometimes convenient to permit f to be a nonnegative extended-real valued function on X in (15.1), where the sum is interpreted as being $+\infty$ when $f(x) = +\infty$ for any $x \in X$.

If X is the set $\mathbf{Z}_+$ of positive integers, then (15.1) can also be obtained by taking the supremum of the partial sums $\sum_{j=1}^n f(j)$ over $n \in \mathbf{Z}_+$. In this case, the finiteness of the sum (15.1) corresponds to the convergence of the infinite series $\sum_{j=1}^\infty f(j)$ in $\mathbf{R}$. Similarly, if X is the set $\mathbf{Z}_+ \cup \{0\}$ of nonnegative integers, then (15.1) can be obtained by taking the supremum of the partial sums $\sum_{j=0}^n f(j)$ over $n \ge 0$.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k with a q_V-seminorm N_V with respect to $|\cdot|$ on k for some $q_V > 0$. Also let r be a positive real number, and let f be a V-valued function on a nonempty set X. If $N_V(f(x))^r$ is summable as a nonnegative real-valued function on X, then f is said to be *r-summable* on X with respect to N_V. In this case, we put

$$\|f\|_r = \|f\|_{\ell^r(X,V)} = \Big(\sum_{x \in X} N_V(f(x))^r\Big)^{1/r}, \tag{15.5}$$

which is defined as a nonnegative real number. Let $\ell^r(X, V)$ be the space of V-valued functions on X that are r-summable with respect to N_V.

If $f \in \ell^r(X, V)$ and $t \in k$, then it is easy to see that $t\, f \in \ell^r(X, V)$, with

$$\|t\, f\|_r = |t|\, \|f\|_r. \tag{15.6}$$

If $g \in \ell^r(X, V)$ too, then one can check that $f + g \in \ell^r(X, V)$. More precisely, we have that

$$\|f + g\|_r^r \le \|f\|_r^r + \|g\|_r^r \tag{15.7}$$

when $r \le q_V$, and

$$\|f + g\|_r^{q_V} \le \|f\|_r^{q_V} + \|g\|_r^{q_V} \tag{15.8}$$

when $q_V \le r$. To get (15.7), one can use the fact that N_V may be considered as an r-seminorm on V with respect to $|\cdot|$ on k when $r \le q_k$, as in Section 7. If $q_V \le r$, then (15.8) can be obtained from Minkowski's inequality (4.13), with exponent $r/q_V \ge 1$. Although Minkowski's inequality was previously stated for finite sums, it can be extended to arbitrary sums in a standard way. Thus $\ell^r(X, V)$ is a linear subspace of the space $c(X, V)$ of all V-valued functions on X, (15.5) is an r-seminorm on $\ell^r(X, V)$ with respect to $|\cdot|$ on k when $r \le q_V$, and (15.5) is a q_V-seminorm on $\ell^r(X, V)$ with respect to $|\cdot|$ on k when $q_V \le r$.

Suppose for the moment that N_V is a q_V-norm on V with respect to $|\cdot|$ on k. In this case, (15.5) is an r-norm on $\ell^r(X, V)$ with respect to $|\cdot|$ on k when $r \le q_V$, and (15.5) is a q_V-norm on $\ell^r(X, V)$ with respect to $|\cdot|$ on k when $q_V \le r$. Suppose that V is also complete with respect to the q_V-metric associated to N_V. Under these conditions, one can show that $\ell^r(X, V)$ is complete with respect to the r or q_V-metric associated to (15.5), as appropriate, for every $r > 0$, using standard arguments.

16 Comparing ℓ^r spaces

Let X be a nonempty set, and let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$. Also let V be a vector space over k with a q_V-seminorm N_V with respect to $|\cdot|$ on k for some $q_V > 0$. If a V-valued function f on X is r-summable with respect to N_V for some positive real number r, then it is easy to see that f is bounded with respect to N_V, with

$$\|f\|_\infty \le \|f\|_r. \tag{16.1}$$

Note that (16.1) corresponds to (4.3), applied to $N_V(f(x))$ as a nonnegative real-valued function on X. In Section 4 we considered nonnegative real-valued functions on X with finite support, but the same argument works for r-summable functions.

Suppose now that $0 < r_1 \le r_2 < \infty$, and that f is a V-valued r_1-summable function on X with respect to N_V. Under these conditions, we have that

$$\sum_{x \in X} N_V(f(x))^{r_2} \le \|f\|_\infty^{r_2 - r_1} \sum_{x \in X} N_V(f(x))^{r_1} = \|f\|_\infty^{r_2 - r_1} \, \|f\|_{r_1}^{r_1}, \tag{16.2}$$

as in (4.4). This implies that f is r_2-summable on X with respect to N_V, with

$$\|f\|_{r_2} \le \|f\|_\infty^{1 - (r_1/r_2)} \, \|f\|_{r_1}^{r_1/r_2} \le \|f\|_{r_1}, \tag{16.3}$$

as in (4.5). In particular,

$$\ell^{r_1}(X, V) \subseteq \ell^{r_2}(X, V). \tag{16.4}$$

This inclusion also holds when $r_2 = \infty$, as in the preceding paragraph.

A V-valued function f on X is said to *vanish at infinity* with respect to N_V if for every $\epsilon > 0$ we have that

$$N_V(f(x)) < \epsilon \tag{16.5}$$

for all but finitely many $x \in X$. Let $c_0(X, V)$ be the space of all V-valued functions on X that vanish at infinity with respect to N_V. It is easy to see that

$c_0(X, V)$ is a linear subspace of the space $c(X, V)$ of all V-valued functions on X. More precisely,

$$c_0(X, V) \subseteq \ell^\infty(X, V). \tag{16.6}$$

One can also check that

$$\ell^r(X, V) \subseteq c_0(X, V) \tag{16.7}$$

for every positive real number r.

One can verify that $c_0(X, V)$ is a closed set in $\ell^\infty(X, V)$ with respect to the corresponding supremum q_V-semimetric, using standard arguments. If N_V is a q_V-norm on V, and V is complete with respect to the associated q_V-metric, then $\ell^\infty(X, V)$ is complete with respect to the supremum q_V-metric, as before. In this case, we get that $c_0(X, V)$ is complete with respect to the restriction of the supremum q_V-metric to $c_0(X, V)$.

If f is a V-valued function on X such that $N_V(f(x)) = 0$ for all but finitely many $x \in X$, then f vanishes at infinity on X with respect to N_V, and f is r-summable on X with respect to N_V for every $r > 0$. In particular, this condition holds when f has finite support in X. If N_V is a q_V-norm on V, then this condition implies that f has finite support in X.

It is not difficult to check that $c_{00}(X, V)$ is dense in $c_0(X, V)$ with respect to the supremum q_V-semimetric. Thus $c_0(X, V)$ is the same as the closure of $c_{00}(X, V)$ in $\ell^\infty(X, V)$, with respect to the supremum q_V-semimetric.

If r is a positive real number, then one can verify that $c_{00}(X, V)$ is dense in $\ell^r(X, V)$, with respect to the r or q_V-semimetric associated to $\|\cdot\|_r$, as appropriate. More precisely, if f is an r-summable V-valued function on X with respect to N_V, then $\|f\|_r^r$ can be approximated by the sum of $N_V(f(x))^r$ over suitable nonempty finite subsets of X. One can use these finite subsets of X to approximate f by V-valued functions on X with finite support with respect to the r or q_V-semimetric associated to $\|\cdot\|_r$, as appropriate.

If f is a V-valued function on X that vanishes at infinity with respect to N_V, then there are only finitely or countably many $x \in X$ such that $N_V(f(x)) > 0$. More precisely, for each positive integer j, there are only finitely many $x \in X$ such that $N_V(f(x)) \geq 1/j$, and the previous statement follows by taking the union of these finite sets. If N_V is a q_V-norm on V, and f vanishes at infinity on X with respect to N_V, then the support of f has only finitely or countably many elements.

17 Doubly-infinite series

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k with a q_V-norm N_V with respect to $|\cdot|$ on k for some $q_V > 0$. Also let

$$\sum_{j=-\infty}^{\infty} v_j \tag{17.1}$$

be a doubly-infinite series with terms in V, so that $v_j \in V$ for every $j \in \mathbf{Z}$. Let us say that (17.1) converges in V with respect to N_V if

$$\sum_{j=0}^{\infty} v_j \quad \text{and} \quad \sum_{j=1}^{\infty} v_{-j} \tag{17.2}$$

converge as ordinary infinite series in V with respect to N_V. In this case, the value of the sum (17.1) is defined by

$$\sum_{j=-\infty}^{\infty} v_j = \sum_{j=0}^{\infty} v_j + \sum_{j=1}^{\infty} v_{-j}. \tag{17.3}$$

One could look at the convergence of (17.1) in terms of the convergence of the partial sums

$$\sum_{j=-l}^{n} v_j \tag{17.4}$$

as $l, n \to \infty$. If $l \geq 1$ and $n \geq 0$, then (17.4) can be expressed as

$$\sum_{j=0}^{n} v_j + \sum_{j=1}^{l} v_{-j}. \tag{17.5}$$

Of course, the convergence of the series in (17.2) is the same as the convergence of these two sequences of partial sums.

Alternatively, one can consider

$$\sum_{j=1}^{\infty} (v_j + v_{-j}) \tag{17.6}$$

as an infinite series with terms in V. If this series converges, then one might interpret (17.1) as being

$$v_0 + \sum_{j=1}^{\infty} (v_j + v_{-j}). \tag{17.7}$$

Note that

$$\sum_{j=-n}^{n} v_j = v_0 + \sum_{j=1}^{n} (v_j + v_{-j}) \tag{17.8}$$

for every positive integer n. If the series in (17.2) converge in V with respect to N_V, then (17.6) converges in V with respect to N_V as well. Under these conditions, (17.3) is equal to (17.7).

Suppose for the moment that $k = \mathbf{R}$ with the standard absolute value function, and that $V = \mathbf{R}$, with $N_V = |\cdot|$. If v_j is a nonnegative real number for each $j \in \mathbf{Z}$, then the convergence of each of the infinite series in (17.2) and (17.6) is equivalent to the boundedness of the corresponding sequences of partial sums.

In this situation, the partial sums of (17.6) are bounded if and only if the partial sums of both series in (17.2) are bounded. Thus (17.6) converges if and only if both series in (17.2) converge, in which case (17.3) is equal to (17.7), as before. One can also define

$$\sum_{j \in \mathbf{Z}} v_j \tag{17.9}$$

as a nonnegative extended real number as in Section 15. This is equivalent to taking the supremum of the partial sums in (17.4) over $l, n \geq 0$, or with $l = n$, as in (17.8). This means that (17.9) is finite exactly when the series in (17.2) or (17.6) converge, with the same value of the sum as in (17.3) and (17.7).

Let k be any field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, let V be a vector space over k with a q_V-norm N_V with respect to $|\cdot|$ on k for some $q_V > 0$, and let (17.1) be a doubly-infinite series with terms in V. Suppose for the moment that $q_V < \infty$. Let us say that (17.1) converges q_V*-absolutely* with respect to N_V if

$$\sum_{j=-\infty}^{\infty} N_V(v_j)^{q_V} \tag{17.10}$$

converges as a doubly-infinite series of nonnegative real numbers. This is the same as saying that the series in (17.2) converge q_V-absolutely with respect to N_V. If V is complete with respect to the q_V-metric associated to N_V, then it follows that the series in (17.2) converge in V, as in Section 9. This means that (17.1) converges in V, and it is easy to see that

$$N_V\Big(\sum_{j=-\infty}^{\infty} v_j\Big)^{q_V} \leq \sum_{j=-\infty}^{\infty} N_V(v_j)^{q_V}. \tag{17.11}$$

This uses the analogous statements for the series in (17.2), as in (9.9).

Suppose now that $q_V = \infty$, and that V is complete with respect to the ultrametric associated to N_V. If

$$N_V(v_j) \to 0 \quad \text{as } j \to \pm\infty, \tag{17.12}$$

then the series in (17.2) converge in V, as in Section 9. This means that (17.1) converges in V, and we have that

$$N_V\Big(\sum_{j=-\infty}^{\infty} v_j\Big) \leq \max_{j \in \mathbf{Z}} N_V(v_j). \tag{17.13}$$

This follows from the analogous statements for the series in (17.2), as in (9.11). In particular, (17.12) implies that the maximum on the right side of (17.13) is attained, as before.

18 Sums over sets

Let X be a nonempty set, let k be a field, and let V be a vector space over k. Remember that

$$f \mapsto \sum_{x \in X} f(x) \tag{18.1}$$

defines a linear mapping from $c_{00}(X, V)$ into V, as in Section 1. Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k for some $q_V > 0$. It is easy to see that

$$N_V\Big(\sum_{x \in X} f(x)\Big) \le \|f\|_{q_V} \tag{18.2}$$

for every $f \in c_{00}(X, V)$, where $\|f\|_{q_V}$ is as in Sections 13 and 15. Note that the restriction of $\|f\|_{q_V}$ to $c_{00}(X, V)$ defines a q_V-seminorm on $c_{00}(X, V)$ with respect to $|\cdot|$ on k. Thus (18.2) says that (18.1) is bounded as a linear mapping from $c_{00}(X, V)$ into V, with respect to $\|\cdot\|_{q_V}$ on $c_{00}(X, V)$ and N_V on V, with operator q_V-seminorm less than or equal to 1. More precisely, the operator q_V-seminorm is equal to 1 when $N_V(v) > 0$ for some $v \in V$.

Let us suppose from now on in this section that N_V is a q_V-norm on V, and that V is complete with respect to the associated q_V-metric. If $q_V < \infty$, then there is a unique extension of (18.1) to a bounded linear mapping from $\ell^{q_V}(X, V)$ into V, as in Section 10. This uses the fact that $c_{00}(X, V)$ is dense in $\ell^{q_V}(X, V)$ with respect to the q_V-metric associated to $\|\cdot\|_{q_V}$ when $q_V < \infty$, as in Section 16. Similarly, if $q_V = \infty$, then there is a unique extension of (18.1) to a bounded linear mapping from $c_0(X, V)$ into V, with respect to the supremum ultranorm on $c_0(X, V)$ and N_V on V, because $c_{00}(X, V)$ is dense in $c_0(X, V)$ with respect to the supremum ultrametric on $c_0(X, V)$. We can use these extensions to define

$$\sum_{x \in X} f(x) \tag{18.3}$$

as an element of V when $f \in \ell^{q_V}(X, V)$ and $q_V < \infty$, and when $f \in c_0(X, V)$ and $q_V = \infty$. Observe that (18.2) holds for every $f \in \ell^{q_V}(X, V)$ when $q_V < \infty$, and for every $f \in c_0(X, V)$ when $q_V = \infty$, because the operator q_V-norm of the extension is the same as on $c_{00}(X, V)$. Of course, if X has only finitely many elements, then every V-valued function on X has finite support, and no extension is necessary.

If $X = \mathbf{Z}_+$, then we can use the infinite series

$$\sum_{j=1}^{\infty} f(j) \tag{18.4}$$

to define these extensions. More precisely, this series converges in V when f is an element of $\ell^{q_V}(\mathbf{Z}_+, V)$ and $q_V < \infty$, and when $f \in c_0(\mathbf{Z}_+, V)$ and $q_V = \infty$, as in Section 9. This definition of the sum defines a bounded linear mapping into V, and (18.4) reduces to the usual finite sum when f has finite support in

$\mathbf{Z}_+$. Similarly, if $X = \mathbf{Z}_+ \cup \{0\}$, then these extensions can be defined using the infinite series

$$\sum_{j=0}^{\infty} f(j). \tag{18.5}$$

If $X = \mathbf{Z}$, then these extensions can be defined by considering

$$\sum_{j=-\infty}^{\infty} f(j) \tag{18.6}$$

as a doubly-infinite series with terms in V, as in the previous section.

Let X be any countably-infinite set, and let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of X in which every element of X occurs exactly once. Suppose that $f \in \ell^{q_V}(X, V)$ when $q_V < \infty$, and that $f \in c_0(X, V)$ when $q_V = \infty$, as before. In both cases, the infinite series

$$\sum_{j=1}^{\infty} f(x_j) \tag{18.7}$$

converges in V, as in Section 9 again. This can be used as another way to define (18.3) in this situation, as in the preceding paragraph. If X is any infinite set, and X_0 is a countably-infinite subset of X, then one can deal with V-valued functions f on X whose support is contained in X_0 in the same way.

19 Iterated sums

Let I and X be nonempty sets, and let E_j be a nonempty subset of X for every $j \in I$. Suppose that the E_j's are pairwise disjoint, so that

$$E_j \cap E_l = \emptyset \tag{19.1}$$

for every $j, l \in I$ with $j \neq l$. If f is a nonnegative real-valued function on X, then

$$\sum_{x \in E_j} f(x) \tag{19.2}$$

can be defined as a nonnegative extended real number for every $j \in I$. Hence

$$\sum_{j \in I} \Big(\sum_{x \in E_j} f(x) \Big) \tag{19.3}$$

can be defined as an extended real number as well. Put

$$E = \bigcup_{j \in I} E_j, \tag{19.4}$$

so that

$$\sum_{x \in E} f(x) \tag{19.5}$$

can be defined as an extended real number too. One can check that (19.3) and (19.5) are the same in this situation. More precisely, one can verify that (19.3) and (19.5) are each less than or equal to the other.

Let k be a field, and let V be a vector space over k. If f is a V-valued function on X with finite support, then (19.2) can be defined as an element of V for every $j \in I$. We also have that (19.2) is equal to 0 for all but finitely many $j \in I$, so that (19.3) can be defined as an element of V too. Similarly, (19.5) can be defined as an element of V, and is equal to (19.3).

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N_V be a q_V-norm on V with respect to $|\cdot|$ on k for some $q_V > 0$. Suppose for the rest of the section that V is complete with respect to the q_V-metric associated to N_V. Suppose for the moment that $q_V < \infty$, and let $f \in \ell^{q_V}(X, V)$ be given. It is easy to see that the restriction of f to E_j is q_V-summable with respect to N_V for every $j \in I$, so that (19.2) can be defined as an element of V as in the previous section. Note that

$$N_V\Big(\sum_{x \in E_j} f(x)\Big)^{q_V} \le \sum_{x \in E_j} N_V(f(x))^{q_V} \tag{19.6}$$

for every $j \in I$, as in (18.2). This implies that

$$\sum_{j \in I} N_V\Big(\sum_{x \in E_j} f(x)\Big)^{q_V} \le \sum_{j \in I}\Big(\sum_{x \in E_j} N_V(f(x))^{q_V}\Big) = \sum_{x \in E} N_V(f(x))^{q_V}, \tag{19.7}$$

using the earlier remarks for nonnegative real-valued functions in the second step. Thus (19.2) is q_V-summable as a V-valued function of $j \in I$ with respect to N_V, so that (19.3) can be defined as an element of V as in the previous section as well. The sum (19.5) can be defined as an element of V as in the previous section too, because the restriction of f to E is q_V-summable with respect to N_V. One can check that (19.3) is equal to (19.5), by approximating f by V-valued functions on X with finite support.

Suppose now that $q_V = \infty$, and let $f \in c_0(X, V)$ be given. Observe that the restriction of f to E_j vanishes at infinity on E_j with respect to N_V for every $j \in I$, so that (19.2) can be defined as an element of V for every $j \in I$, as in the previous section. In this situation, we have that

$$N_V\Big(\sum_{x \in E_j} f(x)\Big) \le \max_{x \in E_j} N_V(f(x)) \tag{19.8}$$

for every $j \in I$, as in (18.2). Using this, one can verify that (19.2) vanishes at infinity as a V-valued function of $j \in I$ with respect to N_V, so that (19.3) can be defined as an element of V as in the previous section. The restriction of f to E vanishes at infinity with respect to N_V, so that (19.5) can be defined as an element of V as in the previous section as well. One can verify that (19.3) is equal to (19.5), by approximating f by V-valued functions on X with finite support again. Remember that the norm of (19.3) with respect to N_V is less than or equal to the maximum of the left side of (19.8) over $j \in I$, as in (18.2).

This is less than or equal to the maximum of the right side of (19.8) over $j \in I$, which is the same as the maximum of $N_V(f(x))$ over $x \in E$.

20 Cauchy products on $\mathbf{Z}_+ \cup \{0\}$

Let k be a field, and let $\mathcal{A}$ be an algebra over k. Also let $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ be infinite series with terms in $\mathcal{A}$, considered formally for the moment. Put

$$c_n = \sum_{j=0}^n a_j \, b_{n-j} \tag{20.1}$$

for each nonnegative integer n. The infinite series $\sum_{n=0}^\infty c_n$ is called the *Cauchy product* of $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$. It is easy to see that

$$\sum_{n=0}^\infty c_n = \Big(\sum_{j=0}^\infty a_j\Big) \Big(\sum_{l=0}^\infty b_l\Big) \tag{20.2}$$

formally. More precisely, consider

$$\sum_{(j,l) \in (\mathbf{Z}_+ \cup \{0\})^2} a_j \, b_l \tag{20.3}$$

formally, where $(\mathbf{Z}_+ \cup \{0\})^2 = (\mathbf{Z}_+ \cup \{0\}) \times (\mathbf{Z}_+ \cup \{0\})$. This can be identified formally with the iterated sums

$$\sum_{j=0}^\infty \Big(\sum_{l=0}^\infty a_j \, b_l\Big) \tag{20.4}$$

and

$$\sum_{l=0}^\infty \Big(\sum_{j=0}^\infty a_j \, b_l\Big), \tag{20.5}$$

each of which reduces formally to the product of the sums on the right side of (20.2). Put

$$E_n = \{(j,l) \in (\mathbf{Z}_+ \cup \{0\})^2 : j + l = n\} \tag{20.6}$$

for each nonnegative integer n, and observe that the E_n's are pairwise-disjoint finite subsets of $(\mathbf{Z}_+ \cup \{0\})^2$ such that

$$\bigcup_{n=0}^\infty E_n = (\mathbf{Z}_+ \cup \{0\})^2. \tag{20.7}$$

By construction,

$$c_n = \sum_{(j,l) \in E_n} a_j \, b_l \tag{20.8}$$

for every $n \geq 0$, so that the left side of (20.2) is the same as

$$\sum_{n=0}^{\infty} \Big(\sum_{(j,l) \in E_n} a_j\, b_l \Big), \tag{20.9}$$

which corresponds formally to (20.3) as well.

Suppose for the moment that $a_j = 0$ for all but finitely many $j \geq 0$, and that $b_l = 0$ for all but finitely many $l \geq 0$, so that $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ reduce to finite sums. In this case, one can check that $c_n = 0$ for all but finitely many $n \geq 0$, so that $\sum_{n=0}^{\infty} c_n$ reduces to a finite sum too. In fact,

$$a_j\, b_l = 0 \tag{20.10}$$

for all but finitely many $(j,l) \in (\mathbf{Z}_+ \cup \{0\})^2$, so that (20.3) reduces to a finite sum as well. The iterated sums (20.4) and (20.5) also reduce to finite sums of finite sums in this situation. As in the previous section, (20.3) is equal to the iterated sums (20.4), (20.5), and (20.9), which implies that (20.2) holds.

Suppose for the moment again that $k = \mathcal{A} = \mathbf{R}$, and that $a_j, b_l \geq 0$ for all $j, l \geq 0$, so that $c_n \geq 0$ for all $n \geq 0$. The sums $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$, and $\sum_{n=0}^{\infty} c_n$ may be considered as nonnegative extended real numbers, by considering them as sums over $\mathbf{Z}_+ \cup \{0\}$ as in Section 15, or by interpreting an infinite series with nonnegative terms as being $+\infty$ when the series does not converge in $\mathbf{R}$ in the usual sense. The product of sums on the right side of (20.2) can be defined as a nonnegative extended real number in the usual way, except when one of the factors is equal to 0 and the other is equal to $+\infty$. This can only happen when $a_j = 0$ for every $j \geq 0$ or $b_l = 0$ for every $l \geq 0$, in which case $c_n = 0$ for every $n = 0$, so that the left side of (20.2) is equal to 0. The sum (20.3) and the iterated sums (20.4) and (20.5) can also be defined as nonnegative extended real numbers, and we have that (20.3) is equal to the iterated sums (20.4), (20.5), and (20.9), as in the previous section. If $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ are finite, then the iterated sums (20.4) and (20.5) can be reduced to the product on the right side of (20.2). This implies that (20.2) holds, and in particular that $\sum_{n=0}^{\infty} c_n$ is finite. If either of $\sum_{j=0}^{\infty} a_j$ or $\sum_{l=0}^{\infty} b_l$ is $+\infty$, and the other is positive, then the iterated sums (20.4), (20.5) are $+\infty$. This implies that (20.3) is $+\infty$, so that (20.9) is $+\infty$, which is the same as saying that $\sum_{n=0}^{\infty} c_n = +\infty$.

Let k be any field again, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. Also let $\mathcal{A}$ be an algebra over k with a submultiplicative q-norm with respect to $|\cdot|$ on k for some $q > 0$, and suppose for the rest of the section that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Suppose for the moment that $q < \infty$, and that $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ are q-absolutely convergent infinite series with terms in $\mathcal{A}$, with respect to N. If $\sum_{n=0}^{\infty} c_n$ is their Cauchy product, as before, then

$$N(c_n)^q \leq \sum_{j=0}^{n} N(a_j\, b_{n-j})^q \leq \sum_{j=0}^{n} N(a_j)^q\, N(b_{n-j})^q \tag{20.11}$$

for each $n \geq 0$. This uses the q-norm version of the triangle inequality in the first step, and submultiplicativity of N on $\mathcal{A}$ in the second set. The right side of (20.11) is the nth term of the Cauchy product of $\sum_{j=0}^\infty N(a_j)^q$ and $\sum_{l=0}^\infty N(b_l)^q$, as infinite series of nonnegative real numbers. It follows that

$$\sum_{n=0}^\infty N(c_n)^q \leq \sum_{n=0}^\infty \Big(\sum_{j=0}^n N(a_j)^q\, N(b_{n-j})^q\Big) = \Big(\sum_{j=0}^\infty N(a_j)^q\Big)\Big(\sum_{l=0}^\infty N(b_l)^q\Big), \tag{20.12}$$

using the remarks in the preceding paragraph in the second step. In particular, this means that $\sum_{n=0}^\infty c_n$ converges q-absolutely with respect to N. Similarly,

$$\sum_{(j,l)\in(\mathbf{Z}_+\cup\{0\})^2} N(a_j\, b_l)^q \leq \sum_{(j,l)\in(\mathbf{Z}_+\cup\{0\})^2} N(a_j)^q\, N(b_l)^q, \tag{20.13}$$

because of the submultiplicativity of N on $\mathcal{A}$. The right side of (20.13) is the same as the iterated sums

$$\sum_{j=0}^\infty \Big(\sum_{l=0}^\infty N(a_j)^q\, N(b_l)^q\Big) \tag{20.14}$$

and

$$\sum_{l=0}^\infty \Big(\sum_{j=0}^\infty N(a_j)^q\, N(b_l)^q\Big), \tag{20.15}$$

as before. These iterated sums are equal to the right side of (20.12), so that the right side of (20.13) is equal to the right side of (20.12). Hence the left side of (20.13) is finite, which means that $f(j,l) = a_j\, b_l$ is q-summable as an $\mathcal{A}$-valued function on $(\mathbf{Z}_+\cup\{0\})^2$, with respect to N. The sum (20.3) can be defined as an element of $\mathcal{A}$ as in Section 18, and this sum is equal to the iterated sums (20.4) and (20.5), as in the previous section. Of course, $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ converge in $\mathcal{A}$, as in Section 9. The product of these two sums is the same as each of the iterated sums (20.4) and (20.5). The sum (20.3) is equal to the iterated sum (20.9) too, which is the same as $\sum_{n=0}^\infty c_n$. This shows that (20.2) holds in this situation.

Suppose now that $q = \infty$, and that

$$\lim_{j\to\infty} N(a_j) = \lim_{l\to\infty} N(b_l) = 0. \tag{20.16}$$

This implies that $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ converge in $\mathcal{A}$, as in Section 9. If c_n is as in (20.1), then

$$N(c_n) \leq \max_{0\leq j\leq n} N(a_j\, b_{n-j}) \leq \max_{0\leq j\leq n} (N(a_j)\, N(b_{n-j})) \tag{20.17}$$

for every $n \geq 0$, using the ultranorm version of the triangle inequality in the first step, and the submultiplicativity of N on $\mathcal{A}$ in the second step. One can check that

$$\lim_{n \to \infty} N(c_n) = 0, \tag{20.18}$$

using (20.16) and (20.17). We also have that

$$\max_{n \geq 0} N(c_n) \leq \Big(\max_{j \geq 0} N(a_j)\Big) \Big(\max_{l \geq 0} N(b_l)\Big), \tag{20.19}$$

by (20.17). Of course,

$$N(a_j\, b_l) \leq N(a_j)\, N(b_l) \tag{20.20}$$

for all $j, l \geq 0$, because of the submultiplicativity of N on $\mathcal{A}$. One can verify that $f(j,l) = a_j\, b_l$ vanishes at infinity as an $\mathcal{A}$-valued function on $(\mathbf{Z}_+ \cup \{0\})^2$ with respect to N, using (20.16) and (20.20). More precisely, $N(a_j)\, N(b_l)$ vanishes at infinity as a nonnegative real-valued function on $(\mathbf{Z}_+ \cup \{0\})^2$. This implies that the sum (20.3) can be defined as an element of $\mathcal{A}$, as in Section 18. This sum is equal to the iterated sums (20.4) and (20.5), and these iterated sums are both equal to the product of $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$. It follows that (20.2) holds in this situation as well, because (20.3) is also equal to the iterated sum (20.9), which is the same as $\sum_{n=0}^{\infty} c_n$.

21 Cauchy products on Z

Let k be a field again, and let $\mathcal{A}$ be an algebra over k. Let us now consider doubly-infinite series $\sum_{j=-\infty}^{\infty} a_j$ and $\sum_{l=-\infty}^{\infty} b_l$ with terms in $\mathcal{A}$, at least formally at first. Consider

$$c_n = \sum_{j=-\infty}^{\infty} a_j\, b_{n-j} \tag{21.1}$$

for each $n \in \mathbf{Z}$, at least formally. The doubly-infinite series $\sum_{n=-\infty}^{\infty} c_n$ is the formal *Cauchy product* of $\sum_{j=-\infty}^{\infty} a_j$ and $\sum_{l=-\infty}^{\infty} b_l$. As before,

$$\sum_{n=-\infty}^{\infty} c_n = \Big(\sum_{j=-\infty}^{\infty} a_j\Big) \Big(\sum_{l=-\infty}^{\infty} b_l\Big) \tag{21.2}$$

formally. To see this, consider

$$\sum_{(j,l) \in \mathbf{Z}^2} a_j\, b_l \tag{21.3}$$

formally, where $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$. As before, this corresponds formally to the iterated sums

$$\sum_{j=-\infty}^{\infty} \Big(\sum_{l=-\infty}^{\infty} a_j\, b_l\Big) \tag{21.4}$$

and

$$\sum_{l=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{\infty} a_j\, b_l \Big), \tag{21.5}$$

which reduce formally to the product on the right side of (21.2). In this situation, we put

$$E_n = \{(j,l) \in \mathbf{Z}^2 : j + l = n\} \tag{21.6}$$

for every $n \in \mathbf{Z}$, which are pairwise-disjoint subsets of $\mathbf{Z}^2$ such that

$$\bigcup_{n=-\infty}^{\infty} E_n = \mathbf{Z}^2. \tag{21.7}$$

Thus (21.1) is formally the same as

$$c_n = \sum_{(j,l) \in E_n} a_j\, b_l, \tag{21.8}$$

so that the left side of (21.2) corresponds formally to

$$\sum_{n=-\infty}^{\infty} \Big(\sum_{(j,l) \in E_n} a_j\, b_l \Big). \tag{21.9}$$

This can be identified formally with (21.3), because of (21.7).

If $a_j = 0$ for all but finitely many $j \in \mathbf{Z}$, and $b_l = 0$ for all but finitely many $l \in \mathbf{Z}$, then $\sum_{j=-\infty}^{\infty} a_j$ and $\sum_{l=-\infty}^{\infty} b_l$ reduce to finite sums in $\mathcal{A}$. Under these conditions, the right side of (21.1) also reduces to a finite sum, so that c_n is defined as an element of $\mathcal{A}$ for every $n \in \mathbf{Z}$. It is easy to see that $c_n = 0$ for all but finitely many $n \in \mathbf{Z}$, so that $\sum_{n=-\infty}^{\infty} c_n$ reduces to a finite sum in $\mathcal{A}$. Similarly,

$$a_j\, b_l = 0 \tag{21.10}$$

for all but finitely many $(j,l) \in \mathbf{Z}^2$, so that (21.3) reduces to a finite sum in $\mathcal{A}$ as well. The iterated sums (21.4) and (21.5) reduce to finite sums of finite sums, and (21.3) is equal to the iterated sums (21.4), (21.5), and (21.9), so that (21.2) holds.

Suppose that $k = \mathcal{A} = \mathbf{R}$, and that $a_j, b_l \geq 0$ for every $j, l \in \mathbf{Z}$. The sums $\sum_{j=-\infty}^{\infty} a_j$ and $\sum_{l=-\infty}^{\infty} b_l$ may be considered as nonnegative extended real numbers, by considering them as sums over $\mathbf{Z}$, as in Section 15, or by interpreting a doubly-infinite series with nonnegative terms as being $+\infty$ when the series does not converge in $\mathbf{R}$ in the usual sense. Similarly, c_n is defined as a nonnegative extended real number for each $n \in \mathbf{Z}$, so that $\sum_{n=-\infty}^{\infty} c_n$ is defined as a nonnegative extended real number. The sum (21.3) and the iterated sums (21.4) and (21.5) are defined as extended real numbers as well, and (21.3) is equal to the iterated sums (21.4), (21.5), and (21.9), as in Section 19. If $\sum_{j=-\infty}^{\infty} a_j$ and $\sum_{l=-\infty}^{\infty} b_l$ are finite, then the iterated sums (21.4) and (21.5) are the same as the product of these two sums. It follows that (21.2) holds, and in particular that $\sum_{n=-\infty}^{\infty} c_n$ is finite. If either $\sum_{j=-\infty}^{\infty} a_j$ or $\sum_{l=-\infty}^{\infty} b_l$ is equal to 0, then either $a_j = 0$ for every $j \in \mathbf{Z}$ or $b_l = 0$ for every $l \in \mathbf{Z}$. In both cases, $c_n = 0$ for every $n \in \mathbf{Z}$, and the sum (21.3) and the iterated sums (21.4), (21.5), and (21.9) are equal to 0. If $\sum_{j=-\infty}^{\infty} a_j$ and $\sum_{l=-\infty}^{\infty} b_l$ are both positive, and at least one of these sums is $+\infty$, then the iterated sums (21.4) and (21.5) are $+\infty$. This implies that (21.3) is $+\infty$, and hence that (21.9) is $+\infty$, which means that $\sum_{n=-\infty}^{\infty} c_n = +\infty$.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative q-norm N with respect to $|\cdot|$ on k for some $q > 0$. Suppose for the rest of the section that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Suppose for the moment that $q < \infty$, and that

$$\sum_{j=-\infty}^{\infty} N(a_j)^q, \ \sum_{l=-\infty}^{\infty} N(b_l)^q < \infty. \tag{21.11}$$

This implies that $\sum_{j=-\infty}^{\infty} a_j$ and $\sum_{l=-\infty}^{\infty} b_l$ can be defined as elements of $\mathcal{A}$, as q-absolutely convergent doubly-infinite series, as in Section 17, or as sums over $\mathbf{Z}$, as in Section 18. Observe that

$$\sum_{j=-\infty}^{\infty} N(a_j\, b_{n-j})^q \le \sum_{j=-\infty}^{\infty} N(a_j)^q\, N(b_{n-j})^q < \infty \tag{21.12}$$

for every $n \in \mathbf{Z}$. This uses the fact that the terms in the sums in (21.11) are bounded, because the sums are finite. Thus c_n can be defined as an element of $\mathcal{A}$ as in (21.1) for each $n \in \mathbf{Z}$, with

$$N(c_n)^q \le \sum_{j=-\infty}^{\infty} N(a_j\, b_{n-j})^q. \tag{21.13}$$

Of course,

$$\sum_{(j,l)\in\mathbf{Z}^2} N(a_j\, b_l)^q \le \sum_{(j,l)\in\mathbf{Z}^2} N(a_j)^q\, N(b_l)^q, \tag{21.14}$$

by the submultiplicativity of N on $\mathcal{A}$. The right side of (21.14) is equal to the iterated sums

$$\sum_{j=-\infty}^{\infty} \Big(\sum_{l=-\infty}^{\infty} N(a_j)^q\, N(b_l)^q \Big) \tag{21.15}$$

and

$$\sum_{l=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{\infty} N(a_j)^q\, N(b_l)^q \Big), \tag{21.16}$$

as in the preceding paragraph. These iterated sums reduce to the product

$$\Big(\sum_{j=-\infty}^{\infty} N(a_j)^q \Big) \Big(\sum_{l=-\infty}^{\infty} N(b_l)^q \Big), \tag{21.17}$$

which is finite, by (21.11). The right side of (21.14) is also equal to the iterated sum

$$(21.18)\quad \sum_{n=-\infty}^{\infty} \Big(\sum_{(j,l)\in E_n} N(a_j)^q\, N(b_l)^q \Big) = \sum_{n=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{\infty} N(a_j)^q\, N(b_{n-j})^q \Big),$$

as before. The finiteness of the right side of (21.14) implies that $f(j,l) = a_j\, b_l$ is q-summable as an $\mathcal{A}$-valued function on $\mathbf{Z}^2$ with respect to N, so that (21.3) can be defined as an element of $\mathcal{A}$, as in Section 18. The iterated sums (21.4) and (21.5) can also be defined as elements of $\mathcal{A}$, and are equal to (21.3), as in Section 19. Of course, these iterated sums reduce to the product on the right side of (21.2). Similarly, the iterated sum (21.9) can be defined as an element of $\mathcal{A}$, and is equal to (21.3). This includes the fact that

$$(21.19)\qquad \sum_{n=-\infty}^{\infty} N(c_n)^q < \infty,$$

which follows from the finiteness of (21.18). Thus $\sum_{n=-\infty}^{\infty} c_n$ can be defined as an element of $\mathcal{A}$, and satisfies (21.2).

Suppose now that $q = \infty$, and that

$$(21.20)\qquad \lim_{j\to\pm\infty} N(a_j) = \lim_{l\to\pm\infty} N(b_l) = 0.$$

This implies that $\sum_{j=-\infty}^{\infty} a_j$ and $\sum_{l=-\infty}^{\infty} b_l$ can be defined as elements of $\mathcal{A}$, as convergent doubly-infinite series, as in Section 17, or as sums over $\mathbf{Z}$, as in Section 18. Remember that

$$(21.21)\qquad N(a_j\, b_l) \le N(a_j)\, N(b_l)$$

for every $j, l \in \mathbf{Z}$, by the submultiplicativity of N on $\mathcal{A}$. It is easy to see that

$$(21.22)\qquad \lim_{j\to\pm\infty} N(a_j\, b_{n-j}) = 0$$

for each $n \in \mathbf{Z}$, because of (21.20), which implies in particular that $N(a_j)$ and $N(b_l)$ are bounded. Hence c_n can be defined as an element of $\mathcal{A}$ as in (21.1) for each $n \in \mathbf{Z}$, and satisfies

$$(21.23)\qquad N(c_n) \le \max_{j\in\mathbf{Z}} N(a_j\, b_{n-j}).$$

One can check that $N(a_j)\, N(b_l)$ vanishes at infinity as a nonnegative real-valued function on $\mathbf{Z}^2$, using (21.20). This means that $f(j,l) = a_j\, b_l$ vanishes at infinity as an $\mathcal{A}$-valued function on $\mathbf{Z}^2$ with respect to N, by (21.21). Thus the sum (21.3) can be defined as an element of $\mathcal{A}$, as in Section 18. This sum is equal to the iterated sums (21.4) and (21.5), as in Section 19, and these iterated sums reduce to the product of $\sum_{j=-\infty}^{\infty} a_j$ and $\sum_{l=-\infty}^{\infty} b_l$. The iterated sum (21.9) can also be defined as an element of $\mathcal{A}$, and is equal to (21.3), as in Section 19. This includes the fact that

$$\lim_{n \to \pm\infty} N(c_n) = 0, \tag{21.24}$$

which can be obtained from the vanishing at infinity of $f(j,l) = a_j\, b_l$ on $\mathbf{Z}^2$. This means that $\sum_{n=-\infty}^{\infty} c_n$ can be defined as an element of $\mathcal{A}$, and satisfies (21.2).

22 Hilbert spaces

Let V and W be vector spaces over the complex numbers. Thus V and W may also be considered as vector spaces over the real numbers. A mapping T from V into W may be called *complex-linear* to indicate that T is a linear mapping from V into W as vector spaces over $\mathbf{C}$. Similarly, T may be called *real-linear* when T is linear as a mapping from V into W as vector spaces over $\mathbf{R}$. A real-linear mapping T from V into W is complex-linear when

$$T(i\, v) = i\, T(v) \tag{22.1}$$

for every $v \in V$. A real-linear mapping T from V into W is said to be *conjugate-linear* if

$$T(a\, v) = \overline{a}\, T(v) \tag{22.2}$$

for every $a \in \mathbf{C}$ and $v \in V$, where $\overline{a}$ is the complex-conjugate of a. This holds when

$$T(i\, v) = -i\, T(v) \tag{22.3}$$

for every $v \in V$, because T is real-linear.

Let V be a vector space over the real or complex numbers. An *inner product* on V is a real or complex-valued function $\langle v, w \rangle$, as appropriate, defined for $v, w \in V$, that satisfies the following properties. First, for each $w \in V$, $\langle v, w \rangle$ should be real-linear as a function of v into $\mathbf{R}$, or complex-linear as a function of v into $\mathbf{C}$, as appropriate. Second, $\langle v, w \rangle$ should be symmetric in the real case, so that

$$\langle v, w \rangle = \langle w, v \rangle \tag{22.4}$$

for every $v, w \in V$. Combining this with the first property, we get that $\langle v, w \rangle$ is linear in w for each $v \in V$. In the complex case, $\langle v, w \rangle$ should be Hermitian-symmetric, which means that

$$\langle w, v \rangle = \overline{\langle v, w \rangle} \tag{22.5}$$

for every $v, w \in V$. It follows that $\langle v, w \rangle$ is conjugate-linear in w for every $v \in V$. Note that

$$\langle v, v \rangle \in \mathbf{R} \tag{22.6}$$

for every $v \in V$ in the complex case, by (22.5). The third condition in both cases is that

$$\langle v, v \rangle > 0 \tag{22.7}$$

for every $v \in V$ with $v \neq 0$. Of course, $\langle v, w \rangle = 0$ when $v = 0$ or $w = 0$, by the first two conditions.

Let $\langle v, w \rangle$ be an inner product on V, and put

$$\|v\| = \langle v, v \rangle^{1/2} \tag{22.8}$$

for every $v \in V$, using the nonnegative square root on the right side. It is well known that

$$|\langle v, w \rangle| \leq \|v\| \, \|w\| \tag{22.9}$$

for every $v, w \in V$, which is the *Cauchy–Schwarz inequality*. Using this, one can show that

$$\|v + w\| \leq \|v\| + \|w\| \tag{22.10}$$

for every $v, w \in V$. This implies that (22.8) defines a norm on V with respect to the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, because of the other properties of the inner product. If V is complete with respect to the metric associated to this norm, then V is said to be a *Hilbert space*.

Let X be a nonempty set, and let f, g be real or complex-valued functions on X that are *square-summable*, which is to say that $|f|^2$ and $|g|^2$ are summable on X. Remember that

$$a\, b \leq (1/2)\, (a^2 + b^2) \tag{22.11}$$

for all nonnegative real numbers a and b, because $(a - b)^2 \geq 0$. This implies that

$$\sum_{x \in X} |f(x)| \, |g(x)| \leq (1/2) \sum_{x \in X} |f(x)|^2 + (1/2) \sum_{x \in X} |g(x)|^2, \tag{22.12}$$

so that $|f| \, |g|$ is summable on X. Put

$$\langle f, g \rangle = \langle f, g \rangle_{\ell^2(X, \mathbf{R})} = \sum_{x \in X} f(x) \, g(x) \tag{22.13}$$

in the real case, and

$$\langle f, g \rangle = \langle f, g \rangle_{\ell^2(X, \mathbf{C})} = \sum_{x \in X} f(x) \, \overline{g(x)} \tag{22.14}$$

in the complex case. It is easy to see that these define inner products on $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$, respectively. In both cases we have that

$$\langle f, f \rangle = \sum_{x \in X} |f(x)|^2, \tag{22.15}$$

so that the norms corresponding to (22.13) and (22.14) are the same as the usual ℓ^2 norms. Thus $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$ are Hilbert spaces, because they are complete with respect to the corresponding ℓ^2 metrics.

23 Hilbert space adjoints

Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be Hilbert spaces, both real or both complex, and let $\| \cdot \|_V$ and $\| \cdot \|_W$ be the corresponding norms on V and W, respectively. If T is a bounded linear mapping from V into W, then it is well known that there is a unique bounded linear mapping T^* from W into V such that

$$\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V \tag{23.1}$$

for every $v \in V$ and $w \in W$. This defines a mapping $T \mapsto T^*$ from $\mathcal{BL}(V, W)$ into $\mathcal{BL}(W, V)$, which is real-linear in the real case, and conjugate-linear in the complex case. One can check that

$$(T^*)^* = T \tag{23.2}$$

for every $T \in \mathcal{BL}(V, W)$, directly from this characterization of the adjoint. If $V = W$ with the same inner product, and I is the identity operator on V, then I is a bounded linear mapping from V into itself, and $I^* = I$.

If $T \in \mathcal{BL}(V, W)$, then

$$|\langle v, T^*(w) \rangle_V| = |\langle T(v), w \rangle_W| \le \|T\|_{op,VW} \, \|v\|_V \, \|w\|_W \tag{23.3}$$

for every $v \in V$ and $w \in W$, using the Cauchy–Schwarz inequality and the definition of the operator norm $\|T\|_{op,VW}$ in the second step. This implies that

$$\|T^*(w)\|_V \le \|T\|_{op,VW} \, \|w\|_W \tag{23.4}$$

for every $w \in W$, by taking $v = T^*(w)/\|T^*(w)\|_V$ when $T^*(w) \neq 0$. It follows that

$$\|T^*\|_{op,WV} \le \|T\|_{op,VW}. \tag{23.5}$$

The opposite inequality can be obtained similarly, or by applying this inequality to T^* in place of T, and using (23.2). It follows that

$$\|T^*\|_{op,WV} = \|T\|_{op,VW}. \tag{23.6}$$

Let $(Z, \langle \cdot, \cdot \rangle_Z)$ be another Hilbert space, which is real or complex depending on whether V, W are real or complex. Also let T_1 be a bounded linear mapping from V into W, and let T_2 be a bounded linear mapping from W into Z, so that their composition $T_2 \circ T_1$ is a bounded linear mapping from V into Z. Note that the adjoints T_1^* and T_2^* of T_1 and T_2 map W into V and Z into W, respectively. If $v \in V$ and $z \in Z$, then

$$\begin{aligned} \langle (T_2 \circ T_1)(v), z \rangle_Z = \langle T_2(T_1(v)), z \rangle_Z &= \langle T_1(v), T_2^*(z) \rangle_W \\ &= \langle v, T_1^*(T_2^*(z)) \rangle_V. \end{aligned} \tag{23.7}$$

This implies that

$$(T_2 \circ T_1)^* = T_1^* \circ T_2^*. \tag{23.8}$$

Let T be a bounded linear mapping from V into W again, so that T^* is a bounded linear mapping from W into V. Thus $T^* \circ T$ is a bounded linear mapping from V into itself, with

$$\|T^* \circ T\|_{op,VV} \le \|T\|_{op,VW} \, \|T^*\|_{op,WV} = \|T\|_{op,VW}^2, \tag{23.9}$$

using (23.6) in the second step. If $v \in V$, then

$$\langle (T^* \circ T)(v), v \rangle_V = \langle T^*(T(v)), v \rangle_V = \langle T(v), T(v) \rangle_W = \|T(v)\|_W^2. \tag{23.10}$$

This implies that

$$\|T(v)\|_W^2 \le \|T^* \circ T\|_{op,VV} \, \|v\|_V^2, \tag{23.11}$$

using the Cauchy–Schwarz inequality. It follows that

$$\|T\|_{op,VW}^2 \le \|T^* \circ T\|_{op,VV}, \tag{23.12}$$

and hence

$$\|T^* \circ T\|_{op,VV} = \|T\|_{op,VW}^2, \tag{23.13}$$

because of (23.9).

24 Isometric linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k. Also let N_V, N_W be q_V, q_W-seminorms on V, W with respect to $|\cdot|$ on k, respectively, for some $q_V, q_W > 0$. A linear mapping T from V into W is said to be an *isometry* if

$$N_W(T(v)) = N_V(v) \tag{24.1}$$

for every $v \in V$. This implies that T is bounded with respect to N_V and N_W, with the corresponding operator q_W-seminorm $\|T\|_{op,VW}$ equal to 1 when $N_V(v) > 0$ for some $v \in V$. If N_V is a q_V-norm on V, then (24.1) implies that T is injective.

Let Z be another vector space over k, with a q_Z-seminorm N_Z with respect to $|\cdot|$ on k for some $q_Z > 0$. If T_1 is a isometric linear mapping from V into W, and T_2 is an isometric linear mapping from W into Z, then their composition $T_2 \circ T_1$ is an isometric linear mapping from V into Z.

Let T be a one-to-one linear mapping from V onto W. If T^{-1} is a bounded linear mapping from W into V, then

$$N_V(T^{-1}(w)) \le \|T^{-1}\|_{op,WV} \, N_W(w) \tag{24.2}$$

for every $w \in W$. This is the same as saying that

$$N_V(v) \le \|T^{-1}\|_{op,WV} \, N_W(T(v)) \tag{24.3}$$

for every $v \in V$, by taking $w = T(v)$. If T and T^{-1} are both bounded linear mappings, with

$$\|T\|_{op,VW},\ \|T^{-1}\|_{op,WV} \le 1, \tag{24.4}$$

then it is easy to see that T is an isometric linear mapping. Of course, if T is an isometric linear mapping, then T^{-1} is an isometric linear mapping from W onto V.

Let $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function, and suppose that $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ are inner product spaces over k. If a linear mapping T from V into W is an isometry with respect to the corresponding norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively, then one can check that

$$\langle T(v), T(v')\rangle_W = \langle v, v'\rangle_V \tag{24.5}$$

for every $v, v' \in V$, using polarization identities. Conversely, (24.5) implies that T is an isometry, by taking $v = v'$. Suppose now that V and W are Hilbert spaces, and let T be a bounded linear mapping from V into W. Observe that

$$\langle (T^* \circ T)(v), v'\rangle_V = \langle T^*(T(v)), v'\rangle_V = \langle T(v), T(v')\rangle_W \tag{24.6}$$

for every $v, v' \in V$, where T^* is the adjoint of T, as in the previous section. Thus T is an isometry if and only if

$$\langle (T^* \circ T)(v), v'\rangle_V = \langle v, v'\rangle_V \tag{24.7}$$

for every $v, v' \in V$. It follows that T is an isometry if and only if

$$T^* \circ T = I_V, \tag{24.8}$$

where I is the identity mapping on V.

A one-to-one linear mapping T from V onto W is said to be *unitary* if (24.5) holds for every $v, v' \in V$, which is equivalent to asking that T be an isometry, as before. In this case, T^* is a one-to-one linear mapping from W onto V, and

$$T^* = T^{-1}, \tag{24.9}$$

by (24.8).

25 C^* Algebras

Let $\mathcal{A}$ be an algebra over the complex numbers. A conjugate-linear mapping

$$x \mapsto x^* \tag{25.1}$$

from $\mathcal{A}$ into itself is said to be an *involution* on $\mathcal{A}$ if

$$(x\,y)^* = y^*\,x^* \tag{25.2}$$

and

$$(x^*)^* = x \tag{25.3}$$

for every $x, y \in \mathcal{A}$. Let us suppose from now on in this section that $\mathcal{A}$ has a nonzero multiplicative identity element e. It is easy to see that

$$e^* = e, \tag{25.4}$$

using (25.2).

Let $\|\cdot\|$ be a submultiplicative norm on $\mathcal{A}$ with respect to the standard absolute value function on $\mathbf{C}$, and suppose that $\mathcal{A}$ is complete with respect to the metric associated to $\|\cdot\|$. If we also have that

$$\|x^* x\| = \|x\|^2 \tag{25.5}$$

for every $x \in \mathcal{A}$, then $\mathcal{A}$ is said to be a C^* *algebra*. Using (25.5) and submultiplicativity, we get that

$$\|x\|^2 \le \|x^*\| \, \|x\| \tag{25.6}$$

for every $x \in \mathcal{A}$, and hence

$$\|x\| \le \|x^*\|. \tag{25.7}$$

Applying this to x^* instead of x, we get that $\|x^*\| \le \|x\|$, so that

$$\|x^*\| = \|x\| \tag{25.8}$$

for every $x \in \mathcal{A}$. Note that

$$\|e\| = 1, \tag{25.9}$$

by (25.4) and (25.5).

Let X be a nonempty topological space, and consider the space $C_b(X, \mathbf{C})$ of bounded continuous complex-valued functions on X, using the standard absolute value function on $\mathbf{C}$. This is a commutative algebra over $\mathbf{C}$ with respect to pointwise addition and multiplication of functions, and the constant function equal to 1 on X is the multiplicative identity element in $C_b(X, \mathbf{C})$. The supremum norm $\|f\|_{sup}$ on $C_b(X, \mathbf{C})$ associated to the standard absolute value function on $\mathbf{C}$ is a submultiplicative norm on $C_b(X, \mathbf{C})$, and $C_b(X, \mathbf{C})$ is complete with respect to the corresponding supremum metric. If $f \in C_b(X, \mathbf{C})$, then the complex-conjugate $\overline{f}$ of f is an element of $C_b(X, \mathbf{C})$ too, and $f \mapsto \overline{f}$ defines an involution on $C_b(X, \mathbf{C})$. It is easy to see that

$$\|f \, \overline{f}\|_{sup} = \||f|^2\|_{sup} = \|f\|_{sup}^2 \tag{25.10}$$

for every $f \in C_b(X, \mathbf{C})$, so that $C_b(X, \mathbf{C})$ is a C^* algebra with respect to the supremum norm.

Let $(V, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space, let $\|\cdot\|$ be the norm on V corresponding to the inner product, and consider the algebra $\mathcal{BL}(V)$ of bounded linear mappings from V into itself with respect to $\|\cdot\|$. The identity operator I on V is the multiplicative identity element in $\mathcal{BL}(V)$, and let us suppose that $V \ne \{0\}$, so that $I \ne 0$. The mapping from $T \in \mathcal{BL}(V)$ to its adjoint T^* defines an involution on V, as in Section 23. The operator norm $\|T\|_{op}$ on $\mathcal{BL}(V)$ corresponding to $\|\cdot\|$ is submultiplicative on $\mathcal{BL}(V)$, and $\mathcal{BL}(V)$ is complete with respect to the associated metric, because V is complete with respect to the metric associated to $\|\cdot\|$, by hypothesis. Remember that $\|T\|_{op}$ satisfies (25.5), as in (23.13), so that $\mathcal{BL}(V)$ is a C^* algebra with respect to the operator norm.

A famous theorem implies that every commutative C^* algebra can be realized as the algebra $C(X, \mathbf{C})$ of continuous complex-valued functions on a nonempty compact Hausdorff topological space X, using the supremum norm and complex-conjugation as the involution, as before.

26 Some weighted conditions

Let X be a nonempty set, and let w be a positive real-valued function on X. Also let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k with a q_V-seminorm N_V with respect to $|\cdot|$ on k for some $q_V > 0$. A V-valued function f on X is said to be *bounded* with respect to w on X and N_V on V if $N_V(f(x))\, w(x)$ is bounded as a nonnegative real-valued function on X. In this case, we put

$$\|f\|_{\infty,w} = \|f\|_{\ell^\infty_w(X,V)} = \sup_{x \in X}(N_V(f(x))\, w(x)). \tag{26.1}$$

One can check that the space $\ell^\infty_w(X, V)$ of these functions is a linear subspace of the space $c(X, V)$ of all V-valued functions on X, and that (26.1) defines a q_V-seminorm on $\ell^\infty_w(X, V)$ with respect to $|\cdot|$ on k. If N_V is a q_V-norm on V, then (26.1) is a q_V-norm on $\ell^\infty_w(X, V)$. If V is also complete with respect to the q_V-metric associated to N_V, then $\ell^\infty_w(X, V)$ is complete with respect to the q_V-metric associated to (26.1), by standard arguments.

Let r be a positive real number. A V-valued function f on X is said to be *r-summable* with respect to w on X and N_V on V if $N_V(f(x))^r\, w(x)^r$ is summable as a nonnegative real-valued function on X. Let $\ell^r_w(X, V)$ be the space of these functions, and put

$$\|f\|_{r,w} = \|f\|_{\ell^r_w(X,V)} = \Big(\sum_{x \in X} N_V(f(x))^r\, w(x)^r\Big)^{1/r} \tag{26.2}$$

for every $f \in \ell^r_w(X, V)$. One can verify that $\ell^r_w(X, V)$ is a linear subspace of $c(X, V)$, and that (26.2) is an r-seminorm on $\ell^r_w(X, V)$ with respect to $|\cdot|$ on k when $r \le q_V$, and a q_V-seminorm on $\ell^r_w(X, V)$ when $q_V \le r$, as in Section 15. If N_V is a q_V-norm on V, then (26.2) is an r-norm on $\ell^r_w(X, V)$ when $r \le q_V$, and a q_V-norm on $\ell^r_w(X, V)$ when $r \ge q_V$. If V is also complete with respect to the q_V-metric associated to N_V, then $\ell^r_w(X, V)$ is complete with respect to the r or q_V-metric associated to (26.2), as appropriate. If $0 < r_1 \le r_2 \le \infty$, then

$$\ell^{r_1}_w(X, V) \subseteq \ell^{r_2}_w(X, V), \tag{26.3}$$

and

$$\|f\|_{r_2,w} \le \|f\|_{r_1,w} \tag{26.4}$$

for every $f \in \ell_w^{r_1}(X, V)$. This is basically the same as in Section 16. More precisely, if f is a V-valued function on X, then one can reduce to the earlier statements applied to $N_V(f(x))\, w(x)$ as a nonnegative real-valued function on X.

Similarly, a V-valued function f on X is said to *vanish at infinity* with respect to w on X and N_V on V if $N_V(f(x))\, w(x)$ vanishes at infinity as a nonnegative real-valued function on X. It is easy to see that the space $c_{0,w}(X, V)$ of these functions is a linear subspace of $c(X, V)$. Observe that

$$c_{0,w}(X, V) \subseteq \ell_w^\infty(X, V), \tag{26.5}$$

and that

$$\ell_w^r(X, V) \subseteq c_{0,w}(X, V) \tag{26.6}$$

when $0 < r < \infty$, as before. One can check that $c_{0,w}(X, V)$ is a closed set in $\ell_w^\infty(X, V)$ with respect to the q_V-semimetric associated to (26.1).

Remember that $c_{00}(X, V)$ is the space of V-valued functions on X with finite support. Clearly

$$c_{00}(X, V) \subseteq \ell_w^r(X, V) \tag{26.7}$$

for every $r > 0$, and thus

$$c_{00}(X, V) \subseteq c_{0,w}(X, V) \tag{26.8}$$

in particular. As before, one can verify that $c_{00}(X, V)$ is dense in $c_{0,w}(X, V)$ with respect to the q_V-semimetric associated to (26.1). If $0 < r < \infty$, then $c_{00}(X, V)$ is dense in $\ell_w^r(X, V)$ with respect to the r or q_V-semimetric associated to (26.2), as appropriate.

Let us now take $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function, and $V = k$. Let f, g be real or complex-valued functions on X that are 2-summable or equivalently square-summable with respect to w, so that $|f|^2\, w^2$ and $|g|^2\, w^2$ are summable as nonnegative real-valued functions on X. This implies that $|f|\,|g|\, w^2$ is also summable as a nonnegative real-valued function on X, as in Section 22. As before,

$$\langle f, g\rangle = \langle f, g\rangle_{\ell_w^2(X,\mathbf{R})} = \sum_{x\in X} f(x)\, g(x)\, w(x)^2 \tag{26.9}$$

and

$$\langle f, g\rangle = \langle f, g\rangle_{\ell_w^2(X,\mathbf{C})} = \sum_{x\in X} f(x)\, \overline{g(x)}\, w(x)^2 \tag{26.10}$$

define inner products on $\ell_w^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$, respectively. In both cases,

$$\langle f, f\rangle = \sum_{x\in X} |f(x)|^2\, w(x)^2 = \|f\|_{2,w}^2, \tag{26.11}$$

so that $\|f\|_{2,w}$ is the norm associated to the inner product.

27 Lebesgue spaces

Let $(X, \mathcal{A}, \mu)$ be a *measure space*, so that X is a set, $\mathcal{A}$ is a σ-algebra of measurable subsets of X, and μ is a nonnegative countably-additive measure defined on elements of $\mathcal{A}$. If f is a nonnegative real-valued function on X that is measurable with respect to $\mathcal{A}$, then the Lebesgue integral $\int_X f\,d\mu$ can be defined as a nonnegative extended real number in the usual way. Remember that the Lebesgue spaces $L^q(X, \mathbf{R})$ and $L^q(X, \mathbf{C})$ are defined for $0 < q \le \infty$, and consist of (equivalence classes of) real and complex-valued functions f on X that are measurable with respect to $\mathcal{A}$ (and equal almost everywhere with respect to μ). If $q < \infty$, then $|f(x)|^q$ is integrable on X with respect to μ, so that

$$\|f\|_q = \Big(\int_X |f|^q\,d\mu\Big)^{1/q} \tag{27.1}$$

is defined as a nonnegative real number. If $q = \infty$, then f is essentially bounded on X with respect to μ, and $\|f\|_\infty$ is defined to be the essential supremum of $|f|$ on X with respect to μ. It is well known that $\|f\|_q$ defines a norm on $L^q(X, \mathbf{R})$ and $L^q(X, \mathbf{C})$ with respect to the standard absolute value functions on $\mathbf{R}$ and $\mathbf{C}$, respectively, when $1 \le q \le \infty$. If $0 < q \le 1$, then it is easy to see that (27.1) defines a q-norm on $L^q(X, \mathbf{R})$ and $L^q(X, \mathbf{C})$. It is also well known that $L^q(X, \mathbf{R})$ and $L^q(X, \mathbf{C})$ are complete with respect to the metric or q-metric associated to $\|f\|_q$, as appropriate, for every $q > 0$.

Suppose that $0 < q_1, q_2, q_3 \le \infty$ satisfy

$$1/q_3 = 1/q_1 + 1/q_2, \tag{27.2}$$

where $1/\infty$ is interpreted as being equal to 0, as usual. If $f \in L^{q_1}(X, \mathbf{R})$ or $L^{q_1}(X, \mathbf{C})$ and $g \in L^{q_2}(X, \mathbf{R})$ or $L^{q_2}(X, \mathbf{C})$, then *Hölder's inequality* implies that $f\,g \in L^{q_3}(X, \mathbf{R})$ or $L^{q_3}(X, \mathbf{C})$, as appropriate, with

$$\|f\,g\|_{q_3} \le \|f\|_{q_1}\,\|g\|_{q_2}. \tag{27.3}$$

This is often stated with $q_3 = 1$, and it is easy to reduce to this case. Let us now take $q_1 = q_2 = 2$, so that $q_3 = 1$, and (27.3) is an integral version of the *Cauchy–Schwarz inequality*. It is easy to see that

$$\langle f, g\rangle = \langle f, g\rangle_{L^2(X,\mathbf{R})} = \int_X f\,g\,d\mu \tag{27.4}$$

defines an inner product on $L^2(X, \mathbf{R})$, and that

$$\langle f, g\rangle = \langle f, g\rangle_{L^2(X,\mathbf{C})} = \int_X f\,\overline{g}\,d\mu \tag{27.5}$$

defines an inner product on $L^2(X, \mathbf{C})$. In both cases, the norm associated to the inner product is the L^2 norm. Thus $L^2(X, \mathbf{R})$ and $L^2(X, \mathbf{C})$ are Hilbert spaces over the real and complex numbers, respectively, with respect to these inner products.

Let X be a nonempty set, and let w_0, w_1 be positive real-valued functions on X. If t is a positive real number with $t < 1$, then put

$$w_t(x) = w_0(x)^{1-t}\, w_1(x)^t \tag{27.6}$$

for every $x \in X$. This defines another positive real-valued function on X, which reduces to w_0, w_1 when $t = 0, 1$, respectively. Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let V be a vector space over k, and let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k for some $q_V > 0$. Suppose that f is a V-valued function on X that is bounded with respect to N_V on V and both w_0 and w_1 on X, as in the previous section. If $x \in X$ and $0 < t < 1$, then

$$\begin{aligned} N_V(f(x))\, w_t(x) &= (N_V(f(x))\, w_0(x))^{1-t}\, (N_V(f(x))\, w_1(x))^t \\ &\le \|f\|_{\infty,w_0}^{1-t}\, \|f\|_{\infty,w_1}^t, \end{aligned} \tag{27.7}$$

where $\|f\|_{\infty,w}$ is as in (26.1). This implies that f is bounded with respect to w_t on X and N_V on V, with

$$\|f\|_{\infty,w_t} \le \|f\|_{\infty,w_0}^{1-t}\, \|f\|_{\infty,w_1}^t. \tag{27.8}$$

If we also have that f vanishes at infinity with respect to N_V on V and either w_0 or w_1 on X, then one can check that f vanishes at infinity with respect to N_V on V and w_t on X when $0 < t < 1$, using the first step in (27.7).

Let r be a positive real number, and suppose now that f is a V-valued function on X that is r-summable with respect to N_V on V and both w_0 and w_1 on X, as in the previous section. If $0 < t < 1$, then f is r-summable with respect to w_t on X and N_V on V, with

$$\|f\|_{r,w_t} \le \|f\|_{r,w_0}^{1-t}\, \|f\|_{r,w_1}^t, \tag{27.9}$$

and where $\|f\|_{r,w}$ is as in (26.2). To see this, put $q_1 = r/(1-t)$, $q_2 = r/t$, and $q_3 = r$, which automatically satisfy (27.2). The r-summability of f with respect to w_0 on X implies that $(N_V(f(x))\, w_0(x))^{1-t}$ is q_1-summable on X in the usual sense, with

$$\|f\|_{r,w_0}^{1-t} = \Big(\sum_{x\in X} (N_V(f(x))\, w_0(x))^{(1-t)\, q_1}\Big)^{1/q_1}. \tag{27.10}$$

Similarly, the r-summability of f with respect to w_1 on X is the same as saying that $(N_V(f(x))\, w_1(x))^t$ is q_2-summable on X in the usual sense, with

$$\|f\|_{r,w_1}^t = \Big(\sum_{x\in X} (N_V(f(x))\, w_1(x))^{t\, q_2}\Big)^{1/q_2}. \tag{27.11}$$

The r-summability of f with respect to w_t is the same as the q_3-summability of $N_V(f(x))\, w_t(x)$ on X in the usual sense, with

$$\|f\|_{r,w_t} = \Big(\sum_{x\in X} (N_V(f(x))\, w_t(x))^{q_3}\Big)^{1/q_3}. \tag{27.12}$$

Using the first step in (27.7) and Hölder's inequality for sums, we get that the first two summability conditions just mentioned imply the third one, and that (27.12) is less than or equal to the product of (27.10) and (27.11). This means that f is r-summable on X with respect to w_t, and that (27.9) holds, as desired.

Part III
Power series

28 Some spaces of power series

Let k be a field, and let T be an indeterminate. The space of formal power series

$$f(T) = \sum_{j=0}^{\infty} f_j\, T^j \tag{28.1}$$

with coefficients in k is denoted $k[[T]]$, as in Section 2. This can be identified with the space $c(\mathbf{Z}_+ \cup \{0\}, k)$ of all k-valued functions on the set $\mathbf{Z}_+ \cup \{0\}$ of nonnegative integers, as before. Similarly, the space $k[[T]]$ of formal polynomials in T with coefficients in k corresponds to the space $c_{00}(\mathbf{Z}_+ \cup \{0\}, k)$ of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$ with finite support.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let r be a positive real number. Put

$$w_r(j) = r^j \tag{28.2}$$

for every nonnegative integer j, which defines a positive real-valued function on $\mathbf{Z}_+ \cup \{0\}$. If q is a positive real number, then put

$$k^q_r[[T]] = \biggl\{ f(T) \in k[[T]] : \sum_{j=0}^{\infty} |f_j|^q\, r^{q\,j} < \infty \biggr\}, \tag{28.3}$$

and put

$$\|f(T)\|_{q,r} = \|f(T)\|_{k^q_r[[T]]} = \Bigl(\sum_{j=0}^{\infty} |f_j|^q\, r^{q\,j} \Bigr)^{1/q} \tag{28.4}$$

for every $f(T) \in k^q_r[[T]]$. Equivalently, (28.3) can be identified with the space $\ell^q_{w_r}(\mathbf{Z}_+ \cup \{0\}, k)$ of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$ that are q-summable with respect to w_r on $\mathbf{Z}_+ \cup \{0\}$ and $|\cdot|$ on k, as a linear subspace of $c(\mathbf{Z}_+ \cup \{0\}, k)$, as in Section 26. Similarly, (28.4) corresponds to $\|\cdot\|_{\ell^q_{w_r}(\mathbf{Z}_+ \cup \{0\}, k)}$ applied to $j \mapsto f_j$ as a k-valued function on $\mathbf{Z}_+ \cup \{0\}$. Thus (28.4) defines a q-norm on (28.3) with respect to $|\cdot|$ on k when $q \le q_k$, and a q_k-norm on (28.3) when $q \ge q_k$, as before. Note that

$$k[T] \subseteq k^q_r[[T]], \tag{28.5}$$

and that $k[T]$ is dense in $k_r^q[[T]]$ with respect to the q or q_k-metric associated to (28.4).

Put

$$k_r^\infty[[T]] = \{f(T) \in k[[T]] : |f_j|\, r^j \text{ is bounded on } \mathbf{Z}_+ \cup \{0\}\}, \tag{28.6}$$

and

$$\|f(T)\|_{\infty,r} = \|f(T)\|_{k_r^\infty[[T]]} = \sup_{j \ge 0}(|f_j|\, r^j) \tag{28.7}$$

for every $f(T) \in k_r^\infty[[T]]$. As before, (28.6) can be identified with the space $\ell^\infty_{w_r}(\mathbf{Z}_+ \cup \{0\}, k)$ of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$ that are bounded with respect to w_r on $\mathbf{Z}_+ \cup \{0\}$ and $|\cdot|$ on k, and (28.7) corresponds to $\|\cdot\|_{\ell^\infty_{w_r}(\mathbf{Z}_+ \cup \{0\}, k)}$ applied to $j \mapsto f_j$ as a k-valued function on $\mathbf{Z}_+ \cup \{0\}$. In particular, (28.6) is a linear subspace of $k[[T]]$, and (28.7) is a q_k-norm on (28.6) with respect to $|\cdot|$ on k. If $0 < q \le \widetilde{q} \le \infty$, then

$$k_r^q[[T]] \subseteq k_r^{\widetilde{q}}[[T]], \tag{28.8}$$

and

$$\|f(T)\|_{\widetilde{q},r} \le \|f(T)\|_{q,r} \tag{28.9}$$

for every $f(T) \in k_r^q[[T]]$, as in (26.3) and (26.4).

Put

$$k_{0,r}[[T]] = \big\{f(T) \in k[[T]] : \lim_{j \to \infty} |f_j|\, r^j = 0\big\}, \tag{28.10}$$

which can be identified with the space $c_{0,w_r}(\mathbf{Z}_+ \cup \{0\}, k)$ of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$ that vanish at infinity with respect to w_r on $\mathbf{Z}_+ \cup \{0\}$ and $|\cdot|$ on k. Thus (28.10) is a closed linear subspace of (28.6) with respect to the q_k-metric associated to (28.7). We also have that

$$k[T] \subseteq k_{0,r}[[T]], \tag{28.11}$$

and that $k[T]$ is dense in $k_{0,r}[[T]]$ with respect to the q_k-metric associated to (28.7). If $0 < q < \infty$, then

$$k_r^q[[T]] \subseteq k_{0,r}[[T]], \tag{28.12}$$

as in (26.6).

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k. In this case,

$$k_r^q[[T]] = k[[T]] \tag{28.13}$$

when $0 < q < \infty$ and $0 < r < 1$, and when $q = \infty$ and $0 < r \le 1$. Otherwise,

$$k_r^q[[T]] = k[T] \tag{28.14}$$

when $0 < q < \infty$ and $r \ge 1$, and when $q = \infty$ and $r > 1$. Similarly,

$$k_{0,r}[[T]] = k[[T]] \tag{28.15}$$

when $0 < r < 1$, and

$$k_{0,r}[[T]] = k[T] \tag{28.16}$$

when $r \ge 1$.

29 Submultiplicativity

Let k be a field, and let T be an indeterminate. Also let $f(T) = \sum_{j=0}^\infty f_j\, T^j$ and $g(T) = \sum_{l=0}^\infty g_l\, T^l$ be formal power series in T with coefficients in k. Remember that their product $f(T)\, g(T) = h(T) = \sum_{n=0}^\infty h_n\, T^n$ is defined by

$$h_n = \sum_{j=0}^n f_j\, g_{n-j} \tag{29.1}$$

for every $n \ge 0$, as in Section 2.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let $0 < q \le q_k$ be given. Thus $|\cdot|$ may be considered as a q-absolute value function on k as well. If $q < \infty$, then

$$|h_n|^q \le \sum_{j=0}^n |f_j|^q\, |g_{n-j}|^q \tag{29.2}$$

for every $n \ge 0$. If $q = \infty$, then

$$|h_n| \le \max_{0 \le j \le n} (|f_j|\, |g_{n-j}|) \tag{29.3}$$

for every $n \ge 0$.

Let r be a positive real number. If $q < \infty$, then

$$|h_n|^q\, r^{q\,n} \le \sum_{j=0}^n |f_j|^q\, |g_{n-j}|^q\, r^{q\,n} = \sum_{j=0}^n (|f_j|^q\, r^{q\,j})\, (|g_{n-j}|^q\, r^{q\,(n-j)}) \tag{29.4}$$

for every $n \ge 0$, using (29.2) in the first step. If $q = \infty$, then

$$|h_n|\, r^n \le \max_{0 \le j \le n} (|f_j|\, |g_{n-j}|)\, r^n = \max_{0 \le j \le n} ((|f_j|\, r^j)\, (|g_{n-j}|\, r^{n-j})) \tag{29.5}$$

for every $n \ge 0$, using (29.3) in the first step.

Suppose that $f(T), g(T) \in k_r^q[[T]]$. If $q < \infty$, then

$$\sum_{n=0}^\infty |h_n|^q\, r^{q\,n} \le \sum_{n=0}^\infty \sum_{j=0}^n (|f_j|^q\, r^{q\,j})\, (|g_{n-j}|^q\, r^{q\,(n-j)}), \tag{29.6}$$

by (29.4). The right side is equal to

$$\Big(\sum_{j=0}^\infty |f_j|^q\, r^{q\,j}\Big)\, \Big(\sum_{l=0}^\infty |g_l|^q\, r^{q\,l}\Big). \tag{29.7}$$

More precisely, the right side of (29.4) corresponds to the nth term of the Cauchy product of these two series, as in Section 20. It follows that $f(T)\, g(T) = h(T)$ is an element of $k_r^q[[T]]$ too, with

$$\|f(T)\, g(T)\|_{q,r} \le \|f(T)\|_{q,r}\, \|g(T)\|_{q,r}. \tag{29.8}$$

If $q = \infty$, then we get that

$$|h_n| \, r^n \le \|f(T)\|_{\infty,r} \, \|g(T)\|_{\infty,r} \tag{29.9}$$

for every $n \ge 0$, using (29.5). This implies that $f(T) \, g(T) = h(T) \in k_r^\infty[[T]]$, and satisfies (29.8). If $f(T), g(T) \in k_{0,r}[[T]]$, then one can check that

$$\lim_{n \to \infty} |h_n| \, r^n = 0, \tag{29.10}$$

using (29.5) again. This means that $f(T) \, g(T) = h(T) \in k_{0,r}[[T]]$ in this situation. Thus $k_r^q[[T]]$ is a subalgebra of $k[[T]]$ when $q \le q_k$, and $k_{0,r}[[T]]$ is a subalgebra of $k[[T]]$ when $q_k = \infty$.

30 Radius of convergence

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let T be an indeterminate. Suppose that $f(T) = \sum_{j=0}^\infty f_j \, T^j \in k_r^q[[T]]$ for some $q > 0$ and $0 < r < \infty$. If $0 < r_1 \le r$, then it is easy to see that $f(T) \in k_{r_1}^q[[T]]$, with

$$\|f(T)\|_{q,r_1} \le \|f(T)\|_{q,r}. \tag{30.1}$$

Suppose that $f(T) \in k_r^\infty[[T]]$ and $0 < r_1 < r$, and observe that

$$|f_j| \, r_1^j = |f_j| \, r^j \, (r_1/r)^j \le \|f(T)\|_{\infty,r} \, (r_1/r)^j \tag{30.2}$$

for every $j \ge 0$. This implies that

$$\lim_{j \to \infty} |f_j| \, r_1^j = 0, \tag{30.3}$$

so that $f(T) \in k_{0,r_1}[[T]]$. If $0 < q < \infty$, then we get that

$$\begin{aligned} \sum_{j=0}^\infty |f_j|^q \, r_1^{q\,j} &\le \|f(T)\|_{\infty,r}^q \sum_{j=0}^\infty (r_1/r)^{q\,j} \\ &= \|f(T)\|_{\infty,r}^q \, (1 - (r_1/r)^q)^{-1}. \end{aligned} \tag{30.4}$$

It follows that $f(T) \in k_{r_1}^q[[T]]$, with

$$\|f(T)\|_{q,r_1} \le \|f(T)\|_{\infty,r} \, (1 - (r_1/r)^q)^{-1/q}. \tag{30.5}$$

The *radius of convergence* of $f(T) \in k[[T]]$ may be defined as a nonnegative extended real number as the supremum of the nonnegative real numbers r such that

$$|f_j| \, r^j \text{ is bounded on } \mathbf{Z}_+ \cup \{0\}. \tag{30.6}$$

Of course, (30.6) holds trivially when $r = 0$, so that the set of $r \ge 0$ with this property is nonempty. Equivalently, the radius of convergence of $f(T)$ is the supremum of the set of $r \ge 0$ such that

$$\lim_{j \to \infty} |f_j| \, r^j = 0. \tag{30.7}$$

If q is any positive real number, then the radius of convergence of $f(T)$ can also be characterized as the supremum of the set of $r \geq 0$ such that

$$\sum_{j=0}^{\infty} |f_j|^q \, r^{q\,j} < \infty. \tag{30.8}$$

Indeed, (30.8) implies (30.7), and (30.7) implies (30.6). If (30.6) holds for some $r > 0$, then the analogues of (30.7) and (30.8) hold with r replaced by any $r_1 \in [0, r)$, as in the preceding paragraph. Hence the suprema of these sets are the same. If $r \geq 0$ is strictly less than the radius of convergence of $f(T)$, then $f(T)$ satisfies each of these conditions.

If $t \in k$ and $t \neq 0$, then it is easy to see that the radius of convergence of $t\, f(T)$ is equal to the radius of convergence of $f(T)$. If $g(T) \in k[[T]]$ too, then the radius of convergence of $f(T) + g(T)$ is greater than or equal to the minimum of the radii of convergence of $f(T)$ and $g(T)$. Similarly, the radius of convergence of $f(T)\, g(T)$ is greater than or equal to the minimum of the radii of convergence of $f(T)$ and $g(T)$. This follows from (29.8), where $0 < q \leq q_k$, as before.

If $0 < r \leq \infty$, then we let $k_r[[T]]$ be the space of $f(T) \in k[[T]]$ whose radius of convergence is greater than or equal to r. This is a subalgebra of $k[[T]]$, by the remarks in the preceding paragraph.

31 A multiplicativity condition

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let T be an indeterminate. Suppose that $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ and $g(T) = \sum_{l=0}^{\infty} g_l\, T^l$ are elements of $k_{0,r}[[T]]$ for some positive real number r. This implies that $f(T)\, g(T) \in k_{0,r}[[T]]$, with

$$\|f(T)\, g(T)\|_{\infty,r} \leq \|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r}, \tag{31.1}$$

as in Section 29. In fact, it is well known that

$$\|f(T)\, g(T)\|_{\infty,r} = \|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r} \tag{31.2}$$

under these conditions. To see this, it suffices to verify that

$$\|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r} \leq \|f(T)\, g(T)\|_{\infty,r}. \tag{31.3}$$

Of course, this is trivial when $f(T) = 0$ or $g(T) = 0$. Thus we may suppose that $f(T) \neq 0$ and $g(T) \neq 0$.

By hypothesis, $|f_j|\, r^j \to 0$ as $j \to \infty$, and $|g_l|\, r^l \to 0$ as $l \to \infty$. These conditions imply that the suprema in the definitions of $\|f(T)\|_{\infty,r}$ and $\|g(T)\|_{\infty,r}$ are attained. Let j_0 and l_0 be the smallest nonnegative integers such that

$$|f_{j_0}|\, r^{j_0} = \|f(T)\|_{\infty,r} \tag{31.4}$$

and

$$|g_{l_0}|\,r^{l_0} = \|g(T)\|_{\infty,r}. \tag{31.5}$$

In particular,

$$\|f(T)\|_{\infty,r}\,\|g(T)\|_{\infty,r} = |f_{j_0}|\,|g_{l_0}|\,r^{j_0+l_0}. \tag{31.6}$$

Remember that $f(T)\,g(T) = h(T) = \sum_{n=0}^{\infty} h_n\,T^n$, where $h_n = \sum_{j=0}^{n} f_j\,g_{n-j}$ for each $n \ge 0$. Note that

$$|h_{j_0+l_0}|\,r^{j_0+l_0} \le \|h(T)\|_{\infty,r} = \|f(T)\,g(T)\|_{\infty,r}. \tag{31.7}$$

To get (31.3), it is enough to check that

$$|f_{j_0}|\,|g_{l_0}|\,r^{j_0+l_0} \le |h_{j_0+l_0}|\,r^{j_0+l_0}. \tag{31.8}$$

Clearly

$$|f_j|\,r^j \le \|f(T)\|_{\infty,r} = |f_{j_0}|\,r^{j_0} \tag{31.9}$$

for every $j \ge 0$, with a strict inequality when $j < j_0$, by the minimality of j_0. Similarly,

$$|g_l|\,r^l \le \|g(T)\|_{\infty,r} = |g_{l_0}|\,r^{l_0} \tag{31.10}$$

for every $l \ge 0$, with a strict inequality when $l < l_0$, by the minimality of l_0. It follows that

$$|f_j|\,|g_l|\,r^{j+l} \le |f_{j_0}|\,|g_{l_0}|\,r^{j_0+l_0} \tag{31.11}$$

for every $j, l \ge 0$, with a strict inequality when either $j < j_0$ or $l < l_0$. If $j + l = j_0 + l_0$, then we get that

$$|f_j|\,|g_l| \le |f_{j_0}|\,|g_{l_0}|, \tag{31.12}$$

with a strict inequality when either $j < j_0$ or $l < l_0$.

We would like to use these inequalities to obtain that

$$|h_{j_0+l_0} - f_{j_0}\,g_{l_0}| < |f_{j_0}|\,|g_{l_0}|. \tag{31.13}$$

Remember that $h_{j_0+l_0}$ is equal to the sum of $f_j\,g_l$ over $j, l \ge 0$ with $j+l = j_0+l_0$. This condition on $j + l$ implies that $j \ne j_0$ if and only if $l \ne l_0$, in which case either $j < j_0$ or $l < l_0$. This means that $h_{j_0+l_0} - f_{j_0}\,g_{l_0}$ can be expressed as the sum of finitely many terms, each of which corresponds to a strict inequality in (31.12). Thus (31.13) follows from the ultrametric version of the triangle inequality.

Of course,

$$|f_{j_0}|\,|g_{l_0}| \le \max(|h_{j_0+l_0}|, |h_{j_0+l_0} - f_{j_0}\,g_{l_0}|), \tag{31.14}$$

by the ultrametric version of the triangle inequality. This implies that

$$|f_{j_0}|\,|g_{l_0}| \le |h_{j_0+l_0}|, \tag{31.15}$$

because of (31.13). Hence (31.8) holds, as desired. We also have that $|h_{j_0+l_0}|$ is less than or equal to $|f_{j_0}|\,|g_{l_0}|$, by (31.13) and the ultrametric version of the triangle inequality. Thus we have equality in (31.15), even if this is not needed here.

32 Some continuity properties

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let T be an indeterminate, and let $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ be a formal power series in T with coefficients in k. Thus

$$\sup_{j \ge 0} (|f_j|\, r^j) \tag{32.1}$$

is defined as a nonnegative extended real number for every positive real number r. If $0 < r_1 \le r$, then

$$\sup_{j \ge 0} (|f_j|\, r_1^j) \tag{32.2}$$

is automatically less than or equal to (32.1). In fact, (32.2) tends to (32.1) as $r_1 \to r-$, which is to say as $r_1 \in (0, r)$ approaches r, with suitable interpretations for extended real numbers. This can be obtained from the continuity of r^j on $\mathbf{R}$ for each $j \ge 0$.

Similarly, if q is a positive real number, then

$$\sum_{j=0}^{\infty} |f_j|^q\, r^j \tag{32.3}$$

can be defined as a nonnegative extended real number for every positive real number r. As before,

$$\sum_{j=0}^{\infty} |f_j|^q\, r_1^j \tag{32.4}$$

is automatically less than or equal to (32.3) when $0 < r_1 \le r$. One can check that (32.4) tends to (32.3) as $r_1 \to r-$, with suitable interpretations for extended real numbers. This uses the continuity of the partial sums

$$\sum_{j=0}^{n} |f_j|^q\, r^j \tag{32.5}$$

in r for each nonnegative integer n. Of course, (32.3) is the same as the supremum of (32.5) over all nonnegative integers n.

Suppose now that $|\cdot|$ is an ultrametric absolute value function on k, and that $f(T)$, $g(T)$ are elements of $k_r^{\infty}[[T]]$ for some positive real number r. Thus $f(T)\, g(T) \in k_r^{\infty}[[T]]$, with

$$\|f(T)\, g(T)\|_{\infty,r} \le \|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r}, \tag{32.6}$$

as in Section 29. If $0 < r_1 < r$, then $f(T), g(T) \in k_{0,r_1}[[T]]$, as in Section 30, and

$$\|f(T)\, g(T)\|_{\infty,r_1} = \|f(T)\|_{\infty,r_1}\, \|g(T)\|_{\infty,r_1}, \tag{32.7}$$

as in the previous section. It follows that

$$\|f(T)\, g(T)\|_{\infty,r} = \|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r}, \tag{32.8}$$

by taking $r_1 \to r-$ in (32.7).

33 Functions on Banach algebras

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative q-norm N with respect to $|\cdot|$ on k for some $q > 0$. Suppose that $\mathcal{A}$ has a multiplicative identity element e with $N(e) = 1$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Let T be an indeterminate, and let r be a positive real number. Suppose for the moment that $q < \infty$, and that $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ is an element of $k_r^q[[T]]$. If $a \in \mathcal{A}$ satisfies

$$N(a) \le r, \tag{33.1}$$

then we would like to put

$$f(a) = \sum_{j=0}^{\infty} f_j\, a^j, \tag{33.2}$$

where a^j is interpreted as being equal to e when $j = 0$. Of course,

$$N(a^j) \le N(a)^j \le r^j \tag{33.3}$$

for each $j \ge 0$, so that

$$\sum_{j=0}^{\infty} N(f_j\, a^j)^q = \sum_{j=0}^{\infty} |f_j|^q\, N(a^j)^q \le \sum_{j=0}^{\infty} |f_j|^q\, r^{q\,j} = \|f(T)\|_{q,r}^q, \tag{33.4}$$

where $\|f(T)\|_{q,r}$ is as in (28.4). In particular, this means that the series on the right side of (33.2) converges q-absolutely with respect to N. It follows that this series converges in $\mathcal{A}$, because $\mathcal{A}$ is complete with respect to the q-metric associated to N, as in Section 9. We also get that

$$N(f(a)) \le \Big(\sum_{j=0}^{\infty} N(f_j\, a^j)^q\Big)^{1/q} \le \|f(T)\|_{q,r}, \tag{33.5}$$

using (9.9) in the first step.

Suppose now that $q = \infty$, and that $f(T) \in k_{0,r}[[T]]$. If $a \in \mathcal{A}$ satisfies (33.1), then

$$N(f_j\, a^j) = |f_j|\, N(a^j) \le |f_j|\, r^j \tag{33.6}$$

for each $j \ge 0$, so that

$$\lim_{j \to \infty} N(f_j\, a^j) = 0. \tag{33.7}$$

This implies that the series on the right side of (33.2) converges in $\mathcal{A}$, because $\mathcal{A}$ is complete with respect to the ultrametric associated to N, as in Section 9 again. In this situation, we have that

$$N(f(a)) \le \max_{j \ge 0} N(f_j\, a^j) \le \max_{j \ge 0}(|f_j|\, r^j) = \|f(T)\|_{\infty,r}, \tag{33.8}$$

using (9.11) in the first step.

We may as well ask that $q \le q_k$, as in Section 7. Hence $k_r^q[[T]]$ is a subalgebra of $k[[T]]$, and $k_{0,r}[[T]]$ is a subalgebra of $k[[T]]$ when $q = \infty$, as in Section 29. If $q < \infty$, then

$$f(T) \mapsto f(a) \tag{33.9}$$

defines a homomorphism from $k_r^q[[T]]$ into $\mathcal{A}$, as algebras over k. More precisely, this mapping is clearly linear, and products of elements of $k_r^q[[T]]$ correspond to Cauchy products of series in $\mathcal{A}$. Similarly, if $q = \infty$, then (33.9) defines an algebra homomorphism from $k_{0,r}[[T]]$ into $\mathcal{A}$.

Let $f(T)$ be an element of $k_r^q[[T]]$ when $q < \infty$, or an element of $k_{0,r}[[T]]$ when $q = \infty$. In both cases, (33.2) defines a mapping from the closed ball

$$\{a \in \mathcal{A} : N(a) \le r\} \tag{33.10}$$

in $\mathcal{A}$ centered at 0 with radius r into $\mathcal{A}$. Observe that the partial sums of (33.2) converge uniformly on (33.10), by estimating the errors as before. It is easy to see that the partial sums are uniformly continuous on (33.10), with respect to the q-metric on $\mathcal{A}$ associated to N, and its restriction to (33.10). This implies that (33.2) is uniformly continuous on (33.10) too, by standard arguments.

Suppose now that $0 < r \le \infty$, and that $f(T)$ is an element of the space $k_r[[T]]$ defined in Section 30. If $0 < r_1 < r$, then $f(T)$ is an element of $k_{r_1}^q[[T]]$ and $k_{0,r_1}[[T]]$, as before. If $a \in \mathcal{A}$ satisfies $N(a) < r$, then we can use this to define $f(a)$ as an element of $\mathcal{A}$, as in (33.2). This defines $f(a)$ as a mapping from

$$\{a \in \mathcal{A} : N(a) < r\} \tag{33.11}$$

into $\mathcal{A}$. It is easy to see that this mapping is continuous, because its restriction to any closed ball in $\mathcal{A}$ centered at 0 with radius $r_1 < r$ is continuous, as in the preceding paragraph.

34 Holomorphic functions

In this section, we take k to be the field $\mathbf{C}$ of complex numbers, with the standard absolute value function. If U is a nonempty open subset of the complex plane, then let $H(U)$ be the space of holomorphic functions on U. This is a subalgebra of the algebra $C(U, \mathbf{C})$ of all continuous complex-valued functions on U. If $0 < r \le \infty$, then put

$$U_r = \{z \in \mathbf{C} : |z| < r\}. \tag{34.1}$$

Let T be an indeterminate, and let $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ be an element of the space $\mathbf{C}_r[[T]]$ of formal power series in T with complex coefficients and radius of convergence greater than or equal to r, as in Section 30. If $z \in U_r$, then put

$$f(z) = \sum_{j=0}^{\infty} f_j\, z^j, \tag{34.2}$$

where the series on the right converges absolutely, as in the previous section. It is well known that this defines a holomorphic function on U_r, and that every holomorphic function on U_r can be represented by a unique power series in this way.

If U is a nonempty open subset of $\mathbf{C}$ again, then let $H^\infty(U)$ be the space of bounded holomorphic functions on U. This is a subalgebra of the algebra $C_b(U, \mathbf{C})$ of all bounded continuous complex-valued functions on U. Note that $H^\infty(U)$ contains all constant complex functions on U. It is well known that $H^\infty(U)$ is a closed set in $C_b(U, \mathbf{C})$, with respect to the supremum metric.

Let r be a positive real number, and let

$$\overline{U_r} = \{z \in \mathbf{C} : |z| \le r\} \tag{34.3}$$

be the closed disk in $\mathbf{C}$ centered at 0 with radius r, which is the same as the closure in $\mathbf{C}$ of the open disk U_r centered at 0 with radius r. Let $A(U_r)$ be the space of continuous complex-valued functions on $\overline{U_r}$ that are holomorphic on U_r. This is a subalgebra of the algebra $C(\overline{U_r}, \mathbf{C})$ of continuous complex-valued functions on $\overline{U_r}$. It is well known that $A(U_r)$ is a closed set in $C(\overline{U_r}, \mathbf{C})$ with respect to the supremum metric, as in the previous paragraph.

Let $f(T)$ be an element of the space $\mathbf{C}^1_r[[T]]$ defined in Section 28, so that

$$\|f(T)\|_{1,r} = \sum_{j=0}^{\infty} |f_j|\, r^j \tag{34.4}$$

is finite. If $z \in \overline{U_r}$, then the series on the right side of (34.2) converges absolutely, and (34.2) defines a continuous complex-valued function on $\overline{U_r}$, as in the previous section. We also have that (34.2) is holomorphic on U_r, as before, so that (34.2) defines an element of $A(U_r)$. Note that

$$|f(z)| \le \|f(T)\|_{1,r} \tag{34.5}$$

for every $z \in \overline{U_r}$.

Now let $f(z)$ by any continuous complex-valued function on $\overline{U_r}$. If $t \in \mathbf{C}$ and $|t| \le 1$, then $f(t\,z)$ defines a continuous complex-valued function on $\overline{U_r}$. It is well known that $\overline{U_r}$ is compact with respect to the standard Euclidean metric on $\mathbf{C}$, and hence that $f(z)$ is uniformly continuous on $\overline{U_r}$ with respect to the standard Euclidean metric on $\mathbf{C}$ and its restriction to $\overline{U_r}$. This implies that $f(t\,z)$ tends to $f(z)$ uniformly as a function of z on $\overline{U_r}$, as $t \in \overline{U_1}$ tends to 1. Of course, $f(t\,z)$ is holomorphic as a function of z on $U_{r/|t|}$. If $|t| < 1$, then $r/|t| > r$, and it follows that $f(t\,z)$ can be represented by an absolutely convergent power series in z on $\overline{U_r}$. In particular, one can use this to approximate $f(z)$ by polynomials in z uniformly on $\overline{U_r}$. More precisely, one can first approximate $f(z)$ uniformly on $\overline{U_r}$ by $f(t\,z)$, by taking $t \in U_1$ sufficiently close to 1. One can approximate $f(t\,z)$ uniformly as a function of $z \in \overline{U_r}$ by polynomials when $|t| < 1$, using the absolute convergence of its power series expansion in z on $\overline{U_r}$.

35 Rescaling power series

Let k be a field, and let T be an indeterminate. If $t \in k$ and $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ is a formal power series in T with coefficients in k, then define $R_t(f(T)) \in k[[T]]$

by

$$R_t(f(T)) = \sum_{j=0}^\infty f_j\, t^j\, T^j. \tag{35.1}$$

This may also be denoted $(R_t(f))(T)$, or simply $R_t(f)$. One can check that R_t defines a homomorphism from $k[[T]]$ into itself, as an algebra over k. Of course, R_1 is the identity mapping on $k[[T]]$, and R_0 reduces to the mapping from $f(T)$ to f_0. If u is another element of k, then it is easy to see that

$$R_t \circ R_u = R_{t\,u}. \tag{35.2}$$

If $t \neq 0$, then R_t is a one-to-one mapping from $k[[T]]$ onto itself, with $R_t^{-1} = R_{1/t}$. Note that R_t maps $k[T]$ into itself, and onto itself when $t \neq 0$.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let r be a positive real number. Also let $t \in k \setminus \{0\}$, $f(T) \in k[[T]]$, and $0 < q \le \infty$ be given. It is easy to see that $R_t(f(T))$ is in the space $k_r^q[[T]]$ defined in Section 28 if and only if $f(T) \in k^q_{r\,|t|}[[T]]$. In this case, we have that

$$\|R_t(f(T))\|_{q,r} = \|f(T)\|_{q,r\,|t|}. \tag{35.3}$$

Similarly, $R_t(f(T))$ is an element of the space $k_{0,r}[[T]]$ defined earlier if and only if $f(T) \in k_{0,r\,|t|}[[T]]$. If $0 < r \le \infty$, then $R_t(f(T))$ is in the space $k_r[[T]]$ defined in Section 30 if and only if $f(T) \in k_{r\,|t|}[[T]]$. This follows from the previous statements, applied to radii less than r.

Let $\mathcal{A}$ be an algebra over k with a submultiplicative q-norm N with respect to $|\cdot|$ on k for some $0 < q \le q_k$. Suppose that $\mathcal{A}$ has a multiplicative identity element e, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Let $t \in k \setminus \{0\}$ and a positive real number r be given, and let a be an element of $\mathcal{A}$ with $N(a) \le r$. Suppose for the moment that $q < \infty$, and that $f(T) \in k^q_{r\,|t|}[[T]]$, so that $R_t(f(T)) \in k_r^q[[T]]$. Note that

$$N(t\,a) = |t|\,N(a) \le r\,|t|, \tag{35.4}$$

so that $f(t\,a)$ can be defined as an element of $\mathcal{A}$ as in Section 33. Similarly, $(R_t(f))(a)$ can be defined as an element of $\mathcal{A}$, and in fact

$$(R_t(f))(a) = \sum_{j=0}^\infty f_j\, t^j\, a^j = \sum_{j=0}^\infty f_j\,(t\,a)^j = f(t\,a). \tag{35.5}$$

If $q = \infty$ and $f(T) \in k_{0,r\,|t|}[[T]]$, then $R_t(f(T)) \in k_{0,r}[[T]]$, $f(t\,a)$ and $(R_t(f))(a)$ can be defined as elements in $\mathcal{A}$, and they are equal, as in (35.5).

Suppose now that $0 < r \le \infty$ and $f(T) \in k_{r\,|t|}[[T]]$, so that $R_t(f(T))$ is an element of $k_r[[T]]$. If $a \in \mathcal{A}$ satisfies $N(a) < r$, then $N(t\,a) < r\,|t|$, $f(t\,a)$ and $(R_t(f))(a)$ can be defined as elements of $\mathcal{A}$, and they are equal, as in (35.5) again.

36 Hardy spaces

Let r be a positive real number, and let f be a holomorphic function on the open disk U_r in the complex plane centered at 0 with radius r. If q and r_1 are positive real numbers, with $r_1 < r$, then put

$$M_q(f, r_1) = \Big(\frac{1}{2\,\pi\, r_1} \int_{|z|=r_1} |f(z)|^q\,|dz|\Big)^{1/q}. \tag{36.1}$$

More precisely, the integral is taken over the set of $z \in \mathbf{C}$ with $|z| = r_1$, with respect to arclength. Similarly, put

$$M_\infty(f, r_1) = \sup\{|f(z)| : z \in \mathbf{C},\ |z| = r_1\}. \tag{36.2}$$

Equivalently,

$$M_q(f, r_1) = \Big(\frac{1}{2\,\pi} \int_{|z|=1} |f(r_1\, z)|^q\,|dz|\Big)^{1/q} \tag{36.3}$$

when $0 < q < \infty$, and

$$M_\infty(f, r_1) = \sup\{|f(r_1\, z)| : z \in \mathbf{C},\ |z| = 1\}. \tag{36.4}$$

In particular, $M_q(f, r_1)$ can be defined for $r_1 = 0$, using (36.3) when $q = \infty$. It is well known that $M_q(f, r_1)$ is monotonically increasing in r_1 for each $q > 0$, which follows from the maximum principle when $q = \infty$. Note that $M_q(f, r_1)$ is monotonically increasing in q as well, by the inequalities of Hölder or Jensen.

The *Hardy space* $H^q(U_r)$ is defined for $0 < q \le \infty$ as the space of holomorphic functions f on U_r such that $M_q(f, r_1)$ is bounded for $0 \le r_1 < r$, in which case we put

$$\|f\|_{H^q(U_r)} = \sup_{0 \le r_1 < r} M_q(f, r_1). \tag{36.5}$$

One can check that $H^q(U_r)$ is a linear subspace of the space $H(U_r)$ of all holomorphic functions on U_r. If $q \ge 1$, then (36.5) defines a norm on $H^q(U_r)$, with respect to the standard absolute value function on $\mathbf{C}$. If $0 < q \le 1$, then (36.5) defines a q-norm on $H^q(U_r)$. It is easy to see that f is bounded on U_r if and only if $M_\infty(f, r_1)$ is bounded for $0 \le r_1 < \infty$, so that this definition of $H^\infty(U_r)$ is the same as the space of bounded holomorphic functions on U_r, as in Section 34. If $0 < q_1 \le q_2 \le \infty$, then

$$H^{q_2}(U_r) \subseteq H^{q_1}(U_r), \tag{36.6}$$

with

$$\|f\|_{H^{q_1}(U_r)} \le \|f\|_{H^{q_2}(U_r)} \tag{36.7}$$

for every $f \in H^{q_2}(U_r)$. This follows from the monotonicity of $M_q(f, r_1)$ in q.

Let T be an indeterminate, and let $f(T) = \sum_{j=0}^\infty f_j\, T^j$ be a formal power series in T with complex coefficients. Suppose that $f(T)$ has radius of convergence greater than or equal to r, so that $f(z) = \sum_{j=0}^\infty f_j\, z^j$ defines a holomorphic function on U_r. If $0 \le r_1 < r$, then

$$M_2(f, r_1) = \Big(\sum_{j=0}^{\infty} |f_j|^2 \, r_1^{2\,j}\Big)^{1/2}, \tag{36.8}$$

by a standard computation. If $f \in H^2(U_r)$, then it follows that $f(T)$ is an element of the space $\mathbf{C}_r^2[[T]]$ defined in Section 28, using the standard absolute value function on $\mathbf{C}$, with

$$\|f\|_{H^2(U_r)} = \|f(T)\|_{\mathbf{C}_r^2[[T]]}, \tag{36.9}$$

where the right side is as in (28.4). Conversely, if $f(T) \in \mathbf{C}_r^2[[T]]$, then $f(T)$ has radius of convergence greater than or equal to r, and the corresponding holomorphic function f on U_r is an element of $H^2(U_r)$.

Suppose that $f(T)$ is an element of the space $\mathbf{C}_r[[T]]$ defined in Section 30 again, so that $f(z)$ defines a holomorphic function on U_r, as before. If j is a nonnegative integer and $0 < r_1 < r$, then

$$f_j \, r_j^j = \frac{1}{2\,\pi} \int_{|z|=1} f(r_1\, z)\, z^{-j}\, |dz|, \tag{36.10}$$

by a standard computation. This implies that

$$|f_j|\, r_1^j \le \frac{1}{2\,\pi} \int_{|z|=1} |f(z)|\, |dz| = M_1(f, r_1) \tag{36.11}$$

for each $j \ge 0$. If $f \in H^1(U_r)$, then it follows that

$$|f_j|\, r^j \le \|f\|_{H^1(U_r)} \tag{36.12}$$

for every $j \ge 0$. This means that $f(T) \in \mathbf{C}_r^\infty[[T]]$, with

$$\|f(T)\|_{\mathbf{C}_r^\infty[[T]]} \le \|f\|_{H^1(U_r)}, \tag{36.13}$$

where the left side is as in (28.7). In fact, it is well known that

$$\lim_{j \to \infty} |f_j|\, r^j = 0 \tag{36.14}$$

in this case, so that $f(T) \in \mathbf{C}_{0,r}[[T]]$. This is typically stated with $r = 1$, and one can reduce to that situation by rescaling.

37 Geometric series

Let k be a field, and let T be an indeterminate. The usual geometric series

$$a(T) = \sum_{j=0}^{\infty} T^j \tag{37.1}$$

may be considered as a formal power series in T with coefficients in k, where the coefficient of T^j is the multiplicative identity element 1 in k for each $j \geq 0$. Remember that (37.1) is the multiplicative inverse of $1-T$ in $k[[T]]$, as in Section 2.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let q, r be positive real numbers. If $r < 1$, then (37.1) is an element of the space $k_r^q[[T]]$ defined in Section 28, with

$$\|a(T)\|_{q,r} = \Big(\sum_{j=0}^{\infty} r^{q\,j}\Big)^{1/q} = (1 - r^q)^{-1/q}. \tag{37.2}$$

In particular, (37.1) is in the space $k_{0,r}[[T]]$ defined in Section 28 when $r < 1$. If $0 < r \leq 1$, then (37.1) is in the space $k_r^\infty[[T]]$ defined in Section 28, with

$$\|a(T)\|_{\infty,r} = 1. \tag{37.3}$$

Otherwise, (37.1) is not in $k_r^q[[T]]$ for any $q < \infty$, nor in $k_{0,r}[[T]]$, when $r \geq 1$, and (37.1) is not in $k_r^\infty[[T]]$ when $r > 1$.

Let $\mathcal{A}$ be an algebra over k, and let N be a submultiplicative q-norm N on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. Suppose that $\mathcal{A}$ has a multiplicative identity element e with $N(e) = 1$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Note that

$$N(x^j) \leq N(x)^j \tag{37.4}$$

for every $x \in \mathcal{A}$ and nonnegative integer j, where x^j is interpreted as being equal to e when $j = 0$. In particular, if $N(x) < 1$, then it follows that

$$N(x^j) \to 0 \quad \text{as } j \to \infty. \tag{37.5}$$

Let us check that

$$\sum_{j=0}^{\infty} x^j \tag{37.6}$$

converges in $\mathcal{A}$ when $N(x) < 1$. If $q < \infty$, then

$$\sum_{j=0}^{\infty} N(x^j)^q \leq \sum_{j=0}^{\infty} N(x)^{q\,j} = (1 - N(x)^q)^{-1}, \tag{37.7}$$

so that (37.6) converges q-absolutely with respect to N. If $q = \infty$, then the convergence of (37.6) follows from (37.5), as in Section 9.

If x is any element of $\mathcal{A}$ and n is a nonnegative integer, then

$$(e - x) \sum_{j=0}^{n} x^j = \Big(\sum_{j=0}^{n} x^j\Big)(e - x) = e - x^{n+1}, \tag{37.8}$$

by a standard computation. Suppose that $N(x) < 1$, so that

$$(e - x) \sum_{j=0}^{\infty} x^j = \Big(\sum_{j=0}^{\infty} x^j\Big) (x - e) = e \tag{37.9}$$

by (37.5). Thus (37.6) is the multiplicative inverse of $e - x$ in $\mathcal{A}$. If $q < \infty$, then it follows that

$$N((e - x)^{-1}) = N\Big(\sum_{j=0}^{\infty} x^j\Big) \le \Big(\sum_{j=0}^{\infty} N(x^j)^q\Big)^{1/q} \le (1 - N(x)^q)^{-1/q}, \tag{37.10}$$

using (9.9) in the second step. If $q = \infty$, then

$$N((e - x)^{-1}) = N\Big(\sum_{j=0}^{\infty} x^j\Big) \le \max_{j \ge 0} N(x^j) \le 1, \tag{37.11}$$

using (9.11) in the second step.

38 Boundary values

Let r be a positive real number, let U_r be the open disk in the complex plane centered at 0 with radius r again, and let f be a holomorphic function on U_r. If $t \in \mathbf{C}$ and $t \ne 0$, then

$$f_t(z) = f(t\,z) \tag{38.1}$$

defines a holomorphic function on $U_{r/|t|}$. Let $0 < q \le \infty$ and $0 \le r_1 < r$ be given, and observe that

$$M_q(f_t, r_1/|t|) = M_q(f, r_1), \tag{38.2}$$

where $M_q(\cdot, \cdot)$ is as in Section 36. In particular, if f is an element of the Hardy space $H^q(U_r)$ defined earlier, then $f_t \in H^q(U_{r/|t|})$, and

$$\|f_t\|_{H^q(U_{r/|t|})} = \|f\|_{H^q(U_r)}. \tag{38.3}$$

If $|t| \le 1$, then $U_r \subseteq U_{r/|t|}$, and so the restriction of f_t to U_r defines a holomorphic function on U_r. If $f \in H^q(U_r)$, then the restriction of f_t to U_r is an element of $H^q(U_r)$ too, with

$$\|f_t\|_{H^q(U_r)} \le \|f\|_{H^q(U_r)}. \tag{38.4}$$

Let us now restrict our attention to positive real numbers t less than 1. Let

$$\partial U_r = \{z \in \mathbf{C} : |z| = r\} \tag{38.5}$$

be the boundary of U_r in $\mathbf{C}$, which is the same as the circle centered at 0 with radius r. This may be considered as a measure space, using arclength measure normalized by dividing by $2\,\pi\,r$. If $f \in H^q(U_r)$, then it is well known that

$$\lim_{t \to 1-} f(t\,z) \tag{38.6}$$

exists for almost every $z \in \partial U_r$. Let $f(z)$ denote these radial boundary values of f for almost every $z \in \partial U_r$. It is well known that this defines f as an element of $L^q(\partial U_r) = L^q(\partial U_r, \mathbf{C})$, with

$$\|f\|_{H^q(U_r)} = \|f\|_{L^q(\partial U_r)}. \tag{38.7}$$

If $q < \infty$, then it is also well known that $f(t\,z)$ converges to $f(z)$ in $L^q(\partial U)$ as $t \to 1-$.

It is well known that $H^q(U_r)$ is complete for every $q > 0$, with respect to the metric or q-metric associated to (36.5), as appropriate. This implies that the radial boundary values of elements of $H^q(U_r)$ form a closed linear subspace of $L^q(\partial U_r)$.

Suppose that $0 < q_1, q_2, q_3 \le \infty$ satisfy $1/q_3 = 1/q_1 + 1/q_2$. Let $f \in H^{q_1}(U_r)$, $g \in H^{q_2}(U_r)$ be given, and note that their product $f\,g$ is holomorphic on U_r as well. If $0 \le r_1 < r$, then

$$M_{q_3}(f\,g, r_1) \le M_{q_1}(f, r_1)\, M_{q_2}(g, r_1), \tag{38.8}$$

by Hölder's inequality. It follows that $f\,g \in H^{q_3}(U_r)$, with

$$\|f\,g\|_{H^{q_3}(U_r)} \le \|f\|_{H^{q_1}(U_r)}\, \|g\|_{H^{q_2}(U_r)}. \tag{38.9}$$

Of course, the radial boundary values of $f\,g$ on ∂U_r are equal to the product of the radial boundary values of f and g, when they exist.

39 Some remarks about invertibility

Let k be a field, and let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e. The collection of elements of $\mathcal{A}$ with a multiplicative inverse in $\mathcal{A}$ is a group with respect to multiplication, that we shall denote $\mathcal{G}(\mathcal{A})$. Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a submultiplicative q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. If $a, b \in \mathcal{G}(\mathcal{A})$, then

$$a^{-1} - b^{-1} = a^{-1}\,(b - a)\,b^{-1}, \tag{39.1}$$

and hence

$$N(a^{-1} - b^{-1}) \le N(a^{-1})\,N(b^{-1})\,N(a - b). \tag{39.2}$$

Suppose for the moment that $q < \infty$, so that

$$\begin{aligned} N(b^{-1})^q &\le N(a^{-1})^q + N(a^{-1} - b^{-1})^q \\ &\le N(a^{-1})^q + N(a^{-1})^q\,N(b^{-1})^q\,N(a - b)^q. \end{aligned} \tag{39.3}$$

Thus

$$(1 - N(a^{-1})^q\,N(a - b)^q)\,N(b^{-1})^q \le N(a^{-1})^q. \tag{39.4}$$

If

$$N(a^{-1})\,N(a - b) < 1, \tag{39.5}$$

then it follows that

$$N(b^{-1})^q \le (1 - N(a^{-1})^q\, N(a-b)^q)^{-1}\, N(a^{-1})^q. \tag{39.6}$$

Equivalently, this means that

$$N(b^{-1}) \le (1 - N(a^{-1})^q\, N(a-b)^q)^{-1/q}\, N(a^{-1}) \tag{39.7}$$

when (39.5) holds. Combining this with (39.2), we obtain that

$$N(a^{-1} - b^{-1}) \le (1 - N(a^{-1})^q\, N(a-b)^q)^{-1/q}\, N(a^{-1})^2\, N(a-b) \tag{39.8}$$

when (39.5) holds.

If $q = \infty$, then

$$\begin{aligned} N(b^{-1}) &\le \max(N(a^{-1}), N(a^{-1} - b^{-1})) \\ &\le \max(N(a^{-1}), N(a^{-1})\, N(b^{-1})\, N(a-b)). \end{aligned} \tag{39.9}$$

One can use this to get that

$$N(b^{-1}) \le N(a^{-1}) \tag{39.10}$$

when (39.5) holds. This implies that

$$N(a^{-1} - b^{-1}) \le N(a^{-1})^2\, N(a-b) \tag{39.11}$$

when (39.5) holds, because of (39.2).

Suppose now that N is a submultiplicative q-norm on $\mathcal{A}$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. If $x \in \mathcal{A}$ and $N(x) < 1$, then $e - x$ is invertible in $\mathcal{A}$, as in Section 37. Let $a \in \mathcal{G}(\mathcal{A})$ and $b \in \mathcal{A}$ be given, and observe that

$$b = a - (a-b) = a\,(e - a^{-1}\,(a-b)). \tag{39.12}$$

If (39.5) holds, then $e - a^{-1}\,(a-b)$ is invertible in $\mathcal{A}$, as before. This implies that $b \in \mathcal{G}(\mathcal{A})$ when (39.5) holds, by (39.12).

Let $\{a_j\}_{j=1}^\infty$ be a sequence of invertible elements of $\mathcal{A}$ that converges to an element a of $\mathcal{A}$ with respect to the q-metric associated to N. Suppose also that $\{a_j^{-1}\}_{j=1}^\infty$ is bounded with respect to N, so that $N(a_j^{-1}) \le C$ for some $C \ge 0$ and every $j \ge 1$. It follows that

$$N(a_j^{-1} - a_l^{-1}) \le C^2\, N(a_j - a_l) \tag{39.13}$$

for every $j, l \ge 1$, as in (39.2). This implies that $\{a_j^{-1}\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{A}$ with respect to the q-metric associated to N, because $\{a_j\}_{j=1}^\infty$ is a Cauchy sequence, since it converges. Thus $\{a_j^{-1}\}_{j=1}^\infty$ converges in $\mathcal{A}$, by completeness, and the limit of this sequence is the multiplicative inverse of a.

Part IV
Laurent series

40 Some spaces of Laurent series

Let k be a field, and let T be an indeterminate. Let us now use $LS(k) = LS(k,T)$ to denote the space of formal Laurent series

$$f(T) = \sum_{j=-\infty}^{\infty} f_j\, T^j \tag{40.1}$$

in T with coefficients in k. As in Section 3, $LS(k)$ can be defined precisely as the space $c(\mathbf{Z}, k)$ of all k-valued functions on the set $\mathbf{Z}$ of integers. In particular, $LS(k)$ is a vector space over k with respect to termwise addition and scalar multiplication, which corresponds to pointwise addition and scalar multiplication of functions on $\mathbf{Z}$.

Let $LP(k) = LP(k,T)$ be the space of formal Laurent polynomials in T with coefficients in k. This is a linear subspace of $LS(k)$, as before, which corresponds to the space $c_{00}(\mathbf{Z}, k)$ of k-valued functions on $\mathbf{Z}$ with finite support. Remember that $k((T))$ is the linear subspace of $LS(k)$ consisting of formal Laurent series in T with coefficients in k such that the coefficient of T^j is equal to 0 for all but finitely many $j < 0$. This is a field with respect to formal multiplication of these series, as before. Of course, $LP(k)$ is a subalgebra of $k((T))$, as an algebra over k.

Let $PS(k) = PS(k,T)$ be the space of formal Laurent series in T with coefficients in k such that the coefficient of T^j is equal to 0 when $j < 0$. This is a linear subspace of $LS(k)$, whose elements can be identified with formal power series in T with coefficients in k. Similarly, let $P(k) = P(k,T)$ be the space of formal Laurent polynomials in T with coefficients in k such that the coefficient of T^j is 0 when $j < 0$, which is the linear subspace of $LP(k)$ corresponding to formal polynomials in T with coefficients in k. Equivalently,

$$P(k) = LP(k) \cap PS(k). \tag{40.2}$$

Note that $P(k)$, $PS(k)$ are subalgebras of $LP(k)$, $k((T))$, respectively, as algebras over k.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let r be a positive real number. Put

$$w_r(j) = r^j \tag{40.3}$$

for every integer j, which defines a positive real-valued function on $\mathbf{Z}$. Let q be a positive real number, and put

$$LS_r^q(k) = LS_r^q(k,T) = \Big\{ f(T) \in LS(k) : \sum_{j=-\infty}^{\infty} |f_j|^q\, r^{q\,j} < \infty \Big\}. \tag{40.4}$$

This sum may be considered as a sum over $\mathbf{Z}$, as in Section 15, or as a doubly-infinite series of nonnegative real numbers, as in Section 17. If $f(T) \in LS_r^q(k)$, then we put

$$\|f(T)\|_{q,r} = \|f(T)\|_{LS_r^q(k)} = \Big(\sum_{j=-\infty}^{\infty} |f_j|^q\, r^{q\,j}\Big)^{1/q}. \tag{40.5}$$

Note that (40.4) corresponds to the space $\ell^q_{w_r}(\mathbf{Z}, k)$ of k-valued functions on $\mathbf{Z}$ that are q-summable with respect to w_r on $\mathbf{Z}$ and $|\cdot|$ on k, as in Section 26. Similarly, (40.5) corresponds to $\|\cdot\|_{\ell^q_{w_r}(\mathbf{Z},k)}$ in Section 26, applied to $j \mapsto f_j$ as a k-valued function on $\mathbf{Z}$. In particular, (40.5) defines a q-norm on (40.4) with respect to $|\cdot|$ on k when $q \le q_k$, and a q_k-norm on (40.4) when $q \ge q_k$. As before, $LS_r^q(k)$ is a linear subspace of $LS(k)$,

$$LP(k) \subseteq LS_r^q(k), \tag{40.6}$$

and $LP(k)$ is dense in $LS_r^q(k)$ with respect to the q or q_k-metric associated to (40.5). Put

$$PS_r^q(k) = PS_r^q(k, T) = LS_r^q(k) \cap PS(k), \tag{40.7}$$

which corresponds to $k_r^q[[T]]$ in Section 28.

Put

$$LS_r^\infty(k) = LS_r^\infty(k, T) = \{f(T) \in LS(k) : |f_j|\, r^j \text{ is bounded on } \mathbf{Z}\}, \tag{40.8}$$

and

$$\|f(T)\|_{\infty,r} = \|f(T)\|_{LS_r^\infty(k)} = \sup_{j\in\mathbf{Z}}(|f_j|\, r^j) \tag{40.9}$$

for each $f(T) \in LS_r^\infty(k)$. Thus (40.8) corresponds to the space $\ell^\infty_{w_r}(\mathbf{Z}, k)$ of k-valued functions on $\mathbf{Z}$ that are bounded with respect to w_r on $\mathbf{Z}$ and $|\cdot|$ on k, and (40.9) corresponds to $\|\cdot\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}$ applied to $j \mapsto f_j$ as a k-valued function on $\mathbf{Z}$. As before, (40.8) is a linear subspace of $LS(k)$, and (40.9) defines a q_k-norm on (40.8) with respect to $|\cdot|$ on k. If $0 < q \le \widetilde{q} \le \infty$, then

$$LS_r^q(k) \subseteq LS_r^{\widetilde{q}}(k), \tag{40.10}$$

and

$$\|f(T)\|_{\widetilde{q},r} \le \|f(T)\|_{q,r} \tag{40.11}$$

for every $f(T) \in LS_r^q(k)$, as in Section 26. Put

$$PS_r^\infty(k) = PS_r^\infty(k, T) = LS_r^\infty(k) \cap PS(k), \tag{40.12}$$

which corresponds to $k_r^\infty[[T]]$ in Section 28.

Put

$$LS_{0,r}(k) = LS_{0,r}(k, T) = \big\{f(T) \in LS(k) : \lim_{j\to\pm\infty} |f_j|\, r^j = 0\big\}, \tag{40.13}$$

which corresponds to the space $c_{0,w_r}(\mathbf{Z}, k)$ of k-valued functions on $\mathbf{Z}$ that vanish at infinity with respect to w_r on $\mathbf{Z}$ and $|\cdot|$ on k. This is a closed linear subspace of (40.8) with respect to the q_k-metric associated to (40.9). As before,

$$LP(k) \subseteq LS_{0,r}(k), \tag{40.14}$$

and $LP(k)$ is dense in $LS_{0,r}(k)$ with respect to the q_k-metric associated to (40.9). We also have that

$$LS_r^q(k) \subseteq LS_{0,r}(k) \tag{40.15}$$

when $0 < q < \infty$, as in Section 26. Put

$$PS_{0,r}(k) = PS_{0,r}(k, T) = LS_{0,r}(k) \cap PS(k), \tag{40.16}$$

which corresponds to $k_{0,r}[[T]]$ in Section 28.

If $|\cdot|$ is the trivial absolute value function on k, then $LS_1^\infty(k) = LS(k)$, $LS_1^q(k) = LP(k)$ when $0 < q < \infty$, and $LS_{0,1}(k) = LP(k)$. If $0 < r < 1$, then $LS_r^q(k) = k((T))$ for every $q > 0$, and $LS_{0,r}(k) = k((T))$ as well.

41 Multiplying Laurent series

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let T be an indeterminate. Suppose that k is complete with respect to the q_k-metric associated to $|\cdot|$. Let a positive real number r and $0 < q \le q_k$ be given, so that $|\cdot|$ may be considered as a q-absolute value function on k too. Suppose for the moment that $q < \infty$, and that $f(T) = \sum_{j=-\infty}^\infty f_j\,T^j$, $g(T) = \sum_{l=-\infty}^\infty g_l\,T^l$ are elements of the space $LS_r^q(k)$ defined in (40.4). We would like to define

$$f(T)\,g(T) = h(T) = \sum_{n=-\infty}^\infty h_n\,T^n \tag{41.1}$$

as a formal Laurent series in T with coefficients in k, with

$$h_n = \sum_{j=-\infty}^\infty f_j\,g_{n-j} \tag{41.2}$$

for every $n \in \mathbf{Z}$.

Observe that

$$r^{q\,n} \sum_{j=-\infty}^\infty |f_j\,g_{n-j}|^q = \sum_{j=-\infty}^\infty (|f_j|^q\,r^{q\,j})\,(|g_{n-j}|^q\,r^{q\,(n-j)}) \tag{41.3}$$

for every $n \in \mathbf{Z}$. The right side corresponds to the nth term of the Cauchy product of $\sum_{j=-\infty}^\infty |f_j|^q\,r^{q\,j}$ and $\sum_{l=-\infty}^\infty |g_l|^q\,r^{q\,l}$, as in Section 21. It follows that

$$\begin{aligned} \sum_{n=-\infty}^\infty \Big(r^{q\,n} \sum_{j=-\infty}^\infty |f_j\,g_{n-j}|^q\Big) &\le \Big(\sum_{j=-\infty}^\infty |f_j|^q\,r^{q\,j}\Big)\Big(\sum_{l=-\infty}^\infty |g_l|^q\,r^{q\,l}\Big) \\ &= \|f(T)\|_{q,r}^q\,\|g(T)\|_{q,r}^q, \end{aligned} \tag{41.4}$$

as before. In particular, (41.3) is finite for each $n \in \mathbf{Z}$. This permits us to define (41.2) as an element of k for every $n \in \mathbf{Z}$, as a q-absolutely convergent doubly-infinite series, as in Section 17, or as a sum over $\mathbf{Z}$, as in Section 18. We also get that

$$|h_n|^q \, r^{q\,n} \le r^{q\,n} \sum_{j=-\infty}^{\infty} |f_j \, g_{n-j}|^q \tag{41.5}$$

for every $n \in \mathbf{Z}$, so that

$$\sum_{n=-\infty}^{\infty} |h_n|^q \, r^{q\,n} \le \|f(T)\|_{q,r}^q \, \|g(T)\|_{q,r}^q, \tag{41.6}$$

by (41.4). This implies that $h(T) \in LS_r^q(k)$ too, with

$$\|h(T)\|_{q,r} \le \|f(T)\|_{q,r} \, \|g(T)\|_{q,r}. \tag{41.7}$$

Suppose now that $q = \infty$, so that $q_k = \infty$ in particular. In this case, we ask that $f(T)$ and $g(T)$ be elements of the space $LS_{0,r}(k)$ defined in the previous section. Of course,

$$|f_j \, g_{n-j}| = r^{-n} \, (|f_j| \, r^j) \, (|g_{n-j}| \, r^{n-j}) \tag{41.8}$$

for every $j, n \in \mathbf{Z}$. It is easy to see that

$$\lim_{j \to \pm\infty} |f_j \, g_{n-j}| = 0 \tag{41.9}$$

for every $n \in \mathbf{Z}$, using (41.8) and the hypothesis that $f(T), g(T) \in LS_{0,r}(k)$. Thus we can define (41.2) as an element of k for every $n \in \mathbf{Z}$, as a convergent doubly-infinite series, as in Section 17, or as a sum over $\mathbf{Z}$, as in Section 18. Note that

$$|h_n| \, r^n \le r^n \max_{j \in \mathbf{Z}} |f_j \, g_{n-j}| = \max_{j \in \mathbf{Z}} ((|f_j| \, r^j) \, (|g_{n-j}| \, r^{n-j})) \tag{41.10}$$

for every $n \in \mathbf{Z}$. One can use this to verify that

$$\lim_{n \to \pm\infty} |h_n| \, r^n = 0, \tag{41.11}$$

because of the analogous conditions for $f(T)$ and $g(T)$. This means that $h(T)$ is an element of $LS_{0,r}(k)$, and we can also use (41.10) to get that

$$\|h(T)\|_{\infty,r} \le \|f(T)\|_{\infty,r} \, \|g(T)\|_{\infty,r}. \tag{41.12}$$

One can check that $LS_r^q(k)$ is a commutative algebra over k with respect to this definition of multiplication when $0 < q \le q_k$ and $q < \infty$, and similarly that $LS_{0,r}(k)$ is a commutative algebra over k when $q_k = \infty$. Of course, this definition of (41.1) reduces to the one in Section 3 when $f(T), g(T) \in LP(k)$. Thus $LP(k)$ is a subalgebra of $LS_r^q(k)$ when $0 < q \le q_k$ and $q < \infty$, and a subalgebra of $LS_{0,r}(k)$ when $q_k = \infty$. Remember that k can be indentified with a subalgebra of $LP(k)$, as in Section 3. The multiplicative identity element 1 in k corresponds to the multiplicative identity element in $LS_r^q(k)$ when $0 < q \le q_k$ and $q < \infty$, and to the multiplicative identity element in $LS_{0,r}(k)$ when $q_k = \infty$.

42 Another multiplicativity condition

Let k be a field with an ultrametric absolute value function $|\cdot|$, and suppose that k is complete with respect to the ultrametric associated to $|\cdot|$. Also let T be an indeterminate, and let r be a positive real number. If $f(T) = \sum_{j=-\infty}^{\infty} f_j\, T^j, g(T) = \sum_{l=-\infty}^{\infty} g_l\, T^l \in LS_{0,r}(k)$, then

$$\|f(T)\, g(T)\|_{\infty,r} = \|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r}. \tag{42.1}$$

It suffices to show that

$$\|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r} \le \|f(T)\, g(T)\|_{\infty,r}, \tag{42.2}$$

because of (41.12). We may as well suppose that $f(T), g(T) \neq 0$.

As in Section 31, we would like to choose $j_0, l_0 \in \mathbf{Z}$ such that

$$|f_{j_0}| r^{j_0} = \|f(T)\|_{\infty,r} \tag{42.3}$$

and

$$|g_{l_0}|\, r^{l_0} = \|g(T)\|_{\infty,r}. \tag{42.4}$$

This would imply that

$$|f_j|\, r^j \le \|f(T)\|_{\infty,r} = |f_{j_0}|\, r^{j_0} \tag{42.5}$$

and

$$|g_l|\, r^l \le \|g(T)\|_{\infty,r} = |g_{l_0}|\, r^{l_0} \tag{42.6}$$

for every $j, l \in \mathbf{Z}$. Remember that $|f_j|\, r^j \to 0$ as $j \to \pm\infty$, and $|g_l|\, r^l \to \infty$ as $l \to \pm\infty$, by hypothesis. Hence the suprema in the definitions of $\|f(T)\|_{\infty,r}$ and $\|g(T)\|_{\infty,r}$ are attained. This means that there are $j_0, l_0 \in \mathbf{Z}$ such that (42.3) and (42.4) hold. In fact, there are only finitely many such j_0 and l_0, because $f(T), g(T) \neq 0$. Thus we can choose j_0 and l_0 to be the first integers such that (42.3) and (42.4) hold, so that the inequalities in (42.5) and (42.6) are strict when $j < j_0$ and $l < l_0$, respectively.

Using (42.3) and (42.4), we get that

$$\|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r} = |f_{j_0}|\, |g_{l_0}|\, r^{j_0+l_0}. \tag{42.7}$$

As in the previous section, $f(T)\, g(T) = h(T) = \sum_{n=-\infty}^{\infty} h_n\, T^n$, where $h_n \in k$ is as in (41.2). In particular,

$$|h_{j_0+l_0}|\, r^{j_0+l_0} \le \|h(T)\|_{\infty,r} = \|f(T)\, g(T)\|_{\infty,r}. \tag{42.8}$$

Thus we would like to check that

$$|f_{j_0}|\, |g_{l_0}| r^{j_0+l_0} \le |h_{j_0+l_0}|\, r^{j_0+l_0} \tag{42.9}$$

to get (42.2). Of course, this is the same as saying that

$$|f_{j_0}|\, |g_{l_0}| \le |h_{j_0+l_0}|. \tag{42.10}$$

Note that

$$|f_{j_0}|\,|g_{l_0}| \le \max(|h_{j_0+l_0}|, |h_{j_0+l_0} - f_{j_0}\, g_{l_0}|), \tag{42.11}$$

by the ultrametric version of the triangle inequality. In order to get (42.10), it is enough to verify that

$$|h_{j_0+l_0} - f_{j_0}\, g_{l_0}| < |f_{j_0}|\,|g_{l_0}|. \tag{42.12}$$

Using (42.5) and (42.6), we get that

$$|f_j|\,|g_l|\, r^{j+l} \le |f_{j_0}|\,|g_{l_0}|\, r^{j_0+l_0} \tag{42.13}$$

for every $j, l \in \mathbf{Z}$. If $j + l = j_0 + l_0$, then it follows that

$$|f_j|\,|g_l| \le |f_{j_0}|\,|g_{l_0}|. \tag{42.14}$$

This inequality is strict when either $j < j_0$ or $l < l_0$, because of the corresponding strict inequalities in (42.5) and (42.6), mentioned earlier. Observe that $h_{j_0+l_0}$ is basically the sum of $f_j\, g_l$ with $j, l \in \mathbf{Z}$ and $j + l = j_0 + l_0$, as in (41.2). Thus $h_{j_0+l_0} - f_{j_0}\, g_{l_0}$ reduces to the sum of $f_j\, g_l$ over $j, l \in \mathbf{Z}$ with $j+l = j_0+l_0$ and either $j < j_0$ or $l < l_0$. Each of these terms has absolute value strictly less than $|f_{j_0}|\,|g_{l_0}|$, because of the strict inequality in (42.14) in these cases. This implies (42.12), because of the ultrametric version of the triangle inequality, and the convergence of the sum in this situation.

43 Some subsets of algebras

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative q-seminorm N with respect to $|\cdot|$ on k, for some $q > 0$. Suppose that $\mathcal{A}$ has a multiplicative identity element e with $N(e) = 1$, and let $\mathcal{G}(\mathcal{A})$ be the group of invertible elements of $\mathcal{A}$, as before. Let r be a positive real number, and consider the set

$$\mathcal{C}(r) = \mathcal{C}_{\mathcal{A}}(r) = \{a \in \mathcal{G}(\mathcal{A}) : N(a) \le r,\ N(a^{-1}) \le 1/r\}. \tag{43.1}$$

Note that $e \in \mathcal{C}(1)$, and that $\mathcal{C}(1)$ is a subgroup of $\mathcal{G}(\mathcal{A})$. If $t \in k \setminus \{0\}$, then $a \in \mathcal{C}(r)$ if and only if $t\, a \in \mathcal{C}(r\,|t|)$.

If $a \in \mathcal{C}(r)$, then

$$N(a^j) \le r^j \tag{43.2}$$

for every $j \in \mathbf{Z}$. Of course,

$$1 \le N(a^j)\, N(a^{-j}) \tag{43.3}$$

for every $a \in \mathcal{G}(\mathcal{A})$ and $j \in \mathbf{Z}$. It follows that

$$N(a^j) = r^j \tag{43.4}$$

for every $a \in \mathcal{C}(r)$ and $j \in \mathbf{Z}$.

Suppose that N is a q-norm on $\mathcal{A}$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. One can check that $\mathcal{C}(r)$ is a closed set in $\mathcal{A}$ with respect to the q-metric associated to N, using the remarks in Section 39. If $q = \infty$, then $\mathcal{C}(r)$ is an open set in $\mathcal{A}$ with respect to the q-metric associated to N.

If $0 < q \leq q_k$ and $q < \infty$, then $LS_r^q(k)$ is a commutative algebra over k, as in the previous section, and $\| \cdot \|_{q,r}$ is a submultiplicative q-norm on $LS_r^q(k)$. Similarly, if $q_k = \infty$, then $LS_{0,r}(k)$ is a commutative algebra over k, and $\| \cdot \|_{\infty,r}$ is a multiplicative ultranorm on $LS_{0,\infty}(k)$. If $j \in \mathbf{Z}$, then T^j may be considered as a formal Laurent polynomial in T with coefficient equal to the multiplicative identity element 1 in k, and $\|T^j\|_{q,r} = r^j$ for every $q > 0$. Note that $LS_r^q(k)$ is complete with respect to the q or q_k-metric associated to $\| \cdot \|_{q,r}$ when k is complete with respect to the q_k-metric associated to $| \cdot |$, as in Section 26. This implies that $LS_{0,r}(k)$ is complete with respect to the q_k-metric associated to $\| \cdot \|_{\infty,r}$, because $LS_{0,r}(k)$ is a closed set in $LS_r^\infty(k)$ with respect to this q_k-metric, as before.

Let V be a vector space over k, and let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k for some $q_V > 0$. Remember that the space $\mathcal{BL}(V)$ of bounded linear mappings from V into itself with respect to N_V is an algebra over k with respect to composition of mappings, and that the corresponding operator q_V-seminorm $\| \cdot \|_{op}$ is submultiplicative on $\mathcal{BL}(V)$. Suppose that $N_V(v) > 0$ for some $v \in V$, which implies that the identity mapping on V satisfies $\|I\|_{op} = 1$. If A is a one-to-one linear mapping from V onto itself that satisfies

$$N_V(A(v)) = r\, N_V(v) \tag{43.5}$$

for every $v \in V$, then

$$N_V(A^j(v)) = r^j\, N_V(v) \tag{43.6}$$

for every $v \in V$ and $j \in \mathbf{Z}$. This implies that A^j is a bounded linear mapping from V into itself for every $j \in \mathbf{Z}$, with

$$\|A^j\|_{op} = r^j. \tag{43.7}$$

Conversely, if A is a bounded linear mapping from V into itself with a bounded inverse, and if $\|A\|_{op} \leq r$, $\|A^{-1}\|_{op} \leq 1/r$, then one can check that (43.5) holds. If N_V is a q_V-norm on V, then $\| \cdot \|_{op}$ is a q_V-norm on $\mathcal{BL}(V)$, and $\mathcal{BL}(V)$ is complete with respect to the q_V-metric associated to $\| \cdot \|_{op}$ when V is complete with respect to the q_V-metric associated to N_V.

44 Functions on these subsets

Let k be a field with a q_k-absolute value function $| \cdot |$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative q-norm N with respect to $| \cdot |$ on k for some $q > 0$. Suppose that $\mathcal{A}$ has a multiplicative identity element e with $N(e) = 1$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Let T be an indeterminate, let r be a positive real number, and let a

be an element of the set $\mathcal{C}(r)$ defined in (43.1). Suppose for the moment that $q < \infty$, and that $f(T) = \sum_{j=-\infty}^{\infty} f_j\, T^j$ is an element of $LS_r^q(k)$. We would like to define $f(a)$ as an element of $\mathcal{A}$ by

$$f(a) = \sum_{j=-\infty}^{\infty} f_j\, a^j, \tag{44.1}$$

where $a^j = e$ when $j = 0$, as usual. Note that

$$\sum_{j=-\infty}^{\infty} N(f_j\, a^j)^q = \sum_{j=-\infty}^{\infty} |f_j|^q\, N(a^j)^q = \sum_{j=-\infty}^{\infty} |f_j|^q\, r^{q\, j} = \|f(T)\|_{q,r}^q. \tag{44.2}$$

Thus (44.1) can be defined as an element of $\mathcal{A}$, as a q-absolutely convergent doubly-infinite series as in Section 17, or as a sum over $\mathbf{Z}$, as in Section 18. We also have that

$$N(f(a)) \le \Big(\sum_{j=-\infty}^{\infty} N(f_j\, a^j)^q\Big)^{1/q} = \|f(T)\|_{q,r}. \tag{44.3}$$

Suppose now that $q = \infty$, and that $f(T) \in LS_{0,r}(k)$. If $a \in \mathcal{C}(r)$, then

$$N(f_j\, a^j) = |f_j|\, N(a^j) = |f_j|\, r^j \to 0 \quad \text{as } j \to \pm\infty. \tag{44.4}$$

This permits us to define (44.1) as an element of $\mathcal{A}$, as a convergent doubly-infinite series, as in Section 17, or as a sum over $\mathbf{Z}$, as in Section 18. In this case, we get that

$$N(f(a)) \le \max_{j \in \mathbf{Z}} N(f_j\, a^j) = \max_{j \in Z}(|f_j|\, r^j) = \|f(T)\|_{\infty,r}. \tag{44.5}$$

Let $f(T)$ be an element of $LS_r^q(k)$ when $q < \infty$, or an element of $LS_{0,r}(k)$ when $q = \infty$. If $a \in \mathcal{C}(r)$, then

$$\sum_{j=-n}^{n} f_j\, a^j \tag{44.6}$$

converges to $f(a)$ as $n \to \infty$, with respect to the q-metric on $\mathcal{A}$ associated to N. It is easy to see that the convergence is uniform on $\mathcal{C}(r)$ in both cases, using the same type of estimates as before.

If $j \in \mathbf{Z}$, then $a \mapsto a^j$ is uniformly continuous as a mapping from $\mathcal{C}(r)$ into $\mathcal{A}$, with respect to the q-metric associated to N on $\mathcal{A}$ and its restriction to $\mathcal{C}(r)$. This is clear when $j = 0, 1$, and the case where $j = -1$ follows from (39.2). One can reduce to these cases when $|j| \ge 2$, because products of bounded uniformly continuous functions are uniformly continuous as well. It follows that (44.6) is uniformly continuous on $\mathcal{C}(r)$ for every $n \ge 0$. This implies that $f(a)$ is uniformly continuous on $\mathcal{C}(r)$ under the conditions in the preceding paragraph, because of uniform convergence.

As in Section 7, we may as well suppose that $q \le q_k$. Let us also ask that k be complete with respect to the q_k-metric associated to $|\cdot|$. Let $f(T)$, $g(T)$ be elements of $LS_r^q(k)$ when $q < \infty$, or elements of $LS_{0,r}(k)$ when $q = \infty$. Thus $f(T)\, g(T) = h(T)$ is defined as an element of the same space, as in Section 41. If $a \in \mathcal{C}(r)$, then $f(a)$, $g(a)$, and $h(a)$ are defined as elements of $\mathcal{A}$, as before. Observe that

$$h(a) = f(a)\, g(a), \tag{44.7}$$

because $h(a)$ is defined by the sum over $\mathbf{Z}$ corresponding to the Cauchy product of the sums over $\mathbf{Z}$ that define $f(a)$ and $g(a)$, as in Section 21. It follows that $f(T) \mapsto f(a)$ defines an algebra homomorphism from $LS_r^q(k)$ into $\mathcal{A}$ when $q < \infty$, and from $LS_{0,r}(k)$ into $\mathcal{A}$ when $q = \infty$.

45 Rescaling Laurent series

Let k be a field, let T be an indeterminate, and let $t \in k \setminus \{0\}$ be given. If $f(T) = \sum_{j=-\infty}^{\infty} f_j\, T^j$ is a formal Laurent series in T with coefficients in k, then put

$$R_t(f(T)) = \sum_{j=-\infty}^{\infty} f_j\, t^j\, T^j. \tag{45.1}$$

This defines $R_t(f(T))$ as an element of $LS(k)$, and it may also be denoted $(R_t(f))(T)$ or simply $R_t(f)$. Note that R_t defines a one-to-one linear mapping from $LS(k)$ onto itself, which is the identity mapping on $LS(k)$ when $t = 1$. It is easy to see that

$$R_t \circ R_u = R_{t\,u} \tag{45.2}$$

for every $t, u \in k \setminus \{0\}$. Of course, (45.1) reduces to the analogous definition for formal power series in Section 35 when $f(T) \in PS(k)$. Note that R_t maps $LP(k)$ onto itself for each $t \in k \setminus \{0\}$, and that R_t is a homomorphism from $LP(k)$ into itself, as an algebra over k.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, let r be a positive real number, and let $t \in k \setminus \{0\}$, $f(T) \in LS(k)$, and $0 < q \le \infty$ be given. Observe that $R_t(f(T)) \in LS_r^q(k)$ if and only if $f(T) \in LS_{r\,|t|}^q(k)$, in which case

$$\|R_t(f(T))\|_{q,r} = \|f(T)\|_{q,r\,|t|}. \tag{45.3}$$

Similarly, $R_t(f(T)) \in LS_{0,r}(k)$ if and only if $f(T) \in LS_{0,r\,|t|}(k)$.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e and a submultiplicative q-norm N with respect to $|\cdot|$ on k for some $0 < q \le q_k$. Suppose that $N(e) = 1$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. Let $t \in k \setminus \{0\}$ and a positive real number r be given, and let a be an element of the subset $\mathcal{C}(r)$ of $\mathcal{A}$ defined in (43.1). Thus $t\,a \in \mathcal{C}(r\,|t|)$, as before. Suppose for the moment that $q < \infty$ and $f(T) \in LS_{r\,|t|}^q(k)$, so that $R_t(f(T)) \in LS_r^q(k)$. Under these conditions, $f(t\,a)$ and $(R_t(f))(a)$ can be defined as elements of $\mathcal{A}$, as in the previous section, with

$$(R_t(f))(a) = \sum_{j=-\infty}^{\infty} f_j\, t^j\, a^j = \sum_{j=-\infty}^{\infty} f_j\, (t\, a)^j = f(t\, a). \tag{45.4}$$

If $q = \infty$ and $f(T) \in LS_{0,r\,|t|}(k)$, then $R_t(f(T)) \in LS_{0,r}(k)$, $f(t\,a)$ and $(R_t(f))(a)$ can be defined as elements of $\mathcal{A}$ again, and satisfy (45.4).

Suppose that k is complete with respect to the q_k-metric associated to $|\cdot|$. If $0 < q \le q_k$ and $q < \infty$, then $LS^q_r(k)$ is a commutative algebra over k for every positive real number r, as in Section 41. One can check that for every $t \in k \setminus \{0\}$ and $0 < r < \infty$, R_t is a homomorphism from $LS^q_{r\,|t|}(k)$ into $LS^q_r(k)$, as algebras over k. Similarly, if $q_k = \infty$, then $LS_{0,r}(k)$ is a commutative algebra over k for every $0 < r < \infty$. One can verify that for every $t \in k \setminus \{0\}$ and $0 < r < \infty$, R_t is a homomorphism from $LS_{0,r\,|t|}(k)$ into $LS_{0,r}(k)$, as algebras over k.

46 Fourier series

Let r be a positive real number, and let

$$C(r) = \{z \in \mathbf{C} : |z| = r\} \tag{46.1}$$

be the circle in the complex plane centered at 0 with radius r, with respect to the standard absolute value function on $\mathbf{C}$. This may be considered as a measure space, using arclength measure divided by $2\,\pi, r$, as before. Let $L^q(C(r)) = L^q(C(r), \mathbf{C})$ be the corresponding L^q space of complex-valued functions on $C(r)$ for each $q > 0$. If $f \in L^1(C(r))$ and $j \in \mathbf{Z}$, then the *jth Fourier coefficient* of f is defined by

$$\widehat{f}(j) = \frac{1}{2\,\pi\, r} \int_{C(r)} f(w)\, w^{-j}\, |dw|, \tag{46.2}$$

where $|dw|$ refers to arclength measure on $C(r)$. The corresponding *Fourier series* is defined formally by

$$\sum_{j=-\infty}^{\infty} \widehat{f}(j)\, z^j. \tag{46.3}$$

If T is an indeterminate, then

$$\sum_{j=-\infty}^{\infty} \widehat{f}(j)\, T^j \tag{46.4}$$

may be considered as a formal Laurent series in T with complex coefficients. Note that $f \mapsto \widehat{f}$ is linear as a mapping from $L^1(C(r))$ into the space $c(\mathbf{Z}, \mathbf{C})$ of complex-valued functions on $\mathbf{Z}$. One can also consider the mapping from f to (46.4) as a linear mapping from $L^1(C(r))$ into $LS(\mathbf{C})$.

Observe that

$$|\widehat{f}(j)| \le \frac{1}{2\,\pi\, r} \int_{C(r)} |f(w)|\, |w|^{-j}\, |dw| = r^{-j}\, \|f\|_{L^1(C(r))} \tag{46.5}$$

for every $j \in \mathbf{Z}$. Equivalently,

$$|\widehat{f}(j)|\, r^j \le \|f\|_{L^1(C(r))} \tag{46.6}$$

for every $j \in \mathbf{Z}$. Put $w_r(j) = r^j$ for every $j \in \mathbf{Z}$, which defines a positive real-valued function on $\mathbf{Z}$. Thus $\widehat{f}$ is an element of the space $\ell^\infty_{w_r}(\mathbf{Z}, \mathbf{C})$ of complex-valued functions on $\mathbf{Z}$ that are bounded with respect to w_r, as in Section 26, with

$$\|\widehat{f}\|_{\ell^\infty_{w_r}(\mathbf{Z},\mathbf{C})} \le \|f\|_{L^1(C(r))}. \tag{46.7}$$

It is well known that

$$\lim_{j \to \pm\infty} |\widehat{f}(j)|\, r^j = 0, \tag{46.8}$$

by the Riemann–Lebesgue lemma. This means that $\widehat{f}$ vanishes at infinity on $\mathbf{Z}$ with respect to w_r. This is the same as saying that (46.4) is an element of the space $LS_{0,r}(\mathbf{C})$ defined in Section 40.

If $f \in L^2(C(r))$, then it is well known that

$$\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2\, r^{2\,j} = \frac{1}{2\,\pi\, r} \int_{C(r)} |f(w)|^2\, |dw| = \|f\|^2_{L^2(C(r))}, \tag{46.9}$$

by Parseval's theorem. This means that $\widehat{f}$ is in the space $\ell^2_{w_r}(\mathbf{Z}, \mathbf{C})$ defined in Section 26, with

$$\|\widehat{f}\|_{\ell^2_{w_r}(\mathbf{Z},\mathbf{C})} = \|f\|_{L^2(C(r))}. \tag{46.10}$$

In particular, (46.4) is in the space $LS^2_r(\mathbf{C})$ defined in Section 40.

Suppose that $f(T) = \sum_{j=-\infty}^{\infty} f_j\, T^j$ is an element of the space $LS^1_r(\mathbf{C})$ defined in Section 40, so that $\sum_{j=-\infty}^{\infty} |f_j|\, r^j$ is finite. In this case,

$$f(w) = \sum_{j=-\infty}^{\infty} f_j\, w^j \tag{46.11}$$

defines a continuous complex-valued function on $C(r)$, as in Section 44. It is well known that

$$\widehat{f}(l) = f_l \tag{46.12}$$

for every $l \in \mathbf{Z}$ in this situation. If $f(T) \in LS^2_r(\mathbf{C})$, then one can define an element of $L^2(C(r))$ as in (46.11), with convergence with respect to the L^2 norm. This element of $L^2(C(r))$ also satisfies (46.12), and (46.10) is the same as $\|f(T)\|_{LS^2_r(\mathbf{C})}$.

47 Pairs of radii

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let T be an indeterminate, and let r_1, r_2 be positive real numbers with $r_1 \le r_2$. If $0 < q \le \infty$, then put

$$LS^q_{r_1,r_2}(k) = LS^q_{r_1,r_2}(k, T) = LS^q_{r_1}(k) \cap LS^q_{r_2}(k), \tag{47.1}$$

where $LS^q_r(k)$ is as in Section 40. Let w_{r_1,r_2} be the positive real-valued function defined on $\mathbf{Z}$ by

$$w_{r_1,r_2}(j) = \max(r_1^j, r_2^j) \quad = \quad r_1^j \quad \text{when } j \le 0 \tag{47.2}$$
$$\qquad\qquad\qquad\qquad = \quad r_2^j \quad \text{when } j \ge 0.$$

If $f(T) = \sum_{j=-\infty}^{\infty} f_j\, T^j$ is an element of (47.1), then put

$$\|f(T)\|_{q,r_1,r_2} = \|f(T)\|_{LS^q_{r_1,r_2}(k)} = \Big(\sum_{j=-\infty}^{\infty} |f_j|^q\, w_{r_1,r_2}(j)^q\Big)^{1/q} \tag{47.3}$$

when $q < \infty$, and

$$\|f(T)\|_{\infty,r_1,r_2} = \|f(T)\|_{LS^\infty_{r_1,r_2}(k)} = \sup_{j \in \mathbf{Z}}(|f_j|\, w_{r_1,r_2}(j)) \tag{47.4}$$

when $q = \infty$. Note that (47.1) corresponds to the space $\ell^q_{w_{r_1,r_2}}(\mathbf{Z}, k)$ defined in Section 26. Similarly, (47.3) and (47.4) correspond to $\|\cdot\|_{\ell^q_{w_{r_1,r_2}}(\mathbf{Z},k)}$ applied to $j \mapsto f_j$ as a k-valued function on $\mathbf{Z}$. In particular, this is a q-norm with respect to $|\cdot|$ on k when $q \le q_k$, and a q_k-norm when $q_k \le q$.

If $f(T)$ is an element of (47.1), then it is easy to see that

$$\max(\|f(T)\|_{q,r_1}, \|f(T)\|_{q,r_2}) \le \|f(T)\|_{q,r_1,r_2}, \tag{47.5}$$

where $\|f(T)\|_{q,r}$ is as in Section 40. In fact,

$$\|f(T)\|_{\infty,r_1,r_2} = \max(\|f(T)\|_{\infty,r_1}, \|f(T)\|_{\infty,r_2}) \tag{47.6}$$

when $q = \infty$. If $q < \infty$, then we also have that

$$\|f(T)\|_{q,r_1,r_2} \le (\|f(T)\|^q_{q,r_1} + \|f(T)\|^q_{q,r_2})^{1/q}. \tag{47.7}$$

In this case, $LP(k)$ is dense in (47.1) with respect to the q or q_k-metric associated to (47.3), as appropriate.

Put

$$LS_{0,r_1,r_2}(k) = LS_{0,r_1,r_2}(k, T) = LS_{0,r_1}(k) \cap LS_{0,r_2}(k), \tag{47.8}$$

where $LS_{0,r}(k)$ is as in Section 40. Equivalently, $f(T) \in LS(k)$ is an element of (47.8) if and only if

$$\lim_{j \to -\infty} |f_j|\, r_1^j = \lim_{j \to +\infty} |f_j|\, r_2^j = 0. \tag{47.9}$$

Thus (47.8) corresponds to the space $c_{0,w_{r_1,r_2}}(\mathbf{Z}, k)$ of k-valued functions on $\mathbf{Z}$ that vanish at infinity with respect to w_{r_1,r_2} on $\mathbf{Z}$ and $|\cdot|$ on k, as in Section 26. As before, (47.8) is a closed linear subspace of $LS^\infty_{r_1,r_2}(k)$ with respect to the q_k-metric associated to (47.4). Note that $LP(k)$ is dense in (47.8) with respect to the q_k-metric associated to (47.4).

Let $t \in k \backslash \{0\}$ be given, and let R_t be the rescaling operator defined on $LS(k)$ in Section 45. Also let $f(T) \in LS(k)$ and $0 < q \leq \infty$ be given. Observe that $R_t(f(T)) \in LS^q_{r_1,r_2}(k)$ if and only if $f(T) \in LS^q_{r_1\,|t|,r_2\,|t|}(k)$, by the analogous statement for $LS^q_r(k)$ mentioned earlier. In this case, one can check that

$$\|R_t(f(T))\|_{q,r_1,2_2} = \|f(T)\|_{q,r_1\,|t|,r_2\,|t|}. \tag{47.10}$$

Similarly, $R_t(f(T)) \in LS_{0,r_1,r_2}(k)$ if and only if $f(T) \in LS_{0,r_1\,|t|,r_2\,|t|}(k)$.

48 Products with two radii

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and suppose that k is complete with respect to the q_k-metric associated to $|\cdot|$. Also let T be an indeterminate, let r_1, r_2 be positive real numbers with $r_1 \leq r_2$, and let $0 < q \leq q_k$ be given. Suppose for the moment that $q < \infty$, and let $f(T) = \sum_{j=-\infty}^{\infty} f_j\,T^j$ and $g(T) = \sum_{l=-\infty}^{\infty} g_l\,T^l$ be elements of the space $LS^q_{r_1,r_2}(k)$ defined in (47.1). Under these conditions, $f(T)\,g(T) = h(T) = \sum_{n=-\infty}^{\infty} h_n\,T^n$ can be defined as an element of $LS^q_{r_1,r_2}(k)$ as well, as in Section 41. More precisely, if $f(T), g(T) \in LS^q_r(k)$ for some $r > 0$, then we saw that

$$\sum_{j=-\infty}^{\infty} |f_j\,g_{n-j}|^q < \infty \tag{48.1}$$

for every $n \in \mathbf{Z}$. This was used to define

$$h_n = \sum_{j=-\infty}^{\infty} f_j\,g_{n-j} \tag{48.2}$$

as an element of k for every $n \in \mathbf{Z}$. If $f(T), g(T) \in LS^q_{r_1,r_2}(k)$, then this criterion for (48.1) holds for $r = r_1, r_2$, but of course (48.2) does not depend on which criterion for (48.1) was used.

The remarks in Section 41 imply that $h(T) \in LS^q_r(k)$ for $r = r_1, r_2$, so that $h(T) \in LS^q_{r_1,r_2}(k)$. We would like to check that

$$\|h(T)\|_{q,r_1,r_2} \leq \|f(T)\|_{q,r_1,r_2}\,\|g(T)\|_{q,r_1,r_2}. \tag{48.3}$$

Remember that

$$|h_n|^q \leq \sum_{j=-\infty}^{\infty} |f_j|^q\,|g_{n-j}|^q \tag{48.4}$$

for every $n \in \mathbf{Z}$. This uses the fact that $|\cdot|$ may be considered as a q-absolute value function on k, because $q \leq q_k$.

Let w_{r_1,r_2} be defined on $\mathbf{Z}$ as in (47.2), and observe that

$$w_{r_1,r_2}(j+l) = \max(r_1^j\,r_1^l, r_2^j\,r_2^l) \leq w_{r_1,r_2}(j)\,w_{r_1,r_2}(l) \tag{48.5}$$

for every $j, l \in \mathbf{Z}$. This implies that

$$(48.6) \quad |h_n|^q \, w_{r_1,r_2}(n)^q \le \sum_{j=-\infty}^{\infty} (|f_j|^q \, w_{r_1,r_2}(j)^q) \, (|g_{n-j}|^q \, w_{r_1,r_2}(n-j)^q)$$

for every $n \in \mathbf{Z}$, because of (48.4). The right side of (48.6) is the same as the nth term of the Cauchy product of the series $\sum_{j=-\infty}^{\infty} |f_j|^q \, w_{r_1,r_2}(j)^q$ and $\sum_{l=-\infty}^{\infty} |g_l|^q \, w_{r_1,r_2}(l)^q$, as in Section 21. It follows that

$$\begin{aligned} \|h(T)\|_{q,r_1,r_2}^q &= \sum_{n=-\infty}^{\infty} |h_n|^q \, w_{r_1,r_2}(n)^q \\ &\le \Big(\sum_{j=-\infty}^{\infty} |f_j|^q \, w_{r_1,r_2}(j)^q\Big) \Big(\sum_{l=-\infty}^{\infty} |g_l|^q \, w_{r_1,r_2}(l)^q\Big) \qquad (48.7) \\ &= \|f(T)\|_{q,r_1,r_2}^q \, \|g(T)\|_{q,r_1,r_2}^q, \end{aligned}$$

as desired.

Suppose now that $q_k = \infty$. If $f(T), g(T) \in LS_{0,r}(k)$ for some $r > 0$, then we saw previously that

$$(48.8) \qquad \lim_{j \to \pm\infty} |f_j \, g_{n-j}| = 0$$

for every $n \in \mathbf{Z}$, which was used to define (48.2) as an element of k for every n. If $f(T)$, $g(T)$ are elements of the space $LS_{0,r_1,r_2}(k)$ defined in (47.8), then this criterion holds for $r = r_1, r_2$, but (48.2) does not depend on which criterion for (48.8) was used. In this case, we get that $h(T) \in LS_{0,r}(k)$ for $r = r_1, r_2$, as before, so that $h(T) \in LS_{0,r_1,r_2}(k)$. We also have that

$$(48.9) \qquad \|h(T)\|_{\infty,r_1,r_2} \le \|f(T)\|_{\infty,r_1,r_2} \, \|g(T)\|_{\infty,r_1,r_2},$$

using (41.12) with $r = r_1, r_2$, and (47.6).

49 Related sets and functions

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative q-seminorm N with respect to $|\cdot|$ on k for some $q > 0$. Suppose that $\mathcal{A}$ has a multiplicative identity element e with $N(e) = 1$, and remember that $\mathcal{G}(\mathcal{A})$ denotes the group of invertible elements in $\mathcal{A}$. Let r_1, r_2 be positive real numbers with $r_1 \le r_2$, and put

$$(49.1) \quad \mathcal{C}[r_1, r_2] = \mathcal{C}_{\mathcal{A}}[r_1, r_2] = \{a \in \mathcal{G}(\mathcal{A}) : N(a) \le r_2, \, N(a^{-1}) \le 1/r_1\}.$$

This reduces to the set $\mathcal{C}(r)$ defined in (43.1) when $r_1 = r_2 = r$. If $t \in k \setminus \{0\}$, then $a \in \mathcal{C}[r_1, r_2]$ if and only if $t\,a \in \mathcal{C}[r_1 \, |t|, r_2 \, |t|]$.

Suppose from now on in this section that N is a q-norm on $\mathcal{A}$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. As before, one can check that $\mathcal{C}[r_1, r_2]$ is a closed set in $\mathcal{A}$ with respect to the q-metric associated to N, using the remarks in Section 39. Similarly, if $q = \infty$, then $\mathcal{C}[r_1, r_2]$ is an open set in $\mathcal{A}$ with respect to the q-metric associated to N.

Let T be an indeterminate, and let a be an element of (49.1). Suppose for the moment that $q < \infty$, and let $f(T) = \sum_{j=-\infty}^{\infty} f_j\, T^j$ be an element of the space $LS^q_{r_1,r_2}(k)$ defined in (47.1). As before, we would like to define $f(a)$ as an element of $\mathcal{A}$ by

$$f(a) = \sum_{j=-\infty}^{\infty} f_j\, a^j. \tag{49.2}$$

Let w_{r_1,r_2} be defined on $\mathbf{Z}$ as in (47.2) again, and observe that

$$N(a^j) \le w_{r_1,r_2}(j) \tag{49.3}$$

for every $j \in \mathbf{Z}$. This implies that

$$\sum_{j=-\infty}^{\infty} N(f_j\, a^j)^q \le \sum_{j=-\infty}^{\infty} |f_j|^q\, w_{r_1,r_2}(j)^q = \|f(T)\|^q_{q,r_1,r_2}, \tag{49.4}$$

where $\|f(T)\|_{q,r_1,r_2}$ is as in (47.3). It follows that (49.2) can be defined as an element of $\mathcal{A}$, as a q-absolutely convergent doubly-infinite series, as in Section 17, or as a sum over $\mathbf{Z}$, as in Section 18. As usual, we also get that

$$N(f(a)) \le \Big(\sum_{j=-\infty}^{\infty} N(f_j\, a^j)^q\Big)^{1/q} = \|f(T)\|_{q,r_1,r_2}. \tag{49.5}$$

If $f(T) \in LS_{0,r_1,r_2}(k)$, then

$$N(f_j\, a^j) = |f_j|\, N(a^j) \to 0 \quad \text{as } j \to \pm\infty, \tag{49.6}$$

because of (49.3). If $q = \infty$, then we can use this to define (49.2) as an element of $\mathcal{A}$, as a convergent doubly-infinite series, as in Section 17, or as a sum over $\mathbf{Z}$, as in Section 18. In this situation, we obtain that

$$N(f(a)) \le \max_{j \in \mathbf{Z}} N(f_j\, a^j) \le \max_{j \in \mathbf{Z}}(|f_j|\, w_{r_1,r_2}(j)) = \|f(T)\|_{\infty,r_1,r_2}. \tag{49.7}$$

Suppose that $q < \infty$ and $f(T) \in LS^q_{r_1,r_2}(T)$, or that $q = \infty$ and $f(T)$ is an element of $LS_{0,r_1,r_2}(k)$, as in the previous two paragraphs. In both cases,

$$\sum_{j=-n}^{n} f_j\, a^j \tag{49.8}$$

tends to $f(a)$ as $n \to \infty$ with respect to the q-metric on $\mathcal{A}$ associated to N. One can check that the convergence is uniform on $\mathcal{C}[r_1, r_2]$, using the same type of estimates as before. If $j \in \mathbf{Z}$, then $a \mapsto a^j$ is uniformly continuous as a mapping from $\mathcal{C}[r_1, r_2]$ into $\mathcal{A}$, with respect to the q-metric on $\mathcal{A}$ associated to N and its restriction to $\mathcal{C}[r_1, r_2]$, for the same reasons as in Section 44. It follows that

$f(a)$ defines a uniformly continuous mapping from $\mathcal{C}[r_1, r_2]$ into $\mathcal{A}$ under these conditions, because of uniform convergence.

Let $t \in k \setminus \{0\}$ be given, so that $t\,a \in \mathcal{C}[r_1\,|t|, r_2\,|t|]$, as before. Suppose that either $q < \infty$ and $f(T) \in LS^q_{r_1\,|t|, r_2\,|t|}(k)$, or that $q = \infty$ and $f(T)$ is an element of $LS_{0,r_1\,|t|,r_2\,|t|}(k)$. This implies that $R_t(f(T)) \in LS^q_{r_1,r_2}(k)$ when $q < \infty$, and that $R_t(f(T)) \in LS_{0,r_1,r_2}(k)$ when $q = \infty$. In both cases, $f(t\,a)$ and $(R_t(f))(a)$ can be defined as elements of $\mathcal{A}$, as before. We also have that $f(t\,a) = (R_t(f))(a)$, as in (45.4).

Suppose now that k is complete with respect to the q_k-metric associated to $|\cdot|$. We may as well take $q \le q_k$, as in Section 7. Let $f(T)$, $g(T)$ be elements of $LS^q_{r_1,r_2}(k)$ when $q < \infty$, or elements of $LS_{0,r_1,r_2}(k)$ when $q = \infty$, so that $f(T)\,g(T) = h(T)$ is defined as an element of the same space, as in the previous section. Thus $f(a)$, $g(a)$, and $h(a)$ are defined as elements of $\mathcal{A}$, as before, and in fact $h(a) = f(a)\,g(a)$, as in (44.7). This implies that $a \mapsto f(a)$ is an algebra homomorphism from $LS^q_{r_1,r_2}(k)$ into $\mathcal{A}$ when $q < \infty$, and from $LS_{0,r_1,r_2}(k)$ into $\mathcal{A}$ when $q = \infty$.

50 Radii of convergence

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let T be an indeterminate, and let r_0, r_1 be positive real numbers with $r_0 \le r_1$. Suppose that $f(T)$ is an element of the space $LS^q_{r_0,r_1}(k)$ defined in Section 47 for some $q > 0$. If $r_0 \le r \le r_1$, then it is easy to see that $f(T)$ is in the space $LS^q_r(k)$ defined in Section 40, with

$$\|f(T)\|_{q,r} \le \|f(T)\|_{q,r_0,r_1}. \tag{50.1}$$

Let $t \in [0,1]$ be given, and put $r_t = r_0^{1-t}\,r_1^t$, so that $r_0 \le r_t \le r_1$. One can check that

$$\|f(T)\|_{q,r_t} \le \|f(T)\|_{q,r_0}^{1-t}\,\|f(T)\|_{q,r_1}^t, \tag{50.2}$$

using (27.8) when $q = \infty$, and (27.9) when $q < \infty$.

Let r_1, r_2 be positive real numbers with $r_1 < r_2$, and suppose that $f(T)$ is an element of the space $LS^\infty_{r_1,r_2}(k)$. If $r_1 < r < r_2$, then one can verify that $f(T) \in LS^q_r(k)$ for every $q > 0$, using the same type of arguments as in Section 30. In particular, $f(T) \in LS_{0,r}(k)$.

Suppose that $0 \le r_1 < r_2 \le \infty$, and put

$$\begin{aligned} LS_{(r_1,r_2)}(k) &= LS_{(r_1,r_2)}(k,T) \\ &= \{f(T) \in LS(k) : f(T) \in LS^\infty_r(k) \text{ for every } r_1 < r < r_2\}. \end{aligned} \tag{50.3}$$

If $f(T)$ is an element of (50.3) and $r_1 < r < r_2$, then we have that $f(T) \in LS^q_r(k)$ for every $q > 0$, and in particular that $f(T) \in LS_{0,r}(k)$, by the remarks in the preceding paragraph. Note that (50.3) is a linear subspace of $LS(k)$.

Let $t \in k \setminus \{0\}$ be given, so that the rescaling operator R_t can be defined on $LS(k)$ as in Section 45. If $f(T) \in LS(k)$, then $R_t(f(T)) \in LS_{(r_1,r_2)}(k)$ if and only if $f(T) \in LS_{(r_1\,|t|, r_2\,|t|)}$. This follows from the analogous statement for $LS^q_r(k)$.

Let $\mathcal{A}$ be an algebra over k with a submultiplicative q-seminorm N with respect to $|\cdot|$ on k for some $q > 0$, and a multiplicative identity element e with $N(e) = 1$. Put

$$\text{(50.4)}\quad \mathcal{C}(r_1, r_2) = \mathcal{C}_{\mathcal{A}}(r_1, r_2) = \{a \in \mathcal{G}(\mathcal{A}) : N(a) < r_2,\ N(a^{-1}) < 1/r_1\},$$

where $\mathcal{G}(\mathcal{A})$ is the group of invertible elements of $\mathcal{A}$, as before. If $t \in k \setminus \{0\}$, then $a \in \mathcal{C}(r_1, r_2)$ if and only if $t\,a \in \mathcal{C}(r_1\,|t|, r_2\,|t|)$. Suppose from now on in this section that N is a q-norm on $\mathcal{A}$, and that $\mathcal{A}$ is complete with respect to the q-metric associated to N. One can use the remarks in Section 39 to check that $\mathcal{C}(r_1, r_2)$ is an open set in $\mathcal{A}$ with respect to the q-metric associated to N, and a closed set when $q = \infty$.

If $a \in \mathcal{C}(r_1, r_2)$ and $f(T) \in LS_{(r_1, r_2)}(k)$, then $f(a)$ can be defined as an element of $\mathcal{A}$, as in the previous section. The resulting mapping from $\mathcal{C}(r_1, r_2)$ into $\mathcal{A}$ is continuous with respect to the q-metric on $\mathcal{A}$ associated to N and its restriction to $\mathcal{C}(r_1, r_2)$, because of the analogous continuity property mentioned in the previous section.

Suppose that k is complete with respect to the q_k-metric associated to $|\cdot|$, and let $f(T), g(T) \in LS_{(r_1, r_2)}(k)$ be given. If $q_k < \infty$, then $f(T), g(T) \in LS^{q_k}_r(k)$ for every $r_1 < r < r_2$, so that $f(T)\,g(T) = h(T)$ can be defined as an element of $LS^{q_k}_r(k)$ as in Section 41. Similarly, if $q_k = \infty$, then $f(T), g(T) \in LS_{0,r}(k)$ for every $r_1 < r < r_2$, so that $f(T)\,g(T) = h(T)$ can be defined as an element of $LS_{0,r}(k)$, as before. As in Section 48, these definitions of $h(T)$ do not depend on r. Thus $h(T)$ is an element of $LS_{(r_1, r_2)}(k)$ in both cases.

51 The complex case

Let us take k to be the field $\mathbf{C}$ of complex numbers with the standard absolute value function, and let $0 \le r_1 < r_2 \le \infty$ be given again. Thus

$$\text{(51.1)}\qquad C(r_1, r_2) = \{z \in \mathbf{C} : r_1 < |z| < r_2\}$$

is an open set in $\mathbf{C}$, which corresponds to (50.4), with $\mathcal{A} = \mathbf{C}$. Let T be an indeterminate, and let $f(T) = \sum_{j=-\infty}^{\infty} f_j\, T^j$ be a formal Laurent series in T with complex coefficients. If $f(T)$ is an element of the space $LS_{(r_1, r_2)}(\mathbf{C})$ defined in the previous section and $z \in C(r_1, r_2)$, then

$$\text{(51.2)}\qquad f(z) = \sum_{j=-\infty}^{\infty} f_j\, z^j$$

can be defined as a complex number, as an absolutely convergent doubly-infinite series. It is well known that this defines a holomorphic function on $C(r_1, r_2)$, and that every holomorphic function on $C(r_1, r_2)$ corresponds to a unique Laurent series in this way.

Let r_1, r_2 be positive real numbers with $r_1 \le r_2$. Note that

$$C[r_1, r_2] = \{z \in \mathbf{C} : r_1 \le |z| \le r_2\} \tag{51.3}$$

is a closed set in $\mathbf{C}$, which corresponds to (49.1), with $\mathcal{A} = \mathbf{C}$. Let $f(T)$ be an element of the space $LS^1_{r_1,r_2}(\mathbf{C})$ defined in Section 47. If $z \in C[r_1, r_2]$, then $f(z)$ can be defined as a complex number as in (51.2), where the sum is an absolutely convergent doubly-infinite series. This defines a continuous complex-valued function on $C[r_1, r_2]$, as in Section 49. If $r_1 < r_2$, then the restriction of $f(z)$ to $C(r_1, r_2)$ is holomorphic, as in the preceding paragraph. Of course, $C(r_1, r_2)$ is the interior of $C[r_1, r_2]$ in this case.

Let $f(z)$ be a holomorphic function on $C(r_1, r_2)$ for some $0 \le r_1 < r_2 \le \infty$. If $r_1 < r < r_2$, then put

$$M_\infty(f, r) = \sup\{|f(z)| : z \in \mathbf{C},\ |z| = r\}. \tag{51.4}$$

Let $r_1 < \rho_0 \le \rho_1 < r_2$ and $t \in [0, 1]$ be given, and put

$$\rho_t = \rho_0^{1-t}\, \rho_1^t, \tag{51.5}$$

so that $\rho_0 \le \rho_t \le \rho_1$. The maximum principle implies that

$$M_\infty(f, \rho_t) \le \max(M_\infty(f, \rho_0), M_\infty(f, \rho_1)). \tag{51.6}$$

In fact, it is well known that

$$M_\infty(f, \rho_t) \le M_\infty(f, \rho_0)^{1-t}\, M_\infty(f, \rho_1)^t, \tag{51.7}$$

by Hadamard's three-circles theorem.

Let q be a positive real number, and put

$$M_q(f, r) = \Big(\frac{1}{2\,\pi\, r} \int_{|z|=r} |f(z)|^q\, |dz|\Big)^{1/q} \tag{51.8}$$

for $r_1 < r < r_2$, where the integral is taken over the set of $z \in \mathbf{C}$ with $|z| = r$, with respect to arclength. If $r_1 < \rho_0 \le \rho_1 < r_2$, $t \in [0, 1]$, and ρ_t is as in (51.5), then it is well known that

$$M_q(f, \rho_t) \le M_q(f, \rho_0)^{1-t}\, M_q(f, \rho_1)^t. \tag{51.9}$$

If f is given by the Laurent expansion (51.2), then

$$M_2(f, r) = \Big(\sum_{j=-\infty}^{\infty} |f_j|^2\, r^{2\,j}\Big)^{1/2}, \tag{51.10}$$

by a standard argument. This is the same as $\|f(T)\|_{2,r}$ in (40.5), where T is an indeterminate, and $f(T)$ is the formal Laurent series in T corresponding to (51.2), as before. Thus (51.9) corresponds to (50.2) when $q = 2$.

Part V
Some additional topics

52 Cesaro means

Let V be a vector space over the real or complex numbers, and let N be a norm on V, with respect to the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate. Also let $\{s_n\}_{n=0}^\infty$ be a sequence of elements of V, and put

$$\sigma_n = \frac{1}{n+1} \sum_{l=0}^n s_l \tag{52.1}$$

for each nonnegative integer n. If $\{\sigma_n\}_{n=0}^\infty$ converges to an element s of V with respect to the metric associated to N, then s is said to be the *Cesaro limit* of the sequence $\{s_n\}_{n=0}^\infty$. It is well known that if $\{s_n\}_{n=0}^\infty$ converges to $s \in V$ with respect to the metric associated to N, then $\{\sigma_n\}_{n=0}^\infty$ converges to s as well.

Suppose that $\{\sigma_n\}_{n=0}^\infty$ converges to $s \in V$ with respect to the metric associated to N. If $a \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, then

$$a\,\sigma_n = \frac{1}{n+1} \sum_{l=0}^n a\,s_l \tag{52.2}$$

for each $n \ge 0$. Hence the Cesaro limit of $\{a\,s_n\}_{n=0}^\infty$ is equal to $a\,s$. Let $\{t_n\}_{n=0}^\infty$ be another sequence of elements of V, and let

$$\tau_n = \frac{1}{n+1} \sum_{l=0}^n t_l \tag{52.3}$$

be the corresponding sequence of Cesaro means. Thus

$$\sigma_n + \tau_n = \frac{1}{n+1} \sum_{l=0}^n (s_l + t_l) \tag{52.4}$$

for each $n \ge 0$. If $\{\tau_n\}_{n=0}^\infty$ also converges to $t \in V$ with respect to the metric associated to N, then $\{\sigma_n + \tau_n\}_{n=0}^\infty$ converges to $s + v$. This implies that the Cesaro limit of $\{s_n + t_n\}_{n=0}^\infty$ is equal to the sum of the Cesaro limits of $\{s_n\}_{n=0}^\infty$ and $\{t_n\}_{n=0}^\infty$, when they exist.

Let $\sum_{j=0}^\infty v_j$ be an infinite series with terms in V, and let

$$s_n = \sum_{j=0}^n v_j \tag{52.5}$$

be the corresponding partial sums, for each nonnegative integer n. If the sequence of Cesaro means (52.1) converges to $s \in V$ with respect to the metric associated to N, then $\sum_{j=0}^\infty v_j$ is said to be *Cesaro summable*, with Cesaro sum equal to s. If $\sum_{j=0}^\infty v_j$ converges in the usual sense with respect to N, then $\sum_{j=0}^\infty v_j$ is Cesaro summable with the same sum, by the analogous property for convergent sequences. In this situation,

$$\sigma_n = \frac{1}{n+1} \sum_{l=0}^{n} \sum_{j=0}^{l} v_j = \frac{1}{n+1} \sum_{j=0}^{n} \sum_{l=j}^{n} v_j = \sum_{j=0}^{n} \Big(\frac{n-j+1}{n+1}\Big) v_j \tag{52.6}$$

for every $n \ge 0$.

Now let

$$\sum_{j=-\infty}^{\infty} v_j \tag{52.7}$$

be a doubly-infinite series of elements of V. As in Section 17, we say that (52.7) converges in V with respect to N if

$$\sum_{j=0}^{\infty} v_j \quad \text{and} \quad \sum_{j=1}^{\infty} v_{-j} \tag{52.8}$$

converge as ordinary infinite series with terms in V with respect to N, in which case the value of the sum (52.7) is defined by

$$\sum_{j=-\infty}^{\infty} v_j = \sum_{j=0}^{\infty} v_j + \sum_{j=1}^{\infty} v_{-j}. \tag{52.9}$$

If

$$\sum_{j=1}^{\infty} (v_j + v_{-j}) \tag{52.10}$$

converges as an infinite series with terms in V, then one can use

$$v_0 + \sum_{j=1}^{\infty} (v_j + v_{-j}) \tag{52.11}$$

to define the sum (52.7). If the series in (52.8) converge, then $\sum_{j=1}^\infty v_j$ converges, and hence (52.10) converges. Of course, (52.9) and (52.11) are the same in this situation.

Put

$$s_n = \sum_{j=-n}^{n} v_j \tag{52.12}$$

for each nonnegative integer n. Thus $s_0 = v_0$, and

$$s_n = v_0 + \sum_{j=1}^{n} (v_j + v_{-j}) \tag{52.13}$$

for every positive integer n. It follows that $\{s_n\}_{n=0}^\infty$ converges as a sequence of elements in V with respect to the metric associated to N if and only if (52.10) converges, with

$$\lim_{n \to \infty} s_n = v_0 + \sum_{j=1}^\infty (v_j + v_{-j}). \tag{52.14}$$

If the corresponding sequence of Cesaro means (52.1) converges to $s \in V$ with respect to the metric associated to N, then we may say that (52.7) is *Cesaro summable*, with Cesaro sum equal to s. Observe that the Cesaro means can be expressed as

$$\begin{aligned} \sigma_n = \frac{1}{n+1} \sum_{l=0}^n \sum_{j=-l}^l v_j &= \frac{1}{n+1} \sum_{j=-n}^n \sum_{l=|j|}^n v_j \\ &= \sum_{j=-n}^n \Big(\frac{n - |j| + 1}{n+1}\Big) v_j \end{aligned} \tag{52.15}$$

for every $n \ge 0$.

53 Abel sums

Let V be a vector space over the real or complex numbers again, and let N be a norm on V with respect to the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate. In this section, we suppose that V is complete with respect to the metric associated to N. Let $\sum_{j=0}^\infty v_j$ be an infinite series with terms in V. Suppose that for each nonnegative real number r with $r < 1$, we have that

$$\sum_{j=0}^\infty N(v_j)\, r^j \tag{53.1}$$

converges as an infinite series of nonnegative real numbers. This means that

$$\sum_{j=0}^\infty v_j\, r^j \tag{53.2}$$

converges absolutely with respect to N, and hence that this series converges in V, by completeness. If the one-sided limit

$$\lim_{r \to 1-} \sum_{j=0}^\infty v_j\, r^j \tag{53.3}$$

exists in V, with respect to the metric associated to N, then $\sum_{j=0}^\infty v_j$ is said to be *Abel summable* in V with respect to N. In this case, the Abel sum of $\sum_{j=0}^\infty v_j$ is defined to be (53.3). If $\sum_{j=0}^\infty v_j$ converges in the ordinary sense, then it is well known that $\sum_{j=0}^\infty v_j$ is Abel summable, with Abel sum equal to the ordinary sum. Note that (53.1) converges when $0 \le r < 1$ and $N(v_j)$ is bounded.

Suppose that $\sum_{j=0}^\infty v_j$ is Abel summable in V, which includes the convergence of (53.1) when $0 \le r < 1$. If a is a real or complex number, as appropriate, then

$$\sum_{j=0}^\infty a\, v_j\, r^j = a \sum_{j=0}^\infty v_j\, r^j \tag{53.4}$$

for every $0 \le r < 1$. It follows that

$$\lim_{r \to 1-} \sum_{j=0}^\infty a\, v_j\, r^j = a \Big(\lim_{r \to 1-} \sum_{j=0}^\infty v_j\, r^j\Big), \tag{53.5}$$

so that $\sum_{j=0}^\infty a\, v_j$ is Abel summable too, with Abel sum equal to a times the Abel sum of $\sum_{j=0}^\infty v_j$. Let $\sum_{j=0}^\infty w_j$ be another infinite series with terms in V that is Abel summable with respect to N. In particular, this means that $\sum_{j=0}^\infty N(w_j)\, r^j$ converges when $0 \le r < 1$. Thus $\sum_{j=0}^\infty N(v_j + w_j)\, r^j$ converges when $0 \le r < 1$, because of the triangle inequality for N and the comparison test. Of course,

$$\sum_{j=0}^\infty (v_j + w_j)\, r^j = \sum_{j=0}^\infty v_j\, r^j + \sum_{j=0}^\infty w_j\, r^j \tag{53.6}$$

for every $0 \le r < 1$, so that

$$\lim_{r \to 1-} \sum_{j=0}^\infty (v_j + w_j)\, r^j = \lim_{r \to 1-} \sum_{j=0}^\infty v_j\, r^j + \lim_{r \to 1-} \sum_{j=0}^\infty w_j\, r^j. \tag{53.7}$$

This implies that $\sum_{j=0}^\infty (v_j + w_j)$ is Abel summable in V with respect to N, with Abel sum equal to the sum of the Abel sums of $\sum_{j=0}^\infty v_j$ and $\sum_{j=0}^\infty w_j$.

Let $\sum_{j=-\infty}^\infty v_j$ be a doubly-infinite series with terms in V. Suppose that

$$\sum_{j=-\infty}^\infty N(v_j)\, r^{|j|} \tag{53.8}$$

converges as a doubly-infinite series of nonnegative real numbers for every nonnegative real number $r < 1$. Equivalently, this means that

$$\sum_{j=0}^\infty N(v_j)\, r^j \quad \text{and} \quad \sum_{j=1}^\infty N(v_{-j})\, r^j \tag{53.9}$$

converge as infinite series of nonnegative real numbers when $0 \le r < 1$, so that $\sum_{j=0}^\infty v_j\, r^j$ and $\sum_{j=1}^\infty v_{-j}\, r^j$ converge in V with respect to N. As in the previous section, we put

$$\sum_{j=-\infty}^\infty v_j\, r^{|j|} = \sum_{j=0}^\infty v_j\, r^j + \sum_{j=1}^\infty v_{-j}\, r^j \tag{53.10}$$

for every $0 \le r < 1$. Let us say that $\sum_{j=-\infty}^{\infty} v_j$ is *Abel summable* in V with respect to N if the one-sided limit

$$\lim_{r \to 1-} \sum_{j=-\infty}^{\infty} v_j \, r^{|j|} \tag{53.11}$$

exists in V, with respect to the metric associated to N. As before, the Abel sum of $\sum_{j=-\infty}^{\infty} v_j$ is defined to be (53.11) in this case. Note that $\sum_{j=0}^{\infty} v_j$ is Abel summable when $\sum_{j=0}^{\infty} v_j$ and $\sum_{j=1}^{\infty} v_{-j}$ are Abel summable as ordinary infinite series.

Put $w_0 = v_0$ and

$$w_j = v_j + v_{-j} \tag{53.12}$$

for every positive integer j. If the series in (53.9) converge, then

$$\sum_{j=0}^{\infty} N(w_j) \, r^j \tag{53.13}$$

converges as an infinite series of nonnegative real numbers when $0 \le r < 1$, by the triangle inequality for N and the comparison test. Under these conditions, the Abel sums

$$\sum_{j=0}^{\infty} w_j \, r^j \tag{53.14}$$

converge in V for each $0 \le r < 1$, and are equal to (53.10). If $\sum_{j=-\infty}^{\infty} v_j$ is Abel summable in V with respect to N, as in the preceding paragraph, then it follows that $\sum_{j=0}^{\infty} w_j$ is Abel summable in V as an ordinary infinite series, and that the corresponding Abel sums are the same.

54 Abel sums and Cauchy products

Let $\mathcal{A}$ be an algebra over the real or complex numbers, and let N be a submultiplicative norm on $\mathcal{A}$, with respect to the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate. Also let $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ be infinite series with terms in $\mathcal{A}$, and put

$$c_n = \sum_{j=0}^{n} a_j \, b_{n-j} \tag{54.1}$$

for each nonnegative integer n. Thus $\sum_{n=0}^{\infty} c_n$ is the Cauchy product of the series $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$, as in Section 20. We would like to look at the Abel summability of $\sum_{n=0}^{\infty} c_n$ in terms of the Abel summability of $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$. Remember that

$$N(c_n) \le \sum_{j=0}^{n} N(a_j) \, N(b_{n-j}) \tag{54.2}$$

for every nonnegative integer n, by the triangle inequality and submultiplicativity of N.

Let r be a nonnegative real number with $r < 1$, and suppose that

$$\sum_{j=0}^{\infty} N(a_j)\, r^j \quad \text{and} \quad \sum_{l=0}^{\infty} N(b_l)\, r^l \tag{54.3}$$

converge as infinite series of nonnegative real numbers. Observe that

$$N(c_n)\, r^n \le \sum_{j=0}^{n} N(a_j)\, N(b_{n-j})\, r^n = \sum_{j=0}^{n} (N(a_j)\, r^j)\, (N(b_{n-j})\, r^{n-j}) \tag{54.4}$$

for every nonnegative integer n. The right side of (54.4) is the same as the nth term of the Cauchy product of the series in (54.3). It follows that the sum over $n \ge 0$ of the right side of (54.4) converges, and is equal to the product of the sums in (54.3), as in Section 20. This implies that

$$\sum_{n=0}^{\infty} N(c_n)\, r^n \tag{54.5}$$

converges, with sum less than or equal to the product of the sums in (54.3).

Thus

$$\sum_{j=0}^{\infty} a_j\, r^j, \quad \sum_{l=0}^{\infty} b_l\, r^l, \quad \text{and} \quad \sum_{n=0}^{\infty} c_n\, r^n \tag{54.6}$$

converge absolutely with respect to N. Suppose that $\mathcal{A}$ is complete with respect to the metric associated to N, so that these series converge in $\mathcal{A}$ with respect to N. We also get that

$$\sum_{n=0}^{\infty} c_n\, r^n = \Big(\sum_{j=0}^{\infty} a_j\, r^j\Big)\Big(\sum_{l=0}^{\infty} b_l\, r^l\Big), \tag{54.7}$$

as in Section 20.

Suppose that the series in (54.3) converge for every $0 \le r < 1$. This implies that (54.5) converges for every $0 \le r < 1$, as before. Hence the series in (54.6) converge in $\mathcal{A}$ for every $0 \le r < 1$, and satisfy (54.7).

Suppose that $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ are Abel summable in $\mathcal{A}$ with respect to N, so that the one-sided limits

$$\lim_{r \to 1-} \sum_{j=0}^{\infty} a_j\, r^j \quad \text{and} \quad \lim_{r \to 1-} \sum_{l=0}^{\infty} b_l\, r^l \tag{54.8}$$

exist in $\mathcal{A}$ with respect to the metric associated to N. This implies that

$$\lim_{r \to 1-} \sum_{n=0}^{\infty} c_n\, r^n \tag{54.9}$$

is equal to the product of the limits in (54.8), because of (54.7). This means that $\sum_{n=0}^{\infty} c_n$ is Abel summable as well, with Abel sum equal to the product of the Abel sums of $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$.

55 Summing Fourier series

Let r be a positive real number, and let $C(r)$ be the circle in the complex plane centered at 0 with radius r with respect to the standard absolute value function on $\mathbf{C}$, as in Section 46. If $f \in L^1(C(r))$, then the jth Fourier coefficient $\widehat{f}(j)$ of f is defined for every $j \in \mathbf{Z}$ as in (46.2). Thus the Fourier series

$$\sum_{j=-\infty}^{\infty} \widehat{f}(j)\, z^j \tag{55.1}$$

is a doubly-infinite series of complex numbers for every $z \in C(r)$. Put

$$s_n(f, z) = \sum_{j=-n}^{n} \widehat{f}(j)\, z^j \tag{55.2}$$

for every nonnegative integer n and $z \in C(r)$, and let

$$\sigma_n(f, z) = \frac{1}{n+1} \sum_{l=0}^{n} s_l(f, z) = \sum_{j=-n}^{n} \Big(\frac{n - |j| + 1}{n+1}\Big)\, \widehat{f}(j)\, z^j \tag{55.3}$$

be the corresponding Cesaro means, as in Section 52. It is well known that (55.3) converges to $f(z)$ as $n \to \infty$ for almost every $z \in C(r)$ with respect to arclength measure. If $1 \le q < \infty$ and $f \in L^q(C(r))$, then (55.3) converges to f as $n \to \infty$ with respect to the L^q norm. If f is a continuous complex-valued function on $C(r)$, then (55.3) converges to f uniformly on $C(r)$ as $n \to \infty$.

Let $f \in L^1(C(r))$ be given again, and put

$$A_{r_1}(f, z) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)\, r_1^{|j|}\, z^j \tag{55.4}$$

for every nonnegative real number r_1 with $r_1 < 1$ and $z \in C(r)$. This is the Abel sum associated to the Fourier series (55.1) and r_1. Remember that

$$|\widehat{f}(j)\, z^j| = |\widehat{f}(j)|\, r^j \le \|f\|_{L^1(C(r))} \tag{55.5}$$

for every $j \in \mathbf{Z}$ and $z \in C(r)$, so that

$$\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|\, r_1^{|j|}\, r^j \le \|f\|_{L^1(C(r))} \sum_{j=-\infty}^{\infty} r_1^{|j|} < \infty \tag{55.6}$$

when $0 \le r_1 < 1$. Thus the doubly-infinite series in (55.4) converges absolutely for every $z \in C(r)$ and $0 \le r_1 < 1$. As before, it is well known that (55.4) converges to $f(z)$ as $r_1 \to 1-$ for almost every $z \in C(r)$ with respect to arclength measure. If $1 \le q < \infty$ and $f \in L^q(C(r))$, then (55.4) converges to f as $r_1 \to 1-$ with respect to the L^q norm. If f is a continuous complex-valued function on $C(r)$, then (55.4) converges to f uniformly on $C(r)$ as $r_1 \to 1-$.

Suppose that $f \in L^1(C(r))$ again, and let $0 \le r_1 < 1$ be given. Consider the partial sum

$$\sum_{j=-n}^{n} \widehat{f}(j)\, r_1^{|j|}\, z^j \tag{55.7}$$

for each nonnegative integer n and $z \in C(r)$. Note that (55.7) converges to (55.4) uniformly on $C(r)$ as $n \to \infty$, by a standard argument.

Suppose now that f is a continuous complex-valued function on $C(r)$. It is well known that $f(z)$ can be approximated uniformly on $C(r)$ by finite linear combinations of the z^j's. This can be obtained from the famous theorem of Lebesgue, Stone, and Weierstrass. This also follows from the uniform convergence of the Cesaro means (55.3) to f as $n \to \infty$. Similarly, this can be obtained from the uniform convergence of the Abel sums (55.4) to f as $r_1 \to 1-$, using the uniform convergence of the partial sums (55.7) for each $0 \le r_1 < 1$, as in the preceding paragraph.

56 Continuous functions and unitary operators

Let $(V, \langle v, w \rangle)$ be a complex Hilbert space, and let $\|v\|$ be the norm associated to the inner product. Remember that the space $\mathcal{BL}(V)$ of bounded linear mappings from V into itself is an algebra over the complex numbers with respect to composition of mappings, and that the corresponding operator norm $\|\cdot\|_{op}$ is a submultiplicative norm on $\mathcal{BL}(V)$. Of course, $\mathcal{BL}(V)$ is complete with respect to the metric associated to the operator norm, because V is complete with respect to the metric associated to $\|\cdot\|$.

Let T be an indeterminate, and let

$$f(T) = \sum_{j=-n}^{n} a_j\, T^j \tag{56.1}$$

be a formal Laurent polynomial in T with complex coefficients. If $z \in \mathbf{C}$ and $z \neq 0$, then

$$f(z) = \sum_{j=-n}^{n} a_j\, z^j \tag{56.2}$$

is defined as a complex number. Similarly, if A is an invertible element of $\mathcal{BL}(V)$, then

$$f(A) = \sum_{j=-n}^{n} a_j\, A^j \tag{56.3}$$

defines an element of $\mathcal{BL}(V)$. Let A be a unitary operator from V onto itself. It is well known that

$$\|f(A)\|_{op} \le \sup_{z \in C(1)} |f(z)|, \tag{56.4}$$

where $C(1)$ is the unit circle in the complex plane, as before.

Remember that continuous functions on $C(1)$ corresponding to Laurent polynomials as in (56.2) are dense in the space of all continuous complex-valued functions on $C(1)$ with respect to the supremum metric, as in the previous section. It follows that there is a unique extension of

$$f \mapsto f(A) \tag{56.5}$$

to a bounded linear mapping from the space of continuous complex-valued functions on $C(1)$ equipped with the supremum norm into $\mathcal{BL}(V)$, as in Section 10. More precisely, this extension satisfies (56.4) for all continuous complex-valued functions f on $C(1)$. Of course, the space of continuous complex-valued functions on $C(1)$ is a commutative algebra with respect to pointwise multiplication of functions. The extension of (56.5) to continuous complex-valued functions f on $C(1)$ is an algebra homomorphism, because of the analogous property for Laurent polynomials.

If f is a continuous complex-valued function on $C(1)$, then the complex-conjugate $\overline{f}$ is continuous on $C(1)$ too. Similarly, if B is a bounded linear mapping from V into itself, then the Hilbert space adjoint B^* of B is a bounded linear mapping as well, as in Section 23. The hypothesis that A be a unitary operator on V is the same as saying that A is invertible, with inverse equal to A^*. In fact, we have that

$$\overline{f}(A) = f(A)^* \tag{56.6}$$

for every continuous complex-valued function f on $C(1)$. To see this, suppose first that $f(z) = z^j$ on $C(1)$ for some $j \in \mathbf{Z}$. In this case,

$$\overline{f(z)} = \overline{z}^j = z^{-j} \tag{56.7}$$

on $C(1)$, so that

$$\overline{f}(A) = A^{-j} = (A^j)^* = f(A)^*. \tag{56.8}$$

It follows that (56.6) holds when f is a linear combination of z^j's on $C(1)$. If f is any continuous complex-valued function on $C(1)$, then one can get (56.6) by approximating f uniformly by linear combinations of z^j's, as before.

57 Analytic type

Let r be a positive real number, and let $C(r)$ be the circle in the complex plane centered at 0 with radius r with respect to the standard absolute value function on $\mathbf{C}$, as before. Also let $f \in L^1(C(r))$ be given, so that the jth Fourier coefficient $\widehat{f}(j)$ can be defined for every $j \in \mathbf{Z}$ as in (46.2). If

$$\widehat{f}(j) = 0 \tag{57.1}$$

for every $j < 0$, then f is said to be of *analytic type*. This means that the corresponding Fourier series reduces to

$$\sum_{j=0}^{\infty} \widehat{f}(j)\, z^j. \tag{57.2}$$

If T is an indeterminate, then

$$\sum_{j=0}^{\infty} \widehat{f}(j)\, T^j \tag{57.3}$$

may be considered as a formal power series in T with complex coefficients.

Remember that

$$|\widehat{f}(j)|\, r^j \le \|f\|_{L^1(C(r))} \tag{57.4}$$

for each j. If $z \in \mathbf{C}$ satisfies $|z| < r$, then (57.2) converges absolutely. This defines a holomorphic function on the open disk U_r in $\mathbf{C}$ centered at 0 with radius r, as in Section 34.

Let $z \in C(r)$ and $0 \le r_1 < 1$ be given, so that the corresponding Abel sum $A_{r_1}(f, z)$ of the Fourier series of f can be defined as in (55.4). If f is of analytic type, then this reduces to

$$A_{r_1}(f, z) = \sum_{j=0}^{\infty} \widehat{f}(j)\, r_1^j\, z^j. \tag{57.5}$$

Suppose now that f is a continuous complex-valued function on $C(r)$ of analytic type. If $z \in U_r$, then let $f(z)$ be the value of the sum in (57.2). This defines f as a complex-valued function on the closed disk $\overline{U_r}$ in $\mathbf{C}$ centered at 0 with radius r. One can check that f is continuous on $\overline{U_r}$ under these conditions. This uses the fact that (57.5) converges to f uniformly on $C(r)$ as $r_1 \to 1-$, as in Section 55.

Conversely, if f is a continuous complex-valued function on $\overline{U_r}$ that is holomorphic on U_r, then the restriction of f to $C(r)$ is of analytic type. Indeed, if $0 < \rho < r$, then

$$\int_{C(\rho)} f(z)\, z^j\, dz = 0 \tag{57.6}$$

for every nonnegative integer j, by Cauchy's theorem. Equivalently, this means that

$$\int_{C(\rho)} f(z)\, z^{j+1}\, |dz| = 0 \tag{57.7}$$

for every nonnegative integer j. If $0 < r_1 < 1$, then we can apply this to $\rho = r_1\, r$, to get that

$$\int_{C(r)} f(r_1\, z)\, z^{j+1}\, |dz| = 0 \tag{57.8}$$

for every nonnegative integer j. Note that f is uniformly continuous on $\overline{U_r}$, because f is continuous on $\overline{U_r}$, and $\overline{U_r}$ is compact. This implies that $f(r_1\, z)$ tends to $f(z)$ uniformly on $\overline{U_r}$ as $r_1 \to 1-$. It follows that

$$\int_{C(r)} f(z)\, z^{j+1}\, |dz| = 0 \tag{57.9}$$

for every nonnegative integer j, by taking the limit as $r_1 \to 1-$ in (57.8).

58 Hilbert space contractions

Let $(V, \langle v, w\rangle)$ be a complex Hilbert space with norm $\|v\|$ associated to the inner product, and let $\|\cdot\|_{op}$ be the corresponding operator norm on the algebra $\mathcal{BL}(V)$ of bounded linear mappings from V into itself. Also let T be an indeterminate, and let

$$f(T) = \sum_{j=0}^{n} f_j \, T^j \tag{58.1}$$

be a formal polynomial in T with complex coefficients. If $z \in \mathbf{C}$, then

$$f(z) = \sum_{j=0}^{n} f_j \, z^j \tag{58.2}$$

is defined as a complex number, as usual. Similarly, if A is a bounded linear mapping from V into itself, then

$$f(A) = \sum_{j=0}^{n} f_j \, A^j \tag{58.3}$$

is defined as an element of $\mathcal{BL}(V)$.

Suppose that A is a contraction on V, in the sense that

$$\|A\|_{op} \le 1. \tag{58.4}$$

Let $\overline{U_1}$ be the closed unit disk in $\mathbf{C}$, as before. A famous theorem of von Neumann and Heinz states that

$$\|f(A)\|_{op} \le \sup_{z \in \overline{U_1}} |f(z)|. \tag{58.5}$$

Let $A(U_1)$ be the space of continuous complex-valued functions on $\overline{U_1}$ that are holomorphic on the open unit disk U_1, as in Section 34. Remember that elements of $A(U_1)$ can be approximated uniformly on $\overline{U_1}$ by polynomials in z. Hence there is a unique extension of

$$f \mapsto f(A) \tag{58.6}$$

to a bounded linear mapping from $A(U_1)$ into $\mathcal{BL}(V)$, as in Section 10. Of course, this uses the supremum norm on $A(U_1)$, and this extension satisfies (58.5) for every $f \in A(U_1)$.

If $f, g \in A(U_1)$, then their product $f\,g$ is an element of $A(U_1)$ too. One can check that

$$(f\,g)(A) = f(A)\,g(A), \tag{58.7}$$

so that (58.6) is an algebra homomorphism. More precisely, this can be verified directly when f and g are polynomials in z, and otherwise one can reduce to that case by approximation.

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