Some basic topics related to partial differential equations

Stephen Semmes
Rice University

Preface

Some aspects of history related to calculus and the theory behind it may be found in [25, 49, 50, 51, 52, 93, 94, 108, 128, 129, 136, 137, 138, 139, 140, 143, 158, 198, 305]. See also [44, 126, 227], in connection with Fourier series in particular. Some songs related to calculus may be found in [231, 232, 233], and some remarks concerning the clarity of explanations in mathematics may be found in [142]. The reader may be interested in [42] as well.

Some aspects of history related to multivariable calculus and differential equations may be found in [14, 15, 60, 61, 62, 82, 118, 127, 134, 141, 143, 144, 158, 196, 219, 220, 221, 227, 229, 240, 241]. A number of other texts in the bibliography include some discussion of history too.

Some additional perspectives concerning partial differential equations may be found in [3, 45, 46, 79, 150, 154, 181, 187, 188, 192, 200, 264, 269, 272, 281, 302, 328, 345, 351].

The study of differential equations is closely related to that of mathematical analysis, as indicated in particular in some of the references concerning history mentioned earlier. Some familiarity with analysis would be helpful here, although it is not required. Of course, there are many textbooks in analysis, some of which are mentioned in the bibliography. My colleague Frank Jones' book [186] includes some of the theory related to multivariable calculus, which may be helpful. The reader may also be interested in the brief introductions in [145, 341], as well as Gouvêa's recent text [135] on infinite series. Some aspects of analysis that are related to the topics discussed here may be found in the appendices. Some details are included for the sake of completeness, although some results are merely stated, or are briefly explained somewhat informally.

Some basic familiarity with ordinary differential equations and linear algebra would be helpful here too, but is not required. Some relevant notions may be reviewed as needed, and a number of related references may be found in the bibliography as well. Of course, many treatments of ordinary differential equations include some linear algebra, as does Frank Jones' book [186].

Contents

Chapter 1

Some basic notions

1.1 What is a differential equation?

The title of Section 1.1 of [335] is "What is a partial differential equation?" It seems helpful to begin by asking what an ordinary differential equation is. Part of the answer is that ordinary differential equations deal with functions of one independent variable, while partial differential equations are concerned with functions of more than one independent variable.

A function of a single real variable might frequently be taken to be defined on something like an *interval* in the real line $\mathbf{R}$, particularly when considering solutions of an ordinary differential equation. Various types of intervals are mentioned in Subsection A.1.1. Sometimes one might consider functions defined on other subsets of the real line, such as the function $1/x$, defined for real numbers $x \neq 0$.

Let u be a real-valued function defined on an interval I in $\mathbf{R}$, and suppose that u is differentiable at every point in I. If

$$u'(x) = 0 \tag{1.1.1}$$

for every $x \in I$, then it is well known that

$$u \text{ is constant on } I, \tag{1.1.2}$$

by the mean value theorem.

If x is an endpoint of I that is also an element of I, then the derivative $u'(x)$ of u at x should be considered as a one-sided deriviative, defined using a one-sided limit, when it exists. Sometimes we may be concerned with functions that are differentiable at points in the interior of an interval or half-line, and continuous at an endpoint. This is sufficient for the result mentioned in the preceding paragraph.

1.1.1 First-order ordinary differential equations

A first-order ordinary differential equation may be expressed as

$$F(x, u(x), u'(x)) = 0. \tag{1.1.3}$$

Here F is a function of three variables, and u is a real-valued function of one variable.

These remarks are intended to be a bit informal, and in practice one should probably be a bit more precise about where these functions are defined. A basic possibility is that

$$F(x, p, q) \tag{1.1.4}$$

should be defined for x in an interval I in the real line, and for all real numbers p and q. In this case, the differential equation (1.1.3) would make sense for functions u defined on a subinterval of I.

Of course, one would also like u to be differentiable, so that $u'(x)$ is defined. Sometimes one may be interested in functions as in (1.1.4) that are defined on other sets of (x, p, q), and this may lead to additional restrictions on u.

If x_0 is an element of I and p_0 is a real number, then the corresponding *initial value problem* for (1.1.3) concerns the existence and uniqueness of a solution u defined on a subinterval of I that contains x_0 such that

$$u(x_0) = p_0. \tag{1.1.5}$$

Of course, this means in particular that

$$F(x_0, u(x_0), u'(x_0)) = 0. \tag{1.1.6}$$

Let us express this as

$$F(x_0, p_0, q_0) = 0, \tag{1.1.7}$$

where $q_0 = u'(x_0)$.

1.1.2 Some questions about $F(x, p, q)$

In order to have a solution of (1.1.4) that satisfies (1.1.5), there has to be a real number q_0 that satisfies (1.1.7).

Without additional hypotheses about F, it could be that (1.1.4) does not depend on q. This means that

$$F(x, p, q) = \widetilde{F}(x, p) \tag{1.1.8}$$

for some function $\widetilde{F}(x, p)$ of two variables. In this case, the differential equation (1.1.3) reduces to

$$\widetilde{F}(x, u(x)) = 0, \tag{1.1.9}$$

and does not involve the derivative of u.

Similarly, (1.1.4) might not depend on p or q, so that

$$F(x,p,q) = \widehat{F}(x) \tag{1.1.10}$$

for some function $\widehat{F}(x)$ of one variable. This means that the differential equation (1.1.3) reduces to

$$\widehat{F}(x) = 0, \tag{1.1.11}$$

and does not involve u or its derivative.

If (1.1.8) holds, then (1.1.7) reduces to

$$\widetilde{F}(x_0, p_0) = 0. \tag{1.1.12}$$

One can still look for solutions of (1.1.9) that satisfy (1.1.5). This is related to the *implicit function theorem*, at least for x near x_0. Part of the hypothesis of the implicit function theorem is that the derivative of $\widetilde{F}(x,p)$ in p at (x_0, p_0) not be equal to 0.

1.1.3 A simpler type of ordinary differential equation

Often one considers first-order ordinary differential equations of the form

$$u'(x) = f(x, u(x)), \tag{1.1.13}$$

where f is a function of two variables. We may take

$$f(x,p) \tag{1.1.14}$$

to be defined for x in an interval in the real line again, and for all real numbers p, although sometimes one may want to consider other restrictions on (x,p), as before.

Observe that (1.1.13) is the same as (1.1.3), with

$$F(x,p,q) = q - f(x,p). \tag{1.1.15}$$

In this case, (1.1.7) is the same as saying that

$$q_0 = f(x_0, p_0). \tag{1.1.16}$$

Suppose that we have a differential equation of the form (1.1.3) again, with the initial condition (1.1.5), and a real number q_0 that satisfies (1.1.7). Under some conditions, we may be able to transform (1.1.3) into a differential equation of the form (1.1.13), at least for x close to x_0, using the implicit function theorem. Part of the hypothesis of the implicit function theorem is that

$$\frac{\partial F}{\partial q}(x_0, p_0, q_0) \neq 0, \tag{1.1.17}$$

as before.

1.2 Some other types of first-order equations

Consider a first-order ordinary differential equation of the form

$$a(x, u(x))\, u'(x) = b(x, u(x)), \tag{1.2.1}$$

where a and b are functions of two variables. We may take $a(x, p)$ and $b(x, p)$ to be defined for x in an interval in the real line, and all real numbers p, although one may consider other restrictions on (x, p), as usual.

This equation is the same as (1.1.3), with

$$F(x, p, q) = a(x, p)\, q - b(x, p). \tag{1.2.2}$$

Similarly, (1.1.7) is the same as saying that

$$a(x_0, p_0)\, q_0 = b(x_0, p_0). \tag{1.2.3}$$

Of course, (1.2.1) corresponds to (1.1.13) with

$$f(x, p) = b(x, p)/a(x, p), \tag{1.2.4}$$

as long as the denominator is not zero. In particular, if

$$a(x_0, p_0) \neq 0, \tag{1.2.5}$$

then (1.2.3) is the same as saying that

$$q_0 = b(x_0, p_0)/a(x_0, p_0), \tag{1.2.6}$$

as in (1.1.16). Note that (1.2.5) is the same as (1.1.17) when F is as in (1.2.2).

Otherwise, if

$$a(a_0, p_0) = 0, \tag{1.2.7}$$

then (1.2.3) reduces to

$$b(x_0, p_0) = 0. \tag{1.2.8}$$

If

$$a \equiv 0, \tag{1.2.9}$$

then (1.2.1) is the same as (1.1.9), with

$$\widetilde{F}(x, p) = b(x, p). \tag{1.2.10}$$

1.2.1 Linear first-order equations

A *linear* first-order ordinary differential equation may be expressed as

$$\alpha(x)\, u'(x) + \beta(x)\, u(x) = \gamma(x), \tag{1.2.11}$$

where $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ are functions of one variable. We may take these functions to be defined on an interval in $\mathbf{R}$, as before.

This equation is the same as in (1.2.1), with

$$a(x,p) = \alpha(x) \text{ and } b(x,p) = -\beta(x)\,p + \gamma(x). \tag{1.2.12}$$

Thus (1.2.3) is the same as saying that

$$\alpha(x_0)\,q_0 = -\beta(x_0)\,p_0 + \gamma(x_0) \tag{1.2.13}$$

in this case.

Suppose that $\gamma \equiv 0$, so that (1.2.11) becomes

$$\alpha(x)\,u'(x) + \beta(x)\,u(x) = 0. \tag{1.2.14}$$

This type of equation is said to be *homogeneous*. If u satisfies this equation and c is a real number, then $c\,u$ satisfies the analogous equation

$$\alpha(x)\,(c\,u)'(x) + \beta(x)\,(c\,u)(x) = 0. \tag{1.2.15}$$

Similarly, if v is another real-valued function defined and differentiable on the same interval as u that satisfies the analogous equation

$$\alpha(x)\,v'(x) + \beta(x)\,v(x) = 0, \tag{1.2.16}$$

then $u + v$ satisfies the analogous equation

$$\alpha(x)\,(u+v)'(x) + \beta(x)\,(u+v)(x) = 0. \tag{1.2.17}$$

1.2.2 Some first-order differential operators

Put

$$L(u) = \alpha\,u' + \beta\,u. \tag{1.2.18}$$

More precisely, if u is a differentiable real-valued function defined on an interval on which α and β are defined, then this defines a real-valued function on the same interval. We may also use the notation

$$L = \alpha\,\frac{d}{dx} + \beta, \tag{1.2.19}$$

and refer to this as a first-order *differential operator*,

It is easy to see that this defines a *linear mapping* from the space of differentiable real-valued functions on an interval on which α and β are defined into the space of all real-valued functions on that interval. This means that if u is such a function and c is a real number, then

$$L(c\,u) = c\,L(u). \tag{1.2.20}$$

This also means that if v is another such function on the same interval, then

$$L(u+v) = L(u) + L(v). \tag{1.2.21}$$

If one is familiar with the notion of a *vector space* over the real numbers, then one may know that the space of all real-valued functions on an interval is a vector spaces, with respect to the usual definitions of addition and scalar multiplication of such functions, using pointwise addition and scalar multiplication of functions. The space of differentiable real-valued functions on the same interval is a linear subspace of the space of all real-valued functions on that interval, and is another vector space over the real numbers in particular. The differential operator L determines a linear mapping from this subspace into the space of all real-valued functions on the interval, in the sense of linear algebra, as in the previous paragraph.

Using L, the homogeneous equation (1.2.14) may be expressed as

$$L(u) = 0, \tag{1.2.22}$$

and the inhomogeneous equation (1.2.11) may be expressed as

$$L(u) = \gamma. \tag{1.2.23}$$

The space of solutions to the homogeneous equation on a suitable interval is a linear subspace of the space of all differentiable real-valued functions on that interval, as in the previous subsection. This also follows from the linearity of the corresponding mapping between the appropriate vector spaces, by a standard argument in linear algebra.

This is related to some remarks beginning on p2 of [335]. If u satisfies the inhomogeneous equation (1.2.23), and v is a differentiable real-valued functio on the same interval that satisfies the homogeneous equation

$$L(v) = 0, \tag{1.2.24}$$

then

$$L(u + v) = L(u) + L(v) = \gamma, \tag{1.2.25}$$

so that $u + v$ satisfies the analogous inhomogeneous equation, as mentioned on p3 of [335]. We shall consider analogous notions for functions of more variables, as in [335].

1.3 Invariance under translations

Let us say that a first-order ordinary differential equation as in (1.1.3) is *invariant under translations* if the corresponding function (1.1.4) does not depend on x. This means that (1.1.4) may be expressed as

$$F(x, p, q) = \overline{F}(p, q) \tag{1.3.1}$$

for some function $\overline{F}(p, q)$ of two variables. In this case, the corresponding ordinary differential equation (1.1.3) reduces to

$$\overline{F}(u(x), u'(x)) = 0. \tag{1.3.2}$$

Let u be a real-valued function on an interval I in the real line that is differentiable at every point in I. Also let ξ be a real number, and put

$$I + \xi = \{t + \xi : t \in I\}. \tag{1.3.3}$$

This is another interval in $\mathbf{R}$, and

$$u(x - \xi) \tag{1.3.4}$$

defines a real-valued of x on (1.3.3). Note that (1.3.4) is differentiable at every point in (1.3.3), with

$$\frac{d}{dx}(u(x - \xi)) = u'(x - \xi). \tag{1.3.5}$$

Using this, it is easy to see that u satisfies (1.3.2) on I if and only if (1.3.4) satisfies the same equation on (1.3.3), or equivalently

$$\overline{F}(u(x - \xi), u'(x - \xi)) = 0 \tag{1.3.6}$$

on (1.3.3).

If c is any real number, then the first-order ordinary differential equation

$$F(x, u(x), u'(x)) = c \tag{1.3.7}$$

corresponds to the function

$$F(x, p, q) - c \tag{1.3.8}$$

as in Subsection 1.1.1. Invariance under translations in the sense considered here means that this differential equation is invariant under translations as well.

1.3.1 Some equations of this type

Let ϕ be a real-valued function on the real line, and consider the differential equation

$$\phi(u(x))\, u'(x) = 1. \tag{1.3.9}$$

This is the same as (1.3.2), with

$$\overline{F}(p, q) = \phi(p)\, q - 1. \tag{1.3.10}$$

Suppose that Φ is a differentiable real-valued function on the real line such that

$$\Phi' = \phi \tag{1.3.11}$$

on $\mathbf{R}$. Thus (1.3.9) is the same as saying that

$$\frac{d}{dx}(\Phi(u(x))) = 1, \tag{1.3.12}$$

by the chain rule. A differentiable real-valued function on an interval in $\mathbf{R}$ satisfies this differential equation exactly when

$$\Phi(u(x)) = x + \text{ a constant} \tag{1.3.13}$$

on that interval.

1.3.2 Constant coefficients

A differential operator L as in (1.2.19) is said to have *constant coefficients* if the coefficients α and β are constants. If γ is also a constant, then the corresponding inhomogeneous differential equation (1.2.23) is invariant under translations.

Even if γ is not a constant, the inhomogeneous differential equation (1.2.23), or equivalently (1.2.11), may be described as having *constant coefficients* when α and β are constants. One may also be concerned with linear ordinary differential equations of higher order with constant coefficients, as well as linear partial differential equations with constant coefficients. Many of the equations that we shall consider here are of this type.

1.4 First-order partial differential equations

Let $u(x, y)$ be a real-valued function of two variables x, y. More precisely, u should be a real-valued function defined on a subset of the plane $\mathbf{R}^2$. This is the set of ordered pairs of real numbers, as in Section A.2, with $n = 2$.

In fact, we might typically ask that u be defined on an *open set* in $\mathbf{R}^2$. We shall not get into the definition of an open set here, although it is mentioned in Section B.10. The basic idea should be clear in many examples, such as an open disk, an open rectangle, or something like that.

Suppose that the *partial derivatives* $\partial u/\partial x$ and $\partial u/\partial y$ exist at every point in the domain of u. It is often convenient to use subscripts to denote these partial derivatives, so that

$$u_x = \frac{\partial u}{\partial x} \text{ and } u_y = \frac{\partial u}{\partial y}, \tag{1.4.1}$$

as on p1 of [335], and as in Section B.11. Let us also ask that the partial derivatives of u be continuous on the domain of u, so that u is *continuously differentiable*, as in Subsection B.11.1. This implies in particular that u is continuous on its domain, as mentioned in Subsection B.11.1. Note that derivatives under consideration in [335] are normally asked to be continuous, as mentioned in item 2 on p4 of [335].

1.4.1 Equations in two variables

A first-order partial differential equation for functions of two variables may be expressed as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0, \tag{1.4.2}$$

as in (1) on p1 of [335]. Here

$$F(x, y, p, q, r) \tag{1.4.3}$$

is a function of the five real variables x, y, p, q, and r. As in Subsection 1.1.1, a basic possibility is that (1.4.3) is defined for all (x, y) in a subset of $\mathbf{R}^2$ that contains the domain of u, and all real numbers p, q, r. As before, one may be interested in functions as in (1.4.3) that are defined on other sets of (x, y, p, q, r), and this may lead to additional restrictions on u.

It is sometimes convenient to express (1.4.2) more succinctly as

$$F(x, y, u, u_x, u_y) = 0, \tag{1.4.4}$$

as in (1) on p1 of [335] again. It is implicit here that u and its derivatives are supposed to be evaluated at (x, y), as in (1.4.2). In particular, partial differential equations in more variables, or involving more derivatives of u, may be expressed in this way.

1.4.2 Linear first-order equations in two variables

Let $a(x, y)$, $b(x, y)$, $c(x, y)$, and $d(x, y)$ be real-valued functions defined on a subset of $\mathbf{R}^2$. Using these functions, we get the *linear* first-order partial differential equation

$$a(x, y)\, u_x(x, y) + b(x, y)\, u_y(x, y) + c(x, y)\, u(x, y) = d(x, y). \tag{1.4.5}$$

This is the same as (1.4.2), with

$$F(x, y, p, q, r) = a(x, y)\, q + b(x, y)\, r + c(x, y)\, p - d(x, y). \tag{1.4.6}$$

Consider the corresponding first-order *differential operator*

$$L = a\, \frac{\partial}{\partial x} + b\, \frac{\partial}{\partial y} + c, \tag{1.4.7}$$

as in Subsection 1.2.2. If the domain of u is contained in a subset of $\mathbf{R}^2$ on which a, b, and c are defined, then

$$L(u) = a\, u_x + b\, u_y + c\, u \tag{1.4.8}$$

is defined as a real-valued function on the domain of u. This defines a linear mapping between suitable vector spaces of functions, as before.

Using L, (1.4.5) may be expressed as

$$L(u) = d. \tag{1.4.9}$$

If $d \equiv 0$, then we get the corresponding *homogeneous* equation

$$a(x, y)\, u_x(x, y) + b(x, y)\, u_y(x, y) + c(x, y)\, u(x, y) = 0. \tag{1.4.10}$$

This may be expressed as

$$L(u) = 0, \tag{1.4.11}$$

as before. This determines a linear subspace of the space of continuously-differentiable real-valued functions on a fixed open set in $\mathbf{R}^2$, as long as a, b, and c are defined on that open set, as before.

1.5 More on invariance under translations

Let us say that a first-order partial differential equation as in (1.4.2) is *invariant under translations* if the corresponding function (1.4.3) does not depend on x or y, so that it may be expressed as

$$F(x, y, p, q, r) = \overline{F}(p, q, r) \tag{1.5.1}$$

for some function $\overline{F}(p, q, r)$ of three variables. This means that the corresponding partial differential equation (1.4.2) reduces to

$$\overline{F}(u(x, y), u_x(x, y), u_y(x, y)) = 0. \tag{1.5.2}$$

Let u be a continuously-differentiable real-valued function on an open subset U of $\mathbf{R}^2$, and let ξ, η be real numbers. Put

$$U + (\xi, \eta) = \{(x, y) + (\xi, \eta) = (x + \xi, y + \eta) : (x, y) \in U\}, \tag{1.5.3}$$

which is another open set in $\mathbf{R}^2$. If U is an open disk in $\mathbf{R}^2$, for instance, then this is an open disk of the same radius, with the center of the initial disk translated by (ξ, η). Similarly, if U is an open rectangle in $\mathbf{R}^2$, then this is an open rectangle with the same sidelengths, and the center of the initial rectangle translated by (ξ, η).

Observe that

$$u(x - \xi, y - \eta) \tag{1.5.4}$$

defines a real-valued function of (x, y) on (1.5.3). The partial derivatives of this function are equal to the corresponding translates of the partial derivatives of u, so that

$$\frac{\partial}{\partial x}(u(x - \xi, y - \eta)) = u_x(x - \xi, y - \eta) \tag{1.5.5}$$

and

$$\frac{\partial}{\partial y}(u(x - \xi, y - \eta)) = u_y(x - \xi, y - \eta) \tag{1.5.6}$$

for all (x, y) in (1.5.3), as in the one-variable case.

In particular, (1.5.4) is continuously differentiable on (1.5.3), because u is continuously differentiable on U, by hypothesis. Using (1.5.5) and (1.5.6), we also have that u satisfies (1.5.2) on U if and only if (1.5.4) satisfies the same equation on (1.5.3), so that

$$\overline{F}(u(x - \xi, y - \eta), u_x(x - \xi, y - \eta), u_y(x - \xi, y - \eta)) = 0 \tag{1.5.7}$$

on (1.5.3).

If c is any real number, then the first-order partial differential equation

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = c \tag{1.5.8}$$

corresponds to the function

$$F(x, y, p, q, r) - c \tag{1.5.9}$$

in the same way as before. Invariance under translations in the sense considered here means that this partial differential equation is invariant under translations too.

1.5.1 More on constant coefficients

A differential operator L as in (1.4.7) is said to have *constant coefficients* if the coefficients a, b, and c are constants. If d is a constant as well, then the corresponding inhomogeneous partial differential equation (1.4.9) is invariant under translations.

The inhomogeneous equation (1.4.9), or equivalently (1.4.5), may be described as having *constant coefficients* when a, b and c are constants, even if d is not a constant.

1.6 Second-order partial differential equations

Let $u(x, y)$ be a real-valued function of two variables x and y defined on an open subset U of $\mathbf{R}^2$ again. Suppose that the first partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ exist at every point in U, as well as the second derivatives

$$\frac{\partial^2 u}{\partial x^2}, \ \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big), \ \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x}\Big), \text{ and } \frac{\partial^2 u}{\partial y^2}. \tag{1.6.1}$$

More precisely, we ask that all of these second derivatives be continuous on U, so that u is *twice continuously differentiable* on U, as in Subsection B.11.3.

This is the same as saying that the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are continuously differentiable on U, as in Subsection B.11.3. This implies that $\partial u/\partial x$ and $\partial u/\partial y$ are continuous on U, as mentioned in Subsection B.11.1, and near the beginning of the previous section. This means that u is continuously differentiable on U, so that u is continuous on U as well, as before.

It is well known that

$$\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big) = \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x}\Big) \tag{1.6.2}$$

on U, because u is twice continuously differentiable on U, as mentioned in Subsection B.11.3. These second derivatives may be denoted

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x} \tag{1.6.3}$$

or

$$u_{xy} = u_{yx}. \tag{1.6.4}$$

Similarly, we put

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} \text{ and } u_{yy} = \frac{\partial^2 u}{\partial y^2}, \tag{1.6.5}$$

as in Subsection B.11.3.

1.6.1 Equations in two variables again

A second-order partial differential equation for functions of two variables may be expressed as

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{1.6.6}$$

as in (2) on p1 of [335]. Here

$$F(x, y, p, q, r, t, v, w) \tag{1.6.7}$$

is a function of the eight real variables x, y, p, q, r, t, v, and w. As before, a basic possibility is that (1.6.7) is defined for all (x, y) in a subset of $\mathbf{R}^2$ that contains U, and for all real values of the other variables. Sometimes one may be interested in functions as in (1.6.7) that are defined on other sets of (x, y, p, q, r, t, v, w), and this may lead to additional restrictions on u.

As in Subsection 1.4.1, it is implicit here that u and its derivatives are supposed to be evaluated at (x, y) in (1.6.7). Partial differential equations in more variables, or involving more derivatives of u, may be expressed similarly, as before. Of course, for equations involving more derivatives of u, one would normally ask that U satisfy appropriate differentiability properties on U.

Appendix A

A bit of analysis

A.1 The real line

Let $\mathbf{R}$ be the real line, which is to say the set of real numbers. If $x \in \mathbf{R}$, then the *absolute value* is defined by

$$\text{(A.1.1)} \qquad \begin{aligned} |x| &= x \quad \text{when } x \geq 0 \\ &= -x \quad \text{when } x \leq 0. \end{aligned}$$

It is easy to see that

$$\text{(A.1.2)} \qquad |x\,y| = |x|\,|y|$$

for every $x, y \in \mathbf{R}$. One can also check that

$$\text{(A.1.3)} \qquad |x + y| \leq |x| + |y|$$

for every $x, y \in \mathbf{R}$. This is a version of the *triangle inequality.*

If $x, y \in \mathbf{R}$, then the distance from x to y with respect to the *standard Euclidean metric* is defined by

$$\text{(A.1.4)} \qquad d(x, y) = |x - y|.$$

If z is another real number, then

$$\text{(A.1.5)} \qquad \begin{aligned} d(x, z) &= |x - z| = |(x - y) + (y - z)| \\ &\leq |x - y| + |y - z| = d(x, y) + d(y, z). \end{aligned}$$

This is another version of the triangle inequality.

A.1.1 Intervals in R

Let a, b be real numbers with $a < b$. The *open interval* in $\mathbf{R}$ from a to b is defined by

$$\text{(A.1.6)} \qquad (a, b) = \{x \in \mathbf{R} : a < x < b\}.$$

Similarly, the *closed interval* from a to b is defined by

$$[a, b] = \{x \in \mathbf{R} : a \le x \le b\}, \tag{A.1.7}$$

which may also be used when $a = b$. Sometimes one may wish to consider the corresponding half-open, half-closed intervals

$$[a, b) = \{x \in \mathbf{R} : a \le x < b\} \tag{A.1.8}$$

and

$$(a, b] = \{x \in \mathbf{R} : a < x \le b\} \tag{A.1.9}$$

as well.

It is sometimes convenient to use the notation

$$(a, +\infty) = \{x \in \mathbf{R} : a < x\} \tag{A.1.10}$$

and

$$(-\infty, b) = \{x \in \mathbf{R} : x < b\}. \tag{A.1.11}$$

These may be described as *open half-lines* in $\mathbf{R}$. We may also put

$$(-\infty, +\infty) = \mathbf{R}. \tag{A.1.12}$$

These may be considered as unbounded open intervals in $\mathbf{R}$.

Similarly, we may use the notation

$$[a, +\infty) = \{x \in \mathbf{R} : a \le x\} \tag{A.1.13}$$

and

$$(-\infty, b] = \{x \in \mathbf{R} : x \le b\}. \tag{A.1.14}$$

These may be described as *closed half-lines* in $\mathbf{R}$. Sometimes $+\infty$, $-\infty$ are considered as *extended real numbers*, with

$$-\infty < x < +\infty \tag{A.1.15}$$

for every $x \in \mathbf{R}$.

A.2 Euclidean spaces

Let n be a positive integer, and let $\mathbf{R}^n$ be the usual space of (ordered) n-tuples of real numbers. This means that each $x \in \mathbf{R}^n$ may be expressed as

$$x = (x_1, \dots, x_n), \tag{A.2.1}$$

with $x_j \in \mathbf{R}$ for $j = 1, \dots, n$. This is the same as the real line when $n = 1$.

If $x, y \in \mathbf{R}^n$, then $x + y$ is defined as an element of $\mathbf{R}^n$ by

$$x + y = (x_1 + y_1, \dots, x_n + y_n). \tag{A.2.2}$$

If $t \in \mathbf{R}$, then $t\,x$ is defined as an element of $\mathbf{R}^n$ by

$$t\,x = (t\,x_1, \ldots, t\,x_n). \tag{A.2.3}$$

It is well known that $\mathbf{R}^n$ is a *vector space* over the real numbers, with addition and sclalar multiplication defined in this way. This basically means that addition and scalar multiplication on $\mathbf{R}^n$ satisfy some standard properties, such as commutativity and associaitvity of addition and distributivity of scalar multiplication, that follow easily from analogous properties of addition and multiplication of real numbers.

A.2.1 The standard Euclidean norm and metric on $\mathbf{R}^n$

If $x \in \mathbf{R}^n$, then the *standard Euclidean norm* is the nonnegative real number defined by

$$|x| = \Big(\sum_{j=1}^{n} x_j^2\Big)^{1/2}. \tag{A.2.4}$$

The is the same as the absolute value of a real number when $n = 1$. Note that $|x| = 0$ if and only if $x = 0$, which means that $x_j = 0$ for $j = 1, \ldots, n$.

It is easy to see that

$$|t\,x| = |t|\,|x| \tag{A.2.5}$$

for every $t \in \mathbf{R}$, where $|t|$ is the absolute value of t, as in the previous section. If y is another element of $\mathbf{R}^n$, then it is well known that

$$|x + y| \le |x| + |y|. \tag{A.2.6}$$

This is another version of the *triangle inequality*, which is more complicated when $n \ge 2$. A proof of this will be discussed in the next section.

If $x, y \in \mathbf{R}^n$, then the distance from x to y with respect to the *standard Euclidean metric* is defined by

$$d(x, y) = |x - y|, \tag{A.2.7}$$

which is the same as before when $n = 1$. If $z \in \mathbf{R}^n$ too, then

$$d(x, z) \le d(x, y) + d(y, z), \tag{A.2.8}$$

because of the triangle inequality for the standard Euclidean norm on $\mathbf{R}^n$, as before. This is the triangle inequality for the standard Euclidean metric on $\mathbf{R}^n$.

A.2.2 Open and closed balls

Let $x \in \mathbf{R}^n$ and a positive real number r be given. The *open ball* in $\mathbf{R}^n$ centered at x with radius r with respect to the standard Euclidean metric is defined by

$$B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}. \tag{A.2.9}$$

Similarly, the *closed ball* in $\mathbf{R}^n$ centered at x with radius r with respect to the standard Euclidean metric is defined by

$$\overline{B}(x, r) = \{y \in \mathbf{R}^n : |x - y| \le r\}, \tag{A.2.10}$$

which may also be used when $r = 0$.

If $n = 1$, then (A.2.9) is the same as the open interval $(x - r, x + r)$, and (A.2.10) is the same as the closed interval $[x - r, x + r]$.

Observe that the standard Euclidean metric on $\mathbf{R}^n$ is *invariant under translations*, in the sense that

$$d(x + a, y + a) = |(x + a) - (y + a)| = |x - y| = d(x, y) \tag{A.2.11}$$

for all $a, x, y \in \mathbf{R}^n$. This implies that translates of open balls in $\mathbf{R}^n$ are open balls of the same radius. Similarly, translates of closed balls in $\mathbf{R}^n$ are closed balls of the same radius.

A.3 The dot product on $\mathbf{R}^n$

Let n be a positive integer, and let $x, y \in \mathbf{R}^n$ be given. The *dot product* of x and y is the real number defined by

$$x \cdot y = \sum_{j=1}^{n} x_j \, y_j. \tag{A.3.1}$$

This is also known as the *standard inner product* on $\mathbf{R}^n$. Note that the dot product is symmetric in x and y, which is to say that

$$x \cdot y = y \cdot x. \tag{A.3.2}$$

We also have that

$$x \cdot x = \sum_{j=1}^{n} x_j^2 = |x|^2, \tag{A.3.3}$$

where $|x|$ is the standard Euclidean norm of x, as in (A.2.4).

If t is a real number, then it is easy to see that

$$(t\, x) \cdot y = x \cdot (t\, y) = t\, (x \cdot y). \tag{A.3.4}$$

If w is another element of $\mathbf{R}^n$, then

$$(x + w) \cdot y = x \cdot y + w \cdot y \tag{A.3.5}$$

and

$$x \cdot (y + w) = x \cdot y + x \cdot w. \tag{A.3.6}$$

Of course,

$$x \cdot y = 0 \tag{A.3.7}$$

when $x = 0$ or $y = 0$.

A.3.1 The Cauchy–Schwarz inequality

The *Cauchy–Schwarz inequality* states that

$$|x \cdot y| \le |x|\,|y|, \tag{A.3.8}$$

using the absolute value of $x \cdot y$ as a real numbers on the left side, and the standard Euclidean norms on x and y on the right side. Note that (A.3.8) holds trivially when $x = 0$ or $y = 0$. Two proofs of the Cauchy–Schwarz inequality when $x, y \ne 0$ will be given in the next section.

It is easy to see that equality holds in (A.3.8) when $x = t\,y$ for some $t \in \mathbf{R}$. The proof will show that this is the only way to get equality in (A.3.8) when $y \ne 0$.

A.3.2 Using the Cauchy–Schwarz inequality

Let us see how to use the Cauchy–Schwarz inequality to get the triangle inequality (A.2.6) for the standard Euclidean norm on $\mathbf{R}^n$.

Observe that

$$\begin{aligned} |x+y|^2 = (x+y)\cdot(x+y) &= x \cdot x + 2\,x \cdot y + y \cdot y \\ &= |x|^2 + 2\,x \cdot y + |y|^2. \end{aligned} \tag{A.3.9}$$

Using the Cauchy–Schwarz inequality, we get that

$$|x+y|^2 \le |x|^2 + 2\,|x|\,|y| + |y|^2 = (|x| + |y|)^2. \tag{A.3.10}$$

Of course, this implies (A.2.6).

More precisely, equality holds in (A.3.10) if and only if

$$x \cdot y = |x|\,|y|. \tag{A.3.11}$$

This means that equality holds in (A.2.6) if and only if (A.3.11) holds. The proof of the Cauchy–Schwarz inequality will show that (A.3.11) holds only when $y = 0$ or $x = t\,y$ for some nonnegative real number t.

A.4 Proving the Cauchy–Schwarz inequality

Let n be a positive integer again, and let $x, y \in \mathbf{R}^n$ be given. One way to prove the Cauchy–Schwarz inequality is to use the fact that

$$(x - t\,y)\cdot(x - t\,y) = |x - t\,y|^2 \ge 0 \tag{A.4.1}$$

for every $t \in \mathbf{R}$. This implies that

$$x \cdot x - 2\,t\,(x \cdot y) + t^2\,(y \cdot y) \ge 0, \tag{A.4.2}$$

which is the same as saying that

$$|x|^2 - 2\,t\,(x \cdot y) + t^2\,|y|^2 \ge 0. \tag{A.4.3}$$

Equivalently, this means that

$$2\,t\,(x \cdot y) \le |x|^2 + t^2\,|y|^2. \tag{A.4.4}$$

Observe that these inequalities are strict when $x \ne t\,y$.

We may as well suppose that $x, y \ne 0$, as before. If we choose t so that

$$|t| = |x|/|y|, \tag{A.4.5}$$

then we get that

$$2\,(t/|t|)\,(x \cdot y) \le |x|^2/|t| + |t|\,|y|^2 = 2\,|x|\,|y|. \tag{A.4.6}$$

We may also choose t so that

$$(t/|t|)\,(x \cdot y) = |x \cdot y|. \tag{A.4.7}$$

Using this, (A.3.8) follows from (A.4.7).

More precisely, this argument shows that if $x, y \ne 0$ and equality holds in (A.3.8), then

$$x = t\,y \tag{A.4.8}$$

for some $t \in \mathbf{R}$.

A.4.1 Another approach to the Cauchy–Schwarz inequality

If a and b are real numbers, then

$$a^2 - 2\,a\,b + b^2 = (a - b)^2 \ge 0. \tag{A.4.9}$$

This implies the well-known inequality

$$a\,b \le (1/2)\,(a^2 + b^2). \tag{A.4.10}$$

These inequalities are strict when $a \ne b$.

It follows that

$$x \cdot y \le \sum_{j=1}^{n} (1/2)\,(x_j^2 + y_j^2) = (1/2)\,(|x|^2 + |y|^2). \tag{A.4.11}$$

This inequlity is strict when $x_j \ne y_j$ for some j, so that $x \ne y$.

In particular,

$$x \cdot y \le 1 \tag{A.4.12}$$

when

$$|x|, |y| \le 1. \tag{A.4.13}$$

The inequality in (A.4.12) is strict when $x \ne y$, and when $|x| < 1$ or $|y| < 1$.

This shows that

$$x \cdot y \le |x|\,|y| \tag{A.4.14}$$

when $|x| = |y| = 1$. If $x, y \ne 0$, then one can reduce to this case, using $x/|x|$ and $y/|y|$. We also get that the inequality is strict when $x/|x| \ne y/|x|$.

Observe that

$$-x \cdot y = (-x) \cdot y \le |-x|\,|y| = |x|\,|y|. \tag{A.4.15}$$

The Cauchy–Schwarz inequality is the same as the combination of this and the previous inequality.

A.5 Some other norms on $\mathbf{R}^n$

Let n be a positive integer, and let N be a nonnegative real-valued function on $\mathbf{R}^n$. We say that N is a *norm* on $\mathbf{R}^n$ if it satisfies the following three conditions. First, for each $x \in \mathbf{R}^n$, we have that

$$N(x) = 0 \text{ if and only if } x = 0. \tag{A.5.1}$$

Second,

$$N(t\,x) = |t|\,N(x) \tag{A.5.2}$$

for every $t \in \mathbf{R}$ and $x \in \mathbf{R}^n$. Third,

$$N(x + y) \le N(x) + N(y) \tag{A.5.3}$$

for every $x, y \in \mathbf{R}^n$, which is the triangle inequality for a norm.

The standard Euclidean norm on $\mathbf{R}^n$ is a norm on this sense. One can check that

$$\|x\|_1 = \sum_{j=1}^{n} |x_j| \tag{A.5.4}$$

also defines a norm on $\mathbf{R}^n$. One can verify that

$$\|x\|_\infty = \max_{1 \le j \le n} |x_j| \tag{A.5.5}$$

defines a norm on $\mathbf{R}^n$ as well. The standard Euclidean norm on $\mathbf{R}^n$ may be denoted $\|x\|_2$, to be consistent with this notation. Note that these three norms are the same as the absolute value of a real number when $n = 1$.

A.5.1 Comparing these three norms

If $x \in \mathbf{R}^n$, then it is easy to see that

$$\|x\|_\infty \le \|x\|_1 \tag{A.5.6}$$

and

$$\|x\|_\infty \le \|x\|_2. \tag{A.5.7}$$

One can also check that

$$\|x\|_1 \le n\,\|x\|_\infty \tag{A.5.8}$$

and

$$\|x\|_2 \le n^{1/2}\,\|x\|_\infty. \tag{A.5.9}$$

Observe that

$$\|x\|_2^2 = \sum_{j=1}^n x_j^2 \le \|x\|_1\,\|x\|_\infty \le \|x\|_1^2, \tag{A.5.10}$$

using (A.5.6) in the third step. This implies that

$$\|x\|_2 \le \|x\|_1. \tag{A.5.11}$$

One can use the Cauchy–Schwarz inequality to get that

$$\|x\|_1 \le \|x\|_2 \Big(\sum_{j=1}^n 1\Big)^{1/2} = n^{1/2}\,\|x\|_2. \tag{A.5.12}$$

A.6 Metrics associated to norms

Let n be a positive integer, and let N be a norm on $\mathbf{R}^n$. If $x, y \in \mathbf{R}^n$, then the distance from x to y with respect to N may be defined by

$$d_N(x,y) = N(x-y). \tag{A.6.1}$$

Observe that

$$d_N(x,y) = 0 \text{ if and only if } x = y, \tag{A.6.2}$$

because of (A.5.1). Similarly,

$$d_N(x,y) = N(x-y) = N(y-x) = d_N(y,x), \tag{A.6.3}$$

using (A.5.2) in the second step, with $t = -1$. If $z \in \mathbf{R}^n$ too, then

$$d_N(x,z) = N(x-z) \le N(x-y) + N(y-z) = d_N(x,y) + d_N(y,z), \tag{A.6.4}$$

using (A.5.3) in the second step.

In fact, one can define the notion of a metric on any set. The remarks in the preceding paragraph imply that d_N defines a metric on $\mathbf{R}^n$.

Note that d_N is *invariant under translations* on $\mathbf{R}^n$, in the sense that

$$d_N(x+a, y+a) = N((x+a) - (y+a)) = N(x-y) = d_N(x,y) \tag{A.6.5}$$

for all $a, x, y \in \mathbf{R}^n$.

A.6.1 The metrics associated to $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$

In particular,

$$d_1(x,y) = \|x-y\|_1 \tag{A.6.6}$$

and

$$d_\infty(x,y) = \|x-y\|_\infty \tag{A.6.7}$$

are the metrics on $\mathbf{R}^n$ associated to the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ in (A.5.4) and (A.5.5), respectively. The metric

$$d_2(x,y) = \|x-y\|_2 \tag{A.6.8}$$

on $\mathbf{R}^n$ associated to the standard Euclidean norm $\|\cdot\|_2$ is the same as the standard Euclidean metric.

Observe that

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y) \tag{A.6.9}$$

for all $x, y \in \mathbf{R}^n$, because of (A.5.7) and (A.5.11). We also have that

$$d_2(x,y) \le n^{1/2}\, d_\infty(x,y) \tag{A.6.10}$$

and

$$d_1(x,y) \le n^{1/2}\, d_2(x,y) \tag{A.6.11}$$

for every $x, y \in \mathbf{R}^n$, by (A.5.9) and (A.5.12).

A.6.2 Open and closed balls with respect to a norm

Let N be a nrom on $\mathbf{R}^n$ again, and let $x \in \mathbf{R}^n$ and a positive real number r be given. The *open ball* in $\mathbf{R}^n$ centered at x with radius r with respect to N, or equivalently with respect to the metric d_N associated to N, is defined by

$$B_N(x,r) = B_{d_N}(x,r) = \{y \in \mathbf{R}^n : N(x-y) < r\}. \tag{A.6.12}$$

Similarly, the *closed ball* in $\mathbf{R}^n$ centered at x with radius r with respect to N, or equivalently with respect to d_N, is defined by

$$\overline{B}_N(x,r) = \overline{B}_{d_N}(x,r) = \{y \in \mathbf{R}^n : N(x-y) \le r\}. \tag{A.6.13}$$

Let us use $B_{d_1}(x,r)$, $B_{d_2}(x,r)$, and $B_{d_\infty}(x,r)$ for the open balls corresponding to the metrics d_1, d_2, and d_∞ associated to the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$, respectively, and similarly for closed balls. It is easy to see that

$$B_{d_1}(x,r) \subseteq B_{d_2}(x,r) \subseteq B_{d_\infty}(x,r) \tag{A.6.14}$$

and

$$\overline{B}_{d_1}(x,r) \subseteq \overline{B}_{d_2}(x,r) \subseteq \overline{B}_{d_\infty}(x,r), \tag{A.6.15}$$

using (A.6.9).

We also have that

$$B_{d_\infty}(x, r) \subseteq B_{d_2}(x, n^{1/2}\, r) \tag{A.6.16}$$

and

$$\overline{B}_{d_\infty}(x, r) \subseteq \overline{B}_{d_2}(x, n^{1/2}\, r), \tag{A.6.17}$$

because of (A.6.10). Similarly,

$$B_{d_2}(x, r) \subseteq B_{d_1}(x, n^{1/2}\, r) \tag{A.6.18}$$

and

$$\overline{B}_{d_2}(x, r) \subseteq \overline{B}_{d_1}(x, n^{1/2}\, r), \tag{A.6.19}$$

because of (A.6.11).

A.6.3 Convex subsets of $\mathbf{R}^n$

A subset E of $\mathbf{R}^n$ is said to be *convex* if for any pair of elements w, z of E and real number t with $0 < t < 1$, we have that

$$(1 - t)\, w + t\, z \in E. \tag{A.6.20}$$

If N is a norm on $\mathbf{R}^n$, then one can check that the open and closed balls in $\mathbf{R}^n$ with respect to N are convex.

A.7 Complex numbers

A *complex number* z can be expressed in a unique way as $z = x + y\, i$, where $x, y \in \mathbf{R}$ and $i^2 = -1$. In this case, x and y are known as the *real* and *imaginary parts* of z, and may be denoted $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively.

Addition and multiplication of real numbers can be extended to the set $\mathbf{C}$ of complex numbers in a standard way.

A.7.1 Complex conjugates

The *complex conjugate* of z is the complex number defined by

$$\overline{z} = x - y\, i. \tag{A.7.1}$$

If w is another complex number, then one can check that

$$\overline{z + w} = \overline{z} + \overline{w} \tag{A.7.2}$$

and

$$\overline{(z\, w)} = \overline{z}\, \overline{w}. \tag{A.7.3}$$

Note that the complex conjugate of $\overline{z}$ is z.

It is easy to see that

$$\operatorname{Re} z = (1/2)\, (z + \overline{z}) \tag{A.7.4}$$

and

$$\operatorname{Im} z = (1/(2\, i))\, (z - \overline{z}) \tag{A.7.5}$$

A.7.2 Absolute values

The *absolute value* or *modulus* is the nonnegative real number defined by

$$|z| = (x^2 + y^2)^{1/2}. \tag{A.7.6}$$

Clearly

$$|\overline{z}| = |z|. \tag{A.7.7}$$

It is easy to see that

$$z\,\overline{z} = |z|^2. \tag{A.7.8}$$

It follows that

$$|z\,w|^2 = z\;w\,\overline{(z\,w)} = z\,w\,\overline{z}\,\overline{w} = z\,\overline{z}\,w\,\overline{w} = |z|^2\,|w|^2. \tag{A.7.9}$$

This means that

$$|z\,w| = |z|\,|w|. \tag{A.7.10}$$

If $z \neq 0$, then $|z| > 0$, and

$$z\,(\overline{z}/|z|^2) = (z\,\overline{z})/|z|^2 = 1. \tag{A.7.11}$$

This implies that z has a multiplicative inverse in $\mathbf{C}$, with

$$1/z = \overline{z}/|z|^2. \tag{A.7.12}$$

A.7.3 Some inequalities

Observe that

$$|\operatorname{Re} z|,\ |\operatorname{Im} z| \le |z|. \tag{A.7.13}$$

More precisely, $|z|$ is equal to $|\operatorname{Re} z|$ or $|\operatorname{Im} z|$ exactly when z is real or purely imaginary, as appropriate.

It follows that

$$|\operatorname{Re}(z\,\overline{w})| \le |z\,\overline{w}| = |z|\,|\overline{w}| = |z|\,|w|. \tag{A.7.14}$$

Equality holds if and only if $z\,\overline{w}$ is a real number, as in the preceding paragraph.

We also have that

$$\begin{aligned} |z + w|^2 &= (z + w)\,\overline{(z + w)} = (z + w)\,(\overline{z} + \overline{w}) \\ &= z\,\overline{z} + z\,\overline{w} + \overline{z}\,w + w\,\overline{w} = |z|^2 + 2\ \operatorname{Re}(z\overline{w}) + |w|^2. \end{aligned} \tag{A.7.15}$$

Combining this with (A.7.14), we get that

$$|z + w|^2 \le |z|^2 + 2\,|z|\,|w| + |w|^2 = (|z| + |w|)2. \tag{A.7.16}$$

Of course, this means that

$$|z + w| \le |z| + |w|. \tag{A.7.17}$$

One can verify that equality holds if and only if $z\,\overline{w}$ is a nonnegative real number.

A.7.4 Comparison with $\mathbf{R}^2$

If $z = x + y\,i$ with $x, y \in \mathbf{R}$, as before, then z corresponds in a simple way to (x, y) as an element of $\mathbf{R}^2$. This correspondence is compatible with addition on $\mathbf{C}$ and $\mathbf{R}^2$, as well as multiplication by a real number. Similarly, the absolute value of a complex number corresponds exactly to the standard Euclidean norm on $\mathbf{R}^2$ in this way.

Suppose that $w = u + v\,i$ with $u, v \in \mathbf{R}$, so that w corresponds to (u, v) as an element of $\mathbf{R}^2$ in the same way. Observe that

$$\mathrm{Re}(z\overline{w}) = x\,u + y\,v = (x, y) \cdot (u, v), \tag{A.7.18}$$

where the right side uses the dot product on $\mathbf{R}^2$. Thus (A.7.14) corresponds to the Cauchy–Schwarz inequality on $\mathbf{R}^2$.

Note that

$$d(z, w) = |z - w| \tag{A.7.19}$$

corresponds to the standard Euclidean metric on $\mathbf{R}^2$ in this way.

A.8 Convergent sequences

Let n be a positive integer, and let $x(1), x(2), x(3), \ldots$ be an infinite sequence of elements of $\mathbf{R}^n$. Thus, for each positive integer l, $x(l)$ may be expressed as

$$x(l) = (x_1(l), x_2(l), \ldots, x_n(l)), \tag{A.8.1}$$

where $x_j(l)$ is a real number for each $j = 1, 2, \ldots, n$.

A.8.1 Convergent sequences in $\mathbf{R}^n$

We say that $\{x(l)\}_{l=1}^\infty$ *converges* to an element x of $\mathbf{R}^n$ if for every positive real number ϵ there is a positive integer L such that

$$|x(l) - x| < \epsilon \tag{A.8.2}$$

for every $l \geq L$.

In this case, x is said to be the *limit* of $\{x(l)\}_{l=1}^\infty$, which may be expressed by

$$\lim_{l \to \infty} x(l) = x \tag{A.8.3}$$

or

$$x(l) \to \infty \text{ as } l \to \infty. \tag{A.8.4}$$

More precisely, it is well known and not too difficult to show that the limit of a convergent sequence is unique.

In particular, this reduces to the usual definition of a convergent sequence of real numbers when $n = 1$. It is also well known and not too difficult to show that (A.8.3) holds if and only if

$$\lim_{l \to \infty} x_j(l) = x_j \tag{A.8.5}$$

for each $j = 1, 2, \ldots, n$, as a sequence of real numbers.

A.8.2 Bounded sequences in $\mathbf{R}^n$

A sequence $\{x(l)\}_{l=1}^\infty$ of elements of $\mathbf{R}^n$ is said to be *bounded* if there is a nonnegative real number C such that

$$|x(l)| \le C \tag{A.8.6}$$

for every $l \ge 1$. It is well known and not difficult to check that convergent sequences in $\mathbf{R}^n$ are bounded.

If $\{x(l)\}_{l=1}^\infty$ converges to $x \in \mathbf{R}^n$, then it is well known and not too difficult to verify that

$$\lim_{l\to\infty} |x(l)| = |x|, \tag{A.8.7}$$

as a sequence of real numbers.

A.8.3 Monotone sequences of real numbers

Let $\{x_j\}_{j=1}^\infty$ be a sequence of real numbers. A real number a is said to be a *lower bound* for $\{x_j\}_{j=1}^\infty$ if

$$a \le x_j \tag{A.8.8}$$

for each $j \ge 1$. Similarly, a real number b is said to be an *upper bound* for $\{x_j\}_{j=1}^\infty$ if

$$x_j \le b \tag{A.8.9}$$

for each $j \ge 1$. Note that $\{x_j\}_{j=1}^\infty$ is bounded in $\mathbf{R}$, as in the previous subsection, if and only if $\{x_j\}_{j=1}^\infty$ has both an upper and a lower bound in $\mathbf{R}$.

Suppose for the moment that $\{x_j\}_{j=1}^\infty$ is *monotonically increasing*, so that

$$x_j \le x_{j+1} \tag{A.8.10}$$

for each $j \ge 1$. In particular, this implies that x_1 is a lower bound for $\{x_j\}_{j=1}^\infty$. If $\{x_j\}_{j=1}^\infty$ has an upper bound in $\mathbf{R}$, then it is well known that $\{x_j\}_{j=1}^\infty$ converges in $\mathbf{R}$. In this case, the limit of $\{x_j\}_{j=1}^\infty$ is an upper bound for $\{x_j\}_{j=1}^\infty$, and in fact

$$\lim_{j\to\infty} x_j \text{ is the smallest upper bound for } \{x_j\}_{j=1}^\infty. \tag{A.8.11}$$

This is known as the *least upper bound* or *supremum* of the set of x_j's, $j \ge 1$, which may be expressed as

$$\sup_{j\ge 1} x_j. \tag{A.8.12}$$

Suppose now that $\{x_j\}_{j=1}^\infty$ is *monotonically decreasing*, so that

$$x_j \ge x_{j+1} \tag{A.8.13}$$

for each $j \ge 1$. This implies that x_1 is an upper bound for $\{x_j\}_{j=1}^\infty$. If $\{x_j\}_{j=1}^\infty$ has a lower bound in $\mathbf{R}$, then $\{x_j\}_{j=1}^\infty$ converges in $\mathbf{R}$, and

$$\lim_{j\to\infty} x_j \text{ is the largest lower bound for } \{x_j\}_{j=1}^\infty. \tag{A.8.14}$$

This is known as the *greatest lower bound* or of the set of x_j's, $j \geq 1$, which may be expressed as

$$\inf_{j \geq 1} x_j. \tag{A.8.15}$$

These statements can be obtained from those in the preceding paragraph, because $\{-x_j\}_{j=1}^\infty$ is monotonically increasing.

A.8.4 Convergent sequences of complex numbers

An infinite sequence $\{z_j\}_{j=1}^\infty$ of complex numbers is said to *converge* to a complex number z if for every $\epsilon > 0$ there is a positive integer L such that

$$|z_j - z| < \epsilon \tag{A.8.16}$$

for each $j \geq L$. This is equivalent to the convergence of the corresponding sequence in $\mathbf{R}^2$, as in Subsection A.7.4. In this case, we have that

$$\lim_{j \to \infty} |z_j| = |z|, \tag{A.8.17}$$

as before.

Note that

$$\lim_{j \to \infty} z_j = z \tag{A.8.18}$$

if and only if

$$\lim_{j \to \infty} \operatorname{Re} z_j = \operatorname{Re} z \tag{A.8.19}$$

and

$$\lim_{j \to \infty} \operatorname{Im} z_j = \operatorname{Im} z, \tag{A.8.20}$$

as sequences of real numbers, as before. This is also equivalent to

$$\lim_{j \to \infty} \overline{z_j} = \overline{z}. \tag{A.8.21}$$

A.9 More on convergent sequences

Let $\{z_j\}_{j=1}^\infty$ and $\{w_j\}_{j=1}^\infty$ be sequences of complex numbers that converge to $z, w \in \mathbf{C}$, respectively. It is well known and not difficult to show that the corresponding sequence $\{z_j + w_j\}_{j=1}^\infty$ of sums converges to $z + w$. This means that

$$\lim_{j \to \infty} (z_j + w_j) = \lim_{j \to \infty} z_j + \lim_{j \to \infty} w_j, \tag{A.9.1}$$

where more precisely the limits on the right exist by hypothesis, and the existence of the limit on the left is part of the conclusion.

It is also well known that the sequence $\{z_j\, w_j\}_{j=1}^\infty$ of products converges to $z\, w$, although this is a bit more complicated. This means that

$$\lim_{j \to \infty} (z_j\, w_j) = \Big(\lim_{j \to \infty} z_j\Big) \Big(\lim_{j \to \infty} w_j\Big), \tag{A.9.2}$$

where the limits on the right exist by hypothesis, and the existence of the limit on th left is part of the conclusion.

A.9.1 A helpful lemma

Let $\{a_j\}_{j=1}^\infty$ be a sequence of complex numbers that converges to 0, and let $\{b_j\}_{j=1}^\infty$ be a bounded sequence of complex numbers. Under these conditions, one can check that

$$\lim_{j \to \infty} (a_j \, b_j) = 0. \tag{A.9.3}$$

In particular, one can use this to get that

$$\lim_{j \to \infty} ((z_j - z) \, w_j) = 0. \tag{A.9.4}$$

One can also verify that

$$\lim_{j \to \infty} (z \, w_j) = z \, w, \tag{A.9.5}$$

directly from the definitions. It is easy to obtain (A.9.2) from (A.9.4) and (A.9.5).

A.9.2 Sequences of reciprocals

If $z_j \ne 0$ for each j, and $z \ne 0$, then it is well known that $\{1/z_j\}_{j=1}^\infty$ converges to 0. This means that

$$\lim_{j \to \infty} (1/z_j) = 1/\Big(\lim_{j \to \infty} z_j\Big), \tag{A.9.6}$$

whwre the limit on the right exists by hypohtesis, and the existence of the limit on the left is part of the conclusion, as usual.

In order to show this, it is helpful to first use the convergence of $\{z_j\}_{j=1}^\infty$ to z to get that there is a positive integer L_1 such that

$$|z_j - z| < |z|/2 \tag{A.9.7}$$

when $j \ge L_1$. One can check that this implies that

$$|z_j| > |z|/2 \tag{A.9.8}$$

when $j \ge L_1$.

A.9.3 More on sequences in $\mathbf{R}^n$

Let n be a positive integer, and let $\{x(l)\}_{l=1}^\infty$, $\{y(l)\}_{l=1}^\infty$ be sequences of elements of $\mathbf{R}^n$ that converge to $x, y \in \mathbf{R}^n$, respectively. Under these conditions, one can check that $\{x(l) + y(l)\}_{l=1}^\infty$ converges to $x + y$. This means that

$$\lim_{l \to \infty} (x(l) + y(l)) = \lim_{l \to \infty} x(l) + \lim_{l \to \infty} y(l), \tag{A.9.9}$$

where the limits on the right exist by hypothesis, and the existence of the limit on the left is part of the conclusion, as before. This can be shown using an argument like that for sequences of real or complex numbers, or by reducing to the analogous statement for sequences of real numbers.

One can also verify that the sequence $\{x(l) \cdot y(l)\}_{l=1}^{\infty}$ converges to the dot product $x \cdot y$ of the limits. This means that

$$\lim_{l \to \infty} x(l) \cdot y(l) = \Big(\lim_{l \to \infty} x(l)\Big) \cdot \Big(\lim_{l \to \infty} y(l)\Big), \tag{A.9.10}$$

where the limits on the right existe by hypothesis, and the existence of the limit on the left is part of the conclusion. This follows from the earlier statements about sums and products of convergent sequences of real numbers.

Let $\{t_l\}_{l=1}^{\infty}$ be a sequence of real numbers that converges to $t \in \mathbf{R}$. In this case, we have that $\{t_l\, x(l)\}_{l=1}^{\infty}$ converges to $t\, x$ in $\mathbf{R}^n$. This means that

$$\lim_{l \to \infty} (t_l\, x(l)) = \Big(\lim_{l \to \infty} t_l\Big) \Big(\lim_{l \to \infty} x(l)\Big), \tag{A.9.11}$$

where the limits on the right exist by hypothesis, and the existence of the limit on the left is part of the conclusion. This can be obtained from the analogous statement for products of convergent sequences of real numbers, or shown in a similar way.

A.10 Some particular sequences

We consider the convergence of some particular sequences of real and complex numbers in this section. Some of the arguments for obtaining convergence are at least sketched, although other approaches are sometimes used as well.

A.10.1 $\{1/j^p\}_{j=1}^{\infty}$

If p is a positive real number, then it is well known and not difficult to show that

$$\lim_{j \to \infty} 1/j^p = 0. \tag{A.10.1}$$

One might as well take $p = 1/k$ for some positive integer k here, which is a bit simpler.

A.10.2 $\{j^{\alpha}\, a^j\}_{j=1}^{\infty}$

If a is a complex number with

$$|a| < 1 \tag{A.10.2}$$

and α is any real number, then it is well known that

$$\lim_{j \to \infty} j^{\alpha}\, a^j = 0. \tag{A.10.3}$$

Of course, this is trivial when $a = 0$, and one can reduce to the case where a is a positive real number. One can also reduce to the case where $\alpha < 1$, using a positive integer k such that

$$\alpha/k < 1. \tag{A.10.4}$$

More precisely, if one can show that

$$\lim_{j \to \infty} j^{\alpha/k}\, a^{j/k} = 0, \tag{A.10.5}$$

then it follows that

$$j^\alpha\, a^j = (j^{\alpha/k}\, a^{j/k})^k \to 0 \tag{A.10.6}$$

as $j \to \infty$, as before. Note that $a^{1/k} < 1$ when $0 < a < 1$.

A.10.3 $\{p^{1/j}\}_{j=1}^\infty$

If p is a positive real number, then it is well known that

$$\lim_{j \to \infty} p^{1/j} = 1. \tag{A.10.7}$$

One can reduce to the case where $p \ge 1$, using the remarks in Subsection A.9.2. If ϵ is a positive real number, then

$$p^{1/j} < 1 + \epsilon \tag{A.10.8}$$

if and only if

$$p < (1 + \epsilon)^j. \tag{A.10.9}$$

It is not too difficult to show that (A.10.9) holds when j is sufficiently large. This is basically the same as (A.10.3), with $\alpha = 0$, and $a = 1/(1 + \epsilon)$.

A.10.4 $\{j^{1/j}\}_{j=1}^\infty$

It is also well known that

$$\lim_{j \to \infty} j^{1/j} = 1. \tag{A.10.10}$$

If ϵ is a positive real number again, then

$$j^{1/j} < 1 + \epsilon \tag{A.10.11}$$

if and only if

$$j < (1 + \epsilon)^j. \tag{A.10.12}$$

One can check that this holds when j is sufficiently large. Note that (A.10.12) is the same as saying that

$$j\,(1 + \epsilon)^{-j} < 1. \tag{A.10.13}$$

That this holds when j is sufficiently large corresponds to (A.10.3) with $\alpha = 1$ and $a = 1/(1 + \epsilon)$.

A.11 Infinite series

Let $\sum_{j=1}^\infty a_j$ be an infinite series of complex numbers. If n is a positive integer, then

(A.11.1) $$s_n = \sum_{j=1}^n a_j$$

is called the *nth partial sum* of $\sum_{j=1}^\infty a_j$. If the sequence $\{s_n\}_{n=1}^\infty$ of partial sums converges, then we say that $\sum_{j=1}^\infty a_j$ converges as an infinite series, and that the value of the sum is

(A.11.2) $$\sum_{j=1}^\infty a_j = \lim_{n\to\infty} s_n.$$

If $\sum_{j=1}^\infty a_j$ converges, then it is well known that

(A.11.3) $$\lim_{j\to\infty} a_j = 0.$$

One way to see this is to observe that

(A.11.4) $$a_n = s_n - s_{n-1}$$

when $n \geq 2$, and that the right side tends to 0 as $n \to \infty$, because s_n and s_{n-1} converge to the same limit.

A.11.1 Some properties of convergent series

If $\sum_{j=1}^\infty a_j$ converges and $\sum_{j=1}^\infty b_j$ is another convergent series of complex numbers, then $\sum_{j=1}^\infty (a_j + b_j)$ converges too, with

(A.11.5) $$\sum_{j=1}^\infty (a_j + b_j) = \sum_{j=1}^\infty a_j + \sum_{j=1}^\infty b_j.$$

Indeed,

(A.11.6) $$\sum_{j=1}^n (a_j + b_j) = \sum_{j=1}^n a_j + \sum_{j=1}^n b_j$$

for each n, so that

(A.11.7) $$\lim_{n\to\infty} \sum_{j=1}^n (a_j + b_j) = \lim_{n\to\infty} \sum_{j=1}^n a_j + \lim_{n\to\infty} \sum_{j=1}^n b_j.$$

Similarly, if $\sum_{j=1}^\infty a_j$ converges and c is a complex number, then $\sum_{j=1}^\infty c\, a_j$ converges, with

(A.11.8) $$\sum_{j=1}^\infty c\, a_j = c \sum_{j=1}^\infty a_j.$$

This is because

$$\sum_{j=1}^{n} c\, a_j = c \sum_{j=1}^{n} a_j \tag{A.11.9}$$

for each n, so that

$$\lim_{n \to \infty} \sum_{j=1}^{n} c\, a_j = c \Big(\lim_{n \to \infty} \sum_{j=1}^{n} a_j \Big). \tag{A.11.10}$$

A.11.2 Geometric series

If z is a complex number, then $\sum_{j=0}^{\infty} z^j$ is called a *geometric series*. Here z^j is interpreted as being equal to 1 when $j = 0$, even when $z = 0$. Of course, infinite series beginning with $j = 0$, or any other integer, can be handled in the same way as before.

Let n be a nonnegative integr, and observe that

$$(1 - z) \sum_{j=0}^{n} z^j = \sum_{j=0}^{n} z^j - \sum_{j=0}^{n} z^{j+1} = \sum_{j=0}^{n} z^j - \sum_{j=1}^{n+1} z^j = 1 - z^{n+1}. \tag{A.11.11}$$

If $z \neq 1$, then it follows that

$$\sum_{j=0}^{n} z^j = (1 - z^{n+1})/(1 - z). \tag{A.11.12}$$

If $|z| < 1$, then

$$\lim_{n \to \infty} z^n = 0, \tag{A.11.13}$$

as in Subsection A.10.2, with $\alpha = 0$. This implies that $\sum_{j=0}^{\infty} z^j$ converges, with

$$\sum_{j=0}^{\infty} z^j = 1/(1 - z), \tag{A.11.14}$$

because $z^n \to 0$ as $n \to \infty$,

If $|z| \geq 1$, then

$$|z^j| = |z|^j \geq 1 \tag{A.11.15}$$

for each $j \geq 0$. This implies that $\{z^j\}_{j=0}^{\infty}$ does not converge to 0, so that $\sum_{j=0}^{\infty} z^j$ does not converge.

A.11.3 Infinite series with nonnegative terms

If $\sum_{j=1}^{\infty} a_j$ is a convergent series of complex numbers, then the corresponding sequence of partial sums is bounded, because it converges.

Suppose now that $\sum_{j=1}^{\infty} a_j$ is an infinite series whose terms are nonnegative real numbers. This implies that

$$s_n \leq s_n + a_{n+1} = s_{n+1} \tag{A.11.16}$$

for each n, so that the corresponding sequence of partial sums is monotonically increasing. If $\{s_n\}_{n=1}^\infty$ has an upper bound in $\mathbf{R}$, then it follows that $\sum_{j=1}^\infty a_j$ converges, as in Subsection A.8.3.

If p is a positive real number, then it is well known that

$$\sum_{j=1}^\infty 1/j^p \tag{A.11.17}$$

converges if and only if $p > 1$.

A.11.4 Absolute convergence

An infinite series $\sum_{j=1}^\infty a_j$ of complex numbers is said to converge *absolutely* if

$$\sum_{j=1}^\infty |a_j| \tag{A.11.18}$$

converges. It is well known that this implies that $\sum_{j=1}^\infty a_j$ converges. In this case, it is not too difficult to show that

$$\left|\sum_{j=1}^\infty a_j\right| \le \sum_{j=1}^\infty |a_j|. \tag{A.11.19}$$

If $\sum_{j=1}^\infty a_j$ is an absolutely convergent series of real numbers, then one can show that the corresponding sums of the positive and negative parts of the a_j's converge, using the remarks in the previous subsection. One can use this to get that $\sum_{j=1}^\infty a_j$ converges.

If $\sum_{j=1}^\infty a_j$ is an absolutely convergent series of complex numbers, then it is easy to see that the corresponding series of real and imaginary parts of the a_j's are absolutely convergent. One can use this and the remarks in the preceding paragraph to get that $\sum_{j=1}^\infty a_j$ converges.

Let $\sum_{j=1}^\infty a_j$ be an infinite series of complex numbers, and let $\sum_{j=1}^\infty b_j$ be an infinite series of nonnegative real numbers. If

$$|a_j| \le b_j \tag{A.11.20}$$

for each j, and $\sum_{j=1}^\infty b_j$ converges, then the *comparison test* says that $\sum_{j=1}^\infty a_j$ converges absolutely. This can be obtained from the remarks in the previous subsection.

A.12 More on infinite series

Let $\{a_j\}_{j=0}^\infty$ be a sequence of complex numbers such that the sequence of sums

$$A_n = \sum_{j=0}^n a_j \tag{A.12.1}$$

is bounded. Also let $\{b_j\}_{j=0}^{\infty}$ be a monotonically decreasing sequence of nonnegative real numbers that converges to 0. Under these conditions, it is well known that

$$\sum_{j=0}^{\infty} a_j\, b_j \tag{A.12.2}$$

converges. This is Theorem 3.42 on p70 of [309].

If $a_j = (-1)^j$ for each j, then it is easy to see that $A_n = 1$ when n is even, and $A_n = 0$ when n is odd. In this case, this criterion for convergence corresponds to Leibniz' *alternating series test*. This corresponds to Theorem 3.3 A on p72 of [125], and to Theorem 3.43 on p71 of [309].

Suppose that z is a complex number with $|z| = 1$ and $z \neq 1$. If

$$a_j = z^j \tag{A.12.3}$$

for each j, then one can check that $\{A_n\}_{n=0}^{\infty}$ is a bounded sequence in $\mathbf{C}$, using some of the remarks in Section A.11.2. This corresponds to Theorem 3.44 on p71 of [309].

A.12.1 Power series

Let $a_0, a_1, a_2, a_3, \ldots$ be an infinite sequence of complex numbers, and consider the corresponding *power series*

$$\sum_{j=0}^{\infty} a_j\, z^j. \tag{A.12.4}$$

Suppose for the moment that (A.12.4) converges absolutely for some $z \in \mathbf{C}$, and that $w \in \mathbf{C}$ satisfies

$$|w| \le |z|. \tag{A.12.5}$$

One can check that

$$\sum_{j=0}^{\infty} a_j\, w^j \tag{A.12.6}$$

converges absolutely, using the comparison test.

Suppose for the moment again that (A.12.4) converges for some $z \in \mathbf{C}$ with $z \neq 0$. If $w \in \mathbf{C}$ satisifes

$$|w| < |z|, \tag{A.12.7}$$

then it is not too difficult to show that (A.12.6) converges absolutely.

More precisely, the convergence of (A.12.4) implies that

$$\lim_{j \to \infty} a_j\, z^j = 0, \tag{A.12.8}$$

and in particular that $\{a_j\, z^j\}_{j=0}^{\infty}$ is a bounded sequence. This means that there is a nonnegative real number C such that

$$|a_j\, z^j| \le C \tag{A.12.9}$$

for each j. Using this, it is easy to see that

$$|a_j\, w^j| \le C\,(|w|/|z|)^j \tag{A.12.10}$$

for each j. If (A.12.7) holds, then $\sum_{j=0}^{\infty}(|w|/|z|)^j$ is a convergent geometric series. In this case, one can use the comparison test to get that (A.12.6) converges asbolutely.

A.12.2 The radius of convergence

Suppose for the moment that there is a $z \in \mathbf{C}$ such that (A.12.4) does not converge. In this case, it is well known that there is a nonnegative real number ρ with the following two properties. First, if $w \in \mathbf{C}$ and

$$|w| > \rho, \tag{A.12.11}$$

then (A.12.6) does not converge. Second, if $w \in \mathbf{C}$ and

$$|w| < \rho, \tag{A.12.12}$$

then (A.12.6) converges absolutely.

If (A.12.4) converges for every $z \in \mathbf{C}$, then (A.12.6) converges absolutely for every $w \in \mathbf{C}$, as in the previous subsection. In this case, we can take $\rho = +\infty$.

One can check that ρ is uniquely determined by these properties. This is called the *radius of convergence* of the power series (A.12.4).

A.13 Rearrangements of infinite series

Let π be a one-to-one mapping from the set $\mathbf{Z}_+$ of positive integers onto itself. This means that $\pi(j) \in \mathbf{Z}_+$ for every positive inetger j, and that every positive integer k can be expressed as $\pi(j)$ for exactly on $j \in \mathbf{Z}_+$, which may be expressed as $\pi^{-1}(k)$.

If $\sum_{j=1}^{\infty} a_j$ is an infinite series of complex numbers, then the infinite series

$$\sum_{j=1}^{\infty} a_{\pi(j)} \tag{A.13.1}$$

is called a *rearrangement* of $\sum_{j=1}^{\infty} a_j$. If

$$a_j = 0 \text{ for all but finitely many } j, \tag{A.13.2}$$

then it is easy to see that

$$a_{\pi(j)} = 0 \text{ for all but finitely many } j \tag{A.13.3}$$

as well, with

$$\sum_{j=1}^{\infty} a_{\pi(j)} = \sum_{j=1}^{\infty} a_j. \tag{A.13.4}$$

Suppose for the moment that a_j is a nonnegative real number for each j. Under these conditions, it is well known and not too difficult to show that $\sum_{j=1}^{\infty} a_j$ converges if and only if (A.13.1) converges, in which case (A.13.4) holds. This is because

$$\text{every partial sum of } \sum_{j=1}^{\infty} a_j \text{ is less than or equal to some partial sum of (A.13.1),} \tag{A.13.5}$$

and vice-versa. This corresponds to Lemma 3.5 E on p77 of [125].

If $\sum_{j=1}^{\infty} a_j$ is an absolutely convergent series of complex numbers, then it follows that (A.13.1) is absolutely convergent too. More precisely, the convergence of $\sum_{j=1}^{\infty} |a_j|$ implies the convergence of

$$\sum_{j=1}^{\infty} |a_{\pi(j)}|, \tag{A.13.6}$$

as in the preceding paragraph.

If a_j is a real number for each j, then one can get that (A.13.4) holds, by considering the positive and negative parts of a_j separately. This corresponds to Theorem 3.5 F on p77 of [125]. Otherwise, one can reduce to this case, by considering the real and imaginary parts of a_j for each j.

Another argument is used in the proof of Theorem 3.55 on p78 of [309]. Basically, one can show more directly that

$$\lim_{n \to \infty} \Big(\sum_{j=1}^{n} a_j - \sum_{j=1}^{n} a_{\pi(j)} \Big) = 0 \tag{A.13.7}$$

under these conditions.

Suppose now that $\sum_{j=1}^{\infty} a_j$ is an infinite series of real numbers that converges, and does not converge absolutely. In this case, it is well known that there are rearrangements of the series that do not converge, as in Exercises 4, 5 on p80 of [125]. There are also rearrangements that converge, with the sum equal to any real number, as in Theorem 3.5 D on p77 of [125]. See Theorem 3.54 on p76 of [309] as well.

A.14 Cauchy products of infinite series

If $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ are infinite series of complex numbers, then one might like to multiply these two series, and arrange the terms into a single series. A nice way to do this is to take

$$c_n = \sum_{j=0}^{n} a_j \, b_{n-j} \tag{A.14.1}$$

for each nonnegative integer n. The corresponding infinite series $\sum_{n=0}^\infty c_n$ is called the *Cauchy product* of $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$.

It is not difficult to see that

$$\sum_{n=0}^\infty c_n = \Big(\sum_{j=0}^\infty a_j\Big) \Big(\sum_{l=0}^\infty b_l\Big) \tag{A.14.2}$$

formally. In fact, both sides of the equation correspond to

$$\sum_{j,l \ge 0} a_j \, b_l \tag{A.14.3}$$

formally, where the sum is taken over all pairs of nonnegative integers j, l.

Suppose for the moment that $a_j = 0$ for all but finitely many j, and that $b_l = 0$ for all but finitely many l. This implies that $a_j \, b_l = 0$ for all but finitely many pairs of nonnegative integers j, l. One can also check that $c_n = 0$ for all but finitely many n. In this case, one can verify that (A.14.2) holds. One can interpret (A.14.3) as reducing to a finite sum, that is equal to both sides of (A.14.2).

A.14.1 Cauchy products and power series

The definition of the Cauchy product works well with power series. Indeed, the Cauchy product of the power series $\sum_{j=0}^\infty a_j \, z^j$ and $\sum_{l=0}^\infty b_l \, z^l$, is the power series

$$\sum_{n=0}^\infty c_n \, z^n, \tag{A.14.4}$$

where c_n is as in (A.14.1) for each n.

A.14.2 Convergence of Cauchy products

Suppose for the moment that a_j is a nonnegative real number for each j, and that b_l is a nonnegative real number for each l. This implies that c_n is a nonnegative real number for each n. If N is a nonnegative integer, then one can check that

$$\sum_{n=0}^N c_n \le \Big(\sum_{j=0}^N a_j\Big) \Big(\sum_{l=0}^N b_l\Big). \tag{A.14.5}$$

Similarly, if N_1 and N_2 are nonnegative integers, then one can verify that

$$\Big(\sum_{j=0}^{N_1} a_j\Big) \Big(\sum_{l=0}^{N_2} b_l\Big) \le \sum_{n=0}^{N_1+N_2} c_n. \tag{A.14.6}$$

If $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ converge, then one can use these inequalities to get that $\sum_{n=0}^\infty c_n$ converges, with sum as in (A.14.2).

Observe that

$$|c_n| \le \sum_{j=0}^{n} |a_j|\,|b_{n-j}| \tag{A.14.7}$$

for each n when the a_j's and b_l's are any complex numbers. The right side of this inequality is the same as the nth term of the Cauchy product of the series $\sum_{j=0}^{\infty} |a_j|$ and $\sum_{l=0}^{\infty} |b_l|$. If these series converge, then their Cauchy product converges, as in the previous paragraph. This implies that $\sum_{n=0}^{\infty} |c_n|$ converges, by the comparison test.

It is not difficult to show that (A.14.2) holds under these conditions, by reducing to convergent series of nonnegative real numbers. This corresponds to the theorem on p78 of [125], which uses another argument.

If one of $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ converges absolutely, and the other series converges, then it is well known that $\sum_{n=0}^{\infty} c_n$ converges, with sum as in (A.14.2). This is Theorem 3.50 on p74 of [309]. However, the Cauchy product of convergent series may not converge, as in Example 3.49 on p73 of [309].

A.15 The binomial theorem

If j is a positive integer, then $j!$ is j factorial, the product of the positive integers from 1 to j, as usual. This is interpreted as being equal to 1 when $j = 0$.

Let n be a nonnegative integer, and let z, w be complex numbers. The *binomial theorem* states that

$$(z + w)^n = \sum_{j=0}^{n} \binom{n}{j} z^j\, w^{n-j}, \tag{A.15.1}$$

where

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!} \tag{A.15.2}$$

is the usual *binomial coefficient*.

It is easy to see that

$$(z + w)^n \tag{A.15.3}$$

can be expanded into a sum of terms of the form

$$z^j\, w^{n-j}, \tag{A.15.4}$$

$0 \le j \le n$, where each of these terms occurs a positive number of times. One can show that the number of times that these terms occur is equal to (A.15.2), using induction.

These coefficients can also be obtained using differentiation.

Appendix B

A bit of analysis, 2

B.1 The complex exponential function

If z is a complex number, then put

$$E(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \tag{B.1.1}$$

as in (25) on p178 of [309]. One can check that the series on the right converges absolutely, using the ratio test. Alternatively, one can use the comparison test, by comparing this series with a convergent geometri series.

If $z \in \mathbf{R}$, then the right side of (B.1.1) is the usual Taylor series expansion for the exponential function. One can also use this to define the exponential function on the real line, and as a complex-valued function on the complex plane.

Of course, we would like to check that this satisfies the usual properties of the exponential function on the real line. We would like to consider its properties on the complex plane as well.

B.1.1 Exponentials of sums

If w is another complex number, then

$$E(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} \frac{z^j}{j!} \, \frac{w^{n-j}}{(n-j)!} \Big), \tag{B.1.2}$$

using the binomial theorem in the second step, as in Section A.15. The right side is the same as the Cauchy product of the series used to define $E(z)$ and $E(w)$, as in Section A.14. This implies that

$$E(z+w) = E(z)\, E(w), \tag{B.1.3}$$

as in (26) on p178 of [309], because these series converge absolutely, as in Subsection A.14.2.

In particular, we can take $w = -z$, to get that

$$E(z)\, E(-z) = E(z - z) = E(0) = 1, \tag{B.1.4}$$

as in (27) on p178 of [309]. This means that $E(z) \neq 0$, with

$$E(z)^{-1} = E(-z). \tag{B.1.5}$$

If $x \in \mathbf{R}$, then $E(x) \in \mathbf{R}$, and one can check that $E(x)$ is positive and strictly increasing for $x \geq 0$. One can verify that the same properties hold when $x \leq 0$, using (B.1.5).

Similarly, if $x \in \mathbf{R}$, then $E(x) \to +\infty$ as $x \to +\infty$, as on p179 of [309]. It follows that $E(x) \to 0$ as $x \to -\infty$, because of (B.1.5), as in [309].

We may use $E(z)$ as the definition of the *complex exponential function* $e^z = \exp z$ for $z \in \mathbf{C}$, as on p1 of [308]. We shall use this notation from now on here.

B.1.2 Exponentials and complex conjugates

It is easy to see that

$$\overline{(\exp z)} = \exp(\overline{z}) \tag{B.1.6}$$

for every $z \in \mathbf{C}$. This implies that

$$\begin{aligned} |\exp z|^2 &= (\exp z)\, \overline{(\exp z)} = (\exp z)\, (\exp \overline{z}) \\ &= \exp(z + \overline{z}) = \exp(2 \operatorname{Re} z). \end{aligned} \tag{B.1.7}$$

In particular,

$$|\exp(i\, y)| = 1 \tag{B.1.8}$$

for every $y \in \mathbf{R}$. In fact, *Euler's identity* states that

$$\exp(i\, y) = \cos y + i\, \sin y, \tag{B.1.9}$$

as in (46) on p182 of [309], and (5) on p2 of [308].

B.2 Continuous complex-valued functions

Let n be a positive integer, let E be a nonempty subset of $\mathbf{R}^n$, and let f be a complex-valued function on E. We say that f is *continuous* at a point $x \in E$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \tag{B.2.1}$$

for every $y \in E$ such that

$$|x - y| < \delta. \tag{B.2.2}$$

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of E that converges to x, as in Subsection A.8.1. One can check that

$$\lim_{j \to \infty} f(x_j) = f(x) \tag{B.2.3}$$

under these conditions.

It is well known that continuity of f at x is characterized by this property. This can be shown using an argument by contradiction.

If f is continuous at every point in E, then we may simply say that f is *continuous on* E.

B.2.1 More on continuous functions

Let g be another complex-valued function on E. If f and g are both continuous at $x \in E$, then it is well known that

$$f + g \text{ and } f\, g \text{ are continuous at } x, \tag{B.2.4}$$

as complex-valued functions on E. This can be obtained from the characterization of continuity in terms of convergent sequences and the analogous statements for convergent sequences of complex numbers, as in Section A.9.

Suppose for the moment that $f(y) \neq 0$ for each $y \in E$, so that $1/f$ defines a complex-valued function on E. If f is continuous at x, then it is well known that

$$1/f \text{ is continuous at } x \tag{B.2.5}$$

too. This can be obtained from the analogous statement for sequences of complex numbers, as in Subsection A.9.2.

One can use (B.2.4) to get that functions on $\mathbf{R}^n$ defined by polynomials are continuous. Similarly, a rational function is continuous on any set where the denominator does not take the value 0.

B.2.2 Continuity of compositions

Let A be a nonempty subset of the complex plane, and let ϕ be a complex-valued function on A. If

$$f(y) \in A \tag{B.2.6}$$

for every $y \in E$, then the *composition* $\phi \circ f$ of f with ϕ may be defined as another complex-valued function on E, with

$$(\phi \circ f)(y) = \phi(f(y)) \tag{B.2.7}$$

for every $y \in E$, as usual.

If f is continuous at $x \in E$, and ϕ is continuous at $f(x)$, then it is well known that

$$\phi \circ f \text{ is continuous at } x. \tag{B.2.8}$$

This can be verified directly from the definitions, or using the characterization of continuity in terms of convergent sequences.

Of course,

$$z \mapsto 1/z \tag{B.2.9}$$

defines a complex-valued function on the set $\mathbf{C}\backslash\{0\}$ of nonzero complex numbers. If $f(y) \neq 0$ for each $y \in E$, then $1/f$ is the same as the composition of f with (B.2.9).

If f is any complex-valued function on E, then

$$|f(y)| \tag{B.2.10}$$

defines a nonnegative real-valued function on E. This is the same as the composition of f with

$$z \mapsto |z|, \tag{B.2.11}$$

as a nonnegative real-valued function on $\mathbf{C}$. It is well known and not too difficult to show that (B.2.11) is continuous on $\mathbf{C}$, which corresponds to a remark about convergent sequences of complex numbers in Subsection A.8.4. If f is continous at $x \in E$, then it follows that (B.2.10) is continuous at x as well.

Similarly, it is well known and not difficult to show that

$$w \mapsto |w| \tag{B.2.12}$$

is a continuous real-valued function on $\mathbf{R}^n$, which corresponds to a remark about convergent sequences in $\mathbf{R}^n$ in Subsection A.8.2.

B.2.3 Real and imaginary parts

It is easy to see that f is continuous on E if and only if

$$\operatorname{Re} f(y) \text{ and } \operatorname{Im} f(y) \tag{B.2.13}$$

are continuous real-valued functions on E. Of course, these two functions are the same as the compositions of f with

$$z \mapsto \operatorname{Re} z \text{ and } z \mapsto \operatorname{Im} z, \tag{B.2.14}$$

respectively, as real valued functions on $\mathbf{C}$.

Similarly, f is continuous on E if and only if

$$\overline{f(y)} \tag{B.2.15}$$

is continuous on E. This is the same as the composition of f with the complex-valued function defined on $\mathbf{C}$ by complex conjugation.

B.3 Pointwise and uniform convergence

Let E be a nonempty set, let $\{f_j\}_{j=1}^\infty$ be a sequence of complex-valued functions on E, and let f be a complex-valued function on E. We say that $\{f_j\}_{j=1}^\infty$ converges to f *pointwise* on E if for every $y \in E$,

(B.3.1) $$\{f_j(y)\}_{j=1}^\infty \text{ converges to } f(y),$$

as a sequence of complex numbers.

We say that $\{f_j\}_{j=1}^\infty$ converges to f *uniformly* on E if for every $\epsilon > 0$ there is a positive integer L such that

(B.3.2) $$|f_j(y) - f(y)| < \epsilon$$

for every $y \in E$ and $j \ge L$. This implies that $\{f_j\}_{j=1}^\infty$ converges to f pointwise on E. If E has only finitely many elements, and $\{f_j\}_{j=1}^\infty$ converges to f pointwise on E, then one can verify that $\{f_j\}_{j=1}^\infty$ converges to f uniformly on E.

B.3.1 An example with $E = [0, 1]$

Let us take E to be the closed unit interval $[0, 1]$ in the real line, and put

(B.3.3) $$f_j(y) = y^j$$

for each j and $0 \le y \le 1$. This sequence of functions converges pointwise on $[0, 1]$, with

(B.3.4) $$\begin{aligned} \lim_{j\to\infty} f_j(y) &= 0 \quad \text{when } 0 \le y < 1 \\ &= 1 \quad \text{when } y = 1. \end{aligned}$$

If r is a positive real number with $r < 1$, the one can check that

(B.3.5) $$\{f_j\}_{j=1}^\infty \text{ converges to 0 uniformly on } [0, r].$$

However, one can also verify that $\{f_j\}_{j=1}^\infty$ does not converge uniformly on $[0, 1]$. Indeed, for each j, y^j is as close to 1 as we want when y is sufficiently close to 1.

B.3.2 Uniform convergence and continuity

Let n be a posiitve integer, and let E be a nonempty subset of $\mathbf{R}^n$. Also let $\{f_j\}_{j=1}^\infty$ be a sequence of complex-valued functions on E that converges uniformly to a complex-valued function f on E. Suppose that $x \in E$, and that

(B.3.6) $$f_j \text{ is continuous at } x$$

for each j. Under these conditions, it is well known and not too difficult to show that

(B.3.7) $$f \text{ is continuous at } x.$$

B.4 Weierstrass' criterion for uniform convergence

Let E be a nonempty set, and let $a_1(y), a_2(y), a_3(y), \ldots$ be an infinite sequence of complex-valued functions on E. Also let $A_1, A_2, A_3, \ldots$ be an infinite sequence of nonnegative real numbers such that

$$|a_j(y)| \le A_j \tag{B.4.1}$$

for every $y \in E$ and $j \ge 1$. Suppose that

$$\sum_{j=1}^{\infty} A_j \tag{B.4.2}$$

converges. This implies that

$$\sum_{j=1}^{\infty} a_j(y) \tag{B.4.3}$$

converges absolutely for each $y \in E$, by the comparison test.

In fact, the sequence of partial sums

$$\sum_{j=1}^{l} a_j(y) \tag{B.4.4}$$

converges to (B.4.3) uniformly on E under these conditions. This is a well-known criterion of Weierstrass for uniform convergence.

Indeed, if l is a positive integer, than

$$\Big|\sum_{j=1}^{\infty} a_j(y) - \sum_{j=1}^{l} a_j(y)\Big| = \Big|\sum_{j=l+1}^{\infty} a_j(y)\Big| \le \sum_{j=l+1}^{\infty} |a_j(y)| \le \sum_{j=l+1}^{\infty} A_j \tag{B.4.5}$$

for every $y \in E$. The right side tends to 0 as $l \to \infty$, because of the convergence of (B.4.2). This implies that (B.4.4) converges to (B.4.3) uniformly on E, because the right side of (B.4.5) does not depened on y.

B.4.1 Continuity of the sum

Let n be a positive integer, and suppose that E is a subset of $\mathbf{R}^n$. If $a_j(y)$ is continuous on E for each j, then the partial sums (B.4.4) are continuous on E for each l as well.

If (B.4.2) converges, then the uniform convergence of the partial sums implies that (B.4.3) is continuous on E, as in Subsection B.3.2.

B.5 Continuity of functions defined by power series

Let $\sum_{j=0}^{\infty} a_j\, z^j$ be a power series with complex coefficients. Suppose for the moment that r is a positive real number such that

$$\sum_{j=0}^{\infty} |a_j|\, r^j \tag{B.5.1}$$

converges. This implies that $\sum_{j=0}^{\infty} a_j\, z^j$ converges absolutely for each $z \in \mathbf{C}$ with $|z| \le r$, by the comparison test, as in Subsection A.12.1.

Thus

$$f(z) = \sum_{j=0}^{\infty} a_j\, z^j \tag{B.5.2}$$

defines a complex-valued function on the closed disk

$$\{z \in \mathbf{C} : |z| \le r\}. \tag{B.5.3}$$

We can use Weierstrass' criterion to get that the sequence of partial sums

$$\sum_{j=0}^{l} a_j\, z^j \tag{B.5.4}$$

converges uniformly to $f(z)$ on (B.5.3) as $l \to \infty$. This uses the fact that

$$|a_j\, z^j| = |a_j|\, |z|^j \le |a_j|\, r^j \tag{B.5.5}$$

on (B.5.3) for each j.

Note that (B.5.4) is continuous as a complex-valued function on $\mathbf{C}$ for each nonnegative integer l, as in Subsection B.2.1. In particular, the restriction of (B.5.4) to (B.5.3) is continuous for each l. It follows that $f(z)$ is continuous on (B.5.3), as in Subsection B.4.1.

B.5.1 Continuity on open disks

Suppose now that ρ is a positive real number such that (B.5.1) converges when $0 < r < \rho$. It is convenient to include the case where $\rho = +\infty$ here too, so that (B.5.1) converges for all $r > 0$.

Under these conditions, $\sum_{j=0}^{\infty} a_j\, z^j$ converges absoutely for every complex number z with $|z| < \rho$, as before. This means that (B.5.2) defines a complex-valued function on

$$\{z \in \mathbf{C} : |z| < \rho\}. \tag{B.5.6}$$

Of course, (B.5.6) is the same as the complex plane when $\rho = +\infty$.

In fact, $f(z)$ is continuous as a complex-valued function on (B.5.6). To see this, let $z_0 \in \mathbf{C}$ with $|z_0| < \rho$ be given. It suffices to check that f is continuous at z_0, as a function on (B.5.6).

Let r be a positive real number such that

$$|z_0| < r < \rho. \tag{B.5.7}$$

We have already seen that f is continuous at z_0 as a function defined on (B.5.3). One can use this to get that f is continuous at z_0 as a function on (B.5.6), because $|z_0| < r$.

B.6 Bounded functions

Let E be a nonempty set, and let f be a complex-valued function on E. We say that f is *bounded* on E if there is a nonnegative real number C such that

$$|f(y)| \le C \tag{B.6.1}$$

for every $y \in E$.

Suppose that f satisfies (B.6.1), and that g is another bounded complex-valued function on E, with

$$|g(y)| \le C' \tag{B.6.2}$$

for some $C' \ge 0$ and all $y \in E$. In this case, $f + g$ is bounded on E, with

$$|f(y) + g(y)| \le |f(y)| + |g(y)| \le C + C' \tag{B.6.3}$$

for every $y \in E$. Similarly, $f\,g$ is bounded on E, with

$$|f(y)\,g(y)| = |f(y)|\,|g(y)| \le C\,C' \tag{B.6.4}$$

for every $y \in E$.

B.6.1 A version of the extreme value theorem

Let n be a positive integer, let w be an element of $\mathbf{R}^n$, and let r be a positive real number. Remember that $\overline{B}(w,r)$ is the closed ball in $\mathbf{R}^n$ centered at w with radius r, as in Subsection A.2.2. Let f be a continuous real-valued function on $\overline{B}(w,r)$. Under these conditions, the *extreme value theorem* states that f attains its maximum and minimum on $\overline{B}(w,r)$. This means that there are points u, v in $\overline{B}(w,r)$ such that

$$f(u) \le f(y) \le f(v) \tag{B.6.5}$$

for every $y \in E$.

If f is a continuous complex-valued function on $\overline{B}(w,r)$, then $|f|$ is a continuous real-valued function on $\overline{B}(w,r)$, as in Subsection B.2.2. This implies that $|f|$ attains its maximum and minimum on $\overline{B}(w,r)$, as in the preceding paragraph. In particular, this means that f is bounded on $\overline{B}(w,r)$.

B.7 Uniformly bounded sequences of functions

Let E be a nonempty set again, and let $\{f_j\}_{j=1}^\infty$ be a sequence of complex-valued functions on E. We say that $\{f_j\}_{j=1}^\infty$ is *uniformly bounded* if there is a nonnegative real number C such that

$$|f_j(y)| \le C \tag{B.7.1}$$

for every $y \in E$ and positive integer j. Of course, this implies in particular that f_j is bounded nn E for each j.

Suppose that $\{f_j\}_{j=1}^\infty$ satisfies (B.7.1) on E. Suppose also for the moment that $\{f_j\}_{j=1}^\infty$ converges pointwise to a complex-valued function f on E. In this case, one can check that (B.6.1) holds for every $y \in E$. In particular, this means that f is bounded on E.

Let $\{g_j\}_{j=1}^\infty$ be another uniformly bounded sequence of complex-valued functions on E, with

$$|g_j(y)| \le C' \tag{B.7.2}$$

for some $C' \ge 0$ and all $y \in E$ and $j \ge 1$. Observe that $\{f_j + g_j\}_{j=1}^\infty$ is uniformly bounded on E, with

$$|f_j(y) + g_j(y)| \le C + C' \tag{B.7.3}$$

for every $y \in E$ and $j \ge 1$. Similarly, $\{f_j\,g_j\}_{j=1}^\infty$ is uniformly bounded on E, with

$$|f_j(y)\,g_j(y)| \le C\,C' \tag{B.7.4}$$

for each $y \in E$ and $j \ge 1$.

B.7.1 More on uniform convergence

Suppose now that $\{f_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ are sequences of complex-valued functions on E that converge uniformly to complex-valued functions f and g on E, respectively. One can check that

$$\{f_j + g_j\}_{j=1}^\infty \text{ converges uniformly to } f + g \text{ on } E \tag{B.7.5}$$

under these conditions. If a is a complex number, then one can verify that

$$\{a\,f_j\}_{j=1}^\infty \text{ converges uniformly to } a\,f \text{ on } E. \tag{B.7.6}$$

Suppose in addition that $\{f_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ are uniformly bounded on E. This implies that f and g are bounded on E too, as before. In this case, it is well known and not too difficult to show that

$$\{f_j\,g_j\}_{j=1}^\infty \text{ converges uniformly to } f\,g \text{ on } E. \tag{B.7.7}$$

To see this, observe that

$$f_j(y)\,g_j(y) - f(y)\,g(y) = (f_j(y) - f(y))\,g_j(y) + f(y)\,(g_j(y) - g(y)) \tag{B.7.8}$$

for every $y \in E$ and $j \geq 1$. This implies that

$$|f_j(y)\, g_j(y) - f(y)\, g(y)| \leq |f_j(y) - f(y)|\, |g_j(y)| + |f(y)|\, |g_j(y) - g(y)| \tag{B.7.9}$$

for all $y \in E$ and $j \geq 1$. If $C, C' \geq 0$ are as in (B.6.1) and (B.7.2), respectively, then we get that

$$|f_j(y)\, g_j(y) - f(y)\, g(y)| \leq C'\, |f_j(y) - f(y)| + C\, |g_j(y) - g(y)| \tag{B.7.10}$$

for each $y \in E$ and $j \geq 1$. One can use this to get (B.7.7).

B.7.2 Boundedness and uniform convergence

Let $\{f_j\}_{j=1}^\infty$ be a sequence of complex-valued functions on E that converges uniformly to a complex-valued function f on E again. If

$$f_j \text{ is bounded on } E \tag{B.7.11}$$

for each j, then one can check that

$$f \text{ is bounded on } E. \tag{B.7.12}$$

Using this, one can also verify that

$$\{f_j\}_{j=1}^\infty \text{ is uniformly bounded on } E \tag{B.7.13}$$

under these conditions.

B.8 Uniform convergence and integration

Let a and b be real numbers with $a < b$, and let $\{f_j\}_{j=1}^\infty$ be a sequence of continuous real-valued functions on $[a, b]$ that converges uniformly to a real-valued function f on $[a, b]$. Remember that f is also continuous on $[a, b]$, as in Subsection B.3.2. It is well known and not too difficult to show that

$$\lim_{j \to \infty} \int_a^b f_j(x)\, dx = \int_a^b f(x)\, dx. \tag{B.8.1}$$

More precisely,

$$\begin{aligned} \left| \int_a^b f_j(x)\, dx - \int_a^b f(x)\, dx \right| &= \left| \int_a^b (f_j(x) - f(x))\, dx \right| \\ &\leq \int_a^b |f_j(x) - f(x)|\, dx \end{aligned} \tag{B.8.2}$$

for each j. One can use uniform convergence to get that

$$\lim_{j \to \infty} \int_a^b |f_j(x) - f(x)|\, dx = 0. \tag{B.8.3}$$

There are analogous statements for multiple integrals over suitable regions in $\mathbf{R}^n$.

One may also consider the case where the functions f_j are Riemann integrable, instead of continuous. It is well known that uniform convergence implies that f is Riemann integrable as well.

One may consider integrals of complex-valued functions too. The integral of a complex-valued function may be defined under suitable conditions as the complex number whose real and imaginary parts are the corresponding integrals of the real and imaginary parts of the function.

B.8.1 The bounded convergence theorem

Let $\{f_j\}_{j=1}^\infty$ be a sequence of continuous real-valued functions on $[a,b]$ again. Suppose now that $\{f_j\}_{j=1}^\infty$ converges to a continuous real-valued function f pointwise on $[a,b]$, and that $\{f_j\}_{j=1}^\infty$ is uniformly bounded on $[a,b]$. Under these conditions, a classical theorem of Arzelà and Osgood impies that (B.8.1) holds, as in [83, 133, 235, 242, 348]. This also works when the f_j's and f are Riemann integrable on $[a,b]$, instead of continuous.

More results like these can be obtained using Lebesgue integrals.

B.9 Uniform convergence and differentiation

Let a and b be real numbers with $a < b$ again, and let $\{g_j\}_{j=1}^\infty$ be a sequence of continuously-differentiable real-valued functions on $[a,b]$. This means that for each positive integer j, the derivative $g_j'(x)$ of g_j exists at every $x \in [a,b]$, using the appropriate one-sided derivative when $x = a$ or b, and that g_j' is continuous on $[a,b]$.

Suppose that $\{g_j(a)\}_{j=1}^\infty$ converges to a real number $g(a)$, and that $\{g_j'\}_{j=1}^\infty$ converges uniformly to a real-valued function f on $[a,b]$. Note that f is continuous on $[a,b]$, as in Subsection B.3.2.

Of course,

$$g_j(x) = g_j(a) + \int_a^x g_j'(t)\,dt \tag{B.9.1}$$

for each $j \geq 1$ and $x \in [a,b]$, by the fundamental theorem of calculus. Let g be the real-valued function defined on $[a,b]$ by

$$g(x) = g(a) + \int_a^x f(t)\,dt \tag{B.9.2}$$

for each $x \in [a,b]$.

One can check that

$$\{g_j\}_{j=1}^\infty \text{ converges uniformly to } g \text{ on } [a,b], \tag{B.9.3}$$

using the same type of arguments as mentioned at the beginning of the previous section. We also have that g is uniformly continuous on $[a,b]$, with $g' = f$.

There are analogous statements for complex-valued functions, as before. The derivative of a complex-valued function at a point may be defined as the complex number whose real and imaginary parts are the derivatives of the real and imaginary parts of the function at the same point, when they exist.

B.9.1 Differentiating power series

Let $\sum_{j=0}^{\infty} a_j\, x^j$ be a power series with real or complex coefficients, and suppose that

$$\sum_{j=0}^{\infty} j\, |a_j|\, r^j \tag{B.9.4}$$

converges for some positive real number r. It is easy to see that this implies that

$$\sum_{j=0}^{\infty} |a_j|\, r^j \tag{B.9.5}$$

converges.

Put

$$f(x) = \sum_{j=0}^{\infty} a_j\, x^j \tag{B.9.6}$$

and

$$\phi(x) = \sum_{j=1}^{\infty} j\, a_j\, x^{j-1} \tag{B.9.7}$$

for each $x \in \mathbf{R}$ with $|x| \le r$. The partial sums of these series converge uniformly on $[-r, r]$, as in Section B.5. One can use the remarks at the beginning of the section to get that f is differentiable on $[-r, r]$, with

$$f' = \phi \tag{B.9.8}$$

on $[-r, r]$.

B.9.2 Differentiating $\exp(a\, t)$

If a is a complex number, then

$$\exp(a\, t) = \sum_{j=1}^{\infty} \frac{a^j\, t^j}{j!} \tag{B.9.9}$$

defines a complex-valued function of t on the real line, as in Section B.1. One can use the remarks in the previous subsection to get that this function is differentiable on $\mathbf{R}$, with

$$\frac{d}{dt} \exp(a\, t) = a\, \exp(a\, t). \tag{B.9.10}$$

Alternatively, one can check more directly that this holds at $t = 0$. In order to take the derivative at other points in $\mathbf{R}$, one can use the fact that the exponential of a sum if equal to the product of the exponentials, as on p179 of [309], and p2 of [308].

B.9.3 More on differentiating power series

Suppose that $0 < \rho \le +\infty$ has the property that (B.9.5) converges for all positive real numbers r with $r < \rho$. In this case, it is well known and not difficult to show that this implies that (B.9.4) converges when $r < \rho$ too. More precisely, if $r < t < \rho$, then one can use the convergence of

$$\sum_{j=0}^{\infty} |a_j|\, t^j \tag{B.9.11}$$

to get that (B.9.4) converges.

It follows that the series on the right sides of (B.9.6) and (B.9.7) converge absolutely for every $x \in \mathbf{R}$ with $|x| < \rho$, so that $f(x)$ and $\phi(x)$ may be defined on $(-\rho, \rho)$ as before. We also get that f is differentiable on $(-\rho, \rho)$, with derivative as in (B.9.8).

B.10 Open and closed sets

Let n be a positive integer, and let U be a subset of $\mathbf{R}^n$. We say that U is an *open set* in $\mathbf{R}^n$ if for every $x \in U$ there is a positive real number r such that

$$B(x, r) \subseteq U. \tag{B.10.1}$$

Remember that $B(x, r)$ is the open ball in $\mathbf{R}^n$ centered at x with radius r with respect to the standard Euclidean metric, as in Subsection A.2.2.

If $w \in \mathbf{R}^n$ and t is a positive real number, then it is well known and not difficult to show that

$$B(w, t) \text{ is an open set.} \tag{B.10.2}$$

Indeed, if $x \in B(w, t)$, then one can check that

$$B(x, r) \subseteq B(w, t) \tag{B.10.3}$$

when $r = t - d(w, x) > 0$, using the triangle inequality.

If $U_1, \ldots, U_l$ are finitely many open sets in $\mathbf{R}^n$, then one can verify that

$$U_1 \cap \cdots \cap U_n \text{ is an open set.} \tag{B.10.4}$$

It is easy to see that the union of any family of open sets in $\mathbf{R}^n$ is an open set too.

B.10.1 Limit points and closed sets

Let E be a subset of $\mathbf{R}^n$, and let x be an element of $\mathbf{R}^n$. We say that x is a *limit point* of E if for every positive real number r there is a $y \in E$ such that $x \neq y$ and

$$|x - y| < r. \tag{B.10.5}$$

Note that x need not be an element of E, and that elements of E may not be limit points of E.

If x is a limit point of E and t is a positive real number, then it is well known and not difficult to show that

$$\text{there are infinitely many elements of } E \text{ in } B(x, t). \tag{B.10.6}$$

In particular, if E has only finitely many elements, then E has no limit points in $\mathbf{R}^n$.

We say that E is a *closed set* in $\mathbf{R}^n$ if E contains all of its limit points. Equivalently, this means that if $x \in \mathbf{R}^n$ is a limit point of E, then

$$x \in E. \tag{B.10.7}$$

It is well known that

$$\text{closed balls in } \mathbf{R}^n \text{ are closed sets.} \tag{B.10.8}$$

This can be verified directly from the definitions, or using another characterization of closed sets in the next subsection.

B.10.2 Complements of subsets of $\mathbf{R}^n$

If A and B are sets, then put

$$A \setminus B = \{x \in A : x \notin B\}. \tag{B.10.9}$$

If B is a subset of A, then this may be called the *complement* of B in A. Observe that

$$A \setminus (A \setminus B) = B \tag{B.10.10}$$

when $B \subseteq A$.

If E is any subset of $\mathbf{R}^n$, then it is well known and not too difficult to show that E is a closed set if and only if

$$\mathbf{R}^n \setminus E \text{ is an open set.} \tag{B.10.11}$$

Equivalently, this means that a subset U of $\mathbf{R}^n$ is an open set if and only if

$$\mathbf{R}^n \setminus U \text{ is a closed set,} \tag{B.10.12}$$

because of (B.10.10).

If $E_1, \ldots, E_l$ are finitely many closed sets in $\mathbf{R}^n$, then it is well known that

$$E_1 \cup \cdots \cup E_l \text{ is a closed set in } \mathbf{R}^n. \tag{B.10.13}$$

Indeed, if $E_1, \ldots, E_l$ are any subsets of $\mathbf{R}^n$, then one can check that

$$\mathbf{R}^n \setminus \Big(\bigcup_{j=1}^{l} E_j\Big) = \bigcap_{j=1}^{l} (\mathbf{R}^n \setminus E_j). \tag{B.10.14}$$

If $E_1, \ldots, E_l$ are closed sets, then the right side is an open set, because of (B.10.4).

It is also well known that the intersection of any family of closed sets in $\mathbf{R}^n$ is a closed set as well. This can be obtained from the definition of a closed set, or using the fact that the complement of an intersection is equal to the union of the complements of the sets.

B.10.3 Bounded sets and the extreme value theorem

A subset E of $\mathbf{R}^n$ is said to be *bounded* if there is a nonnegative real number C such that

$$|x| \le C. \tag{B.10.15}$$

One can check that open and closed balls in $\mathbf{R}^n$ are bounded sets, using the triangle inequality.

Suppose that E is a nonempty subset of $\mathbf{R}^n$ that is both closed and bounded. Also let f be a continuous real-valued continuous function on E. Another version of the *extreme value theorem* states that f attains its maximum and minimum on E. This was mentioned in Subsection B.6.1 when E is a cosed ball in $\mathbf{R}^n$.

If f is a continuous complex-valued function on E, then one can apply the previous statement to $|f|$, to get that f is bounded on E, as before.

B.10.4 Closures and boundaries

If E is any subset of $\mathbf{R}^n$, then the *closure* of E in $\mathbf{R}^n$ is the subset $\overline{E}$ of $\mathbf{R}^n$ consisting of the elements of E and the limit points of E in $\mathbf{R}^n$. In particular,

$$E \subseteq \overline{E}, \tag{B.10.16}$$

by definition. One can check that $E = \overline{E}$ if and only if E is a closed set, directly from the definitions.

If E is any subset of $\mathbf{R}^n$ again, then it is well known and not too difficult to show that

$$\overline{E} \text{ is a closed set.} \tag{B.10.17}$$

This is the same as saying that

$$\mathbf{R}^n \setminus \overline{E} \text{ is an open set,} \tag{B.10.18}$$

as before.

The *boundary* of E in $\mathbf{R}^n$ is defined to be the set

$$\partial E = \overline{E} \cap \overline{(\mathbf{R}^n \setminus E)}. \tag{B.10.19}$$

This is a closed set in $\mathbf{R}^n$, because it is the intersection of two closed sets.

If U is an open set in $\mathbf{R}^n$ then the boundary of U may be expressed more simply as

$$\partial U = \overline{U} \setminus U. \tag{B.10.20}$$

This is because the complement of U in $\mathbf{R}^n$ is a closed set, and thus equal to its own closure.

B.11 Partial derivatives

Let n be a positive integer, let U be an open subset of $\mathbf{R}^n$, and let f be a real-valued function on U. If x is an element of U and l is a positive integer less than or equal to n, then the *partial derivative* of f at x in the lth variable may be denoted

$$\partial_l f(x) = D_l f(x) = \frac{\partial f}{\partial x_l}(x), \tag{B.11.1}$$

when it exists. Sometimes subscripts are used to indicate partial derivatives, so that (B.11.1) may also be denoted $f_{x_l}(x)$.

If n is not too large, then we may use different letters for the coordinates of a point in $\mathbf{R}^n$, and similar notation for partial derivatives. A point in $\mathbf{R}^2$ may be expressed as (x, y), for instance, and the partial derivatives of f with respect to x and y may be expressed as

$$f_x = \frac{\partial f}{\partial x} \text{ and } f_y = \frac{\partial f}{\partial y}, \tag{B.11.2}$$

respectively.

Sometimes one of the variables may be denoted t, and the other variables may be expressed as in either of the previous two paragraphs. The partial derivative of f with respect to t may be expressed as

$$f_t = \frac{\partial f}{\partial t}, \tag{B.11.3}$$

and the other partial derivatives may be expressed as in (B.11.1) or (B.11.2), as appropriate.

One may also consider complex-valued functions f on U. In this case, the real and imaginary parts of a partial derivative of f are the same as the corresponding partial derivative of the real and imaginary parts of f, respectively, when they exist.

B.11.1 Continuous differentiability

Suppose that f is a real or complex-valued function on U such that the partial derivatives of f in each variable exist at every point in U. If each of the partial derivatives of f is continuous on U, then f is said to be *continuously differentiable* on U.

If f is continuously differentiable on U, then it is well known that

$$f \text{ is continuous on } U. \tag{B.11.4}$$

More precisely, this holds when the partial derivatives of f are locally bounded on U, in the sense that for each $x \in U$ there is a positive real number r such that $B(x,r) \subseteq U$ and the partial derivatives of f are bounded on $B(x,r)$. If $n = 1$, then a function is continuous at any point at which the derivative exists.

Let g be another real or complex-valued function on U, and suppose that f and g are both continuously differentiable on U. It is easy to see that $f + g$ and $f\,g$ are continuously differentiable on U as well.

B.11.2 Directional derivatives

Let f be a real or complex-valued function on U again, and let $x \in U$ and $w \in \mathbf{R}^n$ be given. Also let $U(x,w)$ be the of real numbers t such that

$$x + t\,w \in U. \tag{B.11.5}$$

One can check that this is an open set in the real line that contains 0.

The *directional derivative* of f at x in the direction of w is defined to be the derivative of

$$f(x + t\,w) \tag{B.11.6}$$

as a function of $t \in U(x,w)$ at 0, when it exists. This may be denoted

$$D_w f(x). \tag{B.11.7}$$

If f is continuously differentiable on U, then it is well known that the directional derivative exists, with

$$D_w f(x) = \sum_{j=1}^{n} w_j \, \frac{\partial f}{\partial x_j}(x). \tag{B.11.8}$$

Note that the right side is linear in w.

B.11.3 Twice continuous differentiability

Let f be a real or complex-valued function on U, and suppose for the moment that the partial derivative of f in the lth variable exists at every point in U for some $l \le n$. Consider the partial derivative

$$\frac{\partial}{\partial x_j}\Big(\frac{\partial f}{\partial x_l}\Big) \tag{B.11.9}$$

of $\partial f/\partial x_l$ in the jth variable, $1 \le j \le n$, when it exists.

Suppose now that $\partial f/\partial x_l$ exists at every point in U for each $l = 1, \ldots, n$, and that its partial derivatives (B.11.9) exist at every point in U for each $j = 1, \ldots, n$. If each of these second derivatives of f is continuous on U, then f is said to be *twice continuously differentiable* on U.

Equivalently, this means that

$$\frac{\partial f}{\partial x_l} \text{ is continuously differentiable on } U \tag{B.11.10}$$

for each l. This implies that

$$\frac{\partial f}{\partial x_l} \text{ is continuous on } U \tag{B.11.11}$$

for each l, as before, so that

$$f \text{ is continuously differentiable on } U. \tag{B.11.12}$$

It follows that f is continuous on U, as before.

If f is twice continuously differentiable on U, then it is well known that

$$\frac{\partial}{\partial x_j}\Big(\frac{\partial f}{\partial x_l}\Big) = \frac{\partial}{\partial x_l}\Big(\frac{\partial f}{\partial x_j}\Big) \tag{B.11.13}$$

on U. These second derivatives may be denoted

$$\frac{\partial^2 f}{\partial x_j\, \partial x_l} = \frac{\partial^2 f}{\partial x_l\, \partial x_j} \tag{B.11.14}$$

or

$$f_{x_j\, x_l} = f_{x_l\, x_j}. \tag{B.11.15}$$

Let g be another real or complex-valued function on U, and suppose that f and g are both twice continuously differentiable on U. One can check that $f + g$ and $f\,g$ are also twice continuously differentiable on U.

B.12 Connected sets in $\mathbf{R}^n$

Let n be a positive integer, and let A, B be subsets of $\mathbf{R}^n$. We say that A and B are *separated* if

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset, \tag{B.12.1}$$

where $\overline{A}$, $\overline{B}$ are the closures of A, B in $\mathbf{R}^n$, respectively, as in Subsection B.10.4.

Note that disjoint closed subsets of $\mathbf{R}^n$ are separated. It is well known and not difficult to show that disjoint open subsets of $\mathbf{R}^n$ are separated too.

If a subset E of $\mathbf{R}^n$ can be expressed as

$$E = A \cup B, \tag{B.12.2}$$

where A, B are separated and $A, B \neq \emptyset$, then E is said to be *not connected* in $\mathbf{R}^n$. Otherwise, E is said to be *connected*.

If E is a closed set in $\mathbf{R}^n$, and E is expressed as in (B.12.2), where A, B are separated sets in $\mathbf{R}^n$, then one can check that A and B are closed sets in $\mathbf{R}^n$. In this case, we get that E is connected if and only if E cannot be expressed as the union of two nonempty disjoint closed subsets of $\mathbf{R}^n$.

Similarly, if E is an open set in $\mathbf{R}^n$, and E is expressed as in (B.12.2), where A, B are separated sets in $\mathbf{R}^n$, then one can verify that A and B are open sets in $\mathbf{R}^n$. This means that E is connected if and only if E cannot be expressed as the union of two nonempty disjoint open subsets of $\mathbf{R}^n$.

B.12.1 Connectedness in the real line

It is well known that a subset E of the real line is connected if and only if for every $x, y \in E$ with $x < y$ and every $w \in \mathbf{R}$ with

$$x < w < y, \tag{B.12.3}$$

we have that

$$w \in E. \tag{B.12.4}$$

The necessity of this condition can be obtained directly from the definitions. More precisely, if $w \notin E$, then one can check that

$$A = \{t \in E : t < w\} \tag{B.12.5}$$

and

$$B = \{t \in E : t > w\} \tag{B.12.6}$$

are nonempty separated subsets of $\mathbf{R}$ whose union is equal to E.

One can use this to show that

$$\text{convex subsets of } \mathbf{R}^n \text{ are connected.} \tag{B.12.7}$$

The converse holds when $n = 1$, as in the preceding paragraph.

B.12.2 Locally constant functions on subsets of $\mathbf{R}^n$

Let E be a nonempty subset of $\mathbf{R}^n$, and let f be a function on E with values in any set. We say that

$$f \text{ is } \textit{locally constant} \text{ at a point } x \in E \tag{B.12.8}$$

if there is a positive real number r such that

$$f(x) = f(y) \tag{B.12.9}$$

for every $y \in E$ with

$$|x - y| < r. \tag{B.12.10}$$

Supose for the moment that f is *locally constant* on E, in the sense that f is locally constant at every point in E. Let w be an element of E, and put

$$A = \{x \in E : f(x) = f(w)\} \tag{B.12.11}$$

and

$$B = \{x \in E : f(x) \neq w\}. \tag{B.12.12}$$

Clearly $w \in A$, and E is equal to the union of A and B.

One can check that A and B are separated in $\mathbf{R}^n$, because f is locally constant on E. If E is connected, then it follows that $B = \emptyset$, so that f is constant on E.

Conversely, if E is not connected, then E may be expressed as the union of two nonempty separated sets A and B in $\mathbf{R}^n$. In this case, one can get a locally constant function f on E that is not constant by taking f to be constant on each of A and b, with different constant values on these sets.

B.12.3 Locally constant functions on open sets

Let U be a nonempty open set in $\mathbf{R}^n$, and let f be a real-valued function on U. Suppose that the first partial derivatives of f exist at every point in U, with

$$\frac{\partial f}{\partial x_l} = 0 \tag{B.12.13}$$

on U for each $l = 1, \ldots, n$. Under these cnditions, it is well known and not difficult to show that f is locally constant on U. If U is connected, then it follows that f is constant on U. This also works for complex-valued functions on U.

B.13 Vector-valued functions

Let m and n be positive integers, let E be a nonempty subset of $\mathbf{R}^n$, and let f be a function on E with values in $\mathbf{R}^m$. Thus, if $x \in E$, then

$$f(x) = (f_1(x), \ldots, f_m(x)) \tag{B.13.1}$$

is an element of $\mathbf{R}^m$. This means that $f_1, \ldots, f_m$ are real-valued functions on E.

One can define what it means for f to be *continuous* at a point $x \in E$ in essentially the same way as in Section B.2, using the standard Euclidean metrics on $\mathbf{R}^n$ and $\mathbf{R}^m$. One can check that f is continuous at x if and only if $f_1, \ldots, f_m$ are all continuous at x, as real-valued functions on E.

Continuity of f at x can be characterized in terms of convergence of sequences as before. This can be shown using the same type of arguments as for real or complex-valued functions, or by reducing to the case of real-valued functions.

If f is continuous at every point in E, then we may simply sat that f is *continuous* on E, as before.

B.13.1 More on continuity of compositions

Let A be a nonempty subset of $\mathbf{R}^m$, let p be another positive integer, and let ϕ be a function on A with values in $\mathbf{R}^p$. If

$$f(y) \in A \tag{B.13.2}$$

for every $y \in E$, then the *composition* $\phi \circ f$ of f with ϕ may be defined as a function on E with values in $\mathbf{R}^p$ by

$$(\phi \circ f)(y) = \phi(f(y)), \tag{B.13.3}$$

as usual.

If f is continuous at $x \in E$, and ϕ is continuous at $f(x)$, then $\phi \circ f$ is continuous at x, as in Subsection B.2.2.

Remember that $v \mapsto |v|$ is continuous as a real-valued function on $\mathbf{R}^m$, as in Subsection B.2.2. If f is continuous at $x \in E$, then $|f|$ is continuous at x as a real-valued function on E, as before.

B.13.2 More on partial derivatives

Suppose now that E is an open set in $\mathbf{R}^n$. If $x \in E$ and l is a positive integer less than or equal to n, then the *partial derivative* of f at x in the lth variable may be denoted as in (B.11.1), when it exists. Of course, the partial derivative is an element of $\mathbf{R}^m$, when it exists.

This partial derivative exists exactly when the partial derivatives of the components of f at x in the lth variable exist. In this case, the components of the partial derivative of f at x in the lth variable are equal to the corresponding partial derivatives of the components of f at x.

We say that f is *continuously differentiable* on E if the partial derivatives of f in each variable exist at every point in E, and are continuous as $\mathbf{R}^m$-valued functions on E. This is the same as saying that the components of f are continuously differentiable as real-valued functions on E, as in Subsection B.11.1. This implies that f is continuous as an $\mathbf{R}^m$-valued function on E, as before.

If $x \in E$ and $w \in \mathbf{R}^n$, then the *directional derivative* $D_w f(x)$ of f at x in the direction of w may be defined as in Subsection B.11.2, when it exists. Note that $D_w f(x)$ is an element of $\mathbf{R}^m$, when it exists. This happens exactly when the directional derivatives of the components of f at x in the direction of w exist, in which case the componentis of $D_w f(x)$ are the same as the corresponding directional derivatives of the components of f.

If f is continuously differentiable on E, then $D_w f(x)$ exists and may be expressed as in (B.11.8), as before.

B.13.3 Compositions and continuous differentiability

Suppose that A is an open subset of $\mathbf{R}^m$ too. If f is continuously differentiable on E, and ϕ is continuously differentiable on A, then it is well known that

$$\phi \circ f \text{ is continuously differentiable on } E. \tag{B.13.4}$$

In fact, if $x \in U$ and $w \in \mathbf{R}^n$, then the directional derivative of $\phi \circ f$ at x in the direction of w may be expressed as

$$D_w(\phi \circ f)(x) = (D_{D_w f(x)}\phi)(f(x)), \tag{B.13.5}$$

which is to say that it is equal to the directional derivative of ϕ at $f(x)$ in the direction of $D_w f(x)$. This is the version of the *chain rule* for vector-valued functions of several variables.

Let $e_1, \ldots, e_n$ be the *standard basis vectors* in $\mathbf{R}^n$. This means that the jth component of e_l is equal to 1 when $j = l$ and to 0 otherwise, $1 \le j, l \le n$. The partial derivative at x of a function on E in the lth variable is the same as the directional derivative at x of the function in the direction e_l.

Thus (B.13.5) may be used to express the partial derivatives of $\phi \circ f$ at x in terms of the partial derivatives of ϕ at $f(x)$ and the partial derivatives of f at x. More precisely,

$$\partial_l(\phi \circ f)(x) = (D_{\partial_l f(x)}\phi)(f(x)) = \sum_{k=1}^{m} (\partial_l f_k(x))\,(\partial_k \phi(f(x))) \tag{B.13.6}$$

for each $l = 1, \ldots, n$, where f_k is the kth component of f for each $k = 1, \ldots, m$.

It is easy to reduce to the case where $p = 1$, by considering the components of ϕ separately, as real-valued functions on A. If we also have that $m = 1$, then this version of the chain rule reduces to the usual formulation for real-valued functions of one variable. Indeed, (B.13.5) is the same as saying that

$$D_w(\phi \circ f)(x) = \phi'(f(x))\,(D_w f(x)) \tag{B.13.7}$$

in this case. In particular, (B.13.6) simplifies to

$$\partial_l(\phi \circ f)(x) = \phi'(f(x))\,(\partial_l f(x)). \tag{B.13.8}$$

The continuity of the partial derivatives of $\phi \circ f$ may be obtained more directly from the continuity of f, its first partial derivatives, and ϕ' under these conditions.

B.13.4 More on twice continuous differentiability

If $\partial f/\partial x_l$ exists at every point in E, then we may consider its partial derivatives, as in (B.11.9), when they exist. If $\partial f/\partial x_l$ is continuously differentiable on E for each $l = 1, \ldots, n$, then f is said to be *twice continuously differentiable* on E, as in Subsection B.11.3. This happens exactly when the components of f are continuously differentiable as real-valued functions on E, as usual.

If f is twice continuously differentiable on E, then f is continuously differentiable on E, and thus continuous on E, as before. We also have that (B.11.13) holds in this case, as before.

If f is twice continuously differentiable on E, and ϕ is twice continuously differentiable on A, then

(B.13.9) $$\phi \circ f \text{ is twice continuously differentiable on } E.$$

Equivalently, one can check that the partial derivatives of $\phi \circ f$ are continuously differentiable on E, using (B.13.6). This also uses the fact that

(B.13.10) $$(\partial_k \phi) \circ f \text{ is continuously differentiable on } E,$$

because $\partial_k \phi$ is continuously differentiable on A and f is continuously differentiable on E.

B.14 More on connectedness

Let m and n be positive integers, let E be a nonempty subset of $\mathbf{R}^n$, and let f be a continuous mapping from E into $\mathbf{R}^m$. Put

(B.14.1) $$f(E) = \{f(y) : y \in E\},$$

so that $f(E)$ is a subset of $\mathbf{R}^m$. If E is connected as a subset of $\mathbf{R}^n$, then it is well known that

(B.14.2) $$f(E) \text{ is connected as a subset of } \mathbf{R}^m.$$

More precisely, in order to obtain this from standard results about continuity and connectedness, one also uses the fact that E is connected when it is considered as a subset of itself, with respect to the restriction of the standard Euclidean metric on $\mathbf{R}^n$ to E. This is not needed when f is the restriction to E of a continuous mapping from $\mathbf{R}^n$ into $\mathbf{R}^m$.

The intermediate value theorem basically corresponds to the case where $m = n = 1$.

Suppose for the moment that $n = 1$, and that $E = [a, b]$ for some $a, b \in \mathbf{R}$ with $a \le b$. If f is a continuous mapping from $[a, b]$ into $\mathbf{R}^m$, then f can be extended to a continuous mapping fom $\mathbf{R}$ into $\mathbf{R}^m$, by putting

(B.14.3) $$f(t) = f(a)$$

when $t < a$, and

(B.14.4) $$f(t) = f(b)$$

when $t > b$.

B.14.1 Path connectedness

A subset E of $\mathbf{R}^n$ is said to be *path connected* if for every $x, y \in E$ there is a continuous path in E from x to y. This means that there is a continuous mapping p from the closed unit interval $[0,1]$ in the real line into $\mathbf{R}^m$ such that

$$p(0) = x,\ p(1) = y, \text{ and } p([0,1]) \subseteq E. \tag{B.14.5}$$

It is easy to see that

$$\text{convex sets in } \mathbf{R}^n \text{ are path connected.} \tag{B.14.6}$$

It is well known and not too difficult to show that

$$\text{path-connected sets in } \mathbf{R}^n \text{ are connected.} \tag{B.14.7}$$

This uses the fact that if p is a continuous mapping from $[0,1]$ into $\mathbf{R}^n$, then $p([0,1])$ is a connected set in $\mathbf{R}^n$.

It is also well known that

$$\text{connected open sets in } \mathbf{R}^n \text{ are path connected.} \tag{B.14.8}$$

Suppose that E is a path-connected subset of $\mathbf{R}^n$, and that f is a continuous mapping from E into $\mathbf{R}^m$. Under these conditions, one can check that

$$f(E) \text{ is path connected in } \mathbf{R}^m. \tag{B.14.9}$$

B.14.2 Nonemptiness of the boundary of an open set

Let U be a nonempty open subset of $\mathbf{R}^n$ that is a proper subset of $\mathbf{R}^n$, so that $U \neq \mathbf{R}^n$. One can verify that

$$\partial U \neq \emptyset. \tag{B.14.10}$$

This is the same as saying that U is not a closed set in $\mathbf{R}^n$, because of the description of the boundary of an open set in Subsection B.10.4. Otherwise, one could check that $\mathbf{R}^n$ would not be connected.

Alternatively, let x be an element of U, and let z be an element of $\mathbf{R}^n \setminus U$. One can show that there is a real number t_0 such that $0 < t_0 \leq 1$ and

$$(1 - t_0)\, x + t_0\, z \in \partial U. \tag{B.14.11}$$

In fact, one can take t_0 to be the infimum or greatest lower bound of the set of positive real numbers t such that

$$(1 - t)\, x + t\, z \in \mathbf{R}^n \setminus U. \tag{B.14.12}$$

Appendix C

A bit of analysis, 3

Bibliography

[1] S. Abian and A. Brown, *On the solution of the differential equation* $f(x, y, y') = 0$, American Mathematical Monthly **66** (1959), 192–199.

[2] W. Adkins and M. Davison, *Putzer's algorithm for* $e^{A\,t}$ *via the Laplace transform*, Mathematics Magazine **83** (2010), 267–275.

[3] F. Adler, E. Allman, and L. de Pillis, *A letter from the guest editors*, American Mathematical Monthly **121** (2014), 751–753.

[4] R. Agnew, *Differential Equations*, 2nd edition, McGraw-Hill, 1960.

[5] R. Agnew, *Views and approximations of differential equations*, American Mathematical Monthly **60** (1953), 1–6.

[6] R. Agnew, *Classroom notes: Transformations of the Laplacian*, American Mathematical Monthly **60** (1953), 323–325.

[7] A. Aksoy and M. Martelli, *The wave equation, mixed partial derivatives, and Fubini's theorem*, American Mathematical Monthly **111** (2004), 340–347.

[8] G. Albert, *The closure of systems of orthogonal functions*, American Mathematical Monthly **50** (1943), 163–169.

[9] C. Allendoerfer, *Classroom notes: Editorial — The proof of Euler's equation*, American Mathematical Monthly **55** (1948), 94–95.

[10] S. Antman, *The equations for large vibrations of strings*, American Mathematical Monthly **87** (1980), 359–370.

[11] T. Apostol, *Classroom notes: Some explict formulas for the exponential matrix* $e^{t\,A}$, American Mathematical Monthly **76** (1969), 289–292.

[12] T. Apostol, *Mathematical Analysis*, 2nd edition, Addison-Wesley, 1974.

[13] T. Apostol, *Explicit formulas for solutions of the second-order matrix differential equation* $Y'' = A\,Y$, American Mathematical Monthly **82** (1975), 159–162.

[14] R. Archibald, *Mathematicians and music*, American Mathematical Monthly **31** (1924), 1–25.

[15] T. Archibald, *Connectivity and smoke-rings: Green's second identity in its first fifty years*, Mathematics Magazine **62** (1989), 219–232.

[16] G. Aumann, *Classroom notes: The Lipschitz condition and uniqueness*, American Mathematical Monthly **72** (1965), 652–653.

[17] S. Axler, *Harmonic functions from a complex analysis viewpoint*, American Mathemtical Monthly **93** (1986), 246–258.

[18] S. Axler, P. Bourdon, and W. Ramey, *Bôcher's theorem*, American Mathematical Monthly **99** (1992), 51–55.

[19] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, 2nd edition, Springer-Verlag, 2001.

[20] R. Bagley and J. McKnight, *Differentiation of infinite series of functions*, American Mathematical Monthly **66** (1959), 890–892.

[21] J. Baker, *Integration over spheres and the divergence theorem for balls*, American Mathematical Monthly **104** (1997), 36–47.

[22] J. Baker, *Integration of radial functions*, Mathematics Magazine **72** (1999), 392–395.

[23] J. Baker, *The Dirichlet problem for ellipsoids*, American Mathematical Monthly **106** (1999), 829–834.

[24] J. Ballantine, *Questions, discussions, and notes: Elementary development of certain infinite series*, American Mathematical Monthly **44** (1937), 470–472.

[25] E. Barbeau, *Mathematical education: A course on the development of analysis*, American Mathematical Monthly **83** (1976), 42–43.

[26] R. Beals, *Advanced Mathematical Analysis: Periodic Functions and Distributions, Complex Analysis, Laplace Transform and Applications*, Springer-Verlag, 1973.

[27] R. Beals, *Analysis: An Introduction*, Cambridge University Press, 2004.

[28] R. Beals, *The scope of the power series method*, Mathematics Magazine **86** (2013), 56–62.

[29] R. Beals and R. Wong, *Special Functions and Orthogonal Polynomials*, Cambridge University Press, 2016.

[30] P. Beesack, *On an existence theorem for complex-valued differential equations*, American Mathematical Monthly **65** (1958), 112–115.

[31] P. Beesack, *On improper multiple integrals*, Mathematics Magazine **43** (1970), 113–123.

[32] A. Berger, *A beautiful inequality by Saint-Venant and Pólya revisited*, American Mathematical Monthly **130** (2023), 239–250.

[33] S. Berhanu and A. Mohammed, *A Harnack inequality for ordinary differential equations*, American Mathematica; Monthly **112** (2005), 32–41.

[34] P. Bharatiya, *The inversion of a convolution transform whose kernel is a Bessel function*, American Mathematical Monthly **72** (1965), 393–397.

[35] G. Birkhoff and G.-C. Rota, *On the completeness of Sturm–Liouville expansions*, American Mathematical Monthly **67** (1960), 835–841.

[36] G. Birkhoff and G.-C. Rota, *Ordinary Differential Equations*, 4th edition, Wiley, 1989.

[37] I. Bivens, *When do orthogonal families of curves possess a complex potential?*, Mathematics Magazine **65** (1992), 226–235.

[38] D. Bleecker, *Infiitesimal deformations of portions of the standard sphere in E^3*, American Mathematical Monthly **87** (1980), 175–182.

[39] D. Bleecker and G. Csordas, *Basic Partial Differential Equations*, International Press, 1996.

[40] M. Boas, *A formula for the derivatives of Legendre polynomials*, American Mathematical Monthly **70** (1963), 643–644.

[41] R. Boas, Jr., *Inversion of Fourier and Laplace transforms*, American Mathematical Monthly **69** (1962), 955–960.

[42] R. Boas, Jr., *Lion Hunting & Other Mathematical Pursuits: A Collection of Mathematics, Verse, and Stories*, Edited and with an introduction by G. Alexanderson and D. Mugler, Mathematical Association of America, 1995.

[43] R. Boas, Jr., *A Primer of Real Functions*, 4th edition, revised and updated by H. Boas, Mathematical Association of America, 1996.

[44] S. Bochner, *Fourier series came first*, American Mathematical Monthly **86** (1979), 197–199.

[45] J. Boyd, *Correcting three errors in Kantorovich & Krylov's* Approximate methods of higher analysis, American Mathematical Monthly **123** (2016), 241–257.

[46] J. Boyd and H. Wang, *Convergent power series for boundary value problems and eigenproblems with application to atmospheric and oceanic tides*, American Mathematical Monthly **124** (2-17), 306–323.

[47] J. Bramble and L. Payne, *Mean value theorems for polyharmonic functions*, American Mathematical Monthly **73** (1966), 124–127.

[48] F. Brauer, *On the asymptotic behavior of Bessel functions*, American Mathematical Monthly **70** (1963), 954–957.

[49] D. Bressoud, *A Radical Approach to Real Analysis*, 2nd edition, Mathematical Association of America, 2007.

[50] D. Bressoud, *A Radical Approach to Lebesgue's Theory of Integration*, MAA Textbooks, Cambridge University Press, 2008.

[51] D. Bressoud, *Historical reflections on teaching the fundamental theorem of integral calculus*, American Mthematical Monthly **118** (2011), 99–115.

[52] D. Bressoud, *Calculus Reordered: A History of the Big Ideas*, Princeton University Press, 2019.

[53] A. Browder, *Mathematical Analysis: An Introduction*, Springer-Verlag, 1996.

[54] P. Bullen, *An inequality for variations*, American Mathematical Monthly **90** (1983), 560.

[55] R. Burckel, *A strong converse to Gauss's mean-value theorem*, American Mathematical Monthly **87** (1980), 819–820.

[56] R. Burckel, *Three secrets about harmonic functions*, American Mathematical Monthly **104** (1997), 52–56.

[57] R. Burckel and C. Goffman, *Rectifiable curves are of zero content*, Mathematics Magazine **44** (1971), 179–180.

[58] P. Buser, *Notes: A geometric approach to invariant subspaces of orthogonal matrices*, American Mathematical Monthly **89** (1982), 751.

[59] K. Bush, *Mathematical Notes: A characterization of orthogonal matrices*, American Mathematical Monthly **74** (1967), 1083–1084.

[60] F. Cajori, *The early history of partial differential equations and of partial differentiation and integration*, American Mathematical Monthly **35** (1928), 459–467.

[61] D. Cannell, *Geoge Green: an enigmatic mathematician*, American Mathematical Monthly **106** (1999), 136–151.

[62] D. Cannell, *George Green, Mathematician & Physicist 1793–1841*, 2nd edition, Society for Industrial and Applied Mathematics (SIAM), 2001.

[63] J. Cantarella, D. DeTurck, and H. Gluck, *Vector calculus and the topology of domains in* 3*-space*, American Mathematical Monthly **109** (2002), 409–442.

[64] J. Cell, *Solution of certain linear boundary value problems by use of finite transforms*, Mathematics Magazine **40** (1967), 200–206.

[65] L. Cesari, *Variation, multiplicity, and semicontinuity*, American Mathematical Monthly **65** (1958), 317–332.

[66] L. Cesari, *Rectifiable curves and the Weierstrass integral*, American Mathematical Monthly **65** (1958), 485–500.

[67] L. Cesari, *Recent results in surface area theory*, American Mathematical Monthly **66** (1959), 173–192.

[68] S. Chapman, *Drums that sound the same*, American Mathematical Monthly **102** (1995), 124–138.

[69] P. Chernoff, *Pointwise convergence of Fourier series*, American Mathematical Monthly **87** (1980), 399–400; letter to the editor, **92** (1985), 373–374.

[70] S. Chu, *On a mean value property for solutions of a wave equation*, American Mathematical Monthly **74** (1967), 711–713.

[71] R. Churchill, *Integral transforms and boundary value problems*, American Mathematical Monthly **59** (1952), 149–155.

[72] J. Cid and R. López Pouso, *Does Lipschitz with respect to x imply uniqueness for the differential equation $y' = f(x, y)$?*, American Mathematical Monthly **116** (2009), 61–66.

[73] J. Cid and R. López Pouso, *Integration by parts and by substitution unified with applications to Green's theorem and uniqueness for ODEs*, American Mathematical Monthly **123** (2016), 40–52.

[74] J. Cliborn and B. Jordan, *Minimal surfaces*, American Mathematical Monthly **99** (1992), 376.

[75] E. Coddington and N. Levinson, *Theory of Ordinary Differential quations*, McGraw-Hill, 1955.

[76] A. Coffman and Y. Zhang, *Vector fields with continuous curl but discontinuous partial derivaties*, American Mathematical Monthly **127** (2020), 911–917.

[77] J. Conway, *A First Course in Analysis*, Cambridge University Press, 2018.

[78] J. Cortissoz, *A note on harmonic functions on surfaces*, American Mathematical Monthly **123** (2016), 884–893.

[79] R. Courant, *Soap film experiments with minimal surfaces*, American Mathematical Monthly **47** (1940), 167–174.

[80] A. Czarnecki, *A note on the zeros of Bessel functions*, American Mathematical Monthly **71** (1964), 403–404.

[81] X. Dai, *Continuous differentiability of solutions of ODEs with respect to initial conditions*, American Mathematical Monthly **113** (2006), 66–70.

[82] M. Deakin, *Euler's version of the Laplace transform*, American Mathematical Monthly **87** (1980), 264–269.

[83] N. de Silva, *A concise, elementary proof of Arzelà's bounded convergence theorem*, American Mathematical Monthly **117** (2010), 918–920.

[84] J. Dettman, *The solution of a second order linear differential equation near a regular singular point*, American Mathematical Monthly **71** (1964), 378–385.

[85] J. Dettman, *Power series solutions of ordinary differential equations*, American Mathematical Monthly **74** (1967), 428–430.

[86] A. Devinatz, *Advanced Calculus*, Holt, Rinehart and Winston, 1968.

[87] J. Dodziuk, *Eigenvalues of the Laplacian and the heat equation*, American Mathematical Monthly **88** (1981), 686-695.

[88] T. Drammatico, *A new proof of the equality of mixed second partial derivatives*, American Mathematical Monthly **122** (2015), 602–603.

[89] R. Driver, *Classroom notes: Exponential decay in some linear delay differential equations*, American Mathematical Monthly **85** (1975), 757–760.

[90] R. Driver, *Torricelli's law — an ideal example of an elementary ODE*, American Mathematical Monthly **105** (1998), 453–455.

[91] R. Driver, D. Sasser, and M. Slater, *The equation $x'(t) = a\,x(t) + b\,x(t-\tau)$ with "small" delay*, American Mathematical Monthly **80** (1973), 990–995.

[92] J. Duemmel, *Classroom notes: Continuous dependence on parameters*, American Mathematical Monthly **76** (1969), 294–295.

[93] W. Dunham, *Touring the calculus gallery*, American Mathematical Monthly **112** (2005), 1–19.

[94] W. Dunham, *The Calculus Gallery: Masterpieces from Newton to Lebesgue*, Princeton University Press, 2018.

[95] P. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, 2004.

[96] P. Duren, *Invitation to Classical Analysis*, American Mathematical Society, 2012.

[97] P. Duren, W. Hengartner, and R. Laugesen, *The argument principle for harmonic functions*, American Mathematical Monthly **103** (1996), 411–415.

[98] C. Edstrom, *A boundary value problem*, Mathematics Magazine **45** (1972), 149–150.

[99] C. Edstrom, *A Dirichlet problem*, Mathematics Magazine **45** (1972), 204–205.

[100] C. Edstrom, *A solution of Laplace's equation for a semi-infinite strip*, Mathematics Magazine **45** (1972), 254–259.

[101] C. Efthimiou, *Finding exact values for infinite sums*, Mathematics Magazine **72** (1999), 45–51.

[102] A. Erdélyi, *The inversion of the Laplace transformation*, Mathematics Magazine **24** (1950), 1–6.

[103] L. Evans, *Partial Differential Equations*, 2nd edition, American Mathematical Society, 2010.

[104] J. Fabrey, *Picard's theorem*, American Mathematical Monthly **79** (1972), 1020–1023.

[105] A. Fadell, *Classroom notes: A proof of the chain rule for derivatives in n-space*, American Mathematical Monthly **80**, 1134–1135.

[106] C. Falbo, *Polynomial solutions of certain linear differential equations by differentiation*, Mathematics Magazine **35** (1962), 21–26.

[107] C. Falbo, *The use of Green's functions to solve some general initial-value problems*, American Mathematical Monthly **73** (1966), 146–157.

[108] W. Felscher, *Bolzano, Cauchy, epsilon, delta*, American Mathematical Monthly **107** (2000), 844–862.

[109] F. Ficken, *A derivation of the equation for a vibrating string*, American Mathematical Monthly **64** (1957), 155–157.

[110] F. Ficken, *Some uses of linear spaces in analysis*, American Mathematical Monthly **66** (1959), 259–275.

[111] G. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, 1976. A second edition was published in 1995.

[112] G. Folland, *On characterizations of analyticity*, American Mathematical Monthly **93** (1986), 640–641.

[113] G. Folland, *Fourier Analysis and its Applications*, Wadsworth & Brooks / Cole, 1992.

[114] G. Folland, *How to integrate a polynomial over a sphere*, American Mathematical Monthly **108** (2001), 446–448.

[115] W. Ford and J. Pennline, *When does convergence in the mean imply uniform convergence?*, American Mathematical Monthly **114** (2007), 58–60.

[116] E. Fulmer, *Computation of the matrix exponential*, American Mathematical Monthly **82** (1975), 156–159.

[117] B. Fusaro, *A solution of a singular, mixed problem for the equation of Euler–Poisson-Darboux (EPD)*, American Mathematical Monthly **73** (1966), 610–613.

[118] L. Gårding, *Some Points of Analysis and their History*, American Mathematical Society, 1997.

[119] C. Gardner, *Another elementary proof of Peano's existence theorem*, American Mathematical Monthly **83** (1976), 556-560.

[120] J. Garner, *Classroom notes: Boundary conditions for the linear differential equation*, American Mathematical Monthly **69** (1962), 47–50.

[121] J. Garner, *Mathematical notes: A multiple-point boundary condition for linear differential systems*, American Mathematical Monthly **70** (1963), 850–852.

[122] J. Garner, *Solutions of linear differential systems satisfying multiple-point boundary conditions*, American Mathematical Monthly **73** (1966), 614–616.

[123] J. Garner, *Classroom notes: Uniqueness theorems for a differential equation*, American Mathematical Monthly **73** (1966), 752–754.

[124] J. Garner, *Mathematical notes: Further uniqueness theorems for n-th order fifferential equations*, American Mathematical Monthly **75** (1968), 873–875.

[125] R. Goldberg, *Methods of Real Analysis*, 2nd edition, Wiley, 1976.

[126] E. González-Velasco, *Connections in mathematical analysis: the case of Fourier series*, American Mathematical Monthly **99** (1992), 427–441.

[127] E. González-Velasco, *Fourier Analysis and Boundary Value Problems*, Academic Press, 1996.

[128] E. González-Velasco, *James Gregory's calculus in the* Geometriæ pars universalis, American Mathematical Monthly **114** (2007), 565–576.

[129] E. González-Velasco, *Journey Through Mathematics: Creative Episodes in its History*, Springer, 2011.

[130] C. Gordon, *Sunada's isospectrality technique: two decades later*, in *Spectral Analysis in Geometry and Number Theory*, 45–58, Contemporary Mathematics **484**, American Mathematical Society, 2009.

[131] C. Gordon, D. Webb, and S. Wolpert, *One cannot hear the shape of a drum*, Bulletin of the American Mathematical Society (N.S.) **27** (1992), 134–138.

[132] C. Gordon, D. Webb, and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Inventiones Mathematicae **110** (1992), 1-22.

[133] R. Gordon, *A convergence theorem for the Riemann integral*, Mathematics Magazine **73** (2000), 141–147.

[134] P. Gorkin and J. Smith, *Dirichlet: his life, his principle, and his problem*, Mathematics Magazine **78** (2005), 283–296.

[135] F. Gouvêa, *A Short Book on Long Sums — Infinite Series for Calculus Students*, Springer, 2023.

[136] J. Grabiner, *The Origins of Cauchy's Rigorous Calculus*, MIT Press, 1981.

[137] J. Grabiner, *Who gave you the epsilon? Cauchy and the origins of rigorous calculus*, American Mathematical Monthly **90** (1983), 185–194.

[138] J. Grabiner, *The changing concept of change: the derivative from Fermat to Weierstrass*, Mathematics Magazine **56** (1983), 195–206.

[139] J. Grabiner, *Was Newton's calculus a dead end? The continental influence of Maclaurin's treatise of fluxions*, American Mathematical Monthly **104** (1997), 393–410.

[140] J. Grabiner, *A Historian Looks Back:* The Calculus as Algebra *and Selected Writings*, Mathematical Association of America, 2010.

[141] I. Grattan-Guinness, *Why did George Green write his essay of 1828 on electricity and magnetism?*, American Mathematical Monthly **102** (1995), 387–396.

[142] J. Gray, *Miscellanea: Clear theory*, American Mathematical Monthly **92** (1985), 158–159.

[143] J. Gray, *The Real and the Complex: A History of Analysis in the 19th Century*, Springer, 2015.

[144] J. Gray, *Change and Variations — A History of Differential Equations to 1900*, Springer, 2021.

[145] J. Green, *The theory of functions of a real variable*, Mathematics Magazine **24** (1951), 209–217.

[146] R. Grinberg, *The Real Analysis Lifesaver: All the Tools that You Need to Understand Proofs*, Princeton University Press, 2017.

[147] N. Grossman, *A C^∞ Lagrange inversion theorem*, American Mathematical Monthly **112** (2005), 512–514.

[148] R. Guenther, *Some elementary properties of the fundamental solution of parabolic equations*, Mathematics Magazine **39** (1966), 294–298.

[149] R. Gunning, *An Introduction to Analysis*, Princeton University Press, 2018.

[150] R. Hall and K. Jocić, *The mathematics of musical instruments*, American Mathematical Monthly **108** (2001), 347–354.

[151] W. Hansen, *A strong version of Liouville's theorem*, American Mathematical Monthly **115** (2008), 583–595.

[152] P. Hartman, *A differential equation with non-unique solutions*, American Mathematical Monthly **70** (1963), 255–259.

[153] P. Hartman, *Ordinary Differential Equations*, corrected reprint of the second edition, Society for Industrial and Applied Mathematics (SIAM), 2002.

[154] S. Hastings, *Some mathematical problems from neurobiology*, American Mathematical Monthly **82** (1975), 881–895.

[155] I. Herron, *The radius of convergence of power series solutions to linear differential equations*, American Mathematical Monthly **96** (1989), 824–827.

[156] R. Hersh, *How to classify differential polynomials*, American Mathematical Monthly **80** (1973), 641–654.

[157] G. Herzog, *Polynomials solving Dirichlet boundary value problems*, American Mathematical Monthly **107** (2000), 934–936.

[158] E. Hille, *Mathematics and mathematicians from Abel to Zermelo*, Mathematics Magazine **26** (1953), 127–146.

[159] E. Hille, *Lectures on Ordinary Differential Equations*, Addison-Wesley, 1969.

[160] E. Hille, *Ordinary Differential Equations in the Complex Domain*, Dover, 1997.

[161] M. Hirsch, S. Smale, and R. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd edition, Elsevier / Academic Press, 2013.

[162] I. Hirschman, Jr., *Infinite Series*, Holt, Rinehart and Winston, 1962.

[163] I. Hirschman, Jr. and D. Widder, *A miniature theory in illustration of the convolution transform*, American Mathematical Monthly **57** (1950), 667–674.

[164] P. Hodge, Jr., *On the methods of characteristics*, American mathematical Monthly **57** (1950), 621–623.

[165] K. Hoffman, *Analysis in Euclidean Space*, Prentice-Hall, 1975.

[166] L. Hörmander, *Notions of Convexity*, Birkhäuser, 2007.

[167] C. Horowitz, *Fourier series of functions of bounded variation*, American Mathematical Monthly **82** (1975), 391–392.

[168] J. Horvàth, *An introduction to distributions*, American Mathematical Monthly **77** (1970), 227–240.

[169] A. Howson and C. Brown, *Classroom notes: Orthogonality of Bessel functions*, American Mathematical Monthly **69** (1962), 793–794.

[170] J. Hubbard, S. Habre, and B. West, *The convergence of an Euler approximation of an initial value problem is not always obvious*, American Mathematical Monthly **108** (2001), 326–335.

[171] J. Hubbard and B. Lundell, *A first look at differential algebra*, American Mathematical Monthly **118** (2011), 245–261.

[172] H. Hunzeker, *The separation of partial differential equations with mixed derivatives*, American Mathematical Monthly **68** (1961), 131–132.

[173] W. Hurewicz, *Lectures on Ordinary Differential Equations*, with a preface by N. Levinson and an obituary by S. Lefschetz, Dover, 1990.

[174] D. Jackson, *Questions and discussions: Discussions: Note on the convergence of Fourier series*, American Mathematical Monthly **33** (1926), 39–40.

[175] D. Jackson, *The convergence of Fourier series*, American Mathematical Monthly **41** (1934), 67–84.

[176] D. Jackson, *A proof of Weierstrass's theorem*, American Mathematical Monthly **41** (1934), 309–312.

[177] D. Jackson, *A new class of orthogonal polynomials*, American Mathematical Monthly **46** (1939), 493–497.

[178] D. Jackson, *Legendre functions of the second kind and related functions*, American Mathematical Monthly **50** (1943), 291–302.

[179] D. Jackson, *The harmonic boundary value problem for an ellipse or an ellipsoid*, American Mathematical Monthly **51** (1944), 555–563.

[180] D. Jackson, *Fourier Series and Orthogonal Polynomials*, Dover, 2004.

[181] T. Jackson, N. Komarova, and K. Swanson, *Mathematical oncology: using mathematics to enable cancer discoveries*, American Mathematical Monthly **121** (2014), 840–856.

[182] H. Johnson, *Linear completely integrable partial differential equations*, American Mathematical Monthly **74** (1967), 548–552.

[183] L. Johnson, Jr., *The Cauchy problem for Laplace's equation in three dimensions*, American Mathematical Monthly **42** (1935), 65–74.

[184] P. Johnson and R. Redheffer, *Scrambled series*, American Mathematical Monthly **73** (1966), 822–828.

[185] F. Jones, *Lebesgue Integration on Euclidean Space*, Jones and Bartlett, 1993.

[186] F. Jones, *Honors Calculus III and IV*, available from his website at Rice University (https://mathweb.rice.edu/honors-calculus).

[187] M. Kac, *Random walk and the theory of Brownian motion*, American Mathematical Monthly **54** (1947), 369–391.

[188] M. Kac, *Can one hear the shape of a drum?*, American Mathematical Monthly **73** (1966), No. 4, Part II, 1–23.

[189] J. Karon and J. Ralston, *Double integrals as initial value problems*, American Mathematical Monthly **84** (1977), 201–204.

[190] D. Kearns, *Classroom notes: On solutions of linear differential equations*, American Mathematical Monthly **67** (1960), 369–370.

[191] D. Kearns, *Classroom notes: Elementary uniqueness theorems for differential equations*, American Mathematical Monthly **68** (1961), 275–277.

[192] J. Keener, *Biology in Time and Space — A Partial Differential Equation Modeling Approach*, American Mathematical Society, 2021. A review of this book may be found in [351].

[193] R. Kelman, *Regularity of certain explicit solutions to Laplace's equation at artificial interfaces*, American Mathematical Monthly **73** (1966), 1073–1078.

[194] R. Kelman and T. Rivlin, *Classroom notes: Conditions for integrand of an improper integral to be bounded or tend to zero*, American Mathematical Monthly **67** (1960), 1019–1022.

[195] H. Kennedy, *Research problems: Is there an elementary proof of Peano's existence theorem for first order differential equations?*, American Mathematical Monthly **76** (1969), 1043–1045.

[196] H. Kennedy, *Peano: Life and Works of Giuseppe Peano*, Reidel, 1980.

[197] R. Kirchner, *An explicit formula for* e^{At}, American Mathematical Monthly **74** (1967), 1200–1204.

[198] M. Kline, *Euler and infinite series*, Mathematics Magazine **56** (1983), 307–314.

[199] A. Knapp, *Basic Real Analysis*, Birkhäuser, 2005.

[200] J. Knowles, *Universal states of finite anti-plane shear: Ericksen's problem in miniature*, American Mathematical Monthly **86** (1979), 109–113.

[201] I. Kolodner, *Mathematical notes: On the applicability of a certain formula in the theory of linear differential equations*, American Mathematical Monthly **59** (1952), 168–170.

[202] I. Kolodner and R. Pederson, *A note on first order semi-linear partial differential equations*, American Mathematical Monthly **75** (1968), 275–276.

[203] T. Körner, *Differentiable functions on the rationals*, Bulletin of the London Mathematical Society **23** (1991), 557–562.

[204] T. Körner, *Exercises for* Fourier Analysis ([205]). Cambridge University Press, 1993.

[205] T. Körner, *Butterfly hunting and the symmetry of mixed partial derivativs*, American Mathematical Monthly **105** (1998), 756–758.

[206] T. Körner, *A Companion to Analysis: A Second First and First Second Course in Analysis*, American Mathematical Society, 2004.

[207] T. Körner, *Fourier Analysis*, Cambridge University Press, 2022.

[208] J. Král and W. Pfeffer, *Poisson integrals of Riemann integrable functions*, American Mathematical Monthly **98** (1991), 929–931.

[209] S. Krantz, *You don't need to be a weatherman to know which way the wind blows*, American Mathematical Monthly **106** (1999), 915–918.

[210] S. Krantz, *A Panorama of Harmonic Analysis*, Mathematical Association of America, 1999.

[211] S. Krantz, *A Guide to Real Variables*, Mathematical Association of America, 2009.

[212] S. Krantz, *Foundations of Analysis*, CRC Press, 2015.

[213] S. Krantz, *A calculus approach to complex variables*, Mathematics Magazine **94** (2021), 194–200.

[214] S. Krantz, *Real Analysis and Foundations*, 5th edition, CRC Press, 2022.

[215] S. Krantz, *The Elements of Advanced Mathematics*, 5th edition, CRC Press, 2022.

[216] S. Krantz and H. Parks, *A Primer of Real-Analytic Functions*, 2nd edition, Birkhäuser, 2002.

[217] S. Krantz and H. Parks, *The Implicit Function Theorem: History, Theory, and Applications*, Birkhäuser / Springer, 2013.

[218] K. Kreith, *Geometric interpretation of the Implicit Function Theorem*, Mathematics Magazine **36** (1963), 64–65.

[219] E. Kreyszig, *On the calculus of variations and its major influences on the mathematics of the first half of our century*, in *Proceedings of the Canadian Society for the History and Philosophy of Mathematics*, Volume 6, 119–149, 1993.

[220] E. Kreyszig, *On the calculus of variations and its major influences on the mathematics of the first half of our century, I*, American Mathematical Monthly **101** (1994), 674–678.

[221] E. Kreyszig, *On the calculus of variations and its major influences on the mathematics of the first half of our century, II*, American Mathematical Monthly **101** (1994), 902–908.

[222] A. Krupowies and F. Prus-Wiśniowski, *The character of convergence of the Cauchy product*, American Mathematical Monthly **130** (2023), 671–677.

[223] M. Kushelman and P. McGrath, *On Liouville's theorem for conformal maps*, American Mathematical Monthly **131** (2024), 619–623.

[224] A. Laforgia and M. Muldoon, *Some consequences of the Sturm comparison theorem*, American Mathematical Monthly **93** (1986), 89–94.

[225] S. Lang, *Undergraduate Analysis*, 2nd edition, Springer-Verlag, 1997.

[226] R. Lange and R. Walsh, *The teaching of mathematics: A heuristic for the Poisson integral for the half plane and some caveats*, American Mathematical Monthly **92** (1985), 356–358.

[227] R. Langer, *Fourier's Series: The Genesis and Evolution of a Theory*, The first Herbert Ellsworth Slaught Memorial Paper, American Mathematical Monthly **54** (1947), no. 7, part II.

[228] P. Lax, *Rethinking the Lebesgue integral*, American Mathematical Monthly **116** (2009), 863–881.

[229] A. Leanhardt and A. Parker, *Fontaine's forgotten method for inexact differential equations*, Mathematics Magazine **90** (2017), 208–219.

[230] S. Lefschetz, *Differential Equations: Geometric Theory*, 2nd edition, Dover, 1977.

[231] T. Lehrer, *The Derivative Song*, American Mathematical Monthly **81** (1974), 490.

[232] T. Lehrer, *There's a Delta for Every Epsilon (Calypso)*, American Mathematical Monthly **81** (1974), 612.

[233] T. Lehrer, *The Professor's Song*, American Mathematical Monthly **81** (1974), 745.

[234] J. Lesko and W. Smith, *A Laplace transform technique for evaluating infinite series*, Mathematics Magazine **76** (2003), 394–398.

[235] J. Lewin, *The teaching of mathematics: A truly elementary approach to the bounded convergence theorem*, American Mathematical Monthly **93** (1986), 395–397.

[236] J. Lewin, *The teaching of mathematics: Some applications of the bounded convergence theorem for an introductory course in analysis*, American Mathematical Monthly **94** (1987), 988–993.

[237] H. Logemann and E. Ryan, *Volterra functional differential equations: existence, uniqueness, and continuation of solutions*, American Mathematical Monthly **117** (2010), 490–511.

[238] R. López Pouso, *Fourier series for nonperiodic functions*, American Mathematical Monthly **132** (2025), 134–149.

[239] Z. Lu and J. Rowlett, *The sound of symmetry*, American Mathematical Monthly **122** (2015), 815–835.

[240] J. Lützen, *Euler's vision of a general partial differential calculus for a generalized kind of functions*, Mathematics Magazine **56** (1983), 299–306.

[241] J. Lützen, *Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics*, Springer-Verlag, 1990.

[242] W. Luxemburg, *Arzelà's dominated convergence theorem for the Riemann integral*, American Mathematical Monthly **78** (1971), 970-979.

[243] A. Mackie, *A class of integral equations*, American Mathematical Monthly **72** (1965), 956–960.

[244] S. Mandelbrojt, *Obituary: Emile Picard, 1856–1941*, American Mathematical Monthly **49** (1942), 277–278.

[245] S. Mandelbrojt, *The mathematical work of Jacques Hadamard*, American Mathematical Monthly **60** (1953), 599–604.

[246] W. McEwen, *Spectral theory and its applications to differential eigenvalue problems*, American Mathematical Monthly **60** (1953), 223-233.

[247] P. McGrath, *Bases for second order linear ODEs*, American Mathematical Monthly **127** (2020), 849.

[248] B. Mendelson, *Introduction to Topology*, 3rd edition, Dover, 1990.

[249] F. Metcalf and M. Zlámal, *On the zeros of solutions to Bessel's equation*, American Mathematical Monthly **73** (1966), 746–749.

[250] M. Mezzino and M. Pinsky, *Leibniz's formula, Cauchy majorants, and linear differential equations*, Mathematics Magazine **71** (1998), 360–368.

[251] E. Miles, Jr. and E. Williams, *A basic set of polynomial solutions for the Euler–Poisson–Darboux and Beltrami equations*, American Mathematical Monthly **63** (1956), 401–404.

[252] D. Minda, *The Dirchlet problem for a disk*, American Mathematical Monthly **97** (1990), 220–223.

[253] P. Mohazzabi and T. Fournelle, *Evaluation of ill-behaved power series*, American Mathematical Monthly **111** (2004), 308–321.

[254] S. Moreno and E. García-Caballero, *A characterization of Bernoulli differential equations*, American Mathematical Monthly **125** (2018), 561.

[255] F. Morgan, *Mathematicians, including undergraduates, look at soap bubbles*, American Mathematical Monthly **101** (1994), 343–351.

[256] F. Morgan, *What is a surface?*, American Mathematical Monthly **103** (1996), 369–376.

[257] F. Morgan, *Proof of the double bubble conjecture*, American Mathematical Monthly **108** (2001), 193–205.

[258] F. Morgan, *Real Analysis*, American Mathematical Society, 2005.

[259] F. Morgan, *Real Analysis and Applications: Including Fourier Series and the Calculus of Variations*, American Mathematical Society, 2005.

[260] M. Muldoon and A. Ungar, *Beyond* sin *and* cos, Mathematics Magazine **69** (1996), 3–14.

[261] T. Needham, *The geometry of harmonic functions*, Mathematics Magazine **67** (1994), 92–108.

[262] I. Netuka, *The Dirichlet problem for harmonic functions*, American Mathematical Monthly **87** (1980), 621–628.

[263] I. Netuka and J. Veselý, *Regions of harmonicity*, American Mathematical Monthly **87** (1980), 203–205.

[264] H. Nguyen, R. Cortez, and L. Fauci, *Computing flows around microorganisms: slender-body theory and beyond*, American Mathematical Monthly **121** (2014), 810–823.

[265] L. Ni, *A Perron-type theorem on the principal eigenvalue of nonsymmetric elliptic operators*, American Mathematical Monthly **121** (2014), 903–908.

[266] J. Nitsche, *A remark on Enneper's minimal surface*, American Mathematical Monthly **66** (1959), 295–297.

[267] J. Nitsche, *Mathematical notes: On the module of doubly-connected regions under harmonic mappings*, American Mathematical Monthly **69** (1962), 781–782.

[268] J. Nitsche, *An isoperimetric property of surfaces with moveable boundaries*, American Mathematical Monthly **77** (1970), 359–362.

[269] J. Nitsche, *Plateau's problems and their modern ramifications*, American Mathematical Monthly **81** (1974), 945–968.

[270] K. O'Brien, *Laplace transforms*, Mathematics Magazine **54** (1981), 78.

[271] J. Ovall, *The Laplacian and mean and extreme values*, American Mathematical Monthly **123** (2016), 287–291.

[272] P. Olver, *Dispersive quantization*, American Mathematical Monthly **117** (2010), 599–610.

[273] P. Olver, *Introduction to Partial Differential Equations*, Springer, 2014.

[274] E. Pellicciaro, *A representation formula for the solutions of the second order linear differential equation*, Mathematics Magazine **43** (1970), 77–80.

[275] G. Peyser, *Note on the derivatives of the Legendre polynomials*, Mathematics Magazine **31** (1957 / 58), 210.

[276] G. Peyser, *On the Cauchy–Lipschitz theorem*, American Mathematical Monthly **65** (1958), 760–762.

[277] S. Phull, *Classroom notes: A definite integral for Bessel's function*, American Mathematical Monthly **76** (1969), 549–551.

[278] M. Pinsky, *Mean values and the maximum principle: a proof in search of more theorems*, American Mathematical Monthly **112** (2005), 515–521.

[279] O. Plaat, *Separation of variables*, American Mathematical Monthly **75** (1968), 844–847.

[280] G. Porter, *Classroom notes: An alternative to the integral test for infinite series*, American Mathematical Monthly **79** (1972), 634–635.

[281] J. Potts and M. Lewis, *A mathematical approach to territorial pattern formation*, American Mathematical Monthly **121** (2014), 754–770.

[282] M. Protter, *The characteristic initial value problem for the wave equation and Riemann's method*, American Mathematical Monthly **61** (1954), 702–705.

[283] M. Protter and C. Morrey, Jr., *A First Course in Real Analysis*, 2nd edition, Springer-Verlag, 1991.

[284] M. Protter and H. Weinberger, *Maximum Principles in Differential Equations*, corrected reprint of the 1967 original, Springer-Verlag, 1984.

[285] C. Pugh, *Real Mathematical Analysis*, 2nd edition, Springer, 2015.

[286] E. Putzer, *Avoiding the Jordan canonical form in the discussion of linear systems with constant coefficients*, American Mathematical Monthly **73** (1966), 2–7.

[287] T. Radó, *On semi-continuity*, American Mathematical Monthly **49** (1942), 446–450.

[288] T. Radó, *What is the area of a surface?*, American Mathematical Monthly **50** (1943), 139–141.

[289] T. Radó, *On the Problem of Plateau* and *Subharmonic Functions*, Springer-Verlag, 1971.

[290] P. Ramankutty, *The complementary function and the general solution*, Mathematics Magazine **64** (1991), 124–130.

[291] W. Randels, *Questions, discussions, and notes: A remark on Fourier series of continuous functions*, American Mathematical Monthly **40** (1933), 97–99.

[292] R. Redheffer, *Collegiate articles: Graduate training not required for reading: Derivation of certain partial differential equations*, Mathematics Magazine **27** (1954), 195–200.

[293] R. Redheffer, *An elementary proof of sharp uniqueness theorems for second-order initial-value problems*, American Mathematical Monthly **81** (1974), 377–379.

[294] R. Redheffer, *Some thoughts about limits*, Mathematics Magazine **62** (1989), 176–184.

[295] M. Reed, *Fundamental Ideas of Analysis*, Wiley, 1998.

[296] W. P. Reid, *Classroom notes: On numerical solutions to the one-dimensional wave equation*, American Mathematical Monthly **61** (1954), 115.

[297] W. P. Reid, *On some contradictions in boundary value problems*, Mathematics Magazine **37** (1964), 172–175.

[298] W. T. Reid, *Ordinary Differential Equations*, Wiley, 1971.

[299] W. T. Reid, *Anatomy of the ordinary differential equation*, American Mathematical Monthly **82** (1975), 971–984; addendum, **83** (1976), 718.

[300] B. Rhoades, *On forming partial differential equations*, American Mathematical Monthly **66** (1959), 473–476.

[301] D. Richmond, *Elementary evaluation of Laplace transforms*, American Mathematical Monthly **52** (1945), 481–487.

[302] J. Riley, Jr., *What I heard from the P.D.E.*, Mathematics Magazine **98** (2025), 6–14.

[303] P. Rosenbloom and D. Widder, *A temperature function which vanishes initially*, American Mathematical Monthly **65** (1958), 607–609.

[304] M. Rosenlicht, *Introduction to Analysis*, Dover, 1986.

[305] C. Rosentrater, *Varieties of Integration*, Mathematical Association of America, 2015.

[306] K. Ross, *Elementary Analysis: The Theory of Calculus*, 2nd edition, in collaboration with J. López, Springer, 2013.

[307] J. Rowlett, M. Blom, H. Nordell, O. Thim, and J. Vahnberg, *Crystallographic groups, strictly tessellating polytopes, and analytic eigenfunctions*, American Mathematical Monthly **128** (2021), 387–406.

[308] W. Rudin, *Real and Complex Analysis*, 2nd edition, McGraw-Hill, 1974. A third edition was published in 1987.

[309] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976.

[310] W. Rudin, *Nonuniqueness and growth in first-order differential equations*, American Mathematical Monthly **89** (1982), 241–244.

[311] L. Rudolph, *The structure of orthogonal transformations*, American Mathematical Monthly **98** (1991), 349–352.

[312] A. Russell, *A commutative Banach algebra of functions of bounded variation*, **87** (1980), 39–40.

[313] D. Sánchez, *Classroom notes: The Green's function and determining equations*, American Mathematical Monthly **82** (1975), 747–749.

[314] D. Sánchez, *An alternative to the shooting method for a certain class of boundary value problems*, American Mathematical Monthly **108** (2001), 552–555.

[315] K. Schrader, *A note on second order boundary value problems*, American Mathematical Monthly **75** (1968), 867–869.

[316] D. Scott, *Notes: When is an ordinary differential equation separable?*, American Mathematical Monthly **92** (1985), 422–423.

[317] E. Scott, *Determination of the Riemann function*, American Mathematical Monthly **80** (1973), 906–909.

[318] R. Seeley, *Classroom notes: An undergraduate approach to a boundary value problem*, American Mathematical Monthly **67** (1960), 1008–1011.

[319] R. Seeley, *Spherical harmonics*, American Mathematical Monthly **73** (1966), no. 4, part II, 115–121.

[320] R. Seeley, *An Introduction to Fourier Series and Integrals*, Dover, 2006.

[321] R. Shakarchi, *Problems and Solutions for* Undergraduate Analysis ([225]), Springer-Verlag, 1998.

[322] V. Shapiro, *Harmonic functions in the unit disk*, American Mathematical Monthly **109** (2002), 37–45.

[323] I. Sheffer, *Convergence of multiply-infinite series*, American Mathematical Monthly **52** (1945), 365–376.

[324] I. Sheffer, *Classroom notes: Note on multiple power series*, American Mathematical Monthly **70** (1963), 309–313.

[325] J. Slade, Jr., *Questions, discussions, and notes: Note on the Laplacian of a vector point function*, American Mathematical Monthly **40** (1933), 483–484.

[326] S. Smale, *What is global analysis?*, American Mathematical Monthly **76** (1969), 4-9.

[327] A. Snider, *Classroom notes: Motivating existence-uniqueness theory for applications oriented students*, American Mathematical Monthly **83** (1976), 805–807.

[328] K. Socha, *Circles in circles: creating a mathematical model of surface water waves*, American Mathematical Monthly **114** (2007), 202–216.

[329] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[330] E. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.

[331] E. Stein and R. Shakarchi, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton University Press, 2005.

[332] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[333] A. Stone, *Power series expansions for trigonometric functions via solutions to initial value problems*, Mathematics Magazine **64** (1991), 247–251.

[334] A. Strauss, *Mathematical notes: Continuous dependence of solutions of ordinary differential equations*, American Mathematical Monthly **71** (1964), 649–652.

[335] W. Strauss, *Partial Differential Equations: An Introduction*, 2nd edition, Wiley, 2008.

[336] R. Strichartz, *The Way of Analysis*, Revised edition, Jones and Bartlett, 2000.

[337] R. Strichartz, *Differential Equations on Fractals*, Princeton University Press, 2006.

[338] K. Stromberg, *An Introduction to Classical Real Analysis*, AMS Chelsea, 1981.

[339] R. Struble, it On the differential equation $f'(x) = a\,f(g(x))$, Mathematics Magazine **41** (1968), 260–262.

[340] O. Szasz, *Questions, discussions, and notes: On Fourier series of continuous functions*, American Mathematical Monthly **42** (1935), 37–38.

[341] A. Taylor, *Derivatives in the calculus*, American Mathematical Monthly **49** (1942), 631–642.

[342] A. Taylor, *A note on the Poisson kernel*, American Mathematical Monthly **57** (1950), 478–479.

[343] A. Taylor, *L'Hospital's rule*, American Mathematical Monthly **59** (1952), 20–24.

[344] A. Taylor and W. Mann, *Advanced Calculus*, 3rd edition, Wiley, 1983.

[345] D. Terman, *Mathematical neuroscience*, American Mathematical Monthly **121** (2014), 824–839.

[346] H. Thielman, *A note on the uae of the Laplace transformation*, American Mathematical Monthly **45** (1938), 508–510.

[347] B. Thomson, *Monotone convergence for the Riemann integral*, American Mathematical Monthly **117** (2010), 547–550.

[348] B. Thomson, *The bounded convergence theorem*, American Mathematical Monthly **127** (2020), 483–503.

[349] H. Thurston, *Classroom notes: On Euler's equation*, American Mathematical Monthly **67** (1960), 678–681.

[350] A. Torchinsky, *The Fourier transform and the wave equation*, American Mathematical Monthly **118** (2011), 599–609.

[351] M. Torres, *Review of* Biology in Time and Space: A Partial Differential Equation Modeling Approach ([192]), American Mathematical Monthly **129** (2022), 395–400.

[352] F. Trèves, *Applications of distributions to PDE theory*, American Mathematical Monthly **77** (1970), 241–248.

[353] F. Trèves, *Basic Linear Partial Differential Equations*, Dover, 2006.

[354] A. Ungar, *The Laplace transform inversion by inspection*, American Mathematical Monthly **93** (1986), 786–791.

[355] A. Ungar, *The teaching of mathematics: Addition theorems in ordinary differential equations*, American Mathematical Monthly **94** (1987), 872–875.

[356] R. Vanderbei, *The falling slinky*, American Mathematical Monthly **124** (2017), 24–36.

[357] B. van Rootselaar, *How to solve the system $x' = A\,x$*, American Mathematical Monthly **92** (1985), 321–327.

[358] C. Wagner, *A generic approach to iterative methods*, Mathematics Magazine **55** (1982), 259–273.

[359] J. Walsh, *Lemniscates and equipotential curves of Green's function*, American Mathematical Monthly **42** (1935), 1–17.

[360] J. Walsh, *On the shape of level curves of Green's function*, American Mathematical Monthly **44** (1937), 202–213.

[361] J. Walsh, *Note on the shape of level curves of Green's function*, American Mathematical Monthly **60** (1953), 671–674.

[362] J. Walsh, *The circles of curvature of steepest descent of Green's function*, American Mathematical Monthly **68** (1961), 323–329.

[363] J. Walter, *On elementary proofs of Peano's existence theorems*, American Mathematical Monthly **80** (1973), 282-286.

[364] W. Walter, *There is an elementary proof of Peano's existence theorem*, American Mathematical Monthly **78** (1971), 170–173.

[365] W. Walter, *An elementary proof of the Cauchy–Kowalevsky theorem*, American Mathemtical Monthly **92** (1985), 115–126.

[366] W. Walter, *Ordinary Differential Equations*, translated from the sixth German (1996) edition by R. Thompson, Springer-Verlag, 1998.

[367] S. Warschawski, *On the Green function of a star-shaped three dimensional region*, American Mathematical Monthly **57** (1950), 471–473.

[368] H. Weinberger, *A First Course in Partial Differential Equations with Complex Variables and Transform Methods*, corrected reprint of the 1965 original, Dover, 1995.

[369] G. Weiss, *Complex methods in harmonic analysis*, American Mathematical Monthly **77** (1970), 465–474.

[370] C. Wells, *Separability conditions for some self-adjoint partial differential equations*, American Mathematical Monthly **66** (1959), 684–689.

[371] D. Wend, *Uniqueness of solutions of ordinary differential equations*, American Mathematical Monthly **74** (1967), 948–950.

[372] W. Whyburn, *Over and under functions as related to differential equations*, American Mathematical Monthly **47** (1940), 1–10.

[373] D. Widder, *The Laplace Transform*, Princeton University Press, 1941.

[374] D. Widder, *What is the Laplace transform?*, American Methematical Monthly **52** (1945), 410–425.

[375] D. Widder, *A symbolic form of an inversion formula for a Laplace transform*, American Mathematical Monthly **55** (1948), 489–491.

[376] D. Widder, *A symbolic form of the classical complex inversion formula for a Laplace transform*, American Mathematical Monthly **58** (1951), 179–181.

[377] D. Widder, *The inversion of a convolution transform whose kernel is a Laguerre polynomial*, American Mathematical Monthly **70** (1963), 291–293.

[378] D. Widder, *The Heat Equation*, Academic Press, 1975.

[379] D. Widder, *The Airy transform*, American Mathematical Monthly **86** (1979), 271–277.

[380] D. Widder, *Advanced Calculus*, 3rd edition, Dover, 1989.

[381] C. Wilcox, *Positive temperatures with prescribed initial heat distributions*, American Mathematical Monthly **87** (1980), 183–186.

[382] C. Wilcox, *The Cauchy problem for the wave equation with distribution data: an elementary approach*, American Mathematical Monthly **98** (1991), 401–410.

[383] D. Willett, *The existence-uniqueness theorem for an nth order linear ordinary differential equation*, American Mathematical Monthly **75** (1968), 174–178.

[384] B. Wirth, *Green's function for the Neumann–Poisson problem on n-dimensional balls*, American Mathematical Monthly **127** (2020), 737–743.

[385] A. Wouk, *On the Cauchy–Picard method*, American Mathematical Monthly **70** (1963), 158–162.

[386] K. Yosida, *Lectures on Differential and Integral Equations*, translated from the Japanese, reprint of the 1960 translation, Dover, 1991.

[387] A. Zemanian, *Distribution Theory and Transform Analysis*, Dover, 1987.

[388] A. Ziebur, *Classroom notes: Uniqueness theorems for ordinary differential equations*, American Mathematical Monthly **69** (1962), 421–423.

Index