Some basic aspects of probability theory, using some basic analysis

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Preface

A fundamental question to be addressed in these informal notes is

What is a *probability space*?

A slight variant of this question that one may ask is

What is a *probability measure*?

In fact, a probability measure is included in the definition of a probability space. More precisely, a probability space is defined to be a measurable space with a probability measure. We shall discuss these and related matters in some detail here.

Of course, these questions are related to measure theory and integration more broadly. The reader is not necessarily expected to be familiar with measure theory, and indeed part of the aim here is to give an introduction to some of the relevant notions and results. However, this is not intended to be a detailed treatment of measure theory, and instead we would like to emphasize some aspects related to probability theory.

The reader is expected to be familiar with some basic analysis, along the lines of metric spaces, sequences and series, continuous mappings, and compactness. Some topics may be reviewed briefly, as needed.

Some familiarity with topological structures and spaces could be helpful in some places, although it is not required. There are many connections with Fourier analysis and functional analysis too, and we shall see some aspects of this here.

Contents

Chapter 1

Some basic notions

1.1 Extended real numbers

The set of *extended real numbers* is defined to be the set $\mathbf{R}$ of real numbers together with two additional elements, $+\infty$ and $-\infty$, such that

$$-\infty < x < +\infty \tag{1.1.1}$$

for every $x \in \mathbf{R}$. When we use extended real numbers here, we shall typically only be concerned with nonnegative extended real numbers.

It is customary to put

$$x + (+\infty) = (+\infty) + x = +\infty \tag{1.1.2}$$

when $-\infty < x \le +\infty$, and

$$x + (-\infty) = (-\infty) + x = -\infty \tag{1.1.3}$$

when $-\infty \le x < +\infty$. Similarly, we put

$$x\,(+\infty) = (+\infty)\,x + +\infty, \;\; x\,(-\infty) = (-\infty)\,x = -\infty \tag{1.1.4}$$

when $0 < x \le +\infty$, and

$$x\,(+\infty) = (+\infty)\,x = -\infty, \;\; x\,(-\infty) = (-\infty)\,x = +\infty \tag{1.1.5}$$

when $-\infty \le x < 0$.

We also put $x/(+\infty) = x/(-\infty) = 0$ when $x \in \mathbf{R}$. Although $1/0$ is not normally defined, it may be appropriate to interpret it as being $+\infty$ when dealing with nonnegative extended real numbers.

The product of 0 and $\pm\infty$ is normally not defined. However, in measure theory and integration, it turns out to be convenient to interpret it as being equal to 0.

1.1.1 Some remarks about sequences of real numbers

Let $\{x_j\}_{j=1}^\infty$ be a sequence of real numbers. We say that x_j tends to $+\infty$ as $j \to \infty$, or $x_j \to +\infty$ as $j \to \infty$, if for every nonnegative real number R there is a positive integer L such that

$$x_j > R \text{ for every } j \ge L. \tag{1.1.6}$$

Similarly, we say that x_j tends to $-\infty$ as $j \to \infty$, or $x_j \to -\infty$ as $j \to \infty$, if for every nonnegative real number R there is a positive integer L such that

$$x_j < -R \text{ for every } j \ge L. \tag{1.1.7}$$

Suppose that $x_j \to x$ as $j \to \infty$ for some extended real number x, and similarly that $\{y_j\}_{j=1}^\infty$ is a sequence of real numbers with $y_j \to y$ as $j \to \infty$ for some extended real number y. If $x + y$ is defined as an extended real number, then

$$x_j + y_x \to x + y \text{ as } j \to \infty. \tag{1.1.8}$$

Of course, this is well known when $x, y \in \mathbf{R}$. If $x = +\infty$, for instance, then one can check that

$$x_j + y_j \to +\infty \text{ as } j \to \infty \tag{1.1.9}$$

when the set of y_j's has a lower bound in $\mathbf{R}$. In particular, this holds when $y_j \to y$ as $j \to \infty$, with $y > -\infty$.

Similarly, if $x\,y$ is defined as an extended real number, then

$$x_j\,y_j \to x\,y \text{ as } j \to \infty. \tag{1.1.10}$$

This does not include the case where one of x or y is 0 and the other is $\pm\infty$. If $x, y \in \mathbf{R}$, then (1.1.10) is well known, as before. If $x = +\infty$ and $y > 0$, for example, then

$$x_j\,y_j \to +\infty \text{ as } j \to \infty. \tag{1.1.11}$$

More precisely, this works when there is a positive real number a such that $y_j \ge a$ for all but finitely many j.

If $\{x_j\}_{j=1}^\infty$ is a sequence of nonzero real numbers such that

$$|x_j| \to +\infty \text{ as } j \to \infty, \tag{1.1.12}$$

then it is easy to see that

$$1/x_j \to 0 \text{ as } j \to \infty. \tag{1.1.13}$$

If $\{x_j\}_{j=1}^\infty$ is a sequence of positive real numbers that converges to 0, then one can verify that

$$1/x_j \to +\infty \text{ as } j \to \infty. \tag{1.1.14}$$

1.1.2 A topology on the set of extended real numbers

One might like to have a metric on the set of extended real numbers so that sequences of real numbers tending to $+\infty$ or $-\infty$ correspond to convergent sequences with respect to that metric. It is not too difficult to do that, although one cannot simply extend the standard metric on the real line to the set of extended real numbers in a convenient way. Instead one may consider metrics on the set of extended real numbers whose restriction to $\mathbf{R}$ is equivalent to the standard metric in a suitable sense, so that they determine the same convergent sequences in $\mathbf{R}$ in particular.

It is somewhat simpler to define an appropriate topology on the set of extended real numbers, if one is familiar with the notion of a topological space. This topology is determined by a metric on the set of extended real numbers, as in the preceding paragraph. One can choose this topology so that the corresponding induced topology on the real line is the same as the standard topology on $\mathbf{R}$. One can also choose this topology so that sequences of real numbers tending to $+\infty$ or $-\infty$ correspond to convergent sequences with respect to the topology.

It is sometimes convenient to consider sequences of extended real numbers as well. One can define what it means for a sequence of extended real numbers to tend to $+\infty$ or to $-\infty$ in the same way as for sequences of real numbers. A sequence of extended real numbers can converge to a real number only when all but finitely many terms in the sequence are real numbers, so that convergence is basically the same as for sequences of real numbers.

1.2 More on extended real numbers

If A is any set of extended real numbers, then the notions of upper and lower bounded for A in the set of extended real numbers may be defined in the usual way. Similarly, the notions of supremum or least upper bound and infimum of greatest lower bound for A in the set of extended real numbers may be defined as usual. In fact, the supremum and infimum of A in the set of extended real numbers always exist, as follows.

If $+\infty \in A$, or if $A \cap \mathbf{R}$ has no upper bound in $\mathbf{R}$, then $\sup A = +\infty$. If $+\infty \notin A$, and if $A \cap \mathbf{R}$ is nonempty and has an upper bound in $\mathbf{R}$, then the supremum of A is the same as the supremum of $A \cap \mathbf{R}$ in $\mathbf{R}$. Otherwise, if $A = \emptyset$ or $A = \{-\infty\}$, then $\sup A = -\infty$.

The infimum of A may be characterized analogously. Note that

$$\inf A \le \sup A \tag{1.2.1}$$

when $A \neq \emptyset$.

Let t be a positive real number, and put

$$t\,A = \{t\,x : x \in A\}. \tag{1.2.2}$$

One can check that

$$\sup(t\,A) = t\,(\sup A), \tag{1.2.3}$$

and similarly for the infimum of $t\,A$.

1.2.1 Suprema and infima of functions

Let E be a set, and let f be a function on E with values in the set of extended real numbers. We may use the notation

$$\sup_{x \in E} f(x) = \sup f(E) \tag{1.2.4}$$

and

$$\inf_{x \in E} f(x) = \inf f(E). \tag{1.2.5}$$

If t is a positive real number, then

$$\sup_{x \in E}(t\,f(x)) = t\,\Big(\sup_{x \in E} f(x)\Big) \tag{1.2.6}$$

and

$$\inf_{x \in E}(t\,f(x)) = t\,\Big(\int_{x \in E} f(x)\Big), \tag{1.2.7}$$

as in (1.2.2) and (1.2.3), with $A = f(E)$, respectively.

Let g be another function on E with values in the extended real numbers. One can check that

$$\sup_{x \in E}(f(x) + g(x)) \le \Big(\sup_{x \in E} f(x)\Big) + \Big(\sup_{x \in E} g(x)\Big) \tag{1.2.8}$$

and

$$\inf_{x \in E}(f(x) + g(x)) \ge \Big(\inf_{x \in E} f(x)\Big) + \Big(\inf_{x \in E} g(x)\Big). \tag{1.2.9}$$

Suppose now that $f, g \ge 0$ on E. One can verify that

$$\sup_{x \in E}(f(x)\,g(x)) \le \Big(\sup_{x \in E} f(x)\Big)\,\Big(\sup_{x \in E} g(x)\Big) \tag{1.2.10}$$

and

$$\inf_{x \in E}(f(x)\,g(x)) \ge \Big(\inf_{x \in E} f(x)\Big)\,\Big(\inf_{x \in E} g(x)\Big). \tag{1.2.11}$$

1.2.2 Infinite series with nonnegative terms

Let $\{x_j\}_{j=1}^\infty$ be a sequence of real numbers that is monotonically increasing, so that $x_j \le x_{j+1}$ for each j. If the set of x_j's has an upper bound in $\mathbf{R}$, then it is well known that $\{x_j\}_{j=1}^\infty$ converges to the supremum of this set. Otherwise, if the set of x_j's does not have an upper bound in $\mathbf{R}$, then it is easy to see that $x_j \to +\infty$ as $j \to \infty$. In both cases, if we put

$$x = \sup_{l \ge 1} x_l, \tag{1.2.12}$$

then we have that $x_j \to x$ as $j \to \infty$.

Consider an infinite series $\sum_{j=1}^\infty a_j$ of nonnegative real numbers. Of course, the corresponding sequence of partial sums $\sum_{j=1}^n a_j$ is monotonically increasing. If the sequence of partial sums has an upper bound in **R**, then $\sum_{j=1}^\infty a_j$ converges, with

$$\sum_{j=1}^\infty a_j = \sup_{n \ge 1} \sum_{j=1}^n a_j, \tag{1.2.13}$$

as in the preceding paragraph. Otherwise, if the sequence of partial sums does not have an upper bound on **R**, then we may interpret the value of $\sum_{j=1}^\infty a_j$ as being $+\infty$.

If t is a positive real number, then

$$\sum_{j=1}^n t\, a_j = t \sum_{j=1}^n a_j \tag{1.2.14}$$

for each positive integer n. This implies that

$$\sum_{j=1}^\infty t\, a_j = t \sum_{j=1}^\infty a_j. \tag{1.2.15}$$

More precisely, this also works when the sums are equal to $+\infty$. This works when $t = 0$ too, with the right side interpreted as being equal to 0 even when $\sum_{j=1}^\infty a_j = +\infty$, as in the previous section.

Let $\sum_{j=1}^\infty b_j$ be another infintie series of nonnegative real numbers, and note that

$$\sum_{j=1}^n (a_j + b_j) = \sum_{j=1}^n a_j + \sum_{j=1}^n b_j \tag{1.2.16}$$

for every positive integer n. One can use this to get that

$$\sum_{j=1}^\infty (a_j + b_j) = \sum_{j=1}^\infty a_j + \sum_{j=1}^\infty b_j. \tag{1.2.17}$$

This also works with suitable interpretations when any of the sums are equal to $+\infty$, as before.

1.3 Upper and lower limits of sequences in R

Let $\{x_j\}_{j=1}^\infty$ be a sequence of real numbers, and let E be the set of extended real numbers x such that there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ with

$$x_{j_l} \to x \text{ as } l \to \infty. \tag{1.3.1}$$

It is well known that

$$E \ne \emptyset. \tag{1.3.2}$$

Indeed, if the set of x_j's has upper and lower bounds in $\mathbf{R}$, then $E \cap \mathbf{R} \neq \emptyset$, because closed intervals in $\mathbf{R}$ are sequentially compact. If the set of x_j's does not have an upper bound in $\mathbf{R}$, then it is not too difficult to show that $+\infty \in E$. Similarly, if the set of x_j's does not have a lower bound in $\mathbf{R}$, then $-\infty \in E$.

1.3.1 Defining upper and lower limits

The *upper limit* or *limit superior* of $\{x_j\}_{j=1}^\infty$ may be defined by

$$\limsup_{j \to \infty} x_j = \sup E, \tag{1.3.3}$$

as on p56 of [158]. Similarly, the *lower limit* or *limit inferior* of $\{x_j\}_{j=1}^\infty$ may be defined by

$$\liminf_{j \to \infty} x_j = \inf E, \tag{1.3.4}$$

as in [158]. Some other equivalent formulations are sometimes used, and we shall say more about that soon.

Note that

$$\liminf_{j \to \infty} x_j \le \limsup_{j \to \infty} x_j. \tag{1.3.5}$$

If there is an extended real number x such that $x_j \to x$ as $j \to \infty$, then

$$E = \{x\}, \tag{1.3.6}$$

and

$$\limsup_{j \to \infty} x_j = \liminf_{j \to \infty} x_j = x. \tag{1.3.7}$$

One can check that

$$\limsup_{j \to \infty} x_j \le \sup_{j \ge 1} x_j \tag{1.3.8}$$

and

$$\liminf_{j \to \infty} x_j \ge \inf_{j \ge 1} x_j, \tag{1.3.9}$$

directly from the previous definitions, or using some of the other characterizations of the upper and lower limits that we shall discuss. It is easy to give examples where these inequalities are strict.

1.3.2 Some properties of the upper and lower limits

Put $y = \limsup_{j \to \infty} x_j$ and $u = \liminf_{j \to \infty} x_j$. It is not too difficult to show that

(1.3.10) if $z \in \mathbf{R}$ and $y < z$, then $x_j < z$ for all but finitely many j.

Otherwise, there would be a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that

$$x_{j_l} \ge z \tag{1.3.11}$$

for each l. This would imply that there is an element of E greater than or equal to z, using a subsequence of $\{x_{j_l}\}_{l=1}^{\infty}$ that tends to an extended real number that is necessarily greater than or equal to z. This corresponds to part (b) of Theorem 3.17 on p56 of [158].

Similarly,

$$\text{if } t \in \mathbf{R} \text{ and } t < u, \text{ then } x_j > t \text{ for all but finitely many } j. \tag{1.3.12}$$

This can be obtained using an analogous argument, or by reducing to the previous case, for $\{-x_j\}_{j=1}^{\infty}$ in place of $\{x_j\}_{j=1}^{\infty}$.

We also have that

$$\text{if } w \in \mathbf{R} \text{ and } w < y, \text{ then } x_j > w \text{ for infinitely many } j. \tag{1.3.13}$$

Indeed, if $w < y$, then w is not an upper bound for E, by definition of the supremum, so that there is an $x \in E$ with $w < x$. One can use this to get that $x_j > w$ for infinitely many j. Alternatively, if $x_j \le w$ for all but finitely many j, then one can check that w is an upper bound for E.

Similarly,

$$\text{if } v \in \mathbf{R} \text{ and } u < v, \text{ then } x_j < v \text{ for infinitely many } j. \tag{1.3.14}$$

One can check that y is uniquely determined by (1.3.10) and (1.3.13). Similarly, u is uniquely determined by (1.3.12) and (1.3.14).

1.3.3 Some variants and related properties

Part (a) of Theorem 3.17 on p56 of [158] says that

$$y \in E. \tag{1.3.15}$$

This can be obtained from (1.3.10) and (1.3.13). Note that (1.3.15) implies (1.3.14).

Another part of Theorem 3.17 on p56 of [158] states that y is uniquely determined by (1.3.10) and (1.3.15). This is essentially the same as before.

Similarly,

$$u \in E, \tag{1.3.16}$$

and u is uniquely determined by (1.3.12) and (1.3.16).

Suppose for the moment that

$$\limsup_{j\to\infty} x_j = \liminf_{j\to\infty} x_j, \tag{1.3.17}$$

and let x be their common value. One can check that $x_j \to x$ as $j \to \infty$, using (1.3.10) and (1.3.12).

If $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ are sequences of real numbers, then it is well known that

$$\limsup_{j\to\infty}(a_j + b_j) \le \limsup_{j\to\infty} a_j + \limsup_{j\to\infty} b_j, \tag{1.3.18}$$

as long as the right side is defined as an extended real number, so that it is not a sum of $+\infty$ and $-\infty$. This corresponds to part (b) of Exercise 4 on p32 of [157], and to Exercise 5 on p78 of [158].

1.3.4 Another approach to upper and lower limits

Let $\{x_j\}_{j=1}^\infty$ be a sequence of extended real numbers, and consider

$$\sup_{j \ge l} x_j \tag{1.3.19}$$

for each positive integer l. This is a monotonically decreasing sequence of extended real numbers, and the upper limit of $\{x_j\}_{j=1}^\infty$ is sometimes defined by

$$\limsup_{j \to \infty} x_j = \inf_{l \ge 1} \Big(\sup_{j \ge l} x_j \Big), \tag{1.3.20}$$

as in Definition 1.13 on p14 of [157].

Similarly,

$$\inf_{j \ge l} x_j \tag{1.3.21}$$

is a monotonically increasing sequence of extended real numbers, and the lower limit of $\{x_j\}_{j=1}^\infty$ is sometimes defined by

$$\liminf_{j \to \infty} x_j = \sup_{l \ge 1} \Big(\inf_{j \ge l} x_j \Big), \tag{1.3.22}$$

as in [157]. This is essentially the way that upper and lower limits of sequences of real numbers are defined in Section 2.9 beginning on p46 of [78].

If (1.3.19) is finite for each l, then this sequence tends to the infimum of its terms as $l \to \infty$. This also works when (1.3.19) may be $\pm\infty$, with suitable interpretations. Similarly, the sequence (1.3.21) tends to the supremum of its terms as $l \to \infty$, with suitable interpretations when these terms may be $\pm\infty$.

It is mentioned on p14 of [157] that (1.3.20) satisfies the same conditions as the previous definition of the upper limit, so that the two approaches are equivalent. It is perhaps a bit simper to check that (1.3.20) satisfies (1.3.10) and (1.3.13), and similarly that (1.3.22) satisfies (1.3.12) and (1.3.14). This corresponds to Theorem 2.9 L on p50 of [78] for bounded sequences of real numbers. It is mentioned just after the proof that (1.3.20) and (1.3.22) are characterized by these properties.

The fact that (1.3.20) is an element of the set E defined earlier is shown in the proof of Theorem 2.9 M on p51 of [78] for bounded sequences of real numbers. The analogous statement for (1.3.22) is the first part of Exercise 3 on p51 of [78]. Exercise 2 on p51 of [78] states that (1.3.20) is an upper bound for E, and the second part of Exercise 3 on p51 of [78] is the same as saying that (1.3.22) is a lower bound for E.

1.4 Indicator functions

Let X be a set, and let A be a subset of X. The *indicator function* of A with respect to X is defined by

$$\begin{aligned} \mathbf{1}_A(x) &= 1 \quad \text{when } x \in A \\ &= 0 \quad \text{when } x \in X \setminus A. \end{aligned} \tag{1.4.1}$$

This is also known as the *characteristic function* of A, and denoted χ_A. However, this terminology is often used for something else.

1.4.1 Some upper and lower limits

Let $\{x_j\}_{j=1}^\infty$ be an infinite sequence such that for each j, $x_j = 0$ or 1. One can check that

$$\begin{aligned} \limsup_{j\to\infty} x_j &= 1 \quad \text{when } x_j = 1 \text{ for infinitely many } j \\ &= 0 \quad \text{when } x_j = 0 \text{ for all but finitely many } j. \end{aligned} \tag{1.4.2}$$

Similarly,

$$\begin{aligned} \liminf_{j\to\infty} x_j &= 0 \quad \text{when } x_j = 0 \text{ for infinitely many } j \\ &= 1 \quad \text{when } x_j = 1 \text{ for all but finitely many } j. \end{aligned} \tag{1.4.3}$$

1.4.2 Upper and lower limits of sequences of sets

Let $\{A_j\}_{j=1}^\infty$ be a sequence of subsets of X. The *upper limit* or *limit superior* of $\{A_j\}_{j=1}^\infty$ may be defined as a subset of X by

$$\limsup_{j\to\infty} A_j = \{x \in X : x \in A_j \text{ for infinitely many } j\}, \tag{1.4.4}$$

as in Definition 2.12 B on p65 of [78].

Similarly, the *lower limit* or *limit inferior* of $\{A_j\}_{j=1}^\infty$ may be defined as a subset of X by

$$\liminf_{j\to\infty} A_j = \{x \in X : x \in A_j \text{ for all but finitely many } j\}, \tag{1.4.5}$$

as in [78]. Note that

$$\liminf_{j\to\infty} A_j \subseteq \limsup_{j\to\infty} A_j, \tag{1.4.6}$$

as in Exercise 1 on p65 of [78].

If $x \in X$, then

$$\limsup_{j\to\infty} \mathbf{1}_{A_j}(x) = \mathbf{1}_{\limsup_{j\to\infty} A_j}(x), \tag{1.4.7}$$

as mentioned on p65 of [78]. Similarly,

$$\liminf_{j\to\infty} \mathbf{1}_{A_j}(x) = \mathbf{1}_{\liminf_{j\to\infty} A_j}(x), \tag{1.4.8}$$

as in [78]. This uses the remarks in the previous subsection.

1.4.3 More on sequences of subsets of X

One can show that

$$\limsup_{j\to\infty} A_j = \bigcap_{l=1}^{\infty} \Big(\bigcup_{j=l}^{\infty} A_j\Big), \tag{1.4.9}$$

as in part (b) of Exercise 3 on p66 of [78]. Similarly,

$$\liminf_{j\to\infty} A_j = \bigcup_{l=1}^{\infty} \Big(\bigcap_{j=l}^{\infty} A_j\Big), \tag{1.4.10}$$

as in part (b) of Exercise 4 on p66 of [78].

If $A_j \subseteq A_{j+1}$ for each j, then

$$\limsup_{j\to\infty} A_j = \liminf_{j\to\infty} A_j = \bigcup_{j=1}^{\infty} A_j, \tag{1.4.11}$$

as in part (a) of Exercise 5 on p66 of [78]. Similarly, if $A_{j+1} \subseteq A_j$ for each j, then

$$\limsup_{j\to\infty} A_j = \liminf_{j\to\infty} A_j = \bigcap_{j=1}^{\infty} A_j, \tag{1.4.12}$$

as in part (b) of Exercise 5 on p66 of [78].

If

$$\limsup_{j\to\infty} A_j = \liminf_{j\to\infty} A_j, \tag{1.4.13}$$

then one may say that $\{A_j\}_{j=1}^{\infty}$ *converges*, with limit

$$\lim_{j\to\infty} A_j \tag{1.4.14}$$

equal to (1.4.13). This means that

$$\lim_{j\to\infty} \mathbf{1}_{A_j}(x) = \mathbf{1}_{\lim_{j\to\infty} A_j}(x) \tag{1.4.15}$$

for each $x \in X$. Monotonic sequences of subsets of X converge, as in the preceding paragraph.

1.5 Algebras and σ-algebras of sets

Let X be a set, and let $\mathcal{A}$ be a nonempty collection of subsets of X. We say that $\mathcal{A}$ is an *algebra* of subsets of X if it has the following two properties. First, if $A, B \in \mathcal{A}$, then

$$A \cup B \in \mathcal{A}. \tag{1.5.1}$$

Second, if $A \in \mathcal{A}$, then

$$X \setminus A \in \mathcal{A}. \tag{1.5.2}$$

This corresponds to parts of 11.1 on p118 of [85] and Definition 1.11 on p4 of [86], with some minor differences in the formulation, and it is also mentioned on p10 of [157] and p270 of [180].

If $\mathcal{A}$ is an algebra of subsets of X and $A, B \in \mathcal{A}$, then it is easy to see that

$$A \cap B \in \mathcal{A}. \tag{1.5.3}$$

We also have that

$$X \in \mathcal{A}, \tag{1.5.4}$$

because $\mathcal{A} \neq \emptyset$, by hypothesis. Similarly, the empty set is an element of $\mathcal{A}$.

1.5.1 σ-Algebras of sets

We say that $\mathcal{A}$ is a *σ-algebra* of subsets of X if (1.5.2) holds for every $A \in \mathcal{A}$, and if for every sequence $A_1, A_2, A_3, \ldots$ of elements of $\mathcal{A}$, we have that

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}. \tag{1.5.5}$$

This corresponds to parts of 11.1 on p118 of [85] and Definition 1.13 on p4 of [86], and to part (a) of Definition 1.3 on p9 of [157], and it is mentioned on p23, 263 of [180].

Of course, if $\mathcal{A}$ is a σ-algebra of subsets of X, then $\mathcal{A}$ is an algebra of subsets of X, and (1.5.4) holds in particular. In this case, if $A_1, A_2, A_3, \ldots$ is a sequence of elements of $\mathcal{A}$, then

$$\bigcap_{j=1}^{\infty} A_j \in \mathcal{A}. \tag{1.5.6}$$

The collection of all subsets of X is a σ-algebra, as is the collection consisting only of X and $\emptyset$.

A σ-algebra of subsets of X is called a *Borel family* in Definition 2.2.1 on p59 of [62]. The σ-algebra of *Borel sets* in a metric or topological space will be discussed in Section 2.2.

1.5.2 Rings of sets

We say that $\mathcal{A}$ is a *ring* of subsets of X if for every $A, B \in \mathcal{A}$ we have that (1.5.1) holds, and also

$$A \setminus B \in \mathcal{A}. \tag{1.5.7}$$

This corresponds to parts of 11.1 on p118 of [85] and Definition 1.11 on p4 of [86] again, as well as Definition 11.1 on p301 of [158], although the condition that $\mathcal{A}$ be nonempty does not appear to be included in [158]. The nonemptiness of $\mathcal{A}$ implies that the empty set is an element of $\mathcal{A}$, as mentioned in [86].

If $\mathcal{A}$ is a ring of subsets of $\mathcal{A}$ and $A, B \in \mathcal{A}$, then (1.5.3) holds, because

$$A \cap B = A \setminus (A \setminus B), \tag{1.5.8}$$

as mentioned in [86, 158]. If $X \in \mathcal{A}$, then $\mathcal{A}$ is an algebra of subsets of X, as in [86].

If A and B are subsets of X, then their *symmetric difference* is the subset of X defined by

$$A \triangle B = (A \setminus B) \cup (B \setminus A). \tag{1.5.9}$$

If $\mathcal{A}$ is a ring of subsets of X and $A, B \in \mathcal{A}$, then

$$A \triangle B \in \mathcal{A}, \tag{1.5.10}$$

as mentioned on p4 of [86].

Conversely, suppose that (1.5.3) and (1.5.10) hold for all $A, B \in \mathcal{A}$, and let us check that $\mathcal{A}$ is a ring of subsets of X. If A, B are any subsets of X, then

$$A \setminus B = A \triangle (A \cap B). \tag{1.5.11}$$

This implies that (1.5.7) holds when $A, B \in \mathcal{A}$. We also have that

$$(A \setminus B) \triangle B = ((A \setminus B) \setminus B) \cup (B \setminus (A \setminus B)) = (A \setminus B) \cup B = A \cup B. \tag{1.5.12}$$

It follows that (1.5.1) holds when $A, B \in \mathcal{A}$.

1.5.3 More on rings of sets

If A and B are any subsets of X, then

$$\mathbf{1}_A(x)\, \mathbf{1}_B(x) = \mathbf{1}_{A \cap B}(x) \tag{1.5.13}$$

for every $x \in X$. We also have that

$$\mathbf{1}_A(x) + \mathbf{1}_B(x) = \mathbf{1}_{A \cup B}(x) + \mathbf{1}_{A \cap B}(x) = \mathbf{1}_{A \triangle B}(x) + 2\, \mathbf{1}_{A \cap B}(x) \tag{1.5.14}$$

for every $x \in X$. In particular, this means that

$$\mathbf{1}_A(x) + \mathbf{1}_B(x) \equiv \mathbf{1}_{A \triangle B}(x) \text{ modulo } 2 \tag{1.5.15}$$

for every $x \in X$.

It is well known that $\{0, 1\}$ is a field with respect to addition and multiplication modulo 2. The collection of all functions on X with values in $\{0, 1\}$ may be considered as a commutative ring with respect to pointwise addition and multiplication of functions module 2. Of course, these functions on X are the same as indicator functions of subsets of X.

This means that the collection of all subsets of X may be considered as a commutative ring with respect to symmetric differences as addition and intersections as multiplication. Rings of subsets of X correspond to subrings of this ring, as discussed in [199].

1.5.4 σ-Rings of sets

A ring $\mathcal{A}$ of subsets of X is said to be a *σ-ring of subsets of X* if (1.5.5) holds for every sequence $A_1, A_2, A_3, \ldots$ of elements of $\mathcal{A}$, as in 11.1 on p118 of [85], Definition 1.13 on p4 of [86], and Definition 11.1 on p301 of [158]. If $X \in \mathcal{A}$, then it follows that $\mathcal{A}$ is a σ-algebra of subsets of X. This condition is included in the definition of a measurable space on p310 of [158].

If $\mathcal{A}$ is a σ-ring of subsets of X and $A_1, A_2, A_3, \ldots$ is a sequence of elements of $\mathcal{A}$, then (1.5.5) holds, as mentioned on p301 of [158]. This uses the observation that

$$\bigcap_{j=1}^{\infty} A_j = A_1 \setminus \Big(\bigcup_{j=1}^{\infty} (A_1 \setminus A_j) \Big). \tag{1.5.16}$$

The collection of all finite subsets of X is a ring of subsets of X. If X has infiitely many elements, then this is neither an algebra of subsets of X, nor a σ-ring of subsets of X. This example is mentioned on p4 of [86], when X is the set $\mathbf{Z}_+$ of positive integers.

Similarly, the collection of all subsets of X with only finitely or countably many elements is a σ-ring of subsets of X. If X is uncountable, then this is not a σ-algebra of subsets of X, as mentioned on p4 of [86].

1.6 Some remarks about intervals

Let a and b be extended real numbers with $a < b$. We may use (a, b) for the set of real numbers x such that

$$a < x < b. \tag{1.6.1}$$

Similarly, we may use $[a, b]$ for the set of extended real numbers x such that

$$a \le x \le b. \tag{1.6.2}$$

We may also use this notation when $a = b$, so that the set contains only one element. In the same way, we may use the notation $[a, b)$ for the set of extended real numbers x such that

$$a \le x < b, \tag{1.6.3}$$

and the notation $(a, b]$ for the set of extended real numbers x such that

$$a < x \le b. \tag{1.6.4}$$

A subset of the real line any of these four types may be described as an *interval*. The *length* of such an interval is defined to be $b - a$. Thus the length of the interval is finite exactly when $a, b \in \mathbf{R}$, and we may say that the interval is *bounded* in this case. The empty set may be considered as an interval in $\mathbf{R}$ as well, with length 0.

1.6.1 Some elementary subsets of R

Let us say that a subset of $\mathbf{R}$ is a *bounded elementary set* if it can be expressed as the union of finitely many bounded intervals. This corresponds to an elementary subset of $\mathbf{R}$ as defined on p303 of [158]. Let $\mathcal{E}(\mathbf{R})$ be the collection of all bounded elementary subsets of $\mathbf{R}$. This is a ring of subsets of $\mathbf{R}$, and not a σ-ring, as in (12) on p303 of [158]. Note that $\mathcal{E}(\mathbf{R})$ is not an algebra of subsets of $\mathbf{R}$.

Let us say that a subset of the real line is a *possibly unbounded elementary set* if it can be expressed as the union of finitely many intervals that are not necessarily bounded. Let $\mathcal{E}_1(\mathbf{R})$ be the collection of all possibly unbounded elementary subsets of $\mathbf{R}$. This is an algebra of subsets of $\mathbf{R}$ that is not a σ-algebra.

1.6.2 Intervals in $\mathbf{R}^n$

Let n be a positive integer, and let $A_1, \ldots, A_n$ be n sets. The *Cartesian product* of $A_1, \ldots, A_n$ may be denoted

$$A_1 \times A_2 \times \cdots \times A_n \text{ or } \prod_{j=1}^{n} A_j, \tag{1.6.5}$$

and is the set of n-tuples $a = (a_1, \ldots, a_n)$ with $a_j \in A_j$ for $j = 1, \ldots, n$.

In particular, the space $\mathbf{R}^n$ of n-tuples of real numbers is the same as the Cartesian product of n copies of the real numbers. A subset of $\mathbf{R}^n$ may be called a *bounded interval* if it is the Cartesian product of n bounded intervals in $\mathbf{R}$. This corresponds to the definition of an interval in $\mathbf{R}^n$ in Definition 11.4 on p302 of [158]. The n-dimensional *volume* of such a bounded interval may be defined as the product of the lengths of these n intervals, as on p303 of [158]. Note that the empty set is considered to be a bounded interval in $\mathbf{R}^n$, with n-dimensional volume equal to 0.

Let us say that a subset of $\mathbf{R}^n$ is a *possibly unbounded interval* if it is the Cartesian product of n intervals in the real line that are not necessarily bounded. The n-dimensional *volume* may be defined as the product of the lengths of these n intervals, as before. This is interpreted as being $+\infty$ when each of the intervals has positive length, and at least one of the intervals has length $+\infty$. If at least one of the intervals has length 0, then we interpret the product as being equal to 0, even if some of the other intervals have length $+\infty$.

1.6.3 Some elementary subsets of $\mathbf{R}^n$

Let us say that a subset of $\mathbf{R}^n$ is a *bounded elementary set* if it can be expressed as the union of finitely many bounded intervals in $\mathbf{R}^n$. This corresponds to an elementary subset of $\mathbf{R}^n$ as defined on p303 of [158] again. The collection $\mathcal{E}(\mathbf{R}^n)$ of all bounded elementary subsets of $\mathbf{R}^n$ is a ring of subsets of $\mathbf{R}^n$, and not a σ-ring, as in (12) on p303 of [158]. This is also not an algebra of subsets of $\mathbf{R}^n$, as before.

Let us say that a subset of $\mathbf{R}^n$ is a *possibly unbounded elementary set* if it can be expressed as the union of finitely many possibly unbounded intervals in $\mathbf{R}^n$. The collection $\mathcal{E}_1(\mathbf{R}^n)$ of all possibly unbounded elementary subsets of $\mathbf{R}^n$ is an algebra of subsets of $\mathbf{R}^n$ that is not a σ-algebra.

Sometimes one may be interested in elementary subsets of a subset of $\mathbf{R}^n$, and this will be discussed more broadly in the next subsection.

1.6.4 Some more rings and algebras of sets

Let X be a set, let $\mathcal{A}$ be a nonempty collection of subsets of X, and let Y be a subset of X. Note that

$$\mathcal{A}_Y = \{A \cap Y : A \in \mathcal{A}\} \tag{1.6.6}$$

is a nonempty collection of subsets of Y. If $\mathcal{A}$ is a ring, algebra, σ-ring, or σ-algebra of subsets of X, then one can check that $\mathcal{A}_Y$ has the same property as a collection of subsets of Y.

Suppose for the moment that there is an element A_1 of $\mathcal{A}$ such that

$$Y \subseteq A_1. \tag{1.6.7}$$

This means that

$$Y = A_1 \cap Y \in \mathcal{A}_Y. \tag{1.6.8}$$

If $\mathcal{A}$ is a ring or σ-ring of subsets of X, then it follows that $\mathcal{A}_Y$ is an algebra or σ-algebra of subsets of Y, as appropriate.

Let $\mathcal{A}_{Y,0}$ be the collection of elements of $\mathcal{A}$ that are subsets of Y. Observe that

$$\mathcal{A}_{Y,0} \subseteq \mathcal{A}_Y, \tag{1.6.9}$$

and that $\emptyset \in \mathcal{A}_{Y,0}$ when $\emptyset \in \mathcal{A}$. If $\mathcal{A}$ is a ring or σ-ring of subsets of X, then $\mathcal{A}_{Y,0}$ has the same property, as a collection of subsets of Y or X. We also have that

$$\mathcal{A}_{Y,0} = \mathcal{A}_Y \tag{1.6.10}$$

when $Y \in \mathcal{A}$ and $\mathcal{A}$ is a ring of subsets of X.

1.7 Finitely and countably additive measures

Let X be a set, and let $\mathcal{A}$ be a ring of subsets of X. Also let μ be a function on $\mathcal{A}$ with values in the set of nonnegative extended real numbers. We say that μ is a *finitely additive measure* on X with respect to $\mathcal{A}$ if

$$\mu(\emptyset) = 0 \tag{1.7.1}$$

and

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{1.7.2}$$

for all $A.B \in \mathcal{A}$ with $A \cap B = \emptyset$. If $A_1, \ldots, A_n$ are finitely many pairwise-disjoint elements of $\mathcal{A}$, then it follows that

$$\mu\Big(\bigcup_{j=1}^{n} A_j\Big) = \sum_{j=1}^{n} \mu(A_j). \tag{1.7.3}$$

This corresponds to part of 11.1 on p118 of [85], and to part of Definition 10.3 on p126 of [86] when $\mathcal{A}$ is an algebra of subsets of X.

This basically corresponds to part of Definition 11.2 on p301 of [158] as well. Nonnegativity of μ is not part of this definition, although that is mostly what is considered, as in Definition 11.12 on p310 of [158]. Other types of measures are very interesting and important, but they are not our main focus here. More precisely, in [158], μ is allowed to take the value $+\infty$ on $\mathcal{A}$, or the value $-\infty$, but not both, so that the sum on the right side of (1.7.2) is defined. One also asks that $\mu(A)$ be finite for some $A \in \mathcal{A}$, and this is used to obtain (1.7.1), as in (5) on p301 of [158].

Suppose that μ is a finitely additive measure on X with respect to $\mathcal{A}$, A, B are elements of $\mathcal{A}$, and $A \subseteq B$. Observe that

$$\mu(A) \le \mu(A) + \mu(B \setminus A) = \mu(B), \tag{1.7.4}$$

as in part (c) of Theorem 1.19 on p17 of [157], and (8) on p302 of [158].

1.7.1 Countable additivity

Similarly, we say that μ is a *countably additive measure* on X with respect to $\mathcal{A}$ if (1.7.1) holds, and if

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j) \tag{1.7.5}$$

for every sequence $A_1, A_2, A_3, \ldots$ of pairwise-disjoint elements of $\mathcal{A}$ such that

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}. \tag{1.7.6}$$

This corresponds to parts of 11.1 on p118 of [85], Definition 10.3 on p126 of [86], and Definition 11.2 on p301 of [158], as before. If $\mathcal{A}$ is an algebra of subsets of X, then μ may be called a *premeasure*, as on p270 of [180].

Of course, (1.7.6) holds automatically when $\mathcal{A}$ is a σ-ring of subsets of X. This also corresponds to part (a) of Definition 1.18 on p17 of [157] when $\mathcal{A}$ is a σ-algebra of subsets of X. One may refer to μ as a positive or nonnegative measure, for emphasis, as in [157]. This corresponds to the definition of a measure on p263 of [180] when $\mathcal{A}$ is a σ-algebra of subsets of X as well.

1.7.2 Measurable and measure spaces

If $\mathcal{A}$ is a σ-algebra of subsets of X, then $(X, \mathcal{A})$ may be called a *measurable space*, as on p149 of [86], and in part (b) of Definition 1.3 on p9 of [157]. In this case, the elements of $\mathcal{A}$ may be called *measurable subsets* of X with respect to $\mathcal{A}$.

If $(X, \mathcal{A})$ is a measurable space and μ is a nonnegative countably additive measure on $(X, \mathcal{A})$, then $(X, \mathcal{A}, \mu)$ is called a *measure space*. This corresponds to part of Definition 10.3 on p126 of [86], and to part (b) of Definition 1.18 on p17 of [157], and it is the same as on p263 of [180]. These terms are used a bit differently on p310 of [158].

If $(X, \mathcal{A}, \mu)$ is a measure space and

$$\mu(X) < +\infty, \tag{1.7.7}$$

then $(X, \mathcal{A}, \mu)$ is said to be a *finite measure space*, and μ is said to be a *finite measure*, as in Definition 10.3 on p126 of [86]. If

$$\mu(X) = 1, \tag{1.7.8}$$

then $(X, \mathcal{A}, \mu)$ may be called a *probability space*, and μ a *probability measure*.

1.7.3 Some basic examples of measures

Let X be any set, and let $\mathcal{A}$ be the collection of all subsets of X. If A is any subset of X, then let $\mu(A)$ be the number of elements of A, interpreted as $+\infty$ when A has infinitely many elements. One can check that this is a countably additive measure, called *counting measure* on X. This corresponds to Example 10.4 (a) on p127 of [86] and Example 1.20 (a) on p18 of [157], and it is mentioned on p310 of [158] with $X = \mathbf{Z}_+$. This is also mentioned in (i) on p263 of [180] when X is countable.

Let $x_0 \in X$ be given. If A is any subset of X, then put

$$\begin{aligned} \delta_{x_0}(A) &= 1 \quad \text{when } x_0 \in A \\ &= 0 \quad \text{when } x_0 \notin A. \end{aligned} \tag{1.7.9}$$

It is easy to see that this is a probability measure, as in Example 1.20 (b) on p18 of [157].

1.7.4 Some more examples of measures

Let X be an uncountable set, and let $\mathcal{A}_1$ be the collection of subsets A of X such that either A has only finitely or countably many elements, or $X \setminus A$ has only finitely or countably many elements. Put

$$\mu_1(A) = 0 \tag{1.7.10}$$

in the first case, and

$$\mu_1(A) = 1 \tag{1.7.11}$$

in the second case. One can check that $\mathcal{A}_1$ is a σ-algebra of subsets of X, and that μ_1 defines a probability measure. This corresponds to Exercise 6 on p33 of [157].

Now let X be any infinite set, and let $\mathcal{A}_0$ be the collection of subsets A of X such that either A has only finitely many elements, or $X \setminus A$ has only finitely many elements. Put

$$\mu_0(A) = 0 \tag{1.7.12}$$

in the first case, and

$$\mu_0(A) = 1 \tag{1.7.13}$$

in the second case. One can verify that $\mathcal{A}_0$ is an algebra of subsets of X, and that μ_0 is finitely additive. If X is uncountable, then $\mathcal{A}_0 \subseteq \mathcal{A}_1$, and μ_0 is the same as the restriction of μ_1 to $\mathcal{A}_0$. However, if X is countably infinite, then μ_0 is not countably additive on $\mathcal{A}_0$.

1.8 Measures and nonnegative sums

Let X be a nonempty set, and let $\mathcal{A}_0$ be the collection of all finite subsets of X. This is a ring of subsets of X, as mentioned in Subsection 1.5.4. Also let w be a function on X with values in the set of nonnegative extended real numbers. If $A \in \mathcal{A}_0$, then put

$$\mu_w(A) = \sum_{x \in A} w(x). \tag{1.8.1}$$

This should be interpreted as being equal to 0 when $A = \emptyset$.

It is easy to see that μ_w is a finitely additive measure on X with respect to $\mathcal{A}_0$. Conversely, if μ is any finitely additive measure on X with respect to $\mathcal{A}_0$, then μ is of this form in a unique way. In fact, $\mu = \mu_w$ exactly when

$$w(x) = \mu(\{x\}) \tag{1.8.2}$$

for every $x \in X$.

1.8.1 Nonnegative sums over arbitrary sets

Let f be a nonnegative real-valued function on X. The sum

$$\sum_{x \in X} f(x) \tag{1.8.1}$$

may be defined as a nonnegative extended real number as the supremum of the finite subsums

$$\sum_{x \in A} f(x) \tag{1.8.2}$$

over all nonempty finite subsets A of X. Of course, if X has only finitely many elements, then this is the same as the usual sum over X. We may also consider functions f on X with values in the set of nonnegative extended real numbers, by interpreting (1.8.2) as being $+\infty$ when $f(x) = +\infty$ for some $x \in A$. This means that (1.8.1) is equal to $+\infty$ when $f(x) = +\infty$ for some $x \in X$.

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of distinct elements of X, and suppose for the moment that $f(x) = 0$ when $x \in X$ is not one of the terms of this sequence. One can check that

$$\sum_{x \in X} f(x) = \sum_{j=1}^{\infty} f(x_j) \tag{1.8.3}$$

under these conditions. More precisely, the sum on the right should also be interpreted as being equal to $+\infty$ when any of its terms is $+\infty$. Note that the partial sums

$$\sum_{j=1}^{n} f(x_j) \tag{1.8.4}$$

of the sum on the right side of (1.8.3) are of the form (1.8.2). It is easy to see that any finite subsum of the form (1.8.2) is less than or equal to (1.8.4) when n is sufficiently large.

Let us say that f is *summable* on X when

$$\sum_{x \in X} f(x) < +\infty. \tag{1.8.5}$$

If there is an $\epsilon > 0$ such that

$$f(x) \geq \epsilon \tag{1.8.6}$$

for infinitely many $x \in X$, then one can check that f is not summable on X. If f is summable on X, then it follows that (1.8.6) holds for only finitely many $x \in X$. One can use this to get that

$$f(x) > 0 \tag{1.8.7}$$

for only finitely or countably many $x \in X$ when f is summable on X.

1.8.2 Some properties of the sum

If t is a positive real number, then one can check that

$$\sum_{x \in X} t\, f(x) = t \sum_{x \in X} f(x), \tag{1.8.8}$$

where the right side is interpreted as being $+\infty$ when (1.8.1) is $+\infty$, as usual. This also works with $t = 0$, with the right side interpreted as being 0 even when (1.8.1) is $+\infty$, as in a remark in Section 1.1.

If g is another nonnegative extended real-valued function on X, then

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x). \tag{1.8.9}$$

To see this, let A be a finite subset of X, and observe that

$$\sum_{x \in A} (f(x) + g(x)) = \sum_{x \in A} f(x) + \sum_{x \in A} g(x) \le \sum_{x \in X} f(x) + \sum_{x \in X} g(x). \tag{1.8.10}$$

This implies that

$$\sum_{x \in X} (f(x) + g(x)) \le \sum_{x \in X} f(x) + \sum_{x \in X} g(x). \tag{1.8.11}$$

If B is another finite subset of X, then

$$\sum_{x \in A} f(x) + \sum_{x \in B} g(x) \le \sum_{x \in A \cup B} (f(x) + g(x)) \le \sum_{x \in X} (f(x) + g(x)). \tag{1.8.12}$$

One can use this to get that

$$\sum_{x \in X} f(x) + \sum_{x \in X} g(x) \le \sum_{x \in X} (f(x) + g(x)). \tag{1.8.13}$$

If E is any subset of X, then $\sum_{x \in E} f(x)$ may be defined as before, as the supremum of the sums (1.8.2) over all finite subsets A of E. If E_1 and E_2 are disjoint subsets of X, then we get that

$$\sum_{x \in E_1 \cup E_2} f(x) = \sum_{x \in E_1} f(x) + \sum_{x \in E_2} f(x). \tag{1.8.14}$$

1.8.3 Monotone convergence for nonnegative sums

Let $\{f_j\}_{j=1}^\infty$ be a sequence of functions on X with values in the nonnegative extended real numbers, and suppose that

$$f_j(x) \le f_{j+1}(x) \tag{1.8.15}$$

for each $j \ge 1$ and $x \in X$. Put

$$f(x) = \sup_{j \ge 1} f_j(x) \tag{1.8.16}$$

for all $x \in X$, and note that

$$\sum_{x \in X} f_j(x) \le \sum_{x \in X} f_{j+1}(x) \le \sum_{x \in X} f(x) \tag{1.8.17}$$

for each j. Under these conditions, a version of the *monotone convergence theorem* states that

$$\sum_{x \in X} f_j(x) \to \sum_{x \in X} f(x) \text{ as } j \to \infty. \tag{1.8.18}$$

To see this, let A be a finite subset of X, and observe that

$$\sum_{x \in A} f_j(x) \to \sum_{x \in A} f(x) \text{ as } j \to \infty. \tag{1.8.19}$$

This means that

$$\sum_{x \in A} f(x) \le \sup_{j \ge 1} \sum_{x \in A} f_j(x) \le \sup_{j \ge 1} \sum_{x \in X} f_j(x). \tag{1.8.20}$$

It follows that

$$\sum_{x \in X} f(x) \le \sup_{j \ge 1} \sum_{x \in X} f_j(x), \tag{1.8.21}$$

by the definition of the sum on the left. Of course, the opposite inequality holds because of (1.8.17).

Let f be any nonnegative extended real-valued function on X again, and let $E_1, E_2, E_3, \ldots$ be an infinite sequence of pairwise-disjoint subsets of X. One can use the monotone convergence theorem to get that

$$\sum_{j=1}^{n} \sum_{x \in E_j} f(x) = \sum_{x \in \bigcup_{j=1}^{n} E_j} f(x) \to \sum_{x \in \bigcup_{j=1}^{\infty} E_j} f(x) \text{ as } n \to \infty. \tag{1.8.22}$$

Equivalently,

$$\sum_{j=1}^{\infty} \sum_{x \in E_j} f(x) = \sum_{x \in \bigcup_{j=1}^{\infty} E_j} f(x). \tag{1.8.23}$$

1.8.4 Measures defined using these sums

Let w be a nonnegative extended real-valued function on X again, and now let $\mu_w(A)$ be as in (1.8.1) for every subset A of X. This defines a nonnegative countably additive measure on the collection of all subsets of X. This basically corresponds to the example (i) on p263 of [180] when X is countably infinite.

Let $\mathcal{A}_1$ be the collection of all subsets of X with only finitely or countably many elements, and remember that this is a σ-ring of subsets of X, as in Subsection 1.5.4. If μ is any countably additive measure on X with respect to $\mathcal{A}_1$, then it is easy to see that $\mu = \mu_w$ on $\mathcal{A}_1$ exactly when w is as in (1.8.2).

1.9 Some basic properties of measures

Let X be a set, let $\mathcal{A}$ be a ring of subsets of X, and let μ be a countably additive nonnegative measure on X with respect to $\mathcal{A}$. Also let $B_1, B_2, B_3, \ldots$ be a sequence of elements of $\mathcal{A}$ such that

$$B_j \subseteq B_{j+1} \tag{1.9.1}$$

for each j, and put

$$B = \bigcup_{j=1}^{\infty} B_j. \tag{1.9.2}$$

If $B \in \mathcal{A}$, then it is well known that

$$\mu(B_j) \to \mu(B) \text{ as } j \to \infty. \tag{1.9.3}$$

This corresponds to Theorem 10.13 on p130 of [86], part (d) of Theorem 1.19 on p17 of [157], and Theorem 11.3 on p302 of [158].

To see this, put $A_1 = B_1$, and

$$A_j = B_j \setminus B_{j-1} \tag{1.9.4}$$

for $j \geq 2$. This defines a sequence of pairwise-disjoint elements of $\mathcal{A}$. Observe that

$$B_n = \bigcup_{j=1}^{n} A_j \tag{1.9.5}$$

for each n, and that

$$B = \bigcup_{j=1}^{\infty} A_j. \tag{1.9.6}$$

This implies that

$$\mu(B_n) = \sum_{j=1}^{n} \mu(A_j) \tag{1.9.7}$$

for each n, and

$$\mu(B) = \sum_{j=1}^{\infty} \mu(A_j). \tag{1.9.8}$$

Clearly (1.9.8) is the same as (1.9.3), because of (1.9.7).

1.9.1 A partial converse

Now let $A_1, A_2, A_3, \ldots$ be any sequence of pairwise-disjoint elements of $\mathcal{A}$. Also let B_n be as in (1.9.5) for each n, and let B be as in (1.9.6). Note that $B_1, B_2, B_3, \ldots$ is a sequence of elements of $\mathcal{A}$ that satisfies (1.9.1) and (1.9.2). Of course, (1.9.7) only uses finite additivity of μ on $\mathcal{A}$. In this case, if $B \in \mathcal{A}$, then (1.9.3) implies the countable additivity property (1.9.8).

1.9.2 Decreasing sequences of measurable sets

Let $C_1, C_2, C_3, \ldots$ be a sequence of elements of $\mathcal{A}$ such that

$$C_{j+1} \subseteq C_j \tag{1.9.9}$$

for each j, and put

$$C = \bigcap_{j=1}^{\infty} C_j. \tag{1.9.10}$$

If $C \in \mathcal{A}$ and

$$\mu(C_1) < +\infty, \tag{1.9.11}$$

then it is well known that

$$\mu(C_j) \to \mu(C) \text{ as } j \to \infty. \tag{1.9.12}$$

This corresponds to Theorem 10.15 on p131 of [86], and part (e) of Theorem 1.19 on p17 of [157].

This can be reduced to the previous statement, by taking

$$B_j = C_1 \setminus C_j \tag{1.9.13}$$

for each j. This defines a sequence of elements of $\mathcal{A}$ that satisfies (1.9.1), because of (1.9.9). If B is as in (1.9.6), then

$$B = \bigcup_{j=1}^{\infty} (C_1 \setminus C_j) = C_1 \setminus \Big(\bigcap_{j=1}^{\infty} C_j\Big) = C_1 \setminus C \tag{1.9.14}$$

under these conditions. If $C \in \mathcal{A}$, then $B \in \mathcal{A}$, and (1.9.3) implies that

$$\mu(C_1 \setminus C_j) \to \mu(C_1 \setminus C) \text{ as } j \to \infty. \tag{1.9.15}$$

Of course,

$$\mu(C_1) = \mu(C_1 \setminus C_j) + \mu(C_j) \tag{1.9.16}$$

for each j, and

$$\mu(C_1) = \mu(C_1 \setminus C) + \mu(C), \tag{1.9.17}$$

using finite additivity. If (1.9.11) holds, then (1.9.12) may be obtained from (1.9.15).

1.9.3 Another partial converse

Let $B_1, B_2, B_3, \ldots$ be a sequence of elements of $\mathcal{A}$ that satisfies (1.9.1) again, and let B be as in (1.9.2). Suppose that $B \in \mathcal{A}$, and put

$$C_j = B \setminus B_j \tag{1.9.18}$$

for each j. This implies that (1.9.9) holds, and if C is as in (1.9.10), then

$$C = \bigcap_{j=1}^{\infty} (B \setminus B_j) = \emptyset. \tag{1.9.19}$$

In this case, (1.9.12) says that

$$\mu(B \setminus B_j) = \mu(C_j) \to 0 \text{ as } j \to \infty. \tag{1.9.20}$$

We also have that

$$\mu(B) = \mu(B_j) + \mu(B \setminus B_j) \tag{1.9.21}$$

for each j, using finite additivity, so that (1.9.20) implies (1.9.3).

1.9.4 A counterexample with $\mu(C_1) = +\infty$

Let X be the set $\mathbf{Z}_+$ of positive integers, and let μ be counting measure on X. If C_j is the set of positive integers greater than or equal to j for each j, then (1.9.9) holds, and (1.9.10) is the empty set. However, (1.9.12) does not hold. This is Example 1.20 (c) on p18 of [157].

1.10 Subadditivity and outer measures

Let X be a set, let $\mathcal{A}$ be a ring of subsets of X, and let μ be a nonnegative extended real-valued function on $\mathcal{A}$.

1.10.1 Finite subadditivity

We say that μ is *finitely subadditive* on $\mathcal{A}$ if for any finite sequence $B_1, \ldots, B_n$ of elements of $\mathcal{A}$, we have that

$$\mu\Big(\bigcup_{j=1}^n B_j\Big) \le \sum_{j=1}^n \mu(B_j). \tag{1.10.1}$$

If μ is finitely additive, then it is well known that μ is finitely subadditive. To see this, put $A_1 = B_1$, and

$$A_j = B_j \setminus \Big(\bigcup_{l=1}^{j-1} B_l\Big) \tag{1.10.2}$$

for $j = 2, \ldots, n$. It is easy to see that $A_1, A_2, \ldots, A_n$ are pairwise-disjoint elements of $\mathcal{A}$. We also have that

$$\bigcup_{j=1}^n A_j = \bigcup_{j=1}^n B_j. \tag{1.10.3}$$

More precisely, if $x \in B_l$ for some l, then $x \in A_j$ for the first j such that $x \in B_j$.

Note that $A_j \subseteq B_j$ for each j, so that

$$\mu(A_j) \le \mu(B_j). \tag{1.10.4}$$

Thus

$$\mu\Big(\bigcup_{j=1}^n B_j\Big) = \mu\Big(\bigcup_{j=1}^n A_j\Big) = \sum_{j=1}^n \mu(A_j) \le \sum_{j=1}^n \mu(B_j). \tag{1.10.5}$$

1.10.2 Countable subadditivity

Similarly, we say that μ is *countably subadditive* if for any infinite sequence $B_1, B_2, B_3, \ldots$ of elements of $\mathcal{A}$ with

$$\bigcup_{j=1}^\infty B_j \in \mathcal{A}, \tag{1.10.6}$$

we have that

$$\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) \le \sum_{j=1}^{\infty} \mu(B_j). \tag{1.10.7}$$

If μ is countably additive, then it is well known that μ is countably subbaditive, for essentially the same reasons as before. Indeed, if we put $A_1 = B_1$ and let A_j be as in (1.10.2) for $j \ge 2$, then $A_1, A_2, A_3, \ldots$ is a sequence of pairwise-disjoint elements of $\mathcal{A}$ such that

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} B_j. \tag{1.10.8}$$

It follows that

$$\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j) \le \sum_{j=1}^{\infty} \mu(B_j), \tag{1.10.9}$$

because of (1.10.4).

1.10.3 A criterion for countable additivity

Suppose that μ is finitely additive on $\mathcal{A}$, and let $A_1, A_2, A_3, \ldots$ be a sequence of pairwise-disjoint elements of $\mathcal{A}$ whose union is an element of $\mathcal{A}$. If n is any positive integer, then

$$\sum_{j=1}^{n} \mu(A_j) = \mu\Big(\bigcup_{j=1}^{n} A_j\Big) \le \mu\Big(\bigcup_{j=1}^{\infty} A_j\Big). \tag{1.10.10}$$

This implies that

$$\sum_{j=1}^{\infty} \mu(A_j) \le \mu\Big(\bigcup_{j=1}^{\infty} A_j\Big). \tag{1.10.11}$$

If μ is also countably subadditive, then it follows that μ is countably additive.

1.10.4 Outer measures

Now let μ be a nonnegative extended real-valued function defined on the collection of all subsets of X. We say that μ is a (Carathéodory) *outer measure* on X if it satisfies the following three conditions, as in Definition 10.2 on p126 of [86]. First,

$$\mu(\emptyset) = 0. \tag{1.10.12}$$

Second, if $A \subseteq B \subseteq X$, then

$$\mu(A) \le \mu(B). \tag{1.10.13}$$

Third, μ should be countably subadditive on the collection of all subsets of X.

Outer measures are also known as *exterior measures*, as on p264 of [180]. Sometimes outer measures are simply called *measures*, as on p1 of [61], p53 of [62], and p8 of [134].

1.11 More on outer measures

Let X be a set, and let μ be an outer measure on X.

1.11.1 Measurability with respect to μ

A subset A of X is said to be *measurable* with respect to μ if

$$\mu(Y) = \mu(Y \cap A) + \mu(Y \setminus A) \tag{1.11.1}$$

for every subset Y of X. This is due to Carathéodory, and corresponds to Definition 10.5 on p127 of [86] and Definition 1.3 on p8 of [134], and it is also mentioned on p2 of [61], p54 of [62], and p264 of [180].

Of course,

$$\mu(Y) \le \mu(Y \cap A) + \mu(Y \setminus A) \tag{1.11.2}$$

automatically, by subadditivity. Thus, in order to get (1.11.1), it suffices to check that

$$\mu(Y) \ge \mu(Y \cap A) + \mu(Y \setminus A). \tag{1.11.3}$$

In particular, this implies that A is measurable when $\mu(A) = 0$.

Note that A is measurable with respect to μ if and only if $X \setminus A$ is measurable with respect to μ, by the definition of measurability. It is well known that the collection of subsets of X that are measurable with respect to μ is a σ-algebra, and that the restriction of μ to this σ-algebra is countably additive. This corresponds to parts (i) and (ii) of Theorem 1 on p2 of [61], parts (2) and (3) of Theorem 2.1.3 on p54 of [62], Theorem 10.11 on p129 of [86], and Theorem 1.1 on p265 of [180], and it is mentioned in parts (1) and (3) of Theorem 1.4 on p8 of [134].

1.11.2 Metric outer measures

Suppose for the moment that (X, d) be a metric space. Let A, B be nonempty subsets of X, and suppose that there is a positive real number η such that

$$d(x, y) \ge \eta \tag{1.11.4}$$

for all $x \in A$ and $y \in B$. If

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{1.11.5}$$

under these conditions, then μ is said to be a *metric outer measure* or *metric exterior measure* on X.

In this case, it is well known that

(1.11.6) open and closed sets in X are measurable with respect to μ,

as in Theorem 5 on p9 of [61], Exercise 10.48 on p144 of [86], and Theorem 1.2 on p267 of [180]. This is part of *Carathéodory's criterion*, as in (9) on p75 of [62], and as mentioned in Theorem 1.7 on p10 of [134].

In order to verify (1.11.5), it suffices to check that

$$\mu(A \cup B) \geq \mu(A) + \mu(B), \tag{1.11.7}$$

because of subadditivity, as before. Metric outer measures can often be defined using suitable coverings of subsets of X. More precisely, one may be able to use covering by subsets of X with arbitrarily small diameter. If A and B satisfy (1.11.4), then any subset of X with diameter less than η can intersect at most one of A and B. This means that a covering of $A \cup B$ by sets of diameter less than η can be split into coverings of A and B separately, and this can often be used to get (1.11.7).

1.11.3 Hopf's extension theorem

Let X be any set again, let $\mathcal{A}$ be an algebra of subsets of X, and let μ be a countably additive measure on X with respect to $\mathcal{A}$. If E is any subset of X, then consider sequences $A_1, A_2, A_3, \ldots$ of elements of $\mathcal{A}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} A_j. \tag{1.11.8}$$

Using such a sequence, we get a sum

$$\sum_{j=1}^{\infty} \mu(A_j). \tag{1.11.9}$$

Let $\overline{\mu}(E)$ be the infimum of these sums over all such coverings of E.

It is well known that $\overline{\mu}$ is an outer measure on X such that

$$\overline{\mu}(A) = \mu(A) \tag{1.11.10}$$

for every $A \in \mathcal{A}$, and that the elements of $\mathcal{A}$ are measurable with respect to $\overline{\mu}$. This corresponds to parts (a), (b), and (c) of Exercise 10.36 on p141 of [86], and to Lemma 1.4 on p271 of [180]. This implies *E. Hopf's extension theorem*, that μ can be extended to a countably additive measure on a σ-algebra of subsets of X that contains $\mathcal{A}$, as in part (d) of Exercise 10.36 on p142 of [86], and Theorem 1.5 on p272 of [180].

1.11.4 Another version of Hopf's theorem

Let X be a set, let $\mathcal{A}$ be an algebra of subsets of X, and let μ be a finitely additive measure on X with respect to $\mathcal{A}$. Suppose that for every sequence $C_1, C_2, C_3, \ldots$ of elements of $\mathcal{A}$ such that $C_{j+1} \subseteq C_j$ for each j and

$$\bigcap_{j=1}^{\infty} C_j = \emptyset, \tag{1.11.11}$$

we have that

$$\mu(C_j) \to 0 \text{ as } j \to \infty. \tag{1.11.12}$$

This implies that μ satisfies the condition mentioned at the beginning of Section 1.9, as in Subsection 1.9.3. It follows that μ is countably additive, as in Subsection 1.9.1. Thus the remarks in the previous subsection hold in this case as well, as in Exercise 10.37 on p142 of [86].

1.12 Some regularity conditions for measures

Let X be a metric space, or at least a topological space. Also let $\mathcal{A}$ be a ring of subsets of X, and let μ be a finitely additive measure on X with respect to $\mathcal{A}$. Sometimes μ may satisfy some additional conditions related to the metric or topology on X.

1.12.1 An outer regularity condition

Let us say that μ is *outer regular* if for every $A \in \mathcal{A}$ we have that

$$\mu(A) = \inf\{\mu(U) : U \in \mathcal{A},\ A \subseteq U, \text{ and } U \text{ is an open set in } X\}. \tag{1.12.1}$$

If X is equipped with the discrete metric or topology, then every subset of X is an open set, and this outer regularity condition is trivial.

1.12.2 Two inner regularity conditions

Let us say that μ is *inner regular with respect to closed sets* if for every $A \in \mathcal{A}$ we have that

$$\mu(A) = \sup\{\mu(E) : E \in \mathcal{A},\ E \subseteq A, \text{ and } E \text{ is a closed set in } X\}. \tag{1.12.2}$$

If X is equipped with the discrete metric or topology, then every subset of X is a closed set, and this is trivial, as before.

Similarly, let us say that μ is *inner regular with respect to compact sets* if for every $A \in \mathcal{A}$ we have that

$$\mu(A) = \sup\{\mu(K) : K \in \mathcal{A},\ K \subseteq A, \text{ and } K \text{ is a compact set in } X\}. \tag{1.12.3}$$

It is well known that compact sets are closed sets in metric spaces, and in Hausdorff topological spaces. Thus inner regularity with respect to compact sets implies inner regularity with respect to closed sets in these cases.

If X is compact, then it is well known that closed sets in X are compact too. This means that inner regularity with respect to closed sets implies inner regularity with respect to compact sets in this case.

Suppose for the moment that $\mathcal{A}$ is an algebra of subsets of X, and that $\mu(X) < +\infty$. One can check that inner regularity with respect to closed sets is equivalent to outer regularity under these conditions. More precisely, inner refularity with respect to closed sets for $A \in \mathcal{A}$ corresponds to outer regularity for $X \setminus A$.

1.12.3 Some examples concerning inner regularity

Suppose for the moment that $\mathcal{A}$ contains all one-element subsets of X, and thus all finite subsets of X. Of course, finite subsets of X are automatically compact. If w is a nonnegative extended real-valued function on X and μ_w is defined on $\mathcal{A}$ as in Subsection 1.8.4, then μ_w is inner regular with respect to compact sets, by construction.

Suppose for the moment as well that X is equipped with discrete metric or topology. It is well known that the finite subsets of X are the only compact sets. If μ is inner regular with respect to compact sets, then one can check that $\mu = \mu_w$ on $\mathcal{A}$ exactly when w is obtained from μ as at the beginning of Section 1.8.

Let us continue to suppose for the moment that X is equipped with the discrete metric or topology. If X has infinitely many elements, then the finitely additive measure defined on the algebra $\mathcal{A}_0$ of subsets of X in Subsection 1.7.4 is not inner regular with respect to compact sets. Similarly, if X is uncountable, then the countably additive measure defined on the σ-algebra $\mathcal{A}_1$ of subsets of X in Subsection 1.7.4 is not inner regular with respect to compact sets.

1.12.4 Regularity and elementary sets

Let n be a positive integer, and remember that $\mathcal{E}(\mathbf{R}^n)$ is the ring of bounded elementary subsets of $\mathbf{R}^n$, as in Subsection 1.6.3. Also let μ be a finitely additive measure on $\mathbf{R}^n$ with respect to $\mathcal{E}(\mathbf{R}^n)$ such that $\mu(A) < +\infty$ for every A in $\mathcal{E}(\mathbf{R}^n)$. One can check that μ is both outer regular and inner regular with respect to closed sets if and only if μ is regular in the sense of Definition 11.5 on p303 of [158]. Of course, this uses the standard Euclidean metric on $\mathbf{R}^n$.

If μ is any finitely additive measure on $\mathcal{E}(\mathbf{R}^n)$, then inner regularity with respect to closed sets implies inner regularity with respect to compact sets. This uses the well-known fact that closed and bounded sets in $\mathbf{R}^n$ are compact. This also uses the fact that the elements of $\mathcal{E}(\mathbf{R}^n)$ are bounded subsets of $\mathbf{R}^n$, by construction.

1.13 Regularity conditions and countable additivity

Let X be a metric or a topological space, let $\mathcal{A}$ be a ring of subsets of X, and let μ be a finitely additive measure on X with respect to $\mathcal{A}$. Suppose that μ is outer regular, as well as inner regular with respect to compact sets, as in the previous section. We would like to show that μ is countably additive on $\mathcal{A}$. It suffices to show that μ is countably subadditive on $\mathcal{A}$, as in Subsection 1.10.3.

Let $A_1, A_2, A_3, \ldots$ be a sequence of elements of $\mathcal{A}$ such that $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$, and let us show that

$$\mu\Big(\bigcup_{j=1}^\infty A_j\Big) \le \sum_{j=1}^\infty \mu(A_j). \tag{1.13.1}$$

We may as well suppose that the sum on the right is finite, since otherwise there is nothing to do. In particular, this means that $\mu(A_j) < +\infty$ for each j.

Let $\epsilon > 0$ be given, and let us use outer regularity to get an open set U_j in X for each j such that $A_j \subseteq U_j$, $U_j \in \mathcal{A}$, and

$$\mu(U_j) < \mu(A_j) + 2^{-j}\,\epsilon. \tag{1.13.2}$$

This implies that

$$\sum_{j=1}^{\infty} \mu(U_j) < \sum_{j=1}^{\infty} \mu(A_j) + \epsilon. \tag{1.13.3}$$

Let K be a compact subset of X such that $K \in \mathcal{A}$ and

$$K \subseteq \bigcup_{j=1}^{\infty} A_j. \tag{1.13.4}$$

It follows that

$$K \subseteq \bigcup_{j=1}^{\infty} U_j, \tag{1.13.5}$$

so that

$$K \subseteq \bigcup_{j=1}^{n} U_j \tag{1.13.6}$$

for some positive integer n. This means that

$$\mu(K) \le \sum_{j=1}^{n} \mu(U_j), \tag{1.13.7}$$

by finite subadditivity, as in Subsection 1.10.1. Thus

$$\mu(K) \le \sum_{j=1}^{\infty} \mu(U_j) < \sum_{j=1}^{\infty} \mu(A_j) + \epsilon. \tag{1.13.8}$$

We can now use inner regularity with respect to compact sets to get that

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) \le \sum_{j=1}^{\infty} \mu(A_j) + \epsilon. \tag{1.13.9}$$

This implies (1.13.1), because $\epsilon > 0$ is arbitrary.

1.13.1 Volumes of elementary sets

Let n be a positive integer, and remember that the ring $\mathcal{E}(\mathbf{R}^n)$ of bounded elementary subsets of $\mathbf{R}^n$ and the algebra $\mathcal{E}_1(\mathbf{R}^n)$ of possibly unbounded elementary subsets of $\mathbf{R}^n$ may be defined as in Subsection 1.6.3. More precisely, one can check that the elements of $\mathcal{E}(\mathbf{R}^n)$ may be expressed as the union of finitely many pairwise-disjoint bounded intervals in $\mathbf{R}^n$, as in (13) on p303 of [158]. Similarly, the elements of $\mathcal{E}_1(\mathbf{R}^n)$ may be expressed as the union of finitely many pairwise-disjoint possibly unbounded intervals in $\mathbf{R}^n$. Remember also that the n-dimensional volume of bounded as well as possibly unbounded intervals in $\mathbf{R}^n$ may be defined as in Subsection 1.6.2.

The n-dimensional *volume* of a bounded elementary set in $\mathbf{R}^n$ may be defined as in (11) on p303 of [158]. That is to say, if the bounded elementary set is expressed as the union of finitely many pairwise-disjoint bounded intervals, then the volume of the elementary set is equal to the sum of the volumes of these intervals. This does not depend on the way that the elementary set is expressed as the union of finitely many pairwise-disjoint intervals, as in (14) on p303 of [158]. This defines a finitely additive measure on $\mathbf{R}^n$ with respect to $\mathcal{E}(\mathbf{R}^n)$, as in (15) on p303 of [158]. There are analogous remarks for the n-dimensional *volume* of a possibly unbounded elementary set in $\mathbf{R}^n$.

1.13.2 Regularity of the volume

One can check that n-dimensional volume is outer regular and inner regular with respect to compact sets on $\mathcal{E}(\mathbf{R}^n)$. This corresponds to Example 11.6 (a) on p303 of [158]. More precisely, it suffices to verify these regularity properties for bounded intervals in $\mathbf{R}^n$, and this is easy to do. It follows that n-dimensional volume is countably additive on $\mathcal{E}(\mathbf{R}^n)$, as before.

Similarly, n-dimensional volume is inner regular with respect to compact sets on $\mathcal{E}_1(\mathbf{R}^n)$. As before, it is enough to verify this for possibly unbounded intervals in $\mathbf{R}^n$, and this is also fairly simple. One can check that 1-dimensional volume is outer regular on $\mathcal{E}_1(\mathbf{R})$ as well.

However, if $n \geq 2$, then n-dimensional volume is not outer regular on $\mathcal{E}_1(\mathbf{R}^n)$. This is because of unbounded intervals with n-dimensional volume equal to 0. Outer regularity does work for these sets, because unbounded open sets in $\mathcal{E}_1(\mathbf{R}^n)$ have n-dimensional volume $+\infty$. We shall consider another version of outer regularity in the next section that works in this case.

1.14 Another outer regularity condition

Let X be a metric or topological space again, let $\mathcal{A}$ be a ring of subsets of X, and let μ be a finitely additive measure on $\mathcal{A}$. Let us say that μ is *outer regular with respect to subsets of compact sets* if the outer regularity property (1.12.1) holds when $A \in \mathcal{A}$ and there is a compact subset K of X such that $A \subseteq K$ and $K \in \mathcal{A}$. Of course, this is equivalent to outer regularity when X is compact.

1.14.1 Using this condition to get countable additivity

Suppose that μ is outer regular with respect to subsets of compact sets, and inner regular with respect to compact sets. We would like to show that μ is countably additive on $\mathcal{A}$ under these conditions as well. It is enough to show that μ is countably subadditive on $\mathcal{A}$, as before.

Let $A_1, A_2, A_3, \ldots$ be a sequence of elements of $\mathcal{A}$ such that $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$, so that we would like to show that (1.13.1) holds. To do this, let K be a compact subset of X such that $K \in \mathcal{A}$ and (1.13.4) holds. It suffices to show that

$$\mu(K) \le \sum_{j=1}^\infty \mu(A_j), \tag{1.14.1}$$

because of inner regularity with respect to compact sets.

In fact,

$$\mu(K) \le \sum_{j=1}^\infty \mu(A_j \cap K). \tag{1.14.2}$$

We may as well suppose that the sum on the right is finite, as before. Let $\epsilon > 0$ be given, and let us use outer reguarity with respect to subsets of compact sets to get an open set V_j in X for each j such that $A_j \cap K \subseteq V_j$, $V_j \in \mathcal{A}$, and

$$\mu(V_j) < \mu(A_j \cap K) + 2^{-j}\,\epsilon. \tag{1.14.3}$$

Note that

$$\sum_{j=1}^\infty \mu(V_j) < \sum_{j=1}^\infty \mu(A_j \cap K) + \epsilon. \tag{1.14.4}$$

Using (1.13.4), we get that

$$K = \bigcup_{j=1}^\infty (A_j \cap K) \subseteq \bigcup_{j=1}^\infty V_j. \tag{1.14.5}$$

This implies that

$$K \subseteq \bigcup_{j=1}^n V_j \tag{1.14.6}$$

for some positive integer n, by compactness, so that

$$\mu(K) \le \sum_{j=1}^n \mu(V_j), \tag{1.14.7}$$

because of finite subadditivity. It follows that

$$\mu(K) \le \sum_{j=1}^\infty \mu(V_j) < \sum_{j=1}^\infty \mu(A_j) + \epsilon. \tag{1.14.8}$$

This means that (1.14.1) holds, because $\epsilon > 0$ is arbitrary.

1.14.2 More on the n-dimensional volume on $\mathcal{E}_1(\mathbf{R}^n)$

Let n be a positive integer, and et us continue with the discussion of the n-dimensional volume on $\mathcal{E}_1(\mathbf{R}^n)$ from the previous section. It is easy to see that the n-dimensional volume is outer regular with respect to subsets of compact sets on $\mathcal{E}_1(\mathbf{R}^n)$, because of outer regularity of n-dimensional volume on $\mathcal{E}(\mathbf{R}^n)$.

It follows that n-dimensional volume is countably additive on $\mathcal{E}_1(\mathbf{R}^n)$, using the remarks in the previous subsection. This also uses the inner regularity of n-dimensional volume on $\mathcal{E}(\mathbf{R}^n)$ with respect to compact sets, as in the previous section.

1.15 Lebesgue outer measure on $\mathbf{R}^n$

This suggests a way to define *n-dimensional Lebesgue outer measure*, using the construction discussed in Subsection 1.11.3. Often one defines Lebesgue outer measure on $\mathbf{R}^n$ a bit more directly, and shows that it has the usual properties using related arguments.

If E is any subset of $\mathbf{R}^n$, then in the construction in Subsection 1.11.3, one considers coverings of E by sequences of elements of $\mathcal{E}_1(\mathbf{R}^n)$. As a variant of this, one can consider coverings of E by sequences of possibly unbounded intervals in $\mathbf{R}^n$. It is easy to see that this leads to an equivalent definition of the Lebesgue outer measure of E, because of the way that the volume of an element of $\mathcal{E}_1(\mathbf{R}^n)$ is defined.

As another variant, one can consider coverings of E by sequences of bounded intervals. It is not too difficult to show that this also leads to an equivalent definition of Lebesgue outer measure. This is simpler when $n = 1$, because the 1-dimensional volume of an unbounded interval in $\mathbf{R}$ is $+\infty$. If $n \geq 2$, then the n-dimensional volume of an unbounded interval is either 0 or $+\infty$, as in Subsection 1.6.2. If it is 0, then one should check that its n-dimensional Lebesgue outer measure is equal to 0 when it is defined using coverings by sequences of bounded intervals.

One can also restrict one's attention to coverings of E by sequences of bounded open intervals. This is because any bounded interval in $\mathbf{R}^n$ is contained in a bounded open interval whose n-dimensional volume is arbitrarily close to the volume of the initial interval. This approach to defining Lebesgue outer measure on $\mathbf{R}^n$ is essentially the same as in Definition 11.7 on p304 of [158], where coverings by sequences of bounded open elementary sets is used. More precisely, in the construction in [158], one considers any finitely additive measure on $\mathcal{E}(\mathbf{R}^n)$ that is outer regular, inner regular with respect to closed or equivalently compact sets, and finite on the elements of $\mathcal{E}(\mathbf{R}^n)$.

One could consider coverings of E by sequences of cubes as well. To check that this leads to the same result, one can verify that bounded intervals can be covered by finitely many cubes, with the sum of the volumes of these cubes arbitrarily close to the volume of the interval. One can restrict one's attention to closed cubes too, as on p10 of [180].

One-dimensional Lebesgue outer measure is defined a bit differently on p26 of [61], using coverings of E by sequences of arbitrary subsets of the real line, and taking the infimum of the sum of the diameters of the sets in the covering, over all such coverings. This is equivalent to using coverings by sequences of intervals as before, because the diameter of an interval in $\mathbf{R}$ is the same as its length, or one-dimensional volume. This also uses the fact that any subset of the real line is contained in an interval with the same diameter. Lebesgue outer measure on $\mathbf{R}^n$ is obtained another way in [61] when $n \geq 2$, using the $n = 1$ case.

It is not too difficult to show that one gets an outer measure when defining Lebesgue outer measure using converings in these ways. More precisely, one can check that it is enough to use coverings by sets whose diameter is as small as one likes. This implies that Lebesgue outer measure is a metric outer measure, as in Subsection 1.11.2. This corresponds to Observation 4 on p14 of [180], as mentioned on p267 of [180].

One-dimensional Lebesgue outer measure is obtained using the Riemann integral on p111 of [62], and in Definition 9.19 on p120 of [86]. This is used to get n-dimensional Lebesgue outer measure beginning on p119 of [62].

1.15.1 Lebesgue measure on $\mathbf{R}^n$

One may define *n-dimensional Lebesgue measure* to be the restriction of n-dimensional Lebesgue outer measure to the corresponding σ-algebra of measurable sets, as in Subsection 1.11.1. A subset of $\mathbf{R}^n$ that is measurable with respect to Lebesgue outer measure may be called *Lebesgue measurable*. Another definition of measurability is used on p16 of [180], and Exercise 3 on p312 of [180] asks one to show that this is equivalent to measurability with respect to Lebesgue outer measure, as in Subsection 1.11.1.

Lebesgue measure on $\mathbf{R}^n$ is defined on the σ-algebra of Lebesgue measurable sets another way on p53 of [157].

Chapter 2

More on measures and σ-algebras

2.1 Getting an outer measure from a measure

Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and μ be a nonnegative countably additive measure defined on $\mathcal{A}$. If E is any subset of X, then put

$$\overline{\mu}(E) = \inf\{\mu(A) : A \in \mathcal{A},\ E \subseteq A\}, \tag{2.1.1}$$

as in (1.2) of p8 of [134]. It is easy to see that this is the same as in Subsection 1.11.3 under these conditions.

The properties of $\overline{\mu}$ mentioned in Subsection 1.11.3 are somewhat simpler in this case. In particular, one can check more directly that $\overline{\mu}$ is an outer measure on X, as in Exercise 1 on p22 of [134]. Clearly $\overline{\mu} = \mu$ on $\mathcal{A}$, by construction. One can also verify that the elements of $\mathcal{A}$ are measurable with respect to $\overline{\mu}$, in the sense described in Subsection 1.11.1.

2.1.1 The infimum in the definition of $\overline{\mu}(E)$ is attained

Let E be any subset of X, and let us check that the infimum on the right side of (2.1.1) is attained. This is clear when $\overline{\mu}(E) = +\infty$, and so we suppose that $\overline{\mu}(E) < \infty$. In this case, for each positive integer l there is an $A_l \in \mathcal{A}$ such that $E \subseteq A_l$ and

$$\mu(A_l) < \overline{\mu}(E) + 1/l. \tag{2.1.2}$$

If we put $B = \bigcap_{l=1}^{\infty} A_l$, then $B \in \mathcal{A}$, $E \subseteq B$, and

$$\mu(B) \le \overline{\mu}(E), \tag{2.1.3}$$

because $\mu(B) \le \mu(A_l)$ for each l. It follows that

$$\overline{\mu}(E) = \mu(B), \tag{2.1.4}$$

by the definition of $\overline{\mu}(E)$.

In particular, $\mu(B) = 0$ when $\overline{\mu}(E) = 0$. If E is measurable with respect to $\overline{\mu}$, as in Subsection 1.11.1, then one can use (2.1.4) to get that

$$\overline{\mu}(B \setminus E) = 0. \tag{2.1.5}$$

2.1.2 When E is σ-finite with respect to $\overline{\mu}$

Suppose now that E is *σ-finite* with respect to $\overline{\mu}$, in the sense that there is a sequence $E_1, E_2, E_3, \ldots$ of subsets of X such that $E = \bigcup_{j=1}^{\infty} E_j$ and $\overline{\mu}(E_j) < +\infty$ for each j. Let B_j be an element of $\mathcal{A}$ such that $E_j \subseteq B_j$ and

$$\overline{\mu}(E_j) = \mu(B_j) \tag{2.1.6}$$

for each j, as before. Note that $E_j \subseteq E \cap B_j$ for each j, so that

$$\overline{\mu}(E \cap B_j) = \mu(B_j). \tag{2.1.7}$$

If E is measurable with respect to $\overline{\mu}$, then $E \cap B_j$ is measurable with respect to $\overline{\mu}$ for each j. This implies that

$$\overline{\mu}(B_j \setminus E) = 0 \tag{2.1.8}$$

for each j, as before. It follows that

$$\mu\Big(\Big(\bigcup_{j=1}^{\infty} B_j\Big) \setminus E\Big) = \mu\Big(\bigcup_{j=1}^{\infty} (B_j \setminus E)\Big) = 0. \tag{2.1.9}$$

2.1.3 Complete measure spaces

The measure space $(X, \mathcal{A}, \mu)$ is said to be *complete* if for every subset E of X such that there is a $B \in \mathcal{A}$ with $E \subseteq B$ and $\mu(B) = 0$, we have that $E \in \mathcal{A}$. This corresponds to Definition 11.20 on p155 of [86], and it is also mentioned on p29 of [157], and p266 of [180]. Equivalently, this means that $E \in \mathcal{A}$ when $\overline{\mu}(E) = 0$. Note that any outer measure is complete with respect to the corresponding σ-algebra of measurable sets, as in Subsection 1.11.1.

It is well known that one can get a *completion* of $\mathcal{A}$ as follows. A subset of X is said to measurable with respect to the completion if it can be expressed as $A \cup E$, when $A \in \mathcal{A}$ and $E \subseteq X$ is as in the previous paragraph. In this case, the measure of $A \cup E$ is defined to be $\mu(A)$. One can check that this defines a complete measure space, as in Theorem 11.21 on p155 of [86], Theorem 1.36 on p29 of [157], and Exercise 2 on p312 of [180]. This is related to Exercise 10.38 on p142 of [86].

2.1.4 More on σ-finiteness

We say that $A \in \mathcal{A}$ is *σ-finite* with respect to μ if A can be expressed as the union of a sequence of elements of $\mathcal{A}$ with finite measure, as on p118 of [85], p138 of [86], and p49 of [157]. If X is σ-finite with respect to μ, then $(X, \mathcal{A}, \mu)$ is said to be *σ-finite* as a measure space, as on p127 of [86], and p263 of [180]. Note that σ-finiteness of a set with respect to an outer measure is defined on p4 of [61] to mean that the set may be expressed as the union of a sequence of measurable sets with respect to the outer measure with finite measure.

It is easy to see that the collection of elements of $\mathcal{A}$ that are σ-finite with respect to μ is a σ-ring.

2.2 Borel sets

Let X be a set, and let $\mathcal{E}$ be any collection of subsets of X. Also let $\mathcal{A}(\mathcal{E})$ be the intersection of all of the σ-algebras of subsets of X that contain $\mathcal{E}$. One can check that $\mathcal{A}(\mathcal{E})$ is a σ-algebra of subsets of X as well. Of course, $\mathcal{A}(\mathcal{E})$ is the smallest σ-algebra of subsets of X that contains $\mathcal{E}$, by construction. This is mentioned beginning on p59 of [62], and in Definition 10.19 on p132 of [86], and it corresponds to Theorem 1.10 on p12 of [157].

Suppose now that X is a metric space, or a topological space. The collection of *Borel sets* in X is the smallest σ-algebra of subsets of X that contain the open sets, as on p4 of [61], p60 of [62], p118 of [85], in Definition 10.19 on p132 of [86], p9 of [134], p13 of [157], p309 of [158], and p23, 267 of [180]. Note that some variants of this terminology have sometimes been used.

2.2.1 Borel measures

A nonnegative countably additive measure defined on the σ-algebra of Borel sets in X may be called a *Borel measure*, as in the footnote on p329 of [86], in Definition 2.15 on p49 of [157], and on p269 of [180]. However, this term is sometimes also used for an outer measure μ on X such that the Borel sets in X are measurable with respect to μ, in the sense discussed in Subsection 1.11.1, as on p4 of [61], and in (2) in Definition 1.5 on p9 of [134]. We may use the term *Borel outer measure* for this here.

Of course, if μ is a Borel outer measure on X, then the restriction of μ to the Borel sets is a Borel measure on μ. If X is a metric space, then a metric outer measure on X is a Borel outer measure on X, as in Subsection 1.11.2.

2.2.2 Two basic families of examples of Borel sets

A subset of X is said to be an *F_σ set* if it can be expressed as the union of a sequence of closed sets. Similarly, a subset of X is said to be a *G_δ set* if it can be expressed as an intersection of a sequence of open sets. Clearly F_σ sets and G_δ sets are Borel sets in X. It is easy to see that a subset of X is an F_σ set if and only if its complement in X is a G_δ set.

Suppose that $(X, d(\cdot, \cdot))$ is a metric space. If $x \in X$ and r is a positive real number, then let

$$B(x, r) = \{y \in X : d(x, y) < r\} \tag{2.2.1}$$

be the usual *open ball* in X centered at x of radius r with respect to d. It is well known and not difficult to show that open balls in X are open sets. Similarly, if r is a nonnegative real number, then let

$$\overline{B}(x,r) = \{y \in X : d(x,y) \le r\} \tag{2.2.2}$$

be the *closed ball* in X centered at x of radius r with respect to d. One can check that closed balls in X are closed sets.

It is easy to see that an open ball in X may be expressed as the union of a sequence of closed balls with the same center. This implies that open balls are F_σ sets. Similarly, a closed ball in X may be expressed as the intersection of a sequence of open balls with the same center, and is thus a G_δ set.

If A is any subset of X, then put

$$A_r = \bigcup_{x \in A} B(x,r). \tag{2.2.3}$$

This is an open set in X, because it is a union of open sets. One can check that

$$\overline{A} = \bigcap_{r>0} A_r, \tag{2.2.4}$$

where $\overline{A}$ is the closure of A in X. This implies that

$$\overline{A} = \bigcap_{l=1}^{\infty} A_{1/l}, \tag{2.2.5}$$

so that closed sets in X are G_δ sets. It follows that open sets in X are F_σ sets.

2.2.3 Some examples concerning intervals

Let a and b be real numbers with $a < b$. One can check that $[a,b)$ and $(a,b]$ are each both F_σ sets and G_δ sets in the real line, with respect to the standard Euclidean metric. More precisely, $[a,b)$ and $(a,b]$ may each be expressed as the union of a sequence of closed intervals, and as the intersection of a sequence of open intervals. Similarly, (a,b) may be expressed as the union of a sequence of closed intervals, and $[a,b]$ may be expressed as the intersection of a sequence of open intervals. We also have that an open half-line may be expressed as the union of a sequence of closed half-lines, and a closed half-line may be expressed as the intersection of a sequence of open half-lines.

Let n be a positive integer, and remember that a subset of $\mathbf{R}^n$ may be called an interval if it can be expressed as the Cartesian product of n intervals in the real line, as in Subsection 1.6.2. If a subset of $\mathbf{R}^n$ can be expressed as a product of n open intervals in $\mathbf{R}$, then one can check that this set is an open set in $\mathbf{R}^n$, with respect to the standard Euclidean metric. Similarly, if a subset of $\mathbf{R}^n$ can be expressed as the product of n closed intervals in $\mathbf{R}$, then one can verify that this set is a closed set in $\mathbf{R}^n$. It is well known that the product of n closed intervals in $\mathbf{R}$ that are also bounded is compact in $\mathbf{R}^n$.

One can check that any interval in $\mathbf{R}^n$ is both an F_σ set and a G_δ set. More precisely, it can be expressed as the union of a sequence of closed intervals, and as the intersection of a sequence of open intervals. In particular, intervals in $\mathbf{R}^n$ are Borel sets.

2.3 Separability and countable bases

Let X be a metric space, or a topological space. A collection $\mathcal{B}$ of open subsets of X is said to be a *base for the topology of* X if every open set may be expressed as a union of elements of $\mathcal{B}$. One may consider the empty set as being the union of an empty collection of sets, so that it has this property automatically.

Equivalently, $\mathcal{B}$ is a base for the topology of X if for every open set $U \subseteq X$ and point $x \in U$ there is a $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq U$. If X is a metric space, then this is also equivalent to saying that for every $x \in X$ and $r > 0$ there is a $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq B(x, r)$.

The collection of all open balls in X is a base for the topology of X when X is a metric space. In fact, the collection of all open balls in X with radii of the form $1/l$ for some positive integer l is a base for the topology of X.

Suppose that X is a metric space again, and let E be a dense set in X. One can check that

$$\mathcal{B}_E = \{B(y, 1/l) : y \in E,\, l \in \mathbf{Z}_+\} \tag{2.3.1}$$

is a base for the topology of X. Indeed, if $x \in X$ amd r is a posiitve real number, then let l be a positive integer such that $2/l < r$, and let y be an element of E such that

$$d(x, y) < 1/l. \tag{2.3.2}$$

Thus $B(y, 1/l)$ is an element of $\mathcal{B}_E$ that contains x as an element, and one can check that

$$B(y, 1/l) \subseteq B(x, r), \tag{2.3.3}$$

using the triangle inequality.

2.3.1 Separable metric and topological spaces

A metric or topological space X is said to be *separable* if there is a dense set E in X such that E has only finitely or countably many elements. If X is a metric space, and $E \subseteq X$ has only finitely or countably many elements, then one can verify that (2.3.1) has only finitely or countably many elements.

It is well known that the set $\mathbf{Q}$ of rational numbers is a countable dense set in the real line with respect to the standard Euclidean metric, so that $\mathbf{R}$ is separable. If n is a positive integer, then it is well known and not too difficult to show that the set $\mathbf{Q}^n$ of n-tuples of rational numbers is a countable dense set in $\mathbf{R}^n$ with respect to the standard Euclidean metric, so that $\mathbf{R}^n$.

If X is any set with the discrete metric or topology, then X is the only dense set in itself. It follows that X is separable if and only if X has only finitely or countably many elements.

Let X be a metric or topological space again, and let $\mathcal{B}$ be a base for the topology of X. If $V \in \mathcal{B}$ and $V \neq \emptyset$, then let $x(V)$ be an element of V. Consider the set E of these points $x(V)$. One can check that E is dense in X, because $\mathcal{B}$ is a base for the topology of X. If $\mathcal{B}$ has only finitely or countably many elements, then E has only finitely or countably many elements, so that X is separable.

2.3.2 Bases and Borel sets

If $\mathcal{E}$ is any collection of Borel sets, then the σ-algebra $\mathcal{A}(\mathcal{E})$ of subsets of X generated by $\mathcal{E}$, as in the previous section, is contained in the σ-algebra of Borel sets in X.

If $\mathcal{E}$ is a base for the topology of X with only finitely or countable many elements, then $\mathcal{A}(\mathcal{E})$ contains all Borel sets in X. More precisely, if a subset U of X can be expressed as the union of finitely or countably many elements of $\mathcal{E}$, then $U \in \mathcal{A}(\mathcal{E})$. If this holds for all open sets U in X, then $\mathcal{A}(\mathcal{E})$ contains the Borel sets in X.

2.3.3 Some more examples of bases using intervals

The collection of all open intervals in the real line is a base for the topology of $\mathbf{R}$ with respect to the standard metric. More precisely, the collection of open intervals in $\mathbf{R}$ with rational endpoints is a countable base for the topology of $\mathbf{R}$.

If n is a positive integer, then the collection of all open intervals in $\mathbf{R}^n$ is a base for the topology of $\mathbf{R}^n$ with respect to the standard metric. In fact, the collection of all open intervals in $\mathbf{R}^n$ obtained by taking the Cartesian product of n open intervals in $\mathbf{R}$ with rational endpoints is a countable base for the topology of $\mathbf{R}^n$.

Let $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{E}_1(\mathbf{R}^n)$ be the collections of bounded and possibly unbounded elementary subsets of $\mathbf{R}^n$, as in Subsection 1.6.3. Note that elementary subsets of $\mathbf{R}^n$ are Borel sets, because intervals are Borel sets, as in the previous section.

We also have that $\mathcal{A}(\mathcal{E}(\mathbf{R}^n))$ contains all of the Borel sets in $\mathbf{R}^n$, because the collection of open intervals in $\mathbf{R}^n$ obtained from products of open intervals in $\mathbf{R}$ with rational endpoints is a countable base for the topology of $\mathbf{R}^n$, as before. It follows that $\mathcal{A}(\mathcal{E}(\mathbf{R}^n))$ and $\mathcal{A}(\mathcal{E}(\mathbf{R}^n))$ are both the same as the collection of all Borel sets in $\mathbf{R}^n$.

2.4 Limits of functions

We would like consider some more Borel measures on the real line, related to monotonically increasing real-valued functions on $\mathbf{R}$. In order to do this, we shall first review some facts related to one-sided limits of functions on $\mathbf{R}$.

Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let p be an element of X that is a limit point of E. Also let f be a function on E

with values in Y, and let q be an element of Y. We say that *the limit of $f(x)$, as $x \in E$ approaches p, is equal to q*, if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$d_Y(f(x), q) < \epsilon \tag{2.4.1}$$

for every $x \in E$ with $d(p, x) < \delta$ and $x \neq p$. It is well known and not difficult to show that the limit q is unique when it exists. In this case, we put

$$\lim_{\substack{x \in E \\ x \to p}} f(x) = q, \tag{2.4.2}$$

or simply

$$\lim_{x \to p} f(x) = q \tag{2.4.3}$$

when $E = X$.

It is well known that (2.4.2) holds if and only if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of E that converges to p and satisfies $x_j \neq p$ for each j, we have that $\{f(x_j)\}_{j=1}^\infty$ converges to q in Y. If $E = X$, then it is easy to see that f is continuous at p if and only if

$$\lim_{x \to p} f(x) = f(p). \tag{2.4.4}$$

If p is not a limit point of X, then any mapping from X into Y is continuous at p.

2.4.1 One-sided limits on R

Let us now take $E = X = \mathbf{R}$, with the standard metric. The *limit of $f(x)$ as $x \in \mathbf{R}$ approaches p from the right* is defined by

$$f(p+) = \lim_{x \to p+} f(x) = \lim_{\substack{x \in (p, +\infty) \\ x \to p}} f(x), \tag{2.4.5}$$

when the limit on the right side exists. Similarly, the *limit of $f(x)$ as $x \in \mathbf{R}$ approaces p from the left* is defined by

$$f(p-) = \lim_{x \to p-} f(x) = \lim_{\substack{x \in (-\infty, p) \\ x \to p}} f(x), \tag{2.4.6}$$

when the limit on the right side exists. One can check that

$$\lim_{x \to p} f(x) \tag{2.4.7}$$

exists if and only if the one-sided limits (2.4.5) and (2.4.6) and are equal. Of course, (2.4.7) is equal to the common value of the one-sided limits in this case.

Let us say that f is *continuous at p from the right* if

$$\lim_{x \to p+} f(x) = f(p). \tag{2.4.8}$$

Similarly, we say that f is *continuous at p from the left* if

$$\lim_{x \to p-} f(x) = f(p). \tag{2.4.9}$$

Thus f is continuous at p if and only if f is continuous at p from both the left and the right.

2.4.2 Monotonically increasing functions on R

Let f be a monotonically increasing real-valued function on $\mathbf{R}$, so that

$$f(x) \le f(y) \tag{2.4.10}$$

for every $x, y \in \mathbf{R}$ with $x < y$. It is well known that the limits of f at $p \in \mathbf{R}$ from the left and right exist, with

$$f(p+) = \inf_{p < x < +\infty} f(x) \tag{2.4.11}$$

and

$$f(p-) = \sup_{-\infty < x < p} f(x). \tag{2.4.12}$$

This corresponds to Theorem 4.1 H on p102 of [78] and its proof, the first part of Theorem 8.19 on p111 of [86] and its proof, and to parts of Theorem 4.29 on p95 of [158]. Note that

$$f(p-) \le f(p) \le f(p+), \tag{2.4.13}$$

and that

$$f(p_1+) \le f(p_2-) \tag{2.4.14}$$

when $p_1 < p_2$, as in [158].

It follows from (2.4.13) that f is continuous at p exactly when

$$f(p-) = f(p+). \tag{2.4.15}$$

Thus f is not continuous at p exactly when

$$f(p-) < f(p+). \tag{2.4.16}$$

It is well known that f is continuous at all but finitely or countably many elements of $\mathbf{R}$, as in the first part of Theorem 8.19 on p111 of [86], and Theorem 4.30 on p96 of [158].

It is convenient to put

$$f(+\infty) = \sup_{x \in \mathbf{R}} f(x) \tag{2.4.17}$$

and

$$f(-\infty) = \inf_{x \in \mathbf{R}} f(x). \tag{2.4.18}$$

These satisfy

$$f(x) \to f(+\infty) \text{ as } x \to +\infty \tag{2.4.19}$$

and

$$f(x) \to f(-\infty) \text{ as } x \to -\infty, \tag{2.4.20}$$

with suitable interpretations. This is related to Definition 4.1 E on p101 of [78], Exercise 11 and 21 at the end of Section 4.1 of [78], and some remarks on p98 of [158].

2.5 Lengths of intervals with respect to α

Let α be a monotonically increasing real-valued function on the real line. If a and b are real numbers with $a < b$, then put

$$\lambda_\alpha([a,b)) = \alpha(b-) - \alpha(a-), \tag{2.5.1}$$

$$\lambda_\alpha([a,b]) = \alpha(b+) - \alpha(a-), \tag{2.5.2}$$

$$\lambda_\alpha((a,b]) = \alpha(b+) - \alpha(a+), \tag{2.5.3}$$

$$\lambda_\alpha((a,b)) = \alpha(b-) - \alpha(a+), \tag{2.5.4}$$

as in Example 11.6 (b) on p303 of [158]. We may also use (2.5.2) when $a = b$. These may be described as the *α-lengths* of these intervals. Of course, this is the same as the ordinary length of an interval when $\alpha(x) = x$ for every $x \in \mathbf{R}$.

Similarly, put

$$\lambda_\alpha([a,+\infty)) = \alpha(+\infty) - \alpha(a-), \tag{2.5.5}$$

$$\lambda_\alpha((a,+\infty)) = \alpha(+\infty) - \alpha(a+), \tag{2.5.6}$$

$$\lambda_\alpha((-\infty,b]) = \alpha(b+) - \alpha(-\infty), \tag{2.5.7}$$

$$\lambda_\alpha((-\infty,b)) = \alpha(b-) - \alpha(-\infty), \tag{2.5.8}$$

$$\lambda_\alpha((-\infty,+\infty)) = \alpha(+\infty) - \alpha(-\infty). \tag{2.5.9}$$

Note that (2.5.9) is finite if and only if α is bounded on $\mathbf{R}$. Similarly, (2.5.5) and (2.5.6) are finite exactly when α has a finite upper bound on $\mathbf{R}$, and (2.5.7) and (2.5.8) are finite exactly when α has a finite lower bound on $\mathbf{R}$.

Let $\mathcal{E}(\mathbf{R})$ be the ring of bounded elementary sets in $\mathbf{R}$, and let $\mathcal{E}_1(\mathbf{R})$ be the algebra of possibly unbounded elementary sets in $\mathbf{R}$, as in Subsection 1.6.3. We can extend λ_α to a finitely additive measure on $\mathcal{E}(\mathbf{R})$, in the same way as in Subsection 1.13.1. Similarly, we can extend λ_α to a finitely additive measure on $\mathcal{E}_1(\mathbf{R})$, and we use λ_α to denote this extension as well.

One can check that λ_α is outer regular and inner regular with respect to compact sets on $\mathcal{E}(\mathbf{R})$, as in Example 11.6 (a) on p303 of [158]. Similarly, λ_α is outer regular and inner regular with respect to compact sets on $\mathcal{E}_1(\mathbf{R})$. As in Subsection 1.13.2, it suffices to check these regularity properties for intervals, and one can do this directly from the definitions. This implies that λ_α is countably additive on $\mathcal{E}_1(\mathbf{R})$, by the remarks at the beginning of Section 1.13.

2.5.1 Lebesgue–Stieltjes outer measures

Using λ_α, we get an outer measure $\overline{\lambda_\alpha}$ on the real line as in Subsection 1.11.3. This may be called the *Lebesgue–Stieltjes outer measure associated to α*. Note that $\overline{\lambda_\alpha} = \lambda_\alpha$ on $\mathcal{E}_1(\mathbf{R})$, and that the elements of $\mathcal{E}_1(\mathbf{R})$ are measurable with respect to $\overline{\lambda_\alpha}$, as before. This means that the Borel sets in $\mathbf{R}$ with respect to the standard metric are measurable with respect to $\overline{\lambda_\alpha}$, as in Subsection 2.3.3.

In the construction in Subsection 1.11.3, one considers coverings of a subset E of the real line by sequences of elements of $\mathcal{E}_1\mathbf{R})$. One can consider coverings of E by sequences of possibly unbounded intervals instead, and get an equivalent definition of $\overline{\lambda_\alpha}$, as in Section 1.15. One can consider coverings by sequences of bounded intervals too. In fact, one can consider coverings by sequences of bounded open intervals, and get an equivalent definition of $\overline{\lambda_\alpha}$. This is essentially the same as in Definition 11.7 on p304 of [158], where coverings by sequences of bounded open elementary sets are used.

Alternatively, one can consider coverings by sequences of bounded intervals of the form $(a, b]$, as on p282 of [180]. One can also show that $\overline{\lambda_\alpha}$ is a metric outer measure on $\mathbf{R}$ with respect to the standard metric, as on p283 of [180]. This is another way to see that the Borel sets in $\mathbf{R}$ are measurable with respect to $\overline{\lambda_\alpha}$.

One can obtain Lebesgue–Stieltjes outer measures using Riemann–Stieltjes integrals as well, as on p111 of [62], and in Definition 9.19 on p120 of [86].

One may define the *Lebesgue–Stieltjes measure* associated to α to be the restriction of $\overline{\lambda_\alpha}$ to the corresponding σ-algebra of measurable sets, or simply to the Borel sets. The restriction of $\overline{\lambda_\alpha}$ to the measurable sets may be denoted λ_α again.

2.6 Monotone classes

Let X be a set, and let $\mathcal{M}$ be a collection of subsets of X. We say that $\mathcal{M}$ is a *monotone class* or *monotone family* if it satisfies the following two conditions, as on p380 of [86], and p145 of [157]. First, if $A_1, A_2, A_3, \ldots$ is a sequence of elements of $\mathcal{M}$ such that $A_j \subseteq A_{j+1}$ for each j, then

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}. \tag{2.6.1}$$

Second, if $B_1, B_2, B_3, \ldots$ is a sequence of elements of $\mathcal{M}$ such that $B_{j+1} \subseteq B_j$ for each j, then

$$\bigcap_{j=1}^{\infty} B_j \in \mathcal{M}. \tag{2.6.2}$$

Note that σ-algebras of subsets of X are monotone classes.

Let $\mathcal{E}$ be any collection of subsets of X, and let $\mathcal{M}(\mathcal{E})$ be the intersection of all of the monotone classes of subsets of X that contains X. One can check that $\mathcal{M}(\mathcal{E})$ is also a montone class of subsets of X, as mentioned on p381 of [86], and at the beginning of the proof of Theorem 7.3 on p146 of [157]. This is the smallest monotone class of subsets of X that contains $\mathcal{E}$, by construction.

Let $\mathcal{A}(\mathcal{E})$ be the smallest σ-algebra of subsets of X that contains $\mathcal{E}$, as in Section 2.2. Note that

$$\mathcal{M}(\mathcal{E}) \subseteq \mathcal{A}(\mathcal{E}), \tag{2.6.3}$$

because σ-algebras of subsets of X are monotone classes.

If $\mathcal{E}$ is an algebra of subsets of X, then it is well known that

$$\mathcal{M}(\mathcal{E}) = \mathcal{A}(\mathcal{E}). \tag{2.6.4}$$

This corresponds to Theorem 21.6 on p380 of [86], and Theorem 7.3 on p146 of [157] is a particular case of this.

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