Introductory notes in analysis

Stephen Semmes
Rice University

Contents

Part I
Basic notions and results

1 Metric spaces

A *metric space* is a (nonempty) set M equipped with a distance function or *metric* $d(x, y)$ defined for $x, y \in M$ such that $d(x, y)$ is a nonnegative real number for every $x, y \in M$, $d(x, y) = 0$ if and only if $x = y$,

$$d(y, x) = d(x, y)$$

for every $x, y \in M$, and

$$d(x, z) \leq d(x, y) + d(y, z)$$

for every $x, y, z \in M$. This last condition is known as the *triangle inequality.*

These notes will be largely concerned with metric spaces, loosely following the first four chapters and selections from other parts of Rudin's famous text [20]. Of course, the discussion here will be rather informal, and much more information can be found in Rudin's book. We shall frequently use slightly different definitions, notations, and conventions, and skip over some topics, while reorganizing the presentation of other topics. We may also elaborate a bit more on some points, mention different proofs or examples, and so on. There are numerous other excellent texts on essentially the same subject, a few of which may be found in the references at the end.

1.1 Examples

As usual, the real line is denoted $\mathbf{R}$. If x is a real number, then the *absolute value* of x is denoted $|x|$ and defined to be equal to x when $x \geq 0$ and to $-x$ when $x \leq 0$. One can check that

$$|x\,y| = |x|\,|y|$$

and

$$|x + y| \leq |x| + |y|$$

for all $x, y \in \mathbf{R}$. It follows that $|x - y|$ defines a metric on $\mathbf{R}$, known as the standard metric on the real line. For each positive integer n, $\mathbf{R}^n$ consists of the n-tuples $x = (x_1, \ldots, x_n)$ of real numbers, i.e., $x_i \in \mathbf{R}$ for $i = 1, \ldots, n$. The Euclidean distance between $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$ is

$$\Big(\sum_{i=1}^{n} (x_i - y_i)^2\Big)^{1/2},$$

which reduces to $|x - y|$ when $n = 1$. One can show that this defines a metric on $\mathbf{R}^n$. If $(M, d(x, y))$ is any metric space and $E \subseteq M$, then the restriction of $d(x, y)$ to $x, y \in E$ defines a metric on E, which is to say that E becomes a metric space too.

2 Least upper bounds

Let A be a set of real numbers, i.e., $A \subseteq \mathbf{R}$. We say that a real number b is an *upper bound* for A if $a \le b$ for every $a \in A$. We say that a real number c is the *least upper bound* or *supremum* of A if (i) c is an upper bound of A and (ii) $c \le b$ for every $b \in \mathbf{R}$ which is an upper bound for A. It follows immediately from the definition that the supremum is unique when it exists. For if $c, c' \in \mathbf{R}$ both satisfy the conditions for the supremum of A, then $c' \le c$ because c is an upper bound for A and c' is less than or equal to any upper bound of A, and similarly $c \le c'$ because c' is an upper bound for A and c is less than or equal to any upper bound of A. The supremum of a set $A \subseteq \mathbf{R}$ is denoted $\sup A$ when it exists.

If A is the empty set $\emptyset$, then every $b \in \mathbf{R}$ is an upper bound of A, and the supremum of A does not exist. Some sets of real numbers have no upper bound in $\mathbf{R}$, like $A = \mathbf{R}$. The *completeness property* of the real numbers states that every nonempty $A \subseteq \mathbf{R}$ with an upper bound has a supremum, and one can view the real numbers as a *completion* of the rationals.

2.1 Additional properties

A real number β is said to be a *lower bound* for a set $A \subseteq \mathbf{R}$ if $\beta \le a$ for every $a \in A$. Similarly, $\gamma \in \mathbf{R}$ is the *greatest lower bound* or *infimum* of A if (i) γ is a lower bound for A and (ii) $\beta \le \gamma$ for every $\beta \in \mathbf{R}$ which is a lower bound for A. It is easy to see that the infimum of A is unique when it exists, in which event it is denoted $\inf A$. The completeness property of the real numbers implies that every nonempty set $A \subseteq \mathbf{R}$ with a lower bound has an infimum. For if B is the set of real numbers which are lower bounds for A, then every element of A is an upper bound for B. The hypotheses that A be nonempty and have a lower bound imply that B is nonempty and has an upper bound, and one can check that $\inf A = \sup B$. Alternatively, if $-A$ is the set of real numbers of the form $-a$, $a \in A$, then the negatives of the lower bounds for A are the same as the upper bounds for $-A$. One can use this to show that the infimum of A is equal to the negative of the supremum of $-A$.

Let us mention some other properties of real numbers. First, if $x, y \in \mathbf{R}$ and $x < y$, then there is a rational number r such that $x < r < y$. Second, for any two positive real numbers u, v, there is is a positive integer n such that $u \le n\, v$. These two statements are basically equivalent to each other and can be derived from completeness or considered as consequences of the way in which the real numbers are obtained from the rationals by completion. Third, every positive real number has a positive square root. For if $y > 0$ and

$A_y = \{x \in \mathbf{R} : x > 0 \text{ and } x^2 < y\}$, then one can show that $A_y \neq \emptyset$, A_y has an upper bound in $\mathbf{R}$, and that $(\sup A)^2 = y$.

3 Open sets

Let $(M, d(x, y))$ be a metric space. For each $x \in M$ and positive real number r, the *open ball* in M with center x and radius r is denoted $B(x, r)$ and defined by

$$B(x, r) = \{y \in M : d(x, y) < r\}.$$

A set $U \subseteq M$ is said to be an *open set* if for every $x \in U$ there is an $r > 0$ such that $B(x, r) \subseteq U$. For every $z \in M$ and $t > 0$ the open ball $B(z, t)$ in M is an open set. To see this, let x be any element of $B(z, t)$, which thus satisfies $d(x, z) < t$. Put $r = t - d(x, z)$, and let us check that $B(x, r) \subseteq B(z, t)$. If $y \in B(x, r)$, then the triangle inequality implies that

$$d(y, z) \leq d(y, x) + d(x, z) < r + d(x, z) = t,$$

which is to say that $d(y, z) < t$ and $y \in B(z, t)$, as desired.

Similarly, for $z \in M$ and $t \geq 0$, consider

$$V(z, t) = \{u \in M : d(u, z) > t\}.$$

Let x be an element of $V(z, t)$, and put $r = d(x, z) - t > 0$. We would like to show that $B(x, r) \subseteq V(z, t)$, and hence that $V(z, t)$ is an open set. If $y \in M$, $d(x, y) < r$, and $d(y, z) \leq t$, then the triangle inequality implies that

$$d(x, z) \leq d(x, y) + d(y, z) < r + t = d(x, z),$$

a contradiction. Therefore $d(y, z) > t$ for every $y \in B(x, r)$, as desired.

3.1 Other metrics on $\mathbf{R}^n$

Let $d_2(x, y)$ be the Euclidean metric on $\mathbf{R}^n$, i.e.,

$$d_2(x, y) = \Big(\sum_{i=1}^{n} (x_i - y_i)^2\Big)^{1/2},$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, as usual. One can check that

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

also defines a metric on $\mathbf{R}^n$, and that

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n}\, d_\infty(x, y)$$

for every $x, y \in \mathbf{R}^n$. The open balls in $\mathbf{R}^n$ with respect to the metric $d_\infty(x, y)$ are cubes with sides parallel to the standard axes. It follows from the preceding inequalities that $d_\infty(x,y)$ determines the same class of open subsets of $\mathbf{R}^n$ as the Euclidean metric.

Similarly,

$$d_1(x,y) = \sum_{i=1}^n |x_i - y_i|$$

defines a metric on $\mathbf{R}^n$. It is easy to see that

$$d_\infty(x,y) \le d_1(x,y) \le n\, d_\infty(x,y)$$

for every $x, y \in \mathbf{R}^n$. One can also show that $d_2(x,y) \le d_1(x,y) \le \sqrt{n}\, d_2(x,y)$. At any rate, $d_1(x,y)$ determines the same class of open subsets of $\mathbf{R}^n$ as $d_\infty(x,y)$ and the Euclidean metric.

3.2 Norms

Let V be a vector space over the real numbers. A *norm* on V is a function $N(v)$ defined for $v \in V$ such that $N(v)$ is a nonnegative real number for every $v \in V$, $N(v) = 0$ if and only if $v = 0$,

$$N(t\, v) = |t|\, N(v)$$

for every $t \in \mathbf{R}$ and $v \in V$, and

$$N(v + w) \le N(v) + N(w)$$

for every $v, w \in V$. If $N(v)$ is a norm on V, then

$$d(v,w) = N(v - w)$$

defines a metric on V. For example,

$$\|x\|_2 = \Big(\sum_{i=1}^n x_i^2\Big)^{1/2}$$

defines a norm on $\mathbf{R}^n$, the Euclidean norm, which corresponds to the Euclidean metric on $\mathbf{R}^n$. Similarly,

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

and

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

are norms on $\mathbf{R}^n$, corresponding to the metrics $d_1(x,y)$, $d_\infty(x,y)$, respectively. One can show that

$$\|x\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p}$$

is a norm on $\mathbf{R}^n$ for every real number $p \ge 1$. Moreover, $\|x\|_\infty$ is equal to the limit of $\|x\|_p$ as $p \to \infty$.

4 Closed sets

Let $(M, d(x, y))$ be a metric space. A point $p \in M$ is said to be a *limit point* of a set $E \subseteq M$ if for every positive real number r there is a $q \in E$ such that $d(p, q) < r$ and $p \neq q$.

Suppose that $p \in M$ is a limit point of $E \subseteq M$, and let $r > 0$ be given. Let us check that there are infinitely many elements of E in $B(p, r)$. Otherwise, there is a positive integer n such that there are exactly n elements $x_1, \ldots, x_n$ of E in $B(p, r)$ which are different from p. Let t be the minimum of $d(x_i, p)$, $1 \leq i \leq n$. Hence t is a positive real number and $d(x_i, p) \geq t$ for $i = 1, \ldots, n$. If $q \in E$ and $q \neq p$, then either $d(q, p) \geq r > t$, or $q = x_i$ for some i and $d(q, p) \geq t$. In short, $d(q, p) \geq t$ when $q \in E$ and $q \neq p$, which contradicts the hypothesis that p be a limit point of E. Therefore, there are infinitely many elements of E in $B(p, r)$ for every $r > 0$ when p is a limit point of E. In particular, if E has only finitely many elements, then E has no limit points.

A set $E \subseteq M$ is said to be *closed* if E contains all of its limit points in M, i.e., if for every $p \in M$ which is a limit point of E we have that $p \in E$. If E has only finitely many elements, then E is automatically closed, by the remarks of the previous paragraph. For every $x \in M$ and $t \geq 0$, the *closed ball* $\overline{B}(x, t)$ in M with center x and radius t, defined by

$$\overline{B}(x, t) = \{y \in M : d(x, y) \leq t\},$$

is a closed set. For if p is a limit point of $\overline{B}(x, t)$ and $r > 0$, then there is a $y \in M$ such that $d(x, y) \leq t$ and $d(p, y) < r$, which implies that $d(p, x) < t + r$ by the triangle inequality. Since this holds for every positive real number r, it follows that $d(p, x) \leq t$, as desired.

4.1 Complements of subsets

Let X be a set. If A, B are subsets of X, then $A \backslash B$ denotes the set of $x \in A$ such that $x \notin B$. In particular, if $E \subseteq X$, then the *complement* of E in X is defined to be $X \backslash E$, and may be denoted E^c. Note that $X \backslash (X \backslash E) = E$ when $E \subseteq X$.

Now let $(M, d(x, y))$ be a metric space, and let us show that a set $E \subseteq M$ is closed if and only if its complement $M \backslash E$ in M is an open set. If E is a closed set in M and $p \in M \backslash E$, then p is also not a limit point of E. It follows that there is an $r > 0$ such that $d(p, x) \geq r$ for every $x \in E$. Equivalently, $B(p, r) \subseteq M \backslash E$, which shows that $M \backslash E$ is an open set, since p is an arbitrary element of $M \backslash E$. Next, suppose that $M \backslash E$ is an open set in M, and that $p \in M$ is a limit point of E. If p is not in E, then there is an $r > 0$ such that $B(p, r) \subseteq M \backslash E$, because $M \backslash E$ is an open set. This contradicts the hypothesis that p be a limit point of E. Hence $p \in E$, which implies that E is a closed set. Similarly, $U \subseteq M$ is an open set in M if and only if $M \backslash U$ is a closed set in M, by applying the previous statement to $E = M \backslash U$. As a special case, the empty set and M itself are automatically open and closed subsets of M.

4.2 Unions and intersections

Let A, X be sets, and suppose that for every $\alpha \in A$, E_α is a subset of X, so that $\{E_\alpha\}_{\alpha \in A}$ is a family of subsets of X indexed by A. The union $\bigcup_{\alpha \in A} E_\alpha$ and intersection $\bigcap_{\alpha \in A} E_\alpha$ of the E_α's can be defined in the usual way, and one can check that

$$X \backslash \Big(\bigcup_{\alpha \in A} E_\alpha \Big) = \bigcap_{\alpha \in A} (X \backslash E_\alpha) \quad \text{and} \quad X \backslash \Big(\bigcap_{\alpha \in A} E_\alpha \Big) = \bigcup_{\alpha \in A} (X \backslash E_\alpha).$$

Suppose now that $(M, d(x,y))$ is a metric space. If $\{E_\alpha\}_{\alpha \in A}$ is a family of closed subsets of M, then their intersection $\bigcap_{\alpha \in A} E_\alpha$ is also a closed set. For if $p \in M$ is a limit point of $\bigcap_{\alpha \in A} E_\alpha$, then is a limit point of each E_α, and hence an element of each E_α. Similarly, if $\{U_\alpha\}_{\alpha \in A}$ is a family of open subsets of M, then their union $\bigcap_{\alpha \in A} U_\alpha$ is an open set. For any element p of the union, there is an $\alpha \in A$ such that $p \in U_\alpha$, and hence an $r > 0$ such that $B(p, r)$ is contained in U_α and therefore in the union.

If $U_1, \ldots, U_n$ are finitely many open subsets of M, then their intersection $\bigcap_{i=1}^n U_i$ is also an open set in M. To see this, let p be an element of the intersection, and for each $i = 1, \ldots, n$, let r_i be a positive real number such that $B(p, r_i) \subseteq U_i$. If $r = \min(r_1, \ldots, r_n)$, then $B(p, r) \subseteq \bigcap_{i=1}^n U_i$. It follows that the union of finitely many closed subsets of M is a closed set in M too.

4.3 Closure and dense sets

Let $(M, d(x,y))$ be a metric space. If $E \subseteq M$, then E' denotes the set of $p \in M$ such that p is a limit point of E. The *closure* of E is denoted $\overline{E}$ and defined by $\overline{E} = E \cup E'$. By definition, $E \subseteq M$ is a closed set if and only if $E' \subseteq E$, which is equivalent to $\overline{E} = E$. For any $E \subseteq M$, one can check that $\overline{E}$ is a closed set in M. One can do this by showing that every limit point of $\overline{E}$ is also a limit point of E and hence an element of $\overline{E}$. One can also argue that $M \backslash \overline{E}$ is an open set. Similarly, E' is a closed set in M for any $E \subseteq M$.

One can also characterize $\overline{E}$ as the set of $p \in M$ such that for every positive real number r there is a $q \in E$ with $d(q, p) < r$. Every $p \in E$ automatically has this property, with $q = p$ for each $r > 0$. If $p \in M \backslash E$, then p has this property if and only if p is a limit point of E. One sometimes calls a $p \in M$ with this property an *accumulation point* of E in M.

A set $E \subseteq M$ is said to be *dense* in M if $\overline{E} = M$. Equivalently, E is dense in M if for every $x \in M$ and positive real number t there is a $y \in E$ such that $d(x, y) < t$. For example, the set $\mathbf{Q}$ of rational numbers is dense in the real line with respect to the standard metric.

5 Compactness

Let $(M, d(x,y))$ be a metric space. An *open covering* of a set $K \subseteq M$ is a family $\{U_\alpha\}_{\alpha \in A}$ of open subsets of M, for some indexing set A, such that

$$K \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

We say that $K \subseteq M$ is *compact* if every open covering of K can be reduced to a finite subcovering, i.e., if for every open covering $\{U_\alpha\}_{\alpha \in A}$ of K in M there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}.$$

If K has only finitely many elements, then K is compact.

For example, if $p \in M$, then the family of open balls $B(p,r)$, where r runs through all positive real numbers, or all positive integers, is an open covering of M, and hence of any $K \subseteq M$. If K is compact, then it follows that there are finitely many $r_1, \ldots, r_n > 0$ such that $K \subseteq \bigcup_{i=1}^n B(p, r_i)$. Hence $K \subseteq B(p, R)$, where $R = \max(r_1, \ldots, r_n)$. In general, a set $E \subseteq M$ is said to be *bounded* if E is contained in a ball, and thus we conclude that compact sets are bounded.

Similarly, if $p \in M$, then the sets $V(p,r) = \{x \in M : d(x,p) > r\}$ form an open covering of $M \backslash \{p\}$ as r runs through all positive real numbers, or if we restrict our attention to $r = 1/l$ for positive integers l. If $K \subseteq M$ is compact and $p \notin K$, then we get an open covering of K, and compactness implies that there are finitely many $r_1, \ldots, r_n > 0$ such that $K \subseteq \bigcup_{i=1}^n V(p, r_i)$. Therefore $K \subseteq V(p,t)$ with $t = \min(r_1, \ldots, r_n) > 0$, and p cannot be a limit point of K. It follows that compact sets are closed.

5.1 The limit point property

Let $(M, d(x,y))$ be a metric space. A set $E \subseteq M$ satisfies the *limit point property* if for every set $L \subseteq E$ with infinitely many elements there is a $p \in E$ which is a limit point of L. If E has only finitely many elements, then E has the limit point property automatically. Let us check that a compact set $K \subseteq M$ has the limit point property. Suppose for the sake of a contradiction that $L \subseteq K$ has infinitely many elements and that no $p \in K$ is a limit point of L. This means that for every $p \in K$ there is an $r(p) > 0$ such that $B(p, r(p)) \cap L$ is either empty or contains only p. The family of these open balls $B(p, r(p))$, $p \in K$, forms an open covering of K, since p is an element of $B(p,r)$ for every $r > 0$. Compactness of K implies that there are finitely many elements $p_1, \ldots, p_n$ of K such that $K \subseteq \bigcup_{i=1}^n B(p_i, r(p_i))$. This implies that L has at most n elements, a contradiction.

As a partial converse, suppose that $E \subseteq M$ has the limit point property, and that $U_1, U_2, \ldots$ is a sequence of open subsets of M such that $E \subseteq \bigcup_{i=1}^\infty U_i$. Suppose for the sake of a contradiction that for every $l \geq 1$ there is an x_l in $E \backslash \bigcup_{i=1}^l U_i$. Let L be the set consisting of the x_l's, $l \geq 1$. For every $y \in E$,

$y \in U_k$ for some k, and thus $y \neq x_l$ when $l \geq k$. Hence $y = x_l$ for only finitely many l, and it follows that L has infinitely many elements. Because E satisfies the limit point property, there is a $p \in E$ which is a limit point of L. We also have that $p \in U_i$ for some i, which implies that infinitely many elements of L are in U_i, since U_i is an open set. However, $x_l \notin U_i$ when $l \geq i$, so that there are fewer that i elements of L in U_i, a contradiction. Therefore $E \subseteq \bigcup_{i=1}^{l} U_i$ for some l. One can use this partial converse to show that E is closed and bounded, in practically the same way as for compact sets.

5.2 Unions and closed subsets

Let $(M, d(x, y))$ be a metric space. Suppose that $K_1, K_2 \subseteq M$ are compact, and let us show that $K_1 \cup K_2$ is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary covering of $K_1 \cup K_2$ by open subsets of M. Hence $\{U_\alpha\}_{\alpha \in A}$ is an open covering of K_1 and K_2 individually. Since K_1 is compact, there are finitely many indices $\alpha_1, \ldots, \alpha_l \in A$ such that $K_1 \subseteq \bigcup_{i=1}^{l} U_{\alpha_i}$. Similarly, since K_2 is compact, there are finitely many indices $\beta_1, \ldots, \beta_n \in A$ such that $K_2 \subseteq \bigcup_{j=1}^{n} U_{\beta_j}$. Therefore $K_1 \cup K_2 \subseteq \bigcup_{i=1}^{l} U_{\alpha_i} \cup \bigcup_{j=1}^{n} U_{\beta_j}$, as desired.

Suppose now that $K \subseteq M$ is compact and that $E \subseteq K$ is a closed set in M. We would like to show that E is compact too. Let $\{V_\beta\}_{\beta \in B}$ be an arbitrary open covering of E in M. Because $E \subseteq M$ is closed, $M \backslash E$ is an open set. It follows that the V_β's, $\beta \in B$, together with $M \backslash E$, form an open covering of M and hence of K. Compactness of K implies that there are finitely many indices $\beta_1, \ldots, \beta_n \in B$ such that $K \subseteq \bigcup_{i=1}^{n} V_{\beta_i} \cup (M \backslash E)$, and thus $E \subseteq \bigcup_{i=1}^{n} V_{\beta_i}$.

There are analogous results for the limit point property. Specifically, if $E_1, E_2 \subseteq M$ have the limit point property and $L \subseteq E_1 \cup E_2$ has infinitely many elements, then at least one of $L \cap E_1$, $L \cap E_2$ has infinitely many elements and hence a limit point in $E_1, E_2 \subseteq E$, as appropriate. If $E \subseteq M$ has the limit point property, $A \subseteq E$ is a closed set in M, and $L \subseteq A$ has infinitely many elements, then there is a limit point p of L in E, and $p \in A$ because p is also a limit point of A and A is closed.

6 Relatively open sets

Let $(M, d(x, y))$ be a metric space, and suppose that $Y \subseteq M$. We can consider Y as a metric space too, using the restriction of the metric $d(x, y)$ to $x, y \in Y$. Let us say that $E \subseteq Y$ is *relatively open* in Y if E is an open set as a set contained in Y. Explicitly, this means that for every $p \in E$ there is an $r > 0$ such that

$$\{y \in Y : d(y, p) < r\} \subseteq E.$$

If U is an open set in M and $E = U \cap Y$, then it is easy to see that E is relatively open in Y. Conversely, suppose that $E \subseteq Y$ is relatively open in Y. For every $p \in E$, let $r(p)$ be a positive real number such that E contains every $y \in Y$ with $d(y, p) < r(p)$. Equivalently, $B(p, r(p)) \cap Y \subseteq E$, where $B(x, t)$ is the usual open ball in M with center x and radius t. Consider

$$U = \bigcup_{p \in E} B(p, r(p)).$$

Clearly U is an open set in M, being the union of open subsets of M. Each $p \in E$ is an element of $B(p, r(p))$ and hence of U, and it is not difficult to check that $E = U \cap Y$.

6.1 Compact sets

Let $(M, d(x, y))$ be a metric space, and suppose that $Y \subseteq M$. If $K \subseteq Y$, then K is compact as a set contained in Y, considered as a metric space using the restriction of $d(x, y)$ to $x, y \in Y$, if and only if K is compact as a set in M. For suppose that K is compact relative to Y, and let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary covering of K by open subsets of M. Each $U_\alpha \cap Y$ is relatively open in Y, and hence $\{U_\alpha \cap Y\}_{\alpha \in A}$ is a covering of K by relatively open subsets of Y. Compactness relative to Y implies that there are finitely many indices $\alpha_1, \ldots, \alpha_l$ in A such that $K \subseteq \bigcup_{i=1}^{l} U_{\alpha_i} \cap Y$. In particular, $K \subseteq \bigcup_{i=1}^{l} U_{\alpha_i}$. Conversely, suppose that K is compact as a set contained in M, and let $\{V_\beta\}_{\beta \in B}$ be an arbitrary covering of K by relatively open subsets of Y. For every $\beta \in B$, let W_β be an open set in M such that $V_\beta = W_\beta \cap Y$. Thus $\{W_\beta\}_{\beta \in B}$ is an open covering of K in M, and compactness of K in M implies that there are finitely many indices $\beta_1, \ldots, \beta_n$ in B such that $K \subseteq \bigcup_{i=1}^{n} W_{\beta_i}$. Therefore $K \subseteq \bigcup_{i=1}^{n} V_{\beta_i}$, since $K \subseteq Y$ by hypothesis. Similarly, one can check that $E \subseteq Y$ has the limit point property relative to Y if and only if E has the limit point property as a set contained in M.

7 Totally bounded sets

Let $(M, d(x, y))$ be a metric space. A set $E \subseteq M$ is said to be *totally bounded* if for every $r > 0$ there are finitely many elements $x_1, \ldots, x_l$ of E such that

$$E \subseteq \bigcup_{i=1}^{l} B(x_i, r).$$

Totally bounded sets are automatically bounded, since the union of finitely many bounded sets is bounded. If $K \subseteq M$ is compact, then K is totally bounded, because for every $r > 0$ one can cover K by the open balls $B(x, r)$, $x \in K$, and use compactness to cover K by $B(x, r)$ with finitely many $x \in K$. For any $E \subseteq M$ and $r > 0$ one can look for a covering of E by balls with radius r in the following way. Let x_1 be any element of E, assuming that $E \neq \emptyset$. If $E \subseteq B(x_1, r)$, then we stop. Otherwise, let x_2 be an element of E which is not in $B(x_1, r)$. If $E \subseteq B(x_1, r) \cup B(x_2, r)$, then we stop. Otherwise, we continue the process. The process may stop after finitely many steps and yield a covering of E by finitely many open balls centered at elements of E and with radius r.

If not, then there is an infinite sequence $x_1, x_2, \ldots$ of elements of E such that $d(x_i, x_l) \geq r$ when $i < l$. If L is the set of the x_i's, $i \geq 1$, then L is an infinite set contained in E which has no limit point in M. If E has the limit point property, then this is impossible, and E is totally bounded.

8 Closed intervals

If a, b are real numbers with $a \leq b$, then the *closed interval* $[a, b]$ from a to b is the set of real numbers x such that $a \leq x \leq b$. The *length* of this interval is $b - a$. Closed intervals are clearly closed subsets of the real line with respect to the standard metric.

Suppose that $I_1, I_2, \ldots$ is a sequence of closed intervals in the real line such that $I_{j+1} \subseteq I_j$ for every $j \geq 1$. Let a_j, b_j be the endpoints of I_j, so that $a_j \leq b_j$ in particular. The hypothesis that $[a_{j+1}, b_{j+1}] \subseteq [a_j, b_j]$ implies that $a_j \leq a_{j+1}$ and $b_{j+1} \leq b_j$ for every $j \geq 1$. This implies in turn that

$$a_j \leq b_l$$

for all positive integers j, l. For if $j \leq l$, then $a_j \leq a_l \leq b_l$, and if $j \geq l$, then $a_j \leq b_j \leq b_l$. Let A be the set of the a_j's. Since $a_j \leq b_1$ for every $j \geq 1$, A has an upper bound, and hence a supremum. Each b_l is an upper bound for A, and thus $\sup A \leq b_l$ for every $l \geq 1$. Of course the supremum of A is an upper bound for A, which is to say that $a_j \leq \sup A$ for every $j \geq 1$. It follows that

$$\sup A \in I_j$$

for every $j \geq 1$.

8.1 Compactness properties

Let a, b be real numbers with $a \leq b$, and consider the interval $I = [a, b]$ in the real line. Suppose that $E \subseteq I$ has infinitely many elements. Put $I_1 = I$, and choose I_2 to be $[a, (a + b)/2]$ or $[(a + b)/2, b]$, in such a way that $E \cap I_2$ also has infinitely many elements. In general, if a closed interval $I_j = [a_j, b_j] \subseteq I$ has been chosen such that $E \cap I_j$ has infinitely many elements, then let I_{j+1} be one of the two closed intervals $[a_j, (a_j + b_j)/2]$, $[(a_j + b_j)/2, b_j]$ such that $E \cap I_{j+1}$ has infinitely many elements. Hence $I_{j+1} \subseteq I_j$ and the length of I_{j+1} is equal to one-half the length of I_j for every j. We know from a previous argument that there is a real number t such that $t \in I_j$ for every j. Observe that $|x - t| \leq 2^{-j+1} (b - a)$ when $x \in I_j$, since the length of I_j is equal to 2^{-j+1} times the length of I. It follows that for every $r > 0$, there are infinitely many $y \in E$ such that $|y - t| < r$, since $y \in E \cap I_j$ has this property when j is sufficiently large that $2^{-j+1} (b - a) < r$. Therefore, t is a limit point of E, which shows that I has the limit point property.

Similarly, I is compact, as a set contained in the real line equipped with the standard metric. For suppose that $\{U_\alpha\}_{\alpha \in A}$ is any covering of I by open subsets of $\mathbf{R}$. Suppose also for the sake of a contradiction that there is no finite subcovering of I from this covering. If there were a finite subcovering for each of the intervals $[a,(a+b)/2]$, $[(a+b)/2,b]$, then there would be a finite subcovering for I. Hence there is no finite subcovering for at least one of these two subintervals of I. As before, we put $I_1 = I$, and choose closed subintervals I_2, I_3, etc., so that $I_{j+1} \subseteq I_j$, the length of I_{j+1} is equal to one-half the length of I_j, and I_j cannot be covered by finitely many U_α's for any j. There is a real number t in $\bigcap_{j=1}^\infty I_j$, and an $\alpha_0 \in A$ such that $t \in U_{\alpha_0}$. For j sufficiently large, $I_j \subseteq U_{\alpha_0}$, since U_{α_0} is an open set in $\mathbf{R}$. This is a contradiction, since there is not supposed to be a covering of I_j by finitely many U_α's for any j.

8.2 Cells in $\mathbf{R}^n$

Fix a positive integer n. If $a_1,\ldots,a_n$, $b_1,\ldots,b_n$ are real numbers such that $a_i \le b_i$ for $i = 1,\ldots,n$, then the corresponding *cell* is the set $\mathcal{C} \subseteq \mathbf{R}^n$ consisting of $x = (x_1,\ldots,x_n)$ such that $a_i \le x_i \le b_i$, $1 \le i \le n$. One can show that cells satisfy the limit point property and are compact as subsets of $\mathbf{R}^n$ equipped with the Euclidean metric in much the same way as for closed intervals in the real line. There are two main changes in the argument, as follows. First, if $\mathcal{C}$ is a cell in $\mathbf{R}^n$ associated to the real numbers $a_1,\ldots,a_n$, $b_1,\ldots,b_n$ as above, then the diameter of $\mathcal{C}$, or maximal distance between two elements of $\mathcal{C}$, is equal to

$$\sqrt{\sum_{i=1}^n (b_i - a_i)^2}.$$

Just as an interval is the union of two subintervals of half the length, a cell $\mathcal{C} \subseteq \mathbf{R}^n$ is the union of 2^n cells of one-half the diameter of $\mathcal{C}$. Second, if $\mathcal{C}_1, \mathcal{C}_2, \ldots$ is a sequence of cells in $\mathbf{R}^n$ such that $\mathcal{C}_{j+1} \subseteq \mathcal{C}_j$ for every $j \ge 1$, then there is a $t \in \mathbf{R}^n$ such that $t \in \bigcap_{j=1}^\infty \mathcal{C}_j$. This is basically the same as the one-dimensional case repeated n times for the n components of t.

8.3 Another construction

Let $(M, d(x,y))$ be a metric space, let p be an element of M, and let $E_1, E_2, \ldots$ be a sequence of subsets of M. Suppose that $E_n \subseteq B(p, 1/n)$ for each n, and let E be the set consisting of p and the union of the E_n's. One can check that E is a closed set in M when the E_n's are closed subsets of M. If E_n has the limit point property for each n, then E does too. For if $L \subseteq E$ has infinitely many elements, then either p is a limit point of L, or $L \cap E_n$ has infinitely many elements for some n and therefore L has a limit point in the same E_n.

Similarly, E is compact in M when the E_n's are compact subsets of M. To see this, let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of E in M. Since $p \in E$, there is an $\alpha_0 \in A$ such that $p \in U_{\alpha_0}$, and there is an $r > 0$ such that $B(p,r) \subseteq U_{\alpha_0}$ because U_{α_0} is an open set in M. Thus $E_n \subseteq U_{\alpha_0}$ when $n > 1/r$. For each n there is also a set $A_n \subseteq A$ with finitely many elements such that $E_n \subseteq \bigcup_{\alpha \in A_n} U_\alpha$, by the compactness of E_n. If A_0 is the set consisting of α_0 and the union of the A_n's such that $n \leq 1/r$, then it follows that $E \subseteq \bigcup_{\alpha \in A_0} U_\alpha$. Also, A_0 has only finitely many elements, and so $\{U_\alpha\}_{\alpha \in A_0}$ is a finite subcovering of E from $\{U_\alpha\}_{\alpha \in A}$, as desired. One can use these remarks to give examples of infinite-dimensional compact sets, by taking E_n to be n-dimensional for each n.

9 Countable sets

A set A has exactly n elements for some positive integer n if there is a list $x_1, \ldots, x_n$ of n elements of A such that every element of A is on the list exactly once. A set A is said to be *countably infinite* if there is an infinite sequence $x_1, x_2, \ldots$ of elements of A such that for every $a \in A$ there is exactly one positive integer i such that $x_i = a$. By definition, the set $\mathbf{Z}_+$ of positive integers is countably infinite, with $x_i = i$ for each i. The set $\mathbf{Z}$ of all integers is countably infinite, because one can enumerate the integers with the sequence $\{x_i\}_{i=1}^\infty$ defined by $x_i = i/2$ when i is even and $x_i = -(i-1)/2$ when i is odd.

Suppose that A, B are sets and that $\{y_j\}_{j=1}^\infty$ is a sequence of elements of B such that every $a \in A$ is equal to y_j for at least one j. One can list the elements of A by going through the y_j's and keeping the ones that are in A when they occur the first time. The resulting list may be finite or infinite, and the conclusion is that A is either finite or countably infinite. In particular, if $A \subseteq B$ and B is countably infinite, then A is finite or countably infinite.

9.1 Unions and products

Suppose that $A_1, A_2, \ldots$ is a sequence of finite sets and that $A = \bigcup_{i=1}^\infty A_i$. It is easy to make a list $\{x_l\}_{l=1}^\infty$ of the elements of A such that every $a \in A$ is equal to x_l for at least one l by first listing the elements of A_1, then the elements of A_2, and so on. This list of the elements of A may have repetitions, and so A may have finitely or countably-infinitely many elements.

If A, B are sets, then their *Cartesian product* $A \times B$ consists of the ordered pairs (a, b) such that $a \in A$ and $b \in B$. For each positive integer n, let C_n be the set of $(i, j) \in \mathbf{Z}_+ \times \mathbf{Z}_+$ such that $i + j = n + 1$. It is easy to see that C_n has exactly n elements and that $\bigcup_{n=1}^\infty C_n = \mathbf{Z}_+ \times \mathbf{Z}_+$. It follows that $\mathbf{Z}_+ \times \mathbf{Z}_+$ is countably infinite. Hence $A \times B$ is countably infinite whenever A, B are.

If A, B are finite or countably-infinite sets, then their union $A \cup B$ is too. One can first list the elements of $A \cup B$ by a doubly-infinite sequence $\{y_l\}_{l=-\infty}^\infty$, and then convert that into an ordinary sequence using a listing of the elements of $\mathbf{Z}$. Similarly, if $A_1, A_2, \ldots$ is a sequence of finite or countably-infinite sets and $A = \bigcup_{i=1}^\infty A_i$, then A is finite or countably-infinite. For one can list the elements of A with a doubly-indexed sequence $\{z_{p,q}\}_{p,q=1}^\infty$, and then convert that into an ordinary sequence using a listing of the elements of $\mathbf{Z}_+ \times \mathbf{Z}_+$. As an application, the set $\mathbf{Q}$ of rational numbers is countably infinite, since $\mathbf{Q} = \bigcup_{n=1}^\infty D_n$ where D_n is the set of rational numbers of the form k/n, $k \in \mathbf{Z}$.

9.2 Uncountable sets

Let $\mathcal{B}$ be the set of all binary sequences, which is to say the set of sequences $x = \{x_i\}_{i=1}^{\infty}$ such that $x_i = 0$ or 1 for every $i \ge 1$. Suppose that

$$x(1) = \{x_i(1)\}_{i=1}^{\infty}, x(2) = \{x_i(2)\}_{i=1}^{\infty}, \ldots$$

is a sequence of elements of $\mathcal{B}$. If $y = \{y_i\}_{i=1}^{\infty}$ is the binary sequence defined by $y_i = 1 - x_i(i)$, then $y \ne x(l)$ for every positive integer l. In general, a set is said to be *uncountable* if it is neither finite nor countably infinite. The preceding argument shows that $\mathcal{B}$ is uncountable.

Every binary sequence corresponds to a real number in $[0, 1]$ in the usual way, and every element of $[0, 1]$ has at least one binary expansion. Sometimes a real number corresponds to two binary sequences, which are both eventually constant. One can use the uncountability of $\mathcal{B}$ to show that $[0, 1]$ is uncountable too.

10 Separable metric spaces

Let $(M, d(x, y))$ be a metric space. We say that M is *separable* if there is a dense set $E \subseteq M$ such that E has only finitely or countably many elements. For example, the real line equipped with the standard metric is separable, since the rationals are countable and dense in $\mathbf{R}$. We say that $E \subseteq M$ is ϵ*-dense* in M for some $\epsilon > 0$ if for every $x \in M$ there is a $y \in E$ such that $d(x, y) < \epsilon$. Thus E is dense in M if and only if E is ϵ-dense in M for every $\epsilon > 0$. If for every $\epsilon > 0$ there is an ϵ-dense set E_ϵ in M with only finitely or countably many elements, then $\bigcup_{n=1}^{\infty} E_{1/n}$ is a dense set in M with only finitely or countably many elements, and M is separable. Note that M is totally bounded if and only if for every $\epsilon > 0$ there is an ϵ-dense set $E_\epsilon \subseteq M$ with only finitely many elements, in which event M is separable in particular.

A collection $\mathcal{B}$ of open subsets of M is said to be a *base* for the topology of M if for every point $p \in M$ and radius $r > 0$ there is a $V \in \mathcal{B}$ such that $p \in V$ and $V \subseteq B(p, r)$. Equivalently, $\mathcal{B}$ is a base for the topology of M if every open set $W \subseteq M$ can be expressed as a union of elements of $\mathcal{B}$. If $E \subseteq M$ is dense, then the collection of open balls $B(x, 1/n)$ with $x \in E$ and $n \in \mathbf{Z}_+$ is a base for the topology of M. Consequently, if M is separable, then there is a base for the topology of M with only finitely or countably many elements. Conversely, suppose that $\mathcal{B}$ is a base for the topology of M with only finitely or countably many elements. If $V \in \mathcal{B}$ and $V \ne \emptyset$, then let $p(V)$ be an element of V. If E is the set of the points $p(V)$ chosen in this way, then E has only finitely or countably many elements, and E is dense in M, which implies that M is separable.

10.1 Lindelöf's theorem

Let $(M, d(x, y))$ be a metric space, let $\mathcal{B}$ be a base for the topology of M, and let $\{U_\alpha\}_{\alpha \in A}$ be any family of open subsets of M. Put

$$\mathcal{B}_\alpha = \{V \in \mathcal{B} : V \subseteq U_\alpha\}$$

for every $\alpha \in A$, so that

$$U_\alpha = \bigcup_{V \in \mathcal{B}_\alpha} V$$

for every $\alpha \in A$, because $\mathcal{B}$ is a base for the topology of M. If we put

$$\mathcal{B}' = \bigcup_{\alpha \in A} \mathcal{B}_\alpha,$$

then we get that

$$\bigcup_{V \in \mathcal{B}'} V = \bigcup_{\alpha \in A} \Big(\bigcup_{V \in \mathcal{B}_\alpha} V \Big) = \bigcup_{\alpha \in A} U_\alpha.$$

If $V \in \mathcal{B}'$, then $V \in \mathcal{B}_\alpha$ for some $\alpha \in A$, and we let $\alpha(V)$ be an element of A with this property. Thus $V \subseteq U_{\alpha(V)}$ for every $V \in \mathcal{B}'$, and we let A_1 be the set of $\alpha(V) \in A$ chosen in this way. It follows that

$$\bigcup_{V \in \mathcal{B}'} V \subseteq \bigcup_{V \in \mathcal{B}'} U_{\alpha(V)} = \bigcup_{\alpha \in A_1} U_\alpha,$$

so that

$$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{V \in \mathcal{B}'} V \subseteq \bigcup_{\alpha \in A_1} U_\alpha.$$

More precisely, we have that

$$\bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{\alpha \in A} U_\alpha,$$

because the left side is automatically contained in the right side, since $A_1 \subseteq A$ by construction. If $\mathcal{B}$ has only finitely or countably many elements, then $\mathcal{B}' \subseteq \mathcal{B}$ has only finitely or countably many elements, and it is easy to see that A_1 has only finitely or countably many elements as well.

If M has the limit point property, then M is totally bounded, and hence separable. This implies that there is a base for the topology of M with only finitely or countably many elements, as before. It follows that every open covering of M can be reduced to a finite or countable subcover, by the argument in the previous paragraph. The limit point property also implies that every countable open covering of M can be reduced to a finite subcovering, so that M is compact. Similarly, if $E \subseteq M$ has the limit point property, then one can show that E is compact, by reducing to the case where $E = M$.

11 Connected sets

Let $(M, d(x, y))$ be a metric space. A pair of sets $A, B \subseteq M$ are said to be *separated* if

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

A set E is *connected* if there are no nonempty separated sets $A, B \subseteq M$ such that $A \cup B = E$.

Suppose that $Y \subseteq M$, which can also be considered as a metric space using the restriction of $d(x, y)$ to $x, y \in Y$. If $A \subseteq Y$ and $p \in Y$, then p is a limit point of A relative to Y if and only if p is a limit point of A relative to M. Hence the closure of A relative to Y is equal to the intersection of Y with the closure of A relative to M. It follows that $A, B \subseteq Y$ are separated relative to Y if and only if they are separated relative to M, and therefore that $E \subseteq Y$ is connected relative to Y if and only if E is connected relative to M.

Let us say that $E \subseteq M$ is *ϵ-connected* for some $\epsilon > 0$ if for every $p, q \in E$ there is a finite sequence $x_1, \ldots, x_n$ of elements of E such that $x_1 = p$, $x_n = q$, and $d(x_i, x_{i+1}) < \epsilon$ for $1 \leq i < n$. For any $E \subseteq M$, $p \in E$, and $\epsilon > 0$, let $A_\epsilon(p)$ be the set of $q \in E$ which can by connected to p in this way, and put $B_\epsilon(p) = E \backslash A_\epsilon(p)$. It is easy to see that $d(u, v) \geq \epsilon$ when $u \in A_\epsilon(p)$ and $v \in B_\epsilon(p)$, and hence that $A_\epsilon(p)$, $B_\epsilon(p)$ are separated as subsets of M. If E is connected, then it follows that $A_\epsilon(p) = E$ and that E is ϵ-connected for every $\epsilon > 0$. The converse does not work in general, but it does hold for compact sets.

11.1 The real line

Let E be a set contained in the real line. If there are $x, z \in E$ and $y \in \mathbf{R} \backslash E$ such that $x < y < z$, then E is not connected. This is because the sets A, B of $w \in E$ such that $w < y$, $y < w$, respectively, are separated in $\mathbf{R}$.

Conversely, suppose that $E \subseteq \mathbf{R}$ is not connected, and let A, B be nonempty separated subsets of $\mathbf{R}$ such that $A \cup B = E$. Let x, z be elements of A, B, and let us suppose without loss of generality that $x < z$, since we can interchange the roles of A and B. Let y be the supremum of the set of $t \in A$ such that $x \leq t < z$. Observe that $y \in \overline{A}$, and hence $y \notin B$. In particular, $y < z$. If $y \notin A$, then $y \notin E$, and $x < y < z$. If $y \in A$, then $y \notin \overline{B}$, and there is a $y' \in \mathbf{R}$ such that $y < y' < z$ and $y' \notin E$. In both cases there is a point between x and z which is not an element of E. It follows that intervals are connected subsets of the real line, and in particular that the real line is connected.

Part II

Sequences, series, and functions

12 Sequences

Let $(M, d(x, y))$ be a metric space. A sequence $\{x_j\}_{j=1}^\infty$ of elements of M is said to *converge* to $x \in M$ if for every $\epsilon > 0$ there is a positive integer L such that

$$d(x_j, x) < \epsilon \quad \text{for every } j \ge L.$$

In this event we call x the *limit* of the sequence $\{x_j\}_{j=1}^\infty$ and express this by $\lim_{j \to \infty} x_j = x$. The limit of a convergent sequence is unique, for if $\{x_j\}_{j=1}^\infty$ converges to $x, x' \in M$, then

$$d(x, x') \le d(x, x_j) + d(x_j, x') < \epsilon + \epsilon = 2\,\epsilon$$

for every $\epsilon > 0$ and sufficiently large j, depending on ϵ, which implies that $d(x, x') = 0$ and hence $x = x'$. A sequence in M is said to be *bounded* if the terms in the sequence are contained in a bounded set in M. Convergent sequences are bounded, since all but finitely many terms in a convergent sequence are contained in a ball of radius 1. If $\{x_j\}_{j=1}^\infty$ is a sequence of real numbers which is bounded and monotone increasing, which is to say that $x_j \le x_{j+1}$ for every j, then one can check that $\{x_j\}_{j=1}^\infty$ converges to the supremum of the set of x_j's. Similarly, a bounded monotonically decreasing sequence of real numbers converges to the infimum of the set of terms in the sequence.

13 Complex numbers

The complex numbers are denoted $\mathbf{C}$, and every $z \in \mathbf{C}$ can be expressed as $x + y\,i$, where x, y are real numbers and $i^2 = -1$. In this case, x and y are known as the real and imaginary parts of z, and the complex conjugate of z is denoted $\overline{z}$ and defined to be $x - y\,i$. One can check that

$$\overline{z + w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{z\,w} = \overline{z}\,\overline{w}$$

for every $z, w \in \mathbf{C}$. The modulus of a complex number $z = x + y\,i$, $x, y \in \mathbf{R}$, is the nonnegative real number $|z| = \sqrt{x^2 + y^2}$. Observe that $|z|^2 = z\,\overline{z}$, and hence

$$|z\,w| = |z|\,|w|$$

for every $z, w \in \mathbf{C}$. If we think of a complex number $z = x + y\,i$ as corresponding to $(x, y) \in \mathbf{R}^2$, then the modulus of z is the same as the ordinary Euclidean norm of the vector (x, y). It follows that

$$|z + w| \le |z| + |w|$$

for every $z, w \in \mathbf{C}$, since the sum of two complex numbers corresponds to the usual sum of the corresponding vectors in $\mathbf{R}^2$. This implies that $|z - w|$ defines a metric on $\mathbf{C}$, which corresponds exactly to the standard Euclidean metric on $\mathbf{R}^2$.

13.1 Sequences in C

Suppose that $\{z_j\}_{j=1}^\infty$, $\{w_j\}_{j=1}^\infty$ are sequences of complex numbers which converge to $z, w \in \mathbf{C}$, respectively. Let us check that $\{z_j + w_j\}_{j=1}^\infty$ converges to $z + w$. Let $\epsilon > 0$ be given, and let L_1, L_2 be positive integers such that $|z_j - z| < \epsilon/2$ when $j \ge L_1$ and $|w_j - w| < \epsilon/2$ when $j \ge L_2$. If $j \ge \max(L_1, L_2)$, then

$$|(z_j + w_j) - (z + w)| \le |z_j - z| + |w_j - w| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired. Similarly, if $\{z_j\}_{j=1}^\infty$ is a sequence of complex numbers which converges to $z \in \mathbf{C}$, and if $a \in \mathbf{C}$, then $\{a\, z_j\}_{j=1}^\infty$ converges to $a\, z$. If $\{z_j\}_{j=1}^\infty$ is a sequence of complex numbers which converges to 0 and $\{w_j\}_{j=1}^\infty$ is a bounded sequence of complex numbers, then $\{z_j\, w_j\}_{j=1}^\infty$ converges to 0 too. Consequently, if $\{z_j\}_{j=1}^\infty$, $\{w_j\}_{j=1}^\infty$ are sequences of complex numbers converging to $z, w \in \mathbf{C}$, respectively, then $\{z_j\, w_j\}_{j=1}^\infty$ converges to $z\, w$, since $\lim_{j \to \infty} z\, w_j = z\, w$ and $\lim_{j \to \infty} (z_j - z)\, w_j = 0$ by the previous statements. If $\{z_j\}_{j=1}^\infty$ is a sequence of nonzero complex numbers which converges to $z \in \mathbf{C}$ with $z \ne 0$, then $\{1/z_j\}_{j=1}^\infty$ converges to $1/z$. To see this, one can first check that $\{1/z_j\}_{j=1}^\infty$ is bounded, and then apply the previous statements to $1/z_j - 1/z = -(z_j - z)/(z\, z_j)$.

14 Subsequences and sequential compactness

If $\{x_j\}_{j=1}^\infty$ is a sequence of elements of some set, and $\{j_l\}_{l=1}^\infty$ is a strictly increasing sequence of positive integers, then $\{x_{j_l}\}_{l=1}^\infty$ is a *subsequence* of $\{x_j\}_{j=1}^\infty$. Let $(M, d(x, y))$ be a metric space. If $\{x_j\}_{j=1}^\infty$ is a sequence of elements of M which converges to $x \in M$, then every subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ also converges to x.

Observe that a point $p \in M$ is an element of the closure $\overline{E}$ of a set $E \subseteq M$ if and only if there is a sequence $\{p_n\}_{n=1}^\infty$ of elements of E which converges to p. Similarly, $p \in M$ is a limit point of E if and only if there is a sequence $\{p_n\}_{n=1}^\infty$ of elements of E which converges to p and satisfies $p_n \ne p$ for every n. Let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of M, and let Λ be the set of $y \in M$ for which there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ which converges to y. This is somewhat analogous to the closure of a set, or the set of limit points of a set. One can check that $y \in \Lambda$ if and only if for every $r > 0$ there are infinitely many positive integers j such that $d(x_j, y) < r$. For if $y \in M$ has this property, then one can choose j_l for $l \ge 1$ in such a way that $d(x_{j_l}, y) < 1/l$ and $j_l > j_1, \ldots, j_{l-1}$ when $l \ge 2$. It follows easily from this characterization that Λ is automatically a closed set in M.

A set $K \subseteq M$ is said to be *sequentially compact* if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of K there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ which converges to an element of K. If $K \subseteq M$ is sequentially compact, then K has the limit point property. For if $L \subseteq K$ has infinitely many elements, and if $\{x_j\}_{j=1}^{\infty}$ is any sequence of distinct elements of K, then the limit of any convergent subsequence of $\{x_j\}_{j=1}^{\infty}$ is a limit point of L. Conversely, if $K \subseteq M$ has the limit point property, then K is sequentially compact. To see this, let $\{x_j\}_{j=1}^{\infty}$ be any sequence of elements of K. If there is an $x \in K$ such that $x_j = x$ for infinitely many j, then $\{x_j\}_{j=1}^{\infty}$ has a constant subsequence which converges trivially. Otherwise, the set L of the x_j's has infinitely many elements, and hence a limit point in K. By the remarks of the previous paragraph, a limit point of L is also the limit of a subsequence of $\{x_j\}_{j=1}^{\infty}$, as desired.

15 Cauchy sequences

Let $(M, d(x,y))$ be a metric space. A sequence $\{x_j\}_{j=1}^{\infty}$ of elements of M is said to be a *Cauchy sequence* if for every $\epsilon > 0$ there is a positive integer L such that $d(x_j, x_l) < \epsilon$ for every $j, l \geq L$. If $\{x_j\}_{j=1}^{\infty}$ converges to a point $x \in M$, then $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence. For let $\epsilon > 0$ be given, and let L be a positive integer such that $d(x_j, x) < \epsilon/2$ when $j \geq L$. If $j, l \geq L$, then

$$d(x_j, x_l) \leq d(x_j, x) + d(x, x_l) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in M and $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ which converges to some $x \in M$, then $\{x_n\}_{n=1}^{\infty}$ converges to x. For let $\epsilon > 0$ be given, and let L be a positive integer such that $d(x_j, x_l) < \epsilon/2$ when $j, l \geq L$. If $j \geq L$, then

$$d(x_j, x) \leq d(x_j, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

when k is sufficiently large that $n_k \geq L$ and $d(x_{n_k}, x) < \epsilon/2$. It follows that $d(x_j, x) < \epsilon$ for every $j \geq L$, as desired. In particular, a Cauchy sequence of elements of a compact set $K \subseteq M$ converges to an element of K.

We say that M is *complete* if every Cauchy sequence in M converges to an element of M. Cauchy sequences are always bounded, and hence the real line is complete since bounded sets are contained in compact sets. Alternatively, suppose that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence, and let a_j, b_j be the infimum and supremum of x_l, $l \geq j$, respectively. Thus $a_j \leq a_{j+1} \leq b_{j+1} \leq b_j$ for every j, and hence if $I_j = [a_j, b_j]$, then $I_{j+1} \subseteq I_j$ for every j. Because $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence, the lengths of the I_j's tend to 0 as $j \to \infty$. Therefore $\bigcap_{j=1}^{\infty} I_j$ contains exactly one element, which is the limit of $\{x_j\}_{j=1}^{\infty}$. Similarly, the complex numbers and $\mathbf{R}^n$ equipped with the Euclidean metric are complete. Note that sequences of complex numbers or elements of $\mathbf{R}^n$ are Cauchy sequences or converge if and only if the corresponding sequences of components in $\mathbf{R}$ are Cauchy sequences or converge.

16 Infinite series

Let $\{a_l\}_{l=1}^{\infty}$ be a sequence of complex numbers. We say that the infinite series $\sum_{l=1}^{\infty} a_l$ converges if the partial sums $s_n = \sum_{l=1}^{n} a_l$ converge as a sequence of complex numbers, in which event the sum $\sum_{l=1}^{\infty} a_l$ is defined to be $\lim_{n\to\infty} s_n$. It will sometimes be convenient to have an a_0 term in the series, which is not important for matters of convergence. The Cauchy criterion states that $\sum_{l=1}^{\infty} a_l$ converges if and only if for every $\epsilon > 0$ there is a positive integer L such that

$$\left|\sum_{j=l}^{n} a_j\right| < \epsilon$$

when $n \ge l \ge L$. Indeed, this condition holds exactly when the corresponding sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence. If $\sum_{l=1}^{\infty} a_l$ converges, then $\{a_l\}_{l=1}^{\infty}$ converges as a sequence of complex numbers to 0, as one can see by taking $n = l$ in the Cauchy criterion. If $\sum_{l=1}^{\infty} a_l$, $\sum_{l=1}^{\infty} b_l$ are convergent series of complex numbers and $\alpha, \beta \in \mathbf{C}$, then $\sum_{l=1}^{\infty}(\alpha\, a_l + \beta\, b_l)$ converges, and

$$\sum_{l=1}^{\infty}(\alpha\, a_l + \beta\, b_l) = \alpha \sum_{l=1}^{\infty} a_l + \beta \sum_{l=1}^{\infty} b_l,$$

by the analogous results for sequences applied to the partial sums.

A series $\sum_{l=1}^{\infty} a_l$ is said to converge absolutely if $\sum_{l=1}^{\infty} |a_l|$ converges. Absolute convergence implies ordinary convergence, as a consequence of the Cauchy criterion. If $\{b_l\}_{l=1}^{\infty}$ is a sequence of nonnegative real numbers, then the partial sums for the series $\sum_{l=1}^{\infty} b_l$ are monotone increasing and converge if and only if they are bounded. The comparison test states that if $\sum_{l=1}^{\infty} a_l$ is a series of complex numbers, $\sum_{l=1}^{\infty} b_l$ is a convergent series of nonnegative real numbers, and $|a_l| \le b_l$ for every l, then $\sum_{l=1}^{\infty} a_l$ converges absolutely.

16.1 The Cauchy condensation test

Let z be a complex number, and consider the series $\sum_{l=0}^{\infty} z^l$. Here z^l is interpreted as being equal to 1 for every $z \in \mathbf{C}$ when $l = 0$. It is well known and easy to check that

$$(1-z)\sum_{l=0}^{n} z^l = 1 - z^{n+1}$$

for each nonnegative integer n, so that

$$\sum_{l=0}^{n} z^l = \frac{1 - z^{n+1}}{1 - z}$$

when $z \ne 1$. If $|z| < 1$, then $\lim_{n\to\infty} z^n = 0$, and it follows that $\sum_{l=0}^{\infty} z^l$ converges, with

$$\sum_{l=0}^{\infty} z^l = \frac{1}{1-z}.$$

If $|z| \geq 1$, then $|z^l| = |z|^l \geq 1$ for every l, which implies that $\{z^l\}_{l=0}^\infty$ does not converge to 0, and hence that $\sum_{l=0}^\infty z^l$ diverges.

Suppose now that $\{a_l\}_{l=1}^\infty$ is a monotone decreasing sequence of nonnegative real numbers. The Cauchy condensation test states that $\sum_{l=1}^\infty a_l$ converges if and only if $\sum_{k=0}^\infty 2^k\, a_{2^k}$ converges. More precisely, an infinite series of nonnegative real numbers converges if and only if the partial sums are bounded, and so the point of the Cauchy condensation test is that the partial sums of $\sum_{l=1}^\infty a_l$ are bounded if and only if the partial sums of $\sum_{k=0}^\infty 2^k\, a_{2^k}$ are bounded. In one direction, we have that

$$\sum_{l=2^k}^{2^{k+1}-1} a_l \leq 2^k\, a_{2^k}$$

for each $k \geq 0$, because there are 2^k terms on the left side, each of which is bounded by a_{2^k}, by monotonicity. This implies that

$$\sum_{l=1}^{2^{n+1}-1} a_l = \sum_{k=0}^{n} \Big(\sum_{l=2^k}^{2^{k+1}-1} a_l\Big) \leq \sum_{k=0}^{n} 2^k\, a_{2^k}$$

for each $n \geq 0$, which shows that the partial sums of $\sum_{l=1}^\infty a_l$ are bounded when the partial sums of $\sum_{k=0}^\infty 2^k\, a_{2^k}$ are bounded. In the other direction, we have that

$$2^k\, a_{2^k} \leq 2 \sum_{l=2^{k-1}+1}^{2^k} a_l$$

for each $k \geq 1$, because there are 2^{k-1} terms on the right side, and $a_{2^k} \leq a_l$ when $l \leq 2^k$ by monotonicity again. Hence

$$\sum_{k=0}^{n} 2^k\, a_{2^k} = a_1 + \sum_{k=1}^{n} 2^k\, a_{2^k} \leq a_1 + 2\sum_{k=1}^{n} \Big(\sum_{l=2^{k-1}+1}^{2^k} a_l\Big) = a_1 + 2\sum_{l=2}^{2^n} a_l \leq 2\sum_{l=1}^{2^n} a_l,$$

which shows that the partial sums of $\sum_{k=0}^\infty 2^k\, a_{2^k}$ are bounded when the partial sums of $\sum_{l=1}^\infty a_l$ are bounded. As an application, for each positive real number p, $\sum_{n=1}^\infty 1/n^p$ converges if and only if $\sum_{k=0}^\infty 2^{(1-p)\,k}$ converges, which happens exactly when $p > 1$.

17 Alternating series

Let $\{b_j\}_{j=1}^\infty$ be a monotone decreasing sequence of nonnegative real numbers which converges to 0. The Leibniz alternating series test states that

$$\sum_{j=1}^\infty (-1)^j\, b_j$$

converges under these conditions. To see this, observe that

$$\left|\sum_{j=l}^{n}(-1)^j\, b_j\right| \le b_l$$

when $n \ge l$, which implies that $\sum_{j=1}^{\infty}(-1)^j\, b_j$ satisfies the Cauchy criterion. For example, it follows that $\sum_{n=1}^{\infty}(-1)^n/n^p$ converges for every positive real number p. More generally, suppose that $\{a_j\}_{j=1}^{\infty}$ is a sequence of complex numbers for which the partial sums $A_n = \sum_{j=1}^{n} a_j$ are bounded. If $\{b_j\}_{j=1}^{\infty}$ is as before, then $\sum_{j=1}^{\infty} a_j\, b_j$ converges. For if $A_0 = 0$, then

$$\begin{aligned}\sum_{j=1}^{n} a_j\, b_j &= \sum_{j=1}^{n}(A_j - A_{j-1})\, b_j = \sum_{j=1}^{n} A_j\, b_j - \sum_{j=1}^{n} A_{j-1}\, b_j \\ &= \sum_{j=1}^{n} A_j\, b_j - \sum_{j=0}^{n-1} A_j\, b_{j+1} \\ &= \sum_{j=1}^{n} A_j\,(b_j - b_{j+1}) + A_n\, b_{n+1}.\end{aligned}$$

It suffices to show that $\sum_{j=1}^{\infty} A_j\,(b_j - b_{j+1})$ converges, since $\{A_n\, b_{n+1}\}_{n=1}^{\infty}$ converges to 0 by hypothesis. For each $n \ge 1$, $\sum_{j=1}^{n}(b_j - b_{j+1}) = b_1 - b_n$, which implies that $\sum_{j=1}^{\infty}(b_j - b_{j+1})$ converges. The convergence of $\sum_{j=1}^{\infty} A_j\,(b_j - b_{j+1})$ now follows from the comparison test, since the A_j's are bounded and the b_j's are monotone decreasing. For example, if $z \in \mathbf{C}$ satisfies $|z| = 1$ and $z \ne 1$, then $a_l = z^l$ has bounded partial sums, and therefore $\sum_{l=1}^{\infty} z^l/l^p$ converges for every $p > 0$.

18 Power series

Let $\{a_l\}_{l=0}^{\infty}$ be a sequence of complex numbers, and consider the corresponding power series $\sum_{l=0} a_l\, z^l$. Suppose that $\sum_{l=0}^{\infty} a_l\, w^l$ converges for some $w \in \mathbf{C}$, $w \ne 0$. This implies that $\lim_{l\to\infty} a_l\, w^l = 0$, and in particular that $\{a_l\, w^l\}_{l=0}^{\infty}$ is a bounded sequence of complex numbers. Let A be a nonnegative real number such that $|a_l\, w^l| \le A$ for every $l \ge 0$. For every $z \in \mathbf{C}$,

$$|a_l\, z^l| \le A\left(\frac{|z|}{|w|}\right)^l,$$

and it follows from the comparison test that $\sum_{l=0}^{\infty} a_l\, z^l$ converges absolutely when $|z| < |w|$. Depending on the coefficients a_l, it may be that $\sum_{l=0}^{\infty} a_l\, z^l$ only converges in the trivial case where $z = 0$. At the other extreme, it may be that $\sum_{l=0}^{\infty} a_l\, z^l$ converges for every $z \in \mathbf{C}$, in which event it converges absolutely for every $z \in \mathbf{Z}$ by the previous remarks. Otherwise, the set of positive real numbers r for which there is a $w \in \mathbf{C}$ such that $|w| = r$ and $\sum_{l=0}^{\infty} a_l\, w^l$ converges is nonempty and bounded from above. The supremum R of this set is known as the radius of convergence of the power series $\sum_{l=0}^\infty a_l\, z^l$, and we put $R = 0$ when the series converges only when $z = 0$ and $R = +\infty$ when the series converges for every $z \in \mathbf{C}$. Thus $\sum_{l=0}^\infty a_l\, z^l$ converges absolutely when $|z| < R$ and diverges when $|z| > R$, and moreover $\{a_l\, z^l\}_{l=0}^\infty$ is unbounded when $|z| > R$ by similar reasoning. The convergence of $\sum_{l=0}^\infty a_l\, z^l$ when $|z| = R$ depends on the situation.

19 Extended real numbers

The extended real numbers consist of the real numbers together with two additional elements, $\pm\infty$, such that $-\infty < x < +\infty$ for every $x \in \mathbf{R}$. One can check that every nonempty set of extended real numbers has an infimum and supremum in the extended real line. Let us put $x + \infty = +\infty$ for $x \in \mathbf{R}$, $\infty + \infty = +\infty$, and so on, but leave the sum of $-\infty$ and $+\infty$ undefined. Similarly, the product of a nonzero real number x and $+\infty$ is equal to $+\infty$ when $x > 0$ and to $-\infty$ when $x < 0$, etc. We can also set $1/+\infty = 1/-\infty = 0$, and leave $1/0$ undefined, although it makes sense to let $1/0 = +\infty$ in situations where the quantities are nonnegative.

If $\{x_j\}_{j=1}^\infty$ is a sequence of real numbers, then $x_j \to +\infty$ as $j \to +\infty$ means that for every $N \ge 1$ there is an $L \ge 1$ such that $x_j \ge N$ when $j \ge L$. Similarly, $x_j \to -\infty$ as $j \to +\infty$ means that for every $N \ge 1$ there is an $L \ge 1$ such that $x_j \le -N$ when $j \ge L$. The usual results about sums and products of limits also work for infinite limits when the sum or product of limits makes sense. If $\{x_j\}_{j=1}^\infty$ is any sequence of real numbers, then there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ and an extended real number x such that $x_{j_l} \to x$ as $l \to \infty$, where this refers to ordinary convergence when $x \in \mathbf{R}$. If $\{x_j\}_{j=1}^\infty$ is a bounded sequence of real numbers, then there is a subsequence which converges to a real number by compactness of closed intervals. Otherwise, there may not be an upper bound for the x_j's, or a lower bound, or either one. One can check that there is no upper bound for the x_j's if and only if there is a subsequence which tends to $+\infty$, and that there is no lower bound if and only if there is a subsequence which tends to $-\infty$.

20 Upper and lower limits

Let $\{x_j\}_{j=1}^\infty$ be a sequence of real numbers, and let E be the set of extended real numbers x for which there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that $x_{j_l} \to x$ as $l \to \infty$. We have seen that $E \ne \emptyset$. The upper limit of $\{x_j\}_{j=1}^\infty$ is denoted $\limsup_{j\to\infty} x_j$ and defined to be the supremum of E, and the lower limit of $\{x_j\}_{j=1}^\infty$ is denoted $\liminf_{j\to\infty} x_j$ and defined to be the infimum of E. Hence

$$\liminf_{j\to\infty} x_j \le \limsup_{j\to\infty} x_j$$

automatically. If $x_j \to x$ as $j \to \infty$ for some extended real number x, then every subsequence also tends to x, x is the only element of E, and

$$x = \liminf_{j \to \infty} x_j = \limsup_{j \to \infty} x_j.$$

Put $u = \limsup_{j \to \infty} x_j$. If v is a real number such that $v > u$, then $x_j < v$ for all but finitely many j. Otherwise there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that $x_{j_l} \ge v$ for every l, and a subsubsequence $\{x_{j_{l_n}}\}_{n=1}^\infty$ which tends to an extended real number z. This subsubsequence is also a subsequence of $\{x_j\}_{j=1}^\infty$, and we get a contradiction since $z \ge v > u$, the supremum of the subsequential limits of $\{x_j\}_{j=1}^\infty$. If $t < u$, then $x_j > t$ for infinitely many j, because otherwise t is an upper bound for the subsequential limits of $\{x_j\}_{j=1}^\infty$ which is strictly less than the supremum. If u', u'' are extended real numbers which satisfy both of these properties and $u' < u''$, then we can get a contradiction by considering a real number y such that $u' < y < u''$ and observing that either $x_j > y$ for only finitely many j or for infinitely many j but not both. Thus u is uniquely characterized by these two properties. There is an analogous characterization for the lower limit.

20.1 Sequences of sets

Let X be a set, and let $A_1, A_2, \ldots$ be a sequence of subsets of X. The upper and lower limits of $\{A_n\}_{n=1}^\infty$ are defined by

$$\limsup_{n \to \infty} A_n = \bigcap_{n=1}^\infty \bigcup_{l=n}^\infty A_l$$

and

$$\liminf_{n \to \infty} A_n = \bigcup_{n=1}^\infty \bigcap_{l=n}^\infty A_l.$$

By definition, $\limsup_{n \to \infty} A_n$ consists of the $x \in X$ such that $x \in A_n$ for infinitely many n, while $\liminf_{n \to \infty} A_n$ consists of the $x \in X$ such that $x \in A_n$ for all but finitely many n, which is the same as saying that $x \in A_n$ for all sufficiently large n. In particular,

$$\liminf_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n.$$

One might say that $\{A_n\}_{n=1}^\infty$ converges to $A \subseteq X$ when

$$\limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n = A.$$

If $A_n \subseteq A_{n+1}$ for each n, then $\{A_n\}_{n=1}^\infty$ converges in this sense to $A = \bigcup_{n=1}^\infty A_n$, and if $A_{n+1} \subseteq A_n$ for each n, then $\{A_n\}_{n=1}^\infty$ converges in this sense to $A = \bigcap_{n=1}^\infty A_n$. For any set $E \subseteq X$, let $\mathbf{1}_E(x)$ be the indicator function of E on X, which is equal to 1 when $x \in E$ and to 0 when $x \in X \backslash E$. It is easy to check that the upper and lower limits of $\mathbf{1}_{A_n}(x)$ as a sequence of real numbers are the same as the indicator functions associated to the upper and lower limits of $\{A_n\}_{n=1}^\infty$ evaluated at x for every $x \in X$, respectively.

21 Root and ratio tests

Let $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers, and consider

$$\alpha = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}.$$

The root test asserts that $\sum_{n=1}^\infty a_n$ converges absolutely when $\alpha < 1$ and diverges when $\alpha > 1$. For if $\alpha < 1$, then we can let β be a real number such that $\alpha < \beta < 1$, and we get that $|a_n|^{1/n} < \beta$ for all but finitely many n. Hence $|a_n| < \beta^n$ for all but finitely many n, and $\sum_{n=1}^\infty a_n$ converges absolutely by comparison with $\sum_{n=1}^\infty \beta^n$. If $\alpha > 1$, then the a_n's are unbounded.

As an application, consider the power series $\sum_{n=1}^\infty a_n\, z^n$. If α is as above, then one can check that $R = 1/\alpha$ is the radius of convergence of this power series. When $\alpha = 0$, this means that $R = +\infty$, which is to say that the power series converges for every $z \in \mathbf{C}$.

Suppose now that $a_n \neq 0$ for every $n \geq 1$, and consider

$$\gamma = \limsup_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}.$$

If $\gamma < 1$, then the ratio test asserts that $\sum_{n=1}^\infty a_n$ converges absolutely. For if $\gamma < \delta < 1$, then there is a positive integer N such that $|a_{n+1}|/|a_n| < \delta$ when $n \geq N$. This implies that $|a_n| \leq \delta^{n-N}\, |a_N|$ when $n \geq N$, and hence that $\sum_{n=1}^\infty a_n$ converges absolutely since $\sum_{n=1}^\infty \delta^n$ converges. In the other direction, if

$$\liminf_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} > 1,$$

then $|a_n| \to \infty$ as $n \to \infty$, and $\sum_{n=1}^\infty a_n$ diverges.

22 Continuous mappings

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces, and let f be a function defined on M with values in N. We say that f is *continuous* as a point $x \in M$ if for every positive real number ϵ there is a positive real number δ such that

$$\rho(f(y), f(x)) < \epsilon$$

for every $y \in M$ with $d(y, x) < \delta$. Suppose that $f : M \to N$ is continuous at $x \in M$, and that $\{x_j\}_{j=1}^\infty$ is a sequence of elements of M that converges to x. Let $\epsilon > 0$ be given, and let $\delta > 0$ be as in the definition of continuity. Because $\{x_j\}_{j=1}^\infty$ converges to x, there is a positive integer L such that $d(x_j, x) < \delta$ when $j \geq L$. It follows that $\rho(f(x_j), f(x)) < \epsilon$ when $j \geq L$, and hence that $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in N. Conversely, suppose that f is not continuous at $x \in M$. This means that there is an $\epsilon > 0$ such that for every $\delta > 0$ there is a $y \in M$ with $d(y, x) < \delta$ and $\rho(f(y), f(x)) \geq \epsilon$. In particular, for each positive integer l there is a $y_l \in M$ which satisfies $d(y_l, x) < 1/l$ and $\rho(f(y_l), f(x)) \geq \epsilon$. Thus $\{y_l\}_{l=1}^\infty$

is a sequence of elements of M which converges to x for which $\{f(y_l)\}_{l=1}^\infty$ does not converge to $f(x)$ in N. This shows that f is continuous at x if and only if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of M that converges to x, $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in N.

We say that f is a continuous mapping from M to N if f is continuous at every element of M. Equivalently, $f : M \to N$ is continuous if for every convergent sequence $\{x_j\}_{j=1}^\infty$ of elements of M, $\{f(x_j)\}_{j=1}^\infty$ converges in N, and

$$\lim_{j\to\infty} f(x_j) = f\Big(\lim_{j\to\infty} x_j\Big).$$

One can check that sums and products of continuous complex-valued functions are continuous, using the analogous results for sequences and the characterization of continuity in terms of sequences, or using the same arguments as for sequences directly.

22.1 Another characterization

Suppose that X, Y are sets, and that f is a mapping from X to Y. For each $A \subseteq X$, put $f(A) = \{y \in Y : y = f(a) \text{ for some } a \in A\}$. Similarly, for $E \subseteq Y$, put $f^{-1}(E) = \{x \in X : f(x) \in E\}$. Note that $X\backslash f^{-1}(E) = f^{-1}(Y\backslash E)$ for every $E \subseteq Y$.

Let $(M, d(x, y))$, $(N, \rho(u, v))$ be metric spaces, and let f be a mapping from M to N. If $f : M \to N$ is continuous and U is an open set in N, then $f^{-1}(U)$ is an open set in M. To see this, let x be any element of $f^{-1}(U)$. Thus $f(x) \in U$, and there is an $\epsilon > 0$ such that U contains every $z \in N$ which satisfies $\rho(z, f(x)) < \epsilon$. Let δ be a positive real number such that $\rho(f(y), f(x)) < \epsilon$ when $y \in M$ satisfies $d(y, x) < \delta$, as in the definition of continuity. It follows that $y \in f^{-1}(U)$ when $y \in M$ and $d(y, x) < \delta$, so that $f^{-1}(U)$ contains an open ball in M centered at x, as desired. Conversely, if $f^{-1}(U)$ is an open set in M whenever U is an open set in N, then $f : M \to N$ is continuous. For let $x \in M$ and $\epsilon > 0$ be given, and let U be the set of $z \in N$ such that $\rho(z, f(x)) < \epsilon$. Clearly $x \in f^{-1}(U)$ and U is an open set in N, and so there is a $\delta > 0$ such that $y \in f^{-1}(U)$ when $y \in M$ and $d(y, x) < \delta$, by hypothesis. This is the same as saying that $\rho(f(y), f(x)) < \epsilon$ when $y \in M$ and $d(y, x) < \delta$, as required by the definition of continuity. As a corollary, $f : M \to N$ is continuous if and only if $f^{-1}(E)$ is a closed set in M for every closed set E in N.

Let $(M_1, d_1(x, y))$, $(M_2, d_2(u, v))$, and $(M_3, d_3(w, z))$ be metric spaces, and suppose that $f_1 : M_1 \to M_2$, $f_2 : M_2 \to M_3$ be continuous mappings. As usual, the composition $f_2 \circ f_1$ is the mapping from $M_1 \to M_2$ defined by

$$(f_2 \circ f_1)(x) = f_2(f_1(x)), \quad x \in M_1.$$

One can show that $f_2 \circ f_1$ is a continuous mapping from M_1 to M_3 using the definition of continuity in terms of ϵ's and δ's, or using the characterization of continuity in terms of convergent sequences, or using the characterization of continuity in terms of open sets.

23 Continuity and compactness

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces, and suppose that f is a continuous mapping from M to N. If $K \subseteq M$ is compact, then $f(K)$ is a compact set in N. For suppose that $\{V_\alpha\}_{\alpha \in A}$ is an open covering of $f(K)$ in N. If $U_\alpha = f^{-1}(V_\alpha)$ for every $\alpha \in A$, then $\{U_\alpha\}_{\alpha \in A}$ is an open covering of K in M. Since K is compact in M, there are finitely many indices $\alpha_1, \dots, \alpha_l \in A$ such that $K \subseteq \bigcup_{i=1}^{l} U_{\alpha_i}$. This implies that $f(K) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$, as desired. As a consequence, if f is a continuous real-valued function on M and $K \subseteq M$ is nonempty and compact, then there are $p, q \in K$ such that

$$f(p) \le f(x) \le f(p)$$

for every $x \in K$, which is to say that the maximum and minimum of f on K are attained. More precisely, this follows from the fact that $f(K)$ is closed and bounded, since it is compact. This statement is known as the extreme value theorem.

If $f : M \to N$ is continuous and $K \subseteq M$ is sequentially compact, then we can also argue directly that $f(K)$ is sequentially compact in N. Let $\{w_j\}_{j=1}^{\infty}$ be any sequence of elements of $f(K)$. For each j, let x_j be an element of K such that $f(x_j) = w_j$. Because K is sequentially compact, there is a subsequence $\{x_{j_l}\}_{l=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ which converges to a point $x \in K$. The continuity of f implies that the corresponding subsequence $\{w_{j_l}\}_{l=1}^{\infty}$ of $\{w_j\}_{j=1}^{\infty}$ converges to $f(x)$ in N.

23.1 One-to-one mappings

Let X, Y be sets, and let f be a function defined on X with values in Y. We say that $f : X \to Y$ is *one-to-one* if for every $x, x' \in X$ with $x \ne x'$ we have that $f(x) \ne f(x')$. We say that f maps X *onto* Y if for every $y \in Y$ there is an $x \in X$ such that $f(x) = y$. If both conditions hold, then there is an inverse mapping $f^{-1} : Y \to X$ such that $f^{-1}(y) = x$ when $y = f(x)$.

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces, and let f be a one-to-one continuous mapping from M onto N. If M is compact, then f^{-1} is continuous as a mapping from N onto M. For if $E \subseteq M$ is closed, then E is compact in M, and $f(E)$ is compact in N. This implies that $f(E)$ is a closed set in N, which is to say that $(f^{-1})^{-1}(E)$ is a closed set in N for every closed set $E \subseteq M$, and hence f^{-1} is continuous. Alternatively, let $\{w_j\}_{j=1}^{\infty}$ be a sequence of elements of N which converges to a point $w \in N$. Put $x_j = f^{-1}(w_j)$ and $x = f^{-1}(w)$, and suppose for the sake of a contradiction that $\{x_j\}_{j=1}^{\infty}$ does not converge to x in M. This means that there is an $\epsilon > 0$ such that

$$d(x_j, x) \ge \epsilon$$

for infinitely many j, and we may as well suppose that this inequality holds for all j by passing to a subsequence if necessary. Because M is sequentially compact, there is a subsequence $\{x_{j_l}\}_{l=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ which converges to a point

$y \in M$, $y \neq x$. Thus $w_{j_l} = f(x_{j_l}) \to f(y) \neq f(x) = w$ as $l \to \infty$ by the continuity of f, a contradiction.

24 Continuity and connectedness

Let $(M, d(x,y))$ and $(N, \rho(u,v))$ be metric spaces, and suppose that f is a continuous mapping from M into N. If $E \subseteq M$ is connected, then $f(E)$ is a connected set in N. For suppose to the contrary that there are nonempty separated sets $A, B \subseteq N$ such that $f(E) = A \cup B$. Using the continuity of f, one can show that $A_1 = f^{-1}(A) \cap E$, $B_1 = f^{-1}(B) \cap E$ are nonempty separated sets in M such that $E = A_1 \cup B_1$, a contradiction. As an application, if a, b are real numbers with $a < b$, f is a continuous real-valued function on the interval $[a,b] \subseteq \mathbf{R}$, and c is a real number such that $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then there is an $x \in \mathbf{R}$ such that $a < x < b$ and $f(x) = c$. Otherwise, $f([a,b])$ would not be a connected set in $\mathbf{R}$, a contradiction. This fact is known as the intermediate value theorem. For another application, let us say that a set $E \subseteq M$ is *pathwise connected* if for every $y, z \in E$ there are $a, b \in \mathbf{R}$ with $a \leq b$ and a continuous mapping $p : [a,b] \to M$ such that $p([a,b]) \subseteq E$, $p(a) = y$, and $p(b) = z$. One can show that pathwise connected sets are connected, using the connectedness of $p([a,b])$ when $p : [a,b] \to M$ is continuous. If $E \subseteq M$ is pathwise connected and $f : M \to N$ is continuous, then it is easy to see that $f(E)$ is pathwise connected in N. One can also show that connected open sets in $\mathbf{R}^n$ are pathwise connected, using the observation that open subsets of $\mathbf{R}^n$ are locally pathwise connected.

25 Uniform continuity

Let $(M, d(x,y))$ and $(N, \rho(u,v))$ be metric spaces. A mapping $f : M \to N$ is said to be *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ for every $x, y \in M$ with $d(x,y) < \delta$. Uniformly continuous mappings are continuous in particular. One can check that $f : M \to N$ is uniformly continuous if and only if for every pair of sequences $\{x_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$ of elements of M such that $\lim_{j\to\infty} d(x_j, y_j) = 0$,

$$\lim_{j\to\infty} \rho(f(x_j), f(y_j)) = 0.$$

It is easy to see that the composition of two uniformly continuous mappings is uniformly continuous, using the definition of uniform continuity in terms of ϵ's and δ's, or the characterization of uniform continuity in terms of sequences.

If $f : M \to N$ is uniformly continuous and $\{w_l\}_{l=1}^\infty$ is a Cauchy sequence of elements of M, then $\{f(w_l)\}_{l=1}^\infty$ is a Cauchy sequence in N. If f is uniformly continuous and $E \subseteq M$ is totally bounded, then $f(E)$ is totally bounded in N. The sum of two uniformly continuous complex-valued functions is uniformly continuous, as is the product of such a function and a constant. The product of two bounded uniformly continuous complex-valued functions is uniformly continuous.

25.1 Compact spaces

Let $(M, d(x,y))$ and $(N, \rho(u,v))$ be metric spaces, and let f be a continuous mapping from M to N. If M is compact, then f is uniformly continuous. To see this, let $\epsilon > 0$ be given. For each $x \in M$, there is a $\delta(x) > 0$ such that

$$\rho(f(y), f(x)) < \frac{\epsilon}{2}$$

when $y \in M$ and $d(y,x) < \delta(x)$, by continuity. If $B(x)$ is the open ball in M with center x and radius $\delta(x)/2$, then the open balls $B(x)$, $x \in M$, cover M. By compactness, there are finitely many elements $x_1, \ldots, x_k$ of M such that $M \subseteq \bigcup_{i=1}^k B(x_i)$. Put $\delta = \min(\delta(x_1)/2, \ldots, \delta(x_k)/2)$, and let x, y be arbitrary elements of M such that $d(x,y) < \delta$. There is an i, $1 \le i \le k$, such that $x \in B(x_i)$, and for which $d(y, x_i) < \delta(x_i)/2 + \delta \le \delta(x_i)$, by the triangle inequality. It follows that

$$\rho(f(x), f(y)) \le \rho(f(x), f(x_i)) + \rho(f(x_i), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired.

Alternatively, suppose that $\{x_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$ are sequences of elements of M such that $\lim_{j \to \infty} d(x_j, y_j) = 0$, but $\rho(f(x_j), f(y_j))$ does not converge to 0. This means that there is an $\epsilon > 0$ such that $\rho(f(x_j), f(y_j)) \ge \epsilon$ for infinitely many j. Without loss of generality, we may suppose that this holds for all j, since otherwise we can replace our sequences with the subsequences where it does hold. By compactness, there is a strictly increasing sequence $\{j_l\}_{l=1}^\infty$ of positive integers such that $\{x_{j_l}\}_{l=1}^\infty$ converges to a point $x \in M$, and we also get that $\{y_{j_l}\}_{l=1}^\infty$ converges to x too, since $d(x_{j_l}, y_{j_l}) \to 0$ as $l \to \infty$. Continuity of f at x implies that $\{f(x_{j_l})\}_{l=1}^\infty$ and $\{f(y_{j_l})\}_{l=1}^\infty$ both converge to $f(x)$ in N, and hence that $\lim_{l \to \infty} \rho(f(x_{j_l}), f(y_{j_l})) = 0$, a contradiction.

26 Uniform convergence

Let E be a set, let $(N, \rho(u,v))$ be a metric space, and let f_j, $j \ge 1$, and f be functions on E with values in N. If $\{f_j(x)\}_{j=1}^\infty$ converges to $f(x)$ in N for every $x \in E$, then we say that the sequence of functions $\{f_j\}_{j=1}^\infty$ converges *pointwise* to f on E. We say that $\{f_j\}_{j=1}^\infty$ converges to f *uniformly* on E if for every $\epsilon > 0$ there is a positive integer L such that

$$\rho(f_j(x), f(x)) < \epsilon \quad \text{for every } x \in E$$

when $j \ge L$. It is easy to see that uniform convergence implies pointwise convergence.

Now suppose that $(M, d(x, y))$, $(N, \rho(u, v))$ are metric spaces, and let $\{f_j\}_{j=1}^\infty$ be a sequence of continuous mappings from M to N. If $\{f_j\}_{j=1}^\infty$ converges uniformly to a mapping $f : M \to N$, then f is continuous too. For let $x \in M$ and $\epsilon > 0$ be given. Since $\{f_j\}_{j=1}^\infty$ converges uniformly to f, there is a positive integer L such that

$$\rho(f_j(y), f(y)) < \frac{\epsilon}{3} \quad \text{for every } y \in M$$

when $j \geq L$. Because f_L is continuous at x, there is a $\delta > 0$ such that $\rho(f_L(y), f_L(x)) < \epsilon/3$ when $y \in M$ and $d(y, x) < \delta$. Therefore,

$$\begin{aligned}\rho(f(y), f(x)) &\leq \rho(f(y), f_L(y)) + \rho(f_L(y), f_L(x)) + \rho(f_L(x), f(x)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

when $y \in M$ and $d(y, x) < \delta$, as desired. The same argument shows that f is uniformly continuous if the f_j's are. As another variant, if $\{x_j\}_{j=1}^\infty$ is a sequence of elements of M that converges to $x \in M$, then one can show that $\{f_j(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in N under these conditions.

26.1 Complex-valued functions

Let E be a set, and let us consider complex-valued functions on E. If a pair of sequences $\{f_j\}_{j=1}^\infty$, $\{\widetilde{f}_j\}_{j=1}^\infty$ of functions on E converge uniformly to functions f, $\widetilde{f}$ on E, respectively, then it is easy to see that $\{f_j + \widetilde{f}_j\}_{j=1}^\infty$ converges uniformly to $f + \widetilde{f}$ on E. If $\{f_j\}_{j=1}^\infty$ converges uniformly to f on E and c is a complex number, then $\{c\, f_j\}_{j=1}^\infty$ converges uniformly to $c\, f$ on E. If a pair of uniformly bounded sequences of functions on E converge uniformly, then the corresponding sequence of products converges uniformly to the product of limits.

Suppose that $\{a_j\}_{j=1}^\infty$ is a sequence of functions on E and $\{A_j\}_{j=1}^\infty$ is a sequence of nonnegative real numbers such that $|a_j(x)| \leq A_j$ for every $j \geq 1$ and $x \in E$. If $\sum_{j=1}^\infty A_j$ converges, then $\sum_{j=1}^\infty a_j(x)$ converges absolutely for every $x \in E$ by the comparison test. Weierstrass made the nice observation that the partial sums of $\sum_{j=1}^\infty a_j$ converge uniformly on E under these conditions. The main points are that

$$\left|\sum_{j=1}^\infty a_j(x) - \sum_{j=1}^n a_j(x)\right| = \left|\sum_{j=n+1}^\infty a_j(x)\right| \leq \sum_{j=n+1}^\infty |a_j(x)| \leq \sum_{j=n+1}^\infty A_j$$

for every $x \in E$ and $n \geq 1$, and that $\sum_{j=n+1}^\infty A_j$ does not depend on x and tends to 0 as $n \to \infty$, by hypothesis.

Let $\sum_{l=0}^\infty a_l\, z^l$ be a power series with complex coefficients. If r is a positive real number and $\sum_{l=0} |a_l|\, r^l$ converges, then Weierstrass' observation implies that $\sum_{l=0}^\infty a_l\, z^l$ converges uniformly on the closed disk consisting of the $z \in \mathbf{C}$ with $|z| \leq r$. It follows that $\sum_{l=0}^\infty a_l\, z^l$ defines a continuous function on this closed disk. If $\sum_{l=0}^\infty a_l\, z^l$ has radius of convergence $R > 0$, then one can check that $\sum_{l=0}^{\infty} a_l \, z^l$ defines a continuous function on the open disk consisting of $z \in \mathbf{C}$ such that $|z| < R$. More precisely, one can show that this function is continuous at a point $z_0 \in \mathbf{C}$ such that $|z_0| < R$, by applying the previous argument to a positive real number r such that $|z_0| < r < R$.

27 The supremum metric

Let $(M, d(x, y))$, $(N, \rho(u, v))$ be metric spaces. A mapping $f : M \to N$ is said to be *bounded* if $f(M)$ is a bounded set in N. Let $\mathcal{C}_b(M, N)$ be the space of bounded continuous mappings from M into N. The *supremum metric* on $\mathcal{C}_b(M, N)$ is defined by

$$\theta(f_1, f_2) = \sup\{\rho(f_1(x), f_2(x)) : x \in M\}$$

for $f_1, f_2 \in \mathcal{C}_b(M, N)$. It is easy to see that this is a metric on $\mathcal{C}_b(M, N)$, and that a sequence $\{f_j\}_{j=1}^{\infty}$ of elements of $\mathcal{C}_b(M, N)$ converges to $f \in \mathcal{C}_b(M, N)$ in the supremum metric if and only if $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly.

Suppose that N is complete, and let us show that $\mathcal{C}_b(M, N)$ is complete with respect to the supremum metric. Let $\{f_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $\mathcal{C}_b(M, N)$, so that for every $\epsilon > 0$ there is a positive integer $L(\epsilon)$ such that

$$\theta(f_j, f_l) < \epsilon$$

when $j, l \ge L(\epsilon)$. In particular, $\{f_j(x)\}_{j=1}^{\infty}$ is a Cauchy sequence in N for every $x \in M$, which converges to an element $f(x)$ of N since N is complete. One can check that

$$\rho(f_j(x), f(x)) \le \epsilon \quad \text{for every } x \in M$$

when $j \ge L(\epsilon)$, which means that $\{f_j\}_{j=1}^{\infty}$ converges uniformly to f and that f is continuous, as desired.

27.1 The supremum norm

Let $(M, d(x, y))$ be a metric space, and let $\mathcal{C}_b(M)$ be the space of bounded continuous real-valued functions on M, i.e., $\mathcal{C}_b(M) = \mathcal{C}_b(M, \mathbf{R})$. Thus $\mathcal{C}_b(M)$ is a vector space over the real numbers with respect to the usual operations of pointwise addition of functions and multiplication of functions by constants, and moreover $\mathcal{C}_b(M)$ is a commutative algebra with respect to the operation of pointwise multiplication of functions. The *supremum norm* of a function $f \in \mathcal{C}_b(M)$ is

$$\|f\|_* = \sup\{|f(x)| : x \in M\},$$

and one can check that

$$\|f_1 + f_2\|_* \le \|f_1\|_* + \|f_2\|_*$$

and

$$\|f_1 \, f_2\|_* \le \|f_1\|_* \, \|f_2\|_*$$

for every $f_1, f_2 \in \mathcal{C}_b(M)$. It is easy to see that $\|f_1 - f_2\|_*$ is the same as the supremum metric on $\mathcal{C}_b(M)$. Using the triangle inequality, one can check that $f_p(x) = d(x, p)$ is a continuous function on M for every $p \in M$. These functions are bounded when M is bounded, and otherwise $\min(f_p(x), r)$ is a bounded continuous function on M for every $r \geq 0$. This shows that there are always a lot of bounded continuous real-valued functions on any metric space, and in particular these functions separate elements of M, in the sense that for every $x, y \in M$ with $x \neq y$ there is an $f \in \mathcal{C}_b(M)$ such that $f(x) \neq f(y)$.

28 The contraction mapping theorem

Let $(M, d(x, y))$ be a metric space. Suppose that $\phi : M \to M$ is a strict contraction in the sense that there is a positive real number $c < 1$ such that

$$d(\phi(x), \phi(y)) \leq c\, d(x, y)$$

for every $x, y \in M$. If $x, x' \in M$ are fixed by ϕ, which is to say that $\phi(x) = x$ and $\phi(x') = x'$, then $d(x, x') = d(\phi(x), \phi(x')) \leq c\, d(x, x')$, which implies that $d(x, x') = 0$ since $c < 1$ and hence that $x = x'$. If M is complete, then the contraction mapping theorem states that there is an $x \in M$ such that $\phi(x) = x$. To see this, let z be any point in M, and consider the sequence $\{z_n\}_{n=1}^\infty$ of elements of M defined recursively by $z_1 = z$, $z_{n+1} = \phi(z_n)$. Thus $d(z_{n+2}, z_{n+1}) \leq c\, d(z_{n+1}, z_n)$ for each n. By repeating this, we get

$$d(z_{n+1}, z_n) \leq c^{n-1}\, d(z_2, z_1)$$

for every $n \geq 1$, and hence

$$\begin{aligned} d(z_{n+l}, z_n) &\leq \sum_{i=0}^{l-1} d(z_{n+i+1}, z_{n+i}) \leq \sum_{i=0}^{l-1} c^{n+i-1}\, d(z_2, z_1) \\ &\leq c^{n-1} \Big(\sum_{i=0}^{\infty} c^i \Big)\, d(z_2, z_1) = \frac{c^{n-1}}{1-c}\, d(z_2, z_1) \end{aligned}$$

for $l, n \geq 1$. This implies that $\{z_n\}_{n=1}^\infty$ is a Cauchy sequence in M, which converges because M is complete. Moreover,

$$\phi\Big(\lim_{n\to\infty} z_n\Big) = \lim_{n\to\infty} \phi(z_n) = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} z_n$$

since ϕ is continuous.

29 Limits of functions

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces. Suppose that $E \subseteq M$ and that $p \in M$ is a limit point of E. Suppose also that f is a function defined on

E with values in N and that $z \in N$. We say that the limit of $f(x)$ as $x \to p$ with $x \in E$ is equal to z if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho(f(x), z) < \epsilon$$

for every $x \in E$ with $x \neq p$ and $d(x,p) < \delta$. Note that p may not be an element of E, and the value of f at p is not involved in the limit even when $p \in E$. The limit is denoted

$$\lim_{\substack{x \to p \\ x \in E}} f(x)$$

when it exists. It may also be denoted more simply as

$$\lim_{x \to p} f(x)$$

if $E = M$ or the choice of E is clear from the context.

Suppose that $E = M$. If p is a limit point of M, then

$$\lim_{x \to p} f(x) = f(p)$$

if and only if f is continuous at p. If p is not a limit point of M, then there is a $\delta > 0$ such that $x \in M$ and $d(x,p) < \delta$ imply that $x = p$, and f is automatically continuous at p.

29.1 Limits and sequences

Let $(M, d(x,y))$, $(N, \rho(u,v))$ be metric spaces, let E be a subset of M, and let $p \in M$ be a limit point of E. Also let f be a mapping from E into N, and let z be an element of N. Under these conditions,

$$\lim_{\substack{x \to p \\ x \in E}} f(x) = z$$

if and only if $\{f(x_j)\}_{j=1}^{\infty}$ converges in N to z for every sequence $\{x_j\}_{j=1}^{\infty}$ of elements of E that converges to p and satisfies $x_j \neq p$ for each j. This is analogous to the characterization of continuity in terms of convergence of sequences.

Suppose that N is the space of real or complex numbers, with the standard metric. Let f_1, f_2 be functions on E for which the corresponding limits at p exist. In this case, the limits also exist for the sum $f_1 + f_2$ and product $f_1 \, f_2$, and satisfy

$$\lim_{\substack{x \to p \\ x \in E}} (f_1(x) + f_2(x)) = \lim_{\substack{x \to p \\ x \in E}} f_1(x) + \lim_{\substack{x \to p \\ x \in E}} f_2(x)$$

and

$$\lim_{\substack{x \to p \\ x \in E}} f_1(x) \, f_2(x) = \Big(\lim_{\substack{x \to p \\ x \in E}} f_1(x) \Big) \Big(\lim_{\substack{x \to p \\ x \in E}} f_2(x) \Big).$$

This follows from the characterization of limits in terms of sequences and the analogous statements for sequences, and it can also be shown directly in the same way as for sequences.

30 One-sided limits

Let a, b be real numbers with $a < b$, and let f be a real-valued function on the open interval (a, b). If $a \le p < b$, then the limit of $f(x)$ as $x \to p$ with $p < x < b$ may be denoted

$$\lim_{x \to p+} f(x) \quad \text{or} \quad f(p+)$$

when it exists. Similarly, if $a < p \le b$, then the limit of $f(x)$ as $x \to p$ with $a < x < p$ may be denoted

$$\lim_{x \to p-} f(x) \quad \text{or} \quad f(p-)$$

when it exists. If $a < p < b$, then

$$\lim_{x \to p} f(x)$$

exists if and only if the one-sided limits exist are equal.

Suppose now that $f : (a, b) \to \mathbf{R}$ is also monotone increasing. In this case, the one-sided limits exist at each point in (a, b), with

$$f(p+) = \inf\{f(x) : p < x < b\}$$

and

$$f(p-) = \sup\{f(x) : a < x < p\}.$$

In particular,

$$f(p-) \le f(p) \le f(p+).$$

If f is bounded from below on (a, b), then $f(a+)$ exists and is equal to the infimum of f on (a, b), and if f is bounded from above on (a, b), then $f(b-)$ exists and is equal to the supremum of f on (a, b). Note that

$$f(p+) \le f(t) \le f(q-)$$

when $a \le p < t < q \le b$.

30.1 Monotone functions

Let f be a monotone increasing real-valued function on an open interval (a, b) in the real line. Thus the one-sided limits $f(p+)$, $f(p-)$ exist at each point $p \in (a, b)$ and satisfy

$$f(p-) \le f(p) \le f(p+),$$

and f is continuous at p if and only if equality holds in both of these inequalities. Equivalently, f is not continuous at p if and only if

$$f(p-) < f(p+).$$

Let $I(p)$ be the open interval $(f(p-), f(p+))$ for each $p \in (a, b)$ at which f is discontinuous. If $a < p < q < b$, then $f(p+) \le f(q-)$, and hence

$$I(p) \cap I(q) = \emptyset$$

if f is also discontinuous at each of p and q. For each $p \in (a, b)$ at which f is discontinuous, let $r(p)$ be a rational number contained in $I(p)$. It follows from the previous observation that

$$r(p) < r(q)$$

when $a < p < q < b$ and f is discontinuous at both p and q. This implies that f can be discontinuous at only finitely or countably many elements of (a, b), since the set of rational numbers is countably infinite.

Part III

Some additional topics

31 Cauchy products

If $\sum_{j=0}^\infty a_j$, $\sum_{l=0}^\infty b_l$ are two series of complex numbers, then their Cauchy product is defined to be the series $\sum_{n=0}^\infty c_n$, where

$$c_n = \sum_{j=0}^n a_j \, b_{n-j}. \tag{31.1}$$

For instance, if $a_j = b_l = 0$ for all but finitely many $j, l \ge 0$, then $c_n = 0$ for all but finitely many n,

$$\sum_{n=0}^\infty c_n = \Big(\sum_{j=0}^\infty a_j\Big)\Big(\sum_{l=0}^\infty b_l\Big). \tag{31.2}$$

Another reason for this definition is that the Cauchy product of the power series $\sum_{j=0}^\infty a_j \, z^j$, $\sum_{l=0}^\infty b_l \, z^l$ is the power series $\sum_{n=0}^\infty c_n \, z^n$ when the coefficients c_n are as in (31.1).

Suppose for the moment that a_j, b_l are nonnegative real numbers for each $j, l \ge 0$, so that c_n is also a nonnegative real number for each $n \ge 0$. In this case, it is easy to see that

$$\Big(\sum_{j=0}^{[N/2]} a_j\Big)\Big(\sum_{l=0}^{[N/2]} b_l\Big) \le \sum_{n=0}^N c_n \le \Big(\sum_{j=0}^N a_j\Big)\Big(\sum_{l=0}^N b_l\Big) \tag{31.3}$$

for every nonnegative integer N. Here $[N/2]$ is the integer part of $N/2$, which is the largest integer which is less than or equal to $N/2$. If $\sum_{j=0}^\infty a_j$, $\sum_{l=0}^\infty b_l$

converge, then it follows easily that $\sum_{n=0}^{\infty} c_n$ converges too, and that (31.2) holds.

If a_j, b_l are real or complex numbers such that $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$ converge absolutely, then $\sum_{n=0}^{\infty} c_n$ also converges absolutely, and (31.2) holds again. One way to see this is to reduce to the previous case by expressing $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$ as linear combinations of convergent series of nonnegative real numbers. Alternatively, for each $N \ge 0$, it is easy to see that

$$\sum_{n=0}^{N} |c_n| \le \Big(\sum_{j=0}^{N} |a_j|\Big) \Big(\sum_{l=0}^{N} |b_l|\Big). \tag{31.4}$$

This implies that $\sum_{n=0}^{\infty} c_n$ converges absolutely when $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$ converge absolutely. To get (31.2), one can approximate $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$ by series with only finitely many nonzero terms, and use inequalities like (31.4) to estimate the remainders.

32 The exponential function

Put

$$E(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \tag{32.1}$$

for each complex number z. Here $j!$ is "j factorial", equal to the product of the positive integers from 1 to j when $j \ge 1$, and equal to 1 when $j = 0$. It is easy to see that this series converges absolutely for every $z \in \mathbf{C}$, using the ratio test, for instance. It follows that the partial sums for $E(z)$ converge uniformly on bounded subsets of $\mathbf{C}$, and hence that $E(z)$ is a continuous function on $\mathbf{C}$.

If $z, w \in \mathbf{C}$, then

$$E(z)\, E(w) = \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} \frac{z^j}{j!}\, \frac{w^{n-j}}{(n-j)!}\Big), \tag{32.2}$$

using Cauchy products, as in the previous section. Remember that

$$(z+w)^n = \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\, z^j\, w^{n-j}, \tag{32.3}$$

by the binomial theorem, so that

$$E(z)\, E(w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = E(z+w). \tag{32.4}$$

In particular,

$$E(z)\, E(-z) = E(0) = 1 \tag{32.5}$$

for each $z \in \mathbf{C}$, which implies that $E(z) \ne 0$ and $1/E(z) = E(-z)$.

If x is a real number, then $E(x) \in \mathbf{R}$, and $E(x) \geq 1$ when $x \geq 0$. If $x \leq 0$, then $1/E(x) = E(-x) \geq 1$, and hence $0 < E(x) \leq 1$. It is easy to see from the series expansion that $E(x)$ is strictly increasing for $x \geq 0$, and that $E(x) \to +\infty$ as $x \to +\infty$. It follows that $E(x)$ is also strictly increasing for $x \leq 0$, and hence on the whole real line, and that $E(x) \to 0$ as $x \to -\infty$.

If z is any real number again, then

$$\overline{E(z)} = \sum_{j=0}^{\infty} \frac{\overline{z}^j}{j!} = E(\overline{z}). \tag{32.6}$$

Thus

$$|E(z)|^2 = E(z)\,\overline{E(z)} = E(z)\,E(\overline{z}) = E(z + \overline{z}) = E(2 \operatorname{Re} z), \tag{32.7}$$

where $\operatorname{Re} z$ denotes the real part of z. Of course, $E(2\,rez) = E(\operatorname{Re} z)^2$, by (32.4), so that

$$|E(z)| = E(\operatorname{Re} z). \tag{32.8}$$

33 Diameters of bounded sets

Let $(M, d(x, y))$ be a metric space. The *diameter* of a nonempty bounded set $E \subseteq M$ is defined by

$$\operatorname{diam} E = \sup\{d(p, q) : p, q \in E\}. \tag{33.1}$$

If $A \subseteq B \subseteq M$ are nonempty and bounded, then

$$\operatorname{diam} A \leq \operatorname{diam} B. \tag{33.2}$$

If $E \subseteq M$ is nonempty and bounded, then it is easy to see that the closure $\overline{E}$ of E is also bounded, and one can check that

$$\operatorname{diam} \overline{E} = \operatorname{diam} E. \tag{33.3}$$

Suppose that $\{E_j\}_{j=1}^\infty$ is a sequence of nonempty bounded subsets of M such that $E_{j+1} \subseteq E_j$ for each j and $\operatorname{diam} E_j \to 0$ as $j \to \infty$. If $x_j \in E_j$ for each j, then $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in M. If M is complete, then it follows that $\{x_j\}_{j=1}^\infty$ converges to an element x of M. Note that $x \in \overline{E_l}$ for each l, since $x_j \in E_l$ when $j \geq l$. Conversely, suppose that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in M, and let E_l be the set of x_j with $j \geq l$ for each positive integer l. Thus $E_l \neq \emptyset$ and $E_{l+1} \subseteq E_l$ for each l, and one can check that the E_l's are bounded subsets of M with $\operatorname{diam} E_l \to 0$ as $l \to \infty$, because $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence. If $x \in \bigcap_{l=1}^\infty \overline{E_l}$, then it is easy to see that $\{x_j\}_{j=1}^\infty$ converges to x in M.

Suppose that $\phi : M \to M$ is a strict contraction, in the sense that

$$d(\phi(x), \phi(y)) \leq c\, d(x, y) \tag{33.4}$$

for some nonnegative real number $c < 1$ and every $x, y \in M$. If E is a nonempty bounded subset of M, then it follows that $\phi(E)$ is also nonempty and bounded, and satisfies

$$\operatorname{diam} \phi(E) \le c \operatorname{diam} E. \tag{33.5}$$

Let ϕ^n be the n-fold composition of ϕ for each positive integer n, so that $\phi^1 = \phi$, $\phi^2 = \phi \circ \phi$, and $\phi^{n+1} = \phi \circ \phi^n = \phi^n \circ \phi$ for every $n \ge 1$. Put $E_n = \phi^n(M)$ for each $n \ge 1$, and observe that

$$E_{n+1} = \phi^n(\phi(M)) \subseteq \phi^n(M) = E_n \tag{33.6}$$

for each n. If M is bounded, then

$$\operatorname{diam} E_n \le c^n \operatorname{diam} M \tag{33.7}$$

for each n, and $\operatorname{diam} E_n \to 0$ as $n \to \infty$ in particular. If M is also complete, then there is a point $p \in \bigcap_{n=1}^\infty \overline{E_n}$, as in the previous paragraph. Note that $\phi(\overline{E_n}) \subseteq \overline{\phi(E_n)} = \overline{E_{n+1}}$ for each n, using the continuity of ϕ in the first step. Hence $\phi(p) \in \overline{E_{n+1}}$ for each n as well. This implies that $\phi(p) = p$, since $p, \phi(p) \in \overline{E_{n+1}}$ for each n and $\operatorname{diam} \overline{E_{n+1}} = \operatorname{diam} E_{n+1} \to 0$ as $n \to \infty$. This gives another way to look at the contraction mapping theorem when M is bounded.

34 Compactness and completeness

Let $(M, d(x, y))$ be a metric space. We have seen that compact subsets of M are closed and totally bounded. Conversely, if M is complete and $K \subseteq M$ is closed and totally bounded, then K is compact. One can show this in much the same way as for compactness of closed intervals in the real line, or cells in $\mathbf{R}^n$. More precisely, let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of K in M, and suppose for the sake of a contradiction that K cannot be covered by finitely many U_α's. Because K is totally bounded, K can be expressed as the union of finitely many subsets with diameter less than 1. At least one of these subsets of K cannot be covered by finitely many U_α's, since otherwise K could be covered by finitely many U_α's. Thus we get a subset K_1 of K with $\operatorname{diam} K_1 < 1$ such that K_1 cannot be covered by finitely many U_α's. Repeating the process, we get a sequence $K_1, K_2, \ldots$ of subsets of K such that $K_{n+1} \subseteq K_n$, $\operatorname{diam} K_n < 1/n$, and K_n cannot be covered by finitely many U_α's, for each n. In particular, $K_n \ne \emptyset$ for each n, and it follows that there is an $x \in M$ such that $x \in \overline{K_n}$ for each n, because M is complete, as in the previous section. We also have that $x \in K$, because $K_n \subseteq K$ for each n and K is closed. Hence $x \in U_{\alpha_0}$ for some $\alpha_0 \in A$, since K is covered by the U_α's by hypothesis. As usual, there is an $r > 0$ such that $B(x, r) \subseteq U_{\alpha_0}$, because U_{α_0} is an open set in M, by hypothesis. This implies that

$$K_n \subseteq B(x, r) \subseteq U_{\alpha_0} \tag{34.1}$$

when $n > 1/r$, since $x \in \overline{K_n}$ and $\operatorname{diam} K_n < 1/n$, by construction. Of course, (34.1) contradicts the fact that K_n cannot by covered by finitely many U_α's, from which we conclude that K can be covered by finitely many U_α's, as desired.

34.1 Cauchy subsequences

Alternatively, for any metric space M, a set $E \subseteq M$ is totally bounded if and only if every sequence of elements of E has a subsequence which is a Cauchy sequence. For if E is not totally bounded, then we have seen that there is an $\epsilon > 0$ and a sequence $\{x_l\}_{l=1}^{\infty}$ of elements of E such that $d(x_l, x_n) \geq \epsilon$ for every l, n such that $l \neq n$. Every subsequence of this sequence has the same property and is therefore not Cauchy. Conversely, suppose that E is totally bounded, and that $\{x_l\}_{l=1}^{\infty}$ is a sequence of elements of E. For every $\epsilon > 0$, E can be expressed as the union of finitely many subsets of diameter less than ϵ, from which it follows that there is a subsequence of $\{x_l\}_{l=1}^{\infty}$ whose terms are contained in a set of diameter less than ϵ. By repeating this argument, for every positive integer n there is a subsequence of $\{x_l\}_{l=1}^{\infty}$ whose terms are contained in a set of diameter less than $1/n$, and which is a subsequence of the previous subsequence when $n \geq 2$. The diagonal sequence whose nth term is the nth term of the nth subsequence is a subsequence of $\{x_l\}_{l=1}^{\infty}$ which is a Cauchy sequence. If M is complete and E is totally bounded, then every sequence of elements of E has a Cauchy subsequence which converges to an element of M, and hence to an element of E when E is also closed. Thus E is sequentially compact when M is complete and $E \subseteq M$ is closed and totally bounded.

35 The Baire category theorem

Let $(M, d(x,y))$ be a metric space. A set $E \subseteq M$ is dense in M if and only if for every nonempty open set $V \subseteq M$, $E \cap V \neq \emptyset$. If $E \subseteq M$ is dense in M and $U \subseteq M$ is a dense open set in M, then one can use this to show that $E \cap U$ is dense in M. In particular, the intersection of two dense open subsets of M is a dense open set in M.

The Baire category theorem states that if M is complete and $U_1, U_2, \ldots$ is a sequence of dense open subsets of M, then the intersection $\bigcap_{n=1}^{\infty} U_n$ is dense in M. To see this, let $p \in M$ and $r > 0$ be given, and let us show that there is a $q \in \bigcap_{n=1}^{\infty} U_n$ such that $d(p,q) \leq r$. Since $U_1 \subseteq M$ is dense and open, there is a $q_1 \in U_1$ and an $r_1 > 0$ such that $\overline{B}(q_1, r_1) \subseteq \overline{B}(p, r) \cap U_1$ and $r_1 \leq 1$. Similarly, there is a $q_2 \in U_2$ and an $r_2 > 0$ such that $\overline{B}(q_2, r_2) \subseteq \overline{B}(q_1, r_1) \cap U_2$ and $r_2 \leq 1/2$. By repeating the process, we get for each $n \geq 3$ a point $q_n \in U_n$ and an $r_n > 0$ such that $\overline{B}(q_n, r_n) \subseteq \overline{B}(q_{n-1}, r_{n-1}) \cap U_n$ and $r_n \leq 1/n$. By construction, $\{q_n\}_{n=1}^{\infty}$ is a Cauchy sequence in M, and hence converges to a point q in M. Each q_n is an element of $\overline{B}(p, r)$, and thus $d(p,q) \leq r$ too. Similarly, q_l is an element of $\overline{B}(q_n, r_n)$ when $l \geq n$, which implies that q is an element of $\overline{B}(q_n, r_n)$ and therefore of U_n for every n, as desired.

36 Diameters of compact sets

Let $(M, d(x,y))$ be a metric space, and let K be a nonempty compact subset of M. Also let $\{p_j\}_{j=1}^\infty$, $\{q_j\}_{j=1}^\infty$ be sequences of elements of K such that

$$\operatorname{diam} K < d(p_j, q_j) + 1/j \tag{36.1}$$

for each j. By sequential compactness, there is a strictly increasing sequence $\{j_l\}_{l=1}^\infty$ of positive integers such that the subsequence $\{p_{j_l}\}_{l=1}^\infty$ of $\{p_j\}_{j=1}^\infty$ converges to an element p of K. Similarly, there is a strictly increasing sequence of positive integers $\{l_n\}_{n=1}^\infty$ such that $\{q_{j_{l_n}}\}_{n=1}^\infty$ converges to $q \in K$. Since $\{p_{j_{l_n}}\}_{n=1}^\infty$, $\{q_{j_{l_n}}\}_{n=1}^\infty$ both converge to p, q, respectively, it follows that the diameter of K is equal to $d(p,q)$.

Suppose now that $K_1, K_2, \ldots$ is a sequence of nonempty compact subsets of M such that $K_{n+1} \subseteq K_n$ for each n, and put $K = \bigcap_{n=1}^\infty K_n$. If $\{x_n\}_{n=1}^\infty$ is a sequence of elements of M such that $x_n \in K_n$ for each n, then the limit of any convergent subsequence of $\{x_j\}_{j=1}^\infty$ is an element of K, and $K \neq \emptyset$ in particular. Alternatively, if K is empty, then $\{M \backslash K_n\}_{n \geq 1}$ is an open covering of K_1 for which there is no finite subcovering, a contradiction. Similarly, if U is an open set in M such that $K \subseteq U$, then $K_n \subseteq U$ for some n, since otherwise the $M \backslash K_n$'s together with U form an open covering of K_1 with no finite subcovering, a contradiction. Using this, one can show that

$$\lim_{n \to \infty} \operatorname{diam} K_n = \operatorname{diam} K \tag{36.2}$$

under these conditions. Of course,

$$\operatorname{diam} K \leq \operatorname{diam} K_{n+1} \leq \operatorname{diam} K_n \tag{36.3}$$

for each n, since $K \subseteq K_{n+1} \subseteq K_n$, and so the point is to show that $\operatorname{diam} K_n$ is not too much larger than $\operatorname{diam} K$ when n is sufficiently large. To do this, let $\epsilon > 0$ be given, and consider

$$U(\epsilon) = \bigcup_{x \in K} B(x, \epsilon/2). \tag{36.4}$$

This is an open set in M that contains K and satisfies $\operatorname{diam} U(\epsilon) \leq \operatorname{diam} K + \epsilon$. The previous argument implies that $K_n \subseteq U(\epsilon)$ for some n, which implies that $\operatorname{diam} K_n \leq \operatorname{diam} K + \epsilon$, as desired. One can also get the same conclusion by considering sequences of pairs of elements of the K_n's, and then passing to suitable subsequences to get corresponding pairs of elements of K.

37 Another fixed-point theorem

Let $(M, d(x,y))$ be a metric space, and suppose that $\phi : M \to M$ is a contraction in the sense that

$$d(\phi(x), \phi(y)) < d(x,y) \tag{37.1}$$

for every $x, y \in M$ with $x \neq y$. If $x, x' \in M$, $\phi(x) = x$, $\phi(x') = x'$, and $x \neq x'$, then

$$d(x, x') = d(\phi(x), \phi(x')) < d(x, x'), \tag{37.2}$$

a contradiction, which is to say that there can be at most one fixed point of ϕ in M. If $K \subseteq M$ is compact, then $\phi(K)$ is compact too, since ϕ is continuous. Hence the diameter of $\phi(K)$ is equal to the distance between two of its elements, and one can use this to show that

$$\operatorname{diam} \phi(K) < \operatorname{diam} K \tag{37.3}$$

when K has more than one element.

If M is compact, then ϕ has a fixed point in M. A well-known trick for showing this is to minimize

$$f(x) = d(\phi(x), x). \tag{37.4}$$

Specifically, there is a $p \in M$ such that $f(p) \leq f(x)$ for every $x \in M$, since $f : M \to \mathbf{R}$ is continuous and M is compact. The contractivity property of ϕ implies that $f(\phi(x)) < f(x)$ when $\phi(x) \neq x$, and consequently $\phi(p) = p$, as desired.

As another argument, let ϕ^n be the n-fold composition of ϕ for each positive integer n, so that $\phi^1 = \phi$ and $\phi^{n+1} = \phi \circ \phi^n = \phi^n \circ \phi$ for every $n \geq 1$. Consider the compact sets $K_n = \phi^n(M)$ and $K = \bigcap_{n=1}^\infty K_n$. Observe that

$$K_{n+1} = \phi^n(K_1) \subseteq \phi^n(M) = K_n \tag{37.5}$$

for each n. It follows that $K \neq \emptyset$, since the K_n's are nonempty and compact. Clearly $\phi(K) \subseteq K$, and we would like to show that $\phi(K) = K$. Let x be any element of K. For each n, $x \in K_{n+1}$, which means that

$$\phi^{-1}(\{x\}) \cap K_n \neq \emptyset. \tag{37.6}$$

Note that $\phi^{-1}(\{x\})$ is a closed set in M, because ϕ is continuous, so that $\phi^{-1}(\{x\}) \cap K_n$ is compact for each n. This implies that

$$\bigcap_{n=1}^\infty (\phi^{-1}(\{x\}) \cap K_n) = \phi^{-1}(x) \cap K \neq \emptyset, \tag{37.7}$$

since $\phi^{-1}(\{x\}) \cap K_{n+1} \subseteq \phi^{-1}(\{x\}) \cap K_n$ for each n, and thus $x \in \phi(K)$. Hence $\phi(K) = K$, which implies that K has exactly one element, since otherwise $\operatorname{diam} \phi(K) < \operatorname{diam} K$, as before. If p is this element of K, then it follows that $\phi(p) = p$, as desired. An advantage of this argument is that for each $r > 0$, we also get that $K_n = \phi^n(M) \subseteq B(p, r)$ for all sufficiently large n.

38 An extension theorem

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces, and let E be a dense subset of M. If f, g are continuous mappings from M into N such that $f(x) = g(x)$ for every $x \in E$, then one can check that $f(x) = g(x)$ for every $x \in E$.

Suppose now that f is a uniformly continuous mapping from E into N, and that N is complete. Under these conditions, there is an extension of f to a uniformly continuous mapping from M into N, which is unique by the remark in the previous paragraph. To see this, let p be an element of M, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of E that converges to p, which exists because E is dense in M. If $p \in E$, then $\{f(x_j)\}_{j=1}^\infty$ converges to $f(p)$ in N, because f is continuous at p. Otherwise, $\{x_j\}_{j=1}^\infty$ may be considered as a Cauchy sequence in E. Because $f : E \to N$ is uniformly continuous, it follows that $\{f(x_j)\}_{j=1}^\infty$ is a Cauchy sequence in N. This implies that $\{f(x_j)\}_{j=1}^\infty$ converges to an element of N, since N is supposed to be complete. If $\{y_j\}_{j=1}^\infty$ is another sequence of elements of E that converges to p, then

$$\lim_{j \to \infty} d(x_j, y_j) = 0.$$

Using the uniform continuity of $f : E \to N$ again, one can check that

$$\lim_{j \to \infty} \rho(f(x_j), f(y_j)) = 0.$$

This implies that $\{f(x_j)\}_{j=1}^\infty$ and $\{f(y_j)\}_{j=1}^\infty$ converge to the same element of N. Thus one can define the extension of f at p to be the limit of $\{f(x_j)\}_{j=1}^\infty$, where $\{x_j\}_{j=1}^\infty$ is any sequence of elements of E that converges to p in M. Note that the uniform continuity of $f : E \to N$ may be formulated as saying that for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that

$$\rho(f(x), f(y)) \le \epsilon$$

for every $x, y \in E$ such that $d(x, y) < \delta(\epsilon)$. When formulated this way, one can check that the extension of f to a mapping from M into N just described is also uniformly continuous, with the same choice of $\delta(\epsilon)$ for each $\epsilon > 0$, by approximating elements of M by elements of E as before.

39 An embedding theorem

Let $(M, d(x, y))$ be a metric space, and let $C_b(M)$ be the space of bounded continuous real-valued functions on M, equipped with the supremum metric. Suppose for the moment that M is bounded, and put

$$f_p(x) = d(p, x)$$

for each $p, x \in M$. It is easy to see that $f_p(x)$ is continuous as a function of $x \in M$ for each $p \in M$, using the triangle inequality. Also, $f_p(x)$ is bounded as a function of $x \in M$ for each $p \in M$, because M is bounded by hypothesis. Thus $p \mapsto f_p$ may be considered as a mapping from M into $C_b(M)$. Using the triangle inequality again, one can check that

$$|f_p(x) - f_q(x)| \le d(p, q)$$

for every $p, q, x \in M$, and of course equality holds when $x = p$ or q. This implies that $p \mapsto f_p$ is an isometric embedding of M into $C_b(M)$, in the sense that the distance between f_p and f_q with respect to the supremum metric is equal to $d(p, q)$ for every $p, q \in M$.

If M is not bounded, then we can modify the construction slightly to get the same conclusion. Let a be a fixed element of M, and put

$$\widetilde{f}_p(x) = f_p(x) - f_a(x) = d(p, x) - d(a, x)$$

for every $p, x \in M$. Thus $\widetilde{f}_p(x)$ is continuous as a function of $x \in M$ for each $p \in M$, since it is the difference of two continuous functions on M. It is easy to see that $\widetilde{f}_p(x)$ is bounded on M for each $p \in M$, even when M is unbounded, using the triangle inequality as in the previous paragraph. If $p, q \in M$, then

$$\widetilde{f}_p(x) - \widetilde{f}_q(x) = f_p(x) - f_q(x)$$

for each $x \in M$. This permits one to show that the distance between $\widetilde{f}_p$ and $\widetilde{f}_q$ with respect to the supremum metric is equal to $d(p, q)$ for every $p, q \in M$, for the same reasons as before. Hence $p \mapsto \widetilde{f}_p$ is an isometric embedding of M into $C_b(M)$, even when M is unbounded.

40 Completions

Let $(M, d(x, y))$ and $(M_1, d_1(u, v))$ be metric spaces. As usual, a mapping ϕ_1 from M into M_1 is said to be an isometric embedding if

$$d_1(\phi_1(x), \phi_1(y)) = d(x, y)$$

for every $x, y \in M$, in which case it is obviously uniformly continuous. By a *completion* of M we mean a complete metric space M_1 and an isometric embedding $\phi_1 : M \to M_1$ such that $\phi_1(M)$ is dense in M_1. If $\phi_1(M)$ is not dense in M_1, then one can simply replace M_1 with the closure of $\phi_1(M)$ in M_1, since a closed subset of a complete metric space is also complete as a metric space, using the restriction of the metric from the larger space. In particular, one can get a completion of any metric space M using an isometric embedding of M into $C_b(M)$, as in the previous section.

If M is already complete, and ϕ_1 is an isometric embedding of M onto a dense subset of a metric space M_1, then $\phi_1(M) = M_1$. For if z is any element of M_1, then there is a sequence $\{\phi_1(x_j)\}_{j=1}^{\infty}$ of elements of $\phi_1(M)$ that converges to z, because $\phi_1(M)$ is dense in M_1. In particular, $\{\phi_1(x_j)\}_{j=1}^{\infty}$ is a Cauchy sequence in M_1, which implies that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence in M, because ϕ_1 is an isometric embedding. Hence $\{x_j\}_{j=1}^{\infty}$ converges to an element x of M, because M is complete. This implies that $\{\phi_1(x_j)\}_{j=1}^{\infty}$ converges to $\phi_1(x)$ in M_1, so that $z = \phi_1(x) \in \phi_1(M)$, as desired.

If $\{x_j\}_{j=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$ are Cauchy sequences in M, then one can check that $\{d(x_j, y_j)\}_{j=1}^{\infty}$ is a Cauchy sequence in the real line, with respect to the standard metric. This implies that $\{d(x_j, y_j)\}_{j=1}^\infty$ converges in $\mathbf{R}$, since the real line is complete. The limit of $\{d(x_j, y_j)\}_{j=1}^\infty$ can be used as a measurement of distance between the two Cauchy sequences $\{x_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$. This definition of the distance between two Cauchy sequences satisfies the requirements of a metric, except that the distance between distinct Cauchy sequences may be equal to 0.

Let us say that two Cauchy sequences $\{x_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$ in M are equivalent when

$$\lim_{j\to\infty} d(x_j, y_j) = 0.$$

This defines an equivalence relation on the set of all Cauchy sequences in M, and we let M^* be the corresponding set of equivalence classes of Cauchy sequences in M. If $\{x_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$, $\{x'_j\}_{j=1}^\infty$, and $\{y'_j\}_{j=1}^\infty$ are Cauchy sequences in M such that $\{x_j\}_{j=1}^\infty$ is equivalent to $\{x'_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ is equivalent to $\{y'_j\}_{j=1}^\infty$, then one can check that the distance between $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ mentioned earlier is equal to the distance between $\{x'_j\}_{j=1}^\infty$ and $\{y'_j\}_{j=1}^\infty$. It follows that this distance determines a metric on M^*.

If $x \in M$, then the constant sequence $\{x_j\}_{j=1}^\infty$ with $x_j = x$ for each j is a Cauchy sequence in M. This defines an embedding of M into M^*, which is easily seen to preserve distances. It is also easy to see that the image of M in M^* is dense with respect to the metric on M^* discussed previously. In addition, one can show that M^* is complete, and hence a completion of M.

Suppose that (M_1, d_1) and (M_2, d_2) are complete metric spaces, and that $\phi_1 : M \to M_1$ and $\phi_2 : M \to M_2$ are isometric embeddings of M onto dense subsets of M_1 and M_2, respectively. Consider the mapping ψ from $\phi_1(M)$ onto $\phi_2(M)$ defined by

$$\psi(\phi_1(x)) = \phi_2(x).$$

This is well-defined, because the isometric embeddings ϕ_1, ϕ_2 are automatically one-to-one. This is also an isometry with respect to the restrictions of d_1, d_2 to $\phi_1(M)$, $\phi_2(M)$, since

$$d_2(\phi_2(x), \phi_2(y)) = d(x, y) = d_1(\phi_1(x), \phi_1(y))$$

for every $x, y \in M$.

In particular, $\psi : \phi_1(M) \to \phi_2(M)$ is uniformly continuous with respect to the restrictions of d_1, d_2 to $\phi_1(M)$, $\phi_2(M)$. Because $\phi_1(M)$ is dense in M_1 and M_2 is complete, there is a unique extension of ψ to a uniformly continuous mapping from M_1 into M_2. It is easy to see that this extension is also an isometry, and hence that it maps M_1 onto M_2, since M_1 is complete and $\phi_2(M)$ is dense in M_2. This shows that the completion of a metric space is unique up to isometric equivalence in a natural way.

41 Integral metrics

Let f, g be continuous real-valued functions on the unit interval $[0, 1]$, and put

$$d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx, \tag{41.1}$$

using the standard Riemann integral on the real line. It is easy to check that this defines a metric on the space $C([0,1])$ of continuous real-valued functions on $[0,1]$. Remember that the supremum metric on $C([0,1])$ is defined by

$$d_\infty(f,g) = \sup_{0 \le x \le 1} |f(x) - g(x)|, \tag{41.2}$$

and that $C([0,1])$ is complete with respect to this metric. Of course,

$$d_1(f,g) \le d_\infty(f,g) \tag{41.3}$$

for all continuous functions f, g on $[0,1]$, and one can give examples where $d_1(f,g)$ is much smaller than $d_\infty(f,g)$. One can also give examples to show that $C([0,1])$ is not complete with respect to $d_1(f,g)$.

For instance, let $f_j(x)$ be the continuous function defined on $[0,1]$ by

$$\begin{aligned} f_j(x) &= 0 \quad \text{when } 0 \le x \le 1/2 - 1/(2j), \\ &= 1 \quad \text{when } 1/2 \le x \le 1, \end{aligned} \tag{41.4}$$

and taking $f_j(x)$ to be linear between $1/2 - 1/(2j)$ and $1/2$. If

$$\begin{aligned} f(x) &= 0 \quad \text{when } 0 \le x < 1/2, \\ &= 1 \quad \text{when } 1/2 \le x \le 1, \end{aligned} \tag{41.5}$$

then it is easy to see that

$$\lim_{j \to \infty} \int_0^1 |f_j(x) - f(x)| \, dx = 0. \tag{41.6}$$

This is basically the same as saying that $\{f_j\}_{j=1}^\infty$ converges to f with respect to d_1, even though f is not a continuous function on $[0,1]$. In particular, $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to d_1, but it does not converge to a continuous function on $[0,1]$ with respect to d_1. Note that $\{f_j(x)\}_{j=1}^\infty$ converges to $f(x)$ pointwise on $[0,1]$, but not uniformly.

If a sequence $\{g_j\}_{j=1}^\infty$ of continuous real-valued functions on $[0,1]$ converges uniformly to a continuous function g on $[0,1]$, then it is easy to see that

$$\lim_{j \to \infty} \int_0^1 g_j(x) \, dx = \int_0^1 g(x) \, dx. \tag{41.7}$$

This is basically because of (41.3), and one can give examples to show that (41.7) may not hold if $\{g_j(x)\}_{j=1}^\infty$ only converges to $g(x)$ pointwise on $[0,1]$. If $\{g_j(x)\}_{j=1}^\infty$ converges to $g(x)$ pointwise on $[0,1]$ and $g_j(x)$ is uniformly bounded for $0 \le x \le 1$ and $j \ge 1$, then Lebesgue's dominated convergence theorem implies that (41.7) holds again. Lebesgue's theory of integration also provides a description of the completion of $C([0,1])$ with respect to d_1. Another well-known result in this area states that monotone functions on the real line are differentiable "almost everywhere".

42 Double sums

If $a_{j,l}$ is a nonnegative real number for every pair of positive integers j, l, then we would like to check that

$$\sum_{j=1}^{\infty}\Big(\sum_{l=1}^{\infty} a_{j,l}\Big) = \sum_{l=1}^{\infty}\Big(\sum_{j=1}^{\infty} a_{j,l}\Big). \tag{42.1}$$

More precisely, if $\sum_{l=1}^{\infty} a_{j,l}$ does not converge to a finite real number for some j, then we interpret the sum as being $+\infty$, and we also interpret the left side of (42.1) as being $+\infty$. If $\sum_{l=1}^{\infty} a_{j,l}$ converges for each j, but the sum of these sums over j does not converge, then again we interpret the left side of (42.1) as being $+\infty$. There are similar interpretations for the right side of (42.1), and part of the statement is that one side of (42.1) is finite if the other side is finite.

In order to prove (42.1), it suffices to show that

$$\sum_{j=1}^{\infty}\Big(\sum_{l=1}^{\infty} a_{j,l}\Big) \le \sum_{l=1}^{\infty}\Big(\sum_{j=1}^{\infty} a_{j,l}\Big) \tag{42.2}$$

and

$$\sum_{l=1}^{\infty}\Big(\sum_{j=1}^{\infty} a_{j,l}\Big) \le \sum_{j=1}^{\infty}\Big(\sum_{l=1}^{\infty} a_{j,l}\Big). \tag{42.3}$$

These two inequalities are essentially the same, but with the roles of j and l interchanged, and so it suffices to check the first one. To do this, it is enough to verify that

$$\sum_{j=1}^{N}\Big(\sum_{l=1}^{\infty} a_{j,l}\Big) \le \sum_{l=1}^{\infty}\Big(\sum_{j=1}^{\infty} a_{j,l}\Big) \tag{42.4}$$

for every positive integer N. This follows from

$$\sum_{j=1}^{N}\Big(\sum_{l=1}^{\infty} a_{j,l}\Big) = \sum_{l=1}^{\infty}\Big(\sum_{j=1}^{N} a_{j,l}\Big) \le \sum_{l=1}^{\infty}\Big(\sum_{j=1}^{\infty} a_{j,l}\Big), \tag{42.5}$$

by the usual rules for dealing with finite sums of infinite series.

If $b_{j,l}$ is a real or complex number for each $j, l \in \mathbf{Z}_+$, then (42.1) implies that

$$\sum_{j=1}^{\infty}\Big(\sum_{l=1}^{\infty} |b_{j,l}|\Big) = \sum_{l=1}^{\infty}\Big(\sum_{j=1}^{\infty} |b_{j,l}|\Big), \tag{42.6}$$

with $a_{j,l} = |b_{j,l}|$. In particular, if one side of (42.6) is finite, then the other side of (42.6) is also finite. In this case, $\sum_{l=1}^{\infty} b_{j,l}$ converges absolutely for each j and satisfies

$$\Big|\sum_{l=1}^{\infty} b_{j,l}\Big| \le \sum_{l=1}^{\infty} |b_{j,l}|, \tag{42.7}$$

and hence $\sum_{j=1}^{\infty} \Big(\sum_{l=1}^{\infty} b_{j,l} \Big)$ converges absolutely and satisfies

$$\Big| \sum_{j=1}^{\infty} \Big(\sum_{l=1}^{\infty} b_{j,l} \Big) \Big| \le \sum_{j=1}^{\infty} \Big| \sum_{l=1}^{\infty} b_{j,l} \Big| \le \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} |b_{j,l}|. \tag{42.8}$$

Similarly, $\sum_{j=1}^{\infty} b_{j,l}$ converges absolutely for each l and satisfies

$$\Big| \sum_{j=1}^{\infty} b_{j,l} \Big| \le \sum_{j=1}^{\infty} |b_{j,l}|, \tag{42.9}$$

so that $\sum_{l=1}^{\infty} \Big(\sum_{j=1}^{\infty} b_{j,l} \Big)$ converges absolutely and satisfies

$$\Big| \sum_{l=1}^{\infty} \Big(\sum_{l=1}^{\infty} b_{j,l} \Big) \Big| \le \sum_{l=1}^{\infty} \Big| \sum_{j=1}^{\infty} b_{j,l} \Big| \le \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} |b_{j,l}|. \tag{42.10}$$

We would like to check that

$$\sum_{j=1}^{\infty} \Big(\sum_{l=1}^{\infty} b_{j,l} \Big) = \sum_{l=1}^{\infty} \Big(\sum_{j=1}^{\infty} b_{j,l} \Big), \tag{42.11}$$

under these conditions. If $b_{j,l} \in \mathbf{R}$ for each $j, l \in \mathbf{Z}_+$, then (42.11) can be derived from (42.1) by taking $a_{j,l}$ to be the positive and negative parts of $b_{j,l}$. Similarly, the complex case can be obtained from the real case by passing to the real and imaginary parts of $b_{j,l}$. Alternatively, (42.11) clearly holds when $b_{j,l} = 0$ for all but finitely many $(j, l) \in \mathbf{Z}_+^2$. Otherwise, if (42.6) is finite, then for each $\epsilon > 0$ there is a $J(\epsilon) \ge 1$ such that

$$\sum_{j=J(\epsilon)+1}^{\infty} \Big(\sum_{l=1}^{\infty} |b_{j,l}| \Big) < \frac{\epsilon}{2}. \tag{42.12}$$

One can then get an $L(\epsilon) \ge 1$ such that

$$\sum_{j=1}^{J(\epsilon)} \Big(\sum_{l=L(\epsilon)+1}^{\infty} |b_{j,l}| \Big) < \frac{\epsilon}{2}. \tag{42.13}$$

This permits one to reduce to the case where $b_{j,l} = 0$ for all but finitely many $(j, l) \in \mathbf{Z}_+$, by an approximation argument.

43 Rearrangements

Let $\sum_{j=1}^{\infty} a_j$ be an infinite series of complex numbers, and let π be a permutation on the set $\mathbf{Z}_+$ of positive integers, which is to say a one-to-one mapping from $\mathbf{Z}_+$ onto itself. If $b_j = a_{\pi(j)}$ for each j, then $\sum_{j=1}^{\infty} b_j$ is said to be a *rearrangement* of $\sum_{j=1}^{\infty} a_j$. If $\sum_{j=1}^{\infty} a_j$ converges, then one might like to say that $\sum_{j=1}^{\infty} b_j$ also converges and has the same sum, but this does not always work. Of course, this does work when $a_j = 0$ for all but finitely many j, and when $\pi(j) = j$ for all but finitely many j.

Another case where this works is when a_j is a nonnegative real number for each j. To see this, observe first that

$$\sum_{j=1}^{n} b_j \le \sum_{l=1}^{L(n)} a_l \tag{43.1}$$

for $n \ge 1$, where $L(n) = \max(\pi^{-1}(1), \ldots, \pi^{-1}(n))$. If $\sum_{l=1}^{\infty} a_l$ converges, so that the partial sums of $\sum_{l=1}^{\infty} a_l$ are bounded, then it follows that the partial sums of $\sum_{j=1}^{\infty} b_j$ are bounded, and hence that $\sum_{j=1}^{\infty} b_j$ converges. This argument also shows that

$$\sum_{j=1}^{\infty} b_j \le \sum_{l=1}^{\infty} a_l, \tag{43.2}$$

under these conditions. To get the opposite inequality, observe that

$$\sum_{l=1}^{L} a_l \le \sum_{j=1}^{n(L)} b_j \tag{43.3}$$

for each $L \ge 1$, where $n(L) = \max(\pi(1), \ldots, \pi(L))$.

If a_j are arbitrary real or complex numbers, and $\sum_{l=1}^{\infty} a_l$ converges absolutely, then $\sum_{j=1}^{\infty} b_j$ also converges absolutely, and satisfies

$$\sum_{j=1}^{\infty} |b_j| = \sum_{l=1}^{\infty} |a_j|, \tag{43.4}$$

since one can apply the argument in the previous paragraph to $|a_j|$. Of course, one would also like to say that $\sum_{j=1}^{\infty} b_j = \sum_{l=1}^{\infty} a_l$. If $a_j \in \mathbf{R}$ for each j, then this can be derived from the argument in the previous paragraph, applied to the positive and negative parts of a_j. Otherwise, if the a_j's are complex numbers, then one can reduce to the real case by considering the real and imaginary parts of a_j. Alternatively, one can show that the two infinite sums are the same using absolute convergence to approximate them by finite sums that are the same.

If $\sum_{j=1}^{\infty} a_j$ is an infinite series of real numbers that converges but does not converge absolutely, then it can be shown that there are rearrangements of $\sum_{j=1}^{\infty} a_j$ that do not converge. It can also be shown that there are rearrangements of $\sum_{j=1}^{\infty}$ that converge with sum equal to any given real number.

44 Sums and limits

Let $A_{k,l}$ be a real or complex number for each pair of positive integers k, l, and suppose that $\sum_{l=1}^{\infty} A_{k,l}$ converges for each k. Suppose also that $\{A_{k,l}\}_{k=1}^{\infty}$

converges to a real or complex number A_l, as appropriate, for each l. One might like to say that $\sum_{l=1}^\infty A_l$ converges under these conditions, and that

$$\lim_{k \to \infty} \sum_{l=1}^\infty A_{k,l} = \sum_{l=1}^\infty A_l. \tag{44.1}$$

Of course, we do have that

$$\lim_{k \to \infty} \sum_{l=1}^L A_{k,l} = \sum_{l=1}^L A_l \tag{44.2}$$

for each $L \ge 1$, and the problem is basically to interchange the order of the limits in k and L.

To do this, let us make the additional hypothesis that B_l is a nonnegative real number for each $l \ge 1$ such that

$$|A_{k,l}| \le B_l \tag{44.3}$$

for each $k, l \ge 1$, and that $\sum_{l=1}^\infty B_l$ converges. This implies that

$$|A_l| \le B_l \tag{44.4}$$

for every $l \ge 1$ too, and hence that $\sum_{l=1}^\infty A_l$ converges as well. Of course,

$$\Big|\sum_{l=1}^\infty A_{k,l} - \sum_{l=1}^\infty A_l\Big| = \Big|\sum_{l=1}^\infty (A_{k,l} - A_l)\Big| \le \sum_{l=1}^\infty |A_{k,l} - A_l| \tag{44.5}$$

for each k. Let $\epsilon > 0$ be given, and let us check that

$$\sum_{l=1}^\infty |A_{k,l} - A_l| < \epsilon \tag{44.6}$$

for all sufficiently large k. Because $\sum_{l=1}^\infty B_l$ converges, there is an $L(\epsilon) \ge 1$ such that

$$\sum_{l=L(\epsilon)+1}^\infty B_l = \sum_{l=1}^\infty B_l - \sum_{l=1}^{L(\epsilon)} B_l < \frac{\epsilon}{3}. \tag{44.7}$$

Thus

$$\sum_{l=L(\epsilon)+1}^\infty |A_{k,l} - A_l| \le \sum_{l=L(\epsilon)+1}^\infty (|A_{k,l}| + |A_l|) \le \sum_{l=L(\epsilon)+1}^\infty 2\, B_l < \frac{2\,\epsilon}{3} \tag{44.8}$$

for each k. It remains to observe that

$$\sum_{l=1}^{L(\epsilon)} |A_{k,l} - A_l| < \frac{\epsilon}{3} \tag{44.9}$$

for all sufficiently large k, which follows from the hypothesis that $\{A_{k,l}\}_{k=1}^\infty$ converges to A_l for each k. This is the analogue of Lebesgue's dominated convergence theorem in the context of infinite series.

Alternatively, let E be the set of real numbers of the form $1/k$ for $k \in \mathbf{Z}_+$ together with 0. If l is a positive integer, then let f_l be the real or complex-valued function on E defined by $f_l(1/k) = A_{k,l}$ for each $k \geq 1$ and $f_l(0) = A_l$. Thus the hypothesis that $\{A_{k,l}\}_{k=1}^\infty$ converges to A_l for each l is equivalent to saying that f_l is continuous on E with respect to restriction of the standard metric on the real line to E for each l. The hypothesis (44.3) then says that $|f_l| \leq B_l$ on E for each l. The convergence of $\sum_{l=1}^\infty B_l$ implies that the partial sums of $\sum_{l=1}^\infty f_l$ converge uniformly on E, by the observation of Weierstrass. It follows that $\sum_{l=1}^\infty f_l$ defines a continuous function on E. Hence

$$\lim_{k \to \infty} \sum_{l=1}^\infty f_l(1/k) = \sum_{l=1}^\infty f_l(0), \tag{44.10}$$

as before.

Suppose now that $a_{j,l}$ is a real or complex number for each pair of positive integers j, l, and put

$$A_{k,l} = \sum_{j=1}^k a_{j,l} \tag{44.11}$$

for each $k, l \geq 1$. In this case, the previous question about limits of sums becomes one of interchanging order of summation, as in Section 42. In the proof of Theorem 8.3 on p175 of [20], Rudin essentially uses the argument described in the previous paragraph to deal with interchanging the order of summation of a double sum under suitable conditions of absolute convergence.

44.1 Monotone convergence

Let $A_{k,l}$ be a nonnegative real number for each pair of positive integers k, l, and suppose that $A_{k,l} \leq A_{k+1,l}$ for each k, l. This implies that for each l, $A_{k,l}$ tends to a nonnegative extended real number A_l as $k \to \infty$, which is the same as the supremum of $A_{k,l}$ over k as an extended real number. Under these conditions,

$$\sum_{l=1}^\infty A_{k,l} \to \sum_{l=1}^\infty A_l \quad \text{as } k \to \infty. \tag{44.12}$$

This is the analogue of Lebesgue's monotone convergence theorem for sums. More precisely, if $\sum_{l=1}^\infty A_{k,l}$ does not converge in the usual sense for some k, then we interpret its sum as being equal to $+\infty$. In this case, $\sum_{l=1}^\infty A_{k',l} = +\infty$ when $k' \geq k$, by monotonicity, $\sum_{l=1}^\infty A_l = +\infty$, and the convergence is trivial. Similarly, if $A_{l_0} = +\infty$ for some l_0, then we interpret $\sum_{l=1}^\infty A_l$ as being $+\infty$. This means that $A_{k,l_0} \to +\infty$ as $k \to \infty$, so that $\sum_{l=1}^\infty A_{k,l} \to +\infty$ as $k \to \infty$ too. Otherwise, we may as well suppose that $\sum_{l=1}^\infty A_{k,l}$ is finite for every k, and that A_l is finite for every l. This still leaves the possibility that $\sum_{l=1}^{\infty} A_l = +\infty$, in which case the conclusion is that $\sum_{l=1}^{\infty} A_{k,l} \to +\infty$ as $k \to \infty$.

Of course,

$$\sum_{l=1}^{\infty} A_{k,l} \le \sum_{l=1}^{\infty} A_l \tag{44.13}$$

for each k, because of monotonicity. We also have that

$$\sum_{l=1}^{L} A_{k,l} \to \sum_{l=1}^{L} A_l \quad \text{as } k \to \infty \tag{44.14}$$

for each $L \ge 1$, and that

$$\sum_{l=1}^{L} A_l \to \sum_{l=1}^{\infty} A_l \quad \text{as } L \to \infty. \tag{44.15}$$

It is easy to complete the proof using these remarks, since $\sum_{l=1}^{L} A_{k,l} \le \sum_{l=1}^{\infty} A_{k,l}$ for each k, L.

If $a_{j,l}$ is a nonnegative real number for each pair of positive integers j, l, then $A_{k,l} = \sum_{j=1}^{k} a_{j,l}$ is monotone increasing in k. In this case, the preceding question about limits of sums corresponds to interchanging the order of summation in a double sum of nonnegative real numbers, as in Section 42.

44.2 Fatou's lemma

If $\{a_j\}_{j=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$ are sequences of real numbers, then it is easy to see that

$$\sup_{j \ge 1}(a_j + b_j) \le \Big(\sup_{j \ge 1} a_j\Big) + \Big(\sup_{j \ge 1} b_j\Big). \tag{44.16}$$

More precisely, if $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ have upper bounds in $\mathbf{R}$, then their sum does too, and both sides of (44.16) are finite. Otherwise, the right side of (44.16) is interpreted as being $+\infty$, and the inequality is trivial.

Similarly, one can show that

$$\limsup_{j \to \infty}(a_j + b_j) \le \Big(\limsup_{j \to \infty} a_j\Big) + \Big(\limsup_{j \to \infty} b_j\Big), \tag{44.17}$$

as long as the right side is well-defined, in the sense that it is not of the form $\infty + (-\infty)$ or $(-\infty) + \infty$. One way to do this is to use the definition of $\limsup_{j \to \infty}(a_j + b_j)$ as the supremum of the set of subsequential limits of $\{a_j + b_j\}_{j=1}^{\infty}$. By passing to suitable subsequences, it suffices to consider subsequences of $\{a_j + b_j\}_{j=1}^{\infty}$ for which the corresponding subsequences of $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ also have limits. Alternatively, one can use the fact that if

$$\limsup_{j \to \infty} a_j < v \quad \text{and} \quad \limsup_{j \to \infty} b_j < w \tag{44.18}$$

for some real numbers v, w, then $a_j < v$ and $b_j < w$ for all but finitely many j. This implies that $a_j + b_j < v + w$ for all but finitely many j, and hence

$$\limsup_{j\to\infty}(a_j + b_j) \le v + w. \tag{44.19}$$

In the same way,

$$\Big(\liminf_{j\to\infty} a_j\Big) + \Big(\liminf_{j\to\infty} b_j\Big) \le \liminf_{j\to\infty}(a_j + b_j), \tag{44.20}$$

as long as the left side is not of the form $\infty + (-\infty)$ or $(-\infty) + \infty$. This can also be obtained by applying (44.17) to $-a_j$, $-b_j$. In particular, (44.20) holds when $a_j, b_j \ge 0$ for each j, because

$$0 \le \liminf_{j\to\infty} a_j, \liminf_{j\to\infty} b_j \le +\infty \tag{44.21}$$

in this case.

If $a_{j,l}$ is a nonnegative real number for each pair j, l of positive integers, then

$$\sum_{l=1}^{\infty} \liminf_{j\to\infty} a_{j,l} \le \liminf_{j\to\infty} \sum_{l=1}^{\infty} a_{j,l}. \tag{44.22}$$

This is the analogue of Fatou's lemma for sums. As usual, the sum of a divergent series of nonnegative terms is interpreted as being $+\infty$, as well as the sum on the left when any of its terms is equal to $+\infty$. To get (44.22), observe that

$$\sum_{l=1}^{L} \liminf_{j\to\infty} a_{j,l} \le \liminf_{j\to\infty} \sum_{l=1}^{L} a_{j,l} \tag{44.23}$$

for every $L \ge 1$, by (44.22). This implies that

$$\sum_{l=1}^{L} \liminf_{j\to\infty} a_{j,l} \le \liminf_{j\to\infty} \sum_{l=1}^{\infty} a_{j,l}, \tag{44.24}$$

because $\sum_{l=1}^{L} a_{j,l} \le \sum_{l=1}^{\infty} a_{j,l}$ for each L, and (44.22) follows from this by taking the supremum over L.

A Three homework assignments

These homework assignments should be given in the context of Part I, since they could be handled quite differently using results from Part II.

A.1 Maximizing distances

Let $(M, d(x, y))$ be a metric space, let K be a nonempty set contained in M, and let p be an element of M. If K is compact, then there is a point $q \in K$ such that

$$d(p, x) \le d(p, q)$$

for every $x \in K$, which is to say that q maximizes the distance to p among elements of K. For the homework assignment, one is asked to give two proofs of this statement, one based on the definition of compactness in terms of open coverings and the other on the limit point property.

A.2 Minimizing distances

Let $(M, d(x, y))$ be a metric space, let K be a nonempty set contained in M, and let p be an element of M. If K is compact, then there is a point $z \in K$ such that

$$d(p, z) \leq d(p, y)$$

for every $y \in K$, which is to say that z minimizes the distance to p among elements of K. For the homework assignment, one is asked to give two proofs of this statement, one based on the definition of compactness in terms of open coverings and the other on the limit point property.

A.3 Positive lower bounds

Let $(M, d(x, y))$ be a metric space. Suppose that E, K are nonempty subsets of M such that E is closed, K is compact, and E, K are disjoint, i.e.,

$$E \cap K = \emptyset.$$

Show that there is a positive real number η such that

$$d(x, y) \geq \eta$$

for every $x \in E$ and $y \in K$.

References

[1] R. Beals, *Analysis: An Introduction*, Cambridge University Press, 2004.

[2] R. Boas, *A Primer of Real Functions*, 4th edition, revised and with a preface by H. Boas, Mathematical Association of America, 1996.

[3] A. Browder, *Mathematical Analysis: An Introduction*, Springer-Verlag, 1996.

[4] A. Devinatz, *Advanced Calculus*, Holt, Rinehart, and Winston, 1968.

[5] T. Gamelin and R. Greene, *Introduction to Topology*, 2nd edition, Dover, 1999.

[6] R. Goldberg, *Methods of Real Analysis*, 2nd edition, Wiley, 1976.

[7] I. Kaplansky, *Set Theory and Metric Spaces*, 2nd edition, Chelsea, 1977.

[8] T. Körner, *A Companion to Analysis: A Second First and a First Second Course in Analysis*, American Mathematical Society, 2004.

[9] S. Krantz, *The Elements of Advanced Mathematics*, 2nd edition, Chapman & Hall / CRC, 2002.

[10] S. Krantz, *Real Analysis and Foundations*, 2nd edition, Chapman & Hall / CRC, 2005.

[11] S. Krantz, *A Guide to Real Variables*, Mathematical Association of America, 2009.

[12] S. Krantz, *A Guide to Topology*, Mathematical Association of America, 2009.

[13] S. Krantz, *Essentials of Topology with Applications*, CRC Press, 2010.

[14] S. Lang, *Undergraduate Analysis*, 2nd edition, Springer-Verlag, 1997.

[15] B. Mendelson, *Introduction to Topology*, 3rd edition, Dover, 1990.

[16] F. Morgan, *Real Analysis*, American Mathematical Society, 2005.

[17] F. Morgan, *Real Analysis and Applications: Including Fourier Series and the Calculus of Variations*, American Mathematical Society, 2005.

[18] M. Reed, *Fundamental Ideas of Analysis*, Wiley, 1998.

[19] M. Rosenlicht, *Introduction to Analysis* Dover, 1986.

[20] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976.

[21] R. Strichartz, *The Way of Analysis*, revised edition, Jones and Bartlett, 2000.

[22] K. Stromberg, *Introduction to Classical Real Analysis*, Wadsworth, 1981.