Some notes related to partial differential equations

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Preface

These informal notes are intended to complement more detailed treatments, as in the references. Some familiarity with basic analysis and linear algebra would be helpful, and some definitions and results along these lines are reviewed here. My colleague Frank Jones' book [160] may be a very helpful resource for this. Some familiarity with Lebesgue measure and integration could be helpful as well, but we shall normally not be getting into this too much here.

Of course, there are many connections between complex analysis and partial differential equations. The reader is not necessarily expected to be familiar with complex analysis here, although some familiarity would be helpful in some places.

The subject of partial differential equations is obviously closely related to that of ordinary differential equations. Often only basic facts about ordinary differential equations are used here, but some familiarity with standard results related to existence and uniqueness of solutions would be helpful in some places. More precisely, some familiarity with standard results concerning the dependence of solutions on initial conditions and other parameters would be helpful in some places.

There are many connections between partial differential equations, Fourier analysis, and functional analysis too. We shall not get into this too much here, but some of these connections will be mentioned a bit, or are fairly close.

A number of the texts in the bibliography include some aspects of the history of differential equations and related matters, such as Fourier analysis. In particular, one may be interested in [13, 14, 50, 51, 52, 102, 107, 108, 114, 117, 119, 120, 141, 192, 193, 194, 198, 199, 204, 205, 206] in this regard.

Some additional perspectives concerning partial differential equations may be found in [3, 66, 125, 133, 154, 161, 162, 165, 173, 233, 238, 239, 246, 264, 290, 304, 309].

I would like to dedicate these notes to Eli Stein and Guido Weiss, whose influence should hopefully be clear here, and in a variety of related directions. Of course, Alberto Calderón and Antoni Zygmund are also very important here, in connection with various related aspects of harmonic analysis, for which I have endeavored to include some basic indications.

Contents

Chapter 1

Some basic facts

Some very interesting introductory remarks about partial differential equations can be found in the first chapter of [81]. Another interesting overview with a somewhat different perspective is in Section A of Chapter 1 of [87]. Here we begin with some basic notions related to Euclidean spaces and functions on them, which are helpful for this.

1.1 Some preliminaries about $\mathbf{R}^n$

Let n be a positive integer, and let $\mathbf{R}^n$ be the usual space of n-tuples $x = (x_1, \ldots, x_n)$ of real numbers. If $x, y \in \mathbf{R}^n$ and $t \in \mathbf{R}$, then $x + y$ and $t\,x$ can be defined as elements of $\mathbf{R}^n$ using coordinatewise addition and scalar multiplication, as usual.

1.1.1 The standard Euclidean norm

The *standard Euclidean norm* of $x \in \mathbf{R}^n$ is defined by

$$|x| = \Big(\sum_{j=1}^{n} x_j^2\Big)^{1/2}, \tag{1.1.1}$$

using the nonnegative square root on the right side. This reduces to the usual absolute value of a real number when $n = 1$. Observe that

$$|t\,x| = |t|\,|x| \tag{1.1.2}$$

for every $t \in \mathbf{R}$ and $x \in \mathbf{R}^n$. It is well known that

$$|x + y| \le |x| + |y| \tag{1.1.3}$$

for every $x, y \in \mathbf{R}^n$. This is called the *triangle inequality* for the standard Euclidean norm.

1.1.2 The standard Euclidean metric

The *standard Euclidean metric* on $\mathbf{R}^n$ is defined by

$$d(x,y) = |x-y| \tag{1.1.4}$$

for every $x, y \in \mathbf{R}^n$. This may also be described as the distance between x and y, with respect to the standard Euclidean metric.

Note that the standard Euclidean metric on $\mathbf{R}^n$ is *invariant under translations*, in the sense that

$$d(x+a, y+a) = |(x+a)-(y+a)| = |x-y| = d(x,y) \tag{1.1.5}$$

for all $a, x, y \in \mathbf{R}^n$.

1.1.3 Open sets

If $x \in \mathbf{R}^n$ and r is a positive real number, then the *open ball* in $\mathbf{R}^n$ centered at x with radius r is defined by

$$B(x,r) = \{y \in \mathbf{R}^n : |x-y| < r\}. \tag{1.1.6}$$

Similarly, the *closed ball* in $\mathbf{R}^n$ centered at x with radius r is defined by

$$\overline{B}(x,r) = \{y \in \mathbf{R}^n : |x-y| \le r\}. \tag{1.1.7}$$

A subset U of $\mathbf{R}^n$ is said to be an *open set* with respect to the standard Euclidean metric if for every $x \in U$ there is an $r > 0$ such that

$$B(x,r) \subseteq U. \tag{1.1.8}$$

It is well known and not too difficult to show that

$$\text{every open ball in } \mathbf{R}^n \text{ is an open set} \tag{1.1.9}$$

in this sense. Similarly, if t is a nonnegative real number, then one can check that

$$\{y \in \mathbf{R}^n : |x-y| > t\} \tag{1.1.10}$$

is an open set.

1.1.4 Convergent sequences

Let $\{x_j\}_{j=1}^\infty$ be a sequence of points in $\mathbf{R}^n$. There is a well-known definition of what it means for $\{x_j\}_{j=1}^\infty$ to *converge* to a point $x \in \mathbf{R}^n$ with respect to the standard Euclidean metric, and we shall not repeat this here. In this case, x is said to be the *limit* of $\{x_j\}_{j=1}^\infty$, which may be expressed by

$$\lim_{j\to\infty} x_j = x. \tag{1.1.11}$$

It is well known and not difficult to show that the limit of a convergent sequence is unique, and this works in any metric space.

Let $x_{j,l}$ be the lth coordinate of x_j for each $j \geq 1$ and $l = 1, \ldots, n$, and similarly let x_l be the lth coordinate of x for each $l = 1, \ldots, n$. It is well known and not difficult to show that (1.1.11) holds if and only if

$$\lim_{j \to \infty} x_{j,l} = x_l \tag{1.1.12}$$

for each $l = 1, \ldots, n$. More precisely, this means that $\{x_{j,l}\}_{j=1}^\infty$ converges to x_l as a sequence of real numbers for each $l = 1, \ldots, n$, with respect to the standard Euclidean metric on the real line $\mathbf{R}$.

1.1.5 Closures of subsets of $\mathbf{R}^n$

Let E be a subset of $\mathbf{R}^n$. The *closure* of E in $\mathbf{R}^n$ with respect to the standard Euclidean metric is defined to be the set

$$\overline{E} \tag{1.1.13}$$

of all $x \in \mathbf{R}^n$ with the following property: for every $r > 0$ there is a $y \in E$ such that

$$|x - y| < r. \tag{1.1.14}$$

Equivalently, this means that for every $r > 0$,

$$E \cap B(x, r) \neq \emptyset. \tag{1.1.15}$$

Note that

$$E \subseteq \overline{E} \tag{1.1.16}$$

automatically.

Alternatively, one can check that

$$x \in \overline{E} \tag{1.1.17}$$

if and only if

(1.1.18) there is a sequence $\{x_j\}_{j=1}^\infty$ of elements of E that converges to x.

If $x \in \mathbf{R}^n$ and $r > 0$, then one can check that

(1.1.19) the closure of $B(x, r)$ in $\mathbf{R}^n$ is equal to $\overline{B}(x, r)$.

However, this does not always work in arbitrary metric spaces.

1.1.6 Closed sets

If

$$\overline{E} = E, \tag{1.1.20}$$

then E is said to be a *closed set* in $\mathbf{R}^n$ with respect to the standard Euclidean metric. This is the same as saying that

$$\overline{E} \subseteq E, \tag{1.1.21}$$

because of (1.1.16). Equivalently, E is a closed set in $\mathbf{R}^n$ if and only if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of E that converges to an element x of $\mathbf{R}^n$, we have that

$$x \in E. \tag{1.1.22}$$

It is well known and not too difficult to show that

$$\text{every closed ball in } \mathbf{R}^n \text{ is a closed set.} \tag{1.1.23}$$

If E is any subset of $\mathbf{R}^n$, then it is well known and not too hard to show that

$$\overline{E} \text{ is a closed set.} \tag{1.1.24}$$

1.1.7 Boundaries of subsets of $\mathbf{R}^n$

If U is an open subset of $\mathbf{R}^n$, then the *boundary* may be defined as the set

$$\partial U \tag{1.1.25}$$

of points in the closure of U that are not in U,

$$\partial U = \overline{U} \setminus U. \tag{1.1.26}$$

If $x \in \mathbf{R}^n$ and $r > 0$, then

$$\partial B(x, r) = \{y \in \mathbf{R}^n : |x - y| = r\}, \tag{1.1.27}$$

but this does not always work in arbitrary metric spaces.

If E is any subset of $\mathbf{R}^n$, then the boundary of E is defined by

$$\partial E = \overline{E} \cap \overline{(\mathbf{R}^n \setminus E)}. \tag{1.1.28}$$

One can check that this is equivalent to the definition in the preceding paragraph when E is an open set.

1.2 Some spaces of functions

Let E be a nonempty subset of $\mathbf{R}^n$, for some $n \geq 1$, and let f be a real-valued function on E. It is well known that f is continuous at a point $x \in E$ if and only if for every sequence $\{x_j\}_{j=1}^{\infty}$ of elements of E that converges to x, we have that

$$\lim_{j \to \infty} f(x_j) = f(x). \tag{1.2.1}$$

If f is continuous at every point in E, then f is said to be *continuous on* E. The space of continuous real-valued functions on E may be denoted

$$C(E). \tag{1.2.2}$$

1.2.1 Continuous differentiability

Let U be a nonempty open subset of $\mathbf{R}^n$, let f be a real-valued function on U, let x be an element of U, and let l be a positive integer less than or equal to n. The *partial derivative* of f at x in the lth variable may be denoted

$$\partial_l f(x) = D_l f(x) = \frac{\partial f}{\partial x_l}(x), \tag{1.2.3}$$

when it exists.

If (1.2.3) exists for every $x \in U$ and $l = 1, \ldots, n$, and is continuous on U for each l, then f is said to be *continuously differentiable* on U. It is well known that this implies that

$$f \text{ is continuous on } U, \tag{1.2.4}$$

although this may sometimes be included in the definition, for convenience. The space of continuously-differentiable real-valued functions on U may be denoted

$$C^1(U). \tag{1.2.5}$$

1.2.2 k-Times continuous differentiability

If k is any positive integer, then we may say that f is *k-times continuously differentiable* on U if f is continuous on U, and all derivatives of f up to order k exist at every point in U, and are continuous on U. The space of k-times continuously-differentiable real-valued functions on U may be denoted

$$C^k(U). \tag{1.2.6}$$

More precisely, this may be defined recursively when $k \geq 2$, by saying that $C^k(U)$ consists of all continuously-differentiable real-valued functions f on U such that

$$\frac{\partial f}{\partial x_l} \in C^{k-1}(U) \tag{1.2.7}$$

for each $l = 1, \ldots, n$. It is sometimes convenient to take

$$C^0(U) = C(U). \tag{1.2.8}$$

If the derivatives of f of all orders exist everywhere on U and are continuous on U, then f is said to be *infinitely differentiable*, or *smooth*, on U. The space of infinitely-differentiable real-valued functions on U may be denoted

$$C^\infty(U). \tag{1.2.9}$$

1.2.3 Multi-indices

An n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers is said to be a *multi-index*, of *order*

$$|\alpha| = \sum_{j=1}^n \alpha_j. \tag{1.2.10}$$

Of course, this is not necessarily the same as the standard Euclidean norm of α, as an element of $\mathbf{R}^n$, and it should normally be clear which is intended. If $f \in C^k(U)$ for some $k \ge 1$ and $|\alpha| \le k$, then the corresponding derivative of f of order $|\alpha|$ may be denoted

$$\partial^\alpha f = D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \tag{1.2.11}$$

Note that this function is continuously differentiable of order

$$k - |\alpha| \tag{1.2.12}$$

on U under these conditions.

If f is a twice continuously-differentiable function on U, then it is well known that

$$\frac{\partial^2 f}{\partial x_j \, \partial x_l} = \frac{\partial^2 f}{\partial x_l \, \partial x_j} \tag{1.2.13}$$

on U for every $j, l = 1, \ldots, n$. Similarly, if f is k-times continuously differentiable on U, then derivatives of f up to order k may be taken in any order.

Sometimes derivatives are expressed using subscripts to indicate the variables in which the derivative is taken. Thus one may put

$$f_{x_j} = \frac{\partial f}{\partial x_j}, \quad f_{x_j \, x_l} = \frac{\partial^2 f}{\partial x_j \, \partial x_l}, \tag{1.2.14}$$

and so on, where appropriate.

1.2.4 Monomials

If $x \in \mathbf{R}^n$, then we may put

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{1.2.15}$$

where $x_j^{\alpha_j}$ is interpreted as being equal to 1 when $\alpha_j = 0$, even when $x_j = 0$. This defines a real-valued function on $\mathbf{R}^n$, which is the *monomial* of degree $|\alpha|$ associated to α.

Similarly, (1.2.11) corresponds to

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \tag{1.2.16}$$

or

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \tag{1.2.17}$$

applied to f. More precisely, $\partial_j = D_j$ defines a linear mapping from $C^k(U)$ into $C^{k-1}(U)$ for each $k \ge 1$. Composition of these mappings can be considered as a type of multiplication, with $\partial_j^{\alpha_j} = D_j^{\alpha_j}$ interpreted as being the identity mapping when $\alpha_j = 0$.

1.3 Partial differential equations

Let k and n be positive integers, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let u be a k-times continuously-differentiable real-valued function on U. A kth-order partial differential equation for u on U can be expressed as

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0, \tag{1.3.1}$$

as in Section 1.1 of [81], and Section A of Chapter 1 of [87]. Here $D^l u(x)$ is intended to represent the collection of all possible derivatives of u of order l at x, which may be identified with an element of $\mathbf{R}^{n^l} (= \mathbf{R}^{(n^l)})$. Thus F may be considered as a real-valued function on

$$\mathbf{R}^{n^k} \times \mathbf{R}^{n^{k-1}} \times \cdots \times \mathbf{R}^n \times \mathbf{R} \times U. \tag{1.3.2}$$

A linear kth-order partial differential equation for u on U can be expressed as

$$\sum_{|\alpha| \le k} a_\alpha(x)\, \partial^\alpha u(x) = f(x), \tag{1.3.3}$$

as in Section 1.1 of [81], and Section A of Chapter 1 of [87]. More precisely, the sum is taken over all multi-indices α with $|\alpha| \le k$, which is of course a finite set. Thus $a_\alpha(x)$ should be a function on U for each such α, as well as $f(x)$. If $f(x) = 0$ for every $x \in U$, then (1.3.3) is said to be *homogeneous.*

One may also consider *systems* of partial differential equations, as in [81]. In this case, one can think of u as taking values in $\mathbf{R}^m$ for some positive integer m. Continuous differentiability of u of order k on U means that each of the m components of u is k-times continuously differentiable as a real-valued function on U. One considers finitely many equations involving the components of u and their deriviatives of order up to k on U, as before.

1.3.1 Invariance under translations

Let us say that a partial differential equation as in (1.3.1) is *invariant under translations* if F does not depend on x in the last variable. This means that F may be considered as a real-valued function on

$$\mathbf{R}^{n^k} \times \mathbf{R}^{n^{k-1}} \times \cdots \times \mathbf{R}^n \times \mathbf{R}, \tag{1.3.4}$$

so that (1.3.1) becomes

$$F(D^k u(x), D^{k-1} u(x), \ldots, D u(x), u(x)) = 0. \tag{1.3.5}$$

If u satisfies this equation on U and $a \in \mathbf{R}^n$, then

$$u(x - a) \tag{1.3.6}$$

satisfies the same equation on

$$U + a = \{x + a : x \in U\}. \tag{1.3.7}$$

Note that this is also an open set in $\mathbf{R}^n$. Of course, there are analogous notions for systems. More precisely, invariance under translations in this sense means that $F - c$ has the same property for any constant c. Thus the partial differential equation corresponding to $F - c$ has the same property as before.

The left side of (1.3.3) is said to have *constant coefficients* if $a_\alpha(x)$ is a constant for each multi-index α. If f is also a constant, then (1.3.3) is invariant under translations, as in the preceding paragraph. There are analogous notions for linear systems, as before.

1.3.2 Divergence and directional derivatives

Let v be a continuously-differentiable $\mathbf{R}^n$-valued function on U. The *divergence* of v is the real-valued function on U defined as usual by

$$\operatorname{div} v = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \tag{1.3.8}$$

where $v_j(x)$ is the jth coordinate of $v(x)$ for each $j = 1, \ldots, n$.

Let f be a real-valued function on U, and let $x \in U$ and $w \in \mathbf{R}^n$ be given. It is easy to see that

$$U(x, w) = \{t \in \mathbf{R} : x + t\, w \in U\} \tag{1.3.9}$$

is an open set in the real line that contains 0. The *directional derivative* of f at x in the direction w is defined to be the derivative of

$$f(x + t\, w) \tag{1.3.10}$$

as a function of $t \in U(x, w)$ at $t = 0$, if it exists. This may be denoted

$$D_w f(x). \tag{1.3.11}$$

If f is continuously differentiable on U, then it is well known that the directional derivative exists, and is equal to

$$\sum_{j=1}^{n} w_j \, \frac{\partial f}{\partial x_j}(x). \tag{1.3.12}$$

1.4 Complex numbers

A complex number z can be expressed in a unique way as

$$z = x + y\,i, \tag{1.4.1}$$

where $x, y \in \mathbf{R}$ and $i^2 = -1$. In this case, x and y are called the *real* and *imaginary parts* of z, and may be denoted $\operatorname{Re} z$, $\operatorname{Im} z$, respectively. The *complex conjugate* of z is the complex number

$$\overline{z} = x - y\,i, \tag{1.4.2}$$

and the *absolute value* or *modulus* of z is the nonnegative real number

$$|z| = (x^2 + y^2)^{1/2}. \tag{1.4.3}$$

In particular, the complex conjugate of $\overline{z}$ is z, and $|\overline{z}| = |z|$.

The real line $\mathbf{R}$ may be considered as a subset of the set $\mathbf{C}$ of complex numbers, and addition and multiplication of real numbers can be extended to complex numbers in a standard way. Note that

$$\overline{z + w} = \overline{z} + \overline{w}, \tag{1.4.4}$$

$$\overline{z\,w} = \overline{z}\,\overline{w} \tag{1.4.5}$$

and

$$z\,\overline{z} = |z|^2 \tag{1.4.6}$$

for every $z, w \in \mathbf{C}$. One can use this to get that

$$|z\,w| = |z|\,|w| \tag{1.4.7}$$

for every $z, w \in \mathbf{C}$. If $z \in \mathbf{C}$ and $z \neq 0$, then z has a multiplicative inverse in $\mathbf{C}$, namely,

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}. \tag{1.4.8}$$

Of course, (1.4.3) is the same as the standard Euclidean norm of $(x, y) \in \mathbf{R}^2$. The triangle inequality for the standard Euclidean norm on $\mathbf{R}^2$ is the same as saying that

$$|z + w| \leq |z| + |w| \tag{1.4.9}$$

for every $z, w \in \mathbf{C}$, which can also be verified more directly in this case. The standard metric on $\mathbf{C}$ is defined by

$$d(z, w) = |z - w|, \tag{1.4.10}$$

which corresponds exactly to the standard Euclidean metric on $\mathbf{R}^2$.

1.4.1 Complex-valued functions

Let n be a positive integer, let U be an open subset of $\mathbf{R}^n$, and let f be a complex-valued function on U. Continuity of f on U can be defined in the same way as for real-valued functions, and is equivalent to continuity of the real and imaginary parts of f. Similarly, differentiability properties of f can be defined in the same way as for real-valued functions, and are equivalent to the corresponding differentiability properties of the real and imaginary parts of f. Complex analysis deals with different types of differentiability properties of complex-valued functions on open subsets of $\mathbf{C}$. This is related to the *Cauchy–Riemann equations* for the real and imaginary parts of such a function.

1.5 Complex exponentials

The *exponential* of a complex number z can be defined by

$$\exp z = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \tag{1.5.1}$$

where the absolute convergence of the series can be obtained from the ratio test, for instance. This is equivalent to taking

$$\exp(x + y\,i) = (\exp x)\,(\cos y + i \sin y) \tag{1.5.2}$$

for every $x, y \in \mathbf{R}$.

It is well known that

$$\exp(z + w) = (\exp z)\,(\exp w) \tag{1.5.3}$$

for every $z, w \in \mathbf{C}$. This can be obtained using the binomial theorem, and standard results about products of absolutely convergent series.

In particular, if $z \in \mathbf{C}$, then one can take $w = -z$ in (1.5.3) to get that $\exp z \neq 0$, with

$$1/(\exp z) = \exp(-z). \tag{1.5.4}$$

Of course, if $x \in \mathbf{R}$, then $\exp x \in \mathbf{R}$, with $\exp x \geq 1$ when $x \geq 0$. If $x \leq 0$, then $0 < \exp x = 1/(\exp(-x)) \leq 1$.

It is easy to see that

$$\overline{(\exp z)} = \exp \overline{z} \tag{1.5.5}$$

for every $z \in \mathbf{C}$. One can use this to get that

$$|\exp(i\,y)| = 1 \tag{1.5.6}$$

for every $y \in \mathbf{R}$. Indeed,

$$|\exp(i\,y)|^2 = \exp(i\,y)\,\overline{\exp(i\,y)} = \exp(i\,y)\,\exp(-i\,y) = 1, \tag{1.5.7}$$

using (1.4.6) in the first step, (1.5.5) in the second step, and (1.5.3) in the third step.

1.5.1 Differentiating complex exponentials

It is well known that $\exp z$ is complex-analytic, or equivalently holomorphic, as a complex-valued function of $z \in \mathbf{C}$. Here we shall be more concerned with related complex-valued functions of real variables. If $a \in \mathbf{C}$, then $\exp(a\,t)$ may be considered as a complex-valued function of $t \in \mathbf{R}$. It is well known that this function is differentiable, with

$$\frac{d}{dt}(\exp(a\,t)) = a\,(\exp(a\,t)). \tag{1.5.8}$$

Let n be a positive integer, and let $\mathbf{C}^n$ be the space of n-tuples $a = (a_1, \ldots, a_n)$ of complex numbers. If $a, b \in \mathbf{C}^n$, then put

$$a \cdot b = \sum_{j=1}^{n} a_j\, b_j. \tag{1.5.9}$$

If $a \in \mathbf{C}^n$ and $x \in \mathbf{R}^n$, then $\exp(a \cdot x)$ is a complex number, which defines a complex-valued function of x on $\mathbf{R}^n$. This function is continuously differentiable on $\mathbf{R}^n$, with

$$\frac{\partial}{\partial x_j}\,\exp(a \cdot x) = a_j\,(\exp(a \cdot x)) \tag{1.5.10}$$

for every $j = 1, \ldots, n$.

More precisely, $\exp(a \cdot x)$ is infinitely differentiable as a complex-valued function of x on $\mathbf{R}^n$. If α is a multi-index, then

$$\partial^\alpha \exp(a \cdot x) = a^\alpha\, \exp(a \cdot x). \tag{1.5.11}$$

Here $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$, as in Subsection 1.2.4, which is now a complex number.

1.6 More on complex-valued functions

Let n be a positive integer, and let E be a nonempty subset of $\mathbf{R}^n$. The space of continuous complex-valued functions on E may be denoted

$$C(E, \mathbf{C}), \tag{1.6.1}$$

and we may use

$$C(E, \mathbf{R}) \tag{1.6.2}$$

for the space of continuous real-valued functions on E to be more precise. Remember that a complex-valued function on E is continuous if and only if its real and imaginary parts are continuous. Note that $C(E, \mathbf{R})$ and $C(E, \mathbf{C})$ are *vector spaces* over the real and complex numbers, respectively, with respect to pointwise addition and scalar multiplication of functions.

Similarly, if U is a nonempty open subset of $\mathbf{R}^n$, and k is a positive integer, then we let

$$C^k(U, \mathbf{C}) \tag{1.6.3}$$

be the space of k-times continuously-differentiable complex-valued functions on U. Equivalently, these are the complex-valued functions on U whose real and imaginary parts are k-times continuously differentiable. We may use

$$C^k(U, \mathbf{R}) \tag{1.6.4}$$

for the space of k-times continuously-differentiable real-valued functions on U. As before, we may use the same notation with $k = 0$ for the corresponding spaces of real and complex-valued continuous functions. The space of infinitely-differentiable complex-valued functions on U may be denoted

$$C^\infty(U, \mathbf{C}), \tag{1.6.5}$$

and we may use

$$C^\infty(U, \mathbf{R}) \tag{1.6.6}$$

for the space of smooth real-valued functions on U. We may consider $C^k(U, \mathbf{R})$, $C^k(U, \mathbf{C})$ as linear subspaces of $C(U, \mathbf{R})$, $C(U, \mathbf{C})$, respectively, for each $k \geq 1$. Similarly, $C^\infty(U, \mathbf{R})$, $C^\infty(U, \mathbf{C})$ are linear subspaces of $C^k(U, \mathbf{R})$, $C^k(U, \mathbf{C})$, respectively, for each k.

1.6.1 Some linear mappings and eigenfunctions

If α is a multi-index with $|\alpha| \leq k$, then ∂^α defines a linear mapping from each of $C^k(U, \mathbf{R})$, $C^k(U, \mathbf{C})$ into $C^{k-|\alpha|}(U, \mathbf{R})$, $C^{k-|\alpha|}(U, \mathbf{C})$, respectively. Similarly, ∂^α defines a linear mapping from each of $C^\infty(U, \mathbf{R})$, $C^\infty(U, \mathbf{C})$ into itself.

Let $a \in \mathbf{C}^n$ be given, so that

$$\exp(a \cdot x) \tag{1.6.7}$$

is a smooth complex-valued function on $\mathbf{R}^n$, as in Subsection 1.5.1. This function is an eigenvector for $\partial/\partial x_j$ for each $j = 1, \ldots, n$, as a linear mapping from $C^\infty(\mathbf{R}^n, \mathbf{C})$ into itself, with eigenvalue a_j, as before. Similarly, (1.6.7) is an eigenvector for ∂^α for each muli-index α, as a linear mapping from $C^\infty(\mathbf{R}^n, \mathbf{C})$ into itself, with eigenvalue a^α.

1.7 Polynomials in n variables

Let n be a positive integer, and let us consider polynomials in the n variables $w_1, \ldots, w_n$ with coefficients in $\mathbf{R}$ or $\mathbf{C}$. Such a polynomial can be expressed as

$$p(w) = \sum_{|\alpha| \leq N} a_\alpha \, w^\alpha, \tag{1.7.1}$$

where N is a nonnegative integer, and the sum is taken over all multi-indices α with $|\alpha| \leq N$. The coefficients a_α may be real or complex numbers for each such α, and the monomial w^α is as defined in Subsection 1.2.4. More precisely,

p is said to have degree less than or equal to N in this case. Note that $p(w) \in \mathbf{C}$ when $w \in \mathbf{C}^n$, and $p(w) \in \mathbf{R}$ when $w \in \mathbf{R}^n$ and the coefficients a_α are real numbers.

If p is as in (1.7.1), then put

$$p(\partial) = \sum_{|\alpha| \le N} a_\alpha \, \partial^\alpha, \tag{1.7.2}$$

or equivalently

$$p(D) = \sum_{|\alpha| \le N} a_\alpha \, D^\alpha. \tag{1.7.3}$$

This defines a *differential operator* on $\mathbf{R}^n$ with constant coefficients in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, of order less than or equal to N.

Let U be a nonempty open subset of $\mathbf{R}^n$, and suppose that f is a k-times continuously-differentiable real or complex-valued function on U, with $N \le k$. Under these conditions,

$$p(\partial)(f) = \sum_{|\alpha| \le N} a_\alpha \, \partial^\alpha f \tag{1.7.4}$$

defines a $(k - N)$-times continuously-differentiable real or complex-valued function on U, as appropriate. More precisely, this defines a linear mapping from $C^k(U, \mathbf{R})$ or $C^k(U, \mathbf{C})$ into $C^{k-N}(U, \mathbf{R})$ or $C^{k-N}(U, \mathbf{C})$, respectively, as appropriate. Similarly, this defines a linear mapping from $C^\infty(U, \mathbf{R})$ or $C^\infty(U, \mathbf{C})$ into itself, as appropriate.

If $b \in \mathbf{C}^n$, then $\exp(b \cdot x)$ defines an infinitely-differentiable complex-valued function of x on $\mathbf{R}^n$, as in Subsection 1.5.1. Observe that

$$p(\partial)(\exp(b \cdot x)) = p(b) \, \exp(b \cdot x). \tag{1.7.5}$$

Thus $\exp(b \cdot x)$ is an eigenvector for $p(\partial)$ as a linear mapping from $C^\infty(\mathbf{R}^n, \mathbf{C})$ into itself, with eigenvalue $p(b)$.

1.7.1 Products of polynomials

If α, β are multi-indices, then $\alpha + \beta$ can be defined by coordinatewise addition, as usual, and is another multi-index. Clearly

$$|\alpha + \beta| = |\alpha| + |\beta|, \tag{1.7.6}$$

where $|\cdot|$ refers to the order of the multi-index, as in Subsection 1.2.3. Observe that

$$w^\alpha \, w^\beta = w^{\alpha + \beta}. \tag{1.7.7}$$

Similarly,

$$\partial^\alpha \, \partial^\beta = \partial^{\alpha + \beta}, \tag{1.7.8}$$

because of the commutativity of derivatives under suitable conditions, as in Subsection 1.2.3.

Let $p_1(w)$, $p_2(w)$ be polynomials in $w_1, \ldots, w_n$ with real or complex coefficients, and of degrees less than or equal to nonnegative integers N_1, N_2. The product

$$p(w) = p_1(w)\, p_2(w) \tag{1.7.9}$$

can be defined as a polynomial of degree less than or equal to $N_1 + N_2$ in the usual way, using (1.7.7). Similarly,

$$p(\partial) = p_1(\partial)\, p_2(\partial), \tag{1.7.10}$$

because of (1.7.8).

More precisely, let f be a k-times continuously-differentiable real or complex-valued function on a nonempty open subset U of $\mathbf{R}^n$ again. If α, β are multi-indices with $|\alpha| + |\beta| \le k$, then $\partial^\beta f$ is $(k - |\beta|)$-times continuously differentiable on U, and

$$\partial^\alpha(\partial^\beta f) = \partial^{\alpha+\beta} f \tag{1.7.11}$$

on U. If p_1, p_2, and p are as in the preceding paragraph and $N_1 + N_2 \le k$, then $p_2(\partial)(f)$ is $(k - N_2)$-times continuously differentiable on U, and

$$p_1(\partial)(p_2(\partial)(f)) = p(\partial)(f) \tag{1.7.12}$$

on U.

1.8 Connectedness and convexity

Let n be a positive integer, and let E be a subset of $\mathbf{R}^n$. We say that

$$E \text{ is } \textit{convex} \tag{1.8.1}$$

if for every $x, y \in E$ and $t \in \mathbf{R}$ with $0 \le t \le 1$, we have that

$$(1 - t)\, x + t\, y \in E. \tag{1.8.2}$$

It is well known and not too difficult to show that

$$\text{open and closed balls in } \mathbf{R}^n \text{ are convex.} \tag{1.8.3}$$

More precisely, this means that open and closed balls in $\mathbf{R}^n$ with respect to the standard Euclidean metric are convex, although one could also use a metric associated to any norm.

1.8.1 Path-connected sets

We say that

$$E \text{ is } \textit{path connected} \tag{1.8.4}$$

if for every $x, y \in E$,

$$\text{there is a continuous path in } E \text{ connecting } x \text{ and } y. \tag{1.8.5}$$

More precisely, this means that there is a continuous mapping f from the closed unit interval $[0,1]$ in the real line into $\mathbf{R}^n$ such that

$$f(0) = x, \quad f(1) = y, \tag{1.8.6}$$

and

$$f(t) \in E \tag{1.8.7}$$

for every $t \in [0,1]$. If $f_j(t)$ is the jth coordinate of $f(t)$ for every $j = 1, \ldots, n$ and $t \in [0,1]$, then the continuity of f as a mapping from $[0,1]$ into $\mathbf{R}^n$ is equivalent to the continuity of f_j as a real-valued function on $[0,1]$ for each j. It is easy to see that

(1.8.8) if E is convex, then E is path connected.

1.8.2 Connected sets

The precise definition of *connectedness* of subsets of $\mathbf{R}^n$ is a bit complicated, and although we shall not discuss it here, we shall mention some of its properties. It is well known and not too difficult to show that

(1.8.9) path-connected sets are connected.

It is also well known that

(1.8.10) a subset of the real line is connected if and only if it is convex.

Another well-known theorem states that

(1.8.11) connected open subsets of $\mathbf{R}^n$ are path connected.

Let U be an open subset of $\mathbf{R}^n$. In this case,

(1.8.12) U is not connected

if and only if

(1.8.13) U can be expressed as the union of two nonempty disjoint open subsets of $\mathbf{R}^n$.

This is close to the definition of connectedness, depending on how it is formulated.

1.8.3 Nonemptiness of the boundary

If $U \neq \emptyset, \mathbf{R}^n$, then

$$\partial U \neq \emptyset. \tag{1.8.14}$$

This is the same as saying that U is not a closed set, because of the description of the boundary of an open set in $\mathbf{R}^n$ mentioned in Subsection 1.1.7. This can be obtained from the connectedness of $\mathbf{R}^n$. Alternatively, if $x \in U$ and $z \in \mathbf{R}^n \setminus U$, then one can show that there is a $t_0 \in \mathbf{R}$ such that $0 < t_0 \le 1$ and

$$(1 - t_0)\, x + t_0\, z \in \partial U. \tag{1.8.15}$$

More precisely, one can take t_0 to be the infimum or greatest lower bound of the set of $t > 0$ such that

$$(1 - t)\, x + t\, z \in \mathbf{R}^n \setminus U. \tag{1.8.16}$$

1.8.4 Locally constant functions

Let E be a nonempty subset of $\mathbf{R}^n$, and let f be a function on E with values in any set. Let us say that

$$f \text{ is } \textit{locally constant} \text{ at a point } x \in E \tag{1.8.17}$$

if there is an $r > 0$ such that

$$f(x) = f(y) \tag{1.8.18}$$

for every $y \in E$ with $|x - y| < r$. If E is connected, and f is locally constant at every point in E, then one can show that

$$f \text{ is constant on } E. \tag{1.8.19}$$

One can also show that connectedness is characterized by this property.

Let U be a nonempty open subset of $\mathbf{R}^n$, and let f be a real or complex-valued function on U. Observe that if

$$f \text{ is locally constant at every point in } U, \tag{1.8.20}$$

then

$$\begin{aligned} &f \text{ is continuously differentiable on } U, \text{ with all} \\ &\text{of its first partial derivatives equal to 0 on } U. \end{aligned} \tag{1.8.21}$$

If U is convex, then it is well known and not difficult to show that (1.8.21) implies that

$$f \text{ is constant on } U. \tag{1.8.22}$$

Using this, one can check that (1.8.21) implies (1.8.20) for any open set U in $\mathbf{R}^n$. If U is connected, then it follows that (1.8.22) holds, as in the preceding paragraph, although this is a bit simpler in this case.

1.9 Compactness in $\mathbf{R}^n$

Let n be a positive integer, and let E be a subset of $\mathbf{R}^n$. We say that E is *bounded* if there is a nonnegative real number C such that

$$|x| \le C \tag{1.9.1}$$

for every $x \in E$. It is easy to see that open and closed balls in $\mathbf{R}^n$ with respect to the standard Euclidean metric are bounded sets.

The precise definition of compactness of a subset of $\mathbf{R}^n$, or of an arbitrary metric space, is a bit complicated, and we shall not discuss it here. However, we would like to mention the following two well-known results about compactness. The first is that a subset E of $\mathbf{R}^n$ is *compact* if and only if it is closed and bounded. The second is the *extreme value theorem*, which states that if f is a continuous real-valued function on a nonempty compact set E, then

$$\text{the maximum and minimum of } f \text{ on } E \text{ are attained.} \tag{1.9.2}$$

1.9.1 Relative closure

Let U be an open subset of $\mathbf{R}^n$. The *relative closure* of a subset E of U may be defined to be the intersection of the closure of E in $\mathbf{R}^n$ with U,

$$\overline{E} \cap U. \tag{1.9.3}$$

In particular, E is said to be *relatively closed* in U if

$$E = \overline{E} \cap U. \tag{1.9.4}$$

If E is closed as a subset of $\mathbf{R}^n$, then it follows that E is relatively closed in U. Note that U is automatically relatively closed as a subset of itself.

There is a notion of compactness of a subset E of U relative to U, with respect to the restriction of the standard Euclidean metric on $\mathbf{R}^n$. However, it is well known that this holds if and only if E is compact as a subset of $\mathbf{R}^n$.

1.9.2 Supports of functions

Let f be a real or complex-valued function on $\mathbf{R}^n$, or a function with values in $\mathbf{R}^m$ for some positive integer m. The *support* of f is the subset of $\mathbf{R}^n$ defined by

$$\operatorname{supp} f = \overline{\{x \in \mathbf{R}^n : f(x) \neq 0\}}. \tag{1.9.5}$$

Of course, this is a closed set in $\mathbf{R}^n$, by construction.

Thus the support of f is compact exactly when it is bounded. This is the same as saying that

$$f(x) = 0 \text{ when } |x| \text{ is sufficiently large.} \tag{1.9.6}$$

Suppose now that f is a function defined on an open set $U \subseteq \mathbf{R}^n$. We say that f has *compact support* in U if there is a compact set $E \subseteq \mathbf{R}^n$ such that $E \subseteq U$ and

$$\{x \in U : f(x) \neq 0\} \subseteq E. \tag{1.9.7}$$

1.9.3 Sequential compactness

Let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of $\mathbf{R}^n$, and let $\{j_l\}_{l=1}^\infty$ be a strictly increasing sequence of positive integers. Under these conditions,

$$\{x_{j_l}\}_{l=1}^\infty \tag{1.9.8}$$

is said to be a *subsequence* of $\{x_j\}_{j=1}^\infty$. Note that $\{x_j\}_{j=1}^\infty$ may be considered as a subsequence of itself. If $\{x_j\}_{j=1}^\infty$ converges to a point $x \in \mathbf{R}^n$, then it is easy to see that every subsequence of $\{x_j\}_{j=1}^\infty$ converges to x as well.

A subset E of $\mathbf{R}^n$ is said to be *sequentially compact* every sequence $\{x_j\}_{j=1}^\infty$ of elements of E has a subsequence that converges to an element of E. It is well known that this is equivalent to compactness. More precisely, this works in any metric space.

1.10 Some derivatives

Let n be a positive integer, and let α be a multi-index. It is customary to put

$$\alpha! = \alpha_1! \, \alpha_2! \cdots \alpha_n!, \tag{1.10.1}$$

which is a positive integer. Observe that

$$\partial^\alpha x^\alpha = \alpha!. \tag{1.10.2}$$

Let β be another multi-index. If

$$\beta_j < \alpha_j \text{ for some } j, \tag{1.10.3}$$

then

$$\partial^\alpha x^\beta = 0. \tag{1.10.4}$$

In particular, this holds when $|\alpha| = |\beta|$ and $\alpha \neq \beta$.

Suppose now that $\alpha_j \le \beta_j$ for each $j = 1, \ldots, n$, so that $\beta - \alpha$ is a multi-index. Of course, $\partial^\alpha x^\beta$ is a multiple of $x^{\beta-\alpha}$ in this case. If $\alpha \neq \beta$, so that $\alpha_j < \beta_j$ for some j, then we get that $\partial^\alpha x^\beta$ is equal to 0 at 0.

1.10.1 Taylor polynomials

Let U be a nonempty open subset of $\mathbf{R}^n$, let k be a positive integer, and let f be a k-times continuously-differentiable real-valued function on U. The degree k Taylor polynomial of f at a point $w \in U$ may be expressed as

$$P(x) = \sum_{|\beta| \le k} \frac{1}{\beta!} \, \partial^\beta f(w) \, x^\beta, \tag{1.10.5}$$

where the sum is taken over all multi-indices β with $|\beta| \le k$. Using the remarks in the previous paragraphs, we get that

$$\partial^\alpha P(0) = \partial^\alpha f(w) \tag{1.10.6}$$

for every multi-index α with $|\alpha| \le k$.

Put

$$g(x) = f(w + x) - P(x) \tag{1.10.7}$$

for $x \in U - w$. Here $U - w = U + (-w)$ is as in Subsection 1.3.1. This is a k-times continuously-differentiable function on $U - w$, with

$$\partial^\alpha g(0) = \partial^\alpha P(0) - \partial^\alpha f(w) = 0 \tag{1.10.8}$$

for every multi-index α with $|\alpha| \le k$.

If $x \in \mathbf{R}^n$ and $|x|$ is sufficiently small, then

$$t\,x \in U - w \tag{1.10.9}$$

for all $t \in [0, 1]$. More precisely, this uses the fact that $U - w$ is an open set in $\mathbf{R}^n$ that contains 0, by hypothesis. In this case,

$$g(t\,x) \tag{1.10.10}$$

may be considered as a k-times continuously-differentiable function of t on an open set in the real line that contains $[0, 1]$. The derivatives of $g(t\,x)$ in t up to order k can be expressed in terms of derivatives of g, as a function on $U - w$, of the same order. These derivatives are equal to 0 at $t = 0$, because of (1.10.8).

One can use this to show that

$$\lim_{x \to 0} |x|^{-k}\, g(x) = 0, \tag{1.10.11}$$

which is Taylor's theorem in n dimensions. This uses the fact that $\partial^\alpha g$ is small near 0 when $|\alpha| = k$, because of (1.10.8) and the continuity of $\partial^\alpha g$ on $U - w$. More precisely, this implies that the kth derivative of (1.10.10) in t is small when $|x|$ is small and $t \in [0, 1]$. This permits one to reduce (1.10.11) to standard versions of Taylor's theorem in one variable.

1.11 Some smooth functions

Consider the real-valued function defined on $\mathbf{R}$ by

$$\begin{aligned} \psi(t) &= \exp(-1/t) \quad \text{when } t > 0 \\ &= 0 \qquad\qquad\quad \text{when } t \le 0. \end{aligned} \tag{1.11.1}$$

It is well known and not too difficult to show that ψ is infinitely differentiable on $\mathbf{R}$, with all of its derivatives at 0 equal to 0. This uses the fact that

$$\lim_{t \to 0+} t^{-l} \exp(-1/t) = 0 \tag{1.11.2}$$

for every positive integer l.

1.11.1 Some functions associated to intervals

Let a, b be real numbers with $a < b$, and put

$$\psi_{a,b}(t) = \psi(t-a)\,\psi(b-t). \tag{1.11.3}$$

This is an infinitely-differentiable function on $\mathbf{R}$ that is positive on (a,b), and equal to 0 otherwise.

One can integrate $\psi_{a,b}$ to get an infinitely-differentiable function on $\mathbf{R}$ that is equal to 0 when $t \le a$, is a positive constant when $t \ge b$, and strictly increasing on (a,b). Using this, one can get infinitely-differentiable nonnegative real-valued functions on $\mathbf{R}$ that are equal to 1 on any given closed interval, and equal to 0 on the complement of a slightly larger open interval.

Alternatively, observe that

$$\psi(t-a) + \psi(b-t) \tag{1.11.4}$$

is a positive smooth function on $\mathbf{R}$. This implies that

$$\frac{\psi(t-a)}{\psi(t-a) + \psi(b-t)} \tag{1.11.5}$$

and

$$\frac{\psi(b-t)}{\psi(t-a) + \psi(b-t)} \tag{1.11.6}$$

are nonnegative smooth functions on $\mathbf{R}$ that are less than or equal to 1, by construction. It is easy to see that (1.11.5) is equal to 0 when $t \le a$, and to 1 when $t \ge b$. Similarly, (1.11.6) is equal to 0 when $t \ge b$, and to 1 when $t \le a$. Note that the sum of (1.11.5) and (1.11.6) is equal to 1 for every $t \in \mathbf{R}$.

1.11.2 Some smooth functions on $\mathbf{R}^n$

If n is any positive integer, then one can use functions like these to get a lot of infinitely-differentiable nonnegative real-valued functions on $\mathbf{R}^n$ with compact support. One can take products of smooth functions on $\mathbf{R}$ with compact support in each variable, for instance. If $a \in \mathbf{R}^n$, then

$$|x-a|^2 = \sum_{j=1}^{n} (x_j - a_j)^2 \tag{1.11.7}$$

is a polynomial in x, and infinitely differentiable on $\mathbf{R}^n$ in particular. If ϕ is a smooth real-valued function on $\mathbf{R}$, then

$$\phi(|x-a|^2) \tag{1.11.8}$$

is a smooth function on $\mathbf{R}^n$. If $\phi(t) = 0$ when $t \in \mathbf{R}$ is sufficiently large, then (1.11.8) has compact support in $\mathbf{R}^n$.

1.12 Semilinearity and quasilinearity

Let k and n be positive integers, let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a k-times continuously-differentiable real-valued function on U. One may be interested in kth-order partial differential equations for u on U that have some linearity properties, without being linear in u and its derivatives. Such a differential equation is said to be *semilinear* if it can be expressed as

$$\sum_{|\alpha|=k} a_\alpha(x)\,\partial^\alpha u(x) + a_0(D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0, \tag{1.12.1}$$

as in Section 1.1 of [81]. Here the sum is taken over all multi-indices α with $|\alpha| = k$, and $a_\alpha(x)$ should be a real-valued function on U for each such α. As before, a_0 may be considered as a real-valued function on

$$\mathbf{R}^{n^{k-1}} \times \cdots \times \mathbf{R}^n \times \mathbf{R} \times U. \tag{1.12.2}$$

Similarly, a kth order partial differential equation for u on U is said to be *quasilinear* if it can be expressed as

$$\begin{aligned} &\sum_{|\alpha|=k} a_\alpha(D^{k-1}u(x), \ldots, Du(x), u(x), x)\,\partial^\alpha u(x) \\ &+a_0(D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0, \end{aligned} \tag{1.12.3}$$

as in Section 1.1 of [81], and Section A of Chapter 1 of [87]. In this case, the coefficients a_α as well as a_0 may be considered as real-valued functions on (1.12.2).

A kth-order partial differential equation for u on U is said to be *fully nonlinear* if it depends nonlinearly on at least some of the kth-order derivatives of u, as in [81]. Of course, there are analogous notions for systems of partial differential equations.

1.12.1 More on invariance under translations

As in Subsection 1.3.1, one may be interested in partial differential equations that are invariant under translations. In the case of a semilinear equation as in (1.12.1), this means that a_α is a constant for each multi-index α with $|\alpha| = k$, and that a_0 does not depend on x in the last variable. Thus a_0 may considered as a real-valued function on

$$\mathbf{R}^{n^{k-1}} \times \cdots \times \mathbf{R}^n \times \mathbf{R}. \tag{1.12.4}$$

Similarly, a quasilinear equation as in (1.12.3) is invariant under translations when the a_α's and a_0 do not depend on x in the last variable, so that they may be considered as real-valued functions on (1.12.4). There are analogous statements for systems of partial differential equations, as usual.

1.13 More on $\mathbf{R}^n$

Let n be a positive integer, and let U be an open subset of $\mathbf{R}^n$. Suppose that K is a compact subset of $\mathbf{R}^n$ such that

$$K \subseteq U. \tag{1.13.1}$$

Under these conditions, it is well known that there is a positive real number t such that for every $x \in K$, we have that

$$B(x,t) \subseteq U. \tag{1.13.2}$$

1.13.1 Closed balls contained in U

Suppose now that w is an element of U and r is a positive real number such that

$$\overline{B}(w,r) \subseteq U. \tag{1.13.3}$$

Remember that closed balls in $\mathbf{R}^n$ are closed and bounded, as in Subsection 1.1.6 and Section 1.9, and thus compact. It follows that there is a positive real number ϵ such that

$$B(w,r+\epsilon) \subseteq U, \tag{1.13.4}$$

by the remarks in the preceding paragraph.

1.13.2 Open balls of maximal radius

Let $y \in U$ be given, and let A be the set of positive real numbers r such that

$$B(y,r) \subseteq U. \tag{1.13.5}$$

Note that A is nonempty, because U is an open set, by hypothesis. Suppose that

$$U \neq \mathbf{R}^n, \tag{1.13.6}$$

so that there is a point z in the complement of U in $\mathbf{R}^n$. If $r \in A$, then we get that

$$r \le |y-z|, \tag{1.13.7}$$

because $z \notin B(y,r)$. This means that $|y-z|$ is an upper bound for A in $\mathbf{R}$.

It is well known that A has a least upper bound or supremum ρ in $\mathbf{R}$ under these conditions. One can check that

$$B(y,\rho) \subseteq U, \tag{1.13.8}$$

because otherwise A would have an upper bound strictly less that ρ. We also have that

$$B(y,\rho+\epsilon) \not\subseteq U \tag{1.13.9}$$

for every $\epsilon > 0$, because ρ is an upper bound for A.

Using (1.13.9), we obtain that

$$\overline{B}(y,\rho) \not\subseteq U, \tag{1.13.10}$$

because of the earlier remarks. This means that

$$\partial B(y,\rho) \not\subseteq U, \tag{1.13.11}$$

because of (1.13.8).

It is easy to see that

$$\overline{B}(y,\rho) \subseteq \overline{U}, \tag{1.13.12}$$

using (1.13.8). Combining this with (1.13.11), we get that

$$\partial B(y,\rho) \cap \partial U \neq \emptyset. \tag{1.13.13}$$

In particular, $\partial U \neq \emptyset$, as mentioned in Subsection 1.8.3.

1.14 More on complex exponentials

Let a be a complex number. Suppose that f is a differentiable complex-valued function on the real line such that

$$f' = a\, f \tag{1.14.1}$$

on $\mathbf{R}$. This implies that

$$\frac{d}{dt}(\exp(-a\,t)\, f(t)) = 0 \tag{1.14.2}$$

on $\mathbf{R}$. Of course, this means that $\exp(-a\,t)\, f(t)$ is constant on $\mathbf{R}$. It follows that

$$f(t) = f(0)\, \exp(a\,t) \tag{1.14.3}$$

for every $t \in \mathbf{R}$.

Let n be a positive integer, and let b be an element of $\mathbf{C}^n$. Suppose that u is a complex-valued function on $\mathbf{R}^n$ such that for each $j = 1, \ldots, n$, the partial derivative of u in the jth variable exists at every point in $\mathbf{R}^n$, with

$$\frac{\partial u}{\partial x_j} = b_j\, u. \tag{1.14.4}$$

Under these conditions, one can check that

$$u(x) = u(0)\, \exp(b \cdot x) \tag{1.14.5}$$

for every $x \in \mathbf{R}^n$, using the remarks in the preceding paragraph. Alternatively, one can use the same type of argument as before, by verifying that

$$\frac{\partial}{\partial x_j}(\exp(-b \cdot x)\, u(x)) = 0 \tag{1.14.6}$$

for each $j = 1, \ldots, n$ and $x \in \mathbf{R}^n$.

1.14.1 The definition of t^a

Let a be a complex number again. If t is a positive real number, then put

(1.14.7) $$t^a = \exp(a \log t).$$

This is a smooth complex-valued function of t on the set $\mathbf{R}_+$ of positive real numbers, with

(1.14.8) $$\frac{d}{dt}(t^a) = a\, t^{a-1}$$

for every $t > 0$. Note that

(1.14.9) $$t^a \neq 0$$

for each $t > 0$, and that

(1.14.10) $$t^a\, t^b = t^{a+b}$$

for all $t > 0$ and $b \in \mathbf{C}$, because of the properties of the exponential function mentioned in Section 1.5.

Let g be a differentiable complex-valued function on $\mathbf{R}_+$ such that

(1.14.11) $$g'(t) = a\, t^{-1}\, g(t)$$

for every $t > 0$. Using this, we get that

(1.14.12) $$\frac{d}{dt}(t^{-a}\, g(t)) = 0$$

on $\mathbf{R}_+$. This implies that $t^{-a}\, g(t)$ is constant on $\mathbf{R}_+$, so that

(1.14.13) $$g(t) = g(1)\, t^a$$

for every $t > 0$.

1.15 The dot product on $\mathbf{R}^n$

If $x, y \in \mathbf{R}^n$ for some positive integer n, then their *dot product* is defined by

(1.15.1) $$x \cdot y = \sum_{j=1}^{n} x_j\, y_j,$$

which is consistent with the notation in Subsection 1.5.1. This is also known as the *standard inner product* on $\mathbf{R}^n$. Clearly

(1.15.2) $$x \cdot y = y \cdot x$$

for every $x, y \in \mathbf{R}^n$.

Note that

(1.15.3) $$x \cdot x = \sum_{j=1}^{n} x_j^2 = |x|^2$$

for every $x \in \mathbf{R}^n$. This means that the standard Euclidean norm on $\mathbf{R}^n$ is the same as the norm associated to the standard inner product.

It is well known that

$$|x \cdot y| \le |x|\,|y| \tag{1.15.4}$$

for every $x, y \in \mathbf{R}^n$, which is a version of the *Cauchy–Schwarz inequality.* This can be used to obtain the triangle inequality for the standard Euclidean norm on $\mathbf{R}^n$, by a standard argument.

If $x, y \in \mathbf{R}^n$, then

$$\begin{aligned} |x+y|^2 = (x+y)\cdot(x+y) &= x\cdot x + x\cdot y + y\cdot x + y\cdot y \\ &= |x|^2 + 2\,x\cdot y + |y|^2. \end{aligned} \tag{1.15.5}$$

Thus

$$x \cdot y = (1/2)\,(|x+y|^2 - |x|^2 - |y|^2), \tag{1.15.6}$$

which is known as a *polarization identity.*

1.15.1 Orthogonal transformations

Let T be a linear mapping from $\mathbf{R}^n$ into itself. It is easy to see that

$$\ker T = \{x \in \mathbf{R}^n : T(x) = 0\} \tag{1.15.7}$$

is a linear subspace of $\mathbf{R}^n$, which is called the *kernel* of T.

One can check that T is one-to-one on $\mathbf{R}^n$ if and only if $\ker T = \{0\}$, using linearity. It is well known that T is one-to-one on $\mathbf{R}^n$ if and only if T maps $\mathbf{R}^n$ onto itself, which is to say that $T(\mathbf{R}^n) = \mathbf{R}^n$. In this case, the inverse mapping T^{-1} is linear on $\mathbf{R}^n$ too.

A one-to-one linear mapping T from $\mathbf{R}^n$ onto itself is said to be an *orthogonal transformation* if T preserves the standard inner product on $\mathbf{R}^n$. This means that

$$T(x) \cdot T(y) = x \cdot y \tag{1.15.8}$$

for every $x, y \in \mathbf{R}^n$. Under these conditions, the inverse mapping T^{-1} is an orthogonal transformation on $\mathbf{R}^n$ as well.

If we take $x = y$ in (1.15.8), then we get that

$$|T(x)| = |x|. \tag{1.15.9}$$

Conversely, if (1.15.9) holds for every $x \in \mathbf{R}^n$, then (1.15.8) holds for every $x, y \in \mathbf{R}^n$. This uses the linearity of T and the polarization identity (1.15.6).

Of course, if (1.15.9) holds for every $x \in \mathbf{R}^n$, then $\ker T = \{0\}$. This implies that T is one-to-one on $\mathbf{R}^n$, and thus that T maps $\mathbf{R}^n$ onto itself, as before.

1.15.2 The adjoint of T

If T is any linear mapping from $\mathbf{R}^n$ into itself, then it is well known that there is a unique linear mapping T' from $\mathbf{R}^n$ into itself such that

$$T(x) \cdot y = x \cdot T'(y) \tag{1.15.10}$$

for every $x, y \in \mathbf{R}^n$. More precisely, every linear mapping from $\mathbf{R}^n$ into itself corresponds to an $n \times n$ matrix of real numbers in a standard way using the standard basis for $\mathbf{R}^n$. The matrix associated to T' in this way is the transpose of the matrix associated to T.

If T is an orthogonal transformation on $\mathbf{R}^n$, then one can check that T' is the same as the inverse of T. Conversely, if T is an invertible linear mapping on $\mathbf{R}^n$, with inverse equal to T', then one can verify that T is an orthogonal transformation on $\mathbf{R}^n$.

If T is an orthogonal transformation on $\mathbf{R}^n$, then it is well known that the determinant of T is ± 1, because $T^{-1} = T'$. If the determinant of T is equal to 1, then T is said to be a *rotation* on $\mathbf{R}^n$.

Chapter 2

Some related notions

2.1 The Laplacian

Let n be a positive integer. The *Laplacian* on $\mathbf{R}^n$ defined by

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{2.1.1}$$

Let $p(w)$ be the polynomial in n variables $w_1, \ldots, w_n$ with real coefficients defined by

$$p(w) = \sum_{j=1}^{n} w_j^2. \tag{2.1.2}$$

Observe that

$$p(\partial) = \sum_{j=1}^{n} \partial_j^2 = \Delta, \tag{2.1.3}$$

using the notation in Section 1.7.

If $w \in \mathbf{C}^n$, then

$$p(w) = w \cdot w, \tag{2.1.4}$$

using the notation in Subsection 1.5.1. If $w \in \mathbf{R}^n$, then

$$p(w) = |w|^2. \tag{2.1.5}$$

If $b \in \mathbf{C}^n$, then

$$\Delta(\exp(b \cdot x)) = (b \cdot b)\, \exp(b \cdot x), \tag{2.1.6}$$

as in Section 1.7. In particular,

$$\Delta(\exp(b \cdot x)) = 0 \tag{2.1.7}$$

when $b \cdot b = 0$.

2.1.1 Laplace's equation

Let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a twice continuuously-differentiable real or complex-valued function on U. We say that u is *harmonic* on U if it satisfies *Laplace's equation*

$$\Delta u = 0 \tag{2.1.8}$$

on U.

Let T be a linear mapping from $\mathbf{R}^n$ into itself. It is easy to see that T is continuous, so that the inverse image

$$T^{-1}(U) = \{x \in \mathbf{R}^n : T(x) \in U\} \tag{2.1.9}$$

of U under T is an open subset of $\mathbf{R}^n$ too. If u is any twice continuously-differentiable function on U, then the composition $u \circ T$ of T and u is twice continuously differentiable on $T^{-1}(U)$.

If T is an orthogonal transformation on $\mathbf{R}^n$, then one can check that

$$\Delta(u \circ T) = (\Delta u) \circ T \tag{2.1.10}$$

on $T^{-1}(U)$, as on p3 of [18]. In particular, if u is harmonic on U, then $u \circ T$ is harmonic on $T^{-1}(U)$, as in Problem 2 in Section 2.5 of [81].

2.2 Two differential operators on $\mathbf{R}^2$

Consider the differential operators

$$L = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\Big) \tag{2.2.1}$$

and

$$\overline{L} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\Big) \tag{2.2.2}$$

on $\mathbf{R}^2$. Observe that

$$L(x_1 + i\,x_2) = \overline{L}(x_1 - i\,x_2) = 1 \tag{2.2.3}$$

and

$$L(x_1 - i\,x_2) = \overline{L}(x_1 + i\,x_2) = 0. \tag{2.2.4}$$

If $z = x_1 + i\,x_2$ is considered as a complex variable, then L and $\overline{L}$ may be denoted $\partial/\partial z$ and $\partial/\partial \overline{z}$, respectively.

Let U be a nonempty open subset of $\mathbf{R}^2$, and let f be a continuously-differentiable complex-valued function on U. If

$$\overline{L}(f) = 0 \tag{2.2.5}$$

on U, then f is said to be *complex analytic* or *holomorphic* on U, as a function of the complex variable z. More precisely, (2.2.5) is equivalent to the usual *Cauchy–Riemann equations* for the real and imaginary parts of f. In this case,

$$f' = L(f) \tag{2.2.6}$$

is the usual complex derivative of f.

2.2.1 Some connections with the Laplacian

Note that

$$L\,\overline{L} = \overline{L}\,L = \frac{1}{4}\,\Delta. \tag{2.2.7}$$

Let u be a twice continuously-differentiable complex-valued function on U. This implies that $L(u)$ and $\overline{L}(u)$ are continuously differentiable on U, and we have that

$$L(\overline{L}(u)) = \overline{L}(L(u)) = \frac{1}{4}\,\Delta(u) \tag{2.2.8}$$

on U. If u is harmonic on U, then it follows that $L(u)$ is holomorphic on U.

If f is holomorphic on U, then it is well known that f is smooth on U, and twice continuously differentiable in particular. It follows that

$$\Delta(f) = 4\,L(\overline{L}(f)) = 0, \tag{2.2.9}$$

so that f is harmonic on U.

2.2.2 Additional properties of L, $\overline{L}$

If f, g are any continuously-differentiable complex-valued functions on U, then

$$L(f\,g) = L(f)\,g + f\,L(g) \tag{2.2.10}$$

and

$$\overline{L}(f\,g) = \overline{L}(f)\,g + f\,\overline{L}(g) \tag{2.2.11}$$

on U, by the product rule. In particular, if f and g are holomorphic on U, then their product $f\,g$ is holomorphic on U.

If f is any continuously-differentiable complex-valued function on U again, then it is easy to see that

$$\overline{L(f)} = \overline{L}(\overline{f}) \tag{2.2.12}$$

on U. It follows that $L(f) = 0$ on U if and only if $\overline{f}$ is holomorphic on U.

Observe that

$$V = \{(x_1, -x_2) : (x_1, x_2) \in U\} \tag{2.2.13}$$

is an open subset of $\mathbf{R}^2$ as well. If f is a continuously-differentiable complex-valued function on U again, then

$$\widetilde{f}(x_1, x_2) = f(x_1, -x_2) \tag{2.2.14}$$

is a continuously-differentiable complex-valued function on V. One can check that f is holomorphic on U if and only if

$$\overline{\widetilde{f}(x_1, x_2)} = \overline{f(x_1, -x_2)} \tag{2.2.15}$$

is holomorphic on V.

2.3 Some complex first-order operators

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Suppose that $a_1, \ldots, a_n$ are n complex-valued functions on U. Thus

$$a(x) = (a_1(x), \ldots, a_n(x)) \tag{2.3.1}$$

may be considered as a mapping from U into $\mathbf{C}^n$. If u is a continuously-differentiable complex-valued function on U, then

$$(L_a(u))(x) = \sum_{j=1}^{n} a_j(x) \, \frac{\partial u}{\partial x_j}(x) \tag{2.3.2}$$

defines a complex-valued function on U. If a_j is a real-valued function on U for each j, so that (2.3.1) is an element of $\mathbf{R}^n$ at each $x \in U$, then (2.3.2) is the same as the directional derivative of u at x in the direction $a(x)$.

Let v be another continuously-differentiable complex-valued function on U, so that the product of u and v is continuously-differentiable on U as well. Observe that

$$L_a(u\,v) = L_a(u)\,v + u\,L_a(v) \tag{2.3.3}$$

on U, by the product rule. If $L_a(u) = 0$ on U, then

$$L_a(u\,v) = u\,L_a(v) \tag{2.3.4}$$

on U. If $L_a(v) = 0$ on U too, then

$$L_a(u\,v) = 0 \tag{2.3.5}$$

on U.

2.3.1 Some commutators

Suppose now that $a_1, \ldots, a_n$ are continuously differentiable on U, and let

$$b_1, \ldots, b_n \tag{2.3.6}$$

be another n continuously-differentiable complex-valued functions on U. Let b and L_b be as in (2.3.1) and (2.3.2), and put

$$c_j = L_a(b_j) - L_b(a_j) \tag{2.3.7}$$

for $j = 1, \ldots, n$. These are continuous complex-valued functions on U, and we let c and L_c be as in (2.3.1) and (2.3.2) again.

Suppose that u is twice continuously differentiable on U. This implies that

(2.3.8) $L_a(u)$ and $L_b(u)$ are continuously differentiable on U,

because the a_j's and b_j's are continuously differentiable on U, by hypothesis. It is easy to see that

$$L_a(L_b(u)) - L_b(L_a(u)) = L_c(u) \tag{2.3.9}$$

on U. This is because the terms on the left side involving second derivatives of u cancel each other out.

2.3.2 Real and imaginary parts

Put

$$\operatorname{Re} a(x) = (\operatorname{Re} a_1(x), \ldots, \operatorname{Re} a_n(x)) \tag{2.3.10}$$

and

$$\operatorname{Im} a(x) = (\operatorname{Im} a_1(x), \ldots, \operatorname{Im} a_n(x)) \tag{2.3.11}$$

for each $x \in U$, which define mappings from U into $\mathbf{R}^n$. If u is any continuously-differentiable complex-valued function on U, then $L_{\operatorname{Re} a}(u)$ and $L_{\operatorname{Im} a}(u)$ can be defined on U as in (2.3.2), and we have that

$$L_a(u) = L_{\operatorname{Re} a}(u) + i\, L_{\operatorname{Im} a}(u). \tag{2.3.12}$$

Of course, if u is real-valued on U, then $L_{\operatorname{Re} a}(u)$ and $L_{\operatorname{Im} a}(u)$ are real-valued on U as well. Otherwise,

$$\operatorname{Re} L_a(u) = L_{\operatorname{Re} a}(\operatorname{Re} u) - L_{\operatorname{Im} a}(\operatorname{Im} u) \tag{2.3.13}$$

and

$$\operatorname{Im} L_a(u) = L_{\operatorname{Re} a}(\operatorname{Im} u) + L_{\operatorname{Im} a}(\operatorname{Re} u) \tag{2.3.14}$$

on U. In particular, $L_a(u) = 0$ may be considered as a system of first-order homogeneous linear partial differential equations in the real and imaginary parts of u, with real coefficients.

2.4 Linear differential operators

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$ again. Also let N be a nonnegative integer, and for each multi-index α with order $|\alpha| \le N$, let a_α be a real or complex-valued function on U. If u is an N-times continuously-differentiable real or complex-valued function on U, then put

$$L(u) = \sum_{|\alpha| \le N} a_\alpha\, \partial^\alpha u \tag{2.4.1}$$

on U, where the sum is taken over all multi-indices α with $|\alpha| \le N$, as usual. This defines a differential operator on U, which can have variable coefficients.

Let r be a nonnegative integer, and suppose that

$$a_\alpha \text{ is } r\text{-times continuously differentiable on } U \tag{2.4.2}$$

for each multi-index α with $|\alpha| \le N$. If

$$u \text{ is } (N + r)\text{-times continuously differentiable on } U, \tag{2.4.3}$$

then

$$L(u) \text{ is } r\text{-times continuously differentiable on } U. \tag{2.4.4}$$

In this case, L defines a linear mapping from $C^{N+r}(U, \mathbf{C})$ into $C^r(U, \mathbf{C})$. If a_α is real-valued on U for each α, then L defines a linear mapping from $C^{N+r}(U, \mathbf{R})$ into $C^r(U, \mathbf{R})$.

Similarly, suppose that

$$a_\alpha \text{ is infinitely differentiable on } U \tag{2.4.5}$$

for every multi-index α with $|\alpha| \le N$. If

$$u \text{ is infinitely differentiable on } U, \tag{2.4.6}$$

then

$$L(u) \text{ is infinitely differentiable on } U \tag{2.4.7}$$

too. This means that L defines a linear mapping from $C^\infty(U, \mathbf{C})$ into itself. If a_α is real-valued on U for each α, then L defines a linear mapping from $C^\infty(U, \mathbf{R})$ into itself.

One can check that the coefficients a_α are uniquely determined by $L(u)$ for polynomials u of degree less than or equal to N. More precisely, a_0 is the same as $L(u)$ when $u(x) \equiv 1$ on U. If $\beta \ne 0$, then a_β can be obtained from $L(x^\beta)$ and the coefficients a_γ with $|\gamma| < |\beta|$.

2.4.1 Composing linear differential operators

Let $\widetilde{N}$ be another nonnegative integer, and let b_β be a real or complex-valued function on U for each multi-index β with $|\beta| \le \widetilde{N}$. If u is an $\widetilde{N}$-times continuously-differentiable real or complex-valued function on U, then

$$\widetilde{L}(u) = \sum_{|\beta| \le \widetilde{N}} b_\beta \, \partial^\beta u \tag{2.4.8}$$

defines a real or complex-valued function on U, as appropriate.

Suppose that

$$b_\beta \text{ is } N\text{-times continuously differentiable on } U \tag{2.4.9}$$

for each multi-index β with $|\beta| \le \widetilde{N}$. If

$$u \text{ is } (N + \widetilde{N})\text{-times continuously differentiable on } U, \tag{2.4.10}$$

then

$$\widetilde{L}(u) \text{ is } N\text{-times continuously differentiable on } U. \tag{2.4.11}$$

This means that

$$L(\widetilde{L}(u)) = \sum_{|\alpha| \le N} a_\alpha \, \partial^\alpha(\widetilde{L}(u)) = \sum_{|\alpha| \le N} \sum_{|\beta| \le \widetilde{N}} a_\alpha \, \partial^\alpha(b_\beta \, \partial^\beta u) \tag{2.4.12}$$

is defined as a real or complex-valued function on U, as appropriate.

Under these conditions, (2.4.12) may be expressed as

$$\widehat{L}(u) = \sum_{|\gamma| \le N + \widetilde{N}} c_\gamma \, \partial^\gamma u, \tag{2.4.13}$$

where c_γ is a real or complex-valued function on U for every multi-index γ with $|\gamma| \le N + \widetilde{N}$. More precisely, the c_γ's can be expressed as sums of products of the α_α's with the b_β's and their derivatives of order less than or equal to N.

Let r be a nonnegative integer again, and suppose that a_α is r-times continuously differentiable on U for every α with $|\alpha| \le N$. If the b_β's are $(N+r)$-times continuously differentiable on U for every β with $|\beta| \le \widetilde{N}$, then the c_γ's are r-times continuously differentiable on U for every γ with $|\gamma| \le N + \widetilde{N}$. If u is also $(N + \widetilde{N} + r)$-times continuously differentiable on U, then $\widetilde{L}(u)$ is $(N + r)$-times continuously differentiable on U, and $\widehat{L}(u)$ is r-times continuously differentiable on U.

Similarly, if the a_α's and b_β's are infinitely differentiable on U, then the c_γ's are infinitely differentiable on U. If u is infinitely differentiable on U too, then $\widetilde{L}(u)$ and $\widehat{L}(u)$ are infinitely differentiable on U as well.

2.5 Some remarks about polynomials

Let n be a positive integer, and let

$$p(x) = \sum_{|\alpha| \le N} a_\alpha \, x^\alpha \tag{2.5.1}$$

be a polynomial in the n variables $x_1, \ldots, x_n$ with complex coefficients, as in Section 1.7. Thus N is a nonnegative integer, $a_\alpha \in \mathbf{C}$ for each multi-index α with order $|\alpha| \le N$, and the sum is taken over all such multi-indices, as before.

If

$$p(x) = 0 \text{ for every } x \in \mathbf{R}^n, \tag{2.5.2}$$

then $\partial^\beta p(x) = 0$ for every $x \in \mathbf{R}^n$ and multi-index β. In particular, this implies that

$$\partial^\beta p(0) = 0 \text{ for every multi-index } \beta. \tag{2.5.3}$$

In this case, this means that

$$a_\alpha = 0 \text{ for every multi-index } \alpha, \, |\alpha| \le N. \tag{2.5.4}$$

If $x \in \mathbf{C}^n$, then $p(x)$ can be defined as a complex number as in (2.5.1). If (2.5.4) holds, then we get that

$$p(x) = 0 \text{ for every } x \in \mathbf{C}^n. \tag{2.5.5}$$

Let r be a positive real number, and suppose that

$$p(x) = 0 \text{ for every } x \in \mathbf{R}^n \text{ with } |x| < r. \tag{2.5.6}$$

This implies that $\partial^\beta p(x) = 0$ for every $x \in \mathbf{R}^n$ with $|x| < r$, and every multi-index β. It follows that (2.5.3) holds in particular under these conditions.

If $b \in \mathbf{R}^n$ or $\mathbf{C}^n$, then

$$p(x+b) = \sum_{|\alpha| \le N} a_\alpha \, (x+b)^\alpha \tag{2.5.7}$$

can be expressed as a polynomial in x with complex coefficients too. If

$$p(x+b) = 0 \text{ for every } x \in \mathbf{R}^n \text{ with } |x| < r, \tag{2.5.8}$$

then the previous remarks imply that $p(x+b) = 0$ for every $x \in \mathbf{C}^n$. This is the same as saying that (2.5.5) holds.

2.5.1 The zero set of p

Note that

$$\{x \in \mathbf{R}^n : p(x) = 0\} \tag{2.5.9}$$

is a closed set in $\mathbf{R}^n$, because p is continuous on $\mathbf{R}^n$. If this set contains a ball of positive radius, then (2.5.9) is equal to $\mathbf{R}^n$, as in the preceding paragraph.

Equivalently, if (2.5.9) is not all of $\mathbf{R}^n$, then the interior of (2.5.9) in $\mathbf{R}^n$ is the empty set. In this case, (2.5.9) may be considered to be rather sparse in $\mathbf{R}^n$. There are stronger results of this type, although we shall not pursue this here. This is related to the remarks in Section 3.10.

The fact that (2.5.9) is a closed set in $\mathbf{R}^n$ implies that

$$\{x \in \mathbf{R}^n : p(x) \ne 0\} \tag{2.5.10}$$

is an open set in $\mathbf{R}^n$, which could also be verified more directly. If this set is nonempty, then its intersection with any ball in $\mathbf{R}^n$ of positive radius is nonempty, as before. This means that the closure of (2.5.10) in $\mathbf{R}^n$ is equal to $\mathbf{R}^n$ in this case, which is the same as saying that (2.5.10) is dense in $\mathbf{R}^n$, with respect to the standard Euclidean metric. One may consider (2.5.10) as being rather large everywhere as a subset of $\mathbf{R}^n$ under these conditions, and there are other results of this type, as before.

Of course, if $n = 1$ and $a_\alpha \ne 0$ for some α, then it is well known that $p(x) = 0$ for at most N points $x \in \mathbf{C}$.

2.6 Some remarks about $\mathbf{C}^n$

Let n be a positive integer, and consider the space $\mathbf{C}^n$ of n-tuples of complex numbers. If $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, then put

$$|z| = \Big(\sum_{j=1}^n |z_j|^2 \Big)^{1/2}, \tag{2.6.1}$$

using the nonnegative square root on the right side, as usual. Here $|z_j|$ is the modulus of $z_j \in \mathbf{C}$ for each $j = 1, \ldots, n$, as in Section 1.4. We may call (2.6.1) the *standard Euclidean norm* on $\mathbf{C}^n$.

If $z, w \in \mathbf{C}^n$ and $t \in \mathbf{C}$, then $z + w$ and $t\,z$ may be defined as elements of $\mathbf{C}^n$ using coordinatewise addition and scalar multiplication. It is easy to see that

$$|t\,z| = |t|\,|z| \tag{2.6.2}$$

for every $z \in \mathbf{C}^n$ and $t \in \mathbf{C}$. One can check that

$$|z + w| \le |z| + |w| \tag{2.6.3}$$

for every $z, w \in \mathbf{C}^n$, using the analogous statements for the modulus of a complex number and the standard Euclidean norm on $\mathbf{R}^n$, as in Subsection 1.1.1 and Section 1.4. The *standard Euclidean metric* on $\mathbf{C}^n$ is defined by

$$d(z, w) = |z - w| \tag{2.6.4}$$

for every $z, w \in \mathbf{C}^n$.

If $z, w \in \mathbf{C}^n$, then we put

$$\langle z, w \rangle = \langle z, w \rangle_{\mathbf{C}^n} = \sum_{j=1}^{n} z_j\,\overline{w_j}. \tag{2.6.5}$$

This is the *standard inner product* on $\mathbf{C}^n$. Observe that (2.6.5) is *Hermitian symmetric*, in the sense that

$$\langle z, w \rangle = \overline{\langle w, z \rangle} \tag{2.6.6}$$

for every $z, w \in \mathbf{C}^n$.

Of course,

$$\langle z, z \rangle = \sum_{j=1}^{n} |z_j|^2 = |z|^2 \tag{2.6.7}$$

for every $z \in \mathbf{C}^n$. This means that the standard Euclidean norm on $\mathbf{C}^n$ is the same as the norm associated to the standard inner product. It is well known that

$$|\langle z, w \rangle| \le |z|\,|w| \tag{2.6.8}$$

for every $z, w \in \mathbf{C}^n$, which is another version of the *Cauchy–Schwarz inequality*. This can also be used to obtain the triangle inequality for the standard Euclidean norm on $\mathbf{C}^n$.

Every $z \in \mathbf{C}^n$ can be expressed in a unique way as

$$z = x + i\,y, \tag{2.6.9}$$

with $x, y \in \mathbf{R}^n$. One can use this to identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. Using this identification, the standard Euclidean norm and metric on $\mathbf{C}^n$ correspond exactly to their analogues on $\mathbf{R}^{2n}$. Similarly, one can check that the real part of the standard inner product on $\mathbf{C}^n$ corresponds to the standard inner product on $\mathbf{R}^{2n}$.

2.6.1 Holomorphic functions

Consider the differential operators

$$L_j = \frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\Big) \tag{2.6.10}$$

and

$$\overline{L}_j = \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\Big) \tag{2.6.11}$$

on $\mathbf{C}^n$, as identified with $\mathbf{R}^{2n}$, for each $j = 1, \ldots, n$. These are the analogues of the operators $L = \partial/\partial z$ and $\overline{L} = \partial/\partial \overline{z}$ in Section 2.2 with respect to

$$z_j = x_j + i\,y_j \tag{2.6.12}$$

for each $j = 1, \ldots, n$.

Let U be a nonempty open subset of $\mathbf{C}^n$, which may be identified with an open subset of $\mathbf{R}^{2n}$. Also let f be a continuously-differentiable complex valued function on U, as an open subset of $\mathbf{R}^{2n}$. This means that the partial derivatives of f in x_j and y_j exist and are continuous on U for each $j = 1, \ldots, n$. If

$$\overline{L}_j(f) = 0 \tag{2.6.13}$$

on U for each $j = 1, \ldots, n$, then f is said to be *holomorphic* on U. This is the same as saying that

$$f(z) = f(z_1, \ldots, z_n) \tag{2.6.14}$$

is holomorphic as a function of z_j for each $j = 1, \ldots, n$, with z_l fixed for $j \neq l$.

It is easy to see that products of holomorphic functions on U are holomorphic. The coordinate functions z_l are holomorphic on $\mathbf{C}^n$ for each $l = 1, \ldots, n$. Of course, constant functions are holomorphic on $\mathbf{C}^n$. It follows that polynomials in $z_1, \ldots, z_n$ with complex coefficients are holomorphic on $\mathbf{C}^n$.

2.7 Polynomials on $\mathbf{C}^n$

Let n be a positive integer, and let $p(z)$ be a polynomial in n complex variables $z_1, \ldots, z_n$ with complex coefficients on $\mathbf{C}^n$. If $n = 1$, and $p(z)$ is not constant, then it is well known that $p(z) = 0$ for some $z \in \mathbf{C}$. More precisely, the number of zeros of p, counted with their multiplicities, is equal to the degree of p.

Suppose now that $n \geq 2$, and let us identify $\mathbf{C}^n$ with $\mathbf{C}^{n-1} \times \mathbf{C}$. If $z = (z_1, \ldots, z_n)$ is an element of $\mathbf{C}^n$, then $z' = (z_1, \ldots, z_{n-1}) \in \mathbf{C}^{n-1}$, and we identify z with $(z', z_n) \in \mathbf{C}^{n-1} \times \mathbf{C}$. Using this, we may express $p(z)$ as

$$p(z) = p(z', z_n) = \sum_{l=0}^{r} p_l(z')\, z_n^l, \tag{2.7.1}$$

where r is a nonnegative integer, and $p_l(z')$ is a polynomial on $\mathbf{C}^{n-1}$ for each $l = 0, \ldots, r$.

Suppose that $r \geq 1$, and that $p_r(z')$ is not identically 0 on $\mathbf{C}^{n-1}$. Otherwise, if $p_l(z')$ is identically 0 on $\mathbf{C}^{n-1}$ for each $l \geq 1$, then $p(z)$ would not depend on z_n, and we could consider it as a polynomial in a smaller number of variables.

Let $z' \in \mathbf{C}^{n-1}$ be given, and suppose that $p_r(z') \neq 0$. Under these conditions, (2.7.1) may be considered as a polynomial of degree r in z_n, which has r roots, with multiplicities, as before. There is an analogous statement as long as $p_l(z') \neq 0$ for some $l \geq 1$.

2.8 The Euler operator

Let n be a positive integer, and put

$$a_j(x) = x_j \tag{2.8.1}$$

for each $j = 1, \ldots, n$ and $x \in \mathbf{R}^n$. In this case,

$$a(x) = (a_1(x), \ldots, a_n(x)) = (x_1, \ldots, x_n) \tag{2.8.2}$$

is the identity mapping on $\mathbf{R}^n$.

Let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuously-differentiable real or complex-valued function on U. Let $L_a(u)$ be the continuous real or complex-valued function on U, as appropriate, defined by

$$(L_a(u))(x) = \sum_{j=1}^{n} x_j \, \frac{\partial u}{\partial x_j}(x) \tag{2.8.3}$$

for every $x \in U$, as in Section 2.3. The differential operator L_a is known as the *Euler operator*.

In this case, $(L_a(u))(x)$ is equal to the directional derivative of u at x in the direction x. Alternatively, if $x \in U$, then $u(t\,x)$ may be considered as a continuously-differentiable real or complex-valued function of t in an open subset of $\mathbf{R}$ that contains 1. The derivative of $u(t\,x)$ in t at 1 is equal to $(L_a(u))(x)$.

2.8.1 Homogeneous functions

Let b be a complex number. A real or complex-valued function u on $\mathbf{R}^n \setminus \{0\}$ is said to be *homogeneous of degree* b if

$$u(t\,x) = t^b\, u(x) \tag{2.8.4}$$

for every $x \in \mathbf{R}^n \setminus \{0\}$ and $t \in \mathbf{R}_+$. If u is continuously differentiable on $\mathbf{R}^n \setminus \{0\}$, then (2.8.4) implies that

$$L_a(u) = b\, u \tag{2.8.5}$$

on $\mathbf{R}^n \setminus \{0\}$.

Let $x \in \mathbf{R}^n \setminus \{0\}$ be given. If u is continuously differentiable on $\mathbf{R}^n \setminus \{0\}$, then $u(t\,x)$ is continuously differentiable as a function of $t \in \mathbf{R}_+$. In this case,

$$\frac{d}{dt}(u(t\,x)) = \sum_{j=1}^{n} x_j\,(\partial_j u)(t\,x) = t^{-1}\,(L_a(u))(t\,x) \tag{2.8.6}$$

for every $t > 0$. If (2.8.5) holds, then we get that

$$\frac{d}{dt}(u(t\,x)) = b\,t^{-1}\,u(t\,x) \tag{2.8.7}$$

for every $t > 0$. This implies that (2.8.4) holds, as in Subsection 1.14.1.

2.8.2 Differentiating homogeneous functions

Let U be a nonempty open subset of $\mathbf{R}^n$, and let t be a positive real number. Observe that

$$t^{-1}\,U = \{t^{-1}\,x : x \in U\} \tag{2.8.8}$$

is an open set in $\mathbf{R}^n$ too. If u is a continuously-differentiable real or complex-valued function on U, then $u(t\,x)$ is continuously differentiable as a function of x on $t^{-1}\,U$. The partial derivatives of $u(tx)$ are equal to

$$\frac{\partial}{\partial x_j}(u(t\,x)) = t\,(\partial_j u)(t\,x) \tag{2.8.9}$$

for each $j = 1, \ldots, n$ and $x \in t^{-1}\,U$.

Let us now take $U = \mathbf{R}^n \setminus \{0\}$, so that $t^{-1}\,U = U$ for every $t > 0$. If u is homogeneous of degree $b \in \mathbf{C}$ on $\mathbf{R}^n \setminus \{0\}$, then

$$\frac{\partial}{\partial x_j}(u(t\,x)) = \frac{\partial}{\partial x_j}(t^b\,u(x)) = t^b\,(\partial_j u)(x) \tag{2.8.10}$$

for each $j = 1, \ldots, n$. It follows that

$$t\,(\partial_j u)(t\,x) = t^b\,(\partial_j u)(x) \tag{2.8.11}$$

for each j, so that $\partial_j u$ is homogeneous of degree $b - 1$ on $\mathbf{R}^n \setminus \{0\}$ under these conditions.

2.8.3 More on homogeneous functions

One can check that

$$|t^b| = t^{\operatorname{Re} b} \tag{2.8.12}$$

for every $t > 0$ and $b \in \mathbf{C}$. Suppose that $\operatorname{Re} b > 0$, and let us interpret t^b as being equal to 0 when $t = 0$. Let us say that a real or complex-valued function u on $\mathbf{R}^n$ is *homogeneous of degree* b if (2.8.4) holds for every $x \in \mathbf{R}^n$ and nonnegative real number t. This means that $u(0) = 0$, and that u is homogeneous of degree b on $\mathbf{R}^n \setminus \{0\}$.

It is customary to interpret t^0 as being equal to 1 for every $t \in \mathbf{R}$, including $t = 0$. Using this, one may interpret a real or complex-valued function u on $\mathbf{R}^n$ as being *homogeneous of degree* 0 on $\mathbf{R}^n$ when u is constant on $\mathbf{R}^n$.

Let u, v be real or complex-valued functions on $\mathbf{R}^n \setminus \{0\}$ that are homogeneous of degrees $b, c \in \mathbf{C}$, respectively. It is easy to see that

(2.8.13) their product $u\,v$ is homogeneous of degree $b + c$ on $\mathbf{R}^n \setminus \{0\}$.

Of course, there is an analogous statement for homogeneous functions on $\mathbf{R}^n$.

2.9 Some spaces of polynomials

Let n be a positive integer, and let

$$\mathcal{P}(\mathbf{R}^n, \mathbf{R}) \text{ and } \mathcal{P}(\mathbf{R}^n, \mathbf{C}) \tag{2.9.1}$$

be the spaces of polynomials on $\mathbf{R}^n$ with real and complex coefficients, respectively. These are linear subspaces of the spaces $C^\infty(\mathbf{R}^n, \mathbf{R})$ and $C^\infty(\mathbf{R}^n, \mathbf{C})$ of smooth real and complex-valued functions on $\mathbf{R}^n$.

Let N be a nonnegative integer, and suppose that a_α is a polynomial on $\mathbf{R}^n$ for each multi-index α of order $|\alpha| \le N$. Under these conditions,

$$L = \sum_{|\alpha| \le N} a_\alpha \, \partial^\alpha \tag{2.9.2}$$

defines a differential operator on $\mathbf{R}^n$ with polynomial coefficients. Of course, the sum is taken over all mutli-indices α with $|\alpha| \le N$, as usual.

It is easy to see that L defines a linear mapping from $\mathcal{P}(\mathbf{R}^n, \mathbf{C})$ into itself. If a_α is a polynomial with real coefficients for each α, then L maps $\mathcal{P}(\mathbf{R}^n, \mathbf{R})$ into itself.

The composition of two differential operators on $\mathbf{R}^n$ with polynomial coefficients is a differential operator with polynomial coefficients too, as in Subsection 2.4.1.

2.9.1 Homogeneous polynomials

Let k be a nonnegative integer. If α is a multi-index of order $|\alpha| = k$, then

(2.9.3) the monomial x^α is homogeneous of degree k

as a real-valued function on $\mathbf{R}^n$. If a polynomial p on $\mathbf{R}^n$ can be expressed as a finite linear combination of monomials x^α with $|\alpha| = k$, then it follows that

(2.9.4) p is homogeneous of degree k on $\mathbf{R}^n$.

Conversely, if a polynomial on $\mathbf{R}^n$ is homogeneous of degree k on $\mathbf{R}^n$, then one can check that it is of this form.

Let

$$\mathcal{P}_k(\mathbf{R}^n, \mathbf{R}) \text{ and } \mathcal{P}_k(\mathbf{R}^n, \mathbf{C}) \tag{2.9.5}$$

be the space of polynomials on $\mathbf{R}^n$ with real and complex coefficients, respectively, that are homogeneous of degree k. These are linear subspaces of $\mathcal{P}(\mathbf{R}^n, \mathbf{R})$ and $\mathcal{P}(\mathbf{R}^n, \mathbf{C})$, respectively.

Let N be a nonnegative integer, and let p be a polynomial on $\mathbf{R}^n$ with real or complex coefficients of degree less than or equal to N. It is easy to see that p can be expressed in a unique way as a sum of homogeneous polynomials of degrees from 0 to N.

If a real or complex-valued function u on $\mathbf{R}^n$ is k-times continuously differentiable and homogeneous of degree k, then one can check that u is equal to its degree k Taylor approximation at the origin.

2.10 Polynomials on $\mathbf{R}^2$

Of course,

$$z = x_1 + i\, x_2, \ \overline{z} = x_1 - i\, x_2 \tag{2.10.1}$$

are homogeneous polynomials of degree 1 with complex coefficients on $\mathbf{R}^2$. We also have that

$$x_1 = (1/2)\,(z + \overline{z}), \ x_2 = (-i/2)\,(z - \overline{z}). \tag{2.10.2}$$

This means that every polynomial on $\mathbf{R}^2$ with complex coefficients corresponds to a polynomial in z and $\overline{z}$ with complex coefficients, and that every polynomial in z and $\overline{z}$ with complex coefficients determines a polynomial in x_1, x_2 with complex coefficients. More precisely, homogeneous polynomials in x_1, x_2 correspond to homogeneous polynomials in z, $\overline{z}$ of the same degree in this way.

2.10.1 The Laplacian of $z^j\,\overline{z}^l$

Let $\partial/\partial z$ and $\partial/\partial \overline{z}$ be as in Section 2.2, and remember that $(\partial/\partial z)(z) = (\partial/\partial \overline{z})(\overline{z}) = 1$ and $(\partial/\partial z)(\overline{z}) = (\partial/\partial \overline{z})(z) = 0$. If j is a positive integer, then it follows that

$$\frac{\partial}{\partial z}(\overline{z}^j) = \frac{\partial}{\partial \overline{z}}(z^j) = 0, \tag{2.10.3}$$

by the product rules for these operators. Similarly,

$$\frac{\partial}{\partial z}(z^j) = j\, z^{j-1} \tag{2.10.4}$$

and

$$\frac{\partial}{\partial \overline{z}}(\overline{z}^j) = j\, \overline{z}^{j-1}. \tag{2.10.5}$$

If l is another positive integer, then we get that

$$\Delta(z^j\,\overline{z}^l) = 4\,\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}(z^j\,\overline{z}^l) = 4\Big(\frac{\partial}{\partial z}(z^j)\Big)\Big(\frac{\partial}{\partial \overline{z}}(\overline{z}^l)\Big) = 4\,j\,l\,z^{j-1}\,\overline{z}^{l-1}. \tag{2.10.6}$$

Note that $z^j\,\overline{z}^l$ is harmonic when j or l is equal to 0.

2.10.2 Homogeneous polynomials on $\mathbf{R}^2$

If k is a nonnegative integer, then a homogeneous polynomial of degree k on $\mathbf{R}^2$ with complex coefficients may be expressed as

$$\sum_{j=0}^{k} c_j\,z^j\,\overline{z}^{k-j} \tag{2.10.7}$$

for some complex coefficients c_j, $0 \le j \le k$. If $j \le k/2$, then

$$z^j\,\overline{z}^{k-j} = |z|^{2j}\,\overline{z}^{k-2\,j}. \tag{2.10.8}$$

If $j \ge k/2$, then

$$z^j\,\overline{z}^{k-j} = z^{2\,j-k}\,|z|^{2\,k-2\,j} \tag{2.10.9}$$

If

$$|z|^2 = x_1^2 + x_2^2 = 1, \tag{2.10.10}$$

then we get that

$$z^j\,\overline{z}^{k-j} = \overline{z}^{k-2\,j} \tag{2.10.11}$$

when $j \le k/2$, and that

$$z^j\,\overline{z}^{k-j} = z^{2\,j-k} \tag{2.10.12}$$

when $j \ge k/2$. It follows that there is a harmonic polynomial on $\mathbf{R}^2$ of degree less than or equal to k that is equal to (2.10.7) on the unit circle.

Using this, it is easy to see that every polynomial on $\mathbf{R}^2$ agrees with a harmonic polynomial on the unit circle. This corresponds to some remarks on p138 of [297].

2.10.3 The Dirichlet problem

Let U be a nonempty bounded open subset of $\mathbf{R}^n$ for some positive integer n. If f is a continuous real or complex-valued function on ∂U, then the *Dirichlet problem* asks one to find a continuous real or complex-valued function u on $\overline{U}$, as appropriate, such that

$$u = f \text{ on } \partial U, \tag{2.10.13}$$

and u is harmonic on U.

The remarks in the previous subsection show that if $n = 2$, U is the open unit disk in $\mathbf{R}^2$, and f is the restriction to the unit circle of a polynomial on $\mathbf{R}^2$, then one can take u to be the restriction to $\overline{U}$ of a harmonic polynomial on $\mathbf{R}^2$.

2.11 Poisson's equation

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. If f is a real or complex-valued function on U, then one might like to find a real or complex-valued function u on U, as appropriate, such that

$$\Delta u = f \tag{2.11.1}$$

on U. This is *Poisson's equation*, as on p193 of [18], and p20 of [81].

Of course, one might like u to be twice continuously-differentiable on U, which would mean that f should be continuous on U. There are extended formulations of the equation, which allow for less regularity. One may also be interested in additional boundary conditions on u.

If f is a homogeneous polynomial of degree $k \geq 0$ on $\mathbf{R}^2$, then one can find a homogeneous polynomial u of degree $k+2$ on $\mathbf{R}^2$ that satisfies (2.11.1) on $\mathbf{R}^2$, using (2.10.6). It follows that if f is any polynomial on $\mathbf{R}^2$, then one can find a polynomial u on $\mathbf{R}^2$ that satisfies (2.11.1).

2.11.1 Dirichlet boundary conditions

Let g be a real or complex-valued function on ∂U. Another version of the *Dirichlet problem* asks one to find a real or complex-valued function u on $\overline{U}$, as appropriate, such that (2.11.1) holds on U and

$$u = g \text{ on } \partial U, \tag{2.11.2}$$

as in Section C of Chapter 2 of [87]. One might like u to be continuous on $\overline{U}$, so that g should be continuous on ∂U. There are extended versions of this too.

The case where

$$u = 0 \text{ on } \partial U \tag{2.11.3}$$

is known as *Dirichlet boundary conditions*. If one can solve Poisson's equation (2.11.1) without restrictions on u on ∂U, and if one can solve the Dirchlet problem for harmonic functions on U with arbitrary boundary values, then one can get a solution to Poisson's equation on U with prescribed boundary values. Similarly, if one can solve Poisson's equation U with Dirichlet boundary conditions, then one can try to use that to solve the Dirichlet problem for harmonic functions on U.

2.12 An interesting inner product

Let n be a positive integer, and let p, q be polynomials on $\mathbf{R}^n$ with complex coefficients. Note that the complex conjugate $\overline{q}$ of q is a polynomial on $\mathbf{R}^n$ too. Put

$$\langle p, q\rangle = \langle p, q\rangle_{\mathcal{P}(\mathbf{R}^n, \mathbf{C})} = (p(\partial)(\overline{q}))(0), \tag{2.12.1}$$

where $p(\partial)$ is as in Section 1.7.

If p and q are homogeneous polynomials of the same degree k, then $p(\partial)(\overline{q})$ is a constant, and (2.12.1) is the same as

$$\langle p, q \rangle = \langle p, q \rangle_{\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})} = p(\partial)(\overline{q}). \tag{2.12.2}$$

This is the definition that is used in the proof of Proposition 2.47 in Section G of Chapter 2 of [87], on p175 of [182], on p69 of [291], and on p139 of [297]. If p and q are homogeneous polynomials of different degrees, then it is easy to see that

$$\langle p, q \rangle = 0. \tag{2.12.3}$$

More precisely, if α, β are multi-indices, then

$$\begin{aligned} \langle x^\alpha, x^\beta \rangle &= \alpha! \quad \text{when } \alpha = \beta \\ &= 0 \quad \text{when } \alpha \neq \beta. \end{aligned} \tag{2.12.4}$$

Using (2.12.4), one can check that

$$\langle p, q \rangle = \overline{\langle q, p \rangle} \tag{2.12.5}$$

for all polynomials p, q on $\mathbf{R}^n$ with complex coefficients. Of course, (2.12.1) is linear in p over the complex numbers, and conjugate-linear in q. If $p(x) = \sum_{|\alpha| \le N} a_\alpha \, x^\alpha$ for some nonnegative integer N and complex coefficients a_α, then

$$\langle p, p \rangle = \sum_{|\alpha| \le N} |a_\alpha|^2 \, \alpha!. \tag{2.12.6}$$

In particular, this is strictly positive, except when $p = 0$. It follows that

$$\text{(2.12.1) defines an inner product on } \mathcal{P}(\mathbf{R}^n, \mathbf{C}), \tag{2.12.7}$$

as a vector space over the complex numbers, as in the proof of Proposition 2.47 in Section G of Chapter 2 of [87], and on p176 of [182], p69 of [291], and p139 of [297].

2.12.1 Laplacians of polynomials

Note that the Laplacian maps $\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$ into $\mathcal{P}_{k-2}(\mathbf{R}^n, \mathbf{C})$ for every integer $k \ge 2$. Let us use this inner product to show that

$$\Delta(\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})) = \mathcal{P}_{k-2}(\mathbf{R}^n, \mathbf{C}) \tag{2.12.8}$$

when $k \ge 2$, as in the proof of Proposition 2.47 in Section G of Chapter 2 of [87], and of Theorem 2.1 on p139 of [297]. Suppose that $q \in \mathcal{P}_{k-2}(\mathbf{R}^n, \mathbf{C})$ is orthogonal to every element of $\Delta(\mathcal{P}_k(\mathbf{R}^n, \mathbf{C}))$ with respect to this inner product, so that

$$\langle q, \Delta(p) \rangle = 0 \tag{2.12.9}$$

for every $p \in \mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$. This means that

$$q(\partial)(\Delta(\overline{p})) = 0, \tag{2.12.10}$$

because $\overline{\Delta(p)} = \Delta(\overline{p})$. This is the same as saying that

$$\Delta(q(\partial)(\overline{p})) = 0. \tag{2.12.11}$$

If we take $p(x) = |x|^2 \, q(x)$, then we get that

$$\langle p, p \rangle = p(\partial)(\overline{p}) = \Delta(q(\partial)(\overline{p})) = 0. \tag{2.12.12}$$

This implies that $p = 0$, as before. This means that $q = 0$, because of the way that we chose p. It follows that (2.12.8) holds, by standard arguments in linear algebra. This also uses the fact that $\mathcal{P}_{k-2}(\mathbf{R}^n, \mathbf{C})$ has finite dimension, as a vector space over $\mathbf{C}$.

2.13 An orthogonality argument

Let us continue with the same notation as in the previous section. If k is any nonnegative integer, then let

$$\mathcal{A}_k = \{p \in \mathcal{P}_k(\mathbf{R}^n, \mathbf{C}) : \Delta(p) = 0\} \tag{2.13.1}$$

be the space of homogeneous polynomials on $\mathbf{R}^n$ of degree k with complex coefficients that are harmonic, which is a linear subspace of $\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$. Of course, this is the same as $\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$ when $k = 0$ or 1. If $k \ge 2$, then put

$$\mathcal{B}_k = \{|x|^2 \, q(x) : q \in \mathcal{P}_{k-2}(\mathbf{R}^n, \mathbf{C})\}, \tag{2.13.2}$$

which is also a linear subspace of $\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$.

Let $k \ge 2$ and $p \in \mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$ be given, and put

$$r_q(x) = |x|^2 \, q(x) \tag{2.13.3}$$

for every $q \in \mathcal{P}_{k-2}(\mathbf{R}^n, \mathbf{C})$. Thus $r_q \in \mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$, and

$$\langle r_q, p \rangle = r_q(\partial)(\overline{p}) = q(\partial)(\Delta(\overline{p})) = \langle q, \Delta(p) \rangle. \tag{2.13.4}$$

Observe that

$$\langle q, \Delta(p) \rangle = 0 \tag{2.13.5}$$

for every $q \in \mathcal{P}_{k-2}(\mathbf{R}^n, \mathbf{C})$ if and only if $\Delta(p) = 0$. It follows that

$$\langle r_q, p \rangle = 0 \tag{2.13.6}$$

for every $q \in \mathcal{P}_{k-2}(\mathbf{R}^n, \mathbf{C})$ if and only if $\Delta(p) = 0$. This means that $\mathcal{A}_k$ is the orthogonal complement of $\mathcal{B}_k$ in $\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$ with respect to this inner product, as in the proof of Proposition 2.47 in Section G of Chapter 2 of [87], on p69 of [291], and p140 of [297].

This implies that

(2.13.7) every element of $\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$ can be expressed in a unique way as a sum of elements of $\mathcal{A}_k$ and $\mathcal{B}_k$,

by standard arguments in linear algebra. More precisely, this uses the fact that $\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$ is a finite-dimensional vector space over $\mathbf{C}$. This also corresponds to Proposition 5.5 on p76 of [18].

2.13.1 Repeating the process

We can repeat the process, to get that every element of $\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$ can be expressed as

$$\sum_{j=0}^{l} |x|^{2j}\, p_j(x), \tag{2.13.8}$$

where $2l \le k$, and $p_j \in \mathcal{P}_{k-2j}(\mathbf{R}^n, \mathbf{C})$ is harmonic for each $j = 1, \dots, l$. This corresponds to Theorem 5.7 on p77 of [18], Corollary 2.48 in Section G of Chapter 2 of [87], Proposition 4.1.1 on p176 of [182], some remarks on p70 of [291], and Theorem 2.1 on p139 of [297].

One can use this to get that every polynomial on $\mathbf{R}^n$ agrees with a harmonic polynomial on the unit sphere, as in some remarks on p77 of [18], Corollary 2.50 in Section G of Chapter 2 of [87], Corollary 4.1.2 on p177 of [182], mentioned on p70 of [291], and Corollary 2.2 on p140 of [297]. This corresponds to the Dirichlet problem on the open unit ball in $\mathbf{R}^n$, for the restriction to the unit sphere of a polynomial on $\mathbf{R}^n$.

2.14 The binomial theorem

If m is a positive integer and x, y are real or complex numbers, then the *binomial theorem* states that

$$(x+y)^m = \sum_{j=0}^{m} \binom{m}{j} x^j\, y^{m-j}, \tag{2.14.1}$$

where

$$\binom{m}{j} = \frac{m!}{j!\,(m-j)!} \tag{2.14.2}$$

is the usual *binomial coefficient* for each $j = 0, 1, \dots, m$. Note that

$$\binom{m}{j} = \binom{m}{m-j} \tag{2.14.3}$$

for $j = 0, 1, \ldots, m$. If we take $y = 1$ in (2.14.1), then we get that

$$(x+1)^m = \sum_{j=0}^{m} \binom{m}{j} x^j. \tag{2.14.4}$$

Conversely, (2.14.1) can be obtained from (2.14.4) by replacing x with x/y when $y \neq 0$.

Of course, it is easy to see that $(x+1)^m$ can be expressed as a sum of positive integer multiples of x^j, $0 \leq j \leq m$. To get that the multiples are given by binomial coefficients as before, one can look at the jth derivative of $(x+1)^m$ at 0 for each $j = 0, 1, \ldots, m$. In particular, this shows that the binomial coefficients are positive integers. Alternatively, one can verify (2.14.4) more directly, using induction on m.

One can expand $(x+1)^m$ into a sum of 2^m terms, each of which is a product of m factors, where every factor is equal to x or to 1. The coefficient of x^j in $(x+1)^m$ is the same as the number of these terms with exactly j factors of x, and $m - j$ factors of 1. This is also the same as the number of subsets of $\{1, \ldots, m\}$ with exactly j elements.

2.14.1 Multi-indices of order k

Let k and n be positive integers. It is well known that the number of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with order $|\alpha| = k$ is equal to

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}. \tag{2.14.5}$$

This corresponds to Problem 2 in Section 1.5 of [81]. Equivalently, this is the dimension of the spaces $\mathcal{P}_k(\mathbf{R}^n, \mathbf{R})$, $\mathcal{P}_k(\mathbf{R}^n, \mathbf{C})$ of homogeneous polynomials of degree k on $\mathbf{R}^n$ with real or complex coefficients, as vector spaces over $\mathbf{R}$ or $\mathbf{C}$, as appropriate. This is mentioned on p78 of [18], in Proposition 2.52 of Section G of Chapter 2 of [87], on p174f of [182], and on p139 of [297].

2.14.2 The multinomial theorem

If $x \in \mathbf{R}^n$ or $\mathbf{C}^n$, then the *multinomial theorem* states that

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha| = k} \binom{|\alpha|}{\alpha} x^\alpha, \tag{2.14.6}$$

as in Problem 3 in Section 1.5 of [81]. More precisely, the sum is taken over all multi-indices α with order $|\alpha| = k$, and we put

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}, \tag{2.14.7}$$

where $\alpha!$ is as in Section 1.10. One may refer to this as the *multinomial coefficient* associated to α. Note that (2.14.6) is trivial when $n = 1$, and that the $n = 2$ case is the same as the binomial theorem.

2.14.3 Another interesting identity

Let α be a multi-index, and let $x, y \in \mathbf{R}^n$ or $\mathbf{C}^n$ be given. One can check that

$$(x+y)^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!}\, x^\beta\, y^\gamma, \tag{2.14.8}$$

where the sum is taken over all multi-indices β, γ with $\beta + \gamma = \alpha$. This is the same as the binomial theorem when $n = 1$.

If $p(x)$ is a polynomial in $x_1, \ldots, x_n$ with real or complex coefficients and $b \in \mathbf{R}^n$ or $\mathbf{C}^n$, then $p(x+b)$ can be expressed as a polynomial in x with real or complex coefficients, as appropriate, as in Section 2.5. One can use (2.14.8) to get a more precise version of this.

2.15 Leibniz' formula

Let n be a positive integer, and let α, β be multi-indices. If

$$\beta_j \le \alpha_j \tag{2.15.1}$$

for each $j = 1, \ldots, n$, then put

$$\beta \le \alpha, \tag{2.15.2}$$

as in Problem 4 in Section 1.5 of [81]. Equivalently, this means that $\alpha - \beta$ is a multi-index too. In this case, we put

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!}, \tag{2.15.3}$$

as in [81].

Let u, v be smooth real-valued functions on $\mathbf{R}^n$. *Leibniz' formula* states that

$$\partial^\alpha(u\,v) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\partial^\beta u)\,(\partial^{\alpha-\beta} v), \tag{2.15.4}$$

as in Problem 4 in Section 1.5 of [81]. More precisely, the sum is taken over all multi-indices β with $\beta \le \alpha$. Of course, this also works when u, v are $|\alpha|$-times continuously-differentiable real or complex-valued functions on a nonempty open subset of $\mathbf{R}^n$. If $|\alpha| = 1$, then this reduces to the usual product rule for partial derivatives.

2.15.1 More on composing differential operators

Let U be a nonempty open subset of $\mathbf{R}^n$, and let N, $\widetilde{N}$ be nonnegative integers. Suppose that for each multi-index α with order $|\alpha| \le N$, a_α is a real or complex-valued function on U. This permits us to define the corresponding differential operator

$$L = \sum_{|\alpha|\le N} a_\alpha\, \partial^\alpha, \tag{2.15.5}$$

as in Section 2.4. Similarly, suppose that b_β is a real or complex-valued function on U for each multi-index β with $|\beta| \le \widetilde{N}$. This permits us to define the differential operator

$$\widetilde{L} = \sum_{|\beta| \le \widetilde{N}} b_\beta \, \partial^\beta, \tag{2.15.6}$$

as before.

Suppose that b_β is N-times continuously differentiable on U for each multi-index β with $|\beta| \le \widetilde{N}$. If u is an $(N+\widetilde{N})$-times continuously-differentiable real or complex-valued function on U, then $\widetilde{L}(u)$ is N-times continuously differentiable on U, so that $L(\widetilde{L}(u))$ is defined as a real or complex-valued function on U, as appropriate, as in Subsection 2.4.1. In fact, this can be expressed as $\widehat{L}(u)$, where

$$\widehat{L} = \sum_{|\gamma| \le N+\widetilde{N}} c_\gamma \, \partial^\gamma, \tag{2.15.7}$$

and c_γ is a real or complex-valued function on U for every multi-index γ with $|\gamma| \le N + \widetilde{N}$, as before. Remember that the c_γ's can be expressed in terms of sums of products of the a_α's with the b_β's and their derivatives of order less than or equal to N. This can be described more precisely using (2.15.4).

Chapter 3

Some integrals and other matters

3.1 Eigenfunctions of differential operators

Let n be a positive integer, and let U be a nonempty open set in $\mathbf{R}^n$. Also let N be a positive integer, and let a_α be a complex-valued function on U for each multi-index α with order $|\alpha| \le N$. If f is an N-times continuously-differentiable complex-valued function on U, then put

$$L(f) = \sum_{|\alpha| \le N} a_\alpha \, \partial^\alpha f \tag{3.1.1}$$

on U, as in Section 2.4.

We say that f is an *eigenfunction* for L with *eigenvalue* $\lambda \in \mathbf{C}$ if

$$L(f) = \lambda \, f \tag{3.1.2}$$

on U. One may wish to ask that f satisfy additional boundary conditions or other restrictions, depending on the circumstances.

3.1.1 A related partial differential equation

Let us identify $\mathbf{R}^n \times \mathbf{R}$ with $\mathbf{R}^{n+1}$, so that

$$V = U \times \mathbf{R} \tag{3.1.3}$$

may be considered as an open subset of $\mathbf{R}^{n+1}$. Put

$$u(x, t) = \exp(\lambda \, t) \, f(x) \tag{3.1.4}$$

for every $x \in U$ and $t \in \mathbf{R}$, which defines an N-times continuously-differentiable complex-valued function on V. It is easy to see that

$$\frac{\partial u}{\partial t} = L(u) \tag{3.1.5}$$

on V. We also have that

$$u(x,0) = f(x) \tag{3.1.6}$$

for every $x \in U$.

3.1.2 Two derivatives in t

Suppose that $\mu \in \mathbf{C}$ satisfies

$$\mu^2 = \lambda, \tag{3.1.7}$$

and put

$$v(x,t) = \exp(\mu\, t)\, f(x) \tag{3.1.8}$$

for every $x \in U$ and $t \in \mathbf{R}$. Observe that v and all of its partial derivatives in t are N-times continuously-differentiable on V, and that

$$\frac{\partial^2 v}{\partial t^2} = L(v) \tag{3.1.9}$$

on V. In addition,

$$v(x,0) = f(x) \tag{3.1.10}$$

and

$$\frac{\partial v}{\partial t}(x,0) = \mu\, f(x) \tag{3.1.11}$$

for every $x \in U$.

Similarly,

$$w(x,t) = \exp(-\mu\, t)\, f(x) \tag{3.1.12}$$

and all of its partial derivatives in t are N-times continuously-differentiable on V. As before,

$$\frac{\partial^2 w}{\partial t^2} = L(w) \tag{3.1.13}$$

on V, because $(-\mu)^2 = \lambda$ too. In this case,

$$w(x,0) = f(x) \tag{3.1.14}$$

and

$$\frac{\partial w}{\partial t}(x,0) = -\mu\, f(x) \tag{3.1.15}$$

for every $x \in U$.

Of course, one may consider multiple eigenfunctions of L on U, with possibly different eigenvalues, to get more solutions of partial differential equations like these on V. One may also consider infinite sums, under suitable conditions.

3.1.3 Hearing shapes of drums

Mark Kac' famous question of whether one can hear the shape of a drum involves eigenvalues for the Laplacian. See [55, 111, 112, 113, 162, 203] for more information.

3.2 The spherical Laplacian

Let n be a positive integer, and let

$$S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\} \tag{3.2.1}$$

be the unit sphere in $\mathbf{R}^n$, with respect to the standard Euclidean norm. If u is a complex-valued function on $\mathbf{R}^n \setminus \{0\}$ that is homogeneous of degree $b \in \mathbf{C}$, then

$$u(x) = |x|^b \, u(|x|^{-1} \, x) \tag{3.2.2}$$

for every $x \in \mathbf{R}^n \setminus \{0\}$, as in Subsection 2.8.1. In particular, this means that u is uniquely determined by its restriction to the unit sphere. Similarly, any real or complex-valued function on the unit sphere can be extended to a function on $\mathbf{R}^n \setminus \{0\}$ that is homogeneous of any given degree in $\mathbf{C}$.

Suppose that $n \geq 2$, and let u be a twice continuously-differentiable real or complex-valued function on $\mathbf{R}^n \setminus \{0\}$ that is homogeneous of degree 0. The restriction of u to the unit sphere may be considered as a twice continuously-differentiable function on S^{n-1}. Smoothness of functions on S^{n-1} can be defined in terms of suitable local coordinates, but it is more convenient for us to look at it here in terms of smoothness of homogeneous extensions to $\mathbf{R}^n \setminus \{0\}$.

3.2.1 Defining the spherical Laplacian

The *spherical Laplacian* of u is the function $\Delta_S u$ defined on S^{n-1} by

$$\Delta_S u = \Delta u \text{ on } S^{n-1}. \tag{3.2.3}$$

Note that Δu is homogeneous of degree -2 on $\mathbf{R}^n \setminus \{0\}$, as in Subsection 2.8.2. Thus

$$|x|^2 \, (\Delta u)(x) \tag{3.2.4}$$

is homogeneous of degree 0 on $\mathbf{R}^n \setminus \{0\}$. Of course, this is the same as (3.2.3) on S^{n-1}.

Now let v be a twice continuously-differentiable complex-valued function on $\mathbf{R}^n \setminus \{0\}$ that is homogeneous of degree $b \in \mathbf{C}$. Observe that

$$|x|^{-b} \, v(x) \tag{3.2.5}$$

is a twice continuously-differentiable function on $\mathbf{R}^n \setminus \{0\}$ that is homogeneous of degree 0 and equal to v on S^{n-1}. The spherical Laplacian of the restriction of v to S^{n-1} is

$$(\Delta_S v)(x) = \Delta(|x|^{-b} \, v(x)) \text{ on } S^{n-1}. \tag{3.2.6}$$

3.2.2 Some eigenfunctions

Suppose that p is a homogeneous polynomial of degree $k \geq 0$ on $\mathbf{R}^n$, and that p is harmonic on $\mathbf{R}^n$. It is well known that the spherical Laplacian of the restriction of p to S^{n-1} satisfies

$$\Delta_S p = -k \, (k + n - 2) \, p \text{ on } S^{n-1}, \tag{3.2.7}$$

as in Lemma 2.61 in Section G of Chapter 2 of [87], and on p70 of [291].

3.3 Connected components

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. It is well known that U can be expressed in a unique way as a union of a family of pairwise-disjoint nonempty connected open subsets of $\mathbf{R}^n$. These nonempty connected open sets are called the *connected components* of U. If U is connected, then U is the only connected component of itself.

In fact, one can define the notion of connected components for any subset E of $\mathbf{R}^n$, as well as subsets of arbitrary metric spaces or topological spaces. One can show that the connected components of E are relatively closed in E, but they are not necessarily relatively open in E.

The connected component of E that contains a point $x \in E$ can be obtained by taking the union of all of the connected subsets of E that contain x. One can verify that this is a connected set too. By construction, this is the largest possible connected subset of E that contains x.

Connected components of open subsets of $\mathbf{R}^n$ are open sets, basically because $\mathbf{R}^n$ is locally connected. This follows from the connectedness of open balls in $\mathbf{R}^n$.

More precisely, $\mathbf{R}^n$ is locally path connected, because every point in an open ball in $\mathbf{R}^n$ can be connected to the center of the ball by a line segment, which is contained in the ball. Because of this, the connected components of U are the same as the *path-connected components*. These can be defined by saying that $x, y \in U$ are in the same path-connected component of U when there is a continuous path in U connecting x and y.

3.3.1 Some properties of connected components

If V is a connected component of U, then it is easy to see that

$$\overline{V} \subseteq \overline{U}, \tag{3.3.1}$$

because $V \subseteq U$. In particular, this implies that

$$\partial V \subseteq \overline{U}. \tag{3.3.2}$$

One can check that

$$\partial V \subseteq \partial U \tag{3.3.3}$$

under these conditions. Otherwise, if there is an element of ∂V in U, then one can show that that point should be in V, to get a contradiction.

Suppose that $U \neq \mathbf{R}^n$, so that $V \neq \mathbf{R}^n$. This implies that $\partial V \neq \emptyset$, and thus that ∂V has an element in ∂U.

3.4 Smoothness near the boundary

Let n be a positive integer, and let V be a nonempty open subset of $\mathbf{R}^n$. We shall sometimes be concerned with smoothness properties of functions on $\overline{V}$, including on the boundary. Suppose that U is an open subset of $\mathbf{R}^n$ with

$$\overline{V} \subseteq U. \tag{3.4.1}$$

If u is a function on U with some smoothness property, then the restriction of u to $\overline{V}$ may be considered as having that property on $\overline{V}$.

However, we may also be concerned with functions that are defined only on $\overline{V}$. If k is a positive integer, then we let $C^k(\overline{V}, \mathbf{R})$, $C^k(\overline{V}, \mathbf{C})$ be the spaces of k-times continuously-differentiable real or complex-valued functions u on V, respectively, such that u and all of its derivatives $\partial^\alpha u$ with $|\alpha| \leq k$ can be extended continuously to $\overline{V}$, as in Section A of Chapter 0 of [87]. A continuous extension of a function on V to $\overline{V}$ is unique when it exists, by standard arguments, and so we may consider u and its derivatives of order less than or equal to k as being defined on $\overline{V}$ in this case.

This is initially defined a bit differently in Appendix A.3 of [81], where one considers k-times continuously-differentiable functions u on V such that u and its derivatives $\partial^\alpha u$ with $|\alpha| \leq k$ are uniformly continuous on every bounded subset of V. It is well known that continuous functions on compact sets are uniformly continuous, which implies that continuous functions of $\overline{V}$ are uniformly continuous on bounded subsets of $\overline{V}$. Conversely, if a function on V is uniformly continuous on all bounded subsets of V, then it is well known and not too difficult to show that the function has a continuous extension to $\overline{V}$.

If m is a positive integer, then we may also be concerned with continuity or smoothness properties of functions with values in $\mathbf{R}^m$ or $\mathbf{C}^m$. Such a function may be considered as an m-tuple of real or complex-valued functions, and the continuity or smoothness properties of the function are equivalent to the analogous properties holding for each of the corresponding m components.

We may be concerned with smoothness properties of the boundary of V as well. Properties like these are discussed in Appendix C.1 of [81], and Section B of Chapter 0 of [87].

3.5 The divergence theorem

Let $n \geq 2$ be an integer, although one could include $n = 1$, with suitable interpretations. Also let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary. Thus we may consider n-dimensional integrals over V, and surface integrals over ∂V, of suitable functions on V and ∂V, respectively.

Let w be a continuously-differentiable function on $\overline{V}$ with values in $\mathbf{R}^n$ or $\mathbf{C}^n$. The *divergence theorem* states that

$$\int_V \operatorname{div} w(x)\, dx = \int_{\partial V} w(y') \cdot \nu(y')\, dy', \tag{3.5.1}$$

where dy' is the element of surface area on ∂V, and $\nu(y')$ is the outward-pointing unit normal to ∂V at $y' \in \partial V$.

3.5.1 Using the divergence theorem

Let u be a twice continuously-differentiable real or complex-valued function on $\overline{V}$. If we take

$$w_j = \frac{\partial u}{\partial x_j} \tag{3.5.2}$$

for each $j = 1, \ldots, n$, then w defines a continuously-differentiable function on $\overline{V}$ with values in $\mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate. In this case, the divergence theorem implies that

$$\int_V (\Delta u)(x)\, dx = \int_{\partial V} (D_{v(y')} u)(y')\, dy', \tag{3.5.3}$$

where $D_{\nu(y')}$ denotes the directional derivative in the direction of $\nu(y')$. In particular, if u is also harmonic on V, then

$$\int_{\partial V} (D_{\nu(y')} u)(y')\, dy' = 0. \tag{3.5.4}$$

3.5.2 Using the divergence theorem again

Suppose that v is a continuously-differentiable real or complex-valued function on $\overline{V}$, as appropriate. Under these conditions,

$$\begin{aligned} \int_V (\Delta u)(x)\, v(x)\, dx + \int_V \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)\, \frac{\partial v}{\partial x_j}(x)\, dx \\ = \int_{\partial V} (D_{\nu(y')} u)(y')\, v(y')\, dy'. \end{aligned} \tag{3.5.5}$$

This follows from the divergence theorem, with

$$w_j(x) = v(x)\, \frac{\partial u}{\partial x_j}(x) \tag{3.5.6}$$

for each $j = 1, \ldots, n$. In particular, we can take $v = \overline{u}$, to get that

$$\int_V (\Delta u)(x)\, \overline{u(x)}\, dx + \int_V \sum_{j=1}^n \left|\frac{\partial u}{\partial x_j}(x)\right|^2 dx = \int_{\partial V} (D_{\nu(y')} u)(y')\, \overline{u(y')}\, dy'. \tag{3.5.7}$$

3.5.3 The Dirichlet integral

If u is any continuously-differentiable real or complex-valued function on $\overline{V}$, then

$$\int_V \sum_{j=1}^n \left|\frac{\partial u}{\partial x_j}(x)\right|^2 dx \tag{3.5.8}$$

is called the *Dirichlet integral* of u on V, as in Section E of Chapter 2 of [87], and Section 4 of Chapter 5 of [295]. This is equal to 0 exactly when all of the first partial derivatives of u are equal to 0 on V. This happens if and only if u is constant on each of the connected components of V.

3.6 Some consequences

Let n be a positive integer, and let V be a nonempty proper open subset of $\mathbf{R}^n$. Suppose that u is a continuous real or complex-valued function on $\overline{V}$ that satisfies Dirichlet boundary conditions, so that $u = 0$ on ∂V. If u is constant on any connected component of V, then it is easy to see that $u = 0$ on that component. If u is constant on every connected component of V, then it follows that $u = 0$ on V.

3.6.1 Using Dirichlet boundary conditions

Suppose now that V is bounded, with reasonably smooth boundary, and that u is twice continuously differentiable on $\overline{V}$. If u satisfies Dirichlet boundary conditions on $\overline{V}$, then (3.5.7) reduces to

$$\int_V (\Delta u)(x)\,\overline{u(x)}\,dx + \int_V \sum_{j=1}^n \left|\frac{\partial u}{\partial x_j}(x)\right|^2 dx = 0. \tag{3.6.1}$$

If u is harmonic on V, then it follows that $u = 0$ on V, as in the preceding paragraph. This corresponds to Theorem 16 in Section 2.2.5 a of [81]. The same conclusion could also be obtained using the maximum principle, as in Section 6.7.

3.6.2 Neumann boundary conditions

Let $\nu(y')$ be the outward-pointing unit normal to ∂V at $y' \in \partial V$, as in the previous section. If

$$(D_{\nu(y')}u)(y') = 0 \tag{3.6.2}$$

for every $y' \in \partial V$, then u is said to satisfy *Neumann boundary conditions* on $\overline{V}$. Note that (3.5.7) reduces to (3.6.1) in this case too. If u is harmonic on V, then this implies that u is constant on every connected component of V. This corresponds to part (a) of Problem 10 in Section 6.6 of [81] and Proposition 3.3 in Section A of Chapter 3 of [87], and is related to part (a) of Exercise 18 on p108 of [18].

Part (b) of Problem 10 in Section 6.6 of [81] asks one to show the same statement using the maximum principle, under suitable smoothness conditions on V. This is related to Exercise 27 on p29 of [18].

3.6.3 Eigenvalues of the Laplacian

Suppose that u is an eigenfunction for the Laplacian on V with eigenvalue $\lambda \in \mathbf{C}$, so that

$$\Delta u = \lambda\, u \tag{3.6.3}$$

on V. If u satisfies Dirichlet or Neumann boundary conditions on $\overline{V}$, then we get that

$$\lambda \int_V |u(x)|^2\, dx + \int_V \sum_{j=1}^n \left|\frac{\partial u}{\partial x_j}(x)\right|^2 dx = 0, \tag{3.6.4}$$

by (3.6.1). If $u \neq 0$ somewhere on V, then

$$\int_V |u(x)|^2\, dx > 0. \tag{3.6.5}$$

Under these conditions, we obtain that $\lambda \in \mathbf{R}$, and that $\lambda \leq 0$. More precisely, if u satisfies Dirichlet boundary conditions on $\overline{V}$, then we get that $\lambda < 0$.

3.7 Some more consequences

Let $n \geq 2$ be an integer, and let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary again. Also let u be a twice continuously-differentiable real or complex-valued function on $\overline{V}$, and let v be a continuously-differentiable real or complex-valued function on $\overline{V}$, as appropriate. Suppose that v satisfies Dirichlet boundary conditions on $\overline{V}$, so that

$$v(y') = 0 \text{ for every } y' \in \partial V. \tag{3.7.1}$$

In this case, (3.5.5) reduces to

$$\int_V (\Delta u)(x)\, v(x)\, dx + \int_V \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)\, \frac{\partial v}{\partial x_j}(x)\, dx = 0. \tag{3.7.2}$$

This means that

$$\int_V \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)\, \frac{\partial v}{\partial x_j}(x)\, dx = 0 \tag{3.7.3}$$

if and only if

$$\int_V (\Delta u)(x)\, v(x)\, dx = 0. \tag{3.7.4}$$

Of course, (3.7.4) holds when u is harmonic on V. Conversely, if (3.7.4) holds for every smooth function v on $\mathbf{R}^n$ with compact support contained in V, then one can check that u is harmonic on V, as follows.

3.7.1 Checking that u is harmonic on V

More precisely, suppose that u is a twice continuously-differentiable real or complex-valued function on V. If v is a continuous real or complex-valued function on $\mathbf{R}^n$ with compact support contained in V, then

$$(\Delta u)(x)\, v(x) \tag{3.7.5}$$

may be extended to a continuous function on $\mathbf{R}^n$ with support contained in the support of v, by setting it equal to 0 on $\mathbf{R}^n \setminus V$. Note that this implies that the integral of (3.7.5) over V may be defined in the usual way. If u is not harmonic on V, then there is a point $x_0 \in V$ such that

$$(\Delta u)(x_0) \neq 0. \tag{3.7.6}$$

One can find a nonnegative real-valued smooth function v_0 on $\mathbf{R}^n$ such that

$$v_0(x_0) > 0 \tag{3.7.7}$$

and the support of v_0 is contained in a ball centered at x_0 with arbitrarily small radius $r_0 > 0$, as in Section1.11. If r_0 is sufficiently small, then

(3.7.8) u is always positive or always negative on the support of v_0,

because u is continuous at x_0. One can also take r_0 small enough so that the support of v_0 is contained in V. It is easy to see that

$$\int_V (\Delta u)(x)\, v_0(x)\, dx \neq 0 \tag{3.7.9}$$

under these conditions. It follows that u is harmonic on V when (3.7.4) holds for all smooth functions v on $\mathbf{R}^n$ with compact support contained in V.

3.7.2 Another version of (3.7.2)

Suppose that u is a twice continuously-differentiable real or complex-valued function on V again, and now let v be a continuously-differentiable real or complex-valued function on $\mathbf{R}^n$ with compact support contained in V. In this case, for each $j = 1, \ldots, n$, we can define w_j as a continuously-differentiable real or complex-valued function on $\mathbf{R}^n$ with compact support contained in V, using (3.5.6) on V, and putting

$$w_j(x) = 0 \text{ when } x \in \mathbf{R}^n \setminus V. \tag{3.7.10}$$

Similarly, $(\Delta u)(x)\, v(x)$ and

$$\sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x)\, \frac{\partial v}{\partial x_j}(x) \tag{3.7.11}$$

can be extended to continuous real or complex-valued functions on $\mathbf{R}^n$ with compact support contained in V. One can use the divergence theorem to get that (3.7.2) holds in this case as well. Thus (3.7.3) is equivalent to (3.7.4) under these conditions too.

3.8 The Dirichlet principle

Let $n \geq 2$ be an integer, and let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary. Suppose that u and v are continuously-differentiable complex-valued functions on $\overline{V}$. Put

$$D(u,v) = \int_V \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)\, \overline{\frac{\partial v}{\partial x_j}(x)}\, dx, \tag{3.8.1}$$

as in Section E of Chapter 2 of [87]. It is easy to see that this is Hermitian symmetric, in the sense that

$$D(u,v) = \overline{D(v,u)}. \tag{3.8.2}$$

Of course, if u and v are real-valued, then $D(u,v)$ is a real number, and symmetric in u and v.

If $u = v$, then (3.8.1) is the same as the Dirichlet integral (3.5.8), which is a nonnegative real number. One might be interested in trying to minimize this quantity, under suitable conditions.

3.8.1 Computing $D(v,v)$ when $u = v$ on ∂V

We can express v as

$$v = u + (v-u), \tag{3.8.3}$$

to get that

$$\begin{aligned} D(v,v) &= D(u+(v-u), u+(v-u)) \\ &= D(u,u) + D(u,v-u) + D(v-u,u) + D(v-u,v-u) \\ &= D(u,u) + 2\,\mathrm{Re}\, D(u,v-u) + D(v-u,v-u). \end{aligned} \tag{3.8.4}$$

Suppose now that u is twice continuously differentiable on $\overline{V}$, and that

$$u = v \text{ on } \partial V. \tag{3.8.5}$$

This means that

$$v - u = 0 \text{ on } \partial V, \tag{3.8.6}$$

so that

$$D(u, v-u) = -\int_V (\Delta u)(x)\, \overline{(v(x) - u(x))}\, dx, \tag{3.8.7}$$

as in (3.7.2).

3.8.2 Minimizing the Dirichlet integral

If u is harmonic on V, then it follows that

$$D(u, v-u) = 0. \tag{3.8.8}$$

This implies that

$$D(v, v) = D(u, u) + D(v - u, v - u), \tag{3.8.9}$$

by (3.8.4). In particular, we get that

$$D(u, u) \le D(v, v), \tag{3.8.10}$$

which is part of the *Dirichlet principle*.

More precisely, equality holds in (3.8.10) if and only if

$$D(v - u, v - u) = 0. \tag{3.8.11}$$

This condition holds if and only if

$$u = v \text{ on } \overline{V}, \tag{3.8.12}$$

because of (3.8.5), as in the remarks in Subsection 3.5.3 and at the beginning of Section 3.6.

3.8.3 Minimizers are harmonic

Conversely, suppose that (3.8.10) holds whenever (3.8.5) holds. If $t \in \mathbf{C}$, then

$$w = u + t\,(v - u) \tag{3.8.13}$$

is another continuously-differentiable complex-valued function on $\overline{V}$, and

$$u = w \text{ on } \partial V, \tag{3.8.14}$$

by (3.8.5). This means that

$$D(u, u) \le D(w, w), \tag{3.8.15}$$

by hypothesis. Note that

$$D(w, w) = D(u, u) + 2\,\mathrm{Re}\,\overline{t}\,D(u, v - u) + |t|^2\,D(v - u, v - u), \tag{3.8.16}$$

as in (3.8.4), because

$$w - u = t\,(v - u). \tag{3.8.17}$$

One can use this and (3.8.15) to get that (3.8.8) holds, because $t \in \mathbf{C}$ is arbitrary. This is a bit simpler when u and v are real-valued, in which case one may as well take $t \in \mathbf{R}$. This implies that u is harmonic on U, because of (3.8.7), as in Subsection 3.7.1. This is another part of the Dirichlet principle. See also Section 4 of Chapter 5 of [295].

3.8.4 Minimizing sequences

Let f be a continuously-differentiable real or complex-valued function on $\overline{V}$, and let $\mathcal{E}_f$ be the collection of continuously-differentiable real or complex-valued functions on $\overline{V}$, as appropriate, such that

$$u = f \text{ on } \partial V. \tag{3.8.18}$$

Note that $f \in \mathcal{E}_f$, by construction. In connection with the Dirichlet principle, one may be interested in the infimum or greatest lower bound

$$\inf_{u \in \mathcal{E}_f} D(u, u) \tag{3.8.19}$$

of the Dirichlet integral over all of the elements of $\mathcal{E}_f$, and whether the infimum is attained.

It is well known and not too difficult to show that there is always a *minimizing sequence*, namely, a sequence $\{u_j\}_{j=1}^\infty$ of elements of $\mathcal{E}_f$ such that the corresponding sequence of Dirichlet integrals

$$D(u_j, u_j) \tag{3.8.20}$$

converges to the infimum (3.8.19). More precisely, one can choose u_j such that

$$\inf_{u \in \mathcal{E}_f} D(u, u) \le D(u_j, u_j) < \inf_{u \in \mathcal{E}_j} D(u, u) + 1/j \tag{3.8.21}$$

for each j.

3.9 Another helpful fact about integrals

Let $n \ge 2$ be an integer, and let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary. One could also include $n = 1$, with suitable interpretations, as before. If u, v are twice continuously-differentiable real or complex-valued functions on $\overline{V}$, then

$$\begin{aligned} \int_V (u(x)\,(\Delta v)(x) - v(x)\,(\Delta u)(x))\,dx \\ = \int_{\partial V} (u(y')\,(D_{\nu(y')} v)(y') - v(y')\,(D_{\nu(y')} u)(y'))\,dy'. \end{aligned} \tag{3.9.1}$$

Here $\nu(y')$ is the outward-pointing unit normal to ∂V at $y' \in \partial V$, and $D_{\nu(y')}$ denotes the directional derivative in the direction of $\nu(y')$, as usual. This can be obtained from the divergence theorem, with

$$w_j = u\,\frac{\partial v}{\partial x_j} - v\,\frac{\partial u}{\partial x_j} \tag{3.9.2}$$

for each $j = 1, \ldots, n$.

3.9.1 Dirichlet or Neumann boundary conditions

Suppose for the moment that u and v both satisfy Dirichlet boundary conditions on $\overline{V}$, so that

$$u = v = 0 \text{ on } \partial V. \tag{3.9.3}$$

In this case, (3.9.1) reduces to

$$\int_V (u(x)\,(\Delta v)(x) - v(x)\,(\Delta u)(x))\,dx = 0. \tag{3.9.4}$$

Similarly, suppose that u and v both satisfy Neumann boundary conditions on $\overline{V}$, which is to say that

$$(D_{\nu(y')}u)(y') = (D_{\nu(y')}v)(y') = 0 \tag{3.9.5}$$

for every $y' \in \partial V$. Clearly (3.9.1) reduces to (3.9.4) in this case as well.

3.9.2 Eigenfunctions for the Laplacian

Suppose now that u and v are eigenfunctions for the Laplacian on V with eigenvalues λ and μ, respectively. This means that

$$\Delta u = \lambda\, u \tag{3.9.6}$$

and

$$\Delta v = \mu\, v \tag{3.9.7}$$

on V. Suppose also that either u and v both satisfy Dirichlet boundary conditions on $\overline{V}$, or that they both satisfy Neumann boundary conditions on $\overline{V}$, so that (3.9.4) holds. This means that

$$(\mu - \lambda) \int_V u(x)\,v(x)\,dx = 0. \tag{3.9.8}$$

If $\lambda \neq \mu$, then it follows that

$$\int_V u(x)\,v(x)\,dx = 0. \tag{3.9.9}$$

3.10 Some remarks about zero sets

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let ϕ be a continuous real-valued function on U, and consider the corresponding zero set of ϕ in U,

$$\{x \in U : \phi(x) = 0\}. \tag{3.10.1}$$

This is a relatively closed set in U.

Suppose now that ϕ is continuously-differentiable on U. Let w be an element of U such that

$$\phi(w) = 0 \tag{3.10.2}$$

and

$$\frac{\partial \phi}{\partial x_l}(w) \neq 0 \tag{3.10.3}$$

for some $l \in \{1, \ldots, n\}$. Under these conditions, the *implicit function theorem* implies that near w, the zero set (3.10.1) can be represented as the graph of a continuously-differentiable real-valued function of the other variables x_j, $j \neq l$. Note that the implicit function theorem for real-valued functions can be shown more directly than for $\mathbf{R}^m$-valued functions with $m \geq 2$, as in Theorem 3.2.1 on p36 of [191].

One can also look at this in terms of the *inverse function theorem.* Consider the mapping Φ from U into $\mathbf{R}^n$ defined by

$$\Phi(x) = (x_1, \ldots, x_{l-1}, \phi(x), x_{l+1}, \ldots, x_n) \tag{3.10.4}$$

for each $x \in U$. Equivalently, the jth coordinate of $\Phi(x)$ is defined to be x_j when $j \neq l$, and to be $\phi(x)$ when $j = l$. This mapping is continuously differentiable on U, because ϕ is continuously differentiable on U, by hypothesis.

The differential of Φ at a point $x \in U$ is the linear mapping from $\mathbf{R}^n$ into itself that sends $v \in \mathbf{R}^n$ to the directional derivative $(D_v \Phi)(x)$ of Φ in the direction v at x. This linear mapping corresponds to the matrix of partial derivatives of the components of Φ. One can check that the differential of Φ at w is invertible as a linear mapping on $\mathbf{R}^n$, because of (3.10.3).

Under these conditions, the inverse function theorem implies that the restriction of Φ to a small neighborhood of w is invertible, where the inverse mapping is continuously differentiable too. Of course, the zero set (3.10.1) is the same as the inverse image of the $x_l = 0$ hyperplane under Φ.

3.11 The Neumann problem

Let n be a positive integer, and let U be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary. If $y' \in \partial U$, then we let $\nu(y')$ be the outward-pointing unit normal to ∂U at y', and we let $D_{\nu(y')}$ denote the directional derivative in the direction $\nu(y')$, as usual.

Let f be a real or complex-valued function on U, and let g be a real or complex-valued function on ∂U. A version of the *Neumann problem* asks one to find a real or complex-valued function u on $\overline{U}$, as appropriate, such that

$$\Delta u = f \tag{3.11.1}$$

on U, and

$$(D_{\nu(y')} u)(y') = g(y') \tag{3.11.2}$$

for every $y' \in \partial U$. This is discussed in Section C of Chapter 2 of [87].

3.11.1 Uniqueness for the Neumann problem

Of course, one could add a constant to u without affecting (3.11.1) or (3.11.2). More precisely, one could add a different constant to u on each connected component of U, without affecting these conditions.

If u is twice-continuously differentiable on $\overline{U}$, harmonic on U, and satisfies Neumann boundary conditions on $\overline{U}$, then u is constant on every connected component of U, as in Subsection 3.6.2. This implies an appropriate uniqueness result for the Neumann problem.

3.11.2 A necessary condition for the existence of solutions

Suppose that u is a twice continuously-differentiable real or complex-valued function on $\overline{U}$ that satisfies (3.11.1) and (3.11.2). If V is a connected component of U, then the restriction of u to $\overline{V}$ satisfies the analogous conditions there. It follows that

$$\int_V f(x)\,dx = \int_{\partial V} g(y')\,dy', \tag{3.11.3}$$

as in Subsection 3.5.1.

3.11.3 Two particular cases

One may be particularly concerned with the Neumann problem with $f = 0$ on U, which may be described as the Neumann problem for harmonic functions. Of course, (3.11.3) reduces to

$$\int_{\partial V} g(y')\,dy' = 0 \tag{3.11.4}$$

in this case. Alternatively, one may be particularly concerned with the case where $g = 0$ on ∂U, so that u satisfies Neumann boundary conditions on $\overline{U}$. In this case, (3.11.3) reduces to

$$\int_V f(x)\,dx = 0. \tag{3.11.5}$$

As with the Dirichlet problem, these two cases of the Neumann problem are related to each other. If one can solve the Poisson equation (3.11.1) without (3.11.2), then a solution to a Neumann problem for harmonic functions could be used to obtain (3.11.2). If one can solve the Poisson equation with Neumann boundary conditions, then one can use that to try to solve the Neumann problem for harmonic functions.

The Dirichlet and Neumann problems for harmonic functions are also discussed in Chapter 3 of [87]. Another approach is discussed in Chapter 7 of [87]. See also Problem 4 in Section 6.6 of [81].

3.12 The unit ball in $\mathbf{R}^n$

Let n be a positive integer, and let us consider the case where

$$U = B(0,1) \tag{3.12.1}$$

is the open unit ball in $\mathbf{R}^n$ in the previous section. If y' is an element of $\partial U = \partial B(0,1)$, which is the unit sphere in $\mathbf{R}^n$, then

$$\nu(y') = y' \tag{3.12.2}$$

is the outward-pointing unit normal to $\partial B(0,1)$ at y'. If u is a continuously-differentiable complex-valued function on $\overline{U} = \overline{B}(0,1)$, then

$$(D_{\nu(y')} u)(y') \tag{3.12.3}$$

is the same as the Euler operator applied to u at y', as in Section 2.8.

3.12.1 Homogeneous polynomials and normal derivatives

Suppose that p is a polynomial on $\mathbf{R}^n$ with complex coefficients that is homogeneous of degree k for some nonnegative integer k. If $y' \in \partial B(0,1)$, then

$$(D_{\nu(y')} p)(y') = k\, p(y'), \tag{3.12.4}$$

as in Subsection 2.8.1.

If $k \geq 1$, and

$$q = k^{-1}\, p, \tag{3.12.5}$$

then

$$(D_{\nu(y')} q)(y') = p(y'). \tag{3.12.6}$$

This may be considered as an instance of the Neumann problem for harmonic functions on $\overline{B}(0,1)$ when p is harmonic, so that q is harmonic as well.

If p is a harmonic polynomial on $\mathbf{R}^n$ that is homogeneous of degree $k \geq 1$, then

$$\int_{\partial B(0,1)} p(y')\, dy' = 0. \tag{3.12.7}$$

This follows from (3.5.4) and (3.12.4).

3.12.2 The Neumann problem for polynomials

If g is any polynomial on $\mathbf{R}^n$ with complex coefficients, then g agrees with a harmonic polynomial on $\partial B(0,1)$, as in Subsection 2.13.1. More precisely, g is equal to a sum of homogeneous harmonic polynomials on $\partial B(0,1)$, as before. If these homogeneous harmonic polynomials are all homogeneous of positive degree, then one can get a polynomial solution to the corresponding Neumann problem for harmonic functions on $\overline{B}(0,1)$, using (3.12.6).

If g is of this type, then

$$\int_{\partial B(0,1)} g(y')\,dy' = 0, \tag{3.12.8}$$

because of (3.12.7). Conversely, it is easy to see that g is of this type when (3.12.8) holds, because homogeneous polynomials of degree 0 are constants. This condition is necessary to have a solution of the Neumann problem for harmonic functions on $\overline{B}(0,1)$, as in (3.11.4). This is related to Exercise 18 on p108 of [18].

3.12.3 An orthogonality property

Let p_1, p_2 be harmonic polynomials on $\mathbf{R}^n$ with complex coefficients that are homogeneous of degrees $k_1, k_2 \ge 0$, respectively. Observe that

$$(k_1 - k_2) \int_{\partial B(0,1)} p_1(y')\,\overline{p_2(y')}\,dy' = 0, \tag{3.12.9}$$

because of (3.9.1) and (3.12.4). If $k_1 \neq k_2$, then we get that

$$\int_{\partial B(0,1)} p_1(y')\,\overline{p_2(y')}\,dy' = 0. \tag{3.12.10}$$

Note that this includes (3.12.7) as a particular case. This corresponds to Proposition 5.9 on p79 of [18], part of Theorem 2.51 in Section G of Chapter 2 of [87], Proposition 4.1.5 on p179 of [182], 3.1.1 on p69 of [291], and Corollary 2.4 on p141 of [297].

3.13 Some integrals over spheres

Let n be a positive integer, and let p be a polynomial on $\mathbf{R}^n$ with complex coefficients. If p is harmonic on $\mathbf{R}^n$ and homogeneous of degree $k \ge 1$, then

$$\int_{\partial B(0,r)} p(y')\,dy' = 0 \tag{3.13.1}$$

for every $r > 0$. This is the same as (3.12.7) when $r = 1$. Otherwise, one can reduce to that case using a change of variables, or use an analogous argument for any $r > 0$.

3.13.1 Homogeneous harmonic polynomials

If p is any polynomial on $\mathbf{R}^n$ of degree less than or equal to N for some nonnegative integer N, then p can be expressed in a unique way as a sum of homogeneous polynomials of degrees from 0 to N, as in Subsection 2.9.1. If p is a harmonic polynomial, then one can use this to get that

(3.13.2) p can be expressed as a sum of harmonic homogeneous polynomials of degrees from 0 to N.

This uses the fact that the Laplacian of a homogeneous polynomial of degree l is a homogeneous polynomial of degree $l-2$ when $l \geq 2$. Of course, the Laplacian of a homogeneous polynomial of degree l is 0 when $l = 0$ or 1.

3.13.2 A mean-value property

If p is a harmonic polynomial on $\mathbf{R}^n$, then

$$\frac{1}{|\partial B(0,r)|} \int_{\partial B(0,r)} p(y')\, dy' = p(0) \tag{3.13.3}$$

for every $r > 0$. Here $|\partial B(0,r)|$ denotes the $(n-1)$-dimensional surface area of $\partial B(0,r)$. More precisely, (3.13.3) follows from (3.13.1) when p is homogeneous of degree $k \geq 1$. If p is homogeneous of degree 0, and thus a constant, then (3.13.3) is clear. One can reduce to the case where p is homogeneous of some degree $k \geq 0$, using the remarks in the preceding paragraph.

It follows that

$$\frac{1}{|\partial B(a,r)|} \int_{\partial B(a,r)} p(y')\, dy' = p(a) \tag{3.13.4}$$

for every $a \in \mathbf{R}^n$ and $r > 0$ under these conditions. This uses the fact that $p(x+a)$ is also a harmonic polynomial in x on $\mathbf{R}^n$. If one replaces $p(x)$ with $p(x+a)$ in (3.13.3), then the result is the same as (3.13.4), using a translation by a to go from an integral over $\partial B(0,r)$ to an integral over $\partial B(a,r)$. Of course, the surface area $|\partial B(a,r)|$ of $\partial B(a,r)$ is the same as the surface area of $\partial B(0,r)$.

This is known as the *mean-value property* of p, which will be discuseed further in Chapter 6. Any harmonic function on an open subset of $\mathbf{R}^n$ has a suitable version of this property, as in Section 6.2. A twice continuously-differentiable function with the mean-value property is harmonic, as in Section 6.3. One can also use the mean-value property to get smoothness, as in Section 6.4.

3.14 Some remarks about compositions

Let W be a nonempty open subset of $\mathbf{R}^2$, and suppose that f is a continuously-differentiable complex-valued function on W. If $v \in \mathbf{R}^2$, then the directional derivative of f in the direction v is equal to

$$D_v f = v_1\, \partial_1 f + v_2\, \partial_2 f \tag{3.14.1}$$

on W, as mentioned in Subsection 1.3.2. If we identify v with the complex number $v_1 + i\, v_2$, then it is easy to see that

$$D_v f = v \, \frac{\partial f}{\partial z} + \overline{v} \, \frac{\partial f}{\partial \overline{z}}, \tag{3.14.2}$$

where $\partial f / \partial z$ and $\partial f / \partial \overline{z}$ are as in Section 2.2.

3.14.1 Some derivatives of compositions

Let n be a positive integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuously-differentiable complex-valued function on U. Suppose that

$$u(U) \subseteq W, \tag{3.14.3}$$

where W is considered as a subset of $\mathbf{C}$, so that the composition $f \circ u$ of u and f is defined as a complex-valued function on U. Suppose also that

$$f \text{ is holomorphic on } W. \tag{3.14.4}$$

Under these conditions, we get that

$$\frac{\partial}{\partial x_j}(f(u(x))) = f'(u(x)) \, \frac{\partial u}{\partial x_j}(x) \tag{3.14.5}$$

on U for each $j = 1, \ldots, n$, where $f' = \partial f / \partial z$ is the usual complex derivative of f. This is the same as the directional derivative of f at $u(x)$ in the direction $\partial_j u(x)$, which can be expressed as in (3.14.2).

3.14.2 The $n = 2$ case

If $n = 2$ and (3.14.4) holds, then one can check that

$$\frac{\partial}{\partial \overline{z}}(f \circ u) = (f' \circ u) \, \frac{\partial u}{\partial \overline{z}} \tag{3.14.6}$$

on U, using (3.14.5). Similarly,

$$\frac{\partial}{\partial z}(f \circ u) = (f' \circ u) \, \frac{\partial u}{\partial z} \tag{3.14.7}$$

on U. If

$$u \text{ is holomorphic on } U, \tag{3.14.8}$$

then it follows that

$$f \circ u \text{ is holomorphic on } U, \tag{3.14.9}$$

by (3.14.6). In this case, we also get that

$$(f \circ u)' = (f' \circ u) \, u' \tag{3.14.10}$$

on U, by (3.14.7).

3.14.3 Some more derivatives of compositions

It is well known that and not too difficult to show that

(3.14.11) the complex exponential function is holomorphic on $\mathbf{C}$,

with complex derivative equal to itself. If n is any positive integer again and u is any continuously-differentiable complex-valued function on U, then it follows that

$$\frac{\partial}{\partial x_j}(\exp u(x)) = (\exp u(x))\,\frac{\partial u}{\partial x_j}(x) \tag{3.14.12}$$

on U for each $j = 1, \ldots, n$, as in (3.14.5).

Suppose now that u is a continuously-differentiable real-valued function on U, and that W is an open subset of $\mathbf{R}$ that satisfies (3.14.3). If f is any continuously-differentiable complex-valued function on W, then (3.14.5) holds on U, by the usual chain rule.

3.15 More on first-order operators

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let $a_1, \ldots, a_n$ be n complex-valued functions on U, so that $a = (a_1, \ldots, a_n)$ may be considered as a mapping from U into $\mathbf{C}^n$. If u is a continuously-differentiable complex-valued function on U, then

$$L_a(u) = \sum_{j=1}^{n} a_j\,\frac{\partial u}{\partial x_j} \tag{3.15.1}$$

defines a complex-valued function on U, as in Section 2.3.

Let b be another complex-valued function on U, and put

$$L_{a,b}(u) = L_a(u) + b\,u. \tag{3.15.2}$$

This defines a differential operator on U, as in Section 2.4, with $N = 1$.

3.15.1 An auxiliary function c

If c is a continuously-differentiable complex-valued function on U, then

$$L_a(c\,u) = L_a(c)\,u + c\,L_a(u) \tag{3.15.3}$$

on U, as in Section 2.3. If $c(x) \neq 0$ for every $x \in U$, then we get that

$$c^{-1}\,L_a(c\,u) = L_a(u) + c^{-1}\,L_a(c)\,u \tag{3.15.4}$$

on U. If

$$b = c^{-1}\,L_a(c) \tag{3.15.5}$$

on U, then it follows that

$$L_{a,b}(u) = c^{-1}\, L_a(c\, u) \tag{3.15.6}$$

on U. Of course, (3.15.5) is the same as saying that

$$L_a(c) = b\, c \tag{3.15.7}$$

on U.

If γ is a continuously-differentiable complex-valued function on U, then

$$c(x) = \exp \gamma(x) \tag{3.15.8}$$

is a continuously-differentiable complex-valued function on U with $c(x) \neq 0$ for every $x \in U$. We also have that

$$c^{-1}\, L_a(c) = L_a(\gamma) \tag{3.15.9}$$

on U, as in (3.14.12). If

$$b = L_a(\gamma), \tag{3.15.10}$$

then (3.15.5) holds, so that (3.15.6) holds, as before.

3.15.2 Products of eigenfunctions of L_a

Suppose that u, v are continuously-differentiable complex-valued functions on U that are eigenfunctions for L_a, with eigenvalues $\lambda, \mu \in \mathbf{C}$, respectively. Observe that

$$L_a(u\, v) = L_a(u)\, v + u\, L_a(v) = \lambda\, u\, v + u\, (\mu\, v) = (\lambda + \mu)\, u\, v \tag{3.15.11}$$

on U, so that $u\, v$ is an eigenfunction for L_a with eigenvalue $\lambda + \mu$.

Chapter 4

First-order equations

4.1 Some real first-order operators

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let $a_1, \ldots, a_n$ be n real-valued functions on U. Alternatively,

$$a(x) = (a_1(x), \ldots, a_n(x)) \tag{4.1.1}$$

defines a mapping from U into $\mathbf{R}^n$.

If u is a continuously-differentiable real-valued function on U, then put

$$L_a(u) = \sum_{j=1}^{n} a_j \, \frac{\partial u}{\partial x_j}. \tag{4.1.2}$$

This defines a real-valued function on U. The value of this function at $x \in U$ is the directional derivative of u at x in the direction $a(x)$, as in Subsection 1.3.2.

4.1.1 Some related functions $w(t)$, $z(t)$

Let I be a nonempty open interval in the real line, or an open half-line, or the whole real line, and let $w(t)$ be a continuously-differentiable function of $t \in I$ with values in $\mathbf{R}^n$. Equivalently, this means that the jth component $w_j(t)$ of $w(t)$ is a continuously-differentiable real-valued function on I for each $j = 1, \ldots, n$. One could also allow I to contain one or both endpoints, with suitable interpretations using one-sided derivatives at the endpoints.

Suppose that

$$w(t) \in U \text{ for every } t \in I, \tag{4.1.3}$$

so that

$$z(t) = u(w(t)) \tag{4.1.4}$$

defines a real-valued function on I. It is well known that $z(t)$ is continuously differentiable on I under these conditions, with

$$z'(t) = \sum_{j=1}^{n} w_j'(t)\,(\partial_j u)(w(t)) \tag{4.1.5}$$

for every $t \in I$. This is the same as the directional derivative of u at $w(t)$ in the direction $w'(t)$.

4.1.2 A differential equation for $w(t)$

Suppose that

$$w_j'(t) = a_j(w(t)) \tag{4.1.6}$$

for each $j = 1, \ldots, n$ and $t \in I$. This is the same as saying that

$$w'(t) = a(w(t)) \tag{4.1.7}$$

for every $t \in I$, as elements of $\mathbf{R}^n$. In this case, we get that

$$z'(t) = \sum_{j=1}^{n} a_j(w(t))\,(\partial_j u)(w(t)) = (L_a(u))(w(t)) \tag{4.1.8}$$

for every $t \in I$.

Suppose for the moment that we also have that

$$L_a(u) = 0 \text{ on } U. \tag{4.1.9}$$

This implies that

$$z'(t) = 0 \tag{4.1.10}$$

for every $t \in I$, so that $z(t)$ is constant on I.

4.1.3 Semilinear first-order equations

Suppose now that u satisfies the semilinear equation

$$(L_a(u))(x) + b(u(x), x) = 0 \tag{4.1.11}$$

for some real-valued function b on $\mathbf{R} \times U$. Under these conditions, we get that

$$z'(t) + b(z(t), w(t)) = 0 \tag{4.1.12}$$

for every $t \in I$, because of (4.1.8).

The equations (4.1.7) and (4.1.12) are called the *characteristic equations* for (4.1.11). This is related to some remarks in Section 3.2.2 a of [81], and Section B of Chapter 1 of [87].

It is interesting to consider the case where a is a nonzero constant, as in Section 2.1 in [81].

4.2 Quasilinear first-order equations

Let n be a positive integer again, and let U be a nonempty open subset of $\mathbf{R}^n$. In this section, we let $a_1, \ldots, a_n$ and b be real-valued functions on $\mathbf{R} \times U$. Consider the quasi-linear first-order partial differential equation

$$\sum_{j=1}^{n} a_j(u(x), x) \, \frac{\partial u}{\partial x_j}(x) + b(u(x), x) = 0, \tag{4.2.1}$$

where u is a continuously-differentiable real-valued function on U.

4.2.1 Corresponding functions $w(t)$ and $z(t)$

Let I be an interval in the real line with nonempty interior, which may be unbounded, as in Subsection 4.1.1. Also let $w(t)$ be a continuously-differentiable function of $t \in I$ with values in $\mathbf{R}^n$, and in fact in U, as before. If u is a continuously-differentiable real-valued function on U, then

$$z(t) = u(w(t)) \tag{4.2.2}$$

is a continuously-differentiable real-valued function of $t \in I$, with derivative as in (4.1.5).

Suppose that

$$w_j'(t) = a_j(u(w(t)), w(t)) \tag{4.2.3}$$

for each $j = 1, \ldots, n$ and $t \in I$. If we consider $a = (a_1, \ldots, a_n)$ as an $\mathbf{R}^n$-valued function on $\mathbf{R} \times U$, then this is the same as saying that

$$w'(t) = a(u(w(t)), w(t)) \tag{4.2.4}$$

for every $t \in I$, as elements of $\mathbf{R}^n$. In this case,

$$z'(t) = \sum_{j=1}^{n} a_j(u(w(t)), w(t)) \, \frac{\partial u}{\partial x_j}(x) \tag{4.2.5}$$

for every $t \in I$, because of (4.1.5). If (4.2.1) holds, then we get that

$$z'(t) + b(z(t), w(t)) = 0 \tag{4.2.6}$$

for every $t \in I$.

Observe that (4.2.3) is the same as saying that

$$w_j'(t) = a_j(z(t), w(t)) \tag{4.2.7}$$

for each $j = 1, \ldots, n$ and $t \in I$. Equivalently, this means that

$$w'(t) = a(z(t), w(t)) \tag{4.2.8}$$

for every $t \in I$, as elements of $\mathbf{R}^n$. This together with (4.2.6) forms a coupled system of ordinary differential equations for $w(t)$ and $z(t)$ that does not depend on u. These are the *characteristic equations* for (4.2.1). This is related to some remarks in Section 3.2.2 b of [81], and in Section B of Chapter 1 of [87].

4.2.2 Comparison with the previous case

There is an important difference between this case and the one discussed in the previous section. It is well known that solutions of the initial value problem for systems of ordinary differential equations are unique under suitable conditions. This implies that different curves corresponding to solutions of (4.1.7) cannot cross each other, under suitable conditions. Although one also has uniqueness for the initial value problem for the system (4.2.6), (4.2.8) under suitable conditions, it is possible for the curves corresponding to the $w(t)$'s to cross each other. This corresponds to some remarks in Sections 3.2.5 a, b of [81].

4.3 Fully nonlinear first-order equations

Let n be a positive integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let

$$F(q, y, x) \tag{4.3.1}$$

be a real-valued function on

$$\mathbf{R}^n \times \mathbf{R} \times U. \tag{4.3.2}$$

Consider the fully nonlinear first-order partial differential equation

$$F(Du(x), u(x), x) = 0, \tag{4.3.3}$$

where u is a continuously-differentiable real-valued function on U.

4.3.1 The functions $w(t)$, $z(t)$, $p(t)$

As in the previous sections, we would like to find some systems of ordinary differential equations that are related to (4.3.3). Let I be an interval in the real line with nonempty interior, and which may be unbounded, and let $w(t)$ be a continuously-differentiable function of $t \in I$ with values in U again. Suppose that u is a continuously-differentiable real-valued function on U, so that

$$z(t) = u(w(t)) \tag{4.3.4}$$

is a continuously-differentiable real-valued function of $t \in I$, as before.

If $t \in I$, then let $p(t) \in \mathbf{R}^n$ be defined by

$$p(t) = Du(w(t)), \tag{4.3.5}$$

so that

$$p_j(t) = (\partial_j u)(w(t)) \tag{4.3.6}$$

for each $j = 1, \ldots, n$. We would like to find a nice system of ordinary differential equations for $w(t)$, $z(t)$, and $p(t)$ related to (4.3.3), as before. To do this, we suppose that u is twice continuously-differentiable on U, so that $p(t)$ is continuously differentiable on I. More precisely,

$$p_j'(t) = \sum_{l=1}^{n} w_l'(t)\, (\partial_j \partial_l u)(w(t)) \tag{4.3.7}$$

for every $j = 1, \ldots, n$ and $t \in I$.

4.3.2 Differentiating the equation

Suppose that u satisfies (4.3.3) on U, and that F is continuously differentiable on (4.3.2). If we differentiate the left side of (4.3.3) with respect to x_j, then we get that

$$\begin{aligned} (4.3.8) \qquad 0 &= \frac{\partial}{\partial x_j}(F(Du(x), u(x), x)) \\ &= \sum_{l=1}^{n} \frac{\partial F}{\partial q_l}(Du(x), u(x), x)\, \frac{\partial^2 u}{\partial x_j\, \partial x_l}(x) \\ &\quad + \frac{\partial F}{\partial y}(Du(x), u(x), x)\, \frac{\partial u}{\partial x_j}(x) + \frac{\partial F}{\partial x_j}(Du(x), u(x), x) \end{aligned}$$

on U. If $t \in I$, then we can take $x = w(t)$ and use the definition (4.3.6) of $p_j(t)$, to get that

$$\begin{aligned} (4.3.9) \qquad & \sum_{l=1}^{n} \frac{\partial F}{\partial q_l}(p(t), z(t), w(t))\, (\partial_j \partial_l u)(w(t)) + \frac{\partial F}{\partial y}(p(t), z(t), w(t))\, p_j(t) \\ & + \frac{\partial F}{\partial x_j}(p(t), z(t), w(t)) = 0 \end{aligned}$$

for each $j = 1, \ldots, n$.

4.3.3 The characteristic equations

Suppose that

$$(4.3.10) \qquad w_l'(t) = \frac{\partial F}{\partial q_l}(p(t), z(t), w(t))$$

for each $l = 1, \ldots, n$ and $t \in I$. If we substitute this into (4.3.7), then we get that

$$\begin{aligned} (4.3.11) \quad p_j'(t) &= \sum_{l=1}^{n} \frac{\partial F}{\partial q_l}(p(t), z(t), w(t))\, \frac{\partial^2 u}{\partial x_j\, \partial x_l}(w(t)) \\ &= -\frac{\partial F}{\partial y}(p(t), z(t), w(t))\, p_j(t) - \frac{\partial F}{\partial x_j}(p(t), z(t), w(t)) \end{aligned}$$

for each $j = 1, \ldots, n$ and $t \in I$, using (4.3.9) in the second step.

Remember that $z'(t)$ can be expressed as in (4.1.5). Using (4.3.10) and the definition (4.3.6) of $p_j(t)$, we get that

$$(4.3.12) \qquad z'(t) = \sum_{j=1}^{n} \frac{\partial F}{\partial q_j}(p(t), z(t), w(t))\, p_j(t)$$

for every $t \in I$.

Thus (4.3.10), (4.3.11), and (4.3.12) form a coupled system of ordinary differential equations for $w(t)$, $z(t)$, and $p(t)$ that does not depend on u. These are the *characteristic equations* for (4.3.3). This corresponds to some remarks in Section 3.2.1 of [81], and Section B of Chapter 1 of [87].

4.4 More on fully nonlinear equations

Let n be a positive integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let $F(q, y, x)$ be a continuously-differentiable real-valued function on $\mathbf{R}^n \times \mathbf{R} \times U$ again. If $w(t)$, $z(t)$, and $p(t)$ are any continuously-differentiable functions on I with values in $\mathbf{R}^n$, $\mathbf{R}$, and U, respectively, then

$$(4.4.1) \qquad \frac{d}{dt} F(p(t), z(t), w(t)) = \sum_{j=1}^{n} \frac{\partial F}{\partial q_j}(p(t), z(t), w(t))\, p_j'(t) + \frac{\partial F}{\partial y}(p(t), z(t), w(t))\, z'(t) + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(p(t), z(t), w(t))\, w_j'(t)$$

on I.

If $w(t)$, $z(t)$, and $p(t)$ satisfy the characteristic equations (4.3.10), (4.3.11), and (4.3.12) on I, then it is easy to see that

$$(4.4.2) \qquad \frac{d}{dt} F(p(t), z(t), w(t)) = 0$$

on I. Of course, this means that $F(p(t), z(t), w(t))$ is constant on I.

4.4.1 Related initial value problems

If

$$(4.4.3) \qquad F(p(t), z(t), w(t)) = 0$$

for some $t \in I$, then it follows that this holds for all $t \in I$ under these conditions. This corresponds to step 2 in the proof of Theorem 2 in Section 3.2.4 of [81], and a remark near the end of Section B of Chapter 1 of [87].

Of course, if $w(t)$, $p(t)$, and $z(t)$ are associated to a solution u of (4.3.3) as in the previous section, then (4.4.3) holds by construction. Alternatively, one may consider initial value problems for the characteristic equations (4.3.10), (4.3.11), and (4.3.12) that satisfy (4.4.3) for some t, and thus for all t. Solutions to initial value problems of this type can be used to try to find solutions of (4.3.3), as in the next section.

4.4.2 The quasilinear case

In the quasilinear case, we have that

$$(4.4.4) \qquad F(q, y, x) = \sum_{j=1}^{n} a_j(y, x)\, q_j + b(y, x)$$

for some real-valued functions $a_j(y, x)$, $1 \le j \le n$, and $b(y, x)$ on $\mathbf{R} \times U$. Note that (4.3.10) is the same as (4.2.7) in this case. If (4.4.3) holds, then it is easy to see that (4.3.12) is the same as (4.2.6). This corresponds to a remark in Section 3.2.2 b of [81], and just after (1.14) in Section B of Chapter 1 of [87].

4.4.3 A related partial differential equation

If c is a real number, then

$$\widehat{F}(q, y, x) = F(q, y, x) - c \tag{4.4.5}$$

is another continuously-differentiable real-valued function on $\mathbf{R}^n \times \mathbf{R} \times U$. If u is a continuously-differentiable real-valued function on U, then the first-order partial differential equation

$$\widehat{F}(Du(x), u(x), x) = 0 \tag{4.4.6}$$

on U is the same as saying that

$$F(Du(x), u(x), x) = c \tag{4.4.7}$$

on U. Note that the characteristic equations associated to $\widehat{F}$ as in the previous section are the same as for F, because they only involve the derivatives of F, $\widehat{F}$.

If F is as in (4.4.4), then

$$\widehat{F}(q, y, x) = \sum_{j=1}^{n} a_j(y, x)\, q_j + \widehat{b}(y, x), \tag{4.4.8}$$

with $\widehat{b}(y, x) = b(y, x) - c$. However, the characteristic equations associated to the quasilinear equations corresponding to F and $\widehat{F}$ as in Subsection 4.2.1 are not the same when $c \neq 0$. The equations for $w_j'(t)$ are the same, but the analogue of the equation (4.2.6) for $z'(t)$ with $\widehat{b}(y, x)$ instead of $b(y, x)$ is a bit different. The conditions under which this equation is supposed to be the same as in the previous section are also a bit different.

4.5 Non-characteristic conditions

Let n be a positive integer, and let U be a nonempty open set in $\mathbf{R}^n$. In Sections 4.1 – 4.3, we started with a solution u of a first-order partial differential equation on U, and found systems of ordinary differential equations that described the behavior of u along certain curves. These systems of ordinary differential equations do not depend on u, and may be used to try to find solutions of the partial differential equation, at least locally, as in Section 3.2.4 of [81], and Section B of Chapter 1 of [87].

4.5.1 The corresponding Cauchy problem

More precisely, this normally involves additional regularity conditions on the functions used to define the original partial differential equation, in order to use appropriate results about systems of ordinary differential equations. One might suppose that

(4.5.1) u is given along a nice $(n-1)$-dimensional submanifold Σ of $\mathbf{R}^n$,

with suitable regularity on Σ, as in [81, 87]. One would like to

(4.5.2) find a solution to the partial differential equation near Σ, with the given values on Σ,

perhaps at least near a given point on Σ. This may be considered as an *initial value problem* or *Cauchy problem* for the partial differential equation.

4.5.2 Initial value problems for the characteristic equations

Remember that we considered systems of ordinary differential equations for $w(t)$, $z(t)$, and possibly $p(t)$ defined on an interval I in the real line. To deal with the initial value problem for the partial differential equation along Σ, we want to consider suitable initial value problems for these systems of ordinary differential equations associated to points in Σ. Let

$$\sigma \in \Sigma \text{ and } t_0 \in \mathbf{R} \tag{4.5.3}$$

be given, although one might normally simply take $t_0 = 0$. The initial conditions for $w(t)$ and $z(t)$ at $t = t_0$ are

$$w(t_0) = \sigma \tag{4.5.4}$$

and

$$z(t_0) = u(\sigma), \tag{4.5.5}$$

where $u(\sigma) \in \mathbf{R}$ should be given as in (4.5.1). In the fully nonlinear case, we would also need to specify $p(t_0) \in \mathbf{R}^n$, and we shall return to that later.

4.5.3 The non-characteristic condition

We would like to define u near Σ in such a way that

$$u(w(t)) = z(t) \tag{4.5.6}$$

on I. In particular, we would like to be able to reach points in U near Σ by such a path $w(t)$. In order to do this, there is an additional *non-characteristic condition*, as in Section 3.2.3 c of [81], and Section B of Chapter 1 of [87]. The non-characteristic condition at σ asks that

$$w'(t_0) \text{ not be tangent to } \Sigma \text{ at } w(t_0). \tag{4.5.7}$$

If $\nu(\sigma)$ is a nonzero element of $\mathbf{R}^n$ that is normal to Σ at σ, then this means that

$$w'(t_0) \cdot \nu(w(t_0)) = w'(t_0) \cdot \nu(\sigma) \neq 0. \tag{4.5.8}$$

4.5.4 The semilinear and quasilinear cases

In Section 4.1, the ordinary differential equations for w are given in terms of the functions $a_j(x)$, $1 \le j \le n$, and the non-characteristic condition at $\sigma \in \Sigma$ can be expressed as

$$a(\sigma) \cdot \nu(\sigma) = \sum_{j=1}^{n} a_j(\sigma)\, \nu_j(\sigma) \neq 0. \tag{4.5.9}$$

In Subsection 4.2.1, the ordinary differential equations for w are coupled with those for z, and the non-characteristic condition at σ can be expressed as

$$a(u(\sigma), \sigma) \cdot \nu(\sigma) = \sum_{j=1}^{n} a_j(u(\sigma), \sigma)\, \nu_j(\sigma) \neq 0. \tag{4.5.10}$$

In particular, this depends on the value of u at σ.

4.5.5 The fully nonlinear case

In Subsection 4.3.3, the ordinary differential equations for w are coupled with those for z and p, and the non-characteristic condition at σ can be expressed as

$$\begin{aligned} &\sum_{j=1}^{n} \frac{\partial F}{\partial q_j}(p(t_0), z(t_0), w(t_0))\, \nu_j(w(t_0)) \\ &\qquad = \sum_{j=1}^{n} \frac{\partial F}{\partial q_j}(p(t_0), u(\sigma), \sigma)\, \nu_j(\sigma) \neq 0. \end{aligned} \tag{4.5.11}$$

This depends on the value of u at σ, and $p(t_0)$, which is supposed to represent the values of the first partial derivatives of u at σ.

4.5.6 Choosing $p(t_0)$

Observe that

(4.5.12) the directional derivative of u at σ in a direction that is tangent to Σ at σ is determined by the restriction of u to Σ.

Another condition on the first partial derivatives of u at σ is given by the partial differential equation (4.3.3) at $x = \sigma$, i.e.,

$$F(Du(\sigma), u(\sigma), \sigma) = 0. \tag{4.5.13}$$

One basically needs to be able to choose $p(t_0)$ in a way that is compatible with these conditions, and the non-characteristic condition (4.5.11) depends on the choice of $p(t_0)$.

In particular, $p(t_0)$ should satisfy

$$F(p(t_0), z(t_0), w(t_0)) = F(p(t_0), u(\sigma), \sigma) = 0, \tag{4.5.14}$$

where the second step is as in (4.5.13). This implies that (4.4.3) holds along the curve, as before.

4.5.7 Initial conditions for p corresponding to other points in Σ

In Section 3.2.3 c of [81], one starts with a suitable choice of $p(t_0)$ for a point σ in Σ. If the non-characteristic condition (4.5.11) holds at σ, then one can use the implicit function theorem to get suitable initial conditions for p corresponding to other points in Σ that are close to σ.

In Section B of Chapter 1 of [87], one simply asks to have suitable initial conditions for p corresponding to points along Σ. An important case where this is easy to get will be discussed in Section 4.10.

4.6 More on the Euler operator

Let n be a positive integer, and put $a_j(x) = x_j$ on $\mathbf{R}^n$ for each $j = 1, \ldots, n$. Thus $a = (a_1, \ldots, a_n)$ is the identity mapping on $\mathbf{R}^n$, and

$$L_a = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \tag{4.6.1}$$

is the Euler operator, as in Section 2.8. In this case, (4.1.6) reduces to

$$w_j'(t) = w_j(t). \tag{4.6.2}$$

This is solved on the real line by

$$w_j(t) = c_j \exp t, \tag{4.6.3}$$

with $c_j \in \mathbf{R}$ for $j = 1, \ldots, n$. Equivalently, (4.1.7) reduces to

$$w'(t) = w(t), \tag{4.6.4}$$

which is solved on the real line by

$$w(t) = (\exp t)\, c, \tag{4.6.5}$$

where $c = (c_1, \ldots, c_n) \in \mathbf{R}^n$.

4.6.1 Relation with homogeneous functions

Let u be a continuously-differentiable real or complex-valued function on

$$\mathbf{R}^n \setminus \{0\}. \tag{4.6.6}$$

Suppose that $c \neq 0$, so that (4.6.5) is nonzero for each $t \in \mathbf{R}$. Observe that

$$\frac{d}{dt}(u((\exp t)\, c)) = \sum_{j=1}^{n} (\exp t)\, c_j\, (\partial_j u)((\exp t)\, c) = (L_a(u))((\exp t)\, c) \tag{4.6.7}$$

for every $t \in \mathbf{R}$. This is the same as (4.1.8) in this case. Of course, this is analogous to considering $u(\tau\, x)$ for $x \in \mathbf{R}^n \setminus \{0\}$ and $\tau > 0$, and differentiating in τ, as in Section 2.8.

If

$$L_a(u) = b\, u \tag{4.6.8}$$

on $\mathbf{R}^n \setminus \{0\}$ for some $b \in \mathbf{C}$, then (4.6.7) implies that

$$\frac{d}{dt}(u((\exp t)\, c)) = b\, u((\exp t)\, c) \tag{4.6.9}$$

for every $t \in \mathbf{R}$. This means that

$$u((\exp t)\, c) = u(c)\, \exp(b\, t) \tag{4.6.10}$$

for every $t \in \mathbf{R}$, as in Section 1.14, which holds automatically when $t = 0$. One can use this to get that u is homogeneous of degree b on $\mathbf{R}^n \setminus \{0\}$, as in Subsection 2.8.1. Conversely, if u is homogeneous of degree b on $\mathbf{R}^n \setminus \{0\}$, then (4.6.10) holds, which implies that (4.6.9) holds, and thus (4.6.8) holds.

4.7 Angular derivatives in the plane

Let $a_1(x)$, $a_2(x)$ be the real-valued functions on $\mathbf{R}^2$ defined by

$$a_1(x) = -x_2, \ a_2(x) = x_1. \tag{4.7.1}$$

Thus

$$a(x) = (a_1(x), a_2(x)) = (-x_2, x_1) \tag{4.7.2}$$

defines a mapping from $\mathbf{R}^2$ onto itself. If we identify $x = (x_1, x_2) \in \mathbf{R}^2$ with $x_1 + x_2\, i \in \mathbf{C}$, then

$$a(x) = -x_2 + x_1\, i = i\, x. \tag{4.7.3}$$

The corresponding system of ordinary differential equations (4.1.6) reduces to

$$w_1'(t) = -w_2(t), \ w_2'(t) = w_1(t) \tag{4.7.4}$$

in this case. If we identify $w(t) = (w_1(t), w_2(t))$ with

$$w_1(t) + w_2(t)\, i, \tag{4.7.5}$$

as before, then this is the same as saying that

$$w'(t) = i\, w(t), \tag{4.7.6}$$

as in (4.1.7). This is solved on the real line by

$$w(t) = (\exp(i\, t))\, c, \tag{4.7.7}$$

where $c = (c_1, c_2) \in \mathbf{R}^2$ is identified with $c_1 + c_2\, i \in \mathbf{C}$, as usual. Note that $w(0) = c$.

4.7.1 Angular derivatives and $\exp(i\,t)$

Let U be a nonempty open subset of $\mathbf{R}^2$, and suppose that u is a continuously-differentiable real or complex-valued function on U. Let $L_a(u)$ be the continuous real or complex-valued function on U, as appropriate, defined by

$$(L_a(u))(x) = -x_2\,\frac{\partial u}{\partial x_1}(x) + x_1\,\frac{\partial u}{\partial x_2}(x) \tag{4.7.8}$$

for every $x \in U$, as in Section 4.1. This is the same as the directional derivative of u at x, in the direction corresponding to $i\,x$, because of (4.7.3).

Let $x \in \mathbf{R}^2$ be given, and consider

$$\{t \in \mathbf{R} : \exp(i\,t)\,x \in U\}, \tag{4.7.9}$$

where $\mathbf{R}^2$ is identified with $\mathbf{C}$ as before. This is an open subset of $\mathbf{R}$, and

$$u(\exp(i\,t)\,x) \tag{4.7.10}$$

may be considered as a continuously-differentiable real or complex-valued function of t in (4.7.9). Observe that

$$\frac{d}{dt}(u(\exp(i\,t)\,x)) = (L_a(u))(\exp(i\,t)\,x) \tag{4.7.11}$$

for every t in (4.7.9), as in (4.1.8).

A nice example related to this case is discussed in Section 3.2.2 a of [81].

4.8 Another example on $\mathbf{R}^2$

Now let $a_1(x)$, $a_2(x)$ be the real-valued functions on $\mathbf{R}^2$ defined by

$$a_1(x) = x_1,\ a_2(x) = -x_2, \tag{4.8.1}$$

and put

$$a(x) = (a_1(x), a_2(x)) = (x_1, -x_2) \tag{4.8.2}$$

for every $x \in \mathbf{R}^2$. This leads to the system of ordinary differential equations

$$w_1'(t) = w_1(t),\ w_2'(t) = -w_2(t), \tag{4.8.3}$$

as in (4.1.6) again. These equations are solved on the real line by

$$w_1(t) = c_1\,\exp t,\ w_2(t) = c_2\,\exp(-t), \tag{4.8.4}$$

with $c_1, c_2 \in \mathbf{R}$. If we put $w(t) = (w_1(t), w_2(t))$ and $c = (c_1, c_2)$, then we get that $w(0) = c$. It follows from (4.8.4) that

$$w_1(t)\,w_2(t) = c_1\,c_2 \tag{4.8.5}$$

for every $t \in \mathbf{R}$.

4.8.1 The corresponding operator L_a

Let U be a nonempty subset of $\mathbf{R}^2$ again, and suppose that u is a continuously-differentiable real or complex-valued function on U. Also let $L_a(u)$ be the continuous real or complex-valued function defined on U by

$$(L_a(u))(x) = x_1 \frac{\partial u}{\partial x_1}(x) - x_2 \frac{\partial u}{\partial x_2}(x), \tag{4.8.6}$$

as in Section 4.1.

Let $x \in \mathbf{R}^2$ be given, and note that

$$\{r \in \mathbf{R}_+ : (r\, x_1, r^{-1}\, x_2) \in U\} \tag{4.8.7}$$

is an open subset of $\mathbf{R}$. We may consider

$$u(r\, x_1, r^{-1}\, x_2) \tag{4.8.8}$$

as a continuously-differentiable real or complex-valued function of r on (4.8.7). If r is in (4.8.7), then

$$\begin{aligned} \frac{d}{dr}(u(r\, x_1, r^{-1}\, x_2)) &= x_1\, (\partial_1 u)(r\, x_1, r^{-1}\, x_2) - r^{-2}\, x_2\, (\partial_2 u)(r\, x_1, r^{-1}\, x_2) \\ &= r^{-1}\, (L_a(u))(r\, x_1, r^{-1}\, x_2). \end{aligned} \tag{4.8.9}$$

4.8.2 Using $w(t)$

Alternatively,

$$\{t \in \mathbf{R} : ((\exp t)\, x_1, (\exp(-t))\, x_2) \in U\} \tag{4.8.10}$$

is an open subset of $\mathbf{R}$, and

$$u((\exp t)\, x_1, (\exp(-t))\, x_2) \tag{4.8.11}$$

may be considered as a continuously-differentiable real or complex-valued function of t on (4.8.10). If t is in (4.8.10), then

$$\frac{d}{dt}(u((\exp t)\, x_1, (\exp(-t))\, x_2)) = (L_a(u))((\exp t)\, x_1, (\exp(-t))\, x_2), \tag{4.8.12}$$

as in (4.1.8).

This is related to Exercise (3) in Section B of Chapter 1 of [87]. In particular, if f is a continuously-differentiable real or complex-valued function on an open subset V of the real line, then

$$u(x_1, x_2) = f(x_1\, x_2) \tag{4.8.13}$$

is a continuously-differentiable function on the open set

$$\{(x_1, x_2) \in \mathbf{R}^n : x_1\, x_2 \in V\} \tag{4.8.14}$$

in the plane, and

$$L_a(u) = 0 \tag{4.8.15}$$

on (4.8.14).

4.9 Some simpler quasilinear equations

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let $a_1, \ldots, a_n$ be real-valued functions on $\mathbf{R} \times U$, and let b be a real-valued function on $\mathbf{R}$. Consider the quasilinear first-order partial differential equation

$$\sum_{j=1}^{n} a_j(u(x), x) \, \frac{\partial u}{\partial x_j}(x) + b(u(x)) = 0, \tag{4.9.1}$$

where u is a continuously-differentiable real-valued function on U. This is the same as in Section 4.2, with b not depending on $x \in U$.

4.9.1 A simpler equation for $z(t)$

Let I be an interval in the real line with nonempty interior, and which may be unbounded, and let $w(t)$ be a continuously-differentiable function of $t \in I$ with values in U, as before. We previously considered a system of ordinary differential equations for $w(t)$ and a continuously-differentiable real-valued function $z(t)$ of $t \in I$. The equation for $w(t)$ is

$$w'(t) = a(z(t), w(t)) \tag{4.9.2}$$

for every $t \in I$, as before. In this case, the equation for $z(t)$ is

$$z'(t) + b(z(t)) = 0 \tag{4.9.3}$$

for every $t \in I$. This does not depend on $w(t)$, and so a solution to (4.9.3) can be used to get that (4.9.2) may be considered as a system of ordinary differential equations for $w(t)$ on I.

4.9.2 The case where $b \equiv 0$

If $b \equiv 0$ on $\mathbf{R}$, then (4.9.1) reduces to

$$\sum_{j=1}^{n} a_j(u(x), x) \, \frac{\partial u}{\partial x_j}(x) = 0, \tag{4.9.4}$$

on U. Similarly, (4.9.3) reduces to

$$z'(t) = 0 \tag{4.9.5}$$

for every $t \in I$. Of course, this means that $z(t)$ is constant on I. In this case, (4.9.2) is simpler than before, although it depends on the constant value of $z(t)$ on I.

4.9.3 Another simplification

Suppose now that $a_1, \ldots, a_n$ are real-valued functions on $\mathbf{R}$, which is to say that they do not depend on $x \in U$. This means that (4.9.1) reduces to

$$\sum_{j=1}^{n} a_j(u(x)) \, \frac{\partial u}{\partial x_j}(x) + b(u(x)) = 0 \tag{4.9.6}$$

on U. Similarly, (4.9.2) reduces to

$$w'(t) = a(z(t)) \tag{4.9.7}$$

on I. If $z(t)$ satisfies (4.9.3), then (4.9.7) can be solved more directly than before.

4.9.4 Another simplification with $b \equiv 0$

If we also ask that $b \equiv 0$ on $\mathbf{R}$ again, then (4.9.4) reduces to

$$\sum_{j=1}^{n} a_j(u(x)) \, \frac{\partial u}{\partial x_j}(x) = 0 \tag{4.9.8}$$

on U. Because (4.9.3) reduces to (4.9.5) in this case, as before, $z(t)$ is constant on I, so that the right side of (4.9.7) is constant on I as well. This means that the curve corresponding to $w(t)$ follows a straight line, at constant speed. Of course, the constant value of the right side of (4.9.7) depends on the constant value of $z(t)$ on I.

Thus, although these curves follow straight lines at constant speeds, these lines do not have to be parallel to each other, nor do the constant speeds of the individual curves need to be the same. In particular, it is possible for curves like these to cross each other, as mentioned in Section 3.2.5 b of [81]. This can lead to limitations on continuously-differentiable solutions of (4.9.8), as in [81].

Some equations like these will be mentioned in Section 4.12.

4.10 A simplification with x_n

Let n be an integer greater than or equal to 2, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let

$$F(q, y, x) = F(q_1, \ldots, q_n, y, x) \tag{4.10.1}$$

be a real-valued function on $\mathbf{R}^n \times \mathbf{R} \times U$, as in Section 4.3. If u is a continuously-differentiable real-valued function on U, then the first-order partial differential equation corresponding to $F(q, y, x)$ can be expressed as

$$F\Big(\frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_n}(x), u(x), x\Big) = 0. \tag{4.10.2}$$

Suppose that $F(q, y, x)$ can be expressed as

$$F(q_1, \ldots, q_{n-1}, q_n, y, x) = q_n + \widetilde{F}(q_1, \ldots, q_{n-1}, y, x) \tag{4.10.3}$$

for some real-valued function $\widetilde{F}(q_1, \ldots, q_{n-1}, y, x)$ on $\mathbf{R}^{n-1} \times \mathbf{R} \times U$. In this case, (4.10.2) is the same as saying that

$$\frac{\partial u}{\partial x_n}(x) + \widetilde{F}\Big(\frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_{n-1}}(x), u(x), x\Big) = 0. \tag{4.10.4}$$

4.10.1 The corresponding characteristic equations

Suppose that $\widetilde{F}$ is continuously differentiable on $\mathbf{R}^{n-1} \times \mathbf{R} \times U$, so that F is continuously differentiable on $\mathbf{R}^n \times \mathbf{R} \times U$. This leads to a coupled system of ordinary differential equations for $w(t)$, $z(t)$, and $p(t)$ as in Subsection 4.3.3. The differential equation for the nth component $w_n(t)$ of $w(t)$ reduces to

$$w_n'(t) = 1 \tag{4.10.5}$$

for every t in the interval I.

In the quasilinear case, as in Section 4.2, the condition analogous to (4.10.3) is that

$$a_n \equiv 1 \tag{4.10.6}$$

on $\mathbf{R} \times U$. In this case, we have a coupled system of ordinary differential equations for $w(t)$ and $z(t)$, as before. The differential equation for $w_n(t)$ reduces to (4.10.5) again.

Similarly, one may consider the condition (4.10.6) in Section 4.1, where a_n is a real-valued function on U. The system of ordinary differential equations for $w(t)$ depends only on a, and the differential equation for $w_n(t)$ reduces to (4.10.5).

4.10.2 Some non-characteristic conditions

Suppose that the hypersurface Σ mentioned in Subsection 4.5.1 is contained in a hyperplane

$$\{x \in \mathbf{R}^n : x_n = c\} \tag{4.10.7}$$

for some $c \in \mathbf{R}$. Note that the non-characteristic condition holds when the differential equation for $w_n(t)$ is as in (4.10.5) and Σ is of this type.

The directional derivatives of u at a point in Σ in directions tangent to Σ are determined by the restriction of u to Σ, as before. In this case, this means that the partial derivative of u with respect to x_j on Σ is determined by the restriction of u to Σ for $j = 1, \ldots, n-1$. If u satisfies a partial differential equation as in (4.10.4), then it follows that the partial derivative of u with respect to x_n on Σ is determined by the restriction of u to Σ as well.

If u is given on Σ, then this makes it easy to get initial conditions for p at points in Σ, as in Subsection 4.5.5. More precisely, the initial condition for p_j at a point in Σ is given by the partial derivative of u with respect to x_j at the point when $j = 1, \ldots, n-1$, and is determined by (4.10.4) when $j = n$.

4.11 Some simpler fully nonlinear equations

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let

$$F(q, x) \tag{4.11.1}$$

be a real-valued function on $\mathbf{R}^n \times U$, and consider the fully nonlinear first-order partial differential equation

$$F(Du(x), x) = 0, \tag{4.11.2}$$

where u is a continuously-differentiable real-valued function on U. This is the same as in Section 4.3, where the function $F(q, y, x)$ on $\mathbf{R}^n \times \mathbf{R} \times U$ does not depend on $y \in \mathbf{R}$.

4.11.1 Simpler characteristic equations

Let I be an interval in the real line with nonempty interior, and which may be unbounded, and let $w(t)$, $z(t)$, and $p(t)$ be continuously-differentiable functions of $t \in I$ with values in U, $\mathbf{R}$, and $\mathbf{R}^n$, respectively. If $F(q, x)$ is continuously differentiable on $\mathbf{R}^n \times U$, then we get a system of ordinary differential equations for $w(t)$, $z(t)$, and $p(t)$, as in Subsection 4.3.3, which has some simplifications in this case. The equations for $w'(t)$ are now

$$w_l'(t) = \frac{\partial F}{\partial q_l}(p(t), w(t)) \tag{4.11.3}$$

for each $l = 1, \ldots, n$ and $t \in I$. The equations for $p'(t)$ reduce to

$$p_j'(t) = -\frac{\partial F}{\partial x_j}(p(t), w(t)) \tag{4.11.4}$$

for each $j = 1, \ldots, n$ and $t \in I$. The equation for $z'(t)$ is

$$z'(t) = \sum_{j=1}^{n} \frac{\partial F}{\partial q_j}(p(t), w(t))\, p_j(t) \tag{4.11.5}$$

on I.

The right sides of these equations do not involve $z(t)$. Thus (4.11.3) and (4.11.4) form a system of ordinary differential equations for $w(t)$ and $p(t)$. If one has solutions for these equations, then (4.11.5) can be solved directly.

4.11.2 Another simplification with x_n

Suppose now that $n \geq 2$, and that $F(q, x)$ can be expressed as

$$F(q_1, \ldots, q_{n-1}, q_n, x) = q_n + \widetilde{F}(q_1, \ldots, q_{n-1}, x) \tag{4.11.6}$$

for some real-valued function $\widetilde{F}(q_1, \ldots, q_{n-1}, x)$ on $\mathbf{R}^{n-1} \times U$, as in the previous section. This means that (4.11.2) is the same as saying that

$$\frac{\partial u}{\partial x_n}(x) + \widetilde{F}\Big(\frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_{n-1}}(x), x\Big) = 0, \tag{4.11.7}$$

on U, as before. Of course, (4.11.3) reduces to (4.10.5) when $l = n$. Similarly, (4.11.5) can be reexpressed as

$$z'(t) = \sum_{j=1}^{n-1} \frac{\partial \widetilde{F}}{\partial q_j}(p_1(t), \ldots, p_{n-1}(t), w(t))\, p_j(t) + p_n(t) \tag{4.11.8}$$

in this case.

4.11.3 The Hamilton–Jacobi equation

If

$$\widetilde{F}(q_1, \ldots, q_{n-1}, x) = \widetilde{F}(q_1, \ldots, q_{n-1}, x_1, \ldots, x_{n-1}, x_n) \tag{4.11.9}$$

does not depend on x_n, then (4.11.7) is the same as the *Hamilton–Jacobi equation.* Under these conditions, (4.11.4) says that

$$p_n'(t) = 0 \tag{4.11.10}$$

for every $t \in I$ when $j = n$, so that p_n is constant on I. This type of equation is discussed in [81], starting in Section 3.2.5 c. These equations are normally expressed a bit differently, as in the next section.

4.12 Other notation in $n + 1$ variables

Let n be a positive integer, and let us identify $\mathbf{R}^n \times \mathbf{R}$ with $\mathbf{R}^{n+1}$ in the usual way. An element of $\mathbf{R}^n \times \mathbf{R}$ may be expressed as (x, τ), where $x \in \mathbf{R}^n$ and $\tau \in \mathbf{R}$.

Let U be a nonempty open subset of $\mathbf{R}^n \times \mathbf{R}$, and let

$$F(q_1, \ldots, q_n, q_{n+1}, y, x, \tau) \tag{4.12.1}$$

be a real-valued function on $\mathbf{R}^{n+1} \times \mathbf{R} \times U$. This means that (4.12.1) is defined for $q_1, \ldots, q_n, q_{n+1}, y \in \mathbf{R}$ and $(x, \tau) \in U$. If $u(x, \tau)$ is a continuously-differentiable real-valued function on U, then the first-order partial differential equation corresponding to (4.12.1) can be expressed as

$$F\Big(\frac{\partial u}{\partial x_1}(x, \tau), \ldots, \frac{\partial u}{\partial x_n}(x, \tau), \frac{\partial u}{\partial \tau}(x, \tau), u(x, \tau), x, \tau\Big) = 0. \tag{4.12.2}$$

Suppose that (4.12.1) can be expressed as

$$q_{n+1} + \widetilde{F}(q_1, \ldots, q_n, y, x, \tau) \tag{4.12.3}$$

for some real-valued function $\widetilde{F}(q_1, \ldots, q_n, y, x, \tau)$ on $\mathbf{R}^n \times \mathbf{R} \times U$. Under these coditions, (4.12.2) is the same as saying that

$$\frac{\partial u}{\partial \tau}(x, \tau) + \widetilde{F}\Big(\frac{\partial u}{\partial x_1}(x, \tau), \ldots, \frac{\partial u}{\partial x_n}(x, \tau), u(x, \tau), x, \tau\Big) = 0. \tag{4.12.4}$$

This corresponds to (4.10.4), in this notation.

If $\widetilde{F}$ is continuously differentiable on $\mathbf{R}^n \times \mathbf{R} \times U$, then we can consider the associated system of characteristic equations, as before. The analogue of (4.10.5) with n replaced by $n+1$ permits us to identify t with τ, perhaps with a suitable translation. Of course, an equation of the form (4.12.4) is often expressed with t in place of τ.

If $\widetilde{F}(q_1, \ldots, q_n, y, x, \tau)$ does not depend on y or τ, then (4.12.4) is the same as the Hamilton–Jacobi equation, as in the previous section, with slightly different notation.

4.12.1 Some quasilinear equations

Let Φ be a continuously-differentiable function on the real line with values in $\mathbf{R}^n$. The partial differential equation

$$\frac{\partial u}{\partial \tau} + \operatorname{div} \Phi(u) = 0 \tag{4.12.5}$$

is called a *scalar conservation law*, as in Example 5 in Section 3.2.5 b of [81]. More precisely, the divergence is taken in the x variables here. Equivalently, this can be expressed as

$$\frac{\partial u}{\partial \tau} + \sum_{j=1}^{n} \Phi_j'(u)\, \frac{\partial u}{\partial x_j} = 0, \tag{4.12.6}$$

where Φ_j is the jth component of Φ for each $j = 1, \ldots, n$. This may be considered as an equation of the type mentioned in Subsection 4.9.4.

If $n = 1$ and $b \in \mathbf{R}$, then

$$\frac{\partial u}{\partial \tau}(x, \tau) + u(x, \tau)\, \frac{\partial u}{\partial x}(x, \tau) = b \tag{4.12.7}$$

is a quasilinear first-order equation that is a simpler version of the type mentioned in Subsection 4.9.3. This is the inviscid form of *Burger's equation* when $b = 0$, which is discussed in Section 3.4.1 of [81]. In this case, this equation is an example of a scalar conservation law. This equation with $b = 1$ is mentioned in Problem 5 (c) in Section 3.5 of [81], as well as Example 2 and Exercise (4) in Section B of Chapter 1 of [87].

4.13 Some other fully nonlinear equations

Let n be a positive integer, and let

$$F(q, y) \tag{4.13.1}$$

be a real-valued function on $\mathbf{R}^n \times \mathbf{R}$. Also let U be a nonempty open subset of $\mathbf{R}^n$, and consider the fully nonlinear first-order partial differential equation

$$F(Du(x), u(x)) = 0, \tag{4.13.2}$$

where u is a continuously-differentiable real-valued function on U. This is the same as in Section 4.3 again, where the function $F(q, y, x)$ on $\mathbf{R}^n \times \mathbf{R} \times U$ does not depend on $x \in U$.

4.13.1 Simpler characteristic equations again

Let I be an interval in the real line with nonempty interior, and which may be unbounded, and let $w(t)$, $z(t)$, and $p(t)$ be continuously-differentiable functions of $t \in I$ with values in U, $\mathbf{R}$, and $\mathbf{R}^n$, respectively, as before. If $F(q, y)$ is continuously differentiable on $\mathbf{R}^n \times \mathbf{R}$, then the system of ordinary differential equations for $w(t)$, $z(t)$, and $p(t)$ discussed in Subsection 4.3.3 can be simplified in this case too. The equations for $w'(t)$ are

$$w_l'(t) = \frac{\partial F}{\partial q_l}(p(t), z(t)) \tag{4.13.3}$$

for each $l = 1, \ldots, n$ and $t \in I$. The equations for $p'(t)$ now reduce to

$$p_j'(t) = -\frac{\partial F}{\partial y}(p(t), z(t))\, p_j(t) \tag{4.13.4}$$

for each $j = 1, \ldots, n$ and $t \in I$. The equation for $z'(t)$ reduces to

$$z'(t) = \sum_{j=1}^{n} \frac{\partial F}{\partial q_j}(p(t), z(t))\, p_j(t) \tag{4.13.5}$$

for every $t \in I$.

The right sides of these equations do not involve $w(t)$, so that (4.13.4) and (4.13.5) form a system of ordinary differential equations for $p(t)$ and $z(t)$. If one has solutions to these equations, then (4.13.3) can be solved directly, as before.

4.13.2 An additional simplification

Suppose for the moment that the derivative of $F(q, y)$ in y does not depend on y. This means that

$$F(q, y) = F(q, 0) + \frac{\partial F}{\partial y}(q, 0)\, y \tag{4.13.6}$$

for every $q \in \mathbf{R}^n$ and $y \in \mathbf{R}$. Equivalently, if we put

$$F_1(q) = F(q, 0) \text{ and } F_2(q) = (\partial F/\partial y)(q, 0), \tag{4.13.7}$$

then

$$F(q, y) = F_1(q) + F_2(q)\, y \tag{4.13.8}$$

for every $q \in \mathbf{R}^n$ and $y \in \mathbf{R}$. Note that $F_1(q)$ and $F_2(q)$ can be arbitrary continuously-differentiable real-valued functions of $q \in \mathbf{R}^n$ here. In this case, (4.13.2) reduces to

$$F_1(Du(x)) + F_2(Du(x))\, u(x) = 0. \tag{4.13.9}$$

Similarly, (4.13.4) reduces to

$$p_j'(t) = -F_2(p(t))\, p_j(t) \tag{4.13.10}$$

under these conditions. The right side of this equation depends only on $p(t)$, so that one gets a system of ordinary differential equations for $p(t)$. If one has a solution for this system, then (4.13.5) gives an ordinary differential equation

$$z'(t) = \sum_{j=1}^{n} \Big(\frac{\partial F_1}{\partial q_j}(p(t)) + \frac{\partial F_2}{\partial q_j}(p(t))\, z(t)\Big)\, p_j(t) \tag{4.13.11}$$

for $z(t)$, that is linear in $z(t)$. Observe that (4.13.3) reduces to

$$w_l'(t) = \frac{\partial F_1}{\partial q_l}(p(t)) + \frac{\partial F_2}{\partial q_l}(p(t))\, z(t) \tag{4.13.12}$$

under these conditions. This can be solved directly using solutions to (4.13.10) and (4.13.11), as before.

4.13.3 Taking $F_2(q)$ to be constant

Suppose now that the derivative of $F(q, y)$ in y is a constant $c \in \mathbf{R}$, so that

$$F(q, y) = F(q, 0) + c\, y \tag{4.13.13}$$

for every $q \in \mathbf{R}^n$ and $y \in \mathbf{R}$. This is the same as saying that

$$F_2(q) = c \tag{4.13.14}$$

on $\mathbf{R}^n$ in the notation of (4.13.8), so that

$$F(q, y) = F_1(q) + c\, y \tag{4.13.15}$$

for every $q \in \mathbf{R}^n$ and $y \in \mathbf{R}$, as in (4.13.8). Thus (4.13.9) reduces to

$$F_1(Du(x)) + c\, u(x) = 0. \tag{4.13.16}$$

In this case, (4.13.10) reduces to

$$p_j'(t) = -c\, p_j(t) \tag{4.13.17}$$

on I for each $j = 1, \ldots, n$. This is solved by taking

$$p_j(t) = a_j\, \exp(-c\, t) \tag{4.13.18}$$

for some real numbers $a_1, \ldots, a_n$. Similarly, (4.13.11) reduces to

$$z'(t) = \sum_{j=1}^{n} \frac{\partial F_1}{\partial q_j}(p(t))\, p_j(t), \tag{4.13.19}$$

and (4.13.12) reduces to

$$w_l'(t) = \frac{\partial F_1}{\partial q_l}(p(t)). \tag{4.13.20}$$

Note that the right sides of these equation do not involve $z(t)$ under these conditions. Example 3 in Section 3.2.2 c of [81] is a nice example of this type.

4.14 A simpler case

Let n be a positive integer again, and let

$$F(q) \tag{4.14.1}$$

be a real-valued function on $\mathbf{R}^n$. Consider the fully nonlinear first-order partial differential equation

$$F(Du(x)) = 0, \tag{4.14.2}$$

where u is a continuously-differentiable real-valued function on a nonempty open subset U of $\mathbf{R}^n$. This is the same as in Section 4.3, where the function $F(q, y, x)$ on $\mathbf{R}^n \times \mathbf{R} \times U$ does not depend on either $y \in \mathbf{R}$ or $x \in U$. This may also be considered as a particular case of the classes of fully nonlinear equations discussed in each of Sections 4.11 and 4.13. If $a \in \mathbf{R}^n$ and $b \in \mathbf{R}$, then

$$u(x) = a \cdot x + b \tag{4.14.3}$$

satisfies (4.14.2) on $\mathbf{R}^n$ if and only if

$$F(a) = 0. \tag{4.14.4}$$

4.14.1 Much simpler characteristic equations

Let I be an interval in the real line with nonempty interior, and which may be unbounded, and let $w(t)$, $z(t)$, and $p(t)$ be continuously-differentiable functions on I with values in U, $\mathbf{R}$, and $\mathbf{R}^n$, respectively, as usual. If $F(q)$ is continuously differentiable on $\mathbf{R}^n$, then the system of ordinary differential equations for $w(t)$, $z(t)$, and $p(t)$ discussed in Subsection 4.3.3 can be simplified further, as follows. The equations for $w'(t)$ are

$$w_l'(t) = \frac{\partial F}{\partial q_l}(p(t)) \tag{4.14.5}$$

for each $l = 1, \ldots, n$ and $t \in I$. The equations for $p'(t)$ are simply

$$p_j'(t) = 0 \tag{4.14.6}$$

for each $j = 1, \ldots, n$ and $t \in I$. The equation for $z'(t)$ is

$$z'(t) = \sum_{j=1}^{n} \frac{\partial F}{\partial q_j}(p(t))\, p_j(t) \tag{4.14.7}$$

for every $t \in I$.

Of course, (4.14.6) implies that $p(t)$ is constant on I. This means that the right sides of (4.14.5) and (4.14.7) are constant on I as well.

4.14.2 The eikonal equation

The *eikonal equation*

$$|\nabla u(x)| = 1 \tag{4.14.8}$$

is a partial differential equation of this type. More precisely, this is equivalent to saying that

$$|\nabla u(x)|^2 = 1 \tag{4.14.9}$$

on U. This corresponds to taking

$$F(q) = |q|^2 - 1 = \sum_{j=1}^{n} q_j^2 - 1, \tag{4.14.10}$$

which is a smooth function on $\mathbf{R}^n$.

4.14.3 More on Hamilton–Jacobi equations

Suppose that $n \geq 2$, and that $F(q)$ can be expressed as

$$F(q_1, \ldots, q_n) = q_n + \widetilde{F}(q_1, \ldots, q_{n-1}) \tag{4.14.11}$$

for some real-valued function $\widetilde{F}(q_1, \ldots, q_{n-1})$ on $\mathbf{R}^{n-1}$, as in Section 4.10. In this case, (4.14.2) is the same as saying that

$$\frac{\partial u}{\partial x_n}(x) + \widetilde{F}\Big(\frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_{n-1}}(x)\Big) = 0 \tag{4.14.12}$$

on U, as before. Remember that (4.14.5) reduces to (4.10.5) when $l = n$. Similarly, (4.14.7) reduces to

$$z'(t) = \sum_{j=1}^{n-1} \frac{\partial \widetilde{F}}{\partial q_j}(p_1(t), \ldots, p_{n-1}(t))\, p_j(t) + p_n(t) \tag{4.14.13}$$

under these conditions. Of course, (4.14.12) is a type of Hamilton–Jacobi equation, as in Subsection 4.11.3. This may normally be expressed a bit differently, as in Section 4.12. This type of Hamilton–Jacobi equation is discussed in Section 3.3 of [81].

4.15 Quasilinearity and derivatives

Let n be a positive integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let $F(q,y,x)$ be a continuously-differentiable real-valued function on $\mathbf{R}^n \times \mathbf{R} \times U$. Also let u be a twice continuously-differentiable real-valued function on U, and suppose that

$$F(Du(x), u(x), x) \text{ is constant on } U. \tag{4.15.1}$$

This implies that

$$\frac{\partial}{\partial x_j}(F(Du(x), u(x), x)) = 0 \tag{4.15.2}$$

on U for each $j = 1, \ldots, n$. This can be expanded using the chain rule to get partial differential equations that are linear in the second derivatives of u, as in Subsection 4.3.2.

4.15.1 A simplification in y

Suppose that there is a real number c such that

$$F(q,y,x) = F(q,0,x) + c\,y \tag{4.15.3}$$

on $\mathbf{R}^n \times \mathbf{R} \times U$. Equivalently,

$$F(q,y,x) = F_0(q,x) + c\,y \tag{4.15.4}$$

on $\mathbf{R}^n \times \mathbf{R} \times U$, where $F_0(q,x) = F(q,0,x)$ is a continuously-differentiable real-valued function on $\mathbf{R}^n \times U$. In this case, (4.15.1) is the same as saying that

$$F_0(Du(x), x) + c\,u(x) \text{ is constant on } U. \tag{4.15.5}$$

This implies that

$$\frac{\partial}{\partial x_j}(F_0(Du(x), x)) + c\,\frac{\partial u}{\partial x_j}(x) = 0 \tag{4.15.6}$$

on U for each $j = 1, \ldots, n$, as before. It is easy to see that these equations only involve the first and second derivatives of u, and not u itself.

4.15.2 Some related quasilinear equations

Suppose now that $n \geq 2$, and that $F(q,y,x)$ can be expressed as

$$F(q,y,x) = \widehat{F}(q_1, x) + \sum_{l=2}^{n} a_l\,q_l + c\,y \tag{4.15.7}$$

on $\mathbf{R}^n \times \mathbf{R} \times U$. Here $\widehat{F}(q_1, x)$ is a continuously-differentiable real-valued function on $\mathbf{R} \times U$, and $a_2, \ldots, a_n$ and c are real numbers. Under these conditions, (4.15.1) is the same as saying that

$$\widehat{F}\Big(\frac{\partial u}{\partial x_1}(x), x\Big) + \sum_{l=2}^{n} a_l \frac{\partial u}{\partial x_l}(x) + c\,u(x) \text{ is constant on } U. \tag{4.15.8}$$

In this case, the equation (4.15.2) with $j = 1$ reduces to

$$\frac{\partial \widehat{F}}{\partial q_1}\Big(\frac{\partial u}{\partial x_1}(x), x\Big) \frac{\partial^2 u}{\partial x_1^2}(x) + \sum_{l=2}^{n} a_l \frac{\partial^2 u}{\partial x_1 \, \partial x_l}(x) + c \frac{\partial u}{\partial x_1}(x) + \frac{\partial \widehat{F}}{\partial x_1}\Big(\frac{\partial u}{\partial x_1}(x), x\Big) = 0. \tag{4.15.9}$$

This may be considered as a first-order quasilinear partial differential equation in $\partial u / \partial x_1$ on U.

Suppose that $n = 2$, and that $F(q, y, x)$ can be expressed as

$$F(q, y, x) = \widetilde{F}(q_1) + q_2, \tag{4.15.10}$$

where $\widetilde{F}$ is a continuously-differentiable real-valued function on $\mathbf{R}$. This corresponds to taking $\widehat{F}(q_1, x) = \widetilde{F}(q_1)$, $a_2 = 1$, and $c = 0$ in (4.15.7). This means that (4.15.9) reduces to

$$\widetilde{F}'\Big(\frac{\partial u}{\partial x_1}(x)\Big) \frac{\partial^2 u}{\partial x_1^2}(x) + \frac{\partial^2 u}{\partial x_1 \, \partial x_2}(x) = 0. \tag{4.15.11}$$

This may be considered as a scalar conservation law in $\partial u / \partial x_1$, as in Subsection 4.12.1. This corresponds to a remark about the initial value problem (26) in Section 3.4.2 in [81].

Chapter 5

Some flows and exponentials

5.1 Some flows on $\mathbf{R}^n$

Let n be a positive integer, and let us identify $\mathbf{R}^n \times \mathbf{R}$ with $\mathbf{R}^{n+1}$, as in Section 4.12. An element of $\mathbf{R}^n \times \mathbf{R}$ may be expressed as (x, τ), with $x \in \mathbf{R}^n$ and $\tau \in \mathbf{R}$, as before. Let I be an interval in $\mathbf{R}$ with nonempty interior, which may be unbounded, and let W be a nonempty open subset of $\mathbf{R}^n$.

Suppose that for each $t \in I$,

(5.1.1) ϕ_t is a mapping from W into itself.

Typically we might have that $0 \in I$, and that ϕ_0 is the identity mapping on W. If $\xi \in W$, then we ask that

(5.1.2) $\phi_t(\xi)$ be differentiable as a function of $t \in I$ with values in $\mathbf{R}^n$.

This should be interpreted in terms of one-sided derivatives at any endpoints of I that are contained in I.

5.1.1 Functions on $W \times I$

Let $u(x, \tau)$ be a continuously-differentiable real or complex-valued function on $W \times I$. If $x \in W$ and τ is an endpoint of I that is contained in I, then the partial derivative of u at (x, τ) in x_j can be defined in the usual way for $j = 1, \ldots, n$, and the partial derivative in τ can be defined as a one-sided derivative.

If $\xi \in W$, then $u(\phi_t(\xi), t)$ is differentiable as a real or complex-valued function of $t \in I$, with

$$\frac{d}{dt}(u(\phi_t(\xi), t)) = \sum_{j=1}^{n} \frac{d\phi_{t,j}(\xi)}{dt} \frac{\partial u}{\partial x_j}(\phi_t(\xi), t) + \frac{\partial u}{\partial \tau}(\phi_t(\xi), t). \tag{5.1.3}$$

Here $\phi_{t,j}(\xi)$ is the jth coordinate of $\phi_t(\xi)$ for each $j = 1, \ldots, n$.

5.1.2 An additional hypothesis on ϕ_t

Note that

$$\Phi(\xi, t) = (\phi_t(\xi), t) \tag{5.1.4}$$

defines a mapping from $W \times I$ into itself. Suppose now that for each $t \in I$,

(5.1.5) ϕ_t is a one-to-one mapping from W onto itself.

Equivalently, this means that

(5.1.6) Φ is a one-to-one mapping from $W \times I$ onto itself.

If $1 \le j \le n$, then let a_j be the real-valued function on $W \times I$ such that

$$a_j(\phi_t(\xi), t) = \frac{d\phi_{t,j}(\xi)}{dt} \tag{5.1.7}$$

for every $\xi \in W$ and $t \in I$. Also put

$$a_{n+1} \equiv 1 \tag{5.1.8}$$

on $W \times I$, so that

$$a = (a_1, \ldots, a_n, a_{n+1}) \tag{5.1.9}$$

defines a mapping from $W \times I$ into $\mathbf{R}^{n+1}$.

5.1.3 The associated operator L_a

Put

$$L_a(u) = \sum_{j=1}^n a_j \frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial \tau} \tag{5.1.10}$$

on $W \times I$, as in Section 4.1. By construction,

$$(L_a(u))(\phi_t(\xi), t) = \frac{d}{dt}(u(\phi_t(\xi), t)) \tag{5.1.11}$$

for every $\xi \in W$ and $t \in I$. Similarly, if $\xi \in W$, then

$$(\phi_t(\xi), t) \tag{5.1.12}$$

satisfies the system of ordinary differential equations associated to a as a function of $t \in I$ as for $w(t)$ in Subsection 4.1.2.

5.1.4 Another additional hypothesis on ϕ_t

Suppose for the moment that $I = \mathbf{R}$, and that

$$\phi_{r+t}(\xi) = \phi_r(\phi_t(\xi)) \tag{5.1.13}$$

for every $\xi \in W$ and $r, t \in \mathbf{R}$. This implies that the derivative of $\phi_t(\xi)$ in t at t is the same as the derivative of $\phi_r(\phi_t(\xi))$ in r at $r = 0$. This means that

$$a(\phi_t(\xi), t) = a(\phi_t(\xi), 0), \tag{5.1.14}$$

so that $a(x, \tau)$ does not depend on τ. Note that (5.1.13) implies that ϕ_0 is the identity mapping on W, because ϕ_0 is supposed to map W onto itself.

5.2 A more local version

Let n be a positive integer, and let us identify $\mathbf{R}^n \times \mathbf{R}$ with $\mathbf{R}^{n+1}$ again. Let U be an open subset of $\mathbf{R}^n \times \mathbf{R}$, and put

$$U_t = \{x \in \mathbf{R}^n : (x, t) \in U\} \tag{5.2.1}$$

for each $t \in \mathbf{R}$, which is an open set in $\mathbf{R}^n$. Let V be another open subset of $\mathbf{R}^n \times \mathbf{R}$, and let V_t be as in (5.2.1) for each $t \in \mathbf{R}$. If $\xi \in \mathbf{R}^n$, then

$$\{t \in \mathbf{R} : (\xi, t) \in V\} \tag{5.2.2}$$

is an open subset of $\mathbf{R}$.

Suppose that for each $t \in \mathbf{R}$,

$$\phi_t \text{ is a mapping from } V_t \text{ into } U_t. \tag{5.2.3}$$

This means that

$$\Phi(\xi, t) = (\phi_t(\xi), t) \tag{5.2.4}$$

defines a mapping from V into U. If $\xi \in \mathbf{R}^n$, then we ask that

$$\phi_t(\xi) \text{ be differentiable as a function of } t \tag{5.2.5}$$

in (5.2.2) with values in $\mathbf{R}^n$.

5.2.1 Functions on U

Let u be a continuously-differentiable real or complex-valued function on U, and let $\xi \in \mathbf{R}^n$ be given. If t is an element of (5.2.2), then $(\xi, t) \in V$, $\xi \in V_t$, $\phi_t(\xi) \in U_t$, and thus

$$(\phi_t(\xi), t) \in U. \tag{5.2.6}$$

This means that

$$u(\phi_t(\xi), t) \tag{5.2.7}$$

is defined as a real or complex-valued function on (5.2.2). In fact, (5.2.7) is differentiable as a real or complex-valued function of t in (5.2.2), with derivative in t as in (5.1.3).

5.2.2 An additional bijectivity condition

Suppose now that for each $t \in \mathbf{R}$,

$$\phi_t \text{ is a one-to-one mapping from } V_t \text{ onto } U_t. \tag{5.2.8}$$

Equivalently, this means that the mapping Φ in (5.2.4) is a one-to-one mapping from V onto U. If $1 \le j \le n$, then let a_j be the real-valued function on U such that

$$a_j(\phi_t(\xi), t) = \frac{d\phi_{t,j}(\xi)}{dt} \tag{5.2.9}$$

for every $(\xi, t) \in V$. Also put $a_{n+1} \equiv 1$ on U, so that $a = (a_1, \ldots, a_n, a_{n+1})$ defines a mapping from U into $\mathbf{R}^{n+1}$.

Let $L_a(u)$ be defined on U as in (5.1.10). If $(\xi, t) \in V$, then

$$(L_a(u))(\phi_t(\xi), t) = \frac{d}{dt}(u(\phi_t(\xi), t)), \tag{5.2.10}$$

as before. Similarly, if $\xi \in \mathbf{R}^n$, then

$$(\phi_t(\xi), t) \tag{5.2.11}$$

satisfies the system of ordinary differential equations associated to a as a function of t in (5.2.2) as for $w(t)$ in Subsection 4.1.2.

Let $(\xi, t) \in V$ be given, and suppose that

$$(\phi_t(\xi), 0) \in V. \tag{5.2.12}$$

This implies that

$$(\phi_t(\xi), r) \in V \tag{5.2.13}$$

for every $r \in \mathbf{R}$ with $|r|$ sufficiently small. Of course, we also have that

$$(\xi, t + r) \in V \tag{5.2.14}$$

when $|r|$ is sufficiently small.

5.2.3 An additional condition on ϕ_{r+t}

Suppose that

$$\phi_{r+t}(\xi) = \phi_r(\phi_t(\xi)) \tag{5.2.15}$$

when r is sufficiently small. This implies that the derivative of $\phi_t(\xi)$ in t at t is equal to the derivative of $\phi_r(\phi_t(\xi))$ in r at $r = 0$, as in the previous section. This means that

$$a(\phi_t(\xi), t) = a(\phi_t(\xi), 0), \tag{5.2.16}$$

as before.

5.3 Some basic first-order operators

Let n be a positive integer, and suppose that $a_j(x)$ is a real-valued linear function on $\mathbf{R}^n$ for each $j = 1, \ldots, n$. This can be expressed as

$$a_j(x) = \sum_{l=1}^{n} a_{j,l}\, x_l \tag{5.3.1}$$

for $x \in \mathbf{R}^n$ and $j = 1, \ldots, n$, where $(a_{j,l}) = (a_{j,l})_{j,l=1}^n$ is an $n \times n$ matrix of real numbers. Equivalently,

$$a(x) = (a_1(x), \ldots, a_n(x)) \tag{5.3.2}$$

is a linear mapping from $\mathbf{R}^n$ into itself, which corresponds to this matrix in the usual way.

Let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuously-differentiable real or complex-valued function on U. Thus

$$(L_a(u))(x) = \sum_{j=1}^{n} a_j(x) \, \frac{\partial u}{\partial x_j}(x) \tag{5.3.3}$$

defines a continuous real or complex-valued function on U, as appropriate. Note that the examples mentioned in Sections 2.8, 4.7, and 4.8 are of this form.

5.3.1 Some homogeneity conditions

Suppose for the moment that $U = \mathbf{R}^n \setminus \{0\}$, and that

$$u \text{ is homogeneous of degree } b \in \mathbf{C}. \tag{5.3.4}$$

It is easy to see that

$$L_a(u) \text{ is homogeneous of degree } b \tag{5.3.5}$$

as well, because the partial derivatives of u are homogeneous of degree $b - 1$, as in Subsection 2.8.2. Similarly, if p is a polynomial on $\mathbf{R}^n$ with real or complex coefficients that is homogeneous of degree k for some nonnegative integer k, then $L_a(p)$ is a homogeneous polynomial of degree k on $\mathbf{R}^n$ too.

5.3.2 Commutators in this case

Let $b_1(x), \ldots, b_n(x)$ be n more real-valued linear functions on $\mathbf{R}^n$, and let b and L_b be as before. Observe that

$$c_j = L_a(b_j) - L_b(a_j) \tag{5.3.6}$$

is a real-valued linear function on $\mathbf{R}^n$ for each $j = 1, \ldots, n$, as in the preceding paragraph. If c and L_c are as before, then L_c corresponds to the commutator of L_a and L_b, as in Subsection 2.3.1.

Let $(b_{j,l})$ and $(c_{j,l})$ be the matrices corresponding to b and c, respectively, as before. Clearly

$$L_a(b_j) = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{k,l} \, x_l \, \frac{\partial b_j}{\partial x_k} = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{k,l} b_{j,k} \, x_l \tag{5.3.7}$$

for each $j = 1, \ldots, n$. Similarly,

$$L_b(a_j) = \sum_{k=1}^{n} \sum_{l=1}^{n} b_{k,l} \, a_{j,k} \, x_l \tag{5.3.8}$$

for each $j = 1, \ldots, n$. It follows that

$$c_{j,l} = \sum_{k=1}^{n} b_{j,k}\, a_{k,l} - \sum_{k=1}^{n} a_{j,k}\, b_{k,l} \tag{5.3.9}$$

for each $j, l = 1, \ldots, n$.

5.4 Exponentiating real matrices

Let n be a positive integer, and let A be a linear mapping from $\mathbf{R}^n$ into itself. This corresponds to an $n \times n$ matrix of real numbers in a standard way, as in the previous section. Of course, the composition of two linear mappings on $\mathbf{R}^n$ is another linear mapping on $\mathbf{R}^n$. It is well known and not difficult to see that this corresponds to matrix multiplication of the corresponding matrices.

If j is a positive integer, then A^j denotes the composition of A with itself a total of $j - 1$ times, so that there are j factors of A. This is interpreted as being the identity mapping I on $\mathbf{R}^n$ when $j = 0$. One would like to define the exponential of A by

$$\exp A = \sum_{j=0}^{\infty} (1/j!)\, A^j, \tag{5.4.1}$$

as another linear mapping on $\mathbf{R}^n$.

5.4.1 Absolute convergence of the sum

More precisely, it is well known and not difficult to show that there is a nonnegative real number C such that

$$|A(v)| \le C\, |v| \tag{5.4.2}$$

for every $v \in \mathbf{R}^n$. The smallest such C is known as the *operator norm* of A with respect to the standard Euclidean norm on $\mathbf{R}^n$. It follows that

$$|A^j(v)| \le C^j\, |v| \tag{5.4.3}$$

for every $j \ge 1$ and $v \in \mathbf{R}^n$. This also works with $j = 0$, and C^j interpreted as being equal to 1, as usual.

If $v \in \mathbf{R}^n$, then

$$\sum_{j=0}^{\infty} (1/j!)\, C^j\, |v| \tag{5.4.4}$$

is a convergent series of nonnegative real numbers, with sum equal to

$$(\exp C)\, |v|, \tag{5.4.5}$$

because of the usual series expansion for $\exp C$. It follows that

$$\sum_{j=0}^{\infty} (1/j!)\, |A^j(v)| \tag{5.4.6}$$

is a convergent series of nonnegative real numbers, with sum less than or equal to (5.4.5), because of (5.4.3) and the comparison test. Let $(A^j(v))_l$ be the lth coordinate of $A^j(v) \in \mathbf{R}^n$ for every $l = 1, \ldots, n$, so that

$$|(A^j(v))_l| \le |A^j(v)| \tag{5.4.7}$$

for each $j \ge 0$ and $l = 1, \ldots, n$. Thus

$$\sum_{j=0}^{\infty} (1/j!)\, |(A^j(v))_l| \tag{5.4.8}$$

is a convergent series of nonnegative real numbers for every $l = 1, \ldots, n$. This means that

$$\sum_{j=0}^{\infty} (1/j!)\, (A^j(v))_l \tag{5.4.9}$$

is an absolutely convergent series of real numbers for every $l = 1, \ldots, n$.

We would like to put

$$(\exp A)(v) = \sum_{j=0}^{\infty} (1/j!)\, A^j(v), \tag{5.4.10}$$

as an element of $\mathbf{R}^n$. The lth coordinate of the right side is equal to (5.4.9) for every $l = 1, \ldots, n$. It is easy to see that this defines a linear mapping from $\mathbf{R}^n$ into itself. One could also look at this in terms of matrices, where the entries of the matrix corresponding to $\exp A$ can be expressed as absolutely convergent series of real numbers.

5.4.2 Exponentials and eigenvectors

Suppose that v is an eigenvector of A with eigenvalue $\lambda \in \mathbf{R}$, so that

$$A(v) = \lambda\, v. \tag{5.4.11}$$

This implies that

$$A^j(v) = \lambda^j\, v \tag{5.4.12}$$

for every $j \ge 0$. It follows that

$$(\exp A)(v) = (\exp \lambda)\, v. \tag{5.4.13}$$

5.4.3 Exponentials and conjugations

Let T be a one-to-one linear mapping from $\mathbf{R}^n$ onto itself, so that the inverse mapping T^{-1} is linear on $\mathbf{R}^n$ too. It is easy to see that

$$T \circ A^j \circ T^{-1} = (T \circ A \circ T^{-1})^j \tag{5.4.14}$$

for every $j \ge 0$. This means that

$$T \circ (\exp A) \circ T^{-1} = \exp(T \circ A \circ T^{-1}). \tag{5.4.15}$$

5.5 Exponentials of sums

Let n be a positive integer, and let A, B be linear mappings from $\mathbf{R}^n$ into itself. Suppose that A and B commute on $\mathbf{R}^n$, so that

$$A \circ B = B \circ A. \tag{5.5.1}$$

If l is a positive integer, then one can check that

$$(A + B)^l = \sum_{j=0}^{l} \binom{l}{j} A^j \circ B^{l-j}, \tag{5.5.2}$$

as in the binomial theorem.

This implies that

$$\begin{aligned} \exp(A + B) &= \sum_{l=0}^{\infty} (1/l!)\,(A + B)^l \\ &= \sum_{l=0}^{\infty} \Big(\sum_{j=0}^{l} (1/j!)\,(1/(l-j)!)\,A^j \circ B^{l-j} \Big). \end{aligned} \tag{5.5.3}$$

The right side corresponds to the Cauchy product of the series used to define $\exp A$ and $\exp B$. In particular, this means that the same terms are being summed, but in different ways. One can use this to show that

$$\exp(A + B) = (\exp A) \circ (\exp B) \tag{5.5.4}$$

under these conditions. More precisely, this also uses absolute convergence of the sums, to ensure that the different ways of arranging the sums lead to the same results.

5.5.1 Invertibility of $\exp A$

Note that $\exp A$ automatically commutes with A. Similarly, if A commutes with B, then $\exp A$ commutes with B. Of course, if $A = 0$, then $\exp A = I$. If A is any linear mapping on $\mathbf{R}^n$, then

$$(\exp A) \circ (\exp(-A)) = (\exp(-A)) \circ (\exp A) = I, \tag{5.5.5}$$

by (5.5.4). This implies that $\exp A$ is invertible on $\mathbf{R}^n$, with inverse equal to $\exp(-A)$.

5.5.2 The exponential of A'

Let A, B be any two linear mappings on $\mathbf{R}^n$, and let A', B' be the linear mappings corresponding to them as in Subsection 1.15.2. If $v, w \in \mathbf{R}^n$, then

$$\begin{aligned} (A \circ B)(v) \cdot w = A(B(v)) \cdot w &= B(v) \cdot A'(w) \\ &= v \cdot B'(A'(w)) = v \cdot (B' \circ A')(w). \end{aligned} \tag{5.5.6}$$

This means that

$$(A \circ B)' = B' \circ A'. \tag{5.5.7}$$

In particular,

$$(A^j)' = (A')^j \tag{5.5.8}$$

for each $j \geq 0$. It follows that

$$(\exp A)' = \exp(A'). \tag{5.5.9}$$

If

$$A' = -A, \tag{5.5.10}$$

then we get that

$$(\exp A)' = \exp(A') = \exp(-A) = (\exp A)^{-1}. \tag{5.5.11}$$

This means that $\exp A$ is an orthogonal transformation on $\mathbf{R}^n$, as in Subsection 1.15.1.

5.6 The exponential of $t\,A$

Let n be a positive integer, let A be a linear mapping from $\mathbf{R}^n$ into itself, and let t be a real number. Of course, $t\,A$ may be considered as a linear mapping on $\mathbf{R}^n$, with $(t\,A)(v) = t\,A(v)$ for every $v \in \mathbf{R}^n$. Thus the exponential of $t\,A$ may be defined as before, so that

$$\exp(t\,A) = \sum_{j=0}^{\infty} (1/j!)\, t^j\, A^j. \tag{5.6.1}$$

This may be considered as a power series in t, whose coefficients are linear mappings on $\mathbf{R}^n$. If $v \in \mathbf{R}^n$, then

$$(\exp(t\,A))(v) = \sum_{j=0}^{\infty} (1/j!)\, t^j\, A^j(v) \tag{5.6.2}$$

may be considered as a power series in t, with coefficients in $\mathbf{R}^n$.

More precisely, for each $l = 1, \ldots, n$, the lth coordinate of $(\exp(t\,A))(v)$ is

$$((\exp(t\,A))(v))_l = \sum_{j=0}^{\infty} (1/j!)\, t^j\, (A^j(v))_l. \tag{5.6.3}$$

This is an absolutely convergent power series in t with coefficients in $\mathbf{R}$. Similarly, the entries of the matrix associated to $\exp(t\,A)$ may be expressed as absolutely convergent power series in t with real coefficients.

In particular, these are smooth functions of t on $\mathbf{R}$, by standard results about power series. We can differentiate these series termwise, to get that

$$\frac{d}{dt}((\exp(t\,A))(v)) = A((\exp(t\,A))(v)) \tag{5.6.4}$$

for every $v \in \mathbf{R}^n$. This can be expressed by

$$\frac{d}{dt}(\exp(t\,A)) = A \circ (\exp(t\,A)). \tag{5.6.5}$$

5.6.1 Differential equations related to $\exp(t\,A)$

Let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a real or complex-valued function on U that is continuously-differentiable on U. Put

$$(L_A(u))(x) = \sum_{l=1}^{n} (A(x))_l \, \frac{\partial u}{\partial x_l}(x) \tag{5.6.6}$$

for each $x \in U$. This is the same as in Section 5.3, with different notation. This is related to the system of ordinary differential equations

$$w'(t) = A(w(t)), \tag{5.6.7}$$

as in Subsection 4.1.2, where $w(t)$ is a continuously-differentiable function on an interval in the real line with nonempty interior, and with values in $\mathbf{R}^n$. If $v \in \mathbf{R}^n$, then

$$w(t) = (\exp(t\,A))(v) \tag{5.6.8}$$

satisfies (5.6.7), as in (5.6.4).

Let I be an open interval in the real line, which may be unbounded, with

$$(\exp(t\,A))(v) \in U \tag{5.6.9}$$

for each $t \in I$. Under these conditions,

$$\frac{d}{dt} u((\exp(t\,A))(v)) = (L_A(u))((\exp(t\,A))(v)) \tag{5.6.10}$$

on I, as in Subsection 4.1.2. This can be used to analyze first-order semilinear equations on U involving L_A, as before.

5.7 Traces and determinants

Let n be a positive integer, and let $(a_{j,l})$ be an $n \times n$ matrix of real numbers. The *trace* of this matrix is defined as usual as

$$\sum_{j=1}^{n} a_{j,j}. \tag{5.7.1}$$

The *determinant* of $(a_{j,l})$ is defined in a standard way, that we shall not repeat here.

If A is a linear mapping from $\mathbf{R}^n$ into itself, then A corresponds to an $n \times n$ matrix $(a_{j,l})$ of real numbers in a standard way. The *trace* $\operatorname{tr} A$ and *determinant* $\det A$ of A are defined as the trace and determinant of $(a_{j,l})$, respectively.

Let B be another linear mapping from $\mathbf{R}^n$ into itself. It is well known and not difficult to verify that

$$\mathrm{tr}(A \circ B) = \mathrm{tr}(B \circ A). \tag{5.7.2}$$

It is also well known that

$$\det(A \circ B) = (\det A)\,(\det B). \tag{5.7.3}$$

If t is a real number, then $I + t\,A$ is another linear mapping from $\mathbf{R}^n$. It is clear from the definition of the determinant that

$$\det(I + t\,A) \tag{5.7.4}$$

is a polynomial in t of degree at most n. One can check that this polynomial is of the form

$$1 + (\mathrm{tr}\,A)\,t + \cdots, \tag{5.7.5}$$

where the additional terms are multiples of t^j, $2 \le j \le n$. This means that the derivative of (5.7.4) in t at $t = 0$ is equal to $\mathrm{tr}\,A$.

5.7.1 A connection with the exponential

It is well known that

$$\det(\exp A) = \exp(\mathrm{tr}\,A). \tag{5.7.6}$$

One way to see this is to use calculus to show that

$$\det(\exp(t\,A)) = \exp(t\ \mathrm{tr}\,A) \tag{5.7.7}$$

for every $t \in \mathbf{R}$. Note that both sides of this equation are equal to 1 at $t = 0$.

The right side of (5.7.7) satisfies the differential equation

$$f'(t) = (\mathrm{tr}\,A)\,f(t) \tag{5.7.8}$$

on $\mathbf{R}$. We would like to check that the left side of (5.7.7) satisfies the same differential equation. If we can do that, then (5.7.7) follows, by standard arguments.

One can verify directly that the left side of (5.7.7) satisfies (5.7.8) at $t = 0$. Let $t_0 \in \mathbf{R}$ be given, and observe that

$$\exp(t\,A) = (\exp((t - t_0)\,A)) \circ (\exp(t_0\,A)) \tag{5.7.9}$$

for every $t \in \mathbf{R}$, as in Section 5.5. One can use this to obtain that the left side of (5.7.7) satisifes (5.7.8) at t_0 from the analogous statement at 0.

5.8 Exponentiating complex matrices

Let m be a positive integer, and let A be a linear mapping from $\mathbf{C}^m$ into itself, as a vector space over the complex numbers. This corresponds to an $m \times m$ matrix of complex numbers in the usual way. The composition of two linear mappings on $\mathbf{C}^m$ corresponds to matrix multiplication of the corresponding matrices of complex numbers.

If j is a positive integer, then A^j denotes the composition of A with itself a total of $j - 1$ times, so that there are j factors of A, and which is interpreted as being the identity mapping I on $\mathbf{C}^m$ when $j = 0$. As in the real case, it is well known and not difficult to show that there is a nonnegative real number C such that

$$|A(v)| \le C\,|v| \tag{5.8.1}$$

for every $v \in \mathbf{C}^m$, and the smallest such C is the *operator norm* of A with respect to the standard Euclidean norm on $\mathbf{C}^m$. This implies that

$$|A^j(v)| \le C^j\,|v| \tag{5.8.2}$$

for every $j \ge 0$ and $v \in \mathbf{C}^m$.

One would like to define the exponential of A as another linear mapping on $\mathbf{C}^m$ by

$$\exp A = \sum_{j=0}^{\infty} (1/j!)\, A^j, \tag{5.8.3}$$

as in Section 5.4. More precisely, if $v \in \mathbf{C}^m$, then we would like to put

$$(\exp A)(v) = \sum_{j=0}^{\infty} (1/j!)\, A^j(v), \tag{5.8.4}$$

as an element of $\mathbf{C}^m$, as before. This means that for each $l = 1, \ldots, m$, the lth coordinate of $(\exp A)(v)$ is equal to

$$((\exp A)(v))_l = \sum_{j=0}^{\infty} (1/j!)\, (A^j(v))_l. \tag{5.8.5}$$

The right side is an absolutely convergent series of complex numbers, by the comparison test. This defines a linear mapping on $\mathbf{C}^m$, and the entries of the corresponding matrix can be expressed as absolutely convergent series of complex numbers in an analogous way.

Note that a linear mapping from $\mathbf{R}^m$ into itself, as a vector space over the real numbers, has a unique extension to a linear mapping from $\mathbf{C}^m$ into itself, as a vector space over the complex numbers. Both linear mappings correspond to the same $m \times m$ matrix of real numbers, which may be considered as an $m \times m$ matrix of complex numbers too. The exponential of the linear mapping on $\mathbf{C}^m$ is the same as the extension of the exponential of the linear mapping on $\mathbf{R}^m$ to a linear mapping on $\mathbf{C}^m$.

5.8.1 Some additional properties of $\exp A$

Suppose that $v \in \mathbf{C}^m$ is an eigenvector of A with eigenvalue $\lambda \in \mathbf{C}$, so that

$$A(v) = \lambda\, v. \tag{5.8.6}$$

It is easy to see that

$$(\exp A)(v) = (\exp \lambda)\, v, \tag{5.8.7}$$

as before. If T is a one-to-one linear mapping from $\mathbf{C}^m$ onto itself, then

$$T \circ (\exp A) \circ T^{-1} = \exp(T \circ A \circ T^{-1}), \tag{5.8.8}$$

as before.

Let B be another linear mapping from $\mathbf{C}^m$ into itself, and suppose that A and B commute on $\mathbf{C}^m$, so that

$$A \circ B = B \circ A. \tag{5.8.9}$$

Under these conditions,

$$\exp(A + B) = (\exp A) \circ (\exp B), \tag{5.8.10}$$

as in Section 5.5. We also have that $\exp A$ commutes with B in this case, as before. If we take $B = -A$, then we get that $\exp A$ is invertible on $\mathbf{C}^m$, with inverse equal to $\exp(-A)$, as in Subsection 5.5.1.

The trace and determinant of an $m \times m$ matrix of complex numbers can be defined in the same way as for real numbers. Similarly, the trace and determinant of A are defined to be the trace and determinant of the matrix corresponding to A, respectively. These satisfy the same basic properties as in the real case. In particular, it is well known that

$$\det(\exp A) = \exp(\operatorname{tr} A), \tag{5.8.11}$$

which can be shown using an argument like the one in Subsection 5.7.1. Alternatively, one can use results from linear algebra to reduce to the case where A corresponds to an upper triangular matrix, for which (5.8.11) can be verified more directly.

5.9 More on $\mathbf{C}^m$

Let m be a positive integer, and let $\langle v, w\rangle = \langle v, w\rangle_{\mathbf{C}^m}$ be the standard inner product on $\mathbf{C}^m$, as in Section 2.6. If $v, w \in \mathbf{C}^m$, then

$$\begin{aligned} |v + w|^2 = \langle v + w, v + w\rangle &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= |v|^2 + 2\,\mathrm{Re}\langle v, w\rangle + |w|^2. \end{aligned} \tag{5.9.1}$$

If we replace w with $i\, w$, then we get that

$$|v + iw|^2 = |v|^2 + 2\,\mathrm{Re}(-i\,\langle v, w\rangle) + |w|^2 = |v|^2 + 2\,\mathrm{Im}\langle v, w\rangle + |w|^2. \tag{5.9.2}$$

It follows that

(5.9.3) $\langle v, w \rangle = (1/2)\,(|v + w|^2 - |v|^2 - |w|^2) + (i/2)\,(|v + i\,w|^2 - |v|^2 - |w|^2).$

This is another *polarization identity.*

Let T be a linear mapping from $\mathbf{C}^m$ into itself, as a vector space over the complex numbers. As in the real case,

$$\ker T = \{v \in \mathbf{C}^m : T(v) = 0\} \tag{5.9.4}$$

is a linear subspace of $\mathbf{C}^m$, called the *kernel* of T. This is equal to $\{0\}$ if and only if T is one-to-one, as before. It is is well known that T is one-to-one on $\mathbf{C}^m$ if and only if T maps $\mathbf{C}^m$ onto itself, in which case the inverse mapping T^{-1} is linear on $\mathbf{C}^m$ too.

5.9.1 Unitary transformations

A one-to-one linear mapping T from $\mathbf{C}^m$ onto itself is said to be *unitary* if

$$\langle T(v), T(w) \rangle = \langle v, w \rangle \tag{5.9.5}$$

for every $v, w \in \mathbf{C}^m$. Note that this implies that T^{-1} is unitary as well. In this case, we can take $v = w$ in (5.9.5), to get that

$$|T(v)| = |v|. \tag{5.9.6}$$

Conversely, if (5.9.5) holds for every $v \in \mathbf{C}^m$, then (5.9.5) holds for every $v.\ w$ in $\mathbf{C}^m$, because of the polarization identity (5.9.3). Of course, (5.9.6) implies that $\ker T = \{0\}$.

If T is any linear mapping from $\mathbf{C}^m$ into itself, then it is well known that there is a unique linear mapping T^* from $\mathbf{C}^m$ into itself such that

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle \tag{5.9.7}$$

for every $v, w \in \mathbf{C}^m$. This is called the *adjoint* of T. As in the real case, every linear mapping from $\mathbf{C}^m$ into itself corresponds to an $m \times m$ matrix of complex numbers in a standard way. The matrix associated to T^* is obtained by taking the complex conjugates of the entries of the transpose of the matrix associated to T.

If T is a unitary transformation on $\mathbf{C}^m$, then one can verify that T^* is the same as the inverse of T. Conversely, if T is an invertible linear mapping on $\mathbf{C}^m$, with inverse equal to T^*, then T is a unitary transformation on $\mathbf{C}^m$.

5.9.2 Some additional properties of adjoints

Let A, B be linear mappings from $\mathbf{C}^m$ into itself, and let t be a complex number. Under these conditions, $A + B$ and $t\,A$ are linear mappings on $\mathbf{C}^m$, and one can check that

$$(A + B)^* = A^* + B^* \tag{5.9.8}$$

and

$$(t\,A)^* = \bar{t}\,A^*. \tag{5.9.9}$$

One can also verify that

$$(A \circ B)^* = B^* \circ A^*. \tag{5.9.10}$$

This implies that

$$(A^j)^* = (A^*)^j \tag{5.9.11}$$

for each nonnegative integer j, so that

$$(\exp A)^* = \exp(A^*). \tag{5.9.12}$$

If

$$A^* = -A, \tag{5.9.13}$$

then it follows that

$$(\exp A)^* = \exp(A^*) = \exp(-A) = (\exp A)^{-1}, \tag{5.9.14}$$

so that $\exp A$ is a unitary transformation on $\mathbf{C}^m$.

5.9.3 Self-adjoint linear mappings

A linear mapping A on $\mathbf{C}^m$ is said to be *self-adjoint* with respect to the standard inner product on $\mathbf{C}^m$ if

$$A^* = A. \tag{5.9.15}$$

If T is any linear mapping on $\mathbf{C}^m$, then it is easy to see that

$$(T^*)^* = T. \tag{5.9.16}$$

One can use this to check that

$$A = (1/2)\,(T + T^*) \tag{5.9.17}$$

and

$$B = (-i/2)\,(T - T^*) \tag{5.9.18}$$

are self-adjoint. Note that

$$T = A + i\,B. \tag{5.9.19}$$

5.10 The exponential of $z\,A$

Let m be a positive integer, let A be a linear mapping from $\mathbf{C}^m$ into itself, and let z be a complex number. Thus $z\,A$ is another linear mapping from $\mathbf{C}^m$ into itself, whose exponential

$$\exp(z\,A) = \sum_{j=0}^{\infty} (1/j!)\,z^j\,A^j \tag{5.10.1}$$

may be considered as a power series in z, with coefficients that are linear mappings on $\mathbf{C}^m$. If $v \in \mathbf{C}^m$, then

$$(\exp(z\,A))(v) = \sum_{j=0}^{\infty} (1/j!)\, z^j\, A^j(v) \tag{5.10.2}$$

may be considered as a power series in z, with coefficients in $\mathbf{C}^m$.

As in Section 5.6, the lth coordinate of $(\exp(z\,A))(v)$ is

$$((\exp(z\,A))(v))_l = \sum_{j=0}^{\infty} (1/j!)\, z^j\, (A^j(v))_l \tag{5.10.3}$$

for each $l = 1, \ldots, m$, which is an absolutely convergent power series in z with complex coefficients. Similarly, the entries of the matrix associated to $\exp(z\,A)$ may be expressed as absolutely convergent power series in z with complex coefficients. One can differentiate these series termwise, to get that they are holomorphic functions of z, with

$$\frac{\partial}{\partial z}((\exp(z\,A))(v)) = A((\exp(z\,A))(v)) \tag{5.10.4}$$

for every $v \in \mathbf{C}^m$. This can be expressed by

$$\frac{\partial}{\partial z}(\exp(z\,A)) = A \circ (\exp(z\,A)), \tag{5.10.5}$$

as before.

5.10.1 Nilpotent linear mappings

Let r be a nonnegative integer, and suppose that

$$A^{r+1} = 0 \tag{5.10.6}$$

on $\mathbf{C}^m$. In this case, A is said to be *nilpotent* on $\mathbf{C}^m$. It is well known that if A is nilpotent on $\mathbf{C}^m$, then one can take $r \le m - 1$. Of course, if (5.10.6) holds, then $A^j = 0$ when $j \ge r + 1$. This means that

$$\exp(z\,A) = \sum_{j=0}^{r} (1/j!)\, z^j\, A^j \tag{5.10.7}$$

is a polynomial in z, with coefficients that are linear mappings on $\mathbf{C}^m$.

Note that

$$\exp(c\,z\,I) = (\exp(c\,z))\, I \tag{5.10.8}$$

for every $c, z \in \mathbf{C}$, where I is the identity maping on $\mathbf{C}^m$. If A is any linear mapping on $\mathbf{C}^m$, then A commutes with $c\,I$ on $\mathbf{C}^m$. This implies that

$$\exp(z\,(c\,I + A)) = (\exp(c\,z\,I)) \circ (\exp(z\,A)) = (\exp(c\,z))\, \exp(z\,A). \tag{5.10.9}$$

5.11 Polynomials and differential operators

Let n be a positive integer, and remember that $\mathcal{P}(\mathbf{R}^n, \mathbf{R})$ and $\mathcal{P}(\mathbf{R}^n, \mathbf{C})$ are the spaces of polynomials on $\mathbf{R}^n$ with real and complex coefficients, respectively, as in Section 2.9. If k is a nonnegative integer, then let $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ and $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$ be the spaces of polynomials on $\mathbf{R}^n$ with real and complex coefficients and degree less than or equal to k, respectively. These are linear subspaces of $\mathcal{P}(\mathbf{R}^n, \mathbf{R})$ and $\mathcal{P}(\mathbf{R}^n, \mathbf{C})$, as vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively.

Consider the collection of monomials x^β, where β is a multi-index with order $|\beta| \le k$. This collection is a basis for $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ and $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively. In particular, $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ and $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$ have the same finite dimension, as vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively.

Let N be a nonnegative integer, and suppose that a_α is a polynomial on $\mathbf{R}^n$ with real or complex coefficients for each multi-index α with $|\alpha| \le N$, so that

$$L = \sum_{|\alpha| \le N} a_\alpha \, \partial^\alpha \tag{5.11.1}$$

defines a differential operator on $\mathbf{R}^n$ with polynomial coefficients, as in Section 2.9. Remember that L maps $\mathcal{P}(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}(\mathbf{R}^n, \mathbf{C})$ into itself, as appropriate. Suppose that

$$\deg a_\alpha \le |\alpha| \tag{5.11.2}$$

for each α, $|\alpha| \le N$. If p is a polynomial on $\mathbf{R}^n$ with real or complex coefficients, as appropriate, then

$$\deg L(p) \le \deg p. \tag{5.11.3}$$

This means that L maps $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$ into itself for each $k \ge 0$, as appropriate.

5.11.1 A more precise condition

Similarly, let c be a nonnegative integer, and suppose that

$$\deg a_\alpha \le |\alpha| - c \tag{5.11.4}$$

for each α, $|\alpha| \le N$. This is interpreted to mean that

$$a_\alpha = 0 \text{ when } |\alpha| < c. \tag{5.11.5}$$

If p is a polynomial on $\mathbf{R}^n$ with real or complex coefficients, as appropriate, then

$$\deg L(p) \le \deg p - c. \tag{5.11.6}$$

As before, this means that

$$L(p) = 0 \text{ when } \deg p < c. \tag{5.11.7}$$

If j is a positive integer, then we get that

$$\deg L^j(p) \le \deg p - c\, j. \tag{5.11.8}$$

This means that

$$L^j(p) = 0 \text{ when } \deg p < c\,j, \tag{5.11.9}$$

as usual. Suppose that $c \geq 1$, and let k be a nonnegative integer. If

$$k < c\,j, \tag{5.11.10}$$

then it follows that

$$L^j = 0 \text{ on } \mathcal{P}^k(\mathbf{R}^n, \mathbf{R}) \text{ or } \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}), \tag{5.11.11}$$

as appropriate. Thus the restriction of L to $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as appropriate, is nilpotent under these conditions.

5.12 Some related differential equations

Let n be a positive integer, let N be a nonnegative integer, and let L be a differential operator of order less than or equal to N on $\mathbf{R}^n$ with polynomial coefficients, as in the previous section. Suppose that the coefficients satisfy (5.11.2) for each α, $|\alpha| \leq N$, and let k be a nonnegative integer. Thus L maps $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$ into itself, as before. Let L_k be the restriction of L to $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as appropriate.

Let $m = m(k)$ be the number of multi-indices β with order $|\beta| \leq k$. We can identify $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ and $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$ with $\mathbf{R}^m$ and $\mathbf{C}^m$, respectively, by listing the coefficients of a polynomial on $\mathbf{R}^n$ with degree less than or equal to k in any reasonable way. This means that we can identify L_k with a linear mapping from $\mathbf{R}^m$ or $\mathbf{C}^m$ into itself, as appropriate.

If $t \in \mathbf{R}$, then we can define the exponential of $t\,L_k$ as a linear mapping on $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as appropriate, as before. Let p be a polynomial on $\mathbf{R}^n$ with real or complex coefficients, as appropriate, and of degree less than or equal to k. Thus

$$(\exp(t\,L_k))(p) \tag{5.12.1}$$

is another polynomial on $\mathbf{R}^n$ with real or complex coefficients, as appropriate, and degree less than or equal to k. Of course, the coefficients of (5.12.1), as a polynomial on $\mathbf{R}^n$, depend on t, and in fact they are smooth functions of t. It follows that

$$u(x,t) = ((\exp(t\,L_k))(p))(x) \tag{5.12.2}$$

is smooth as a function of (x,t) on $\mathbf{R}^n \times \mathbf{R}$, which we can identify with $\mathbf{R}^{n+1}$.

Note that

$$u(x,0) = p(x) \tag{5.12.3}$$

for every $x \in \mathbf{R}^n$. We also have that

$$\frac{\partial}{\partial t}((\exp(t\,L_k))(p)) = L_k((\exp(t\,L_k))(p)), \tag{5.12.4}$$

as before. This means that

$$\frac{\partial u}{\partial t} = L(u) \tag{5.12.5}$$

on $\mathbf{R}^n \times \mathbf{R}$.

5.12.1 Some nilpotency conditions

Suppose for the moment that the coefficients of L satisfy (5.11.4) for some $c \geq 1$. This implies that L_k is nilpotent on $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as appropriate, as in the previous section. It follows that $\exp(t\, L_k)$ is a polynomial in t with coefficients that are linear mappings on $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as appropriate, as in Subsection 5.10.1. This means that (5.12.2) is a polynomial in x and t in this case.

5.13 Some additional related equations

Let us continue with the same notation and hypotheses as at the beginning of the previous section. Suppose now that we are interested in the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = L(u) \tag{5.13.1}$$

on $\mathbf{R}^n \times \mathbf{R}$. If we put

$$v = \frac{\partial u}{\partial t}, \tag{5.13.2}$$

then (5.13.1) is the same as saying that

$$\frac{\partial v}{\partial t} = L(u). \tag{5.13.3}$$

Let us consider (5.13.2) and (5.13.3) as a system of partial differential equations in u and v on $\mathbf{R}^n \times \mathbf{R}$.

Of course, we can identify $\mathbf{R}^m \times \mathbf{R}^m$ and $\mathbf{C}^m \times \mathbf{C}^m$ with $\mathbf{R}^{2m}$ and $\mathbf{C}^{2m}$, respectively. Similarly, we can identify

$$\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}) \times \mathcal{P}^k(\mathbf{R}^n, \mathbf{R}) \tag{5.13.4}$$

and

$$\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}) \times \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}) \tag{5.13.5}$$

with $\mathbf{R}^{2m}$ and $\mathbf{C}^{2m}$, respectively, using the analogous identifications mentioned in the previous section. Let T_k be the mapping from (5.13.4) or (5.13.5) into itself, as appropriate, defined by

$$T_k(p, q) = (q, L_k(p)) \tag{5.13.6}$$

for every $p, q \in \mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as appropriate. Observe that

$$T_k^2(p, q) = T_k(T_k(p, q)) = T_k(q, L_k(p)) = (L_k(p), L_k(q)) \tag{5.13.7}$$

for all such p, q. We can identify T_k with a linear mapping from $\mathbf{R}^{2m}$ or $\mathbf{C}^{2m}$ into itself, as before.

If $t \in \mathbf{R}$, then we can define the exponential of $t\,T_k$ as a linear mapping on (5.13.4) or (5.13.5), as appropriate, in the usual way. Let p, q be elements of $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as appropriate, so that

$$(\exp(t\,T_k))(p, q) \tag{5.13.8}$$

is an element of (5.13.4) or (5.13.5), as appropriate. Let $u(\cdot, t)$, $v(\cdot, t)$ be the elements of $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as appropriate, such that

$$(\exp(t\,T_k))(p, q) = (u(\cdot, t), v(\cdot, t)). \tag{5.13.9}$$

The coefficients of $u(x, t)$ and $v(x, t)$, as polynomials in x on $\mathbf{R}^n$, are smooth functions of t, as before. This implies that $u(x, t)$ and $v(x, t)$ are smooth as functions of (x, t) on $\mathbf{R}^n \times \mathbf{R}$, which we can identify with $\mathbf{R}^{n+1}$, as usual.

Note that

$$\frac{\partial}{\partial t}((\exp(t\,T_k))(p, q)) = T_k((\exp(t\,T_k))(p, q)), \tag{5.13.10}$$

as before. This means that

$$\frac{\partial}{\partial t}(u(\cdot, t), v(\cdot, t)) = T_k(u(\cdot, t), v(\cdot, t)) = (v(\cdot, t), L(u(\cdot, t))), \tag{5.13.11}$$

which is the same as saying that u and v satisfy (5.13.2) and (5.13.3). We also have that

$$u(\cdot, 0) = p, \;\; v(\cdot, 0) = q. \tag{5.13.12}$$

5.13.1 Some more nilpotency conditions

Suppose that the coefficients of L satisfy (5.11.4) for some $c \geq 1$, so that L_k is nilpotent on $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, as appropriate, as before. This implies that T_k is nilpotent on (5.13.4) or (5.13.5), as appropriate, because of (5.13.7). This means that $\exp(t\,T_k)$ is a polynomial in t with coefficients that are linear mappings on (5.13.4) or (5.13.5), as appropriate, as in Subsection 5.10.1. It follows that $u(x, t)$ and $v(x, t)$ are polynomials in x and t under these conditions.

5.14 Some products with $\exp(b \cdot x)$

Let n be a positive integer, and let $b \in \mathbf{R}^n$ or $\mathbf{C}^n$ be given. Also let N be a nonnegative integer, and let p be a polynomial on $\mathbf{R}^n$ with real or complex coefficients of degree less than or equal to N. Thus

$$p_b(x) = p(x + b) \tag{5.14.1}$$

can be expressed as a polynomial of degree less than or equal to N with real or complex coefficients, as appropriate, as in Section 2.5.

Let $p(\partial)$ and $p_b(\partial)$ be the differential operators corresponding to p and p_b as in Section 1.7, respectively. If f is a continuously-differentiable real or complex-valued function on $\mathbf{R}^n$, then

$$\frac{\partial}{\partial x_j}(f(x)\,\exp(b \cdot x)) = \Big(\frac{\partial f}{\partial x_j}(x) + b_j\,f(x)\Big)\,\exp(b \cdot x). \tag{5.14.2}$$

If f is N-times continuously differentiable on $\mathbf{R}^n$, then we get that

$$p(\partial)(f(x)\,\exp(b \cdot x)) = (p_b(\partial)(f))(x)\,\exp(b \cdot x). \tag{5.14.3}$$

If $b \in \mathbf{R}^n$, then let

$$\mathcal{P}(\mathbf{R}^n, \mathbf{R})\,\exp(b \cdot x) \tag{5.14.4}$$

be the space of functions on $\mathbf{R}^n$ of the form

$$q(x)\,\exp(b \cdot x), \tag{5.14.5}$$

where $q \in \mathcal{P}(\mathbf{R}^n, \mathbf{R})$. This is a linear subspace of $C^\infty(\mathbf{R}^n, \mathbf{R})$, as a vector space over the real numbers. If p is a polynomial with real coefficients, then p_b is a polynomial with real coefficients as well. In this case, $p(\partial)$ maps (5.14.4) into itself, because of (5.14.3).

Similarly, if $b \in \mathbf{C}^n$, then let

$$\mathcal{P}(\mathbf{R}^n, \mathbf{C})\,\exp(b \cdot x) \tag{5.14.6}$$

be the space of functions on $\mathbf{R}^n$ of the form (5.14.5), with $q \in \mathcal{P}(\mathbf{R}^n, \mathbf{C})$. This is a linear subspace of $C^\infty(\mathbf{R}^n, \mathbf{C})$, as a vector space over the complex numbers. We also have that $p(\partial)$ maps (5.14.6) into itself, because of (5.14.3), as before.

Let k be a nonnegative integer, and if $b \in \mathbf{R}^n$, then let

$$\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})\,\exp(b \cdot x) \tag{5.14.7}$$

be the space of functions on $\mathbf{R}^n$ of the form (5.14.5), with $q \in \mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$. This is a linear subspace of (5.14.4), as a vector space over the real numbers. If p is a polynomial with real coefficients, then $p(\partial)$ maps (5.14.7) into itself, because of (5.14.3) again.

If $b \in \mathbf{C}^n$, then let

$$\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})\,\exp(b \cdot x) \tag{5.14.8}$$

be the space of functions on $\mathbf{R}^n$ of the form (5.14.5), with $q \in \mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$. This is a linear subspace of (5.14.6), as a vector space over the complex numbers. As usual, $p(\partial)$ maps (5.14.8) into itself, because of (5.14.3).

5.14.1 Another nilpotency condition

Suppose that

$$p_b(0) = p(b) = 0. \tag{5.14.9}$$

If q is a polynomial on $\mathbf{R}^n$ with real or complex coefficients, then

$$\deg(p_b(\partial))(q) \le \deg q - 1. \tag{5.14.10}$$

This implies that the restriction of $p_b(\partial)$ to $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$ is nilpotent, as in Subsection 5.11.1. It follows that the restriction of $p(\partial)$ to (5.14.8) is nilpotent, because of (5.14.3). If $b \in \mathbf{R}^n$, and p is a polynomial with real coefficients, then the restriction of $p(\partial)$ to (5.14.7) is nilpotent, for the same reasons.

5.15 Some remarks about derivatives

Let m be a positive integer, and let I be an interval in the real line, which may be unbounded, and which has nonempty interior. One can define continuity of a mapping from I into $\mathbf{C}^m$ in the usual way, using the restriction of the standard Euclidean metric on $\mathbf{R}$ to I, and the standard Euclidean metric on $\mathbf{C}^m$. It is well known and not difficult to see that this is equivalent to the continuity of the corresponding m component functions, as complex-valued functions on I. Similarly, a complex-valued function on I is continuous if and only if its real and imaginary parts are continuous.

Suppose that for each $t \in I$, $A(t)$ is a linear mapping from $\mathbf{C}^m$ into itself, as a vector space over the complex numbers. The continuity of $A(t)$ as a function on I with values in the space $\mathcal{L}(\mathbf{C}^m)$ of linear mappings from $\mathbf{C}^m$ into itself can also be defined in the usual way, using the restrction of the standard Euclidean metric on $\mathbf{R}$, and a suitable version of the standard Euclidean metric on $\mathcal{L}(\mathbf{C}^m)$. More prcisely, we can use the standard correspondence between elements of $\mathcal{L}(\mathbf{C}^m)$ and $m \times m$ matrices of complex numbers to identify $\mathcal{L}(\mathbf{C}^m)$ with $\mathbf{C}^{m^2}$, and use the standard Euclidean metric there. The continuity of $A(t)$ on I is equivalent to the continuity of the m^2 complex-valued functions on I corresponding to the matrix entries of $A(t)$. This is equivalent to the continuity of

$$(A(t))(v) \tag{5.15.1}$$

for each $v \in \mathbf{C}^m$, as a function of $t \in I$ with values in $\mathbf{C}^m$.

One can define differentiability of a mapping from I into $\mathbf{C}^m$ directly, using one-sided derivatives at any endpoints of I. This is equivalent to the differentiability of the m component functions, as complex-valued functions on I. The differentiability of a complex-valued function on I is equivalent to the differentiability of its real and imaginary parts.

Differentiability of $A(t)$ on I can be defined directly, and is equivalent to the differentiability of the m^2 complex-valued functions on I corresponding to the matrix entries of $A(t)$. This is equivalent as well to the differentiability of (5.15.1) for each $v \in \mathbf{C}^m$, as a function of $t \in I$ with values in $\mathbf{C}^m$.

Let $v(t)$ be a function on I with values in $\mathbf{C}^m$, so that

$$(A(t))(v(t)) \tag{5.15.2}$$

is an element of $\mathbf{C}^m$ for each $t \in I$. Of course, the components of (5.15.2) can be expressed as a sum or products of matrix entries of $A(t)$ and components of $v(t)$ in the usual way. If $v(t)$ is continuous at a point $t_0 \in I$, and if $A(t)$ is continuous at t_0, then (5.15.2) is continuous at t_0 too, as a function of $t \in I$ with values in $\mathbf{C}^m$. If $v(t)$ is differentiable at t_0, and $A(t)$ is differentiable at t_0, then (5.15.2) is differentiable at t_0, with derivative equal to

$$(A'(t_0))(v(t_0)) + (A(t_0))(v'(t_0)). \tag{5.15.3}$$

This is basically another version of the product rule.

5.15.1 A particular case for $A(t)$

Let B be a linear mapping from $\mathbf{C}^m$ into itself, and consider

$$A(t) = \exp(-t\, B). \tag{5.15.4}$$

This is a differentiable function of $t \in \mathbf{R}$ with values in $\mathcal{L}(\mathbf{C}^m)$, with derivative

$$A'(t) = -B \circ A(t) = -A(t) \circ B. \tag{5.15.5}$$

Suppose that $v(t)$ is differentiable on I, and put

$$w(t) = (A(t))(v(t)) = (\exp(-t\, B))(v(t)) \tag{5.15.6}$$

for each $t \in I$. Thus $w(t)$ is differentiable on I, with

$$w'(t) = -B(w(t)) + (A(t))(v'(t)), \tag{5.15.7}$$

as before, using (5.15.5). Note that

$$v(t) = (\exp(t\, B))(w(t)) \tag{5.15.8}$$

for each $t \in I$.

Suppose for the moment that

$$v'(t) = B(v(t)) \tag{5.15.9}$$

on I. In this case,

$$w'(t) = 0 \tag{5.15.10}$$

on I, by (5.15.7). This means that $w(t)$ is constant on I, because of the analogous statement for real-valued functions.

Similarly, consider the differential equation

$$v'(t) = B(v(t)) + z(t), \tag{5.15.11}$$

where $z(t)$ is a function of $t \in I$ with values in $\mathbf{C}^m$. This corresponds to the differential equation

$$w'(t) = (\exp(-t\, B))(z(t)) \tag{5.15.12}$$

on I. Note that the right side is continuous on I when $z(t)$ is continuous on I.

Chapter 6

More on harmonic functions

Some nice references concerning harmonic functions include [18, 81, 87, 297], and some additional information may be found in [291]. See also [7, 275, 295], for instance.

6.1 Some particular harmonic functions

It is well known and not difficult to verify that

$$|x|^{2-n} \tag{6.1.1}$$

is harmonic on $\mathbf{R}^n \setminus \{0\}$ when $n \geq 3$. This implies that

$$|x - a|^{2-n} \tag{6.1.2}$$

is harmonic on $\mathbf{R}^n \setminus \{a\}$ for every $a \in \mathbf{R}^n$ when $n \geq 3$. Of course, this is much simpler when $n = 1$.

Similarly, one can check that

$$\log |x| = (1/2) \log |x|^2 \tag{6.1.3}$$

is harmonic on $\mathbf{R}^2 \setminus \{0\}$. This means that

$$\log |x - a| \tag{6.1.4}$$

is harmonic on $\mathbf{R}^2 \setminus \{a\}$ for every $a \in \mathbf{R}^2$, as before.

6.1.1 Using complex variables when $n = 2$

If we put $z = x_1 + i\, x_2$, then we can express (6.1.3) as

$$(1/2) \log |z|^2. \tag{6.1.5}$$

Let $\partial/\partial z$ and $\partial/\partial \overline{z}$ be the differential operators defined in Section 2.2. Observe that

$$\frac{\partial}{\partial z}((1/2)\,\log|z|^2) = \frac{1}{2\,|z|^2}\,\frac{\partial}{\partial z}(|z|^2) = \frac{1}{2\,|z|^2}\,\frac{\partial}{\partial z}(z\,\overline{z}) = \frac{1}{2\,|z|^2}\,\overline{z} = \frac{1}{2\,z} \tag{6.1.6}$$

when $z \neq 0$.

It is well known and not difficult to check that

$$\frac{\partial}{\partial \overline{z}}\Big(\frac{1}{z}\Big) = 0 \tag{6.1.7}$$

for $z \neq 0$, which is to say that $1/z$ is holomorphic for $z \neq 0$. It follows that (6.1.3) is harmonic on $\mathbf{R}^2 \setminus \{0\}$, as in Subsection 2.2.1.

6.1.2 Some more harmonic functions

If $n \geq 3$ and $1 \leq j \leq n$, then

$$\begin{aligned} \frac{\partial}{\partial x_j}(|x|^{2-n}) &= \frac{\partial}{\partial x_j}((|x|^2)^{(2-n)/2}) \\ &= ((2-n)/2)\,(|x|^2)^{((2-n)/2)-1}\,(2\,x_j) = (2-n)\,\frac{x_j}{|x|^n} \end{aligned} \tag{6.1.8}$$

on $\mathbf{R}^n \setminus \{0\}$. Similarly,

$$\frac{\partial}{\partial x_j}(\log|x|) = \frac{\partial}{\partial x_j}((1/2)\,\log|x|^2) = (1/2)\,|x|^{-2}\,(2\,x_j) = \frac{x_j}{|x|^2} \tag{6.1.9}$$

on $\mathbf{R}^2 \setminus \{0\}$ for $j = 1, 2$, which is basically the same as (6.1.6). Note that these are harmonic functions too, because the partial derivatives of three-times continuously-differentiable harmonic functions are harmonic.

6.2 The mean-value property

Let $n \geq 2$ be an integer, and let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary. It is convenient to use $|V|$ for the n-dimensional volume of V, and $|\partial V|$ for the $(n-1)$-dimensional surface area of ∂V.

In particular, if $a \in \mathbf{R}^n$ and $r > 0$, then $|B(a, r)|$ denotes the volume of $B(a, r)$, and $|\partial B(a, r)|$ denotes the surface area of $\partial B(a, r)$. Note that

$$|B(a, r)| = r^n\,|B(0, 1)| \tag{6.2.1}$$

and

$$|\partial B(a, r)| = r^{n-1}\,|\partial B(0, 1)|. \tag{6.2.2}$$

Let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a twice continuously-differentiable real or complex-valued function on U that is harmonic on U. Also let $a \in U$ and $r > 0$ be given, with

$$\overline{B}(a, r) \subseteq U. \tag{6.2.3}$$

Under these conditions, it is well known that

$$u(a) = \frac{1}{|\partial B(a, r)|} \int_{\partial B(a,r)} u(y')\, dy'. \tag{6.2.4}$$

This works when $n = 1$ as well, with suitable interpretations, and is much simpler.

6.2.1 Some preliminary steps

To see this, it suffices to show that

$$\frac{1}{|\partial B(a, t)|} \int_{\partial B(a,t)} u(y')\, dy' = \frac{1}{|\partial B(a, r)|} \int_{\partial B(a,r)} u(y')\, dy' \tag{6.2.5}$$

when $0 < t < r$. Indeed, one can check that

$$\lim_{t \to 0+} \frac{1}{|\partial B(a, t)|} \int_{\partial B(a,t)} u(y')\, dy' = u(a), \tag{6.2.6}$$

because u is continuous at a. This permits one to obtain (6.2.4) from (6.2.5).

Note that

$$\int_{\partial B(a,r)} (D_{\nu(y')} u)(y')\, dy' = 0, \tag{6.2.7}$$

where $\nu(y')$ is the outward-pointing unit normal to $\partial B(a, r)$ at a point y' in $\partial B(a, r)$. This follows from (3.5.4), with $V = B(a, r)$. Similarly,

$$\int_{\partial B(a,t)} (D_{\nu(y')} u)(y')\, dy' = 0, \tag{6.2.8}$$

where $\nu(y')$ is the outward-pointing unit normal to $\partial B(a, t)$ at a point y' in $\partial B(a, t)$.

6.2.2 Using a previous integral identity

To get (6.2.5), consider

$$V = B(a, r) \setminus \overline{B}(a, t) = \{x \in \mathbf{R}^n : t < |x| < r\}, \tag{6.2.9}$$

which is a nonempty bounded open subset of $\mathbf{R}^n$. Observe that

$$\partial V = (\partial B(a, r)) \cup (\partial B(a, t)). \tag{6.2.10}$$

The outward-pointing unit normal to ∂V is the same as the outward-pointing unit normal to $\partial B(a, r)$ at points in $\partial B(a, r)$, and it is -1 times the outward-pointing unit normal to $\partial B(a, t)$ at points in $\partial B(a, t)$.

Put

$$v(x) = |x - a|^{2-n} \tag{6.2.11}$$

on $\mathbf{R}^n \setminus \{a\}$ when $n \geq 3$, and

$$v(x) = \log |x - a| \tag{6.2.12}$$

on $\mathbf{R}^2 \setminus \{a\}$ when $n = 2$. In both cases, $v(x)$ is harmonic on $\mathbf{R}^n \setminus \{a\}$, as in the previous section.

We would like to use (3.9.1) in this case. The left side of that equation is equal to 0, because u and v are harmonic on V. One can check that

$$\int_{\partial V} v(y')\,(D_{\nu(y')}u)(y')\,dy' = 0 \tag{6.2.13}$$

under these conditions, because of (6.2.7) and (6.2.8). This also uses the fact that v is constant on $\partial B(a, r)$ and $\partial B(a, t)$.

It follows that

$$\int_{\partial V} u(y')\,(D_{\nu(y')}v)(y')\,dy' = 0. \tag{6.2.14}$$

One can use this to get (6.2.5), as desired.

6.2.3 Another approach

Alternatively,

$$\frac{1}{|\partial B(a, t)|} \int_{\partial B(a,t)} u(y')\,dy' = \frac{1}{|\partial B(0, 1)|} \int_{\partial B(0,1)} u(a + t\,z')\,dz' \tag{6.2.15}$$

when $0 < t \leq r$. This uses the change of variables where

$$y' \in \partial B(a, t) \tag{6.2.16}$$

corresponds to

$$a + t\,z' \text{ with } z' \in \partial B(0, 1). \tag{6.2.17}$$

Note that the derivative of $u(a+t\,z')$ in t is the same as the directional derivative of u in the direction z' at $a + t\,z'$. The derivative of the right side of (6.2.15) in t is equal to

$$\frac{1}{|\partial B(0, 1)|} \int_{\partial B(0,1)} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(a + t\,z')\,z'_j\,dz' \tag{6.2.18}$$

More precisely, one can verify that differentiation under the integral sign is permitted here, using the continuous differentiability of u.

We can change variables again, to get that (6.2.18) is equal to

$$\frac{1}{|\partial B(a,t)|} \int_{\partial B(a,t)} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(y')\, t^{-1}\, (y'_j - a_j)\, dy' \tag{6.2.19}$$
$$= \frac{1}{|\partial B(a,t)|} \int_{\partial B(a,t)} (D_{\nu(y')} u)(y')\, dy',$$

where $\nu(y')$ is the outward-pointing unit normal to $\partial B(a,t)$ at $y' \in \partial B(a,t)$ again. If u is harmonic on U, then the right side of (6.2.19) is equal to 0, as in (6.2.8). This implies that the left side of (6.2.15) is constant for $0 < t \le r$, so that (6.2.5) holds.

6.3 More on mean values

Let n be a positive integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuous real or complex-valued function on U. Let us say that u has the *mean-value property* on U if for every $a \in U$ and $r > 0$ such that $\overline{B}(a,r) \subseteq U$, we have that (6.2.4) holds. Equivalently, this means that

$$\int_{\partial B(a,r)} u(y')\, dy' = |\partial B(a,r)|\, u(a) = r^{n-1}\, |\partial B(0,1)|\, u(a). \tag{6.3.1}$$

In this case, we get that

$$\int_{B(a,r)} u(x)\, dx = |B(a,r)|\, u(a) = r^n\, |B(0,1)|\, u(a), \tag{6.3.2}$$

by integrating in r. Of course, this is the same as saying that

$$u(a) = \frac{1}{|B(a,r)|} \int_{B(a,r)} u(x)\, dx. \tag{6.3.3}$$

Conversely, one can get (6.3.1) from (6.3.2), by differentiating in r.

6.3.1 Some basic integrals

One can check that

$$\int_{\partial B(a,r)} (y'_j - a_j)\, dy' = \int_{B(a,r)} (x_j - a_j)\, dx = 0 \tag{6.3.4}$$

for every $a \in \mathbf{R}^n$, $r > 0$, and $j = 1, \ldots, n$. Similarly,

$$\int_{\partial B(a,t)} (y'_j - a_j)\, (y'_l - a_l)\, dy' = \int_{B(a,t)} (x_j - a_j)\, (x_l - a_l)\, dx = 0 \tag{6.3.5}$$

when $j \ne l$. We also have that

$$\int_{\partial B(a,r)} (y'_j - a_j)^2\, dy' = \int_{\partial B(a,r)} (y'_l - a_l)^2\, dy' \tag{6.3.6}$$

and

$$\int_{B(a,r)} (x_j - a_j)^2 \, dx = \int_{B(a,r)} (x_l - a_l)^2 \, dx \tag{6.3.7}$$

for every $j, l = 1, \ldots, n$. One can use these remarks to show directly that a polynomial on $\mathbf{R}^n$ of degree less than or equal to 2 satisfies the mean-value property if and only if it is harmonic.

6.3.2 Using the mean-value property

If u is twice continuously differentiable on U, and u has the mean-value property on U, then u is harmonic on U. This can be seen using the Taylor approximation to u at a point $a \in U$ of degree 2, to estimate the difference between the average of u on balls or spheres centered at a with small radius and $u(a)$.

Alternatively, the mean-value property implies that the right side of (6.2.19) is 0 when $\overline{B}(a, t) \subseteq U$, using the same type of argument as before. This means that

$$\int_{B(a,t)} (\Delta u)(x) \, dx = 0, \tag{6.3.8}$$

because of (3.5.3). One can use this to get that $(\Delta u)(a) = 0$, by taking t to be sufficiently small.

6.4 Mean values and smoothness

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let u be a continuous real or complex-valued function on U with the mean-value property. Let $r > 0$ be given, and let ϕ be a continuous real-valued function on $\mathbf{R}^n$ supported in $\overline{B}(0, r)$. Suppose too that ϕ is a radial function on $\mathbf{R}^n$, so that $\phi(x)$ depends only on $|x|$.

Let $a \in U$ be given, and suppose that $\overline{B}(a, r) \subseteq U$. If $0 < t \leq r$, then

$$\int_{\partial B(a,t)} u(y') \, \phi(y' - a) \, dy' = \Big(\int_{\partial B(a,t)} \phi(y' - a) \, dy' \Big) \, u(a). \tag{6.4.1}$$

This uses the mean-value property of u, and the fact that $\phi(y' - a)$ is constant as a function of y' on $\partial B(a, t)$, because ϕ is radial on $\mathbf{R}^n$. It follows that

$$\int_{\partial B(a,t)} u(y') \, \phi(y' - a) \, dy' = \Big(\int_{\partial B(0,t)} \phi(z') \, dz' \Big) \, u(a). \tag{6.4.2}$$

We can integrate over t to get that

$$\int_{B(a,r)} u(x) \, \phi(x - a) \, dx = \Big(\int_{B(0,r)} \phi(w) \, dw \Big) \, u(a). \tag{6.4.3}$$

If

$$\int_{B(0,r)} \phi(w) \, dw = 1, \tag{6.4.4}$$

then we get that

$$\int_{B(a,r)} u(x)\,\phi(x-a)\,dx = u(a). \tag{6.4.5}$$

Of course, we can get (6.4.4) by dividing ϕ by its integral over $B(0,r)$, as long as the integral is not zero. It is easy to see that the integral is positive when ϕ is nonnegative and not equal to 0 at every point in $B(0,r)$.

6.4.1 Points b near a

Remember that $\overline{B}(a,r) \subseteq U$ implies that

$$\overline{B}(a,r+\epsilon) \subseteq U \tag{6.4.6}$$

for some $\epsilon > 0$, as in Section 1.13. If $b \in \mathbf{R}^n$ and $|a-b| \le \epsilon$, then it follows that

$$\overline{B}(b,r) \subseteq \overline{B}(a,r+\epsilon) \subseteq U, \tag{6.4.7}$$

using the triangle inequality in the first step. This means that

$$u(b) = \int_{B(b,r)} u(x)\,\phi(x-b)\,dx, \tag{6.4.8}$$

as before. This can also be expressed as

$$u(b) = \int_{B(a,r+\epsilon)} u(x)\,\phi(x-b)\,dx, \tag{6.4.9}$$

because ϕ is supported in $\overline{B}(0,r)$.

Suppose that ϕ is a smooth function on $\mathbf{R}^n$ too, which can be arranged by taking a suitable smooth function of $|x|^2$. Under these conditions, one can differentiate under the integral sign in (6.4.9), to get that u is smooth near a. More precisely, this shows that the derivatives of u may be expressed in terms of suitable integrals of u.

6.4.2 Harmonicity and smoothness

One can use this type of argument at every point in U, to get that u is smooth on U. It follows that u is harmonic on U, as in Subsection 6.3.2.

If u is twice continuously differentiable and harmonic on U, then u has the mean-value property, as in Section 6.2. This implies that u is smooth on U, as in the preceding paragraph.

6.5 Uniform convergence

Let E be a nonempty set, let $\{f_j\}_{j=1}^\infty$ be a sequence of real or complex-valued functions on E, and let f be another real or complex-valued function on E. We say that

$$\{f_j\}_{j=1}^\infty \text{ converges to } f \textit{ pointwise} \text{ on } E \tag{6.5.1}$$

if for every $x \in E$, $\{f_j(x)\}_{j=1}^\infty$ converges to $f(x)$ in the usual sense, as a sequence of real or complex numbers, as appropriate. We say that

$$\{f_j\}_{j=1}^\infty \text{ converges } \textit{uniformly} \text{ to } f \text{ on } E \tag{6.5.2}$$

if for every $\epsilon > 0$ there is a positive integer L such that

$$|f_j(x) - f(x)| < \epsilon \tag{6.5.3}$$

for every $x \in E$ and $j \geq L$. Uniform convergence on E clearly implies pointwise convergence on E, because L is only supposed to depend on ϵ, and not on x.

6.5.1 Uniform convergence and continuity

Let n be a positive integer, and suppose now that E is a nonempty subset of $\mathbf{R}^n$. If $\{f_j\}_{j=1}^\infty$ is a sequence of continuous real or complex-valued functions on E that converges uniformly to a real or complex-valued function f on E, as appropriate, then it is well known that

$$f \text{ is continuous on } E \tag{6.5.4}$$

too. This does not always work for pointwise convergence, as one can see by taking

$$f_j(x) = x^j \tag{6.5.5}$$

on $[0, 1]$.

6.5.2 Uniform convergence on compact subsets

Let U be a nonempty open subset of $\mathbf{R}^n$, let $\{f_j\}_{j=1}^\infty$ be a sequence of real or complex-valued functions on U, and let f be a real or complex-valued function on U. We say that $\{f_j\}_{j=1}^\infty$ converges to f *uniformly on compact subsets of* U if

$$\begin{aligned} &\text{for every compact subset } E \text{ of } \mathbf{R}^n \text{ such that } E \subseteq U, \\ &\{f_j\}_{j=1}^\infty \text{ converges to } f \text{ uniformly on } E. \end{aligned} \tag{6.5.6}$$

Uniform convergence on U implies uniform convergence on compact subsets of U, and uniform convergence on compact subsets of U implies pointwise convergence on U.

If $\{f_j\}_{j=1}^\infty$ is a sequence of continuous real or complex-valued functions on U that converges to f uniformly on compact subsets of U, then

$$f \text{ is continuous on } U \tag{6.5.7}$$

as well. More precisely, in order to show that f is continuous at a point $x \in U$, one can use the uniform convergence of $\{f_j\}_{j=1}^\infty$ to f on a closed ball centered at x with sufficientlly small radius so that the ball is contained in U.

6.5.3 Uniform convergence of harmonic functions

Let $\{u_j\}_{j=1}^{\infty}$ be a sequence of harmonic functions on U that converges to a function u on U, uniformly on compact sets contained in U. This implies that u is continuous on U, as before. One can use the mean-value property for u_j for each j to get that

$$u \text{ has the mean-value property on } U \tag{6.5.8}$$

too, because of standard results about uniform convergence and integration. This means that u is harmonic on U, as in the previous section. One can also show that derivatives of the u_j's converge to the corresponding derivatives of u, uniformly on compact subsets of U, by expressing the derivatives in terms of suitable integrals of the functions, as in the previous section.

6.6 Liouville's theorem

Let n be a positive integer, and let u be a bounded harmonic function on $\mathbf{R}^n$. Under these conditions, *Liouville's theorem* states that

$$u \text{ is a constant function on } \mathbf{R}^n. \tag{6.6.1}$$

6.6.1 Differences of averages

To see this, let $x, y \in \mathbf{R}^n$ and $r > 0$ be given, so that

$$\begin{aligned} u(x) - u(y) &= \frac{1}{|B(x,r)|} \int_{B(x,r)} u(w)\,dw - \frac{1}{|B(y,r)|} \int_{B(y,r)} u(w) \\ &= \frac{1}{|B(0,1)|\,r^n} \Big(\int_{B(x,r)} u(w)\,dw - \int_{B(y,r)} u(w)\,dw \Big) \\ &= \frac{1}{|B(0,1)|\,r^n} \int_{B(x,r)\setminus \overline{B}(y,r)} u(w)\,dw \\ &\quad - \frac{1}{|B(0,1)|\,r^n} \int_{B(y,r)\setminus \overline{B}(x,r)} u(w)\,dw \end{aligned} \tag{6.6.2}$$

If $r > |x-y|$, then one can check that

$$\begin{aligned} B(x,r) \setminus \overline{B}(y,r) &\subseteq B(x,r) \setminus \overline{B}(x, r - |x-y|) \\ &= \{z \in \mathbf{R}^n : r - |x-y| < |x-z| < r\}, \end{aligned} \tag{6.6.3}$$

and similarly with the roles of x and y interchanged. The n-dimensional volume of the right side is equal to

$$\begin{aligned} |B(x,r)| - |B(x, r - |x-y|)| &= |B(0,1)|\,(r^n - (r - |x-y|)^n) \\ &= |B(0,1)| \sum_{j=0}^{n-1} \binom{n}{j} (-1)^{n-j+1}\, r^j\, |x-y|^{n-j}, \end{aligned} \tag{6.6.4}$$

and similarly with the roles of x and y interchanged. It follows that the n-dimensional volume of the left side of (6.6.3) is less than or equal to this, and similarly with the roles of x and y exchanged.

The right side of (6.6.4) is small compared to r^n when r is large. If u is bounded on $\mathbf{R}^n$, then one can use this to check that each of the two terms on the right side of (6.6.2) tends to 0 as $r \to \infty$. This implies that $u(x) = u(y)$, as desired.

6.6.2 Estimating first derivatives

Alternatively, we can use arguments like those in Section 6.4 to estimate first derivatives of harmonic functions. These estimates will show that bounded harmonic functions on $\mathbf{R}^n$ have all of their first derivatives equal to 0.

Let ϕ be a smooth real-valued radial function on $\mathbf{R}^n$ supported on the closed unit ball $\overline{B}(0,1)$, and with

$$\int_{B(0,1)} \phi(w)\, dw = 1. \tag{6.6.5}$$

Put

$$\phi_r(w) = r^{-n}\, \phi(r^{-1}\, w) \tag{6.6.6}$$

for every $w \in \mathbf{R}^n$ and $r > 0$. It is easy to see that ϕ_r is a smooth real-valued radial function on $\mathbf{R}^n$ that is supported on $\overline{B}(0,r)$ and satisfies

$$\int_{B(0,r)} \phi_r(w)\, dw = 1. \tag{6.6.7}$$

Let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a real or complex-valued harmonic function on U. If $a \in U$, $r > 0$, and $\overline{B}(a,r) \subseteq U$, then

$$u(a) = \int_{B(a,r)} u(x)\, \phi_r(x-a), \tag{6.6.8}$$

as in (6.4.5). If $\epsilon > 0$ is as in (6.4.6), $b \in \mathbf{R}^n$, and $|a-b| \le \epsilon$, then we get that

$$u(b) = \int_{B(a,r+\epsilon)} u(x)\, \phi_r(x-b)\, dx, \tag{6.6.9}$$

as in (6.4.9).

Observe that

$$\frac{\partial}{\partial w_j}(\phi_r(w)) = r^{-n-1}(\partial_j \phi)(r^{-1} w) \tag{6.6.10}$$

for each $j = 1, \ldots, n$. We can differentiate under the integral sign in (6.6.9) to get that

$$(\partial_j u)(a) = -r^{-n-1} \int_{B(a,r)} u(x)\, (\partial_j \phi)(r^{-1}\, (x-a))\, dx \tag{6.6.11}$$

for each $j = 1, \ldots, n$. This also uses the fact that $\partial_j \phi$ is supported in $\overline{B}(0,1)$.

If $U = \mathbf{R}^n$ and u is bounded on $\mathbf{R}^n$, then one can check that the right side of (6.6.11) tends to 0 as $r \to \infty$. This implies that $\partial_j u = 0$ on $\mathbf{R}^n$ for each $j = 1, \ldots, n$, so that u is constant on $\mathbf{R}^n$.

6.7 The maximum principle

Let n be a positive integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuous real-valued function on U. Suppose that for every $a \in U$ there is an $r > 0$ such that $\overline{B}(a, r) \subseteq U$ and

(6.7.1) the average of u on $B(a, r)$ is equal to $u(a)$,

as in (6.3.3). In particular, this happens when u is harmonic on U, as in Section 6.2.

6.7.1 The strong maximum principle

Let A be a real number such that

$$u(x) \leq A \tag{6.7.2}$$

for every $x \in U$. Note that

$$\{x \in U : u(x) = A\} \tag{6.7.3}$$

is a relatively closed set in U, because u is continuous on U. Equivalently,

$$\{x \in U : u(x) < A\} \tag{6.7.4}$$

is an open set.

Suppose that

$$u(a) = A \tag{6.7.5}$$

for some $a \in U$. If $\overline{B}(a, r) \subseteq U$ and (6.7.1) holds, then one can verify that

$$u(x) = A \tag{6.7.6}$$

for every $x \in B(a, r)$. Equivalently, if $u(x) < A$ for some $x \in B(a, r)$, then one can check that the average of u on $B(a, r)$ is strictly less than A.

This shows that (6.7.3) is an open set under these conditions. If (6.7.3) is nonempty, and U is connected, then it follows that (6.7.3) is equal to U, so that

$$u \equiv A \text{ on } U. \tag{6.7.7}$$

This is often called the *strong maximum principle*.

6.7.2 Bounded open sets

Suppose now that U is also bounded, and let u be a continuous real-valued function on $\overline{U}$. As before, we ask that for each $a \in U$ there be an $r > 0$ such that $\overline{B}(a, r) \subseteq U$ and (6.7.1) holds. Note that $\overline{U}$ is a nonempty compact subset of $\mathbf{R}^n$, so that u attains its maximum on $\overline{U}$, by the extreme value theorem.

Suppose that u attains its maximum on $\overline{U}$ at a point $a \in U$. If V is the connected component of U that contains a, then it follows that u is constant on V, as before. Of course, if u is constant on V, then u is constant on $\overline{V}$, by continuity. This implies that u attains its maximum on $\overline{U}$ at a point in ∂V, which is contained in ∂U, as in Subsection 3.3.1.

Otherwise, if u does not attain its maximum on $\overline{U}$ at a point in U, then

$$u \text{ attains its maximum on } \overline{U} \text{ at a point in } \partial U. \tag{6.7.8}$$

It follows that this holds in either case, which is another version of the *maximum principle*.

In particular, if

$$u(x) = 0 \text{ for every } x \in \partial U, \tag{6.7.9}$$

then we get that $u(x) \le 0$ for every $x \in \overline{U}$. The same argument can be used for $-u$, to obtain that

$$u(x) = 0 \text{ for every } x \in \overline{U}. \tag{6.7.10}$$

6.7.3 Some simple variants

Let U be any nonempty open set in $\mathbf{R}^n$ again, and let v be a continuous real or complex-valued function on $\overline{U}$. Suppose that v is harmonic on U, which is the same as saying that the restriction of v to u satisfies the mean-value property. Thus, if $a \in U$, $r > 0$, and $\overline{B}(a, r) \subseteq U$, then

$$\text{the averages of } v \text{ on } B(a, r) \text{ and } \partial B(a, r) \text{ are equal to } v(a). \tag{6.7.11}$$

In fact, one can check that this holds when

$$B(a, r) \subseteq U \tag{6.7.12}$$

under these conditions. Note that this implies that

$$\overline{B}(a, r) \subseteq \overline{U}. \tag{6.7.13}$$

Indeed, if $0 < t < r$, then

$$\overline{B}(a, t) \subseteq B(a, r) \subseteq U. \tag{6.7.14}$$

This implies that the averages of v on $B(a, t)$ and $\partial B(a, t)$ are equal to $v(a)$, by hypothesis. One can use this to get (6.7.11), by considering the limit as $t \to r-$.

Let u be a continuous real-valued function on $\overline{U}$, and suppose that for each $a \in U$ there is an $r > 0$ such that (6.7.12) and (6.7.1) hold. More precisely, (6.7.12) imples (6.7.13), so that the integral of u on $B(a, r)$ may be defined in the usual way in this case. If u attains its maximum on $\overline{U}$ at $a \in U$, and if V is the connected component of V that contains a, then u is constant on V, for essentially the same reasons as before. This implies that (6.7.8) holds, as before. If U is bounded, then we get that (6.7.8) holds without asking that u attain its maximum on $\overline{U}$ at a point in U, as before.

6.8 A helpful integral formula

Let $n \geq 2$ be an integer, and let $N(x)$ be the real-valued function defined on $\mathbf{R}^n \setminus \{0\}$ by

$$N(x) = \frac{|x|^{2-n}}{(2-n)\,|\partial B(0,1)|} \quad \text{when } n \geq 3 \tag{6.8.1}$$
$$= \frac{1}{2\pi}\,\log|x| \quad \text{when } n = 2.$$

Thus $N(x)$ is harmonic on $\mathbf{R}^n \setminus \{0\}$, as in Section 6.1.

Let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary, and let u be a twice continuously-differentiable real or complex-valued function on $\overline{V}$, as in Section 3.4. Also let $a \in V$ be given, and suppose that

$$\overline{B}(a,r) \subseteq V \tag{6.8.2}$$

for some $r > 0$. Put

$$V_r = V \setminus \overline{B}(a,r), \tag{6.8.3}$$

which is an open subset of $\mathbf{R}^n$. Note that

$$\overline{V_r} = \overline{V} \setminus B(a,r) \tag{6.8.4}$$

and that

$$\partial V_r = (\partial V) \cup (\partial B(a,r)). \tag{6.8.5}$$

6.8.1 Using a previous identity again

We would like to use (3.9.1), with V replaced with V_r, and $v(x) = N(x-a)$. This implies that

$$-\int_{V_r} N(x-a)\,(\Delta u)(x)\,dx$$
$$= \int_{\partial V_r} (u(y')\,(D_{\nu_r(y')}N)(y'-a) - N(y'-a)\,(D_{\nu_r(y')}u)(y'))\,dy', \tag{6.8.6}$$

where $D_{\nu_r(y')}$ denoted the directional derivative in the direction $\nu_r(y')$ of the outward-pointing unit normal to ∂V_r at $y' \in \partial V_r$. It follows that

$$-\int_{V_r} N(x-a)\,(\Delta u)(x)\,dx$$
$$= \int_{\partial V} (u(y')\,(D_{\nu(y')}N)(y'-a) - N(y'-a)\,(D_{\nu(y')}u)(y'))\,dy'$$
$$-\int_{\partial B(a,r)} (u(y')\,(D_{\mu(y')}N)(y'-a) - N(y'-a)\,(D_{\mu(y')}u)(y'))\,dy', \tag{6.8.7}$$

where $\nu(y')$ is the outward-pointing unit normal to ∂V at $y' \in \partial V$, and $\mu(y')$ is the outward-pointing unit normal to $\partial B(a,r)$ at $y' \in \partial B(a,r)$. Of course,

$$\begin{aligned} \nu_r(y') &= \nu(y') && \text{when } y' \in \partial V \\ &= -\mu(y') && \text{when } y' \in \partial B(a,r). \end{aligned} \tag{6.8.8}$$

One can check that

$$\int_{\partial B(a,r)} u(y')\,(D_{\mu(y')}N)(y'-a)\,dy' = \frac{1}{|\partial B(a,r)|}\int_{\partial B(a,r)} u(y')\,dy', \tag{6.8.9}$$

by expressing the partial derivatives of N as in Section 6.1. This tends to $u(a)$ as $r \to 0+$, because u is continuous at a.

One can also verify that

$$\lim_{r\to 0+}\int_{\partial B(a,r)} N(y'-a)\,(D_{\mu(y')}u)(y')\,dy' = 0. \tag{6.8.10}$$

This uses the continuous differentiability of u to get that the first derivatives of u are bounded near a.

6.8.2 The integral formula

This implies that

$$\begin{aligned} &\lim_{r\to 0+}\int_{V_r} N(x-a)\,(\Delta u)(x)\,dx \\ &= \int_{\partial V}(N(y'-a)\,(D_{\nu(y')}u)(y') - u(y')\,(D_{\nu(y')}N)(y'-a))\,dy' + u(a). \end{aligned} \tag{6.8.11}$$

The left side may be considered as

$$\int_V N(x-a)\,(\Delta u)(x)\,dx, \tag{6.8.12}$$

defined as an improper integral, because $N(x-a)$ is unbounded near a. One can check that

$$\int_{V_r} |N(x-a)|\,|(\Delta u)(x)|\,dx \tag{6.8.13}$$

stays bounded as $r \to 0+$, using polar coordinates near a, and the fact that $|(\Delta u)(x)|$ is bounded near a, because Δu is continuous, by hypothesis. In fact, if one uses polar coordinates centered at a, then one does not really need to use an improper integral. In particular, (6.8.12) can be defined as a Lebesgue integral.

6.8.3 A formula for harmonic functions

If u is harmonic on V, then we get that

$$u(a) = \int_{\partial V}(u(y')\,(D_{\nu(y')}N)(y'-a) - N(y'-a)\,(D_{\nu(y')}u)(y'))\,dy'. \tag{6.8.14}$$

In particular, this gives another way to look at the regularity of u on V.

6.9 Poisson's equation on $\mathbf{R}^n$

Let $n \geq 2$ be an integer, and let $N(x)$ be as in (6.8.1). If f is a real or complex-valued function on $\mathbf{R}^n$, then one might like to define u as a real or complex-valued function on $\mathbf{R}^n$, as appropriate, by

$$u(x) = \int_{\mathbf{R}^n} N(x-y)\, f(y)\, dy = \int_{\mathbf{R}^n} N(y)\, f(x-y)\, dy. \tag{6.9.1}$$

If f is a continuous function on $\mathbf{R}^n$ with compact support, then the integral on the right may be considered as an improper integral over a bounded region for each $x \in \mathbf{R}^n$. This is similar to the earlier remarks about (6.8.13), and one does not really need to use an improper integral if one uses polar coordinates centered at x for the first integral, or polar coordinates centered at 0 for the second integral, as before. One can use the Lebesgue integral to define u as a locally integrable function on $\mathbf{R}^n$ under suitable integrability conditions on f.

6.9.1 The Laplacian of u

One would like to have

$$\Delta u = f \tag{6.9.2}$$

on $\mathbf{R}^n$, under suitable conditions, or interpreted in a suitable way. Suppose for the moment that f is twice continuously differentiable on $\mathbf{R}^n$, with compact support. In this case, one can show that u is twice continuously differentiable on $\mathbf{R}^n$, by differentiating the second integral in (6.9.1) in x under the integral sign. In particular, one gets

$$(\Delta u)(x) = \int_{\mathbf{R}^n} N(y)\, (\Delta f)(x-y)\, dy \tag{6.9.3}$$

for each $x \in \mathbf{R}^n$.

One can check that

$$\int_{\mathbf{R}^n} N(y)\, (\Delta f)(x-y)\, dy = f(x) \tag{6.9.4}$$

for every $x \in \mathbf{R}^n$, usng the remarks in the previous section. More precisely, one can take V large enough so that the support of f is contained in V. It follows that (6.9.2) holds on $\mathbf{R}^n$. This corresponds to some remarks on p193 of [18], and to Theorem 1 in Section 2.2.1 b in [81].

6.9.2 A distributional-type version

Let v be a twice continuously-differentiable real or complex-valued function on $\mathbf{R}^n$ with compact support. Thus

$$\int_{\mathbf{R}^n} N(x-y)\, (\Delta v)(x)\, dx = v(y) \tag{6.9.5}$$

for every $y \in \mathbf{R}^n$, as in the previous section, with V taken large enough to contain the support of v. Of course, this is very similar to (6.9.4).

Observe that

$$\int_{\mathbf{R}^n} u(x)\,(\Delta v)(x)\,dx = \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} N(x-y)\,f(y)\,dy\Big)\,(\Delta v)(x)\,dx, \tag{6.9.6}$$

by the definition (6.9.1) of $u(x)$. Under suitable integrability conditions on f, we can interchange the order of integration on the right side, to get that

$$\int_{\mathbf{R}^n} u(x)\,(\Delta v)(x)\,dx = \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} N(x-y)\,(\Delta v)(x)\,dx\Big)\,f(y)\,dy. \tag{6.9.7}$$

This implies that

$$\int_{\mathbf{R}^n} u(x)\,(\Delta v)(x)\,dx = \int_{\mathbf{R}^n} f(y)\,v(y)\,dy, \tag{6.9.8}$$

by (6.9.5). This means that u satisfies (6.9.2) in the sense of distributions, as in Theorem 2.16 in Section B of Chapter 2 of [87].

6.9.3 More on Δu

If u is twice continuously differentiable on $\mathbf{R}^n$, then one can verify that

$$\int_{\mathbf{R}^n} u(x)\,(\Delta v)(x)\,dx = \int_{\mathbf{R}^n} (\Delta u)(x)\,v(x)\,dx. \tag{6.9.9}$$

In this case, (6.9.8) implies that

$$\int_{\mathbf{R}^n} (\Delta u)(x)\,v(x)\,dx = \int_{\mathbf{R}^n} f(y)\,v(y)\,dy. \tag{6.9.10}$$

One can use this to get that (6.9.2) holds on $\mathbf{R}^n$ when f is continuous on $\mathbf{R}^n$. If f has a bit more regularity, then one can get that u is twice continuously differentiable under suitable conditions, as in Theorem 2.17 in Section B of Chapter 2 of [87].

Some topics related to integrals like those in (6.9.1) are discussed in Chapter 5 of [291].

6.10 The Poisson kernel

Let $n \geq 2$ be an integer, and put

$$p(w', x) = \frac{1}{|\partial B(0,1)|}\,\frac{(1-|x|^2)}{|x-w'|^n} \tag{6.10.1}$$

for every $w', x \in \mathbf{R}^n$ with $|w'| = 1$ and $x \neq w'$. This is the *Poisson kernel* associated to the unit ball in $\mathbf{R}^n$.

6.10.1 Harmonicity in x

Let $w' \in \mathbf{R}^n$ with $|w'| = 1$ be given, and let us check that

$$p(w', x) \text{ is harmonic as a function of } x \text{ for } x \neq w'. \tag{6.10.2}$$

Observe that

$$\begin{aligned} |x|^2 &= |(x - w') + w'|^2 = |x - w'|^2 + 2\,(x - w') \cdot w' + |w'|^2 \\ &= |x - w'|^2 + 2\,(x - w') \cdot w' + 1. \end{aligned} \tag{6.10.3}$$

Thus

$$p(w', x) = \frac{1}{|\partial B(0,1)|} \Big(\frac{-1}{|x - w'|^{n-2}} - 2\, \frac{(x - w') \cdot w'}{|x - w'|^n} \Big). \tag{6.10.4}$$

The first term on the right is harmonic in x for $x \neq w'$, as mentioned in Section 6.1 when $n \geq 3$, and trivially when $n = 2$. The second term on the right can be expressed as a linear combination of derivatives of harmonic functions in x for $x \neq w'$, as in Subsection 6.1.2, and these are harmonic too, as before.

6.10.2 A symmetry property

Suppose that $w', x' \in \mathbf{R}^n$, $|w'| = |x'|$, and $r \in \mathbf{R}$. It is easy to see that

$$|r\, x' - w'| = |x' - r\, w'|, \tag{6.10.5}$$

because

$$\begin{aligned} |r\, x' - w'|^2 &= r^2\, |x'|^2 - 2\, r\, x' \cdot w' + |w'|^2 \\ &= r^2\, |w'|^2 - 2\, x' \cdot (r\, w') + |x'|^2 = |x' - r\, w'|^2. \end{aligned} \tag{6.10.6}$$

If $|w'| = |x'| = 1$ and either $x' \neq w'$ or $r \neq 1$, then $r\, x' \neq w'$, $r\, w' \neq x'$, and it follows that

$$\begin{aligned} p(w', r\, x') &= \frac{1}{|\partial B(0,1)|}\, \frac{1 - r^2}{|r\, x' - w'|^n} \\ &= \frac{1}{|\partial B(0,1)|}\, \frac{1 - r^2}{|x' - r\, w'|^n} = p(x', r\, w'). \end{aligned} \tag{6.10.7}$$

6.10.3 Integrating the Poisson kernel

If $x' \in \mathbf{R}^n$, $|x'| = 1$, and $0 \leq r < 1$, then the mean-value property for harmonic functions implies that

$$\frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} p(x', r\, w')\, dw' = p(x', 0) = \frac{1}{|\partial B(0,1)|}, \tag{6.10.8}$$

because $p(x', z)$ is harmonic as a function of z for $z \neq x'$, as before. More precisely, the first step is clear when $r = 0$, and if $0 < r < 1$, then the left side is the same as the average of $p(x', z)$ over $z \in \partial B(0, r)$. This is equal to $p(x', 0)$, because $p(x', z)$ is harmonic as a function of z on $B(0, 1)$.

This means that

$$\int_{\partial B(0,1)} p(x', r\, w')\, dw' = 1. \tag{6.10.9}$$

It follows that

$$\int_{\partial B(0,1)} p(w', r\, x')\, dw' = 1, \tag{6.10.10}$$

because of (6.10.7).

6.10.4 Positivity of the Poisson kernel

Note that

$$p(w', x) > 0 \tag{6.10.11}$$

for every $w', x \in \mathbf{R}^n$ with $|w'| = 1$ and $|x| < 1$. We also have that

$$p(w', x) = 0 \tag{6.10.12}$$

for every $w', x \in \mathbf{R}^n$ with $|w'| = |x| = 1$ and $x \neq w'$.

6.11 More on the Poisson kernel

Let us continue with the same notation and hypothese as in the previous section.

6.11.1 Some simple estimates

If $x, y \in \mathbf{R}^n$, then

$$|x| \le |y| + |x - y| \tag{6.11.1}$$

and

$$|y| \le |x| + |x - y|, \tag{6.11.2}$$

by the triangle inequality for the standard Euclidean norm on $\mathbf{R}^n$. This implies that

$$\big||x| - |y|\big| = \max(|x| - |y|, |y| - |x|) \le |x - y|. \tag{6.11.3}$$

Suppose for the moment that

$$|x - y| \le |y|/2, \tag{6.11.4}$$

so that

$$|y| \le |x| + |y|/2, \tag{6.11.5}$$

by (6.11.2). This implies that

$$|y|/2 \le |x|, \tag{6.11.6}$$

or equivalently

$$|y| \le 2\,|x|. \tag{6.11.7}$$

6.11.2 Some additional simple estimates

Let $w', x, y' \in \mathbf{R}^n$ and $\eta > 0$ be given, and suppose that

$$|w' - y'| \ge \eta \tag{6.11.8}$$

and

$$|x - y'| \le \eta/2, \tag{6.11.9}$$

so that

$$|x - y'| \le |w' - y'|/2, \tag{6.11.10}$$

Under these conditions,

$$|w' - y'| \le |w' - x| + |x - y'| \le |w' - x| + |w' - y'|/2. \tag{6.11.11}$$

It follows that

$$|w' - y'|/2 \le |w' - x|, \tag{6.11.12}$$

or equivalently

$$|w' - y'| \le 2\,|w' - x|. \tag{6.11.13}$$

In particular,

$$\eta/2 \le |w' - x|, \tag{6.11.14}$$

by (6.11.8) and (6.11.12).

6.11.3 A localization property

Suppose now that we also have that

$$|w'| = |y'| = 1 \tag{6.11.15}$$

and

$$|x| < 1, \tag{6.11.16}$$

in addition to (6.11.8) and (6.11.9). In this case, we obtain that

$$\begin{aligned} p(w', x) = \frac{1}{|\partial B(0,1)|} \frac{(1 - |x|^2)}{|x - w'|^n} &\le \frac{2^n}{|\partial B(0,1)|} \frac{(1 - |x|^2)}{|w' - y'|^n} \\ &\le \frac{2^n\, \eta^{-n}\, (1 - |x|^2)}{|\partial B(0,1)|}, \end{aligned} \tag{6.11.17}$$

using (6.11.12) in the second step, and (6.11.8) in the third step.

It follows that

$$\int_{(\partial B(0,1) \setminus B(y', \eta))} p(w', x)\, dw' \le 2^n\, \eta^{-n}\, (1 - |x|^2). \tag{6.11.18}$$

Of course, the right side tends to 0 as $|x| \to 1$.

Observe that

$$1 - |x|^2 = (1 + |x|)\,(1 - |x|) \le 2\,(1 - |x|) \le 2\,|x - y'|, \tag{6.11.19}$$

where the third step is as in (6.11.3). Combining this with (6.11.18), we get that

$$\int_{(\partial B(0,1) \setminus B(y', \eta))} p(w', x)\, dw' \le 2^{n+1}\, \eta^{-n}\, |x - y'|. \tag{6.11.20}$$

Thus

$$\int_{(\partial B(0,1) \setminus B(y', \eta))} p(w', x)\, dw' \to 0 \text{ as } x \to y'. \tag{6.11.21}$$

6.12 The Poisson integral

Let us continue with the same notation and hypotheses as in the previous two sections. Let f be a continuous complex-valued function on the unit sphere $\partial B(0,1)$. Consider the complex-valued function u defined on the closed unit ball $\overline{B}(0,1)$ by

$$\begin{aligned} u(x) &= \int_{\partial B(0,1)} f(w')\, p(w', x)\, dw' && \text{when } |x| < 1 \\ &= f(x) && \text{when } |x| = 1. \end{aligned} \tag{6.12.1}$$

This is the *Poisson integral* of f at x when $|x| < 1$.

6.12.1 Harmonicity of the Poisson integral

It is not too difficult to show that

$$u \text{ is harmonic on } B(0,1), \tag{6.12.2}$$

because $p(w', x)$ is harmonic in x on $B(0,1)$ for every $w' \in \partial B(0,1)$, as in the previous section. One way to do this is to use standard results about differentiation under the integral sign. Another way to do this is to check that u is continuous and satisfies the mean-value property on $B(0,1)$. This uses the mean-value property for $p(w', x)$ in x for each w', and well known results about interchanging the order of integration.

6.12.2 Continuity at the boundary

One can also show that

$$u \text{ is continuous on } \overline{B}(0,1). \tag{6.12.3}$$

The continuity of u on $B(0,1)$ is reasonably straightforward, as in the preceding paragraph. If $y' \in \partial B(0,1)$, then one would like to verify that

$$u \text{ is continuous at } y', \text{ as a function on } \overline{B}(0,1). \tag{6.12.4}$$

Equivalently, this means that

$$u(x) \to u(y') = f(y') \tag{6.12.5}$$

as $x \in \overline{B}(0,1)$ tends to y'. More precisely, it suffices to consider only $x \in B(0,1)$ here, because f is continuous on $\partial B(0,1)$, by hypothesis.

Note that

$$\int_{\partial B(0,1)} p(w', x)\, dw' = 1 \tag{6.12.6}$$

for every $x \in B(0,1)$, by (6.10.10). This implies that

$$\begin{aligned} u(x) - f(y') &= \int_{\partial B(0,1)} p(w',x) f(w')\, dw' - \int_{\partial B(0,1)} p(w',x)\, f(y')\, dw' \\ &= \int_{\partial B(0,1)} p(w',x)\, (f(w') - f(y'))\, dw' \end{aligned} \tag{6.12.7}$$

for every $x \in B(0,1)$. It follows that

$$\begin{aligned} |u(x) - f(y')| &= \left| \int_{\partial B(0,1)} p(w',x)\, (f(w') - f(x))\, dw' \right| \\ &\le \int_{\partial B(0,1)} p(w',x)\, |f(w') - f(y')|\, dw' \end{aligned} \tag{6.12.8}$$

for every $x \in B(0,1)$, because of (6.10.11).

6.12.3 Estimating two terms

We would like to get that the right side of (6.12.8) is as small as we like when x is sufficiently close to y'. If $\eta > 0$, then the right side of (6.12.8) can be expressed as the sum of

$$\int_{(\partial B(0,1)) \cap B(y',\eta)} p(w',x)\, |f(w') - f(y')|\, dw' \tag{6.12.9}$$

and

$$\int_{(\partial B(0,1)) \setminus B(y',\eta)} p(w',x)\, |f(w') - f(y')|\, dw'. \tag{6.12.10}$$

If η is sufficiently small, then

$$|f(w') - f(y')| \tag{6.12.11}$$

is as small as we like when $|w' - y'| < \eta$, because f is continuous at y', by hypothesis. We can use this to get that (6.12.9) is as small as we like, because of (6.12.6). Let us now fix $\eta > 0$ in this way.

With η fixed, we can get that (6.12.10) is as small as we like when x is sufficiently close to y', as in (6.11.21). This also uses the fact that f is bounded on $\partial B(0,1)$, because f is continuous on $\partial B(0,1)$, and $\partial B(0,1)$ is compact.

6.12.4 Uniqueness of the Poisson integral

If v is any continuous complex-valued function on $\overline{B}(0,1)$ that is harmonic on $B(0,1)$ and equal to f on $\partial B(0,1)$, then

$$v = u \text{ on } \overline{B}(0,1), \tag{6.12.12}$$

as in Subsection 6.7.2.

6.13 Some more integral formulas

Let n be a positive integer, and let $N(x)$ be the real-valued function defined on $\mathbf{R}^n \setminus \{0\}$ as in Section 6.8. Put

$$\begin{aligned} c_r &= \frac{r^{2-n}}{(2-n)\,|\partial B(0,1)|} && \text{when } n \geq 3 \\ &= \frac{1}{2\pi}\,\log r && \text{when } n = 2 \end{aligned} \tag{6.13.1}$$

for each $r > 0$, so that $N(x) = c_r$ when $|x| = r$. Let $a \in \mathbf{R}^n$ and $r > 0$ be given, and suppose that u is a twice continuously-differentiable real or complex-valued function on $\overline{B}(a,r)$, as in Section 3.4.

6.13.1 Using the earlier identity again

Let $0 < t < r$ be given, and put

$$V = B(a,r) \setminus \overline{B}(a,t). \tag{6.13.2}$$

If $y' \in \partial V = (\partial B)(a,r)) \cup (\partial B)(a,t))$, then let $\nu(y')$ be the outward pointing unit normal to ∂V at y', as usual. We would like to use (3.9.1), with

$$v(x) = N(x-a) - c_r. \tag{6.13.3}$$

This implies that

$$\begin{aligned} -\int_V & v(x)\,(\Delta u)(x)\,dx \\ &= \int_{\partial V} (u(y')\,(D_{\nu(y')}v)(y') - v(y')\,(D_{\nu(y')}u)(y'))\,dy'. \end{aligned} \tag{6.13.4}$$

6.13.2 The integral over ∂V

If $\rho > 0$ and $y' \in \partial B(a,\rho)$, then put

$$\mu_\rho(y') = \rho^{-1}\,(y' - a), \tag{6.13.5}$$

which is the outward-pointing unit normal to $\partial B(a,\rho)$ at y'. Thus

$$\begin{aligned}\nu(y') &= \mu_r(y') \quad \text{when } y' \in \partial B(a,r) \\ &= -\mu_t(y') \quad \text{when } y' \in \partial B(a,t).\end{aligned} \tag{6.13.6}$$

Using this and (6.13.4), we get that

$$\begin{aligned}&-\int_V v(x)\,(\Delta u)(x)\,dx \\ &= \int_{\partial B(a,r)} u(y')\,(D_{\mu_r(y')}v)(y')dy' \\ &\quad -\int_{\partial B(a,t)} (u(y')\,(D_{\mu_t(y')}v)(y') - v(y')\,(D_{\mu_t(y')}u)(y'))\,dy',\end{aligned} \tag{6.13.7}$$

because $v = 0$ on $\partial B(a,r)$, by construction.

6.13.3 Some simplifications and modifications

It follows from this and (6.13.3) that

$$\begin{aligned}&\int_V (c_r - N(x-a))\,(\Delta u)(x)\,dx \\ &= \frac{1}{|\partial B(a,r)|}\int_{\partial B(a,r)} u(y')\,dy' - \frac{1}{|\partial B(a,t)|}\int_{\partial B(a,t)} u(y')\,dy' \\ &\quad +(c_t - c_r)\int_{\partial B(a,t)} (D_{\mu_t(y')}u)(y')\,dy',\end{aligned} \tag{6.13.8}$$

using also (6.8.9) and its analogue for $\partial B(a,t)$. Remember that

$$\int_{\partial B(a,t)} (D_{\mu_t(y')}u)(y')\,dy' = \int_{B(a,t)} (\Delta u)(x)\,dx, \tag{6.13.9}$$

as in Subsection 3.5.1. Using this, we can reexpress (6.13.8) as

$$\begin{aligned}&\int_{B(a,r)} \min(c_r - N(x-a), c_r - c_t)\,(\Delta u)(x)\,dx \\ &= \frac{1}{|\partial B(a,r)|}\int_{\partial B(a,r)} u(y')\,dy' - \frac{1}{|\partial B(a,t)|}\int_{\partial B(a,t)} u(y')\,dy'.\end{aligned} \tag{6.13.10}$$

We can take the limit as $t \to 0+$ on both sides of (6.13.8) or (6.13.10) to get that

$$\begin{aligned}&\int_{B(a,r)} (c_r - N(x-a))\,(\Delta u)(x)\,dx \\ &= \frac{1}{|\partial B(a,r)|}\int_{\partial B(a,r)} u(y')\,dy' - u(a),\end{aligned} \tag{6.13.11}$$

as in Section 6.8. More precisely, the left side of the equation should be considered as an improper integral, or a Lebesgue integral, as before.

6.13.4 Another argument

Alternatively, if $0 < \rho \le r$, then

$$\begin{aligned}(6.13.12) \qquad & \frac{d}{d\rho}\Big(\frac{1}{|\partial B(a,\rho)|}\int_{\partial B(a,\rho)} u(y')\,dy'\Big) \\ & = \frac{1}{|\partial B(a,\rho)|}\int_{\partial B(a,\rho)} (D_{\mu_\rho(y')}u)(y')\,dy',\end{aligned}$$

as in Subsection 6.2.3. This means that

$$\begin{aligned}(6.13.13) \qquad & \frac{d}{d\rho}\Big(\frac{1}{|\partial B(a,\rho)|}\int_{\partial B(a,\rho)} u(y')\,dy'\Big) \\ & = \frac{1}{|\partial B(a,\rho)|}\int_{B(a,\rho)} (\Delta u)(x)\,dx,\end{aligned}$$

by (6.13.9).

One can get (6.13.10) by integrating both sides of (6.13.13) in ρ from t to r. This also involves interchanging the order of integration on the right side. In some cases, we may be particularly interested simply in the nonnegativity of some of these integrals of Δu when $\Delta u \ge 0$, as in the next section.

6.14 Subharmonic functions

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. A twice continuously-differentiable real-valued function u on U is said to be *subharmonic* if

$$(6.14.1) \qquad \Delta u \ge 0$$

on U. Equivalently, u may be considered as a *subsolution* of the Laplace equation in this case. If $n = 1$, then this corresponds to convexity of u.

6.14.1 Sub-mean-value inequalities

Let $a \in U$ and $r > 0$ be given, with $\overline{B}(a,r) \subseteq U$. If u is subharmonic on U, then it is well known that

$$(6.14.2) \qquad u(a) \le \frac{1}{|\partial B(a,r)|}\int_{\partial B(a,r)} u(y')\,dy'.$$

This can be obtained from suitable integral formulas, as in the previous section. This corresponds to Theorem 4.3 on p76 of [297]. This is also related to Exercises 5 and 8 on p236 of [18], and we shall discuss this further in Chapter 11.

One can use this to get that

$$(6.14.3) \qquad u(a) \le \frac{1}{|B(a,r)|}\int_{B(a,r)} u(x)\,dx,$$

as before. Conditions like these may be used to extend the notion of subharmonicity to functions with less regularity. This will be discussed further in Chapter 11 as well.

6.14.2 Points at which the maximum is attained

Suppose that u is a continuous real-valued function on U, and that there is a real number A such that

$$u(x) \le A \tag{6.14.4}$$

for every $x \in U$. Suppose also for the moment that

$$u(a) = A \tag{6.14.5}$$

for some $a \in U$, and that (6.14.3) holds for some $r > 0$ such that $\overline{B}(a,r) \subseteq U$.

Under these conditions, one can check that

$$u(x) = A \tag{6.14.6}$$

for every $x \in B(a,r)$. More precisely, $A - u(x) \ge 0$ for every $x \in U$, and

$$\int_{B(a,r)} (A - u(x))\, dx \le 0, \tag{6.14.7}$$

because of (6.14.3) and (6.14.5).

6.14.3 Maximum principles for subharmonic functions

Suppose now that for every $a \in U$ there is an $r > 0$ such that $\overline{B}(a,r) \subseteq U$ and (6.14.3) holds. This implies that the set of $x \in U$ such that (6.14.6) holds is an open set, as in the preceding paragraph. This set is relatively closed in U as well, because u is continuous on U. If this set is nonempty, and U is connected, then this set is equal to U, so that

$$u \equiv A \text{ on } U. \tag{6.14.8}$$

This is another version of the strong maximum principle.

Suppose that U is bounded, and that u is a continuous real-valued function on $\overline{U}$ such that for every $a \in U$ there is an $r > 0$ with $\overline{B}(a,r) \subseteq U$ and for which (6.14.3) holds. The extreme value theorem implies that u attains its maximum on $\overline{U}$. In fact, the maximum of u on $\overline{U}$ is attained as a point in ∂U, as in Subsection 6.7.2. More precisely, this uses the remarks in the previous paragraph too. This is another version of the maximum principle.

6.15 Another approach to local maxima

Let n be a positive integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a twice continuously-differentiable real-valued function on U. If u has a local maximum at a point $a \in U$, then

$$\frac{\partial u}{\partial x_l}(a) = 0 \tag{6.15.1}$$

for each $l = 1, \ldots, n$, and

$$\frac{\partial^2 u}{\partial x_l^2}(a) \le 0 \tag{6.15.2}$$

for each $l = 1, \ldots, n$, by the second-derivative test.

In particular, this means that

$$(\Delta u)(a) \le 0. \tag{6.15.3}$$

If

$$(\Delta u)(x) > 0 \tag{6.15.4}$$

for every $x \in U$, then it follows that

$$u \text{ has no local maxima in } U. \tag{6.15.5}$$

6.15.1 Positive Laplacian on U

Suppose from now on in this section that U is bounded, and that u is a continuous real-valued function on $\overline{U}$ that is twice continuously differentiable on U. The extreme value theorem implies that u attains its maximum on $\overline{U}$. If (6.15.4) holds at every point in U, then the maximum of u on $\overline{U}$ cannot be attained at a point in U, as in the preceding paragraph. This implies that the maximum of u on $\overline{U}$ is attained at a point in ∂U.

6.15.2 An approximation argument

Suppose that u is subharmonic on U, so that $\Delta u \ge 0$ on U. Let $\epsilon > 0$ be given, and put

$$v_\epsilon(x) = u(x) + \epsilon\, |x|^2 \tag{6.15.6}$$

for every $x \in \overline{U}$. Note that v_ϵ is continuous on $\overline{U}$, twice continuously differentiable on U, and that

$$(\Delta v_\epsilon)(x) \ge 2\, n\, \epsilon > 0 \tag{6.15.7}$$

for every $x \in U$. It follows that

$$\text{the maximum of } v_\epsilon \text{ on } \overline{U} \text{ is attained at a point in } \partial U, \tag{6.15.8}$$

as in the previous paragraph.

6.15.3 The maximum principle for u

Of course, $\overline{U}$ is bounded in $\mathbf{R}^n$, because U is bounded, so that there is a nonnegative real number R such that

$$|x| \le R \tag{6.15.9}$$

for every $x \in \overline{U}$. This means that

$$v_\epsilon(x) \le u(x) + \epsilon\, R^2 \tag{6.15.10}$$

for every $x \in \overline{U}$. It follows that

$$\max_{x \in \overline{U}} v_\epsilon(x) = \max_{x \in \partial U} v_\epsilon(x) \le \max_{x \in \partial U} u(x) + \epsilon\, R^2, \tag{6.15.11}$$

using (6.15.8) in the first step.

This implies that

$$\max_{x \in \overline{U}} u(x) \le \max_{x \in \partial U} u(x) + \epsilon\, R^2, \tag{6.15.12}$$

because $u \le v_\epsilon$ on $\overline{U}$, by construction. Thus

$$\max_{x \in \overline{U}} u(x) \le \max_{x \in \partial U} u(x), \tag{6.15.13}$$

because $\epsilon > 0$ is arbitrary.

This is the same as saying that the maximum of u on $\overline{U}$ is attained at a point in ∂U. This is another approach to the maximum principle under these conditions.

6.15.4 Some milder differentiability conditions

The argument mentioned at the beginning of this section works if instead of asking that u be twice continuously differentiable on U, we ask that

(6.15.14) the first and second derivatives of u in each variable exist at every point in U.

Similarly, the arguments in the previous subsections work if we ask that u be continuous on $\overline{U}$, and satisfy (6.15.14) on U, instead of twice continuous differentiability on U.

If u satisfies (6.15.14), then we can define the Laplacian of u on U in the usual way, as in Section 2.1. If u is also continuous on U and satisfies the Laplace equation on U, then u is in fact harmonic on U. This is mentioned on p243 of [7] when $n = 2$, and the same argument, due to Carathéodory, works in all dimensions.

More precisely, if u is a continuous real-valued function on the closed unit ball $\overline{B}(0,1)$ that satisfies (6.15.14) on $U = B(0,1)$, and if u satisfies the Laplace equation on $B(0,1)$, then u is equal to the Poisson integral of its restriction to $\partial B(0,1)$, as in Section 6.12.4. This uses the fact that the difference of u and the Poisson integral of its restriction to $\partial B(0,1)$ is a continuous function on $\overline{B}(0,1)$ that is equal to 0 on $\partial B(0,1)$ and satisfies the Laplace equation in the same sense on $B(0,1)$. One can use translations and dilations to get the analogous statement for any closed ball in $\mathbf{R}^n$. If u is continuous on U and satisfies (6.15.14) and the Laplace equation on U, then we can get that u is harmonic on U, by considering closed balls contained in U.

Chapter 7

The heat equation

7.1 Some basic solutions

Let n be a positive integer, and let us identify $\mathbf{R}^n \times \mathbf{R}$ with $\mathbf{R}^{n+1}$, as usual. Let U be a nonempty open subset of $\mathbf{R}^n \times \mathbf{R}$, and let u be a twice continuously-differentiable real or complex-valued function on U. We shall use $\Delta u = \Delta_x u$ to refer to the Laplacian of $u(x,t)$ as a function of x, with t fixed.

We say that $u(x,t)$ satisfies the *heat equation* on U if

$$\frac{\partial u}{\partial t} = \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \tag{7.1.1}$$

on U. One may also consider continuously-differentiable functions $u(x,t)$ on U whose second derivatives in x exist and are continuous on U.

Let V be a nonempty open subset of $\mathbf{R}^n$, and let v be a twice continuously-differentiable real or complex-valued function on V. Thus $W = V \times \mathbf{R}$ is an open set in $\mathbf{R}^n \times \mathbf{R}$, and

$$w(x,t) = v(x) \tag{7.1.2}$$

is twice continuously-differentiable on W. Clearly w satisfies the heat equation on W if and only if v is harmonic on V.

Let $a \in \mathbf{C}$ and $b \in \mathbf{C}^n$ be given, and put

$$u(x,t) = \exp(a\,t + b \cdot x) \tag{7.1.3}$$

for every $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. This satisfies the heat equation on $\mathbf{R}^n \times \mathbf{R}$ if and only if

$$a = b \cdot b. \tag{7.1.4}$$

If $b \in \mathbf{R}^n$, then it follows that $a \geq 0$. If $b = i\,c$ for some $c \in \mathbf{R}^n$, then (7.1.4) implies that

$$a = -c \cdot c \leq 0. \tag{7.1.5}$$

7.1.1 The heat kernel

Put

$$K(x,t) = (4\pi\, t)^{-n/2} \exp(-|x|^2/(4\, t)) \tag{7.1.6}$$

for $x \in \mathbf{R}^n$ and $t > 0$. One can check directly that this satisfies the heat equation on $\mathbf{R}^n \times \mathbf{R}_+$. This is known as the *Gauss–Weierstrass* or *heat kernel*, as in Section A of Chapter 4 of [87] and p8 of [297]. This is also discussed in Section 2.3.1 a of [81].

With this normalization, we have that

$$\int_{\mathbf{R}^n} K(x,t)\, dx = 1 \tag{7.1.7}$$

for every $t > 0$. The integral on the left may be considered as an improper integral or as a Lebesgue integral, and this will be discussed in the next two sections.

Put

$$K(x,t) = 0 \tag{7.1.8}$$

when $t = 0$ and $x \neq 0$, and for every $x \in \mathbf{R}^n$ when $t < 0$. This together with (7.1.6) defines $K(x,t)$ on $(\mathbf{R}^n \times \mathbf{R}) \setminus \{(0,0)\}$. One can verify that $K(x,t)$ is smooth on this set, and satisfies the heat equation there, as in Section 2.3.1 b of [81], and Section A of Chapter 4 of [87].

Of course, the heat equation is invariant under translations. In particular, if $y \in \mathbf{R}^n$ and $r \in \mathbf{R}$, then

$$K(x-y, t-r) \tag{7.1.9}$$

is smooth as a function of (x,t) on $(\mathbf{R}^n \times \mathbf{R}) \setminus \{(y,r)\}$, and satisfies the heat equation there.

Note that

$$(-t)^{-n/2} \exp(-|x|^2/(4t)) \tag{7.1.10}$$

satisfies the heat equation on $\mathbf{R}^n \times (-\infty, 0)$, for the same reasons as before. If $y \in \mathbf{R}^n$ and $r \in \mathbf{R}$, then it follows that

$$(r-t)^{-n/2} \exp(|x-y|^2/(4(r-t))) \tag{7.1.11}$$

satisfies the heat equation as a function of (x,t) on $\mathbf{R}^n \times (-\infty, r)$.

7.2 Integrable continuous functions

Let f be a nonnegative real-valued continuous function on the real line. If a, b are real numbers with $a \leq b$, then

$$\int_a^b f(x)\, dx \tag{7.2.1}$$

is defined as a Riemann integral, and is a nonnegative real number. Let us say that f is *integrable* on $\mathbf{R}$ if the integrals (7.2.1) are bounded. In this case,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{\mathbf{R}} f(x)\,dx \tag{7.2.2}$$

may be defined as the supremum or least upper bound of the integrals in (7.2.1) over all $a, b \in \mathbf{R}$ with $a \leq b$. This could also be considered as an improper integral, which is obtained by taking a suitable limit of (7.2.1) as $a \to -\infty$ and $b \to \infty$.

One could use the Lebesgue integral to define (7.2.2) as a nonnegative extended real number for any nonnegative continuous function on $\mathbf{R}$. Integrability of f in the sense considered in the preceding paragraph is the same as the finiteness of (7.2.2) as a Lebesgue integral, which implies that f is Lebesgue integrable on $\mathbf{R}$.

7.2.1 Integrable real-valued functions

If f is a real-valued continuous function on $\mathbf{R}$, then

$$f_+ = \max(f, 0),\ f_- = \max(-f, 0) \tag{7.2.3}$$

are nonnegative continuous functions on $\mathbf{R}$ such that

$$f = f_+ - f_-,\ |f| = f_+ + f_-. \tag{7.2.4}$$

Let us say that f is *integrable* on $\mathbf{R}$ if

(7.2.5) $|f|$ is integrable as a nonnegative continuous function on $\mathbf{R}$,

which happens if and only if f_+ and f_- are integrable as nonnegative continuous functions on $\mathbf{R}$. This permits us to define the integral (7.2.2) as the difference of the integrals of f_+ and f_- on $\mathbf{R}$. This could also be considered as an improper integral, as before. This is the same as the Lebesgue integral of f on $\mathbf{R}$ as well.

7.2.2 Integrable complex-valued functions

Similarly, a complex-valued continuous function f on $\mathbf{R}$ is said to be *integrable* on $\mathbf{R}$ if (7.2.5) holds. This happens if and only if the real and imaginary parts of f are integrable as real-valued continuous functions on $\mathbf{R}$, and the real and imaginary parts of the integral (7.2.2) are defined as the integrals of the real and imaginary parts of f on $\mathbf{R}$. This could be considered as an improper integral too, and it is the same as the Lebesgue integral of f on $\mathbf{R}$.

7.2.3 Integrability on $\mathbf{R}^n$

There are analogous notions on $\mathbf{R}^n$ for any positive integer n. If f is a nonnegative real-valued continuous function on $\mathbf{R}^n$, then the *integrability* of f on

$\mathbf{R}^n$ can be defined in terms of the boundedness of the integrals of f over any reasonable family of balls, cubes, or other regions that exhaust $\mathbf{R}^n$. Under these conditions, the integral

$$\int_{\mathbf{R}^n} f(x)\,dx \tag{7.2.6}$$

of f on $\mathbf{R}^n$ can be defined as the supremum or least upper bound of these integrals, and this could also be considered as a limit of these integrals.

If f is a real or complex-valued continuous function on $\mathbf{R}^n$, then the integrability of f is defined to mean that

(7.2.7) $|f|$ is integrable as a nonnegative continuous function on $\mathbf{R}^n$,

and this can be used to define the integral of f on $\mathbf{R}^n$ as before. This implies that f is Lebesgue integrable on $\mathbf{R}^n$, and the integral of f on $\mathbf{R}^n$ is the same as the Lebesgue integral. In this case, it is well known and not too difficult to show that

$$\left|\int_{\mathbf{R}^n} f(x)\,dx\right| \le \int_{\mathbf{R}^n} |f(x)|\,dx. \tag{7.2.8}$$

One might like to express an integral over $\mathbf{R}^n$ as an iterated integral too. There are standard results for doing this when using Lebesgue integrals. Otherwise, one can use standard results for doing this over nice bounded regions in $\mathbf{R}^n$, and pass to a limit under suitable conditions.

7.3 Some examples of integrable functions

Let n be a positive integer, and let α be a positive real number. Note that

$$\min(1, |x|^{-\alpha}) \tag{7.3.1}$$

is continuous on $\mathbf{R}^n$, which is interpreted as being equal to 1 at $x = 0$. One can check that this function is integrable on $\mathbf{R}^n$ exactly when

$$\alpha > n. \tag{7.3.2}$$

7.3.1 Integrating Gaussians

It is easy to see that $\exp(-|x|^2)$ is integrable on $\mathbf{R}^n$. It is well known that

$$\int_{\mathbf{R}^n} \exp(-|x|^2)\,dx = \pi^{n/2}. \tag{7.3.3}$$

More precisely, the $n = 2$ case can be obtained using polar coordinates. The 2-dimensional integral is the same as the square of the one-dimensional integral, which can be used to get the $n = 1$ case. Similarly, the n-dimensional integral is equal to the nth power of the one-dimensional integral.

If a is a positive real number again, then $\exp(-a\,|x|^2)$ is integrable on $\mathbf{R}^n$. One can check that

$$\int_{\mathbf{R}^n} \exp(-a\,|x|^2)\,dx = (\pi/a)^{n/2}, \tag{7.3.4}$$

using a change of variables.

7.3.2 Linear terms in the exponential

If $b \in \mathbf{R}^n$, then it is easy to see that

$$\exp(-a\,|x|^2 + b \cdot x) \tag{7.3.5}$$

is integrable on $\mathbf{R}^n$. Observe that

$$\exp(-a\,|x|^2 + b \cdot x) = \exp(-a\,|x - (2\,a)^{-1}\,b|^2 + (4\,a)^{-1}|b|^2) \tag{7.3.6}$$

for every $x \in \mathbf{R}^n$. It follows that

$$\begin{aligned} \int_{\mathbf{R}^n} \exp(-a\,|x|^2 + b \cdot x)\,dx &= \int_{\mathbf{R}^n} \exp(-a\,|x - (2\,a)^{-1}\,b|^2 + (4\,a)^{-1}\,|b|^2)\,dx \\ &= \int_{\mathbf{R}^n} \exp(-a\,|x|^2 + (4\,a)^{-1}\,|b|^2)\,dx \\ &= (\pi/a)^{n/2}\,\exp((4\,a)^{-1}\,|b|^2), \end{aligned} \tag{7.3.7}$$

using a change of variables in the second step, and (7.3.4) in the third step.

7.3.3 Linear terms with complex coefficients

In fact, (7.3.5) is integrable on $\mathbf{R}^n$ when $b \in \mathbf{C}^n$. It is well known that

$$\int_{\mathbf{R}^n} \exp(-a\,|x|^2 + b \cdot x)\,dx = (\pi/a)^{n/2}\,\exp((4\,a)^{-1}\,b \cdot b) \tag{7.3.8}$$

for every $b \in \mathbf{C}^n$, which is the same as (7.3.7) when $b \in \mathbf{R}^n$. One can first reduce to the case where $n = 1$, because the left side is the same as the product of n analogous integrals over $\mathbf{R}$. If $n = 1$, then one can show that both sides of (7.3.8) are holomorphic functions of $b \in \mathbf{C}$. This permits one to reduce to the case where $b \in \mathbf{R}$, using standard results in complex analysis.

Alternatively, one can use the fact that

$$\exp(-a\,z^2 + b\,z) = \exp(-a\,(z - (b/(2\,a)))^2 + b^2/(4\,a)) \tag{7.3.9}$$

is a holomorphic function of $z \in \mathbf{C}$. One can reduce to the case where $b \in \mathbf{R}$ again, using Cauchy's theorem to make a suitable change of contour.

As another approach, one can reduce to the case where b is purely imaginary, using a change of variables in x, as before. This corresponds to a Fourier transform, as on p105f of [182], and Theorem 1.4 on p138 of [293].

7.4 Some integral solutions

Let n be a positive integer, and let f be a continuous real or complex-valued function on $\mathbf{R}^n$. If $x \in \mathbf{R}^n$ and $t > 0$, then we would like to put

$$u(x,t) = \int_{\mathbf{R}^n} K(x-y,t)\, f(y)\, dy = \int_{\mathbf{R}^n} (4\pi\, t)^{-n/2} \exp(-|x-y|^2/(4\,t))\, f(y)\, dy. \tag{7.4.1}$$

This is called the *Gauss–Weierstrass integral* of f. The integral on the right is defined as long as

$$\exp(-|x-y|^2/(4\,t))\, f(y) \tag{7.4.2}$$

is integrable as a function of y on $\mathbf{R}^n$, as in Subsection 7.2.3. Equivalently, this means that

$$\exp((2x\cdot y - |y|^2)/(4\,t))\, |f(y)| \tag{7.4.3}$$

is integrable as a function of y on $\mathbf{R}^n$, because

$$|x-y|^2 = |x|^2 - 2\,x\cdot y + |y|^2 \tag{7.4.4}$$

for all $x, y \in \mathbf{R}^n$.

7.4.1 A sufficient condition for integrability

Let τ be a positive real number, and suppose that there is a nonnegative real number $C(\tau)$ such that

$$|f(y)| \le C(\tau)\, \exp(|y|^2/(4\,\tau)) \tag{7.4.5}$$

for every $y \in \mathbf{R}^n$. If $x \in \mathbf{R}^n$ and $t > 0$, then (7.4.5) implies that

$$\begin{aligned} &\exp((2\,x\cdot y - |y|^2)/(4\,t))\, |f(y)| \\ &\quad \le C(\tau)\, \exp((2\,x\cdot y)/(4\,t) - |y|^2\, ((4\,t)^{-1} - (4\,\tau)^{-1})) \end{aligned} \tag{7.4.6}$$

for every $y \in \mathbf{R}^n$. If $t < \tau$, then

$$(4\,\tau)^{-1} < (4\,t)^{-1}, \tag{7.4.7}$$

and one can check that (7.4.3) is integrable as a function of y on $\mathbf{R}^n$. Thus $u(x,t)$ can be defined as in (7.4.1) in this case.

One can differentiate under the integral sign, to show that

$$u(x,t) \text{ is smooth on } \mathbf{R}^n \times (0,\tau) \tag{7.4.8}$$

under these conditions. In particular, one can check that any number of derivatives of (7.4.2) in x and t is integrable as a function of y on $\mathbf{R}^n$ when $0 < t < \tau$, because of (7.4.5). We also get that

$$u(x,t) \text{ satisfies the heat equation on } \mathbf{R}^n \times (0,\tau), \tag{7.4.9}$$

because $K(x-y,t)$ satisfies the heat equation as a function of (x,t) for every $y \in \mathbf{R}^n$.

7.4.2 Some convergence properties

It is easy to see that

$$\int_{\mathbf{R}^n} K(x-y,t)\,dy = 1 \tag{7.4.10}$$

for every $x \in \mathbf{R}^n$ and $t > 0$, using (7.1.7). If $z \in \mathbf{R}^n$ and $t < \tau$, then it follows that

$$\begin{aligned} u(x,t) - f(z) &= \int_{\mathbf{R}^n} K(x-y,t)\,f(y)\,dy - \int_{\mathbf{R}^n} K(x-y,t)\,f(z)\,dy \\ &= \int_{\mathbf{R}^n} K(x-y,t)\,(f(y)-f(z))\,dy. \end{aligned} \tag{7.4.11}$$

This implies that

$$\begin{aligned} |u(x,t) - f(z)| &= \left|\int_{\mathbf{R}^n} K(x-y,t)\,(f(y)-f(z))\,dy\right| \\ &\le \int_{\mathbf{R}^n} K(x-y,t)\,|f(y)-f(z)|\,dy, \end{aligned} \tag{7.4.12}$$

because $K(x-y,t) \ge 0$. One can use this to show that

$$u(x,t) \to f(z) \tag{7.4.13}$$

as $(x,t) \to (z,0)$ in $\mathbf{R}^n \times \mathbf{R}$.

More precisely, if $\eta > 0$, then the right side of (7.4.12) is equal to the sum of

$$\int_{B(z,\eta)} K(x-y,t)\,|f(y)-f(z)|\,dy \tag{7.4.14}$$

and

$$\int_{\mathbf{R}^n \setminus B(z,\eta)} K(x-y,t)\,|f(y)-f(z)|\,dy. \tag{7.4.15}$$

If η is sufficiently small, then we can get that (7.4.14) is as small as we like, because f is continuous at z, and using (7.4.10) again. If we fix $\eta > 0$ with this property, then we can get that (7.4.15) is as small as we like when (x,t) is sufficiently close to $(z,0)$. This uses (7.4.5) and the definition (7.1.6) of the heat kernel.

This means that

$$u(x,t) \text{ extends continuously to } 0 \le t < \tau, \tag{7.4.16}$$

by taking it to be equal to $f(x)$ when $t = 0$. Properties like these are mentioned in Theorem 1 in Section 2.3.1 b of [81], and Theorem 4.3 in Section A of Chapter 4 of [87].

A related convergence property is that

$$u(x,t) \to f(x) \tag{7.4.17}$$

uniformly on bounded subsets of $\mathbf{R}^n$ as $t \to 0$. This can be obtained using the uniform continuity of f on compact subsets of $\mathbf{R}^n$. The previous convergence property can be obtained from this one and the continuity of f on $\mathbf{R}^n$. This convergence property could also be obtained from the continuous extension of $u(x,t)$ to $t \geq 0$, and the uniform continuity of this extension on compact sets.

7.4.3 Integrability for all $t > 0$

Suppose now that for every $\tau > 0$ there is a nonnegative real number $C(\tau)$ such that (7.4.5) holds. This implies that $u(x,t)$ can be defined as in (7.4.1) for every $x \in \mathbf{R}^n$ and $t > 0$.

Of course, this condition holds when f is bounded on $\mathbf{R}^n$. This condition also holds when f is the exponential of a linear function on $\mathbf{R}^n$.

7.5 Some related integrability conditions

Let n be a positive integer, and let f be a continuous real or complex-valued function on $\mathbf{R}^n$. Also let τ_1 be a positive real number, and suppose that

$$\exp(-|y|^2/(4\,\tau_1))\,|f(y)| \tag{7.5.1}$$

is integrable on $\mathbf{R}^n$, as in Subsection 7.2.3. This implies that (7.4.3) is integrable as a function of y on $\mathbf{R}^n$ when $0 < t < \tau_1$ and $x \in \mathbf{R}^n$. This means that $u(x,t)$ can be defined as in (7.4.1) under these conditions.

Of course, (7.4.5) is the same as saying that

$$\exp(-|y|^2/(4\,\tau))\,|f(y)| \tag{7.5.2}$$

is bounded on $\mathbf{R}^n$, where τ is a positive real number, as before. This implies that (7.5.1) is integrable on $\mathbf{R}^n$ for $0 < \tau_1 < \tau$, as mentioned in the previous section. If (7.5.1) is integrable on $\mathbf{R}^n$ for some $\tau_1 > 0$, then $u(x,t)$ satisfies the same properties on $\mathbf{R}^n \times (0,\tau_1)$ as mentioned in the previous section when (7.4.5) holds.

If (7.5.1) is integrable on $\mathbf{R}^n$ for every $\tau_1 > 0$, then $u(x,t)$ can be defined as in (7.4.1) for every $x \in \mathbf{R}^n$ and $t > 0$. In particular, this holds when (7.5.2) is bounded on $\mathbf{R}^n$ for every $\tau > 0$. Of course, if f is integrable on $\mathbf{R}^n$, then (7.5.1) is integrable on $\mathbf{R}^n$ for every $\tau_1 > 0$.

7.5.1 Using Lebesgue integrals

If one is familiar with Lebesgue integrals, then one may consider real or complex-valued Lebesgue measurable functions f on $\mathbf{R}^n$. The integral on the right side of (7.4.1) can be defined as a Lebesgue integral when (7.4.2) is Lebesgue integrable as a function of y on $\mathbf{R}^n$. This is equivalent to the Lebesgue integrability of (7.4.3) as a function of y on $\mathbf{R}^n$, as before. Note that this implies that f is locally integrable with respect to Lebesgue measure on $\mathbf{R}^n$.

If (7.5.1) is Lebesgue integrable on $\mathbf{R}^n$ for some $\tau_1 > 0$, then (7.4.3) is Lebesgue integrable as a function of y on $\mathbf{R}^n$ when $0 < t < \tau_1$ and $x \in \mathbf{R}^n$, as before. This implies that $u(x,t)$ can be defined as in (7.4.1) on $\mathbf{R}^n \times (0, \tau_1)$. One can differentiate under the integral sign under these conditions too, to get that $u(x,t)$ is smooth on $\mathbf{R}^n \times (0, \tau_1)$. Note that any number of derivatives of (7.4.2) in x and t is Lebesgue integrable as a function of y on $\mathbf{R}^n$ when $0 < t < \tau_1$, because of the Lebesgue integrability of (7.5.1) on $\mathbf{R}^n$. We also have that $u(x,t)$ satisfies the heat equation on $\mathbf{R}^n \times (0, \tau_1)$, as before.

However, the convergence of $u(x,t)$ to $f(x)$ as $t \to 0+$ is more complicated in this case. Some results along these lines are mentioned in Theorem 4.3 in Section A of Chapter 4 of [87], and Theorems 1.18 and 1.25 on p10, 13 of [297].

There are continuity and convergence results like those mentioned in the previous section at points where f is continuous. One can also use Riemann integrals on suitable regions in $\mathbf{R}^n$, and corresponding improper integrals on $\mathbf{R}^n$, to deal with some types of functions that may not be continuous, instead of Lebesgue integrals.

7.6 Translations and integrability

Let n be a positive integer, and let f be a continuous real or complex-valued function on $\mathbf{R}^n$. If $a \in \mathbf{R}^n$, then

$$f_a(x) = f(x - a) \tag{7.6.1}$$

is a continuous function on $\mathbf{R}^n$ as well. Note that f is integrable on $\mathbf{R}^n$ if and only if f_a is integrable on $\mathbf{R}^n$, in which case

$$\int_{\mathbf{R}^n} f_a(x)\,dx = \int_{\mathbf{R}^n} f(x)\,dx. \tag{7.6.2}$$

Of course, this also holds with $|f|$ in place of f, so that

$$\int_{\mathbf{R}^n} |f_a(x)|\,dx = \int_{\mathbf{R}^n} |f(x)|\,dx. \tag{7.6.3}$$

7.6.1 Some related properties of translates

Let $x \in \mathbf{R}^n$ and $t > 0$ be given. Observe that

$$\begin{aligned} &\exp(-|x-y|^2/(4\,t))\,f(y-a) \\ &\quad = \exp(-|(x-a)-(y-a)|^2/(4\,t))\,f(y-a) \end{aligned} \tag{7.6.4}$$

is the same as

$$\exp(-|(x-a)-y|^2/(4\,t))\,f(y) \tag{7.6.5}$$

with y replaced by $y - a$. Thus (7.6.4) is integrable on $\mathbf{R}^n$ if and only if (7.6.5) is integrable on $\mathbf{R}^n$, as in the preceding paragraph. In this case, we get that

$$u(x-a,t) = \int_{\mathbf{R}^n} (4\pi\,t)^{-n/2}\,\exp(-|x-y|^2/(4\,t))\,f(y-a)\,dy, \tag{7.6.6}$$

where the left side is defined as in (7.4.1).

Note that

$$\exp(-|y|^2/(4\,t))\,|f(y-a)| \tag{7.6.7}$$

is integrable on $\mathbf{R}^n$ if and only if

$$\exp(-|y+a|^2/(4\,t))\,|f(y)| \tag{7.6.8}$$

is integrable on $\mathbf{R}^n$. This is equivalent to the integrability of

$$\exp((-2\,a\cdot y-|y|^2)/(4\,t))\,|f(y)| \tag{7.6.9}$$

If (7.5.1) is integrable on $\mathbf{R}^n$ for some $\tau_1 > 0$, then (7.6.9) is integrable on $\mathbf{R}^n$ when $0 < t < \tau_1$, as in the previous section. This means that

$$\text{(7.6.7) is integrable on } \mathbf{R}^n \text{ when } 0 < t < \tau_1. \tag{7.6.10}$$

Of course, there are analogous statements for Lebesgue measurable functions f using Lebesgue integrability.

Similarly, (7.6.7) is bounded on $\mathbf{R}^n$ if and only if (7.6.8) is bounded on $\mathbf{R}^n$, which is equivalent to the boundedness of (7.6.9) on $\mathbf{R}^n$. Suppose that (7.5.2) is bounded on $\mathbf{R}^n$ for some $\tau > 0$, which is the same as saying that (7.4.5) holds for some $C(\tau) \ge 0$. It follows that (7.6.9) is bounded on $\mathbf{R}^n$ when $0 < t < \tau$, so that

$$\text{(7.6.7) is bounded on } \mathbf{R}^n \text{ when } 0 < t < \tau. \tag{7.6.11}$$

Let $0 < \tau_0 \le +\infty$ be given. Consider the condition that

$$\text{(7.5.1) be integrable on } \mathbf{R}^n \text{ for every positive real number } \tau_1 < \tau_0. \tag{7.6.12}$$

This implies that f_a satisfies the analogous condition, as in (7.6.10). Similarly, consider the condition that

$$\text{(7.5.2) be bounded on } \mathbf{R}^n \text{ for every } 0 < \tau < \tau_0. \tag{7.6.13}$$

This implies that f_a satisfies the analogous condition, as in (7.6.11).

7.7 Some properties of these solutions

Let n be a positive integer, and let f be a continuous real or complex-valued function on $\mathbf{R}^n$. Also let $x \in \mathbf{R}^n$ and $t > 0$ be given, and suppose for the moment that (7.4.2) or equivalently (7.4.3) is integrable as a function of y on $\mathbf{R}^n$. Thus $u(x,t)$ may be defined as in (7.4.1), and we have that

$$\begin{aligned} |u(x,t)| &\le \int_{\mathbf{R}^n} K(x-y,t)\,|f(y)|\,dy \\ &= \int_{\mathbf{R}^n} (4\pi\,t)^{-n/2}\,\exp(-|x-y|^2/(4\,t))\,|f(y)|\,dy. \end{aligned} \tag{7.7.1}$$

This also works when f is Lebesgue measurable on $\mathbf{R}^n$, and (7.4.2) or equivalently (7.4.3) is Lebesgue integrable as a function of y on $\mathbf{R}^n$.

7.7.1 Upper and lower bounds

Of course, if f is real-valued on $\mathbf{R}^n$, then $u(x,t) \in \mathbf{R}$. If f is also nonnegative on $\mathbf{R}^n$, then

$$u(x,t) \geq 0. \tag{7.7.2}$$

Similarly, if $f(y) \geq a$ for some $a \in \mathbf{R}$ and every $y \in \mathbf{R}^n$, then

$$u(x,t) \geq a, \tag{7.7.3}$$

because of (7.4.10). If $f(y) \leq b$ for some $b \in \mathbf{R}$ and every $y \in \mathbf{R}^n$, then

$$u(x,t) \leq b, \tag{7.7.4}$$

for basically the same reasons.

7.7.2 Bounded complex-valued functions

If f is a bounded continuous complex-valued function on $\mathbf{R}^n$, then $u(x,t)$ is defined for every $x \in \mathbf{R}^n$ and $t > 0$, as in Subsection 7.4.3. More precisely, suppose that

$$|f(y)| \leq C \tag{7.7.5}$$

for some $C \geq 0$ and every $y \in \mathbf{R}^n$. This implies that

$$|u(x,t)| \leq C \tag{7.7.6}$$

for every $x \in \mathbf{R}^n$ and $t > 0$, because of (7.4.10) and (7.7.1). This works when f is a bounded Lebesgue measurable function on $\mathbf{R}^n$ as well. If f is a constant on $\mathbf{R}^n$, then $u(x,t)$ is equal to the same constant for every $x \in \mathbf{R}^n$ and $t > 0$.

7.7.3 Integral bounds and convergence

Note that

$$\int_{\mathbf{R}^n} K(x-y,t)\,dx = 1 \tag{7.7.7}$$

for every $y \in \mathbf{R}^n$ and $t > 0$, because of (7.1.7). Of course, this is essentially the same as (7.4.10).

Suppose now that f is a real or complex-valued function on $\mathbf{R}^n$ that is continuous and integrable, or simply Lebesgue integrable. This implies that $u(x,t)$ may be defined as in (7.4.1) for every $x \in \mathbf{R}^n$ and $t > 0$. In this case, $u(x,t)$ is integrable as a function of x on $\mathbf{R}^n$ for every $t > 0$, with

$$\int_{\mathbf{R}^n} |u(x,t)|\,dx \leq \int_{\mathbf{R}^n} |f(y)|\,dy. \tag{7.7.8}$$

This can be obtained from (7.7.1) by interchanging the order of integration, and using (7.7.7). Similarly,

$$\int_{\mathbf{R}^n} u(x,t)\,dx = \int_{\mathbf{R}^n} f(y)\,dy \tag{7.7.9}$$

for every $t > 0$.

One can also show that

$$\lim_{t \to 0+} \int_{\mathbf{R}^n} |u(x,t) - f(x)|\, dx = 0. \tag{7.7.10}$$

This corresponds to taking $p = 1$ in Theorem 4.3 in Section A of Chapter 4 of [87], and Theorem 1.18 on p10 of [297]. This is simpler when f is a continuous function on $\mathbf{R}^n$ with compact support. Otherwise, one can approximate f by such functions.

7.8 Parabolic boundaries and maxima

Let n be a positive integer, let V be a nonempty bounded open subset of $\mathbf{R}^n$, and let T be a positive real number. Thus

$$U = V \times (0, T) \tag{7.8.1}$$

is a bounded open subset of $\mathbf{R}^n \times \mathbf{R}$, which we identify with $\mathbf{R}^{n+1}$, as usual. The closure $\overline{U}$ of U in $\mathbf{R}^n \times \mathbf{R}$ is given by

$$\overline{U} = \overline{V} \times [0, T], \tag{7.8.2}$$

where $\overline{V}$ is the closure of V in $\mathbf{R}^n$. The boundary ∂U of U in $\mathbf{R}^n \times \mathbf{R}$ is given by

$$\partial U = (\overline{V} \times \{0\}) \cup ((\partial V) \times [0, T]) \cup (\overline{V} \times \{T\}), \tag{7.8.3}$$

where ∂V is the boundary of V in $\mathbf{R}^n$.

Note that

$$(\overline{V} \times \{0\}) \cup ((\partial V) \times [0, T]) \tag{7.8.4}$$

is a closed set in $\mathbf{R}^n \times \mathbf{R}$ that is contained in ∂U. This may be called the *parabolic boundary* of U, as in Section 2.3.2 of [81], although the term is used slightly differently there. This is the same as

$$(V \times \{0\}) \cup ((\partial V) \times [0, T]), \tag{7.8.5}$$

because $(\partial V) \times \{0\}$ is contained in the second part of the union.

7.8.1 Some related maximum principles

Let u be a continuous real-valued function on $\overline{U}$, and suppose that u is continuously differentiable on U, and that the second derivatives of $u(x,t)$ in x exist and are continuous on U. Remember that u attains its maximum on $\overline{U}$, by the extreme value theorem. If u satisfies the heat equation on U, then it is well known that

(7.8.6) the maximum of u on $\overline{U}$ is attained on the parabolic boundary of U.

This is the *maximum principle* for the heat equation, as in Theorem 4.10 in Section B of Chapter 4 of [87].

This corresponds to part (i) of Theorem 4 in Section 2.3.3 a of [81]. Part (ii) of that theorem is a version of the strong maximum principle for the heat equation. The proof uses a mean-value property for the heat equation in Theorem 3 of Section 2.3.2 of [81]. A more direct approach to the first part is indicated in Problem 16 in Section 2.5 of [81], which is similar to the argument in [87], that we shall follow here. This version of the maximum principle also works for subsolutions of the heat equation, which will be discussed in the next section.

7.8.2 Uniqueness and the parabolic boundary

If $u = 0$ on the parabolic boundary of U, then the maximum principle implies that

$$u \le 0 \text{ on } \overline{U}. \tag{7.8.7}$$

The same argument could be applied to $-u$, to get that

$$u \equiv 0 \text{ on } \overline{U}. \tag{7.8.8}$$

This corresponds to Theorem 5 in Section 2.3.3 a of [81], and to Corollary 4.11 in Section B of Chapter 4 of [87].

7.8.3 A remark about the maximum of u on $\overline{U}$

Of course, the parabolic boundary (7.8.4) of U is closed and bounded in $\mathbf{R}^n \times \mathbf{R}$, and thus compact. If u is any continuous real-valued function on $\overline{U}$, then the maximum of u on the parabolic boundary of U is attained, by the extreme value theorem. In order to show that the maximum of u on $\overline{U}$ is attained on the parabolic boundary of U, it suffices to show that for each $(x,t) \in \overline{U}$, $u(x,t)$ is less than or equal to the maximum of u on the parabolic boundary of U.

7.8.4 The maximum of $|u|$ when u is complex-valued

Suppose now that u is a continuous complex-valued function on $\overline{U}$ that is continuously differentiable on U, and that the second derivatives of $u(x,t)$ in x exist and are continuous on U. Thus the previous statements for real-valued functions can be applied to the real and imaginary parts of u. Similarly, if $\alpha \in \mathbf{C}$, then the previous statements can be applied to

$$\operatorname{Re}(\alpha\, u(x,t)). \tag{7.8.9}$$

If $w \in \mathbf{C}$, then it is easy to see that

$$|w| = \max\{\operatorname{Re}(\alpha\, w) : \alpha \in \mathbf{C},\ |\alpha| = 1\}. \tag{7.8.10}$$

One can use this to show that the maximum of $|u|$ on $\overline{U}$ is attained on the parabolic boundary of U, because of the analogous statement for (7.8.9).

7.9 Subsolutions of the heat equation

Let n be a positive integer, let U be a nonempty open subset of $\mathbf{R}^n \times \mathbf{R}$, and let u be a real-valued function on U. Suppose that u is continuously differentiable on U, and that the second derivatives of $u(x,t)$ in x exist and are continuous on U. If

$$\frac{\partial u}{\partial t} \le \Delta u \tag{7.9.1}$$

on U, then u is said to be a *subsolution* of the heat equation, as in Problem 17 in Section 2.5 of [81]. Let us say that u is a *strict subsolution* of the heat equation if

$$\frac{\partial u}{\partial t} < \Delta u \tag{7.9.2}$$

on U.

7.9.1 Strict subsolutions and local maxima

Suppose that u has a local maximum at $(\xi, \tau) \in U$. This implies that (ξ, τ) is a critical point of u, and that the second derivative of u at (ξ, τ) in x_j is less than or equal to 0 for each $j = 1, \ldots, n$. It follows that u is not a strict subsolution of the heat equation on U.

7.9.2 An argument for strict subsolutions

Now let V, T, and U be as in the previous section, and let u be a continuous real-valued function on $\overline{U}$. Suppose that u is continuously differentiable on U again, and that the second derivatives of $u(x,t)$ in x exist and are continuous on U. Suppose for the moment that u is a strict subsolution of the heat equation on U.

Let R be a positive real number with $R < T$, and note that

$$\overline{V} \times [0, R] \tag{7.9.3}$$

is closed and bounded in $\mathbf{R}^n \times \mathbf{R}$, and thus compact. This means that the maximum of u on (7.9.3) is attained, by the extreme value theorem. The maximum of u on (7.9.3) cannot be attained at a point in

$$V \times (0, R), \tag{7.9.4}$$

as before.

Suppose for the sake of a contradiction that u has a local maximum at (ξ, R) for some $\xi \in V$, as a function on (7.9.3). In particular, (ξ, R) is a local maximum for u as a function on

$$V \times \{R\}, \tag{7.9.5}$$

so that ξ is a critical point for $u(x, R)$ as a function of x, and the second derivative of u at (ξ, R) in x_j is less than or equal to 0 for each $j = 1, \ldots, n$.

We also get that the derivative of u in t at (ξ, R) is greater than or equal to 0, because (ξ, R) is a local maximum for u on (7.9.3). This is not possible, because u is supposed to be a strict subsolution of the heat equation on U.

It follows that the maximum of u on (7.9.3) can only be attained at a point in

$$(\overline{V} \times \{0\}) \cup ((\partial V) \times [0, R]). \tag{7.9.6}$$

This is the parabolic boundary of (7.9.4), as in the previous section.

Remember that the maximum of u on the parabolic boundary (7.8.4) of U is attained, by the extreme value theorem. Of course, the maximum of u on (7.9.6) is less than or equal to the maximum of u on (7.8.4), because (7.9.6) is contained in (7.8.4). This implies that the maximum of u on (7.9.3) is less than or equal to the maximum of u on (7.8.4), by the statement in the preceding paragraph. One can use this to get that the maximum of u on $\overline{U}$ is attained on (7.8.4), because the previous statement holds for all $R \in (0, T)$.

In [81], one typically asks that the regularity properties of u extend to the "parabolic cylinder", which includes

$$V \times \{T\}. \tag{7.9.7}$$

In this case, one can get directly that the maximum of u on $\overline{U}$ can only be attained on the parabolic boundary of U, as before.

7.9.3 Non-strict subsolutions

Suppose now that u is a non-strict subsolution of the heat equation on U, and let $\epsilon > 0$. It is easy to see that

$$u_\epsilon(x, t) = u(x, t) - \epsilon\, t \tag{7.9.8}$$

and

$$v_\epsilon(x, t) = u(x, t) + \epsilon\, |x|^2 \tag{7.9.9}$$

are strict subsolutions of the heat equation on U. Thus the maxima of u_ϵ and v_ϵ on $\overline{U}$ are attained on the parabolic boundary of U, as before. One can use either of these to get that the maximum of u on $\overline{U}$ is attained on the parabolic boundary of U. This is a version of the maximum principle for subsolutions of the heat equation, as mentioned in the previous section.

7.10 Another approach to uniqueness

Let n be a positive integer, let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary, and let T be a positive real number. Put $U = V \times (0, T)$, which is a bounded open subset of $\mathbf{R}^n \times \mathbf{R}$, as before. Let u be a continuously-differentiable real or complex-valued function on $\overline{U}$, which is twice continuously differentiable in x. More precisely, this means that $u(x, t)$ is twice continuously differentiable as a function of x on $\overline{V}$ for each $t \in [0, T]$, and that all of the second derivatives of $u(x, t)$ in x are continuous on $\overline{U}$.

7.10.1 A related function $e(t)$

If $0 \le t \le T$, then put

$$e(t) = \int_V |u(x,t)|^2 \, dx. \tag{7.10.1}$$

Observe that

$$\begin{aligned} \frac{\partial}{\partial t}(|u(x,t)|^2) &= \frac{\partial}{\partial t}(u(x,t)\,\overline{u(x,t)}) = \frac{\partial u}{\partial t}(x,t)\,\overline{u(x,t)} + u(x,t)\,\overline{\frac{\partial u}{\partial t}(x,t)} \\ &= 2\ \mathrm{Re}\Big(\overline{u(x,t)}\,\frac{\partial u}{\partial t}(x,t)\Big). \end{aligned} \tag{7.10.2}$$

We can differentiate under the integral sign under these conditions, to get that

$$\frac{de}{dt}(t) = 2\ \mathrm{Re} \int_V \overline{u(x,t)}\,\frac{\partial u}{\partial t}(x,t)\,dx. \tag{7.10.3}$$

If u satisfies the heat equation, then this implies that

$$\frac{de}{dt}(t) = 2\ \mathrm{Re} \int_V \overline{u(x,t)}\,(\Delta u)(x,t)\,dx. \tag{7.10.4}$$

7.10.2 Some boundary conditions

Suppose that either

$$u(y',t) = 0 \text{ on } (\partial V) \times [0,T] \tag{7.10.5}$$

or

$$(D_{\nu(y')} u)(y',t) = 0 \text{ on } (\partial V) \times [0,T], \tag{7.10.6}$$

where $\nu(y')$ is the outward-pointing unit normal to ∂V in $\mathbf{R}^n$ at $y' \in \partial V$, as usual, and $D_{\nu(y')}$ indicates the directional derivative in the direction $\nu(y')$. In both cases, we can use the divergence theorem, as in Subsection 3.5.2, to get that

$$\frac{de}{dt}(t) = -2 \int_V |\nabla u(x,t)|^2 \, dx. \tag{7.10.7}$$

More precisely, $\nabla u(x,t) = \nabla_x u(x,t)$ refers to the gradient of $u(x,t)$ in x. In particular, the right side of (7.10.7) is less than or equal to 0, so that $e(t)$ decreases monotonically on $[0,T]$.

7.10.3 Some initial conditions

If we also have that

$$u(x,0) = 0 \text{ on } V, \tag{7.10.8}$$

then we get that $e(0) = 0$. This implies that $e(t) = 0$ for every $t \in [0,T]$, because $e(t)$ decreases monotonically on $[0,T]$. This means that

$$u(x,t) = 0 \text{ on } \overline{U} = \overline{V} \times [0,T]. \tag{7.10.9}$$

This corresponds to Theorem 10 in Section 2.3.4 a of [81] in the case of the Dirichlet boundary conditions (7.10.5), and Problem 1 in Section 7.5 of [81] for the Neumann boundary conditions (7.10.6).

7.11 Some integrals of products

Let n be a positive integer, let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary, and let a, b be real numbers with $a < b$. Thus $U = V \times (a, b)$ is a bounded open subset of $\mathbf{R}^n \times \mathbf{R}$, with closure $\overline{U} = \overline{V} \times [a, b]$. Let u, v be continuously-differentiable real or complex-valued functions on $\overline{U}$ that are twice continuously differentiable in x. Suppose that u satisfies the heat equation, and that v satisfies the "backwards" heat equation

$$\frac{\partial v}{\partial t} = -\Delta v. \tag{7.11.1}$$

Equivalently, this means that $v(x, -t)$ satisfies the heat equation.

7.11.1 Some derivatives in t

Observe that

$$\frac{\partial}{\partial t}(u\,v) = \frac{\partial u}{\partial t}\,v + u\,\frac{\partial v}{\partial t} = (\Delta u)\,v - u\,(\Delta v). \tag{7.11.2}$$

If $a \le t \le b$, then one can differentiate under the integral sign to get that

$$\frac{d}{dt}\int_V u(x,t)\,v(x,t)\,dx = \int_V ((\Delta u)(x,t)\,v(x,t) - u(x,t)\,(\Delta v)(x,t))\,dx. \tag{7.11.3}$$

This implies that

$$\begin{aligned} &\frac{d}{dt}\int_V u(x,t)\,v(x,t)\,dx \\ &\quad = \int_{\partial V} ((D_{\nu(y')}u)(y',t)\,v(y',t) - u(y',t)\,(D_{\nu(y')}v)(y',t))\,dy', \end{aligned} \tag{7.11.4}$$

as in Section 3.9. Here $\nu(y')$ is the outward-pointing unit normal to ∂V in $\mathbf{R}^n$ at $y' \in \partial V$, and $D_{\nu(y')}$ indicates the directional derivative in the direction $\nu(y')$, as before.

If we integrate in t over $[a, b]$, then we get that

$$\begin{aligned} &\int_V u(x,b)\,v(x,b)\,dx - \int_V u(x,a)\,v(x,a)\,dx \\ &\quad = \int_a^b \int_{\partial V} ((D_{\nu(y')}u)(y',t)\,v(y',t) - u(y',t)\,(D_{\nu(y')}v)(y',t)))\,dy'\,dt. \end{aligned} \tag{7.11.5}$$

This corresponds to Problem 3 in Section 7.5 of [81]. This could also be obtained from the divergence theorem on U, as in the proof of Theorem 4.4 in Section A of Chapter 4 of [87].

7.11.2 An interesting v

Let $K(x,t)$ be the heat kernel as defined on $(\mathbf{R}^n \times \mathbf{R}) \setminus \{(0,0)\}$ in Subsection 7.1.1, so that $K(x,t)$ is smooth and satisfies the heat equation on this set. Let $z \in \mathbf{R}^n$ and $t_1 \in \mathbf{R}$ with $t_1 > b$ be given, and consider

$$v(x,t) = K(x-z, t_1 - t), \tag{7.11.6}$$

which is a smooth function on $(\mathbf{R}^n \times \mathbf{R}) \setminus \{(z,t_1)\}$ that satisfies the backward heat equation. In particular, (7.11.6) is smooth and satisfies the backward heat equation on $\overline{U}$, so that (7.11.4) and (7.11.5) hold in this case.

If $z \in V$, then

$$\int_V u(x,b)\, K(x-z, t_1 - b)\, dx \to u(z,b) \tag{7.11.7}$$

as $t_1 \to b+$, as in Subsection 7.4.2. This implies that

$$\begin{aligned} u(z,b) \quad = \quad & \int_V u(x,a)\, K(x-z, b-a)\, dx \\ & + \int_a^b \int_{\partial V} (D_{\nu(y')} u)(y')\, K(y'-z, b-t)\, dy'\, dt \\ & - \int_a^b \int_{\partial V} u(y',t)\, (D_{\nu(y')} K)(y'-z, b-t)\, dy'\, dt, \end{aligned} \tag{7.11.8}$$

by taking the limit as $t_1 \to b+$ in the other terms in (7.11.5).

7.11.3 Integrals over $\mathbf{R}^n$

Suppose now that u is a continuously-differentiable real or complex-valued function on $\mathbf{R}^n \times [a,b]$ that is twice continuously differentiable in x and satisfies the heat equation. If $z \in \mathbf{R}^n$, then we would like to use (7.11.8) to get that

$$\begin{aligned} u(z,b) \quad &= \quad \int_{\mathbf{R}^n} u(x,a)\, K(x-z, b-a)\, dx \\ &= \quad \int_{\mathbf{R}^n} u(x,a)\, (4\pi\,(b-a))^{-n/2}\, \exp(-|x-z|^2/(4\,(b-a)))\, dx \end{aligned} \tag{7.11.9}$$

under suitable conditions, as in the proof of Theorem 4.4 in Section A of Chapter 4 of [87].

Suppose that there are real numbers $b_1 > b$ and $C, C' \ge 0$ such that

$$|u(x,t)| \le C\, \exp(|x|^2/(4\,(b_1 - t))) \tag{7.11.10}$$

and

$$|\nabla u(x,t)| \le C'\, \exp(|x|^2/(4\,(b_1 - t))) \tag{7.11.11}$$

for every $x \in \mathbf{R}^n$ and $a \le t \le b$. Here $\nabla u(x,t) = \nabla_x u(x,t)$ refers to the gradient of $u(x,t)$ in x, as before. In particular, if we take $t = a$ in (7.11.10), then we get that the integrand on the right side of (7.11.9) is integrable on $\mathbf{R}^n$.

If r is a positive real number with $|z| < r$, then we can take $V = B(0, r)$ in (7.11.8). The second and third terms on the right side of (7.11.8) tend to 0 as $r \to \infty$, because of (7.11.10) and (7.11.11). The first term on the right side of (7.11.8) tends to the right side of (7.11.9) as $r \to \infty$, because of (7.11.10) with $t = a$. Thus (7.11.9) holds, as desired.

7.11.4 An integral representation

Suppose that $0 < T \leq \infty$, and let $u(x, t)$ is a continuously-differentiable real or complex-valued function on $\mathbf{R}^n \times (0, T)$ that is twice continuously differentiable in x and satisfies the heat equation. If $0 < a < b < T$, then (7.11.9) holds for every $z \in \mathbf{R}^n$ under the conditions mentioned earlier. If $u(x, t)$ has boundary values as $t \to 0+$ in an appropriate sense, then one can use (7.11.9) to express u as the Gauss–Weierstrass integral of its boundary values, under suitable conditions. A version of this is given by Theorem 4.4 in Section A of [87] and its proof.

7.12 Upper bounds and $t = 0$

Let n be a positive integer, and let T be a positive real number. Also let u be a continuous real-valued function on $\mathbf{R}^n \times [0, T]$. Suppose that on $\mathbf{R}^n \times (0, T)$, $u(x, t)$ is continuously differentiable, twice continuously differentiable in x, and satisfies the heat equation. If

$$u(x, 0) \leq 0 \text{ for every } x \in \mathbf{R}^n, \tag{7.12.1}$$

then we would like to be able to say that

$$u(x, t) \leq 0 \text{ for every } (x, t) \in \mathbf{R}^n \times [0, T], \tag{7.12.2}$$

at least under suitable conditions.

We shall do this here when

$$|x|^{-2} \max(u(x, t), 0) \to 0 \text{ as } |x| \to \infty, \tag{7.12.3}$$

uniformly over $t \in [0, T]$. An analogous statement with a much weaker condition on $u(x, t)$ is given in Theorem 6 in Section 2.3.3 a of [81], which will be discussed in the next section. More precisely, it suffices to ask that u be a subsolution of the heat equation on $\mathbf{R}^n \times (0, T)$, instead of satisfying the heat equation there.

7.12.1 Approximation by other subsolutions

Let $\epsilon > 0$ be given, and observe that

$$|x|^2 + 2\,n\,t \tag{7.12.4}$$

satisfies the heat equation on $\mathbf{R}^n \times \mathbf{R}$. Thus

$$v_\epsilon(x, t) = u(x, t) - \epsilon\,(|x|^2 + 2\,n\,t) \tag{7.12.5}$$

is a subsolution of the heat equation on $\mathbf{R}^n \times (0,T)$. Note that

$$v_\epsilon(x,0) \le u(x,0) \le 0 \tag{7.12.6}$$

for every $x \in \mathbf{R}^n$. We also have that

$$v_\epsilon(x,t) \le 0 \tag{7.12.7}$$

for every $t \in [0,T]$ when $|x|$ is sufficiently large, by hypothesis.

It follows that (7.12.7) holds for every $(x,t) \in \mathbf{R}^n \times [0,T]$, by the maximum principle. This implies (7.12.2), because $\epsilon > 0$ is arbitrary.

7.12.2 The corresponding uniqueness statement

Suppose now that u satisfies the heat equation on $\mathbf{R}^n \times (0,T)$, and that

$$u(x,0) = 0 \text{ for every } x \in \mathbf{R}^n. \tag{7.12.8}$$

If

$$|x|^{-2}\, u(x,t) \to 0 \text{ as } |x| \to \infty, \tag{7.12.9}$$

then

$$u(x,t) = 0 \text{ for every } (x,t) \in \mathbf{R}^n \times [0,T], \tag{7.12.10}$$

by the previous argument for u and $-u$. This corresponds to Theorem 7 in Section 2.3.3 a of [81], which has a much weaker condition on the size of $u(x,t)$, as before.

7.13 A weaker condition on $u(x,t)$

Let n, T, and u be as at the beginning of the previous section, and suppose that (7.12.1) holds. Suppose also that there are nonnegative real numbers a, A such that

$$u(x,t) \le A\, \exp(a\,|x|^2) \tag{7.13.1}$$

for every $(x,t) \in \mathbf{R}^n \times [0,T]$. Under these conditions, we have that (7.12.2) holds, as in Theorem 6 in Section 2.3.3 a of [81]. As in the previous section, it suffices to ask that u be a subsolution of the heat equation on $\mathbf{R}^n \times (0,T)$, instead of satisfying the heat equation there.

7.13.1 An initial step

As in [81], we suppose first that

$$4\,a\,T < 1. \tag{7.13.2}$$

This implies that

$$4\,a\,(T + \eta) < 1 \tag{7.13.3}$$

for some $\eta > 0$. Note that

$$(T + \eta - t)^{-n/2} \exp(|x|^2/(4\,(T + \eta - t))) \tag{7.13.4}$$

satisfies the heat equation on $\mathbf{R}^n \times (-\infty, T + \eta)$, as in Subsection 7.1.1. It follows that for each $\mu > 0$,

$$w(x,t) = u(x,t) - \mu\,(T + \eta - t)^{-n/2} \exp(|x|^2/(4\,(T + \eta - t))) \tag{7.13.5}$$

is a subsolution of the heat equation on $\mathbf{R}^n \times (0, T)$.

Clearly

$$w(x, 0) \le u(x, 0) \le 0 \tag{7.13.6}$$

for every $x \in \mathbf{R}^n$. One can check that

$$w(x,t) \le 0 \tag{7.13.7}$$

for every $t \in [0, T]$ when $|x|$ is sufficiently large, using (7.13.1) and (7.13.3). This implies that (7.13.7) holds for every $(x,t) \in \mathbf{R}^n \times [0, T]$, by the maximum principle. It follows that (7.12.2) holds, because $\mu > 0$ is arbitrary.

7.13.2 Repeating the argument

If (7.13.2) does not hold, then we can use the same argument on smaller intervals that satisfy this condition. One can use this repeatedly to get the same conclusion as before, as in [81].

7.13.3 Another uniqueness statement

If u satisfies the heat equation on $\mathbf{R}^n \times (0, T)$, (7.12.8), and

$$|u(x,t)| \le A\,\exp(a\,|x|^2) \tag{7.13.8}$$

for some $a, A \ge 0$ and all $(x,t) \in \mathbf{R}^n \times [0, T]$, then (7.12.10) holds, as in the previous section. This corresponds to Theorem 7 in Section 2.3.3 a of [81], as before.

7.14 Some more integrals of products

Let n be a positive integer, let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary, and let a, b be real numbers with $a < b$, as in Section 7.11. Put $U = V \times (a, b)$, and let u, v be continuously-differentiable real or complex-valued functions on $\overline{U}$ that are twice continuously differentiable in x.

7.14.1 Some more derivatives in t

If $a \le t \le b$, then

$$\frac{d}{dt} \int_V u(x,t)\, v(x,t)\, dx = \int_V \Big(\frac{\partial u}{\partial t}(x,t)\, v(x\, t) + u(x,t)\, \frac{\partial v}{\partial t}(x,t)\Big)\, dx. \tag{7.14.1}$$

We can combine this with an identity from Section 3.9 to get that

$$\begin{aligned} \frac{d}{dt} \int_V & u(x,t)\, v(x,t)\, dx \\ = & \int_V \Big(\frac{\partial u}{\partial t}(x,t) - (\Delta u)(x,t)\Big)\, v(x,t)\, dx \\ & + \int_V u(x,t)\, \Big(\frac{\partial v}{\partial t}(x,t) + (\Delta v)(x,t)\Big)\, dx \\ & + \int_{\partial V} (u(y',t)\, (D_{\nu(y')} v)(y',t) - v(y',t) (D_{\nu(y')} u)(y',t))\, dy'. \end{aligned} \tag{7.14.2}$$

Here $\nu(y')$ is the outward-pointing unit normal to ∂V in $\mathbf{R}^n$ at $y' \in \partial V$, and $D_{\nu(y')}$ indicates the directional derivative in the direction $\nu(y')$, as usual.

Let us integrate in t over $[a,b]$, to get that

$$\begin{aligned} \int_V & u(x,b)\, v(x,b)\, dx - \int_V u(x,a)\, v(x,a)\, dx \\ = & \int_a^b \int_V \Big(\frac{\partial u}{\partial t}(x,t) - (\Delta u)(x,t)\Big)\, v(x,t)\, dx\, dt \\ & + \int_a^b \int_V u(x,t)\, \Big(\frac{\partial v}{\partial t}(x,t) + (\Delta v)(x,t)\Big)\, dx\, dt \\ & + \int_a^b \int_{\partial V} (u(y',t)\, (D_{\nu(y')} v)(y',t) - v(y',t)\, (D_{\nu(y')} u)(y',t))\, dy'\, dt. \end{aligned} \tag{7.14.3}$$

Of course, this can be simplified when u satisfies the heat equation, or v satisfies the backward heat equation.

7.14.2 Using an interesting v

Let $K(x,t)$ be the heat kernel as defined on $(\mathbf{R}^n \times \mathbf{R}) \setminus \{(0,0)\}$ in Subsection 7.1.1, which is smooth and satisfies the heat equation on this set. If $z \in \mathbf{R}^n$ and $t_1 \in \mathbf{R}$, then

$$v(x,t) = K(x - z, t_1 - t) \tag{7.14.4}$$

is a smooth function on $(\mathbf{R}^n \times \mathbf{R}) \setminus \{(z,t_1)\}$ that satisfies the backward heat equation, as in Subsection 7.11.2. If $t_1 > b$, then (7.14.4) is smooth and satisfies the backward heat equation on $\overline{U}$, as before. In this case, we get that

$$\int_V u(x,b)\, K(x - z, t_1 - b)\, dx - \int_V u(x,a)\, K(x - z, t_1 - a)\, dx$$

$$
\begin{aligned}
&= \int_a^b \int_V \Big(\frac{\partial u}{\partial t}(x,t) - (\Delta u)(x,t)\Big)\, K(x-z, t_1 - t)\, dx\, dt \\
&\quad + \int_a^b \int_{\partial V} u(y',t)\, (D_{\nu(y')} K)(y'-z, t_1 - t)\, dy'\, dt \\
&\quad - \int_a^b \int_{\partial V} K(y'-z, t_1 - t)\, (D_{\nu(y')} u)(y',t)\, dy'\, dt.
\end{aligned} \tag{7.14.5}
$$

Suppose that $z \in V$, and consider the limit as $t_1 \to b+$ of both sides of the equation, as in Subsection 7.11.2. We would like to say that

$$
\begin{aligned}
&u(z,b) - \int_V u(x,a)\, K(x-z, b-a)\, dx \\
&= \int_a^b \int_V \Big(\frac{\partial u}{\partial t}(x,t) - (\Delta u)(x,t)\Big)\, K(x-z, b - t)\, dx\, dt \\
&\quad + \int_a^b \int_{\partial V} u(y',t)\, (D_{\nu(y')} K)(y'-z, b - t)\, dy'\, dt \\
&\quad - \int_a^b \int_{\partial V} K(y'-z, b - t)\, (D_{\nu(y')} u)(y',t)\, dy'\, dt.
\end{aligned} \tag{7.14.6}
$$

More precisely, the first term on the right side should be handled a bit carefully, as in the next section.

7.14.3 Using an interesting u

Similarly, if $z \in \mathbf{R}^n$ and $t_0 \in \mathbf{R}$, then

$$
u(x,t) = K(x-z, t-t_0) \tag{7.14.7}
$$

is a smooth function on $(\mathbf{R}^n \times \mathbf{R}) \setminus \{(z,t_0)\}$ that satisfies the heat equation. If $t_0 < a$, then (7.14.7) is smooth and satisfies the heat equation on $\overline{U}$. If v is as at the beginning of the section again, then we obtain that

$$
\begin{aligned}
&\int_V K(x-z, b-t_0)\, v(x,b)\, dx - \int_V K(x-z, a-t_0)\, v(x,a)\, dx \\
&= \int_a^b \int_V K(x-z, t-t_0)\, \Big(\frac{\partial v}{\partial t}(x,t) + (\Delta v)(x,t)\Big)\, dx\, dt \\
&\quad + \int_a^b \int_{\partial V} K(y'-z, t-t_0)\, (D_{\nu(y')} v)(y',t)\, dy'\, dt \\
&\quad - \int_a^b \int_{\partial V} v(y',t)\, (D_{\nu(y')} K)(y'-z, t-t_0)\, dy'\, dt.
\end{aligned} \tag{7.14.8}
$$

Suppose that $z \in V$ again, and consider the limit as $t_0 \to a-$ of both sides of the equation. Of course, this is basically the same as the previous version, and we get that

$$
\begin{aligned}
&\int_V K(x-z,b-a)\,v(x,b)\,dx - v(z,a) \\
&\quad = \int_a^b \int_V K(x-z,t-a)\,\Big(\frac{\partial v}{\partial t}(x,t) + (\Delta v)(x,t)\Big)\,dx\,dt \\
&\qquad + \int_a^b \int_{\partial V} K(y'-z,t-a)\,(D_{\nu(y')}v)(y',t)\,dy'\,dt \\
&\qquad - \int_a^b \int_{\partial V} v(y',t)\,(D_{\nu(y')}K)(y'-z,t-a)\,dy'\,dt.
\end{aligned}
\tag{7.14.9}
$$

7.15 Some integrals with $K(x,t)$

Let $K(x,t)$ be the heat kernel as defined on $(\mathbf{R}^n \times \mathbf{R}) \setminus \{(0,0)\}$, as in Subsection 7.1.1. Remember that

$$\int_{\mathbf{R}^n} K(x,t)\,dx = 1 \tag{7.15.1}$$

for every $t > 0$. This implies that

$$\int_{r_1}^{r_2} \int_{\mathbf{R}^n} K(x,t)\,dx\,dt = r_2 - r_1 \tag{7.15.2}$$

when r_1, r_2 are positive real numbers with $r_1 \le r_2$. One could also allow $r_1 = 0$ here, by considering the integral over t as an improper integral, or defining the integrand at $t = 0$, or using Lebesgue integrals.

One may consider the left side of (7.15.2) as an $(n+1)$-dimensional integral over $\mathbf{R}^n \times [r_1, r_2]$, even when $r_1 = 0$, using suitable improper integrals, or Lebesgue integrals. In particular, $K(x,t)$ is locally integrable on $\mathbf{R}^n \times \mathbf{R}$, with respect to $(n+1)$-dimensional Lebesgue measure.

7.15.1 Some integrals on bounded sets

Let W be a nonempty bounded open subset of $\mathbf{R}^n$, let T be a positive real number, and let f be a continuous real or complex-valued function on $\overline{W} \times [0,T]$. Note that f is bounded on $\overline{W} \times [0,T]$, so that

$$|f(x,t)| \le C \tag{7.15.3}$$

for some $C \ge 0$ and every $x \in \overline{W}$, $t \in [0,T]$. If one is using Riemann integrals, then one should ask for a bit more regularity of the boundary of W, or that f is equal to 0 on $(\partial W) \times [0,T]$. If one is using Lebesgue integrals, then one might simply ask that f be bounded and measurable on $W \times [0,T]$.

If $0 < t \le T$, then

$$\int_W K(x,t)\,|f(x,t)|\,dx \le C, \tag{7.15.4}$$

by (7.15.1) and (7.15.3). This implies that

$$\int_{r_1}^{r_2} \int_W K(x,t)\, |f(x,t)|\, dx\, dt \le C\,(r_2 - r_1) \tag{7.15.5}$$

when $0 < r_1 \le r_2 \le T$. This also works with $r_1 = 0$, with suitable interpretations when $0 \in \overline{W}$, or using Lebesgue integrals, as before. In particular, this means that $K(x,t)\,|f(x,t)|$ is integrable with respect to $(n+1)$-dimensional Lebesgue measure on $W \times [0,T]$.

Similarly, if η is a positive real number, then

$$\int_W K(x, t+\eta)\, |f(x,t)|\, dx \le C \tag{7.15.6}$$

for every $t \in [0,T]$. It follows that

$$\int_{r_1}^{r_2} \int_W K(x, t+\eta)\, |f(x,t)|\, dx\, dt \le C\,(r_2 - r_1) \tag{7.15.7}$$

when $0 \le r_1 \le r_2 \le T$.

7.15.2 Some more integrals

Of course,

$$\left| \int_W K(x,t)\, f(x,t)\, dx \right| \le C \tag{7.15.8}$$

when $0 < t \le T$, by (7.15.4). If $r \in [0,T]$, then

$$\int_0^r \int_W K(x,t)\, f(x,t)\, dx\, dt \tag{7.15.9}$$

may be defined directly unless $0 \in \overline{W}$, in which case the integral over t may be considered as an improper integral, or defined a bit carefully as a Riemann integral, or using Lebesgue integrals, as before. Note that

$$\left| \int_0^r \int_W K(x,t)\, f(x,t)\, dx\, dt \right| \le C\, r. \tag{7.15.10}$$

If $0 < r_1 \le r_2 \le T$, then

$$\begin{aligned} \lim_{\eta \to 0+} \int_{r_1}^{r_2} \int_W K(x, t+\eta)\, f(x,t)\, dx\, dt \\ = \int_{r_1}^{r_2} \int_W K(x,t)\, f(x,t)\, dx\, dt, \end{aligned} \tag{7.15.11}$$

by standard arguments. It is not too difficult to show that this works with $r_1 = 0$ as well, using the previous remarks.

Chapter 8

Some more equations and solutions

8.1 Another uniqueness argument

Let n be a positive integer, let V be a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary, and let T be a positive real number. Put $U = V \times (0, T)$, which is a bounded open subset of $\mathbf{R}^n \times \mathbf{R}$, with closure $\overline{U} = \overline{V} \times [0, T]$. Let $u(x, t)$ be a twice continuously-differentiable real or complex-valued function on $\overline{U}$.

8.1.1 A related function $E(t)$

If $0 \le t \le T$, then put

$$E(t) = \frac{1}{2} \int_V \left(\left| \frac{\partial u}{\partial t}(x, t) \right|^2 + \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j}(x, t) \right|^2 \right) dx. \tag{8.1.1}$$

We can differentiate under the integral sign in t, to get that

$$\frac{d}{dt} E(t) = \mathrm{Re} \int_V \Big(\overline{\frac{\partial u}{\partial t}(x, t)} \, \frac{\partial^2 u}{\partial t^2}(x, t) + \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x, t) \, \overline{\frac{\partial^2 u}{\partial x_j \, \partial t}(x, t)} \Big) \, dx. \tag{8.1.2}$$

8.1.2 Using some boundary conditions

Suppose that

$$u(y', t) = 0 \text{ on } (\partial V) \times [0, T], \tag{8.1.3}$$

so that

$$\frac{\partial u}{\partial t}(y', t) = 0 \text{ on } (\partial V) \times [0, T]. \tag{8.1.4}$$

Under these conditions,

$$\int_V (\Delta u)(x,t)\,\overline{\frac{\partial u}{\partial t}(x,t)}\,dx + \int_V \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x,t)\,\overline{\frac{\partial^2 u}{\partial x_j\,\partial t}(x,t)}\,dx = 0, \tag{8.1.5}$$

as in Subsection 3.5.2, where $(\Delta u)(x,t)$ refers to the Laplacian of $u(x,t)$ in x. This also works when

$$(D_{\nu(y')}u)(y',t) = 0 \text{ on } (\partial V)\times[0,T], \tag{8.1.6}$$

where $\nu(y')$ is the outward-pointing unit normal to ∂V in $\mathbf{R}^n$ at $y' \in \partial V$, and $D_{\nu(y')}$ indicates the directional derivative in the direction $\nu(y')$. Combining (8.1.2) and (8.1.5), we obtain that

$$\frac{d}{dt}E(t) = \operatorname{Re}\int_V \overline{\frac{\partial u}{\partial t}(x,t)}\,\Big(\frac{\partial^2 u}{\partial t^2}(x,t) - (\Delta u)(x,t)\Big)\,dx. \tag{8.1.7}$$

8.1.3 The wave equation

Suppose now that u satisfies the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \tag{8.1.8}$$

on U. In this case, (8.1.7) reduces to

$$\frac{d}{dt}E(t) = 0. \tag{8.1.9}$$

If

$$E(0) = 0, \tag{8.1.10}$$

then it follows that

$$E(t) = 0 \tag{8.1.11}$$

for every $t \in [0,T]$. This means that the first derivatives of $u(x,t)$ in x and t are equal to 0 on U.

Of course, (8.1.10) holds when

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \tag{8.1.12}$$

for every $x \in V$. If

$$\frac{\partial u}{\partial t}(x,t) = 0 \tag{8.1.13}$$

on U, then (8.1.12) implies that

$$u(x,t) = 0 \tag{8.1.14}$$

on U.

Thus (8.1.14) holds on U when (8.1.12) holds, and (8.1.11) holds for each $t \in [0,T]$. This means that (8.1.14) holds on U when u satisfies (8.1.3), (8.1.8), and (8.1.12). This corresponds to Theorem 5 in Section 2.4.3 a in [81].

A more localized version of this will be discussed in the next section.

8.2 A more localized version

Let n be a positive integer, let T be a positive real number, and let $u(x,t)$ be a twice continuously-differentiable real or complex-valued function on $\mathbf{R}^n \times [0,T]$. Also let $\xi \in \mathbf{R}^n$ and a positive real number $t_0 \le T$ be given, and if $0 \le t \le t_0$, then put

$$e(t) = \frac{1}{2} \int_{B(\xi, t_0 - t)} \left(\left| \frac{\partial u}{\partial t}(x,t) \right|^2 + \sum_{j=1}^n \left| \frac{\partial u}{\partial x_j}(x,t) \right|^2 \right) dx. \tag{8.2.1}$$

Here $B(\xi, t_0 - t)$ is the open ball in $\mathbf{R}^n$ centered at ξ with radius $t_0 - t$, which may be interpreted as the empty set when $t = t_0$, in which case the integral is interpreted as being equal to 0.

8.2.1 Differentiating $e(t)$

Observe that

$$\begin{aligned} \frac{d}{dt} e(t) &= \operatorname{Re} \int_{B(\xi, t_0 - t)} \Big(\overline{\frac{\partial u}{\partial t}(x,t)} \, \frac{\partial^2 u}{\partial t^2}(x,t) + \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x,t) \, \overline{\frac{\partial^2 u}{\partial x_j \, \partial t}(x,t)} \Big) \, dx \\ &\quad - \frac{1}{2} \int_{\partial B(\xi, t_0 - t)} \left(\left| \frac{\partial u}{\partial t}(y',t) \right|^2 + \sum_{j=1}^n \left| \frac{\partial u}{\partial x_j}(y',t) \right|^2 \right) dy'. \end{aligned} \tag{8.2.2}$$

8.2.2 More on differentiating $e(t)$

As in Subsection 3.5.2, we have that

$$\begin{aligned} &\int_{B(\xi, t_0 - t)} (\Delta u)(x,t) \, \overline{\frac{\partial u}{\partial t}(x,t)} \, dx \\ &\qquad + \int_{B(\xi, t_0 - t)} \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x,t) \, \overline{\frac{\partial^2 u}{\partial x_j \, \partial t}(x,t)} \, dx \\ &= \int_{\partial B(\xi, t_0 - t)} (D_{\nu_t(y')} u)(y',t) \, \overline{\frac{\partial u}{\partial t}(y',t)} \, dy'. \end{aligned} \tag{8.2.3}$$

More precisely, if $y' \in \partial B(\xi, t_0 - t)$, then $\nu_t(y')$ denotes the outward-pointing unit normal to $\partial B(\xi, t_0 - t)$ at y', and $D_{\nu_t(y')}$ indicates the directional derivative in the direction $\nu_t(y')$, as usual.

If $u(x,t)$ satisfies the wave equation (8.1.8), then the integral in the first term on the right side of (8.2.2) is the same as the left side of (8.2.3). This means that

$$\begin{aligned} \frac{d}{dt} e(t) &= \operatorname{Re} \int_{\partial B(\xi, t_0 - t)} (D_{\nu_t(y')} u)(y',t) \, \overline{\frac{\partial u}{\partial t}(y',t)} \, dy' \\ &\quad - \frac{1}{2} \int_{\partial B(\xi, t_0 - t)} \left(\left| \frac{\partial u}{\partial t}(y',t) \right|^2 + \sum_{j=1}^n \left| \frac{\partial u}{\partial x_j}(y',t) \right|^2 \right) dy'. \end{aligned} \tag{8.2.4}$$

8.2.3 A monotonicity property

We would like to use this to get that

$$\frac{d}{dt} e(t) \le 0 \tag{8.2.5}$$

To do this, note that

$$\begin{aligned} \text{Re}\Big((D_{\nu_t(y')} u)(y',t)\, \overline{\frac{\partial u}{\partial t}(y',t)}\Big) &\le |(D_{\nu_t(y')} u)(y',t)| \left|\frac{\partial u}{\partial t}(y',t)\right| \\ &\le |(\nabla u)(y',t)| \left|\frac{\partial u}{\partial t}(y',t)\right| \end{aligned} \tag{8.2.6}$$

for every $y' \in \partial B(\xi, t_0 - t)$, where $(\nabla u)(x,t)$ is the gradient of $u(x,t)$ in x, as before. This uses the Cauchy–Schwarz inequality in the second step, as in Section 1.15.

The right side of this inequality is less than or equal to

$$\frac{1}{2}\left(|(\nabla u)(y',t)|^2 + \left|\frac{\partial u}{\partial t}(y',t)\right|^2\right), \tag{8.2.7}$$

because of the well-known fact that $2\,a\,b \le a^2 + b^2$ for all $a, b \in \mathbf{R}$. One can use this to obtain (8.2.5) from (8.2.4).

8.2.4 Using monotonicity to get uniqueness

This shows that $e(t)$ decreases monotonically on $[0, t_0]$. If

$$e(0) = 0, \tag{8.2.8}$$

then we get that

$$e(t) = 0 \tag{8.2.9}$$

when $0 \le t \le t_0$.

Suppose now that

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \tag{8.2.10}$$

for every $x \in B(\xi, t_0)$, which implies that (8.2.8) holds. If

$$0 \le t \le t_0 \text{ and } x \in \overline{B}(\xi, t_0 - t), \tag{8.2.11}$$

then it follows that

$$(\nabla u)(x,t) = \frac{\partial u}{\partial t}(x,t) = 0, \tag{8.2.12}$$

by (8.2.9).

One can use (8.2.10) and (8.2.12) to get that

$$u(x,t) = 0 \tag{8.2.13}$$

when (8.2.11) holds. More precisely, in this argument, we only need that $u(x,t)$ be twice continuously differentiable and satisfy the wave equation on the set where (8.2.11) holds. This corresponds to Theorem 6 in Section 2.4.3 b of [81], and Theorem 5.3 in Section A of Chapter 5 of [87].

8.3 Some differential equations on $\mathbf{R}^2$

Let $w(y_1, y_2)$ be a twice continuously-differentiable real or complex-valued function on $\mathbf{R}^2$, and consider the partial differential equation

$$\frac{\partial^2 w}{\partial y_1 \, \partial y_2} = 0. \tag{8.3.1}$$

This equation obviously holds when $w(y_1, y_2)$ depends only on y_1 or y_2.

Conversely, it is well known and not too difficult to show that if $w(y_1, y_2)$ satisfies (8.3.1) on $\mathbf{R}^2$, then $w(y_1, y_2)$ can be expressed as the sum of a function of y_1 and a function of y_2. More precisely, (8.3.1) implies that $\partial w / \partial y_1$ does not depend on y_2, so that it depends only on y_1. This means that there is a function of y_1 whose derivative is equal to $\partial w / \partial y_1$, so that the derivative with respect to y_1 of w minus this function of y_1 is equal to 0. It follows that w minus this function of y_1 depends only on y_2, so that w is the sum of a function of y_1 and a function of y_2. Alternatively, one could use (8.3.1) to get that $\partial w / \partial y_2$ does not depend on y_1, and use this to get the same type of representation of w.

8.3.1 The wave equation with $n = 1$

Let $u(x, t)$ be a twice continuously-differentiable real or complex-valued function on $\mathbf{R}^2$, and consider the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{8.3.2}$$

This is the same as the wave equation with $n = 1$, and it can also be expressed as

$$\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big) \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\Big) u = 0. \tag{8.3.3}$$

It is easy to see that (8.3.1) corresponds to (8.3.3) under the change of variables

$$y_1 = x + t, \ y_2 = x - t. \tag{8.3.4}$$

Clearly any function of $x + t$ or of $x - t$ satisfies (8.3.2). Conversely, if $u(x, t)$ satisfies (8.3.2) on $\mathbf{R}^2$, then $u(x, t)$ can be expressed as a sum of a function of $x + t$ and a function of $x - t$, as before.

8.3.2 Related first-order equations

Alternatively, put

$$v(x, t) = \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\Big) u(x, t), \tag{8.3.5}$$

which is a continuously differentiable function on $\mathbf{R}^2$. Thus (8.3.3) is the same as saying that

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0. \tag{8.3.6}$$

This is a linear first-order partial differential equation in v, as in Section 4.1, whose solutions are given by functions of $x - t$ on $\mathbf{R}^2$. Given such a solution, (8.3.5) may be considered as a linear first-order partial differential equation in u. This corresponds to some remarks in Section 2.4.1 a in [81]. Of course, we could consider

$$\widetilde{v}(x,t) = \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big)\, u(x,t) \tag{8.3.7}$$

instead, which is another continuously differentiable function on $\mathbf{R}^2$. Using this, (8.3.3) is the same as saying that

$$\frac{\partial \widetilde{v}}{\partial t} - \frac{\partial \widetilde{v}}{\partial x} = 0. \tag{8.3.8}$$

8.3.3 Some remarks about uniqueness

Suppose that

$$u(x,t) = \phi(x+t) + \psi(x-t) \tag{8.3.9}$$

for some continuously-differentiable real or complex-valued functions ϕ, ψ on the real line. This implies that

$$u(x,0) = \phi(x) + \psi(x) \tag{8.3.10}$$

and

$$\frac{\partial u}{\partial t}(x,0) = \phi'(x) - \psi'(x) \tag{8.3.11}$$

for every $x \in \mathbf{R}$. Of course,

$$\frac{\partial u}{\partial x}(x,0) = \phi'(x) + \psi'(x) \tag{8.3.12}$$

for every $x \in \mathbf{R}$, by (8.3.10). It follows that ϕ' and ψ' are uniquely determined by $(\partial u/\partial t)(x,0)$ and $(\partial u/\partial x)(x,0)$ on $\mathbf{R}$.

This means that ϕ and ψ are uniquely determined on $\mathbf{R}$, up to adding a constant to ϕ and subtracting the same constant from ψ, by $u(x,0)$ and $(\partial u/\partial t)(x,0)$ on $\mathbf{R}$. Note that the right side of (8.3.9) is not affected by adding a constant to ϕ, and subtracting the same constant from ψ. Thus we get that $u(x,t)$ is uniquely determined on $\mathbf{R}^2$ by $u(x,0)$ and $(\partial u/\partial t)(x,0)$ on $\mathbf{R}$. This corresponds to some more remarks in Section 2.4.1 a of [81], and some remarks in Section B of Chapter 5 of [87].

8.3.4 Arbitrary initial conditions

Observe that ϕ' and ψ' may be arbitrary continuous real or complex-valued functions on $\mathbf{R}$, so that the right sides of (8.3.11) and (8.3.12) may be arbitrary continuous functions on $\mathbf{R}$. Similarly, the right side of (8.3.10) may be any continuously-differentiable function on $\mathbf{R}$, which can be chosen at the same time as the right side of (8.3.11), as an arbitrary continuous function on $\mathbf{R}$.

If we take ϕ and ψ to be twice continuously-differentiable functions on $\mathbf{R}$, then the right side of (8.3.10) can be any twice continuously-differentiable function on $\mathbf{R}$, which can be chosen at the same time as the right side of (8.3.11), as an arbitrary continuously-differentiable function on $\mathbf{R}$. In this case, (8.3.9) is a twice continuously-differentiable function on $\mathbf{R}^2$ that satisfies the wave equation (8.3.2). This corresponds to Theorem 1 in Section 2.4.1 a of [81], and Theorem 5.6 in Section B of Chapter 5 of [87].

8.4 Some remarks about the Laplacian

Let n be a positive integer, and let a, b be real numbers with

$$0 \le a < b, \tag{8.4.1}$$

although one could also permit $b = +\infty$ here. Also let f be a twice continuously-differentiable real or complex-valued function on (a, b). Note that

$$\{x \in \mathbf{R}^n : a < |x| < b\} \tag{8.4.2}$$

is an open set in $\mathbf{R}^n$. Put

$$F(x) = f(|x|) \tag{8.4.3}$$

on (8.4.2), which is a twice continuously-differentiable function on this set. A function of the form (8.4.3) is said to be *radial* on (8.4.2).

One can check that

$$\Delta F(x) = f''(|x|) + (n-1)\,|x|^{-1}\,f'(|x|) \tag{8.4.4}$$

on this set, as in Lemma 2.60 in Section G of Chapter 2 of [87]. In particular, F is harmonic on (8.4.2) if and only if

$$f''(r) + (n-1)\,r^{-1}\,f'(r) = 0 \tag{8.4.5}$$

on (a, b), as in (5) in Section 2.2.1 a of [81]. This is related to some of the remarks about harmonic functions in Section 6.1.

Let p be a homogeneous polynomial of degree $k \ge 0$ on $\mathbf{R}^n$, and suppose that p is harmonic on $\mathbf{R}^n$. Thus

$$q(x) = |x|^{-k}\,p(x) \tag{8.4.6}$$

is a smooth function on $\mathbf{R}^n \setminus \{0\}$ that is homogeneous of degree 0 and equal to p on the unit sphere. One can check that

$$\Delta q(x) = -k\,(k+n-2)\,|x|^{-2}\,q(x) \tag{8.4.7}$$

on $\mathbf{R}^n \setminus \{0\}$, as in Lemma 2.61 in Section G of Chapter 2 of [87]. This was mentioned in Subsection 3.2.2 when $|x| = 1$.

Under these conditions, one can also verify that

$$\Delta(F\,q)(x) \tag{8.4.8}$$
$$= \big(f''(|x|) + (n-1)\,|x|^{-1}\,f'(|x|) - k\,(k+n-2)\,|x|^{-2}\,f(|x|)\big)\,q(x)$$

on (8.4.2), as in Lemma 2.62 in Section G of Chapter 2 of [87]. More precisely, it is not too difficult to see that

$$\nabla F(x) \cdot \nabla q(x) = 0 \tag{8.4.9}$$

on (8.4.2). This only uses the fact that q is homogeneous of degree 0, and not the hypothesis that p be harmonic. Using this, one can get (8.4.8) from (8.4.4) and (8.4.7).

8.4.1 Some eigenfunctions for the Laplacian

Suppose that

$$f''(r) + (n-1)\,r^{-1}\,f'(r) - k\,(k+n-2)\,r^{-2}\,f(r) = \mu\,f(r) \tag{8.4.10}$$

on (a, b) for some real or complex number μ. Combining this with (8.4.8), we get that

$$\Delta(F\,q) = \mu\,F\,q \tag{8.4.11}$$

on (8.4.2). This is discussed further in Section G of Chapter 2 of [87].

8.5 More on radial functions

Let n be a positive integer, and let b be a positive real number, or $+\infty$. Also let $f(r)$ be a real or complex-valued function defined for $0 \le r < b$, so that F may be defined on

$$\{x \in \mathbf{R}^n : |x| < b\} \tag{8.5.1}$$

as in (8.4.3). If $f(r)$ is continuous for $0 \le r < b$, then F is continuous on (8.5.1). If f is l-times continuously differentiable on $(0, b)$ for some positive integer l, then F is l-times continuously differentiable on

$$\{x \in \mathbf{R}^n : 0 < |x| < b\}. \tag{8.5.2}$$

Suppose that f is differentiable at 0, where the derivative $f'(0)$ of f at 0 is actually a derivative from the right. If

$$f'(0) = 0, \tag{8.5.3}$$

then it is easy to see that F is differentiable at 0, with differential at 0 equal to 0. Equivalently, this means that

$$|x|^{-1}\,F(x) = |x|^{-1}\,f(|x|) \to 0 \text{ as } x \to 0. \tag{8.5.4}$$

In particular, this implies that the partial derivatives of F at 0 are equal to 0, and in fact that the directional derivative of F at 0 in any direction is equal to 0.

If $f(r)$ is continuously differentiable for $0 \le r < b$, and (8.5.3) holds, then one can check that

$$F \text{ is continuously differentiable on (8.5.1).} \tag{8.5.5}$$

More precisely, F is continuously differentiable on (8.5.2), as before. To get (8.5.5), one can verify that the partial derivatives of F are continuous at 0, which is to say that they tend to 0 as $x \to 0$.

8.5.1 Second derivatives

Suppose that $f(r)$ is twice continuously differentiable for $0 \le r < b$, and that (8.5.3) holds, and note that

$$\lim_{r \to 0+} r^{-1} f'(r) = f''(0). \tag{8.5.6}$$

Under these conditions, one can verify that

$$F \text{ is twice continuously differentiable on (8.5.1).} \tag{8.5.7}$$

Of course, F is twice continuously differentiable on (8.5.2), as before. Thus it suffices to show that the second derivatives of F exist at 0, and are continuous at 0.

If

$$f''(0) = 0, \tag{8.5.8}$$

then one can show that the second derivatives of F at 0 are equal to 0, using (8.5.6). One can also check that the second derivatives of F at $x \ne 0$ tend to 0 as $x \to 0$ in this case.

If $f(r) = r^2$, then $F(x) = |x|^2$ is a polynomial. Otherwise, one can reduce to these two cases.

One can verify that (8.4.4) holds at $x = 0$ under these conditions, suitably interpreted, using (8.5.6) to define $r^{-1} f'(r)$ at $r = 0$.

8.5.2 Higher derivatives

Similarly, let l be a positive integer, and suppose that $f(r)$ is l-times continuously differentiable for $0 \le r < b$. If

$$\text{the derivatives of } f \text{ at 0 of odd order less than or equal to } l \text{ are equal to 0,} \tag{8.5.9}$$

then

$$F \text{ is } l\text{-times continuously differentiable on (8.5.1).} \tag{8.5.10}$$

As usual, F is l-times continuously differentiable on (8.5.2), and so it is enough to show that the derivatives of F of order up to l exist at 0, and are continuous at 0.

As before, if all of the derivatives of f at 0 of order less than or equal to l are equal to 0, then the derivatives of F at 0 of order less than or equal to l are equal to 0. One can verify that the derivatives of F at $x \neq 0$ of order less than or equal to l tend to 0 as $x \to 0$ as well.

If $f(r)$ is a polynomial of r^2, then $F(x)$ is a polynomial. Otherwise, one can reduce to these two cases.

8.6 Some spherical means

Let n be a positive integer, and let ϕ be a continuous real or complex-valued function on $\mathbf{R}^n$. If $x \in \mathbf{R}^n$ and $r \in \mathbf{R}$, then the corresponding *spherical mean* of ϕ may be defined by

$$M_\phi(x; r) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} \phi(x + r\, y')\, dy', \tag{8.6.1}$$

as in Section B of Chapter 5 of [87]. Note that

$$M_\phi(x; -r) = M_\phi(x; r), \tag{8.6.2}$$

and that

$$M_\phi(x; 0) = \phi(x), \tag{8.6.3}$$

as in [87]. If $r > 0$, then

$$M_\phi(x; r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \phi(z')\, dz'. \tag{8.6.4}$$

If $x \in \mathbf{R}^n$ is fixed, then

$$M_\phi(x; r) \text{ is continuous as a function of } r \in \mathbf{R}. \tag{8.6.5}$$

Similarly, if

$$\phi \text{ is } k\text{-times continuously differentiable on } \mathbf{R}^n \tag{8.6.6}$$

for some positive integer k, then

$$M_\phi(x; r) \text{ is } k\text{-times continuously differentiable as a function of } r \in \mathbf{R}, \tag{8.6.7}$$

as in [87]. In this case,

$$\frac{\partial^l M_\phi(x; r)}{\partial r^l} = 0 \text{ at } r = 0 \tag{8.6.8}$$

when $l \leq k$ and l is odd, because of (8.6.2).

Let us consider

$$M_\phi(x; |w|) \tag{8.6.9}$$

as a real or complex-valued function of $w \in \mathbf{R}^n$. If (8.6.6) holds, then we get that

$$M_\phi(x; |w|) \text{ is } k\text{-times continuously differentiable as a function of } w \in \mathbf{R}^n, \tag{8.6.10}$$

because of (8.6.8), as in Section 8.5.

8.6.1 Spherical means and orthogonal transformations

Let T be an orthogonal transformation on $\mathbf{R}^n$, and consider

$$\phi(x + T(w)) \tag{8.6.11}$$

as a real or complex-valued function of $w \in \mathbf{R}^n$. We can average (8.6.11) over T in the set $O(n)$ of orthogonal transformations on $\mathbf{R}^n$, as mentioned in Subsection A.5.1. It may be reasonably clear that the average is equal to (8.6.9), without getting into too many details.

Of course, if (8.6.6) holds, then (8.6.11) is k-times continuously differentiable as a function of $w \in \mathbf{R}^n$ for each $T \in O(n)$. This is another way to look at (8.6.10) under these conditions.

8.7 More on spherical means

Let a and b be real numbers with $a < b$, although we could also allow $a = -\infty$ or $b = +\infty$ here. Suppose now that $\phi(x, t)$ is a continuous real or complex-valued function on $\mathbf{R}^n \times (a, b)$. If $x \in \mathbf{R}^n$ and $t \in (a, b)$, then put

$$M_\phi(x; r, t) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} \phi(x + r\, y', t)\, dy' \tag{8.7.1}$$

for each $r \in \mathbf{R}$. This is the same as (8.6.1) for $\phi(x, t)$ as a function of x on $\mathbf{R}^n$ for each $t \in (a, b)$. In particular,

$$M_\phi(x; -r, t) = M_\phi(x; r, t) \tag{8.7.2}$$

for each $r \in \mathbf{R}$,

$$M_\phi(x; 0, t) = \phi(x, t), \tag{8.7.3}$$

and

$$M_\phi(x; r, t) = \frac{1}{|\partial B(x, r)|} \int_{\partial B(x,r)} \phi(z', t)\, dz' \tag{8.7.4}$$

when $r > 0$, as before.

If $x \in \mathbf{R}^n$ is fixed, then

$$M_\phi(x; r, t) \text{ is continuous as a function of } (r, t) \in \mathbf{R} \times (a, b), \tag{8.7.5}$$

as before. Similarly, if

$$\phi(w,t) \text{ is } k\text{-times continuously differentiable on } \mathbf{R}^n \times (a,b) \tag{8.7.6}$$

for some positive integer k, then

$$\begin{aligned} &M_\phi(x;r,t) \text{ is } k\text{-times continuously differentiable} \\ &\text{as a function of } (r,t) \in \mathbf{R} \times (a,b). \end{aligned} \tag{8.7.7}$$

Under these conditions, we get that

$$\frac{\partial^l M_\phi(x;r,t)}{\partial r^l} = 0 \text{ at } r = 0 \tag{8.7.8}$$

when $l \le k$ and l is odd, because of (8.7.2), as before.

8.7.1 The Euler–Poisson–Darboux equation

Let us now take $a = 0$ and $b = +\infty$, and let $u(x,t)$ be a twice continuously-differentiable real or complex-valued function on $\mathbf{R}^n \times \mathbf{R}_+$ that satisfies the wave equation. If $x \in \mathbf{R}^n$, then $M_u(x;r,t)$ satisfies

$$\frac{\partial^2 M_u}{\partial t^2}(x;r,t) = \frac{\partial^2 M_u}{\partial r^2}(x;r,t) + \frac{n-1}{r}\frac{\partial M_u}{\partial r}(x;r,t) \tag{8.7.9}$$

for $r, t > 0$, as in Proposition 5.8 in Section B of Chapter 5 of [87]. This corresponds to part of Lemma 1 in Section 2.4.1 b of [81]. This is known as the *Euler–Poisson–Darboux equation*, as in [81].

More precisely, in [87], $M_u(x;r,t)$ is considered as a function of $r \in \mathbf{R}$ and $t > 0$. Of course, one should be a bit careful about (8.7.9) at $r = 0$, and (8.7.8) is relevant here, with $l = 1$. If $r < 0$, then (8.7.9) corresponds to the analogous statement for $-r > 0$, because of (8.7.2).

The right side of (8.7.9) corresponds to the Laplacian of

$$M_u(x;|w|,t) \tag{8.7.10}$$

as a function of $w \in \mathbf{R}^n \setminus \{0\}$, with $r = |w|$, as in (8.4.4). Thus (8.7.9) is the same as saying that

$$\begin{aligned} &M_u(x;|w|,t) \text{ satisfies the wave equation} \\ &\text{as a function of } w \text{ and } t > 0, \end{aligned} \tag{8.7.11}$$

as in the remark after Proposition 5.8 in [87]. This also corresponds to a remark after the statement of Lemma 1 in Section 2.4.1 b of [81]. More precisely, this works at $w = 0$ too, suitably interpreted, as in the preceding paragraph.

8.7.2 Using orthogonal transformations

Let T be an orthogonal transformation on $\mathbf{R}^n$, and consider

$$u(x + T(w), t) \tag{8.7.12}$$

as a real or complex-valued function of $w \in \mathbf{R}^n$ and $t > 0$. It may be reasonably clear that the average of (8.7.12) over $T \in O(n)$ is equal to (8.7.10), without getting into too many details, as in the previous section. In particular, this is another way to look at the regularity of (8.7.10) as a function of w and t in terms of the regularity of u, as before.

Remember that the Laplacian on $\mathbf{R}^n$ is invariant under orthogonal transformations, as in Subsection 2.1.1. Using this, it is easy to see that

(8.7.13) $u(x + T(w), t)$ satisfies the wave equation, as a function of w and t,

for each $T \in O(n)$. One can use this to get (8.7.11), by averaging over $T \in O(n)$, as before. This is another way to look at (8.7.9).

8.8 The $n = 2, 3$ cases

Suppose for the moment that $n = 3$, and let $u(x, t)$ be a twice continuously-differentiable real or complex-valued function on $\mathbf{R}^3 \times \mathbf{R}_+$ that satisfies the wave equation. Also let $x \in \mathbf{R}^3$ be given, and let $M_u(x; r, t)$ be as in the previous section. This is a twice continuously-differentiable function of $r \in \mathbf{R}$ and $t > 0$, as before. Put

$$v(r, t) = r\, M_u(x; r, t) \tag{8.8.1}$$

for $r \in \mathbf{R}$ and $t > 0$. This corresponds to $\widetilde{U}$ in (17) in Section 2.4.1 c of [81].

Clearly

(8.8.2) $v(r, t)$ is twice continuously differentiable in r and t,

because $M_u(x; r, t)$ has this property. One can check that

$$\frac{\partial^2 v}{\partial r^2} = \frac{\partial^2 v}{\partial t^2}, \tag{8.8.3}$$

using (8.7.9), with $n = 3$. This corresponds to a remark in Section 2.4.1 c of [81], and to a remark after (5.10) in Section B of Chapter 5 of [87], with $n = 3$.

Note that

$$v(0, t) = 0 \tag{8.8.4}$$

for all $t > 0$, by construction. We also have that

$$\lim_{r \to 0} r^{-1}\, v(r, t) = u(x, t) \tag{8.8.5}$$

for every $t > 0$, by construction.

In [81], $v(r,t)$ is considered as a solution of (8.8.3) for $r, t > 0$, and (8.8.4) is considered as a boundary condition. An additional argument is used to reduce to a solution of the wave equation on $\mathbf{R} \times \mathbf{R}_+$. This is used to obtain Kirchhoff's formula for $u(x,t)$ in terms of initial values of u and $\partial u/\partial t$, as in (22) of Section 2.4.1 c of [81]. This corresponds to an analogous formula (5.12) in Section B in Chapter 5 of [87], with $n = 3$.

More precisely, $u(x,t)$ can be obtained from a suitable formula for $v(r,t)$ in terms of initial values of v and $\partial v/\partial t$, using (8.8.5). The initial values of v and $\partial v/\partial t$ are easily obtained from the initial values of u and $\partial u/\partial t$.

8.8.1 Reducing the $n = 2$ case to the $n = 3$ case

A solution to the wave equation on $\mathbf{R}^n \times \mathbf{R}_+$ may be considered as a solution to the wave equation on $\mathbf{R}^{n+1} \times \mathbf{R}_+$, by considering the solution to be constant in x_{n+1}. This can be used to obtain Poisson's formula for solutions of the wave equation when $n = 2$, in terms of the initial values of the function and its derivative in t, from Kirchhoff's formula. This corresponds to (27) in Section 2.4.1 c of [81]. This also corresponds to Theorem 5.14 in Section B in Chapter 5 of [87], with $n = 2$.

If $n \geq 5$ is odd, then there is an analogous although somewhat more complicated way to get solutions of (8.8.3) from solutions of the Euler–Poisson–Darboux equation. This will be discussed in the next section.

If $n \geq 4$ is even, then $n+1$ is odd, and one can use the results for $n+1$, by considering functions that are constant in x_{n+1}, as before. This is discussed in Section 2.4.1 e of [81], and in Section B of Chapter 5 of [87].

8.9 Some helpful identities

Let k be a positive integer, and let ϕ be a $(k+1)$-times continuously-differentiable real or complex-valued function on an open set in $\mathbf{R} \setminus \{0\}$. Under these conditions, it is well known that

$$\frac{d^2}{dr^2}\Big(\frac{1}{r}\,\frac{d}{dr}\Big)^{k-1}(r^{2k-1}\,\phi(r)) = \Big(\frac{1}{r}\,\frac{d}{dr}\Big)^{k}\Big(r^{2k}\,\frac{d\phi}{dr}(r)\Big) \tag{8.9.1}$$

and

$$\Big(\frac{1}{r}\,\frac{d}{dr}\Big)^{k-1}(r^{2k-1}\,\phi(r)) = \sum_{j=0}^{k-1} c_j(k)\,r^{j+1}\,\frac{d^j\phi}{dr^j}(r), \tag{8.9.2}$$

where $c_j(k)$ is a constant that does not depend on ϕ for each j. More precisely,

$$c_0(k) = 1 \cdot 3 \cdot 5 \cdots (2\,k-1). \tag{8.9.3}$$

This also works when ϕ is defined on an open subset of $\mathbf{R}$ that contains 0, with suitable interpretations, because there are sufficiently many factors of r being differentiated that there are not really any factors of $1/r$ left after expanding out the derivatives, as in (8.9.2).

This corresponds to Lemma 2 in Section 2.4.1 d of [81], and to (5.9) and (5.10) in Section B of Chapter 5 of [87]. The proof by induction is left as an exercise in [81], and some additional hints are given in [87].

8.9.1 Using these identities

Let n be an odd integer with $n \geq 3$, so that $n = 2\,k + 1$ for some positive integer k. Also let $u(x,t)$ be a $(k+1)$-times continuously-differentiable real or complex-valued function on $\mathbf{R}^n \times \mathbf{R}_+$ that satisfies the wave equation. Let $x \in \mathbf{R}^n$ be given, and let $M_u(x;r,t)$ be as in Section 8.7, which is a $(k+1)$-times continuously-differentiable function of $r, t > 0$. Put

$$v(r,t) = \Big(\frac{1}{r}\,\frac{d}{dr}\Big)^{k-1} (r^{2k-1}\, M_u(x;r,t)), \tag{8.9.4}$$

as mentioned some time after (5.10) in [87]. This corresponds to $\widetilde{U}(r,t)$ in (28) in Section 2.4.1 d of [81].

Note that $v(r,t)$ is twice continuously differentiable, because $M_u(x;r,t)$ is $(k+1)$-times continuously differentiable in r, t, as in Section 8.7. One can show that

$$\frac{\partial^2 v}{\partial r^2} = \frac{\partial^2 v}{\partial t^2}, \tag{8.9.5}$$

using (8.7.9) and (8.9.1). This corresponds to part of Lemma 3 in Section 2.4.1 d of [81], and some remarks after (5.10) in [87].

In fact, $M_u(x;r,t)$ is $(k+1)$-times continuously differentiable as a function of $(r,t) \in \mathbf{R} \times \mathbf{R}_+$, as in Section 8.7. This implies that $v(r,t)$ may be defined as a twice continuously-differentiable function of $(r,t) \in \mathbf{R} \times \mathbf{R}_+$, because of (8.9.2). It is easy to see that

$$v(0,t) = 0 \tag{8.9.6}$$

for all $t > 0$, using (8.9.2), which is another part of Lemma 3 in Section 2.4.1 d of [81]. Indeed, we have that

$$\lim_{r \to 0} \frac{v(r,t)}{c_0(k)\, r} = \lim_{r \to 0} M_u(x;r,t) = u(x,t), \tag{8.9.7}$$

using (8.9.2) in the first step, and the definition of $M_u(x;r,t)$ in the second step. This is mentioned after Lemma 3 in Section 2.4.1 d of [81], and after (5.11) in Section B of Chapter 5 of [87].

If $n = 3$, then $k = 1$, and (8.9.4) is the same as (8.8.1). If [81], $v(r,t)$ is considered as a solution of (8.9.5) for $r, t > 0$, and (8.9.6) is considered as a boundary condition, as before. An additional argument is used to reduce to a solution to the wave equation on $\mathbf{R} \times \mathbf{R}$, exactly as before. This is used to obtain a formula for $u(x,t)$ in terms of initial values of u and $\partial u/\partial t$, as in (31) in Section 2.4.1 d of [81].

More precisely, $u(x,t)$ can be obtained from a suitable formula for $v(r,t)$ in terms of initial values of v and $\partial v/\partial t$, using (8.9.7), in essentially the same way as before. The initial values for v and $\partial v/\partial t$ may be obtained from the initial values of u and $\partial u/\partial t$, as before. A slightly different version of this formula for $u(x,t)$ may be found in Section B of Chapter 5 of [87], as before.

8.10 An inhomogeneous problem

Let n be a positive integer, and let $[n/2]$ be the integer part of $n/2$, which is equal to $n/2$ when n is even, and to $(n-1)/2$ when n is odd. Also let f be real or complex-valued function on $\mathbf{R}^n \times (\mathbf{R}_+ \cup \{0\})$ that is

$$([n/2]+1)\text{-times continuously differentiable.} \tag{8.10.1}$$

This may be interpreted as in Section 3.4, which amounts in this case to using one-sided derivatives in t from the right at $t = 0$.

Consider the problem of finding a twice continuously-differentiable real or complex-valued function u on $\mathbf{R}^n \times (\mathbf{R}_+ \cup \{0\})$, as appropriate, such that

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \tag{8.10.2}$$

on $\mathbf{R}^n \times \mathbf{R}_+$, with

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \tag{8.10.3}$$

on $\mathbf{R}^n$. To deal with this, one can use a version of *Duhamel's principle*, as in Section 2.4.2 of [81], and Section C of Chapter 5 of [87].

8.10.1 Duhamel's principle

If τ is a nonnegative real number, then let $v(x,t;\tau)$ be the real or complex-valued function of (x,t) on $\mathbf{R}^n \times (\mathbf{R}_+ \cup \{0\})$ that satisfies the wave equation

$$\frac{\partial^2 v}{\partial t^2} - \Delta v = 0 \tag{8.10.4}$$

on $\mathbf{R}^n \times \mathbf{R}_+$, with

$$v(x,0;\tau) = 0 \tag{8.10.5}$$

and

$$\frac{\partial v}{\partial t}(x,0;\tau) = f(x,\tau) \tag{8.10.6}$$

on $\mathbf{R}^n$. More precisely, this may be obtained as in [81, 87], and is twice continuously differentiable under these conditions. Equivalently,

$$v(x,t-\tau;\tau) \tag{8.10.7}$$

satisfies the wave equation on $\mathbf{R}^n \times (\tau,+\infty)$, is equal to 0 when $t = \tau$, and its derivative in t is equal to $f(x,\tau)$ at $t = \tau$, as in [81].

If $x \in \mathbf{R}^n$ and $t \geq 0$, then we take

$$u(x,t) = \int_0^t v(x, t - \tau; \tau)\, d\tau, \tag{8.10.8}$$

as in [87], which is expressed a bit differently in [81]. Of course, this is equal to 0 when $t = 0$. We also have that

$$\frac{\partial u}{\partial t}(x,t) = v(x,0;t) + \int_0^t \frac{\partial v}{\partial t}(x, t-\tau; \tau)\, d\tau = \int_0^t \frac{\partial v}{\partial t}(x, t-\tau;\tau)\, d\tau, \tag{8.10.9}$$

using (8.10.5) in the second step. This is equal to 0 when $t = 0$ as well.

Similarly,

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial v}{\partial t}(x,0;t) + \int_0^t \frac{\partial^2 v}{\partial t^2}(x, t-\tau;\tau)\, d\tau \\ &= f(x,t) + \int_0^t \frac{\partial^2 v}{\partial t^2}(x, t-\tau;\tau)\, d\tau. \end{aligned} \tag{8.10.10}$$

Note that

$$\Delta u(x,t) = \int_0^t \Delta v(x, t-\tau;\tau)\, d\tau = \int_0^t \frac{\partial^2 v}{\partial t^2}(x, t-\tau, \tau)\, d\tau, \tag{8.10.11}$$

using (8.10.4) in the second step. Clearly (8.10.2) follows from these two equations. One can get solutions of (8.10.2) with other initial conditions using this and solutions of the wave equation with prescribed initial conditions, as mentioned in [81, 87].

8.11 More on holomorphic functions

Let n be a positive integer, and remember that $\mathbf{C}^n$ is the space of n-tuples of complex numbers, as in Section 2.6. Every $z \in \mathbf{C}^n$ can be expressed in a unique way as $z = x + i\,y$, with $x, y \in \mathbf{R}^n$, and one can use this to identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, as before. Let U be a nonempty open set in $\mathbf{C}^n$, which may be identified with an open set in $\mathbf{R}^{2n}$, as in Subsection 2.6.1. Equivalently, this means that U is an open set in $\mathbf{C}^n$ with respect to the standard Euclidean metric on $\mathbf{C}^n$.

Let f be a continuously-differentiable complex-valued function on U, as an open subset of $\mathbf{R}^{2n}$. Suppose that

(8.11.1) f is holomorphic on U,

as in Subsection 2.6.1. It is well known that this implies that

(8.11.2) f is smooth on U,

although we shall not get into that here. Let us simply ask for the moment that

(8.11.3) f be twice continuously differentiable as a complex-valued function on U, as an open set in $\mathbf{R}^{2n}$.

This will be clear in many cases of interest, and anyway it holds automatically, as before.

8.11.1 Harmonicity of holomorphic functions

If $n = 1$, then it follows that f is harmonic on U, as an open subset of $\mathbf{R}^2$, as in Subsection 2.2.1. Similarly, for any n, we have that

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} f + \frac{\partial}{\partial y_j}\frac{\partial}{\partial y_j} f = 0 \tag{8.11.4}$$

on U for each $j = 1, \ldots, n$. Of course, this implies that

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} f + \sum_{j=1}^{n} \frac{\partial}{\partial y_j}\frac{\partial}{\partial y_j} f = 0 \tag{8.11.5}$$

on U. This means that

$$f \text{ is harmonic on } U, \text{ as an open subset of } \mathbf{R}^{2n}. \tag{8.11.6}$$

It follows that (8.11.2) holds, as in Subsection 6.4.2. Remember that (8.11.3) is part of the hypothesis here. There are other arguments for obtaining (8.11.2) without this hypothesis, as mentioned earlier.

Observe that

$$\frac{\partial}{\partial \overline{z}_l}\frac{\partial}{\partial z_j} f = \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z}_l} f = 0 \tag{8.11.7}$$

on U for each $j, l = 1, \ldots, n$ under these conditions. This uses the twice continuous differentiability of f in the first step, and the hypothesis that f be holomorphic in the second step. This means that

$$\frac{\partial}{\partial z_j} f \text{ is holomorphic on } U \tag{8.11.8}$$

for each $j = 1, \ldots, n$. Similarly,

$$\frac{\partial}{\partial z_j}\frac{\partial}{\partial z_l} f = \frac{\partial}{\partial z_l}\frac{\partial}{\partial z_j} f \tag{8.11.9}$$

on U for each $j, l = 1, \ldots, n$, because f is twice continuously differentiable on U, as an open set in $\mathbf{R}^{2n}$.

8.11.2 A complex Laplace equation

Consider the partial differential equation

$$\sum_{j=1}^{n} \frac{\partial}{\partial z_j}\frac{\partial}{\partial z_j} f = 0 \tag{8.11.10}$$

on U. This may be considered as a complex version of the *Laplace equation.* This equation clearly holds when f is constant, or

$$f(z) = z_l \tag{8.11.11}$$

for some l. If

$$f(z) = z_l\, z_m, \tag{8.11.12}$$

then this equation holds if and only if $l \neq m$. We also have that (8.11.10) holds when

$$f(z) = z_l^2 - z_m^2. \tag{8.11.13}$$

It is well known that the complex exponential function is holomorphic on $\mathbf{C}$, as mentioned in Subsection 3.14.3. If $a \in \mathbf{C}^n$, then one can use this to check that

$$\exp(a \cdot z) = \exp(a_1\, z_1 + \cdots + a_n\, z_n) \tag{8.11.14}$$

is a holomorphic function on $\mathbf{C}^n$. We also have that

$$\frac{\partial}{\partial z_j} \exp(a \cdot z) = a_j\, \exp(a \cdot z) \tag{8.11.15}$$

for each j. One can use this to get that (8.11.14) satisfies (8.11.10) if and only if

$$a \cdot a = \sum_{j=1}^n a_j^2 = 0. \tag{8.11.16}$$

8.12 More on this differential equation

Let us continue with the same notation and hypotheses as in the previous section. Let f be a holomorphic function on U that satisfies (8.11.3) and thus (8.11.2) again. In particular, this means that f is three-times continuously differentiable on U, as an open set in $\mathbf{R}^{2n}$. In this case, it is easy to see that

$$\frac{\partial}{\partial z_j} \frac{\partial}{\partial z_l} f \text{ is holomorphic on } U \tag{8.12.1}$$

for all j, l, using (8.11.8). It follows that

$$\sum_{j=1}^n \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_j} f \text{ is holomorphic on } U. \tag{8.12.2}$$

8.12.1 Real and complex derivatives

If f is any complex-valued continuously-differentiable function on U, as an open subset of $\mathbf{R}^{2n}$, then

$$\frac{\partial}{\partial z_j} f + \frac{\partial}{\partial \overline{z}_j} j = \frac{\partial}{\partial x_j} f \tag{8.12.3}$$

on U for each j, because of the way that $\partial/\partial z_j$ and $\partial/\partial \overline{z}_j$ are defined, as in Subsection 2.6.1. If f is holomorphic on U, then it follows that

$$\frac{\partial}{\partial z_j} f = \frac{\partial}{\partial x_j} f \tag{8.12.4}$$

on U for each j. Similarly,

$$\frac{\partial}{\partial z_j}\frac{\partial}{\partial z_l}f = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}f \tag{8.12.5}$$

on U for all j, l if we also ask that f be twice continuously differentiable on U, as an open set in $\mathbf{R}^{2n}$, which holds automatically anyway, as before. This implies that

$$\sum_{j=1}^{n}\frac{\partial}{\partial z_j}\frac{\partial}{\partial z_j}f = \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}f \tag{8.12.6}$$

on U under these conditions. Thus (8.11.10) is the same as saying that

$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}f = 0 \tag{8.12.7}$$

on U in this case.

If $y \in \mathbf{R}^n$, then put

$$U_y = \{x \in \mathbf{R}^n : x + i\,y \in U\}. \tag{8.12.8}$$

One can check that this is an open set in $\mathbf{R}^n$, because U is an open set in $\mathbf{C}^n$, which may identified with an open set in $\mathbf{R}^{2n}$, as before. Put

$$f_y(x) = f(x + i\,y) \tag{8.12.9}$$

for each $x \in U_y$, so that f_y defines a complex-valued function on U_y that is twice continuously differentiable, by hypothesis. The condition that (8.12.7) hold on U is the same as saying that f_y is harmonic on U_y for each $y \in \mathbf{R}^n$. This is considered to hold vacuously when $U_y = \emptyset$.

8.12.2 The case of polynomials

Let p be a polynomial in $z_1, \ldots, z_n$ with complex coefficients, as in Section 1.7. Remember that p defines a holomorphic function on $\mathbf{C}^n$, as mentioned in Subsection 2.6.1. Suppose more precisely that the degree of p is less than or equal to N for some nonnegative integer N, as in Section 1.7 again. It is easy to see that

$$\frac{\partial}{\partial z_j}p = \frac{\partial}{\partial x_j}p \tag{8.12.10}$$

is a polynomial in $z_1, \ldots, z_n$ for each j, with degree less than or equal to $N-1$ when $N \geq 1$. Similarly,

$$\frac{\partial}{\partial z_j}\frac{\partial}{\partial z_l}p = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}p \tag{8.12.11}$$

is a polynomial in $z_1, \ldots, z_n$ for each j, l, with degree less than or equal to $N-2$ when $N \geq 2$.

In particular,

$$\sum_{j=1}^{n} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_j} p = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} p \tag{8.12.12}$$

is a polynomial in $z_1, \ldots, z_n$ of degree less than or equal to $N - 2$ when $N \geq 2$. Suppose that this is equal to 0 on $\mathbf{R}^n$, so that p is harmonic on $\mathbf{R}^n$. In this case, (8.12.12) is equal to 0 on all of $\mathbf{C}^n$, as in Section 2.5. More precisely, if $y \in \mathbf{R}^n$, and $p_y(x)$ is defined as in (8.12.9), then $p_y(x)$ can be expressed as a polynomial in $x_1, \ldots, x_n$ with complex coefficients, as mentioned in Section 2.5. If p_y is harmonic on $\mathbf{R}^n$ for some $y \in \mathbf{R}^n$, then (8.12.12) is equal to 0 on all of $\mathbf{C}^n$, as in Section 2.5 again.

8.13 The complex wave equation

Let $n \geq 2$ be an integer, and let U be a nonempty open subset of $\mathbf{C}^n$, which which may be identified with an open set in $\mathbf{R}^{2\,n}$, as usual. Also let f be a twice continuously-differentiable complex-valued function on U, as an open subset of $\mathbf{R}^{2n}$, and suppose that f is holomorphic on U. Consider the partial differential equation

$$\sum_{j=1}^{n-1} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_j} f - \frac{\partial}{\partial z_n} \frac{\partial}{\partial z_n} f = 0 \tag{8.13.1}$$

on U. This may be considered as a complex version of the *wave equation*. This is the same as saying that

$$\sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} f - \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} f = 0 \tag{8.13.2}$$

on U under these conditions, as in Subsection 8.12.1.

If $y \in \mathbf{R}^n$, then let U_y be as in (8.12.8), and let f_y be defined on U_y as in (8.12.9). Of course, (8.13.2) holds on U if and only if

$$\sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} f_y - \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} f_y = 0 \tag{8.13.3}$$

on U_y for each $y \in \mathbf{R}^n$. This is considered to hold vacuously when $U_y = \emptyset$, as before. If f is a polynomial in $z_1, \ldots, z_n$ with complex coefficients, and (8.13.3) holds on $\mathbf{R}^n$ for some $y \in \mathbf{R}^n$, then (8.13.2) holds on $\mathbf{C}^n$, as in Subsection 8.12.2. There are analogous statements for holomorphic functions more broadly, under suitable conditions, that we shall not pursue here.

8.13.1 A simple change of variables

Put

$$\widetilde{U} = \{z \in \mathbf{C}^n : (z_1, \ldots, z_{n-1}, i\, z_n) \in U\}. \tag{8.13.4}$$

One can check that this is an open set in $\mathbf{C}^n$ too, or equivalently that it corresponds to an open set in $\mathbf{R}^{2n}$. Let $\widetilde{f}$ be the complex-valued function defined on $\widetilde{U}$ by

$$\widetilde{f}(z) = f(z_1, \ldots, z_{n-1}, i\, z_n) \tag{8.13.5}$$

for each $z \in \widetilde{U}$. One can verify that $\widetilde{f}$ is twice continuously differentiable on $\widetilde{U}$, as an open subset of $\mathbf{R}^{2n}$, and that $\widetilde{f}$ is holomorphic on $\widetilde{U}$, because of the analogous properties of f on U. We also have that (8.13.2) holds on U if and only if $\widetilde{f}$ satisfies the complex version of the Laplace equation

$$\sum_{j=1}^{n} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_j} \widetilde{f} = 0 \tag{8.13.6}$$

on $\widetilde{U}$.

8.13.2 Some solutions with $n = 2$

Let V be a nonempty open subset of the complex plane, which may be identified with $\mathbf{R}^2$, as before, and let ϕ be a twice continuously-differentiable complex-valued function on V, as an open subset of $\mathbf{R}^2$, that is holomorphic on V, as in Section 2.2. One can check that

$$\{z \in \mathbf{C}^2 : z_1 - z_2 \in V\} \tag{8.13.7}$$

is an open set in $\mathbf{C}^2$, so that it corresponds to an open set in $\mathbf{R}^4$. One can also verify that

$$\phi(z_1 - z_2) \tag{8.13.8}$$

is twice continuously differentiable on (8.13.7), as an open subset of $\mathbf{R}^4$, and that (8.13.8) is holomorphic on (8.13.7), because of the analogous properties of ϕ on V. It is easy to see that (8.13.8) satisfies the complex version of wave equation on (8.13.7). Of course, if $V = \mathbf{C}$, then (8.13.7) is all of $\mathbf{C}^2$.

Similarly, let W be a nonempty open subset of $\mathbf{C}$, which may be identified with $\mathbf{R}^2$, as usual, and let ψ be a twice continuously-differentiable complex-valued function on W, as an open set in $\mathbf{R}^2$, that is holomorphic on W. As before, one can check that

$$\{z \in \mathbf{C}^2 : z_1 + z_2 \in W\} \tag{8.13.9}$$

is an open set in $\mathbf{C}^2$, so that it corresponds to an open set in $\mathbf{R}^4$. We also have that

$$\psi(z_1 + z_2) \tag{8.13.10}$$

is twice continuously differentiable on (8.13.9), as an open subset of $\mathbf{R}^4$, and that (8.13.10) is holomorphic on (8.13.9), because of the corresponding properties of ψ on W. In fact, (8.13.10) satisfies the complex version of the wave equation on (8.13.9), as before. Note that (8.13.9) is all of $\mathbf{C}^2$ when $W = \mathbf{C}$, as before.

Let V and ϕ be as before, and observe that

$$\{z \in \mathbf{C}^2 : z_1 - i\, z_2 \in V\} \tag{8.13.11}$$

is an open set in $\mathbf{C}^2$, or equivalently that it corresponds to an open set in $\mathbf{R}^4$. If (8.13.7) is denoted U, then (8.13.11) is the same as (8.13.4). One can check that

$$\phi(z_1 - i\, z_2) \tag{8.13.12}$$

is twice continuously differentiable on (8.13.11), as an open set in $\mathbf{R}^4$, and that (8.13.12) is holomorphic on (8.13.11). One can also verify that (8.13.12) satisfies the complex version of the Lapalce equation on (8.13.11). If (8.13.8) is denoted f, then (8.13.12) is the same as (8.13.5).

Similarly, if W and ψ are as before, then

$$\{z \in \mathbf{C}^2 : z_1 + i\, z_2 \in W\} \tag{8.13.13}$$

is an open set in $\mathbf{R}^2$, so that it corresponds to an open set in $\mathbf{R}^4$. If (8.13.9) is denoted U, then (8.13.13) is the same as (8.13.4). As usual,

$$\psi(z_1 + i\, z_2) \tag{8.13.14}$$

is twice continuously

8.14 Another inhomogeneous problem

Let n be a positive integer, and let $f(x,t)$ be a real or complex-valued function defined for $x \in \mathbf{R}^n$ and $t \geq 0$. Suppose that $f(x,t)$ is continuously differentiable, and that the second derivatives of $f(x,t)$ in x exist and are continuous as functions of x and t. Suppose also that $f(x,t)$ has compact support, so that

$$f(x,t) = 0 \tag{8.14.1}$$

when $|x|$ or t is large enough. More precisely, it suffices to ask here that for every positive real number T there be a positive real number $R(T)$ such that (8.14.1) holds when

$$|x| \geq R(T) \text{ and } 0 \leq t \leq T. \tag{8.14.2}$$

Consider the problem of finding a real or complex-valued function $u(x,t)$, as appropriate, defined for $x \in \mathbf{R}^n$ and $t \geq 0$, and with the following properties. First, $u(x,t)$ should be continuously differentiable, ans the second derivatives of $u(x,t)$ should exist and be continuous as functions of x and t. Second, we would like to have that

$$\frac{\partial u}{\partial t}(x,t) - (\Delta u)(x,t) = f(x,t) \tag{8.14.3}$$

for all $x \in \mathbf{R}^n$ and $t \geq 0$, where $(\Delta u)(x,t)$ is the Laplacian of $u(x,t)$ in x, as before. Third,

$$u(x,0) = 0 \tag{8.14.4}$$

for every $x \in \mathbf{R}^n$.

We can use another version of *Duhamel's principle* here, as in Section 2.3.1 c of [81]. This is related to Theorem 4.7 in Section A of Chapter 4 of [87].

8.14.1 Another version of Duhamel's principle

Let $K(x,t)$ be the heat kernel, as in Subsection 7.1.1. If $x \in \mathbf{R}^n$, t is a positive real number, and τ is a nonnegative real number, then put

$$v(x,t;\tau) = \int_{\mathbf{R}^n} K(x-y,t)\, f(y,\tau)\, dy. \tag{8.14.5}$$

This satisfies the heat equation as a function of x and t, as in Section 7.4. If we put

$$u(x,0;\tau) = f(x,\tau), \tag{8.14.6}$$

then $u(x,t;\tau)$ is continuous as a function of $x \in \mathbf{R}^n$ and $t \geq 0$, as before. More precisely, it is not too difficult to show that $u(x,t;\tau)$ is continuous as a function of $x \in \mathbf{R}^n$ and $t, \tau \geq 0$, because f is continuous, by hypothesis.

Equivalently,

$$v(x,t-\tau;\tau) \tag{8.14.7}$$

satisfies the heat equation as a function of $x \in \mathbf{R}^n$ and $t > \tau$, and it is equal to $f(x,\tau)$ when $t = \tau$, as in [81]. If $x \in \mathbf{R}^n$ and $t \geq 0$, then put

$$u(x,t) = \int_0^t v(x,t-\tau;\tau)\, d\tau, \tag{8.14.8}$$

which is expressed a bit differently in [81]. Note that this satisfies (8.14.4). If $t > 0$, then

$$u(x,t) = \int_0^t \int_{\mathbf{R}^n} K(x-y,t-\tau)\, f(y,\tau)\, dy\, d\tau, \tag{8.14.9}$$

as in [81].

8.14.2 Differentiating $u(x,t)$

One should be a bit careful about differentiating under the integral sign here, because of the behavior of the heat kernel near $(0,0)$, as mentioned in the proof of Theorem 2 in Section 2.3.1 c of [81]. It is helpful to begin with a change of variables, to get that

$$u(x,t) = \int_0^t \int_{\mathbf{R}^n} K(y,\tau)\, f(x-y,t-\tau)\, dy\, d\tau \tag{8.14.10}$$

when $t > 0$, as in [81]. One can use this to obtain that

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \int_0^t \int_{\mathbf{R}^n} K(y,\tau)\, \frac{\partial f}{\partial t}(x-y,t-\tau)\, dy\, d\tau \\ &\quad + \int_{\mathbf{R}^n} K(y,t)\, f(x-y,0)\, dy, \end{aligned} \tag{8.14.11}$$

as in [81]. In particular, this is continuous as a function of $x \in \mathbf{R}^n$ and $t > 0$.

Similarly, the first and second derivatives of $u(x,t)$ in x may be obtained by differentiating the right side of (8.14.10) under the integral sign, and are continuous functions of $x \in \mathbf{R}^n$ and $t > 0$, as in [81]. This implies that

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,t) - (\Delta u)(x,t) \\ &\quad = \int_0^t \int_{\mathbf{R}^n} K(y,\tau) \Big(\frac{\partial f}{\partial t}(x-y, t-\tau) - (\Delta f)(x-y,t-\tau) \Big) \, dy \, d\tau \\ &\qquad + \int_0^t K(y,t) \, f(x-y,0) \, dy. \end{aligned} \tag{8.14.12}$$

It follows that

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,t) - (\Delta u)(x,t) \\ &\quad = -\int_0^t \int_{\mathbf{R}^n} K(y,\tau) \Big(\Big(\frac{\partial}{\partial \tau} + \Delta_y \Big) f(x-y, t-\tau) \Big) \, dy \, d\tau \\ &\qquad + \int_0^t K(y,t) \, f(x-y,0) \, dy. \end{aligned} \tag{8.14.13}$$

Of course, we would like the right side of (8.14.13) to be equal to $f(x,t)$, as in (8.14.3). This may be obtained as in Subsection 7.14.3. More precisely, remember that V is a nonempty bounded open subset of $\mathbf{R}^n$ with reasonably smooth boundary in Section 7.14. Here we want to take V sufficiently large so that the corresponding integrals over V are the same as integrals over $\mathbf{R}^n$, and the integrals over the boundary of V are equal to 0, because of the hypothesis on the support of f mentioned at the beginning of the section.

8.15 The porous medium equation

Let n be a positive integer, and let $u(x,t)$ be a real-valued function on an open subset of $\mathbf{R}^n \times \mathbf{R}$, which we can identify with $\mathbf{R}^{n+1}$, as usual. Also let γ be a real number, and consider the partial differential equation

$$\frac{\partial u}{\partial t} - \Delta(u^\gamma) = 0, \tag{8.15.1}$$

as in Example 2 in Section 4.1.1 of [81]. This is known as the *porous medium equation*, at least for suitable γ. Of course, this reduces to the heat equation when $\gamma = 1$. If $\gamma = 0$, then this may be interpreted as

$$\frac{\partial u}{\partial t} = 0. \tag{8.15.2}$$

If γ is not an integer, then we ask that $u \geq 0$, so that u^γ is defined. Similarly, if $\gamma < 0$, then we ask that $u \neq 0$.

One way to try to solve this equation is to look for solutions of the form

$$u(x,t) = v(t)\, w(x), \tag{8.15.3}$$

where $v(t)$ and $w(x)$ are real-valued functions defined on open subsets of $\mathbf{R}$ and $\mathbf{R}^n$, respectively. We also ask that $v(t), w(x) \geq 0$ when γ is not an integer, and that $v(t), u(x) \neq 0$ when $\gamma < 0$, as before. In this case, (8.15.1) is the same as saying that

$$v'(t)\, w(x) - v(t)^\gamma\, \Delta(w(x)^\gamma) = 0. \tag{8.15.4}$$

This means that

$$\frac{v'(t)}{v(t)^\gamma} = \frac{\Delta(w(x)^\gamma)}{w(x)} \tag{8.15.5}$$

when $v(t), w(x) \neq 0$. To get this, we need both sides of the equation to be constant, so that

$$\frac{v'(t)}{v(t)^\gamma} = \mu = \frac{\Delta(w(x)^\gamma)}{w(x)}, \tag{8.15.6}$$

where μ is a constant.

Of course, if $\gamma = 0$, then $\mu = 0$. The first part of (8.15.6) is the same as saying that

$$v'(t) = \mu\, v(t)^\gamma, \tag{8.15.7}$$

and the second part is the same as saying that

$$\Delta(w(x)^\gamma) = \mu\, w(x). \tag{8.15.8}$$

One can also consider these equations when $v(t)$ or $w(x)$ is equal to 0, at least if $\gamma \geq 0$. Note that (8.15.4) holds when these two equations hold. These two equations are much simpler when $\mu = 0$, and so we suppose now that $\mu \neq 0$.

The case where $\gamma = 1$ is discussed in Example 1 in Section 4.1.1 of [81], and is related to some of the remarks in Section 3.1. Suppose that $\gamma \neq 1$, and let us solve (8.15.7) to get that

$$v(t) = ((1-\gamma)\, \mu\, t + \lambda)^{1/(1-\gamma)}, \tag{8.15.9}$$

where λ is a constant, as in [81]. More precisely, we ask that

$$(1-\gamma)\, \mu\, t + \lambda \neq 0 \tag{8.15.10}$$

when $\gamma > 1$, and when $\gamma < 0$, as before. We also ask that

$$(1-\gamma)\, \mu\, t + \lambda \geq 0 \tag{8.15.11}$$

when $1/(1-\gamma)$ is not an integer, and when γ is not an integer, as before. Of course, γ and $1/(1-\gamma)$ are both integers only when $\gamma = 0$ or 2.

Let us look for solutions of (8.15.8) of the form

$$w(x) = |x|^\alpha \tag{8.15.12}$$

for some $\alpha \in \mathbf{R}$, as in [81]. One can check that

$$\mu\, w(x) - \Delta(w(x)^\gamma) = \mu\, |x|^\alpha - \alpha\, \gamma\, (\alpha\, \gamma + n - 2)\, |x|^{\alpha\, \gamma - 2}, \tag{8.15.13}$$

as in [81], where $x \neq 0$ when $\alpha < 0$ or $\alpha\, \gamma - 2 < 0$. Let us ask that

$$\alpha\, \gamma - 2 = \alpha, \tag{8.15.14}$$

which means that

$$\alpha = \frac{2}{\gamma - 1}, \tag{8.15.15}$$

as in [81]. In this case, (8.15.13) is the same as saying that

$$\mu\, w(x) - \Delta(w(x)^\gamma) = (\mu - \alpha\, \gamma\, (\alpha\, \gamma + n - 2))\, |x|^\alpha, \tag{8.15.16}$$

where $x \neq 0$ when $\alpha < 0$.

Under these conditions, we get that (8.15.8) holds when

$$\mu = \alpha\, \gamma\, (\alpha\, \gamma + n - 2), \tag{8.15.17}$$

as in [81]. In [81], one takes $\gamma > 1$, which implies that $\alpha > 0$, and that

$$\alpha\, \gamma = \frac{2\, \gamma}{\gamma - 1} > 2. \tag{8.15.18}$$

This means that one can take $w(x)$ as in (8.15.12) on $\mathbf{R}^n$, and that $w(x)^\gamma$ is twice continuously differentiable on $\mathbf{R}^n$. Note that $\mu > 0$ in this case, as in [81].

Of course,

$$(1 - \gamma)\, \mu\, t + \lambda > 0 \tag{8.15.19}$$

when $\gamma > 1$ and

$$t < t_* = \frac{\lambda}{(\gamma - 1)\, \mu}, \tag{8.15.20}$$

because $\mu > 0$. In [81], one also takes $\lambda > 0$, so that $t_* > 0$.

Chapter 9

Some more classes of functions

9.1 Semicontinuity

Let n be a positive integer, let E be a nonempty subset of $\mathbf{R}^n$, and let f be a real-valued function on E. We say that f is *upper semicontinuous* at a point $x \in E$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$f(y) < f(x) + \epsilon \tag{9.1.1}$$

for every $y \in E$ with

$$|x - y| < \delta. \tag{9.1.2}$$

Similarly, f is said to be *lower semicontinuous* at x if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$f(y) > f(x) - \epsilon \tag{9.1.3}$$

for every $y \in E$ that satisfies (9.1.2). It is easy to see that

(9.1.4) f is continuous at x if and only if
f is both upper and lower semicontinuous at x.

Note that f is upper semicontinuous at x if and only if $-f$ is lower semicontinuous at x.

If f is upper semi-continuous at every point in E, then f is said to be *upper semicontinuous* on E. Similarly, if f is lower semicontinuous at every point in E, then f is said to be *lower semicontinuous* on E. As before, f is continuous on E if and only if f if both upper and lower semicontinous on E. We also have that f is upper semicontinuous on E if and only if $-f$ is lower semicontinuous on E.

9.1.1 Uniform semicontinuity?

One might like to say that f is uniformly upper semicontinuous on E if for every $\epsilon > 0$ there is a $\delta > 0$ such that (9.1.1) holds for all $x, y \in E$ that satisfy (9.1.2). Similarly, one might say that f is uniformly lower semicontinuous on E if for every $\epsilon > 0$ there is a $\delta > 0$ such that (9.1.3) holds for all $x, y \in E$ that satisfy (9.1.2). However, one can check that these uniform versions of upper and lower semicontinuity are equivalent to each other, by exchanging the roles of x and y. In fact, these uniform versions of upper and lower semicontinuity of f on E are equivalent to the usual notion of *uniform continuity* of f on E, which says that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \tag{9.1.5}$$

for all $x, y \in E$ that satisfy (9.1.2). It is well known that every continuous function on E is uniformly continuous when E is compact.

9.1.2 Relatively open sets

A subset A of E is said to be *relatively open* in E if for every $x \in A$ there is an $r > 0$ such that

$$B(x, r) \cap E \subseteq A. \tag{9.1.6}$$

If E is an open subset of $\mathbf{R}^n$, then $A \subseteq E$ is relatively open in E if and only if A is an open set in $\mathbf{R}^n$.

It is well known and not difficult to show that f is upper semicontinuous on E if and only if for every real number b,

$$\{x \in E : f(x) < b\} \tag{9.1.7}$$

is a relatively open set in E. Similarly, f is lower semicontinuous on E if and only if for every $a \in \mathbf{R}$,

$$\{x \in E : f(x) > a\} \tag{9.1.8}$$

is a relatively open set in E.

9.1.3 Upper and lower limits

Remember that the continuity of f at $x \in E$ can be characterized in terms of convergent sequences, as mentioned at the beginning of Section 1.2. There are analogues of this for upper and lower semicontinuity, in terms of upper and lower limits of sequences of real numbers, if one is familiar with that. Namely, f is upper semicontinuous at x if and only if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of E that converges to x, we have that

$$\limsup_{j \to \infty} f(x_j) \le f(x). \tag{9.1.9}$$

Similarly, f is lower semicontinuous at x if and only if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of E that converges to x,

$$\liminf_{j \to \infty} f(x_j) \ge f(x). \tag{9.1.10}$$

9.1.4 Limits of functions

A point $z \in \mathbf{R}^n$ is said to be a *limit point* of E if for every $r > 0$ there is a $y \in E$ such that

$$|y - z| < r \tag{9.1.11}$$

and $y \neq z$. Remember that $\overline{E}$ denotes the closure of E in $\mathbf{R}^n$ with respect to the standard Euclidean metric, as in Subsection 1.1.5. One can check that

$$\overline{E} = \{z \in \mathbf{R}^n : z \in E \text{ or } z \text{ is a limit point of } E\}. \tag{9.1.12}$$

Suppose for the moment that $x \in E$ is a limit point of E. In this case, it is well known that f is continuous at x if and only if the limit of $f(y)$ as $y \in E$ approaches x is equal to $f(x)$, so that

$$\lim_{\substack{y \to x \\ y \in E}} f(y) = f(x). \tag{9.1.13}$$

There are also analogues of this for upper and lower semicontinuity, in terms of upper and lower limts of real-valued functions, if one is familiar with that. More precisely, f is upper semicontinuous at x if and only if

$$\limsup_{\substack{y \to x \\ y \in E}} f(y) \leq f(x). \tag{9.1.14}$$

Similarly, f is lower semicontinuous at x if and only if

$$\liminf_{\substack{y \to x \\ y \in E}} f(y) \geq f(x). \tag{9.1.15}$$

9.2 More on semicontinuity

Let us continue with the same notation and hypotheses as in the previous section. However, one could also consider analogous notions for real-valued functions on arbitrary metric spaces, or topological spaces. This is related to the direct method in the calculus of variations, in which one seeks a function or other object that minimizes an expression of interest.

9.2.1 Semicontinuity and compactness

Suppose now that E is a nonempty compact subset of $\mathbf{R}^n$. If f is upper semicontinuous on E, then it is well known that

$$f \text{ attains its maximum on } E. \tag{9.2.1}$$

Similarly, if f is lower semicontinuous on E, then

$$f \text{ attains its minimum on } E. \tag{9.2.2}$$

This extends the *extreme value theorem*, as in Section 1.9.

More precisely, (9.2.1) implies that f has a finite upper bound on E, and (9.2.2) implies that f has a finite lower bound on E. It can be helpful to show these statements first, as appropriate. This means that f has a supremum or infimum over E in $\mathbf{R}$, as appropriate. One can show that the supremum or infimum is attained when f is upper or lower semicontinuous on E, as appropriate.

9.2.2 Combining semicontinuous functions

If t be a positive real number and f is upper or lower semicontinuous on E, then one can check that $t\, f$ has the same property.

Let g be another real-valued function on E, and suppose for the moment that f and g are both upper semicontinuous on E. One can check that

$$f + g \text{ is upper semicontinuous on } E. \tag{9.2.3}$$

One can also verify that

$$\max(f, g) \text{ and } \min(f, g) \text{ are upper semicontinuous on } E. \tag{9.2.4}$$

If $f, g \geq 0$ on E, then one can show that

$$f\, g \text{ is upper semicontinuous on } E. \tag{9.2.5}$$

Suppose for the moment again that f and g are lower semicontinuous on E. One can check that

$$f + g,\ \max(f, g), \text{ and } \min(f, g) \text{ are lower semincontinuous on } E. \tag{9.2.6}$$

If $f, g \geq 0$ on E, then one can verify that

$$f\, g \text{ is lower semicontinuous on } E. \tag{9.2.7}$$

If $f \geq 0$ on E and $f(x) = 0$ for some $x \in E$, then it is easy to see that f is lower semicontinuous at x. This means that f is upper semicontinuous at x if and only if f is continuous at x in this case.

9.2.3 Sequences of semicontinuous functions

Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of real-valued functions on E, and suppose for the moment that $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on E, as in Section 6.5. If

$$f_j \text{ is upper semicontinuous on } E \text{ for each } j, \tag{9.2.8}$$

then one can show that f is upper semicontinuous on E, using an argument like the one for the analogous statement for continuous functions. Similarly, if

$$f_j \text{ is lower semicontinuous on } E \text{ for each } j, \tag{9.2.9}$$

then f is lower semicontinuous on E.

Suppose now that $\{f_j\}_{j=1}^\infty$ converges to f pointwise on E. Suppose also for the moment that

$$f_{j+1}(x) \le f_j(x) \tag{9.2.10}$$

for each $x \in E$ and $j \ge 1$. If (9.2.8) holds, then it is not too difficult to show that f is upper semicontinuous on E. Suppose for the moment again that

$$f_j(x) \le f_{j+1}(x) \tag{9.2.11}$$

for each $x \in E$ and $j \ge 1$. If (9.2.9) holds, then one can verify that f is lower semicontinuous on E.

9.3 Lipschitz functions

Let n be a positive integer, let E be a nonempty subset of $\mathbf{R}^n$, and let f be a real or complex-valued function on E. We say that f is *Lipschitz* if there is a nonnegative real number C such that

$$|f(x) - f(y)| \le C\,|x - y| \tag{9.3.1}$$

for every $x, y \in E$. We may also say that f is Lipschitz with constant C on E in this case, to be more precise. Note that this holds with $C = 0$ if and only if f is constant on E. It is easy to see that Lipschitz functions are uniformly continuous.

9.3.1 Real-valued Lipschitz functions

If f is a real-valued function on E, then one can check that f is Lipschitz with constant C on E if and only if

$$f(x) \le f(y) + C\,|x - y| \tag{9.3.2}$$

for every $x, y \in E$. If $w \in \mathbf{R}^n$, then one can use this to check that

$$f_w(x) = |x - w| \tag{9.3.3}$$

is Lipschitz with constant $C = 1$ on $\mathbf{R}^n$.

9.3.2 Complex-valued Lipschitz functions

Let f be a complex-valued function on E, and for each $a \in \mathbf{C}$ with $|a| = 1$, let f_a be the real-valued function defined on E by

$$f_a(x) = \mathrm{Re}(a\,f(x)) \tag{9.3.4}$$

for every $x \in E$. Note that

$$f_a(x) - f_a(y) = \mathrm{Re}(a\,(f(x) - f(y))) \tag{9.3.5}$$

for every $x, y \in E$. It follows that

$$|f_a(x) - f_a(y)| \le |f(x) - f(y)| \tag{9.3.6}$$

for every $x, y \in E$, and in fact

$$|f(x) - f(y)| = \max\{|f_a(x) - f_a(y)| : a \in \mathbf{C},\ |a| = 1\}. \tag{9.3.7}$$

One can use this to check that f is Lipschitz with constant C on E if and only if

(9.3.8) f_a is Lipschitz with constant C on E for every $a \in \mathbf{C}$ with $|a| = 1$.

9.3.3 Lipschitz conditions and bounded derivatives

Suppose for the moment that $n = 1$, and that E is an open set in $\mathbf{R}$. If f is Lipschitz with constant C on E, and if f is differentiable at a point $x \in E$, then one can check that

$$|f'(x)| \le C. \tag{9.3.9}$$

Suppose now that E is an open interval in $\mathbf{R}$, which may be unbounded, such as $\mathbf{R}$ itself, or an open hal-line in $\mathbf{R}$. If f is a real-valued function on E that is differentiable at every point in E, and if (9.3.9) holds for some nonnegative real number C and every $x \in E$, then one can use the mean value theorem to get that

(9.3.10) f is Lipschitz with constant C on E.

This could also be obtained using the fundamental theorem of calculus when f is continuously differentiable on E.

9.3.4 Bounded derivatives on $\mathbf{R}^n$

Now let E be an open set in $\mathbf{R}^n$ for some n, and suppose for the moment that f is Lipschitz with constant C on E. If $x \in E$, $v \in \mathbf{R}^n$, and the directional derivative $D_v f(x)$ of f at x in the direction v exists, then one can verify that

$$|D_v f(x)| \le C\,|v|. \tag{9.3.11}$$

If f is differentiable in a suitable sense at x, then one can use this to show that

$$|\nabla f(x)| \le C. \tag{9.3.12}$$

It is well known that f is differentiable at every point in E in this sense when f is continuously differentiable on E.

Suppose that f is differentiable in this sense at every point in E, and that (9.3.12) holds for some $C \ge 0$ and every $x \in E$. If E is also convex, then one can show that f is Lipschitz with constant C on E. More precisely, if $x, y \in E$, then this can be obtained by considering

$$f((1-t)\,x + t\,y) \tag{9.3.13}$$

as a function of t on an appropriate open set in $\mathbf{R}$.

If f is Lipschitz on E, then it is well known that f is differentiable in the sense mentioned before at almost every point in E with respect to Lebesgue measure.

9.4 More on Lipschitz functions

Of course, if m and n are positive integers, and E is a nonempty subset of $\mathbf{R}^n$, then the notion of a Lipschitz mapping from E into $\mathbf{R}^m$ may be defined in essentially the same way as before. One can also consider Lipschitz mappings between arbitrary metric spaces. In particular, the contraction mapping theorem deals with the existence and uniqueness of fixed points for a mapping from a metric space into itself under suitable conditions. A contraction in this sense is a Lipschitz mapping with constant strictly less than one.

9.4.1 Distances to subsets of $\mathbf{R}^n$

Let n be a positive integer, and let A be a nonempty subset of $\mathbf{R}^n$. If $x \in \mathbf{R}^n$, then the distance from x to A with respect to the standar Euclidean metric on $\mathbf{R}^n$ is defined by

$$\operatorname{dist}(x, A) = \inf\{|x - a| : a \in \mathbf{R}^n\}, \tag{9.4.1}$$

where the left side is the infimum or greatest lower bound of the set of distances $|x - a|$ from x to elements a of A. Of course,

$$\operatorname{dist}(x, A) = 0 \tag{9.4.2}$$

when $x \in A$. More precisely, one can check that (9.4.2) holds if and only if x is an element of the closure $\overline{A}$ of A in $\mathbf{R}^n$, as in Subsection 1.1.5.

One can verify that

$$\operatorname{dist}(x, A) = \operatorname{dist}(x, \overline{A}) \tag{9.4.3}$$

for every $x \in \mathbf{R}^n$. If A is a closed set in $\mathbf{R}^n$, then it is not too difficult to show that the infimum on the right side of (9.4.1) is attained for each $x \in \mathbf{R}^n$. This uses the fact that closed and bounded sets in $\mathbf{R}^n$ are compact.

If $x, y \in \mathbf{R}^n$ and $a \in A$, then

$$\operatorname{dist}(x, A) \le |x - a| \le |x - y| + |y - a|. \tag{9.4.4}$$

Equivalently, this means that

$$\operatorname{dist}(x, A) - |x - y| \le |y - a|. \tag{9.4.5}$$

It follows that

$$\operatorname{dist}(x, A) - |x - y| \le \operatorname{dist}(y, A), \tag{9.4.6}$$

by the definition of $\operatorname{dist}(y, A)$. This is the same as saying that

$$\operatorname{dist}(x, A) \le \operatorname{dist}(y, A) + |x - y|. \tag{9.4.7}$$

This implies that

$$\mathrm{dist}(x, A) \text{ is Lipschitz with constant } C = 1 \text{ on } \mathbf{R}^n, \tag{9.4.8}$$

as before.

9.4.2 Some combinations of Lipschitz functions

Let E be a nonempty subset of $\mathbf{R}^n$ again, and suppose that f is a real or complex-valued function on E that is Lipschitz with constant $C(f)$. If t is a real or complex number, as appropriate, then it is easy to see that

$$t\, f \text{ is Lipschitz on } E \text{ with constant } |t|\, C(f). \tag{9.4.9}$$

Let g be another real or complex-valued function on E, and suppose that g is Lipschitz on E with constant $C(g)$. One can check that

$$f + g \text{ is Lipschitz on } E \text{ with constant } C(f) + C(g). \tag{9.4.10}$$

Let

$$\mathrm{Lip}(E, \mathbf{R}) \text{ and } \mathrm{Lip}(E, \mathbf{C}) \tag{9.4.11}$$

be the spaces of real and complex-valued Lipschitz functions on E, respectively. These are linear subspaces of the spaces $C(E, \mathbf{R})$ and $C(E, \mathbf{C})$ of continuous real and complex-valued functions on E, respectively.

Suppose for the moment that f and g are real-valued functions on E. One can verify that

$$\begin{gathered} \max(f, g) \text{ and } \min(f, g) \text{ are Lipschitz on } E \\ \text{with constant } \max(C(f), C(g)). \end{gathered} \tag{9.4.12}$$

9.4.3 Products of bounded Lipschitz functions

If $x, y \in E$, then

$$f(x)\, g(x) - f(y)\, g(y) = (f(x) - f(y))\, g(x) + f(y)\, (g(x) - g(y)), \tag{9.4.13}$$

so that

$$|f(x)\, g(x) - f(y)\, g(y)| \le |f(x) - f(y)|\, |g(x)| + |f(y)|\, |g(x) - g(y)|. \tag{9.4.14}$$

Suppose that f and g are also *bounded* on E, so that there are nonnegative real numbers $A(f)$, $A(g)$ such that

$$|f(z)| \le A(f) \tag{9.4.15}$$

and

$$|g(z)| \le A(g) \tag{9.4.16}$$

for every $z \in E$. Note that

(9.4.17) f and g are bounded on E when E is a bounded set in $\mathbf{R}^n$,

because f and g are Lipschitz on E, by hypothesis. Using (9.4.14), we get that

$$|f(x)\, g(x) - f(y)\, g(y)| \le (C(f)\, A(g) + A(f)\, C(g))\, |x - y| \tag{9.4.18}$$

for every $x, y \in E$. This shows that

(9.4.19) $f\, g$ is Lipschitz on E with constant $C(f)\, A(g) + A(f)\, C(g)$

under these conditions.

9.4.4 Sequences of Lipschitz functions

Let E be a nonempty subset of $\mathbf{R}^n$ again, and let $\{f_j\}_{j=1}^\infty$ be a sequence of real or complex-valued functions on E that converges pointwise to a real of complex-valued function f on E, as appropriate. Suppose that there is a nonnegative real number C such that

(9.4.20) f_j is Lipschitz on E with constant C

for each j. One can check that

(9.4.21) f is Lipschitz on E with constant C

in this case.

One can also verify that

(9.4.22) $\{f_j\}_{j=1}^\infty$ converges to f uniformly on bounded subsets of E

under these conditions. This uses the fact that bounded sets in $\mathbf{R}^n$ can be covered by finitely many balls of arbitrarily small radius.

9.5 Convex functions of one variable

Let I be an open interval in the real line, which may be unbounded, such as $\mathbf{R}$ itself, or an open half-line. A real-valued function f on I is said to be *convex* if

$$f(t\, a + (1 - t)\, b) \le t\, f(a) + (1 - t)\, f(b) \tag{9.5.1}$$

for every $a, b \in I$ and $t \in \mathbf{R}$ with $0 \le t \le 1$. Of course, this holds automatically when $t = 0$ or 1, and when $a = b$.

9.5.1 A reformulation of convexity

Suppose that $a < b$ and $0 < t < 1$, and put

$$c = t\,a + (1 - t)\,b, \tag{9.5.2}$$

so that $a < c < b$. It is easy to see that (9.5.1) is the same as saying that

$$t\,(f(c) - f(a)) \le (1 - t)\,(f(b) - f(c)). \tag{9.5.3}$$

Observe that $b - c = t\,(b - a)$, so that

$$t = \frac{b - c}{b - a},\ 1 - t = \frac{c - a}{b - a}. \tag{9.5.4}$$

Using this, one can check that (9.5.3) is equivalent to

$$\frac{f(c) - f(a)}{c - a} \le \frac{f(b) - f(c)}{b - c}. \tag{9.5.5}$$

More precisely, f is convex on I if and only if this holds for all $a, b, c \in I$ with $a < c < b$, because any such c may be expressed as in (9.5.2), as mentioned on p62 of [275].

9.5.2 A refinement of this reformulation

Suppose that f is convex on I, $w, x, y, z \in I$, and

$$w < x < y < z. \tag{9.5.6}$$

Under these conditions, we get that

$$\frac{f(x) - f(w)}{x - w} \le \frac{f(y) - f(x)}{y - x} \le \frac{f(z) - f(y)}{z - y}, \tag{9.5.7}$$

as in Exercise 23 on p101 of [274]. Indeed, each of these inequalities may be considered as an instance of (9.5.5).

9.5.3 Convexity of differentiable functions

Suppose for the moment that f is differentiable on I. If f is convex on I, then one can use (9.5.7) to get that

$$f' \text{ is monotonically increasing on } I. \tag{9.5.8}$$

Conversely, if (9.5.8) holds, then one can use the mean value theorem to get that (9.5.5) holds, so that f is convex on I. This corresponds to Exercise 14 on p115 of [274], and a remark on p63 of [275]. In particular, if f is twice differentiable on I, then it follows that f is convex on I if and only if $f'' \ge 0$ on I.

9.6 More on convex functions

Let us continue with the same notation and hypotheses as in the previous section. If f is convex on I, then Theorem 3.2 on p63 of [275] states that f is continuous on I. This also corresponds to parts of part (b) of Exercise 13.34 on p202 of [139] and Exercise 23 on p101 of [274], and we shall say more about this in a moment. Note that it is important here that I be an open interval, as in part (b) of Exercise 13.34 on p202 of [139], and as mentioned on p63 of [275].

9.6.1 Some continuity conditions

If f is convex on I, then we can use (9.5.7) when $w, x, y, z \in I$ and (9.5.6) holds to get that

$$\frac{f(x) - f(w)}{x - w} \le \frac{f(z) - f(y)}{z - y}. \tag{9.6.1}$$

This also holds when $x = y$, as in (9.5.5).

Similarly, if $u, v \in I$ and

$$u < v \le w, \tag{9.6.2}$$

then

$$\frac{f(v) - f(u)}{v - u} \le \frac{f(x) - f(w)}{x - w}. \tag{9.6.3}$$

It follows that f is Lipschitz on $[v, y]$, with constant

$$\max\Big(- \frac{f(v) - f(u)}{v - u}, \frac{f(z) - f(y)}{z - y}\Big). \tag{9.6.4}$$

Of course, this implies that f is continuous on I in particular.

This is related to part (b) of Exercise 17.37 on p272 of [139]. We shall say more about this in the next section.

9.6.2 Another characterization of convexity

Suppose for the moment that f is continuous on I. If

$$f\Big(\frac{a+b}{2}\Big) \le \frac{f(a) + f(b)}{2} \tag{9.6.5}$$

for all $a, b \in I$, then one can show that f is convex on I, as in Exercise 24 on p101 of [274], and Exercise 3 on p73 of [275].

Suppose for the moment again that f is upper semicontinuous on I. If (9.6.5) holds for all $a, b \in I$, then one can show that f is continuous on I.

9.6.3 Some combinations of convex functions

If f is convex on I and r is a nonnegative real number, then it is easy to see that

$$r\, f \text{ is convex on } I \tag{9.6.6}$$

too. If g is another convex function on I, then

$$f + g \text{ is convex on } I \tag{9.6.7}$$

as well.

In this case, one can also check that

$$\max(f, g) \text{ is convex on } I. \tag{9.6.8}$$

This corresponds to part of Exercise 1 on p73 of [275].

9.6.4 Sequences of convex functions

Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of real-valued functions on I that converges pointwise to a real-valued function f on I. If

$$f_j \text{ is convex on } I \text{ for each } j, \tag{9.6.9}$$

then one can verify that

$$f \text{ is convex on } I. \tag{9.6.10}$$

This is another part of Exrecise 1 on p73 of [275].

9.7 One-sided derivatives

Let I be a nonempty open set in the real line, and let f be a real-valued function on I. The *one-sided derivatives* of f at y are defined by

$$f'_+(y) = \lim_{z \to y+} \frac{f(z) - f(y)}{z - y} \tag{9.7.1}$$

and

$$f'_-(y) = \lim_{x \to y-} \frac{f(y) - f(x)}{y - x}, \tag{9.7.2}$$

when these limits exist. If f is differentiable at y, then these two limits exist, and

$$f_+(y) = f'_-(y) = f'(y). \tag{9.7.3}$$

Conversely, if the one-sided derivatives of f at y exist and are equal, then f is differentiable at y, with derivative as in (9.7.3).

9.7.1 Some more reformulations of convexity

Suppose from now on in this section that I is an open interval in $\mathbf{R}$, which may be unbounded, as before. Also let $a, b \in I$ and $t \in \mathbf{R}$ be given, with $a < b$ and $0 < t < 1$, and put $c = t\,a + (1 - t)\,b \in I$. The usual convexity condition (9.5.1) is the same as saying that

$$f(c) - f(a) \leq (1 - t)\,(f(b) - f(a)). \tag{9.7.4}$$

This is also the same as saying that

$$t\,(f(b) - f(a)) \le f(b) - f(c). \tag{9.7.5}$$

As before, we can express $1 - t$ in terms of a, b, and c to get that (9.7.4) is equivalent to

$$\frac{f(c) - f(a)}{c - a} \le \frac{f(b) - f(a)}{b - a}. \tag{9.7.6}$$

Similarly, (9.7.5) is equivalent to

$$\frac{f(b) - f(a)}{b - a} \le \frac{f(b) - f(c)}{b - c}. \tag{9.7.7}$$

It follows that f is convex on I if and only if (9.7.6) holds for every $a, b, c \in I$ with $a < c < b$. Similarly, f is convex on I if and only if (9.7.7) holds for every $a, b, c \in I$ with $a < c < b$.

9.7.2 One-sided derivatives and convexity

Suppose that f is convex on I. If $x, y, z \in I$ and $x < y < z$, then

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(z) - f(y)}{z - y}, \tag{9.7.8}$$

as in (9.5.5), and this could also be obtained from (9.7.6) and (9.7.7). Observe that

$$\frac{f(y) - f(x)}{y - x} \tag{9.7.9}$$

is monotonically increasing in x, because of (9.7.7). Similarly,

$$\frac{f(z) - f(y)}{z - y} \tag{9.7.10}$$

is monotonically increasing in z, because of (9.7.6).

One can use this to show that the one-sided derivatives of f at y exist, with

$$f'_-(y) = \sup\left\{\frac{f(y) - f(x)}{y - x} : x \in I,\, x < y\right\} \tag{9.7.11}$$

and

$$f'_+(y) = \inf\left\{\frac{f(z) - f(y)}{z - y} : z \in I,\, y < z\right\}. \tag{9.7.12}$$

We also have that

$$f'_-(y) \le f'_+(y), \tag{9.7.13}$$

because of (9.7.8). This corresponds to part of part (a) of Exercise 17.37 on p271 of [139].

If $y_1, y_2 \in I$ and $y_1 < y_2$, then

$$f'_+(y_1) \le \frac{f(y_2) - f(y_1)}{y_2 - y_1} \le f'_-(y_2). \tag{9.7.14}$$

This implies that f'_- and f'_+ are monotonically increasing on I, because of (9.7.13). This is another part of part (a) of Exercise 17.37 on p271 of [139].

9.7.3 Lipschitz conditions on closed subintervals

Let us continue to ask that f be convex on I, and let $v, y \in I$ be given, with $v < y$. One can check that f is Lipschitz on $[v, y]$, with constant

$$\max(-f'_-(v), f'_+(y)). \tag{9.7.15}$$

More precisely, this follows from the analogous statement in Subsection 9.6.1.

9.7.4 A more precise Lipschitz condition

Let $a, b \in I$ be given with $a < b$. In fact, we have that f is Lipschitz on $[a, b]$ with constant

$$\max(-f'_+(a), f_-(b)). \tag{9.7.16}$$

This corresponds to part (b) of Exercise 17.37 on p272 of [139].

To see this, one can first verify that f is Lipschitz on (a, b) with constant (9.7.16). One can use this to get that f is Lipschitz with the same constant on $[a, b]$, because f is continuous on $[a, b]$.

9.8 One-sided limits on R

Let I be an open interval in $\mathbf{R}$ again, which may be unbounded, and let α be a real-valued function on I. If $y \in I$, then the one-sided limits of α at y may be expressed as

$$\alpha(y+) = \lim_{z \to y+} \alpha(z) \tag{9.8.1}$$

and

$$\alpha(y-) = \lim_{x \to y-} \alpha(x), \tag{9.8.2}$$

when these limits exist. It is well known and easy to see that the limit

$$\lim_{w \to y} \alpha(w) \tag{9.8.3}$$

exists if and only if the one-sided limits (9.8.1) and (9.8.2) exist and are equal, in which case (9.8.3) is equal to the common value of (9.8.1) and (9.8.2). Of course, α is continuous at y if and only if the limit (9.8.3) exists and is equal to $\alpha(y)$. This means that α is continuous at y if and only if the one-sided limits (9.8.1) and (9.8.2) exist and are equal to $\alpha(y)$.

If (9.8.1) exists and

$$\alpha(y+) = \alpha(y), \tag{9.8.4}$$

then α is said to be *continuous at y on the right*. Similarly, if (9.8.2) exists and

$$\alpha(y-) = \alpha(y), \tag{9.8.5}$$

then α is said to be *continuous at y on the left*. Thus α is continuous at y if and only if α is continuous at y on both the right and the left.

9.8.1 Monotonically increasing functions on R

Suppose now that α is monotonically increasing on I, so that

$$\alpha(u) \le \alpha(v) \tag{9.8.6}$$

for all $u, v \in I$ with $u < v$. If $y \in I$, then it is well known that the one-sided limits (9.8.1) and (9.8.2) exist, with

$$\alpha(y+) = \inf\{\alpha(z) : z \in I,\, y < z\} \tag{9.8.7}$$

and

$$\alpha(y-) = \sup\{\alpha(x) : x \in I,\, x < y\}. \tag{9.8.8}$$

We also have that

$$\alpha(y-) \le \alpha(y) \le \alpha(y+). \tag{9.8.9}$$

If $y_1, y_2 \in I$ and $y_1 < y_2$, then

$$\alpha(y_1+) \le \alpha(y_2-). \tag{9.8.10}$$

It follows from (9.8.9) that α is continuous at y if and only if

$$\alpha(y-) = \alpha(y+). \tag{9.8.11}$$

Equivalently, α is discontinuous at y if and only if

$$\alpha(y-) < \alpha(y+). \tag{9.8.12}$$

It is well known that this can happen at only finitely or countably many elements of I.

9.8.2 Monotonicity and semicontinuity

Let us continue to ask that α be monotomically increasing on I. If $y \in I$, then one can check that α is upper semicontinuous at y if and only if α is continuous at y on the right. Similarly, one can verify that α is lower semicontinuous at y if and only if α is continuous at y on the left.

9.8.3 Convex functions on I

Let f be a convex real-valued function on I, and remember that f'_+, f'_- are monotonically increasing on I, as in Subsection 9.7.2. Thus the one-sided limits of f'_+ and f'_- exist at every point in I, as before. If $y_1 \in I$, then one can check that

$$f'_+(y_1) \le f'_-(y_1+), \tag{9.8.13}$$

using (9.7.14). Similarly, if $y_2 \in I$, then one can use (9.7.14) to verify that

$$f'_+(y_2-) \le f'_-(y_2). \tag{9.8.14}$$

Remember that $f'_- \le f'_+$ on I, as in (9.7.13). If f'_- is right continuous at $y \in I$, then we get that

$$f'_+(y) = f'_-(y), \tag{9.8.15}$$

because of (9.8.13). Similarly, (9.8.15) holds when f'_+ is left continuous at y, because of (9.8.14). In both cases, it follows that f is differentiable at y, as in the previous section.

This implies that f is differentiable at all but finitely or countably many elements of I, as in part (a) of Exercise 17.37 on p271 of [139].

9.9 Some related inequalities

If p is a real number with $p \ge 1$, then it is easy to see that

$$x^p \text{ is convex on } \mathbf{R}_+, \tag{9.9.1}$$

because its second derivative is positive when $p > 1$. Of course, x^p is also defined and continuous at $x = 0$, and one can check that it satisfies the condition in the definition of convexity on the set of all nonnegative real numbers.

One can verify that

$$|x| \text{ is convex on } \mathbf{R} \tag{9.9.2}$$

as well. Using this and the remarks in the preceding paragraph, one can check that

$$|x|^p \text{ is convex on } \mathbf{R} \text{ when } p \ge 1. \tag{9.9.3}$$

9.9.1 Another reformulation of convexity

Let I be an open interval in $\mathbf{R}$, which may be unbounded, and let f be a real-valued function on I. Suppose for the moment that f is convex on I, and let $c \in I$ be given. Remember that

$$f'_-(c) \le f'_+(c), \tag{9.9.4}$$

as in Subsection 9.7.2. Let α be a real number such that

$$f'_-(c) \le \alpha \le f'_+(c). \tag{9.9.5}$$

Under these conditions, one can check that

$$f(x) \ge f(c) + \alpha\,(x - c) \tag{9.9.6}$$

for every $x \in I$. More precisely,

$$f(x) \ge f(c) + f'_+(c)\,(x - c) \ge f(c) + \alpha\,(x - c) \tag{9.9.7}$$

when $x \ge c$, and

$$f(x) \ge f(c) + f'_-(c)\,(x - c) \ge f(c) + \alpha\,(x - c) \tag{9.9.8}$$

when $x \le c$. Of course, this is trivial when $x = c$. If $x \ne c$, then these inequalities may be obtained from the characterizations of $f'_-(c)$ and $f'_+(c)$ in Subsection 9.7.2.

This corresponds to part (c) of Exercise 13.34 on p202 of [139]. This fact is also mentioned in the proof of Theorem 3.3 on p63 of [275]. Note that one could get α more directly using the characterization of convexity in Subsection 9.5.1.

Conversely, one can check that f is convex on I if for every $c \in I$ there is an $\alpha \in \mathbf{R}$ such that (9.9.6) holds for every $x \in I$.

9.9.2 Jensen's inequalities

Suppose that f is convex on I again, and let $a_1, \ldots, a_n$ be finitely many elements of I. If $t_1, \ldots, t_n$ are positive real numbers, then it is easy to see that

$$\min(a_1, \ldots, a_n) \le \frac{t_1\, a_1 + \cdots + t_n\, a_n}{t_1 + \cdots t_n} \le \max(a_1, \ldots, a_n). \tag{9.9.9}$$

This implies that

$$\frac{t_1\, a_1 + \cdots + t_n\, a_n}{t_1 + \cdots + t_n} \in I. \tag{9.9.10}$$

One can check that

$$f\Big(\frac{t_1\, a_1 + \cdots + t_n\, a_n}{t_1 + \cdots + t_n}\Big) \le \frac{t_1\, f(a_1) + \cdots + t_n\, f(a_n)}{t_1 + \cdots + t_n}, \tag{9.9.11}$$

as in part (a) of Exercise 13.34 on p202 of [139]. This is a version of *Jensen's inequality*. Sometimes this may be stated with the additional condition that

$$\sum_{j=1}^{n} t_j = 1, \tag{9.9.12}$$

and it is easy to reduce to this case. If $n = 2$, then this is essentially the same as the definition of convexity in Section 9.5.

One can get (9.9.11) for all n using induction, as mentioned in [139]. One can also use the characterization of convexity in the previous subsection.

There is a version of Jensen's inequality for averages defined using integrals, as in part (d) of Exercise 13.34 of [139], and Theorem 3.3 on p63 of [275]. This can be obtained by approximating an integral average by averages of finitely many real numbers, and using (9.9.11). One can use the characterization of convexity in the previous subsection more directly, as in [139, 275].

9.10 Lipschitz functions of order $\alpha > 0$

Let n be a positive integer, let α be a positive real number, let E be a nonempty subset of $\mathbf{R}^n$, and let f be a real or complex-valued function on E. We say that f is *Lipschitz of order* α if there is a nonnegative real number C_α such that

$$|f(x) - f(y)| \le C_\alpha\, |x - y|^\alpha \tag{9.10.1}$$

for every $x, y \in E$. This is the same as the definition of a Lipschitz function in Section 9.3 when $\alpha = 1$.

As before, we may say that f is Lipschitz of order α with constant C_α when (9.10.1) holds, to be more precise. One may also say that f is *Hölder continuous of order α with constant C_α* in this case.

Observe that (9.10.1) holds with $C_\alpha = 0$ if and only if f is constant on E. If f is Lipschitz or Hölder continuous on E of any positive order, then

$$f \text{ is uniformly continuous on } E. \tag{9.10.2}$$

Suppose for the moment that f is a complex-valued function on E, and for each $a \in \mathbf{C}$ with $|a| = 1$, let f_a be the real-valued function defined on E by

$$f_a(x) = \mathrm{Re}(a\, f(x)), \tag{9.10.3}$$

as in Subsection 9.3.2. One can check that f is Lipschitz of order α with constant C_α on E if and only if

$$\begin{aligned} &f_a \text{ is Lipschitz of order } \alpha \text{ with constant } C_\alpha \text{ on } E \\ &\text{for every } a \in \mathbf{C} \text{ with } |a| = 1, \end{aligned} \tag{9.10.4}$$

as before.

9.10.1 The case where $\alpha > 1$

Suppose for the moment in this subsection that $\alpha > 1$. If a real or complex-valued function a on an interval I in the real line is Lipschitz of order α, then one can show that

$$a \text{ is constant on } I. \tag{9.10.5}$$

This corresponds to part (a) of Exercise 17.31 on p270 of [139]. One way to do this is to show that the derivative of a is equal to 0 at every point in I when I has more than one element.

Suppose for the moment that E is a nonempty convex subset of $\mathbf{R}^n$. If f is Lipschitz of order $\alpha > 1$ on E, then

$$f \text{ is constant on } E. \tag{9.10.6}$$

Indeed, if $x, y \in E$, then it is easy to see that

$$f((1-t)\,x + t\,y) \tag{9.10.7}$$

is Lipschitz of order α as a function of $t \in [0, 1]$. This implies that (9.10.7) is constant as a function of t on $[0, 1]$, as in the preceding paragraph.

Let U be a nonempty open set in $\mathbf{R}^n$. If a real or complex-valued function b on U is Lipschitz of order $\alpha > 1$, then

$$b \text{ is locally constant on } U, \tag{9.10.8}$$

as in Subsection 1.8.4. In fact, b is constant on any convex set contained in U, as in the previous paragraph. Remember that open balls in $\mathbf{R}^n$ are convex, as mentioned in Section 1.8. Alternatively, it is easy to see that the partial derivatives of b are equal to 0 at every point in U under these conditions.

9.10.2 A helpful inequality

Let a be a positive real number with $a \le 1$. If r, t are nonnegative real numbers, then it is well known that

$$(r+t)^a \le r^a + t^a. \tag{9.10.9}$$

To see this, observe first that

$$\max(r,t) \le (r^a + t^a)^{1/a}. \tag{9.10.10}$$

Using this, we get that

$$r + t \le \max(r,t)^{1-a}\,(r^a + t^a) \le (r^a + t^a)^{((1-a)/a)+1} = (r^a + t^a)^{1/a}. \tag{9.10.11}$$

This is equivalent to (9.10.9).

9.10.3 Real-valued functions on E

Let α be any positive real number again, and suppose that f is a real-valued function on E. One can check that f is Lipschitz of order α with constant C_α on E if and only if

$$f(x) \le f(y) + C_\alpha\,|x-y|^\alpha \tag{9.10.12}$$

for every $x, y \in E$. This was mentioned in Subsection 9.3.1 when $\alpha = 1$, and the same argument works for any $\alpha > 0$.

Suppose now that $f \ge 0$ on E, and that f is Lipschitz of order β with constant C_β on E for some positive real number β. If $0 < a \le 1$ and $x, y \in E$, then

$$f(x)^a \le (f(y) + C_\beta\,|x-y|^\beta)^a \le f(y)^a + C_\beta^a\,|x-y|^{a\,\beta}, \tag{9.10.13}$$

where the first step is as in (9.10.12), and the second step is as in (9.10.9). This implies that

$$f^a \text{ is Lipschitz of order } a\,\beta \text{ with constant } C_\beta^a \text{ on } E. \tag{9.10.14}$$

If A is a nonempty subset of $\mathbf{R}^n$, then $\operatorname{dist}(x, A)$ is Lipschitz of order one with constant $C = 1$, as in Subsection 9.4.1. If $0 < a \le 1$, then it follows that

$$\operatorname{dist}(x,A)^a \text{ is Lipschitz of order } a \text{ with constant } C_a = 1 \text{ on } \mathbf{R}^n, \tag{9.10.15}$$

as in (9.10.14).

9.11 More on these Lipschitz conditions

Let n be a positive integer, let E be a nonempty subset of $\mathbf{R}^n$, and let f be a real or complex-valued function on E again. Suppose for the moment that f is *bounded* on E, so that there is a nonnegative real number A such that

$$|f(x)| \le A \tag{9.11.1}$$

for every $x \in E$. Alternatively, the boundedness of f on E means that there is a nonnegative real number B such that

$$|f(x) - f(y)| \le B \tag{9.11.2}$$

for every $x, y \in E$. More precisely, (9.11.1) implies that (9.11.2) holds, with

$$B = 2\,A. \tag{9.11.3}$$

Conversely, if (9.11.2) holds, and if $w \in E$, then (9.11.1) holds with

$$A = B + |f(w)|. \tag{9.11.4}$$

Let a be a positive real number with $a < 1$. Observe that

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(y)|^{1-a}\,|f(x) - f(y)|^a \\ &\le B^{1-a}\,|f(x) - f(x)|^a \end{aligned} \tag{9.11.5}$$

for every $x, y \in E$.

Let β be a positive real number, and suppose that f is Lipschitz of order β with constant C_β on E. Combining this with (9.11.5), we get that

$$|f(x) - f(y)| \le B^{1-a}\,C_\beta^a\,|x - y|^{a\,\beta} \tag{9.11.6}$$

for every $x, y \in E$. This shows that

(9.11.7) f is Lipschitz of order $a\,\beta$ with constant $B^{1-a}\,C_\beta^a$ on E.

Of course, $a\,\beta$ may be any positive real number strictly less than β here.

9.11.1 Diameters of bounded sets

Suppose for the moment that

(9.11.8) E is a bounded set in $\mathbf{R}^n$.

It is easy to see that this is the same as saying that there is an upper bound for the set of distances between elements of E. In this case, the *diameter* of E is defined to be the least upper bound or supremum of the set of distances between elements of E, i.e.,

$$\operatorname{diam} E = \sup\{|x - y| : x, y \in E\}. \tag{9.11.9}$$

This is a nonnegative real number that is equal to 0 exactly when E has only one element.

Let α and β be positive real numbers with $\alpha < \beta$, and suppose that f is Lipschitz of order β with constant C_β on E again. If $x, y \in E$, then

$$|f(x) - f(y)| \le C_\beta\,|x - y|^\beta \le C_\beta\,(\operatorname{diam} E)^{\beta-\alpha}\,|x - y|^\alpha. \tag{9.11.10}$$

This means that

(9.11.11) f is Lipschitz of order α with constant $C_\beta \, (\operatorname{diam} E)^{\beta-\alpha}$ on E.

In this case, it is easy to see that f is bounded on E, and more precisely that (9.11.2) holds, with

$$B = C_\beta \, (\operatorname{diam} E)^\beta. \tag{9.11.12}$$

If we take $a = \alpha/\beta$, then one can check that (9.11.10) also follows from (9.11.6).

9.11.2 Combining Lipschitz functions

Let E be any nonempty subset of $\mathbf{R}^n$ again, and let f be a real or complex-valued function on E that is Lipschitz if order α on E with constant $C_\alpha(f)$ for some positive real number α. If t is a real or complex number, as appropriate, then it is easy to see that

(9.11.13) $t\,f$ is Lipschitz of order α on E with constant $|t|\,C_\alpha(f)$,

as in Subsection 9.4.2.

Let g be another real or complex-valued function on E, and suppose that g is Lipschitz of order α on E with constant $C_\alpha(g)$. One can check that

(9.11.14) $f + g$ is Lipschitz of order α on E with constant $C_\alpha(f) + C_\alpha(g)$,

as before.

Let

$$\mathrm{Lip}_\alpha(E, \mathbf{R}) \text{ and } \mathrm{Lip}_\alpha(E, \mathbf{C}) \tag{9.11.15}$$

be the spaces of real and complex-valued functions on E that are Lipschitz of order α on E, respectively. These are linear subspaces of $C(E, \mathbf{R})$ and $C(E, \mathbf{C})$, respectively, as before. If E is bounded and β is a real number with $\alpha < \beta$, then

$$\mathrm{Lip}_\beta(E, \mathbf{R}) \subseteq \mathrm{Lip}_\alpha(E, \mathbf{R}) \text{ and } \mathrm{Lip}_\beta(E, \mathbf{C}) \subseteq \mathrm{Lip}_\alpha(E, \mathbf{C}), \tag{9.11.16}$$

as in the previous subsection.

If f and g are real valued on E, then one can verify that

(9.11.17) $\max(f, g)$ and $\min(f, g)$ are Lipschitz of order α on E with constant $\max(C_\alpha(f), C_\alpha(g))$,

as before.

Suppose that f and g are also bounded on E, so that there are nonnegative real numbers $A(f)$ and $A(g)$ such that

$$|f| \le A(f) \text{ and } |g| \le A(g) \tag{9.11.18}$$

on E. One can check that

(9.11.19) $f\,g$ is Lipschitz of order α on E with constant $C_\alpha(f)\,A(g) + A(f)\,C_\alpha(g)$

under these conditions, as in Subsection 9.4.3.

9.11.3 Lipschitz conditions and sequences

Let $\{f_j\}_{j=1}^\infty$ be a sequence of real or complex-valued functions on E that converges pointwise to a real or complex-valued function f on E, as appropriate. Suppose that there is a nonnegative real number C_α such that

(9.11.20) $\quad f_j$ is Lipschitz of order α on E with constant C_α

for each j. One can check that

(9.11.21) $\quad f$ is Lipschitz of order α on E with constant C_α,

as in Subsection 9.4.4.

One can also verify that $\{f_j\}_{j=1}^\infty$ converges to f uniformly on bounded subsets of E, as before.

9.12 Convex functions of several variables

Let n be a positive integer, and let U be a nonempty convex open subset of $\mathbf{R}^n$. Also let f be a real-valued function on U. We say that f is *convex* on U if for every $x, y \in U$ and $t \in \mathbf{R}$ with $0 \le t \le 1$ we have that

$$f(t\,x + (1-t)\,y) \le t\,f(x) + (1-t)\,f(y). \tag{9.12.1}$$

This is the same as the definition in Section 9.5 when $n = 1$. Note that (9.12.1) holds automatically when $t = 0$ or 1, and when $x = y$, as before.

It is easy to see that the standard Euclidean norm is convex as a real-valued function on $\mathbf{R}^n$. In fact, every norm on $\mathbf{R}^n$ is convex as a real-valued function on $\mathbf{R}^n$, and we shall say more about that in Section A.6.

9.12.1 Convexity on intersections with lines

Alternatively, let $z, v \in \mathbf{R}^n$ be given, and put

$$U_{z,v} = \{t \in \mathbf{R} : z + t\,v \in U\}. \tag{9.12.2}$$

It is easy to see that

(9.12.3) $\quad U_{z,v}$ is a convex open subset of $\mathbf{R}$.

This means that $U_{z,v}$ is an open interval in the real line, which may be unbounded, or the empty set. Put

$$F_{z,v}(t) = f(z + t\,v) \tag{9.12.4}$$

for every $t \in U_{z,v}$. One can check that f is convex on U if and only if

(9.12.5) $\quad F_{z,v}$ is convex on $U_{z,v}$

for every $z, v \in \mathbf{R}^n$. This is considered to hold vacuously when $U_{z,v} = \emptyset$. If $z \in U$ and $v = 0$, then $U_{z,v} = \mathbf{R}$, and (9.12.5) holds trivially, because $F_{z,v}$ is a constant function.

9.12.2 Convexity and distance functions

Let A be a nonempty subset of $\mathbf{R}^n$, and let $\mathrm{dist}(x, A)$ be as in Subsection 9.4.1. If A is a convex set, then it is well known that

$$\mathrm{dist}(x, A) \text{ is a convex function on } \mathbf{R}^n. \tag{9.12.6}$$

To see this, let $x \in \mathbf{R}^n$ and $t \in \mathbf{R}^n$ be given, with $0 \le t \le 1$, and let us show that

$$\mathrm{dist}(t\,x + (1-t)\,y, A) \le t\ \mathrm{dist}(x, A) + (1-t)\ \mathrm{dist}(y, A). \tag{9.12.7}$$

To do this, let $a, b \in A$ be given, and note that

$$t\,a + (1-t)\,b \in A, \tag{9.12.8}$$

because A is convex, by hypothesis. This implies that

$$\mathrm{dist}(t\,x + (1-t)\,y, A) \le |t\,x + (1-t)\,y - t\,a - (1-t)\,b|. \tag{9.12.9}$$

It follows that

$$\begin{aligned} \mathrm{dist}(t\,x + (1-t)\,y, A) &\le |t\,(x-a) + (1-t)\,(y-b)| \\ &\le t\,|x-a| + (1-t)\,|y-b|. \end{aligned} \tag{9.12.10}$$

One can use this to get (9.12.7), because $a, b \in A$ are arbitrary.

9.12.3 Convexity and second derivatives

Suppose for the moment in this subsection that f is twice continuously differentiable on U. If $z, v \in \mathbf{R}^n$, then it follows that $F_{z,v}$ is twice continuously differentiable on $U_{z,v}$, which is considered to hold vacuously when $U_{z,v} = \emptyset$. Under these conditions, (9.12.5) holds if and only if

$$F''_{z,v} \ge 0 \text{ on } U_{z,v}, \tag{9.12.11}$$

as in Subsection 9.5.3. If $U_{z,v} = \emptyset$, then (9.12.11) is considered to hold vacuously, as usual. Thus f is convex on U if and only if (9.12.11) holds for every $z, v \in \mathbf{R}^n$.

Observe that

$$F'_{z,v}(t) = \sum_{l=1}^{n} (\partial_l f)(z + t\,v)\,v_l \tag{9.12.12}$$

for every $z, v \in \mathbf{R}^n$ and $t \in U_{z,v}$. This is the same as the directional derivative of f at $z + t\,v$ in the direction v, as in Subsection 1.3.2. Similarly,

$$F''_{z,v}(t) = \sum_{j=1}^{n} \sum_{l=1}^{n} (\partial_j \partial_l f)(z + tv) v_j v_l \tag{9.12.13}$$

for every $z, v \in \mathbf{R}^n$ and $t \in U_{z,v}$. One can use this to get that f is convex on U if and only if

$$\sum_{j=1}^{n} \sum_{l=1}^{n} (\partial_j \partial_l f)(x)\,v_j\,v_l \ge 0 \tag{9.12.14}$$

for every $x \in U$ and $v \in \mathbf{R}^n$.

9.12.4 Some remarks about convex functions

If f is convex on U and r is a nonnegative real number, then it is easy to see that

$$r\,f \text{ is convex on } U. \tag{9.12.15}$$

If g is another real-valued function on U that is convex on U, then

$$f + g \text{ is convex on } U. \tag{9.12.16}$$

We also have that

$$\max(f, g) \text{ is convex on } U \tag{9.12.17}$$

in this case. These statements correspond to those in Subsection 9.6.3 when $n = 1$.

Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of real-valued functions on U that converges pointwise to a real-valued function f on U. If

$$f_j \text{ is convex on } U \text{ for each } j, \tag{9.12.18}$$

then

$$f \text{ is convex on } U. \tag{9.12.19}$$

This is the same as in Subsection 9.6.4 when $n = 1$.

9.13 Some subsets of $\mathbf{R}^n$

Let n be a positive integer, and let A be a nonempty subset of $\mathbf{R}^n$. Remember that $\mathrm{dist}(x, A)$ is defined for $x \in \mathbf{R}^n$ as in Subsection 9.4.1. If r is a positive real number, then put

$$A_r = \{x \in \mathbf{R}^n : \mathrm{dist}(x, A) < r\}. \tag{9.13.1}$$

One can check that this is the same as

$$\bigcup_{a \in A} B(a, r). \tag{9.13.2}$$

Note that this is an open set in $\mathbf{R}^n$ that contains A.

Similarly, if r is a nonnegative real number, then put

$$A^r = \{x \in \mathbf{R}^n : \mathrm{dist}(x, A) \le r\}. \tag{9.13.3}$$

This is the same as the closure $\overline{A}$ of A in $\mathbf{R}^n$ when $r = 0$, as in Subsection 9.4.1. One can verify that

$$A^r \text{ is a closed set in } \mathbf{R}^n \tag{9.13.4}$$

for each $r \ge 0$, because $\mathrm{dist}(x, A)$ is continuous on $\mathbf{R}^n$. If $0 < r < t$, then

$$A_r \subseteq A^r \subseteq A_t. \tag{9.13.5}$$

It is easy to see that

$$\bigcup_{a \in A} \overline{B}(a, r) \subseteq A^r \tag{9.13.6}$$

for each $r \geq 0$, directly from the definition of A^r. If A is a closed set in $\mathbf{R}^n$, then

$$\bigcup_{a \in A} \overline{B}(a, r) = A^r. \tag{9.13.7}$$

This follows from the fact that the infimum in the definition of $\operatorname{dist}(x, A)$ is attained in this case, as mentioned in Section 9.4.1.

9.13.1 Some remarks about bounded sets

Let E_1, E_2 be bounded nonempty subsets of $\mathbf{R}^n$. If

$$E_1 \subseteq E_2, \tag{9.13.8}$$

then it is easy to see that

$$\operatorname{diam} E_1 \leq \operatorname{diam} E_2, \tag{9.13.9}$$

by the definition of the diameter, as in Subsection 9.11.1.

If E is a bounded set in $\mathbf{R}^n$, then one can check that the closure $\overline{E}$ of E is bounded in $\mathbf{R}^n$ too. More precisely, if $E \neq \emptyset$, then

$$\operatorname{diam} \overline{E} = \operatorname{diam} E. \tag{9.13.10}$$

Indeed,

$$\operatorname{diam} E \leq \operatorname{diam} \overline{E}, \tag{9.13.11}$$

because $E \subseteq \overline{E}$, as in (9.13.9). It is not too difficult to show that

$$\operatorname{diam} \overline{E} \leq \operatorname{diam} E, \tag{9.13.12}$$

using the definition of $\overline{E}$, as in Section 1.1.5.

9.13.2 Bounded sets A

If A is a bounded set in $\mathbf{R}^n$, then it is easy to see that

$$A^r \text{ is bounded} \tag{9.13.13}$$

for each $r \geq 0$. More precisely, one can check that

$$\operatorname{diam} A^r \leq \operatorname{diam} A + 2\, r \tag{9.13.14}$$

for each $r \geq 0$. If A is also a closed set in $\mathbf{R}^n$, then this can be obtained from (9.13.7).

If A is not asked to be a closed set, then one can verify (9.13.14) directly, with a slightly more complicated argument. Alternatively, one can reduce to the case of closed sets, using (9.13.10). Note that (9.13.12) is the same as (9.13.14) when $r = 0$.

As another approach, one can check that

$$\operatorname{diam} A_t \le \operatorname{diam} A + 2\,t \tag{9.13.15}$$

for every $t > 0$, using the description of A_t as in (9.13.2). One can use this to get (9.13.14), by considering $t > r$.

9.13.3 Convex sets A

If A is a convex set in $\mathbf{R}^n$, then one can check that

$$A_r \text{ is convex} \tag{9.13.16}$$

for every $r > 0$, and that

$$A^r \text{ is convex} \tag{9.13.17}$$

for every $r \ge 0$. This uses the fact that $\operatorname{dist}(x, A)$ is a convex function on $\mathbf{R}^n$, as in Subsection 9.12.2.

9.14 Some local Lipschitz conditions

Let n be a positive integer, let α be a positive real number, and let E be a nonempty subset of $\mathbf{R}^n$. Also let f be a real or complex-valued function on E, let r be another positive real number, and let C_α be a nonnegative real number. Let us say that f is *Lipschitz of order α on E at the scale r with constant C_α* if

$$|f(x) - f(y)| \le C_\alpha\, |x - y|^\alpha \tag{9.14.1}$$

for every $x, y \in E$ with

$$|x - y| \le r. \tag{9.14.2}$$

Note that f is Lipschitz of order α on E with constant C_α if and only if this condition holds for all $r > 0$. If this condition holds for some $r > 0$, then f is uniformly continuous on E.

Let β be a positive real number, let C_β be a nonnegative real number, and suppose that f is Lipschitz of order β on E at the scale r with constant C_β. If $\alpha < \beta$ and $x, y \in E$ satisfy (9.14.2), then

$$|f(x) - f(y)| \le C_\beta\, |x - y|^\beta \le C_\beta\, r^{\beta - \alpha}\, |x - y|^\alpha. \tag{9.14.3}$$

This means that f is Lipschitz of order α on E at the scale r, with constant

$$C_\beta\, r^{\beta - \alpha}. \tag{9.14.4}$$

If f is Lipschitz of order α on E at the scale r, and if f is bounded on E, then it is easy to see that f is Lipschitz of order α on E.

If E is a bounded set in $\mathbf{R}^n$, then it is well known and not too difficult to show that E can be covered by finitely many sets of arbitrarily small diameter. One can use this to show that f is bounded on E when f is uniformly continuous on E. In particular, this means that f is bounded on E when f is Lipschitz of order α on E at the scale r. If E is compact, then any continuous function on E is bounded.

9.14.1 Local Lipschitz conditions along subsets

Let A be a nonempty subset of E. Let us say that f is Lipschitz of order α at the scale r with constant C_α *along* A if (9.14.1) holds for every $x \in A$ and $y \in E$ that satisfy (9.14.2). Of course, if $A = E$, then this is the same as saying that f is Lipschitz of order α at the scale r with constant C_α on E. Otherwise, this property implies that the restriction of f to A is Lipschitz of order α at the scale r with constant C_α on A. Note that this property also implies that f is continuous at every point in A, as a function on E.

Let A^r be as in (9.13.3), and consider the restriction of f to

$$A^r \cap E. \tag{9.14.5}$$

If the restriction of f to (9.14.5) is Lipschitz of order α at the scale r with constant C_α, then f is Lipschitz of order α at the scale r with constant C_α along A. More precisely, if $x \in A$ and $y \in E$ satisfy (9.14.2), then

$$\mathrm{dist}(y, A) \leq r, \tag{9.14.6}$$

so that

$$y \in A^r \cap E. \tag{9.14.7}$$

If f is Lipschitz of order $\beta > \alpha$ at the scale r with constant $C_\beta \geq 0$ along A, then f is Lipschitz of order α at the scale r along A with constant as in (9.14.4), for the same reasons as before.

9.14.2 Functions on open sets

Let U be a nonempty open subset of $\mathbf{R}^n$, and let f be a real or complex-valued function on U. Let us say that f is *locally Lipschitz of order α on U* if for every nonempty compact set $K \subseteq \mathbf{R}^n$ with

$$K \subseteq U, \tag{9.14.8}$$

we have that

(9.14.9) the restriction of f to K is Lipschitz of order α on K.

If K is a compact subset of $\mathbf{R}^n$ that satisfies (9.14.8), then there is a positive real number $r = r(U, K)$ such that

$$K^r \subseteq U, \tag{9.14.10}$$

where K^r is as in (9.13.3). This follows from the result mentioned at the beginning of Section 1.13. Remember that K^r is a closed set in $\mathbf{R}^n$, as in (9.13.4), and that K^r is a bounded set, as in (9.13.13). This means that K^r is a compact set in $\mathbf{R}^n$, as mentioned in Section 1.9.

If f is locally Lipschitz of order α on U, then it follows that

$$\text{the restriction of } f \text{ to } K^r \text{ is Lipschitz of order } \alpha \text{ on } K^r. \tag{9.14.11}$$

This implies that

$$f \text{ is Lipschitz of order } \alpha \text{ at the scale } r \text{ along } K, \tag{9.14.12}$$

as in the previous subsection. This is essentially the same type of local Lipschitz condition as mentioned in the discussion of "Function Spaces" in Section A of Chapter 0 of [87].

Let

$$C^{0,\alpha}(U,\mathbf{R}) = C^{0,\alpha}_{loc}(U,\mathbf{R}) \text{ and } C^{0,\alpha}(U,\mathbf{C}) = C^{0,\alpha}_{loc}(U,\mathbf{C}) \tag{9.14.13}$$

be the spaces of real and complex-valued functions on U that are locally Lipschitz of order α, respectively. These are linear subspaces of the spaces of all continuous real and complex-valued functions on U, respectively.

If f is locally Lipschitz of order α on U and $\alpha > 1$, then one can check that f is locally constant on U, as in Subsection 1.8.4. This uses the remarks in Subsection 9.10.1.

If f is locally Lipschitz of order $\beta > \alpha$ on U, then f is locally Lipschitz of order α on U. This follows from a remark in Subsection 9.11.1, because compact sets are bounded.

If f is continuously differentiable on U, then f is locally Lipschitz of order one on U. More precisely, let K be a compact subset of $\mathbf{R}^n$ that satisfies (9.14.8), and let r be a positive real number such that (9.14.10) holds. Under these condition, f is Lipschitz of order one at the scale r along K, with constant

$$\sup\{|\nabla f(x)| : x \in K^r\}. \tag{9.14.14}$$

This can be obtained using the remarks in Subsection 9.3.4.

Let k be a positive integer, and let

$$C^{k,\alpha}(U,\mathbf{R}) = C^{k,\alpha}_{loc}(U,\mathbf{R}) \text{ and } C^{k,\alpha}(U,\mathbf{C}) = C^{k,\alpha}_{loc}(U,\mathbf{C}) \tag{9.14.15}$$

be the spaces of k-times continuously-differentiable real and complex-valued functions on U, respectively, whose derivatives of order k are locally Lipschitz of order α on U. These are linear subspaces of the spaces of all k-times continuously-differentiable real and complex-valued functions on U, respectively.

9.15 Some remarks about convexity

Let n be a positive integer, and let U be a convex set in $\mathbf{R}^n$. Also let

$$a(1), \dots, a(m) \tag{9.15.1}$$

be finitely many elements of U, and let $t_1, \ldots, t_m$ be positive real numbers. One can check that

$$\sum_{l=1}^{m} \frac{t_l}{t_1 + \cdots + t_m} \, a(l) \in U, \tag{9.15.2}$$

using induction on m. More precisely, this works when $t_1, \ldots, t_m$ are nonnegative real numbers, at least one of which is positive. Sometimes one may ask that

$$\sum_{l=1}^{m} t_l = 1, \tag{9.15.3}$$

and it is easy to reduce to this case.

The *convex hull* of (9.15.1) may be denoted

$$\mathrm{conv}(a(1), \ldots, a(m)) \tag{9.15.4}$$

and is the subset of $\mathbf{R}^n$ consisting of points of the form

$$\sum_{l=1}^{m} t_l \, a(l), \tag{9.15.5}$$

where $t_1, \ldots, t_m$ are nonnegative real numbers that satisfy (9.15.3). One can verify that

$$\mathrm{conv}(a(1), \ldots, a(m)) \text{ is a convex set in } \mathbf{R}^n. \tag{9.15.6}$$

Note that

$$\mathrm{conv}(a(1), \ldots, a(n)) \subseteq U, \tag{9.15.7}$$

as in (9.15.2).

Suppose from now on in this section that U is also an open set in $\mathbf{R}^n$. If $x \in U$, then it is not too difficult to show that there are finitely many elements of U as in (9.15.1) such that

$$\overline{B}(x, r) \subseteq \mathrm{conv}(a(1), \ldots, a(m)) \tag{9.15.8}$$

for some positive real number r.

9.15.1 Convex functions on U

Let f be a convex function on U, as in Section 9.12, and let (9.15.1) be finitely many elements of U, as before. If $t_1, \ldots, t_m$ are positive real numbers, then one can check that

$$f\Big(\sum_{l=1}^{m} \frac{t_l}{t_1 + \cdots + t_m} \, a(l)\Big) \le \sum_{l=1}^{m} \frac{t_l}{t_1 + \cdots + t_m} \, f(a(l)), \tag{9.15.9}$$

using induction on m. This was mentioned in Subsection 9.9.2 when $n = 1$. This also works when $t_1, \ldots, t_m$ are nonnegative real numbers, at least one of which is positive, as before. It is easy to reduce to the case where (9.15.3) holds, and sometimes this may be stated in this way.

In particular, we get that

$$f \le \max(f(a(1)), \ldots, f(a(m))) \tag{9.15.10}$$

on (9.15.4). If $x \in U$ and $r > 0$ are as in (9.15.8), then it follows that (9.15.10) holds on $\overline{B}(x, r)$. If K is a compact subset of $\mathbf{R}^n$ and $K \subseteq U$, then one can use this to get that there is an upper bound for f on K.

If $x \in U$, $\overline{B}(x, r) \subseteq U$ for some $r > 0$, and f has an upper bound on $\overline{B}(x, r)$, then one can get a lower bound for f on $\overline{B}(x, r)$ too. This means that f is bounded on $\overline{B}(x, r)$ when (9.15.8) holds. One can use this to get that f is bounded on any compact subset K of $\mathbf{R}^n$ that is contained in U.

One can use this type of local boundedness property of f on U to get that f is locally Lipschitz of order one on U. This uses the Lipschitz conditions for convex functions of one variable mentioned in Subsection 9.6.1. See also [266, 326].

Chapter 10

More on harmonic functions, 2

10.1 Removing some isolated singularities

Let $n \ge 2$ be an integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let $a \in U$ be given. Note that

$$U \setminus \{a\} \tag{10.1.1}$$

is an open set too, and let u be a harmonic function on (10.1.1). Under some conditions, one may be able to extend u to a harmonic function on U.

Suppose that

$$\lim_{x \to a} |x - a|^{n-2} \, u(x) = 0 \tag{10.1.2}$$

when $n \ge 3$, and that

$$\lim_{x \to a} \frac{u(x)}{\log |x - a|} = 0 \tag{10.1.3}$$

when $n = 2$. It is well known that

$$u \text{ can be extended to a harmonic function on } U \tag{10.1.4}$$

under these conditions, as in Theorem 2.69 in Section H of Chapter 2 of [87]. This corresponds to Theorem 2.3 on p32 of [18] when u is bounded on U, and the stronger statement is mentioned in Execise 2 on p42 of [18] when U is the open unit ball and $a = 0$.

It is easy to reduce to the case where $a = 0$, using a translation. One can also reduce to the case where

$$\overline{B}(0, 1) \subseteq U, \tag{10.1.5}$$

using a dilation. Thus we may as well suppose that u is a continuous real or complex-valued function on

$$\overline{B}(0, 1) \setminus \{0\} \tag{10.1.6}$$

that is harmonic on

$$B(0,1) \setminus \{0\}. \tag{10.1.7}$$

Because $a = 0$, (10.1.2) means that

$$\lim_{x \to 0} |x|^{n-2}\, u(x) = 0 \tag{10.1.8}$$

when $n \geq 3$, and (10.1.3) means that

$$\lim_{x \to 0} \frac{u(x)}{\log |x|} = 0 \tag{10.1.9}$$

when $n = 2$. We would like to show that u can be extended to a continuous function on $\overline{B}(0,1)$ that is harmonic on $B(0,1)$.

10.1.1 Using a Poisson integral

Let v be the function on $\overline{B}(0,1)$ obtained by taking the Poisson integral of the restriction of u to the unit sphere $\partial B(0,1)$, as in Section 6.12. It suffices to show that

$$u = v \tag{10.1.10}$$

on (10.1.6). We may as well suppose that u is real-valued on (10.1.6), by considering the real and imaginary parts of u separately if necessary. Of course, this means that v is real-valued on $\overline{B}(0,1)$.

Equivalently, we would like to show that

$$u \leq v \tag{10.1.11}$$

and

$$v \leq u \tag{10.1.12}$$

on (10.1.6). In fact, we shall show that (10.1.11) holds on (10.1.6). One can get (10.1.12) using an analogous argument, or by considering $-u$ in place of u.

10.1.2 A helpful family of functions

Let $\epsilon > 0$ be given, and consider the real-valued function w_ϵ defined on (10.1.6) by

$$\begin{aligned} w_\epsilon(x) &= u(x) - v(x) - \epsilon\,(|x|^{2-n} - 1) \quad \text{when } n \geq 3 \\ &= u(x) - v(x) + \epsilon\, \log |x| \qquad \text{when } n = 2. \end{aligned} \tag{10.1.13}$$

Note that

$$w_\epsilon(x) = 0 \text{ when } |x| = 1, \tag{10.1.14}$$

by construction. We also have that w_ϵ is continuous on (10.1.6), and harmonic on (10.1.7). This uses the remarks in Section 6.1.

We would like to show that

$$w_\epsilon \leq 0 \tag{10.1.15}$$

on (10.1.6). If we can do this, then we get (10.1.11), because $\epsilon > 0$ is arbitrary. In order to do this, we shall use the maximum principle.

10.1.3 Using the maximum principle

Let t be a real number with $0 < t < 1$, and put

$$V_t = B(0,1) \setminus \overline{B}(0,t). \tag{10.1.16}$$

This is a bounded open set in $\mathbf{R}^n$, with

$$\overline{V_t} = \overline{B}(0,1) \setminus B(0,t). \tag{10.1.17}$$

Thus the boundary of V_t is

$$\partial V_t = \overline{V_t} \setminus V_t = \partial B(0,1) \cup \partial B(0,t). \tag{10.1.18}$$

It is easy to see that (10.1.15) holds on $\partial B(0,t)$

$$\text{when } t \text{ is sufficiently small,} \tag{10.1.19}$$

depending on ϵ. This uses the hypothesis (10.1.8) or (10.1.9), as appropriate, and the fact that v is bounded on $\overline{B}(0,1)$. It follows that (10.1.15) holds on V_t for the same sufficiently small t, by the maximum principle, because of (10.1.14). This means that (10.1.15) holds on (10.1.6).

10.2 Positive harmonic functions

Let n be a positive integer, and suppose for the moment that u is a positive real-valued harmonic function on $\mathbf{R}^n$. Another version of *Liouville's theorem* states that u has to be constant on $\mathbf{R}^n$. This can be shown in a way that is somewhat analogous to the first proof in Section 6.6, with some adjustments. This is Theorem 3.1 on p45 of [18].

10.2.1 Harnack's inequality

Now let u be a positive harmonic function on a nonempty open subset U of $\mathbf{R}^n$. Suppose that $x, y \in U$ and $r > 0$ satisfy

$$|x - y| \le r \tag{10.2.1}$$

and

$$\overline{B}(x, 2\,r) \subseteq U. \tag{10.2.2}$$

It is easy to see that

$$\overline{B}(y, r) \subseteq \overline{B}(x, 2\,r), \tag{10.2.3}$$

using (10.2.1) and the triangle inequality. It follows that

$$\begin{aligned} u(y) = \frac{1}{|B(y,r)|} \int_{B(y,r)} u(z)\,dz \quad &\le \quad \frac{1}{|B(y,r)|} \int_{B(x,2\,r)} u(z)\,dz \\ &= \quad \frac{2^n}{|B(x,2\,r)|} \int_{B(x,2\,r)} u(z)\,dz = 2^n\,u(x). \end{aligned} \tag{10.2.4}$$

Similarly, if (10.2.1) holds and

$$\overline{B}(y, 2\,r) \subseteq U, \tag{10.2.5}$$

then

$$u(x) \le 2^n\, u(y). \tag{10.2.6}$$

Note that

$$\overline{B}(y, 2\,r) \subseteq \overline{B}(x, 3\,r), \tag{10.2.7}$$

by (10.2.1) and the triangle inequality again. If

$$\overline{B}(x, 3\,r) \subseteq U, \tag{10.2.8}$$

then (10.2.7) implies (10.2.5).

Suppose that U is connected, and that K is compact subset of $\mathbf{R}^n$ that is contained in U. In this case, it is well known that there is a real number $C \ge 1$ such that

$$C^{-1}\, u(x) \le u(y) \le C\, u(x) \tag{10.2.9}$$

for every $x, y \in K$. More precisely, this constant C does not depend on u. This is *Harnack's inequality*, as in Theorem 3.6 on p48 of [18], and Theorem 11 in Section 2.2.3 f of [81].

One can get more precise estimates on balls using the Poisson integral formula, as in 3.4, 3.5 on p47f of [18], and Problem 7 in Section 2.5 of [81]. Inequalities like these are also discussed on p243f of [7] when $n = 2$. Note that the inequalities mentioned in this subsection work as well when u is nonnegative on U, and are sometimes stated this way. However, if U is connected, and u is a nonnegative real-valued harmonic function on U, then u is either strictly positive on U, or identically equal to 0 on U. This follows from the strong maximum principle for $-u$ on U, as in Subsection 6.7.1.

10.3 Some criteria for harmonicity

Let n be a positive integer, let B_0 be an open ball in $\mathbf{R}^n$, and let u be a continuous real or complex-valued function on the closure $\overline{B_0}$ of B_0. Suppose that for every $a \in B_0$ there is an $r(a) > 0$ such that

$$B(a, r(a)) \subseteq B_0 \tag{10.3.1}$$

and

$$\text{the average of } u \text{ on } B(a, r(a)) \text{ is equal to } u(a). \tag{10.3.2}$$

We would like to show that

$$u \text{ is harmonic on } B_0 \tag{10.3.3}$$

under these conditions.

We may as well suppose that

$$B_0 = B(0,1), \tag{10.3.4}$$

because otherwise we can reduce to that case with a translation and dilation. We may suppose that u is real-valued too, by considering the real and imaginary parts of u separately. Using the Poisson integral, as in Section 6.12, we get that there is a continuous real-valued function v on $\overline{B}(0,1)$ that is harmonic on $B(0,1)$ and satisfies

$$v = u \text{ on } \partial B(0,1). \tag{10.3.5}$$

It suffices to verify that

$$u = v \text{ on } \overline{B}(0,1). \tag{10.3.6}$$

Remember that the average of v on any open ball contained in $B(0,1)$ is equal to the value of v at the center of the ball, as in Subsection 6.7.3. If $a \in B(0,1)$ and $r(a)$ is as before, then (10.3.2) implies that

$$\text{the average of } u - v \text{ on } B(a, r(a)) \text{ is equal to } u(a) - v(a). \tag{10.3.7}$$

It follows that the maximum of $u - v$ on $\overline{B}(0,1)$ is attained on $\partial B(0,1)$, as in Subsection 6.7.3. Similarly, the maximum of $v - u$ on $\overline{B}(0,1)$ is attained on $\partial B(0,1)$. This means that (10.3.6) follows from (10.3.5),

10.3.1 Arbitrary open sets in $\mathbf{R}^n$

Now let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuous real or complex-valued function on U. Suppose that for every $a \in U$ and $\epsilon > 0$ there is an $r(a,\epsilon) > 0$ such that

$$r(a,\epsilon) < \epsilon, \tag{10.3.8}$$

$$\overline{B}(a, r(a,\epsilon)) \subseteq U, \tag{10.3.9}$$

and

$$\text{the average of } u \text{ on } B(a, r(a,\epsilon)) \text{ is equal to } u(a). \tag{10.3.10}$$

Of course, (10.3.8) implies (10.3.9) when ϵ is sufficiently small, depending on a. Let us verify that

$$u \text{ is harmonic on } U \tag{10.3.11}$$

in this case.

Let B_0 be an open ball in $\mathbf{R}^n$ such that

$$\overline{B_0} \subseteq U. \tag{10.3.12}$$

It suffices to check that the restriction of u to B_0 is harmonic. To get this, we can use the criterion discussed at the beginning of the section. More precisely, we can take $r(a) = r(a,\epsilon)$ with ϵ small enough so that (10.3.8) impies (10.3.1). This uses the fact that open balls in $\mathbf{R}^n$ are open sets, as mentioned in Subsection 1.1.3.

10.4 The reflection principle

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Suppose that U is symmetric about the $x_n = 0$ hyperplane in $\mathbf{R}^n$, so that

$$(x_1, \ldots, x_{n-1}, x_n) \in U \tag{10.4.1}$$

if and only if

$$(x_1, \ldots, x_{n-1}, -x_n) \in U. \tag{10.4.2}$$

Put

$$U_+ = \{x \in U : x_n > 0\}, \tag{10.4.3}$$

and

$$\widetilde{U}_+ = \{x \in U : x_n \geq 0\}. \tag{10.4.4}$$

One can check that U_+ is also a nonempty open subset of $\mathbf{R}^n$, and that $\widetilde{U}_+$ is relatively closed in U, as in Subsection 1.9.1.

Let u be a continuous real or complex-valued function on $\widetilde{U}_+$, and suppose that

$$u(x) = 0 \text{ when } x \in \widetilde{U}_+ \text{ and } x_n = 0. \tag{10.4.5}$$

If $x \in U$ and $x_n < 0$, then $(x_1, \ldots, x_{n-1}, -x_n) \in U_+$, and we put

$$u(x) = -u(x_1, \ldots, x_{n-1}, -x_n). \tag{10.4.6}$$

This defines an extension of u to a real or complex-valued function on U. One can check that

$$u \text{ is continuous on } U \tag{10.4.7}$$

under these conditions.

If u is harmonic on U_+, then the *reflection principle* states that

$$u \text{ is harmonic on } U. \tag{10.4.8}$$

To see this, it suffices to show that for each $a \in U$, we have that

$$\text{the average of } u \text{ on } B(a, r) \text{ is equal to } u(a) \tag{10.4.9}$$

for all sufficiently small $r > 0$, as in the previous section. If $a \in U_+$, then this holds when

$$\overline{B}(a, r) \subseteq U_+, \tag{10.4.10}$$

because u is harmonic on U_+, by hypothesis. There is an analogous statement when $a_n < 0$, by construction.

Suppose now that $a \in U$ and $a_n = 0$. If

$$\overline{B}(a, r) \subseteq U, \tag{10.4.11}$$

then it is easy to see that

$$\int_{B(a,r)} u(x)\, dx = 0. \tag{10.4.12}$$

This means that (10.4.9) holds in this case too.

10.4.1 A uniqueness result

Suppose that u is also bounded on U_+. This implies that u is bounded on U, by construction. If $U = \mathbf{R}^n$, then it follows that u is a constant function, by Liouville's theorem, as in Section 6.6. This means that

$$u \equiv 0 \text{ on } \mathbf{R}^n, \tag{10.4.13}$$

because of (10.4.5).

10.5 More on Liouville's theorem

Let n be a positive integer, and let u be a real or complex-valued harmonic function on $\mathbf{R}^n$. If

$$\frac{u(x)}{|x|} \to 0 \text{ as } |x| \to \infty, \tag{10.5.1}$$

then a refinement of Liouville's theorem implies that u is constant on $\mathbf{R}^n$. This can be obtained using either of the arguments mentioned in Section 6.6. Of course, (10.5.1) holds when u is bounded on $\mathbf{R}^n$.

10.5.1 Another growth condition

Let k be a positive integer, and suppose now that

$$\frac{u(x)}{|x|^{k+1}} \to 0 \text{ as } |x| \to \infty. \tag{10.5.2}$$

One can show that

$$\frac{\partial_j u(x)}{|x|^k} \to 0 \text{ as } |x| \to \infty \tag{10.5.3}$$

for each $j = 1, \ldots, n$, using the same type of argument as in Subsection 6.6.2.

If α is a multi-index with $|\alpha| \le k$, then one repeat the argument to get that

$$\frac{\partial^\alpha u(x)}{|x|^{k-|\alpha|+1}} \to 0 \text{ as } |x| \to \infty. \tag{10.5.4}$$

In particular, if $|\alpha| = k$, then we get that

$$\frac{\partial^\alpha u(x)}{|x|} \to 0 \text{ as } |x| \to \infty. \tag{10.5.5}$$

This implies that $\partial^\alpha u$ is constant on $\mathbf{R}^n$ when $|\alpha| = k$, as before. Equivalently, if β is a multi-index with $|\beta| = k+1$, then $\partial^\beta u = 0$ on $\mathbf{R}^n$.

This implies that u is a polynomial on $\mathbf{R}^n$ of degree at most k, by standard arguments. This will be discussed further in the next subsections.

10.5.2 Polynomials and derivatives

Let f be a real or complex-valued function on $\mathbf{R}^n$ that is k times condintuously differentiable, and suppose that $\partial^\alpha f$ is constant on $\mathbf{R}^n$ for each multi-index α with $|\alpha| = k$. Equivalently, this means that f is smooth on $\mathbf{R}^n$, and that $\partial^\beta f = 0$ on $\mathbf{R}^n$ for every multi-index β with $|\beta| = k + 1$. It is well known that f is a polynomial of degree at most k on $\mathbf{R}^n$ under these conditions, and we would like to mention a couple fo ways to see that. We might as well take f to be real-valued, since otherwise one can consider the real and imaginary parts of f. This is simpler and more familiar when $n = 1$, and so we may suppose that $n \geq 2$.

If $x \in \mathbf{R}^n$, then $f(t\,x)$ may be considered as a smooth function of $t \in \mathbf{R}$. The derivative of $f(t\,x)$ in t of order $k + 1$ may be expressed in terms of the derivatives of f or oder $k+1$, and is thus equal to 0, by hypothesis. This implies that $f(t\,x)$ is a polynomial of degree at most k in t, with coefficients that may depend on x. More precisely, the coefficient of t^l in $f(t\,x)$ as a polynomial in t may be obtained from the lth derivative of $f(t\,x)$ at $t = 0$ for each l, as usual. One can take $t = 1$ to get that $f(x)$ is a polynomial of degree at most k on $\mathbf{R}^n$.

10.5.3 Using homogeneous polynomials

Suppose again that $\partial^\alpha f$ is constant on $\mathbf{R}^n$ for each multi-index α with $|\alpha| = k$. One can find a homogeneous polynomial P_k of degree k on $\mathbf{R}^n$ such that

$$\partial^\alpha P_k = \partial^\alpha f \tag{10.5.6}$$

for each multi-index α with $|\alpha| = k$. Of course, this means that $\partial^\alpha (f - P_k) = 0$ on $\mathbf{R}^n$ for each multi-index α with $|\alpha| = k$. If $k = 1$, then it follows that $f - P_1$ is a constant on $\mathbf{R}^n$. If $k \geq 2$, then it follows that the derivatives of $f - P_k$ of order $k - 1$ are constant on $\mathbf{R}^n$, and one can repeat the argument.

10.5.4 Reducing the number of variables

Let a be a continuously-differentiable real or complex-valued function on $\mathbf{R}^n$ such that $\partial_j a$ is a polynomial on $\mathbf{R}^n$ for each $j = 1, \ldots, n$. We would like to show that a is a polynomial on $\mathbf{R}^n$ as well under these conditions. This is straightforward when $n = 1$, and for any n it is not too difficult to show that there is a polynomial P on $\mathbf{R}^n$ such that

$$\partial_1 a = \partial_1 P \tag{10.5.7}$$

on $\mathbf{R}^n$. This implies that $a - P$ does not depend on x_1, and its partial derivatives in the other variables are polynomials. One can repeat the argument to get that a is a polynomial on $\mathbf{R}^n$, and one can use this to get another way to show the result mentioned earlier.

10.6 Some more remarks about compositions

Let W be a nonempty open subset of $\mathbf{R}^2$, and suppose that f is a continuously-differentiable complex-valued function on W. Thus $\partial f = \partial f / \partial z$ and $\overline{\partial} f = \partial f / \partial \overline{z}$ are continuous complex-valued functions on W, as in Section 2.2. If $v \in \mathbf{R}^2$ is identified with the complex number $v_1 + i\, v_2$, as usual, then the directional derivative of f in the direction v at a point $z \in W$ may be expressed as

$$(D_v f)(z) = v\,(\partial f)(z) + \overline{v}\,(\overline{\partial} f)(z), \tag{10.6.1}$$

as in Section 3.14.

Let U be another nonempty open subset of $\mathbf{R}^2$, and let u be a continuously-differentiable complex-valued function on U. Suppose that

$$f(W) \subseteq U, \tag{10.6.2}$$

where U is considered as a subset of the complex plane, so that the composition $u \circ f$ of f and u is defined as a complex-valued function on W. Note that $u \circ f$ is continuously differentiable on W, and the directional derivative of $u \circ f$ at z in the direction v is equal to the directional derivative of u at $f(z)$ in the direction $(D_v f)(z)$. This means that

$$(D_v(u \circ f))(z) = (D_v f)(z)\,(\partial u)(f(z)) + \overline{(D_v f)(z)}\,(\overline{\partial} u)(f(z)), \tag{10.6.3}$$

as before.

10.6.1 Composition with holomorphic functions

Suppose now that f is holomorphic on W, so that $\overline{\partial} f = 0$ on W, as in Section 2.2. In this case, we get that

$$(D_v f)(z) = v\,(\partial f)(z) \tag{10.6.4}$$

and

$$(D_v(u \circ f))(z) = v\,(\partial f)(z)\,(\partial u)(f(z)) + \overline{v}\,\overline{(\partial f)(z)}\,(\overline{\partial} u)(f(z)). \tag{10.6.5}$$

One can use this to get that

$$(\partial(u \circ f))(z) = (\partial f)(z)\,(\partial u)(f(z)) \tag{10.6.6}$$

and

$$(\overline{\partial}(u \circ f))(z) = \overline{(\partial f)(z)}\,(\overline{\partial} u)(f(z)). \tag{10.6.7}$$

In particular, if u is also holomorphic on U, then it follows that $u \circ f$ is holomorphic on W, as in Subsection 3.14.2.

10.6.2 The Laplacian of the composition

Suppose that u and f are twice continuously differentiable, although in fact f is automatically smooth, because it is holomorphic, as mentioned in Subsection 2.2.1. Observe that

$$(\Delta(u \circ f))(z) = 4\,((\partial\overline{\partial})(u \circ f))(z) = 4\,\partial(\overline{(\partial f)(z)}\,(\overline{\partial} u)(f(z))), \tag{10.6.8}$$

where the first step is as in Subsection 2.2.1, and the secod step uses (10.6.7). It follows that

$$(\Delta(u \circ f))(z) = 4\,\overline{(\partial f)(z)}\,\partial((\overline{\partial} u)(f(z))), \tag{10.6.9}$$

using the product rule, as in Subsection 2.2.2, and the fact that f is harmonic, as in Subsection 2.2.1. This implies that

$$(\Delta(u \circ f))(z) = 4\,\overline{(\partial f)(z)}\,(\partial f)(z)\,(\partial\overline{\partial} u)(f(z)), \tag{10.6.10}$$

as before. This means that

$$(\Delta(u \circ f))(z) = |(\partial f)(z)|^2\,(\Delta u)(f(z)). \tag{10.6.11}$$

If u is harmonic on U, then it follows that

$$u \circ f \text{ is harmonic on } W. \tag{10.6.12}$$

This also works when $\overline{f}$ is holomorphic on W, so that $\partial f = 0$ on W. This can be obtained from the same type of argument, or using the analogous statement for complex conjugation as a mapping from $\mathbf{C}$ onto itself.

10.6.3 The Kelvin transform

Let $n \ge 2$ be an integer. It is easy to see that

$$x \mapsto x/|x|^2 \tag{10.6.13}$$

is a one-to-one mapping from $\mathbf{R}^n \setminus \{0\}$ onto itself. More precisely, one can check that this mapping is its own inverse.

Let U be a nonempty open set contained in $\mathbf{R}^n \setminus \{0\}$, and let u be a complex-valued function on U. Put

$$\widetilde{U} = \{x/|x|^2 : x \in U\}, \tag{10.6.14}$$

which is another open set in $\mathbf{R}^n \setminus \{0\}$. The *Kelvin transform* of u is the function $\widetilde{u}$ defined on $\widetilde{U}$ by

$$\widetilde{u}(x) = u(x/|x|^2)\,|x|^{2-n}. \tag{10.6.15}$$

If u is harmonic on U, then it is well-known that

$$\widetilde{u} \text{ is harmonic on } \widetilde{U}. \tag{10.6.16}$$

This corresponds to Theorem 4.7 on p63 of [18], Problem 11 in Section 2.5 of [81], and Theorem 2.72 in Section H of Chapter 2 of [87].

10.7 The Green's function

Let $n \geq 2$ be an integer, and let U be a nonempty bounded open subset of $\mathbf{R}^n$. Also let N be the real-valued function defined on $\mathbf{R}^n \setminus \{0\}$ at the beginning of Section 6.8. Note that this corresponds to $-\Phi$ in the definition in Section 2.2.1 a of [81].

Let $x \in U$ be given. Suppose that η^x is a continuous real-valued function on $\overline{U}$ such that

$$\eta^x \text{ is harmonic on } U \tag{10.7.1}$$

and

$$\eta^x(y') = N(y' - x) \text{ for every } y' \in \partial U. \tag{10.7.2}$$

Note that

$$\eta^x \text{ is uniquely determined by these properties,} \tag{10.7.3}$$

as in Subsection 6.7.2.

Let us say that

$$\eta^x \text{ is a } \textit{corrector function} \text{ corresponding to } U \text{ and } x \tag{10.7.4}$$

under these conditions, as in Section 2.2.4 a of [81]. More precisely, this corresponds to $-\phi^x$ in the notation of [81].

In this case, the *Green's function* associated to U and x is

$$G(x, y) = N(y - x) - \eta^x(y). \tag{10.7.5}$$

This is a real-valued function of $y \in \overline{U} \setminus \{x\}$, although

$$G(x, y) - N(y - x) = -\eta^x(y) \tag{10.7.6}$$

is considered to be defined as a function of $y \in \overline{U}$. Of course,

$$G(x, y') = 0 \text{ for every } y' \in \partial U, \tag{10.7.7}$$

by construction. This basically corresponds to the definitions in Section 2.2.4 a in [81] and Section D of Chapter 2 in [87], and we shall say more about this in Section 10.9. This is also discussed on p257 of [7] when $n = 2$.

10.7.1 Another helpful integral formula

Suppose for the moment that U has reasonably smooth boundary, and that η^x is twice continuously differentiable on $\overline{U}$. If u is a twice continuously-differentiable real or complex-valued function on $\overline{U}$, then

$$\begin{aligned} (10.7.8) \qquad & -\int_U \eta^x(z)\,(\Delta u)(z)\,dz \\ & = \int_{\partial U} u(y')\,(D_{\nu(y')}\eta^x)(y') - \eta^x(y')\,(D_{\nu(y')}u)(y')\,dy', \end{aligned}$$

as in Section 3.9, with $v = \eta^x$. Here $\nu(y')$ is the outward-pointing unit normal to ∂U at $y' \in \partial U$, and $D_{\nu(y')}$ denotes the directional derivative in the direction $\nu(y')$, as before.

Remember that

$$\begin{aligned}(10.7.9)\qquad & \int_U N(z-x)\,(\Delta u)(z)\,dz \\ &= \int_{\partial U} (N(y'-x)\,(D_{\nu(y')}u)(y') - u(y')\,(D_{\nu(y')}N)(y'-x))\,dy' + u(x),\end{aligned}$$

as in Subsection 6.8.2, where the integral on the left may be considered as an improper integral, or a Lebesgue integral. We can combine this with (10.7.8) using (10.7.2) to get that

$$\int_U G(x,z)\,(\Delta u)(z)\,dz = -\int_{\partial U} u(y')\,(D_{\nu(y')}G)(x,y')\,dy' + u(x), \tag{10.7.10}$$

where the left side may be considered as an improper integral or a Lebesgue integral, as before. More precisely,

$$(D_{\nu(y')}G)(x,y') \tag{10.7.11}$$

is the directional derivative of $G(x,z)$ as a function of z evaluated at y'. This corresponds to (28) in Section 2.2.4 a of [81], with a different sign because of the slightly different conventions there.

If u is harmonic on U, then we get that

$$u(x) = \int_{\partial U} u(y')\,(D_{\nu(y')}G)(x,y')\,dy'. \tag{10.7.12}$$

In particular, we can take $u \equiv 1$ on $\overline{U}$ to get that

$$\int_{\partial U} (D_{\nu(y')}G)(x,y')\,dy' = 1. \tag{10.7.13}$$

Similarly, if $u = 0$ on ∂U, then (10.7.10) implies that

$$u(x) = \int_U G(x,z)\,(\Delta u)(z)\,dz. \tag{10.7.14}$$

This is related to some remarks in Section D of Chapter 2 in [87].

10.7.2 Another property of $G(x,y)$

Let U be any nonempty bounded open subset of $\mathbf{R}^n$ again, and let η^x be a corrector function corresponding to U and x. Also let r be a positive real number such that

$$\overline{B}(x,r) \subseteq U. \tag{10.7.15}$$

Put

$$U_r = U \setminus \overline{B}(x, r), \tag{10.7.16}$$

and observe that U_r is an open set in $\mathbf{R}^n$ with

$$\overline{U_r} = \overline{U} \setminus B(x, r) \tag{10.7.17}$$

and

$$\partial U_r = \partial U \cup \partial B(x, r). \tag{10.7.18}$$

It is easy to see that

$$G(x, y) = N(y - x) - \eta^x(y) \le 0 \tag{10.7.19}$$

for every $y \in U$ with $x \ne y$ and $|x - y|$ sufficiently small. If r is sufficiently small, then one can use this and (10.7.7) to get that (10.7.19) holds for every $y \in \overline{U}_r$, because of the maximum principle, as in Section 6.7.2. This means that (10.7.19) holds for all $y \in \overline{U} \setminus \{x\}$.

10.7.3 Nonnegativity of the normal derivative

Suppose that U has reasonably smooth boundary, and that η^x is continuously differentiable on $\overline{U}$, so that $G(x, y)$ is continuously differentiable in y on $\overline{U} \setminus \{x\}$. Under these conditions, one can check that

$$(D_{\nu(y')}G)(x, y') \ge 0 \tag{10.7.20}$$

for every $y' \in \partial U$, because of (10.7.7) and (10.7.19).

10.8 Some examples of corrector functions

Let $n \ge 2$ be an integer, let $a \in \mathbf{R}^n$ and $r > 0$ be given. It is easy to see that the corrector function associated to $U = B(a, r)$ and $x = a$ is the constant function

$$\begin{aligned} \eta^a &= \frac{r^{2-n}}{(2-n)\,|\partial B(0,1)|} && \text{when } n \ge 3 \\ &= \frac{1}{2\,\pi}\,\log r && \text{when } n = 2. \end{aligned} \tag{10.8.1}$$

In this case, the right side of (10.7.12) reduces to the usual average of u on $\partial B(a, r)$, which was basically mentioned in Subsection 6.2.2. Similarly, (10.7.10) corresponds to an integral formula in Subsection 6.13.3. More precisely, (10.8.1) is the same as the definition of c_r at the beginning of Section 6.13.

Let us now take $a = 0$ and $r = 1$. We would like to find corrector functions associated to the unit ball $U = B(0, 1)$ and $x \ne 0$.

10.8.1 Some preliminary remarks

If $x, y \in \mathbf{R}^n$ and $x \ne 0$, then

$$|x|\,|(x/|x|^2) - y| = \big|(x/|x|) - |x|\,y\big|. \tag{10.8.2}$$

If $|y| = 1$, then we have that

$$\big|(x/|x|) - |x|\,y\big| = |x - y|. \tag{10.8.3}$$

This follows from an identity mentioned in Subsection 6.10.2, with $x' = x/|x|$, $w' = y$, and $r = |x|$. Combining this with (10.8.2), we get that

$$|x - y| = |x|\,\big|(x/|x|^2) - y\big| \tag{10.8.4}$$

when $|y| = 1$.

In fact, if $y \in \mathbf{R}^n$ and $y \ne 0$, then

$$\big|(x/|x|) - |x|\,y\big| = \big|(x/|x|) - |x|\,|y|\,(y/|y|)\big|. \tag{10.8.5}$$

We also have that

$$\big|(x/|x|) - |x|\,|y|\,(y/|y|)\big| = \big||x|\,|y|\,(x/|x|) - (y/|y|)\big|, \tag{10.8.6}$$

as in Subsection 6.10.2. This implies that

$$\big|(x/|x|) - |x|\,y\big| = \big||y|\,x - (y/|y|)\big|. \tag{10.8.7}$$

10.8.2 The unit ball, $n \ge 3$

Suppose that $n \ge 3$ and $|x| < 1$, so that

$$\big|(x/|x|^2)\big| = 1/|x| > 1. \tag{10.8.8}$$

If $|y| = 1$, then $y \ne x, x/|x|^2$, and (10.8.4) implies that

$$N(x - y) = |x|^{2-n}\,N((x/|x|^2) - y), \tag{10.8.9}$$

where N is as in Section 6.8, as before. Let η^x be the real-valued function defined on $\mathbf{R}^n \setminus \{x/|x|^2\}$ by

$$\eta^x(y) = |x|^{2-n}\,N((x/|x|^2) - y). \tag{10.8.10}$$

Note that $\eta^x(y)$ is harmonic as a function of $y \ne x/|x|^2$, because N is harmonic on $\mathbf{R}^n \setminus \{0\}$, as in Section 6.1. The restriction of η^x to $\overline{B}(0,1)$ is the corrector function corresponding to $U = B(0,1)$ and x, as in the previous section.

Thus the Green's function associated to $U = B(0,1)$ and x is

$$G(x,y) = N(y - x) - \eta^x(y) = N(y - x) - |x|^{2-n}\,N((x/|x|^2) - y), \tag{10.8.11}$$

as in (10.7.5). This corresponds to some remarks in Section 2.2.4 c of [81], and Section G of Chapter 2 in [87]. This is also related to some remarks on p10f of [18].

One can verify that (10.7.12) corresponds to the Poisson integral formula in Section 6.12 in this case. This is how the Poisson kernel is obtained in [18, 81, 87].

Equivalently,

$$\eta^x(y) = N((x/|x|) - |x|\, y). \tag{10.8.12}$$

This also works when $n = 2$, as in the next subsection.

10.8.3 The unit disk

Suppose now that $n = 2$, and that $|x| < 1$ again. If $|y| = 1$, then $y \neq x, x/|x|^2$, and we can use (10.8.4) to get that

$$N(x - y) = \frac{1}{2\,\pi} \log |x - y| = \frac{1}{2\,\pi} \log |x| + \frac{1}{2\,\pi} \log |(x/|x|^2) - y|, \tag{10.8.13}$$

where N is as in Section 6.8. Let η^x be the real-valued function defined on $\mathbf{R}^2 \setminus \{x/|x|^2\}$ by

$$\eta^x(y) = \frac{1}{2\,\pi} \log |x| + \frac{1}{2\,\pi} \log |(x/|x|^2) - y|. \tag{10.8.14}$$

This is harmonic as a function of $y \neq x/|x|^2$, because $\log |\cdot|$ is harmonic on $\mathbf{R}^2 \setminus \{0\}$, as in Section 6.1. It follows that the restriction of η^x to $\overline{B}(0, 1)$ is the corrector function associated to $U = B(0, 1)$ and x.

This means that the Green's function associated to $U = B(0, 1)$ and x is

$$\begin{aligned} G(x, y) &= N(y - x) - \eta^x(y) \\ &= \frac{1}{2\,\pi} \log |x - y| - \frac{1}{2\,\pi} \log |x| - \frac{1}{2\,\pi} \log |(x/|x|^2) - y|. \end{aligned} \tag{10.8.15}$$

This corresponds to some remarks in Section 2.2.4 c of [81], and Section G of Chapter 2 in [87], as before.

One can check that (10.7.12) corresponds to the Poisson integral formula in this case too, as in [81, 87].

10.9 More on the Green's function

Let $n \geq 2$ be an integer, let U be a nonempty bounded open subset of $\mathbf{R}^n$, and let N be as in Section 6.8, as usual. Suppose that for each $x \in U$, η^x is a corrector function associated to U and x, as in Section 10.7. This means that the *Green's function*

$$G(x, y) = N(y - x) - \eta^x(y) \tag{10.9.1}$$

is defined for $x \in U$ and $y \in \overline{U}$ with $x \neq y$. This corresponds to the definitions in Section 2.2.4 a in [81] and Section D of Chapter 2 in [87].

10.9.1 Symmetry of the Green's function

Suppose now that U has reasonably smooth boundary, and let $x, y \in U$ with $x \neq y$ be given. Suppose also that

$$\eta^x,\ \eta^y \text{ are corrector functions associated to } U \text{ and } x,\ y, \text{ respectively,} \tag{10.9.2}$$

and that η^x, η^y are twice continuously differentiable on $\overline{U}$. The corresponding Green's functions $G(x, \cdot)$ and $G(y, \cdot)$ may be defined on $\overline{U} \setminus \{x\}$ and $\overline{U} \setminus \{y\}$, as before. Under these conditions, we would like to show that

$$G(x, y) = G(y, x). \tag{10.9.3}$$

This corresponds to Theorem 13 in Section 2.2.4 a in [81] and Lemma 2.33 in Section D of Chapter 2 in [87], and it is discussed on p258 of [7] when $n = 2$.

Let r be a positive real number that is small enough so that

$$\overline{B}(x, r),\ \overline{B}(y, r) \subseteq U \tag{10.9.4}$$

and $2\,r < |x - y|$, so that

$$\overline{B}(x, r) \cap \overline{B}(y, r) = \emptyset. \tag{10.9.5}$$

Put

$$V_r = U \setminus (\overline{B}(x, r) \cup \overline{B}(y, r)). \tag{10.9.6}$$

This is an open set in $\mathbf{R}^n$, with

$$\overline{V_r} = \overline{U} \setminus (B(x, r) \cup B(y, r)) \tag{10.9.7}$$

and

$$\partial V_r = \partial B(x, r) \cup \partial B(y, r). \tag{10.9.8}$$

If $z \in \overline{U}$, then put

$$v(z) = G(x, z) = N(z - x) - \eta^x(z) \tag{10.9.9}$$

when $x \neq z$, and

$$w(z) = G(y, z) = N(z - y) - \eta^y(z) \tag{10.9.10}$$

when $y \neq z$. These define twice continuously-differentiable real-valued functions on $\overline{V_r}$, by hypothesis. It follows that

$$\begin{aligned} &\int_{V_r} (v(z)\,(\Delta w)(z) - w(z)\,(\Delta v)(z))\,dz \\ &\qquad = \int_{\partial V_r} (v(z')\,(D_{\nu_r(z')} w)(z') - w(z')\,(D_{\nu_r(z')} v)(z'))\,dz', \end{aligned} \tag{10.9.11}$$

as in Section 3.9. Here $\nu_r(z')$ is the outward-pointing unit normal to ∂V_r at $z' \in \partial V_r$, as before.

The left side of (10.9.11) is equal to 0, because v and w are harmonic on V_r, by construction. The contribution to the right side of (10.9.11) from the integral over ∂V is equal to 0 too, because $v = w = 0$ on ∂V. Thus we are left with the contributions to the integral on the right side of (10.9.11) from the integrals over $\partial B(x, r)$ and $\partial B(y, r)$.

Let $\mu_{x,r}(z')$, $\mu_{y,r}(z')$ be the outward-pointing unit normals to $\partial B(x, r)$, $\partial B(y, r)$ at $z' \in \partial B(x, r), \partial B(y, r)$, respectively. Note that

$$\begin{aligned} \nu_r(z') &= -\mu_{x,r}(z') \quad \text{when } z' \in \partial B(x,r) \\ &= -\mu_{y,r}(z') \quad \text{when } z' \in \partial B(y,r). \end{aligned} \tag{10.9.12}$$

Using (10.9.11) and the remarks in the preceding paragraph, we get that

$$\begin{aligned} 0 \;=\; & \int_{\partial B(x,,r)} (v(z')\,(D_{\mu_{x,r}(z')} w)(z') - w(z')\,(D_{\mu_{x,r}(z')} v)(z'))\,dz' \\ & + \int_{\partial B(y,r)} (v(z')\,(D_{\mu_{y,r}(z')} w)(z') - w(z')\,(D_{\mu_{y,r}(z')} v)(z'))\,dz'. \end{aligned} \tag{10.9.13}$$

10.9.2 Some limits as $r \to 0$

In order to get (10.9.3), we would like to consider the limits of the various terms on the right side of (10.9.13) as $r \to 0$, as in [81, 87]. One can check that

$$\lim_{r \to 0} \int_{\partial B(x,r)} v(z')\,(D_{\mu_{x,r}(z')} w)(z'))\,dz' = 0, \tag{10.9.14}$$

because w is smooth near x. This also uses the fact that the $(n-1)$-dimensional volume of $\partial B(x, r)$ is a multiple of r^{n-1}, which takes care of the singularity of v near x. Similarly,

$$\lim_{r \to 0} \int_{\partial B(y,r)} w(z')\,(D_{\mu_{y,r}(z')} v)(z')\,dz' = 0. \tag{10.9.15}$$

It is easy to see that

$$\lim_{r \to 0} \int_{\partial B(x,r)} w(z')\,(D_{\mu_{x,r}(z')} \eta^x)(z')\,dz' = 0, \tag{10.9.16}$$

because w is continuous at x, and η^x is continuously differentiable near x. Similarly,

$$\lim_{r \to 0} \int_{\partial B(y,r)} v(z')\,(D_{\mu_{y,r}(z')} \eta^y)(z')\,dz' = 0. \tag{10.9.17}$$

This leaves terms on the right side of (10.9.13) of the form

$$-\int_{\partial B(x,r)} w(z')\,(D_{\mu_{x,r}(z')} N)(z')\,dz' \tag{10.9.18}$$

and

$$\int_{\partial B(y,r)} v(z')\,(D_{\mu_{y,r}(z')}N)(z')\,dz'. \tag{10.9.19}$$

These are the same as

$$-\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} w(z')\,dz' \tag{10.9.20}$$

and

$$\frac{1}{|\partial B(y,r)|}\int_{\partial B(y,r)} v(z')\,dz', \tag{10.9.21}$$

respectively, as mentioned in Subsection 6.8.1. One could deal with the limits of these integrals as $r \to 0$ in the usual way, or use the mean-value property of harmonic functions to evaluate these integrals. Combining this with (10.9.13) and the previous remarks, we obtain that

$$0 = -w(x) + v(y). \tag{10.9.22}$$

Of course, this is the same as (10.9.3).

10.10 The upper half-space

Let $n \geq 2$ be an integer, and consider the open *upper half-space*

$$\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_n > 0\} \tag{10.10.1}$$

in $\mathbf{R}^n$. This is an open set in $\mathbf{R}^n$, whose closure is the closed upper half-space

$$\overline{\mathbf{R}^n_+} = \{x \in \mathbf{R}^n : x_n \geq 0\}. \tag{10.10.2}$$

Similarly, the boundary of $\mathbf{R}^n_+$ is the $x_n = 0$ hyperplane,

$$\partial\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_n = 0\}. \tag{10.10.3}$$

If $x \in \mathbf{R}^n$, then let $\widetilde{x}$ be the element of $\mathbf{R}^n$ defined by

$$\widetilde{x} = (x_1, \ldots, x_{n-1}, -x_n). \tag{10.10.4}$$

Note that

$$\widetilde{(\widetilde{x})} = x. \tag{10.10.5}$$

It is easy to see that

$$|\widetilde{y} - \widetilde{x}| = |y - x| \tag{10.10.6}$$

for every $y \in \mathbf{R}^n$. In particular, if $y_n = 0$, then $\widetilde{y} = y$, and we get that

$$|y - \widetilde{x}| = |y - x|. \tag{10.10.7}$$

10.10.1 The Green's function for $\mathbf{R}^n_+$

Let N be the real-valued function defined on $\mathbf{R}^n \setminus \{0\}$ at the beginning of Section 6.8 again. Although $\mathbf{R}^n_+$ is unbounded, many of the basic notions in Section 10.7 can also be used here.

If $x \in \mathbf{R}^n_+$, then

$$N(y' - \widetilde{x}) = N(y' - x) \tag{10.10.8}$$

for every $y' \in \mathbf{R}^n$ with $y'_n = 0$, because of (10.10.7). Put

$$\eta^x(y) = N(y - \widetilde{x}) \tag{10.10.9}$$

for $y \in \mathbf{R}^n$ with $y \neq \widetilde{x}$. Note that η^x is harmonic on $\mathbf{R}^n \setminus \{\widetilde{x}\}$, and in particular η^x is continuous on $\overline{\mathbf{R}^n_+}$ and harmonic on $\mathbf{R}^n_+$. This is considered a corrector function corresponding to $\mathbf{R}^n_+$ and x, as in Section 2.2.4 b of [81].

Thus we put

$$G(x, y) = N(y - x) - \eta^x(y) = N(y - x) - N(y - \widetilde{x}) \tag{10.10.10}$$

when $y \in \mathbf{R}^n$ and $y \neq x, \widetilde{x}$, and in particular when $y \in \overline{\mathbf{R}^n_+} \setminus \{x\}$, as in Section 10.7. This is considered as the Green's function associated to $\mathbf{R}^n_+$, as in Sections 10.7 and 10.9, and Section 2.2.4 b of [81].

10.11 The Poisson kernel for $\mathbf{R}^n_+$

Let us continue with the same notation and hypotheses as in the previous section. If $y' \in \mathbf{R}^n$ and $y_n = 0$, then let $\nu(y')$ be the outward-pointing unit normal to the boundary of $\mathbf{R}^n_+$ at y'. In this case, this simply means that

$$\nu(y') = (0, \ldots, 0, -1). \tag{10.11.1}$$

The directional derivative in this direction is the same as -1 times the partial derivative in the nth variable.

In particular, if $x \in \mathbf{R}^n_+$, then

$$(D_{\nu(y')}G)(x, y') = -\frac{\partial G}{\partial y_n}(x, y') = -(\partial_n N)(y' - x) + (\partial_n N)(y' - \widetilde{x}). \tag{10.11.2}$$

Note that

$$(\partial_j N)(z) = \frac{1}{|\partial B(0, 1)|} \frac{z_j}{|z|^n} \tag{10.11.3}$$

for each $j = 1, \ldots, n$, $z \in \mathbf{R}^n \setminus \{0\}$, and $n \geq 2$, by the definition of N in Section 6.8, and the remarks in Subsection 6.1.2. Thus the right side of (10.11.2) is equal to

$$\frac{1}{|\partial B(0, 1)|} \Big(\frac{x_n}{|y' - x|^n} - \frac{\widetilde{x}_n}{|y' - \widetilde{x}|^n} \Big). \tag{10.11.4}$$

This is equal to

$$P(x, y') = \frac{2}{|\partial B(0, 1)|} \frac{x_n}{|x - y'|^n}, \tag{10.11.5}$$

because of (10.10.7) and the definition (10.10.4) of $\widetilde{x}$.

This is the *Poisson kernel* associated to $\mathbf{R}^n_+$, as in Section 2.2.4 b in [81]. This is also mentioned on p145 of [18], where it was obtained another way. Sometimes

$$P(x,0) = \frac{2}{|\partial B(0,1)|}\,\frac{x_n}{|x|^n} \tag{10.11.6}$$

is called the Poisson kernel associated to $\mathbf{R}^n_+$. Note that

$$P(x,y') = P(x-y',0). \tag{10.11.7}$$

The Poisson kernel may also be obtained using Fourier analysis.

10.11.1 Some properties of $P(x,y')$

Observe that

$$P(x,y') \text{ is harmonic as a function of } x \in \mathbf{R}^n_+ \tag{10.11.8}$$

for each $y' \in \mathbf{R}^n$ with $y'_n = 0$. This follows from the remarks in Subsection 6.1.2.

It is well known that

$$\int_{\partial \mathbf{R}^n_+} P(x,y')\,dy' = 1 \tag{10.11.9}$$

for each $x \in \mathbf{R}^n_+$. More precisely, the integral on the left corresponds to an integral over $\mathbf{R}^{n-1}$ in an obvious way. This integral may be defined as in Section 7.2. The integrability of $P(x,y')$ as a function of $y' \in \partial\mathbf{R}^n_+$ corresponds to a remark at the beginning of Section 7.3.

It is easy to see that the left side of (10.11.9) does not depend on x_j for $j = 1,\ldots,n-1$, because of invariance of the integral under translations. One can also check that the left side of (10.11.9) does not depend on x_n, using a change of variables.

One way to get (10.11.9) is discussed on p145 of [18].

Alternatively, (10.11.9) corresponds to a property of Green's functions mentioned in Subsection 10.7.1 for bounded open sets with reasonably nice boundary. One can use an analogous argument here, by considering suitable bounded open sets contained in $\mathbf{R}^n_+$. To do this, one can estimate the partial derivatives of (10.10.10) in y as $|y| \to \infty$.

10.12 Poisson integrals for $\mathbf{R}^n_+$

We continue with the same notation and hypotheses as in the previous two sections. Let f be a continuous real or complex-valued function on $\partial\mathbf{R}^n_+$. We can identify $\partial\mathbf{R}^n_+$ with $\mathbf{R}^{n-1}\times\{0\}$, or simply with $\mathbf{R}^{n-1}$, so that f corresponds to a continuous real or complex-valued function on $\mathbf{R}^{n-1}$.

If $x \in \mathbf{R}^n_+$, then we would like to put

$$(10.12.1)\qquad \begin{aligned} u(x) &= \int_{\partial \mathbf{R}^n_+} P(x, y')\, f(y')\, dy' \\ &= \frac{2}{|\partial B(0,1)|} \int_{\partial \mathbf{R}^n_+} \frac{x_n}{|x - y'|^n}\, f(y')\, dy'. \end{aligned}$$

This is the *Poisson integral* of f on $\mathbf{R}^n_+$. More precisely, the integral on the right is defined as long as

$$(10.12.2)\qquad P(x, y')\, f(y') = \frac{2}{|\partial B(0,1)|}\, \frac{x_n}{|x - y'|^n}\, f(y')$$

is integrable as a function of y' on $\partial \mathbf{R}^n_+$. This corresponds to a continuous real or complex-valued function on $\mathbf{R}^{n-1}$, as before, whose integrability may be defined as in Subsection 7.2.3. One can check that the integrability of (10.12.2) is equivalent to the integrability of

$$(10.12.3)\qquad \min(1, |y'|^{-n})\, |f(y')|$$

on $\partial \mathbf{R}^n_+$.

Suppose that there are nonnegative real numbers a and $C(a)$ such that $a < 1$ and

$$(10.12.4)\qquad |f(y')| \le C(a)\, \max(1, |y'|^a)$$

for all $y' \in \partial \mathbf{R}^n_+$. This implies that

$$(10.12.5)\qquad \min(1, |y'|^{-n})\, |f(y')| \le C(a)\, \min(1, |y'|^{a-n})$$

for all $y' \in \partial \mathbf{R}^n_+$. The right side is integrable on $\partial \mathbf{R}^n_+$, because $n - a > n - 1$, as in Section 7.3. It follows that (10.12.3) is integrable on $\partial \mathbf{R}^{n-1}$ in this case.

10.12.1 Differentiating under the integral sign

Suppose that (10.12.3) is integrable on $\partial \mathbf{R}^n_+$. If α is a multi-index, then one can check that

$$(10.12.6)\qquad \frac{\partial^{|\alpha|}}{\partial x^\alpha}\, P(x, y')\, f(y')$$

is integrable as a function of $y' \in \partial \mathbf{R}^n_+$ for every $x \in \mathbf{R}^n_+$. In fact, it is not too difficult to show that $u(x)$ is smooth on $\mathbf{R}^n_+$, with

$$(10.12.7)\qquad \frac{\partial^{|\alpha|} u}{\partial x^\alpha}(x) = \int_{\partial \mathbf{R}^n_+} \frac{\partial^{|\alpha|}}{\partial x^\alpha}\, P(x, y')\, f(y')\, dy'$$

for every $x \in \mathbf{R}^n_+$.

In particular, this implies that

$$(10.12.8)\qquad u \text{ is harmonic on } \mathbf{R}^n_+,$$

because of (10.11.8). This could also be obtained by verifying that u is continuous on $\mathbf{R}^n_+$ and satisfies the mean-value property, as in Subsections 6.4.2 and 6.12.1.

If one is familiar with Lebesgue integrals, then one may consider real or complex-valued Lebesgue measurable functions f on $\partial\mathbf{R}^n_+$, which correspond to Lebesgue measurable functions on $\mathbf{R}^{n-1}$, as before. In this case, one should ask that (10.12.3) be Lebesgue integrable on $\partial\mathbf{R}^n_+$, in the sense that it corresponds to a Lebesgue integrable function on $\mathbf{R}^{n-1}$. This permits one to define the Poisson integral (10.12.1) as a Lebesgue integral for each $x \in \mathbf{R}^n_+$.

If α is a multi-index, then it follows that (10.12.6) is Lebesgue integrable as a function of $y' \in \partial\mathbf{R}^n_+$ for every $x \in \mathbf{R}^n_+$, and it is not too difficult to show that u is smooth on $\mathbf{R}^n_+$, with derivatives as in (10.12.7), as before. This implies (10.12.8), which could also be obtained using the mean-value property, as before.

10.13 Limits at points in $\partial\mathbf{R}^n_+$

Let $n \geq 2$ be an integer again, and let f be a continuous real or complex-valued function on $\partial\mathbf{R}^n_+$ such that (10.12.3) is integrable on $\partial\mathbf{R}^n_+$. Also let u be the Poisson integral of f on $\mathbf{R}^n_+$, as in (10.12.1), and let $x \in \mathbf{R}^n_+$ and $z' \in \partial\mathbf{R}^n_+$ be given. Observe that

$$\begin{aligned} u(x) - f(z') &= \int_{\partial\mathbf{R}^n_+} P(x,y') f(y')\,dy' - \int_{\partial\mathbf{R}^n_+} P(x,y')\,f(z')\,dz' \\ &= \int_{\partial\mathbf{R}^n_+} P(x,y')\,(f(y') - f(z'))\,dy', \end{aligned} \tag{10.13.1}$$

using (10.11.9) in the first step. It follows that

$$\begin{aligned} |u(x) - f(z')| &= \left| \int_{\partial\mathbf{R}^n_+} P(x,y')\,(f(y') - f(z'))\,dy' \right| \\ &\leq \int_{\partial\mathbf{R}^n_+} P(x,y')\,|f(y') - f(z')|\,dy', \end{aligned} \tag{10.13.2}$$

because

$$P(x,y') \geq 0 \tag{10.13.3}$$

for every $y' \in \partial\mathbf{R}^n_+$, by the definition (10.11.5) of the Poisson kernel.

One can use this to show that

$$u(x) \to f(z') \text{ as } x \to z'. \tag{10.13.4}$$

More precisely, if η is a positive real number, then the right side of (10.13.2) is equal to the sum of

$$\int_{\partial\mathbf{R}^n_+ \cap B(z',\eta)} P(x,y')\,|f(y') - f(z')|\,dy' \tag{10.13.5}$$

and

$$\int_{\partial \mathbf{R}^n_+ \setminus B(z',\eta)} P(x,y')\,|f(y') - f(z')|\,dy'. \tag{10.13.6}$$

If η is sufficiently small, then we can get that (10.13.5) is as small as we like, because f is continuous at z', and using (10.11.9) again. If $\eta > 0$ is fixed with this property, then we can get that (10.13.6) is as small as we like when x is sufficiently close to z'. This uses the integrability of (10.12.3) on $\partial \mathbf{R}^n_+$, by hypothesis, and the definition (10.11.5) of the Poisson kernel.

This means that

$$u \text{ extends continuously to } \overline{\mathbf{R}^n_+}, \tag{10.13.7}$$

by taking it equal to f on $\partial \mathbf{R}^n_+$. This corresponds to parts of 7.3 on p147 of [18], Theorem 14 in Section 2.2.4 b of [81], and Theorem 2.43 in Section F of Chapter 2 of [87], at least is f is bounded on $\partial \mathbf{R}^n_+$.

10.13.1 Some related results about limits

If one is familiar with Lebesgue integrals, then one may consider real and complex-valued Lebesgue measurable functions f on $\partial \mathbf{R}^n_+$ such that (10.12.3) is Lebesgue integrable on $\partial \mathbf{R}^n_+$, as mentioned in Subsection 10.12.1. If f is also continuous at z', then (10.13.4) holds, for essentially the same reasons as before. Otherwise, one may consider the convergence of $u(x)$ to $f(z')$ along the line where

$$x_j = z'_j \text{ for } j = 1, \ldots, n-1, \tag{10.13.8}$$

as $x_n \to 0+$, for almost every $z' \in \partial \mathbf{R}^n_+$ with respect to $(n-1)$-dimensional Lebegue measure. This corresponds to part (b) of Theorem 1 on p62 of [291], and to part of Theorem 1.25 on p13 of [297], at least when f is integrable on $\partial \mathbf{R}^n_+$. It is easy to reduce to this case when (10.12.3) is integrable on $\partial \mathbf{R}^n_+$, by considering separately parts of f supported on a bounded set, and its complement.

10.14 More on these Poisson integrals

Let $n \geq 2$ be an integer, and let f be a contnuous real or complex-valued function on $\partial \mathbf{R}^n_+$ such that (10.12.3) is integrable on $\partial \mathbf{R}^n_+$, as before. Note that the Poisson integral u of f, as in (10.12.1), satisfies

$$\begin{aligned} |u(x)| &\leq \int_{\partial \mathbf{R}^n_+} P(x,y')\,|f(y')|\,dy' \\ &= \frac{2}{|\partial B(0,1)|} \int_{\partial \mathbf{R}^n_+} \frac{x_n}{|x-y'|^n}\,|f(y')|\,dy' \end{aligned} \tag{10.14.1}$$

for every $x \in \mathbf{R}^n_+$, because of (10.13.3). This also works when f is Lebesgue measurable on $\partial \mathbf{R}^n_+$, and (10.12.3) is Lebesgue integrable on $\partial \mathbf{R}^n_+$.

If f is real-valued on $\partial \mathbf{R}^n_+$, then u is real-valued on $\mathbf{R}^n_+$. If f is also nonnegative on $\partial \mathbf{R}^n_+$, then

$$u \ge 0 \tag{10.14.2}$$

on $\mathbf{R}^n_+$, because of (10.13.3).

Similarly, if

$$f \ge a \tag{10.14.3}$$

on $\partial \mathbf{R}^n_+$ for some $a \in \mathbf{R}$, then

$$u \ge a \tag{10.14.4}$$

on $\mathbf{R}^n_+$, because of (10.11.9) and (10.13.3). If

$$f \le b \tag{10.14.5}$$

on $\partial \mathbf{R}^n_+$ for some $b \in \mathbf{R}$, then

$$u \le b \tag{10.14.6}$$

on $\mathbf{R}^n_+$.

If f is a bounded continuous complex-valued function on $\partial \mathbf{R}^n_+$, then (10.12.3) is integrable on $\partial \mathbf{R}^n_+$, as before. More precisely, if

$$|f| \le C \tag{10.14.7}$$

on $\partial \mathbf{R}^n_+$ for some nonnegative real number C, then

$$|u| \le C \tag{10.14.8}$$

on $\mathbf{R}^n_+$, because of (10.11.9) and (10.14.1). This works when f is a bounded Lebesgue measurable function on $\partial \mathbf{R}^n_+$ too. Note that if f is a constant on $\partial \mathbf{R}^n_+$, then u is equal to the same constant on $\mathbf{R}^n_+$, because of (10.11.9).

10.14.1 An integral estimate

If $x \in \mathbf{R}^n$, then put

$$\widetilde{x} = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}, \tag{10.14.9}$$

so that x is determined by $\widetilde{x}$ and $x_n \in \mathbf{R}$. If x_n is a positive real number, then it is easy to see that

$$\int_{\mathbf{R}^{n-1}} P(x, y')\, d\widetilde{x} = 1 \tag{10.14.10}$$

for every $y' \in \partial \mathbf{R}^n_+$, where $P(x, y')$ is considered as a function of $\widetilde{x}$ on $\mathbf{R}^{n-1}$. More precisely, this is equivalent to (10.11.9).

Suppose that f is a real or complex-valued function on $\partial \mathbf{R}^n_+$ that is continuous and integrable, or simply Lebesgue integrable. This implies that $u(x)$ is integrable as a function of $\widetilde{x} \in \mathbf{R}^{n-1}$ for every $x_n > 0$, with

$$\int_{\mathbf{R}^{n-1}} |u(x)|\, d\widetilde{x} \le \int_{\partial \mathbf{R}^n_+} |f(y')|\, dy'. \tag{10.14.11}$$

This can be obtained from (10.14.1) by interchanging the order of integration, and using (10.14.10). Similarly,

$$\int_{\mathbf{R}^{n-1}} u(x)\,d\widetilde{x} = \int_{\partial \mathbf{R}^n_+} f(y')\,dy' \tag{10.14.12}$$

for every $x_n > 0$.

Put

$$\widetilde{x}' = (x_1, \ldots, x_{n-1}, 0) \in \partial \mathbf{R}^n_+. \tag{10.14.13}$$

One can also show that

$$\lim_{x_n \to 0+} \int_{\mathbf{R}^{n-1}} |u(x) - f(\widetilde{x}')|\,d\widetilde{x} = 0. \tag{10.14.14}$$

This corresponds to taking $p = 1$ in Theorem 7.8 on p148 of [18], Theorem 2.43 in Section F of Chapter 2 of [87], part (c) of Theorem 1 on p62 of [291], and Theorem 1.18 on p10 of [297]. This is simpler when f is a continuous function on $\partial \mathbf{R}^n_+$ with compact support, and otherwise one can approximate f by such functions.

10.15 Harnack's principle

Let $n \geq 2$ be an integer, and let U be a nonempty connected open subset of $\mathbf{R}^n$. Also let $\{u_j\}_{j=1}^\infty$ be a sequence of real-valued harmonic functions on U that are monotonically increasing in j, so that

$$u_j(x) \leq u_{j+1}(x) \tag{10.15.1}$$

for every $x \in U$ and $j \geq 1$.

It is well known that a monotonically increasing sequence of real numbers converges in $\mathbf{R}$ if and only if it has an upper bound in $\mathbf{R}$. It follows that for each $x \in U$,

$$\{u_j(x)\}_{j=1}^\infty \text{ converges in } \mathbf{R} \tag{10.15.2}$$

if and only if $\{u_j(x)\}_{j=1}^\infty$ has an upper bound in $\mathbf{R}$.

Harnack's principle states that there are two possibilities in this situation. The first possibility is that there is a real-valued harmonic function u on U such that

$$\{u_j\}_{j=1}^\infty \text{ converges to } u \text{ uniformly on compact subsets of } U, \tag{10.15.3}$$

as in Subsection 6.5.2. The second possibility is that

$$u_j \to +\infty \text{ as } j \to \infty, \text{ uniformly on compact subsets of } U. \tag{10.15.4}$$

More precisely, this means that if K is a nonempty compact set in $\mathbf{R}^n$ such that $K \subseteq U$, and if R is a positive real number, then there is a positive integer $L = L(K, R)$ with the following property. If $x \in K$ and $j \geq L$, then

$$u_j(x) > R. \tag{10.15.5}$$

This corresponds to 3.8 on p49 of [18]. This also corresponds to Theorem 7 on p244 of [7] when $n = 2$, and which is stated somewhat more broadly.

10.15.1 Using Harnack's inequality

To show this, we may as well suppose that

$$u_j(x) \geq 0 \tag{10.15.6}$$

for every $x \in U$ and $j \geq 1$. Otherwise, we can simply replace u_j with $u_j - u_1$ for each j.

Let $x_0 \in U$ be given. If $\{u_j(x_0)\}_{j=1}^{\infty}$ does not have an upper bound in $\mathbf{R}$, then

$$u_j(x_0) \to +\infty \text{ as } j \to \infty, \tag{10.15.7}$$

because of monotonicity. In this case, it is easy to see that (10.15.4) holds, using Harnack's inequality, as in Subsection 10.2.1.

If $\{u_j(x_0)\}_{j=1}^{\infty}$ has an upper bound in $\mathbf{R}$, then

$$\{u_j\}_{j=1}^{\infty} \text{ is uniformly bounded on compact subsets of } U, \tag{10.15.8}$$

because of Harnack's inequality. In particular, for each $x \in U$,

$$\{u_j(x)\}_{j=1}^{\infty} \text{ has an upper bound in } \mathbf{R}, \tag{10.15.9}$$

and thus converges in $\mathbf{R}$, as before. Put

$$u(x) = \lim_{j \to \infty} u_j(x) \tag{10.15.10}$$

for each $x \in U$, which defines a real-valued function on U. We would like to show that (10.15.3) holds under these conditions. This would imply that u is harmonic on U, as in Subsection 6.5.3.

Let K be a nonempty compact subset of $\mathbf{R}^n$ that is contained in U. One can use Harnack's inequality to get that there is a real number $C_0 \geq 1$ such that

$$u_l(x) - u_j(x) \leq C_0\,(u_l(x_0) - u_j(x_0)) \tag{10.15.11}$$

for all $x \in K$ and $l \geq j \geq 1$. One can use this to get that

$$u(x) - u_j(x) \leq C_0\,(u(x_0) - u_j(x_0)) \tag{10.15.12}$$

for all $x \in K$ and $j \geq 1$. This means that (10.15.3) follows from the fact that $\{u_j(x_0)\}_{j=1}^{\infty}$ converges to $u(x_0)$ in $\mathbf{R}$.

Chapter 11

More on subharmonic functions

11.1 Continuous subharmonic functions

Let $n \geq 2$ be an integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuous real-valued function on U. Let us say that u is *subharmonic* on U if for every $a \in U$ and positive real number r with

$$\overline{B}(a, r) \subseteq U, \tag{11.1.1}$$

we have that

$$u(a) \leq \frac{1}{|\partial B(a, r)|} \int_{\partial B(a,r)} u(y')\, dy'. \tag{11.1.2}$$

This corresponds to the definition of subharmonicity on p76 of [297].

There are some well-known variations on this definition, some of which will be discussed here. On p224 of [18], for instance, it is asked that

$$\text{(11.1.2) hold for all sufficiently small } r > 0, \text{ depending on } a. \tag{11.1.3}$$

An exercise is mentioned where the formulation used here can be obtained from that version, and we shall say more about that later.

Sometimes upper semicontinuous functions on U are considered instead of continuous functions, as in Section 9.1. In this case, the appropriate integrals of u may be considered as Lebesgue integrals. One may also consider functions u that take the value $-\infty$ at some points in U. We shall not consider formulations like these here, for the sake of simplicity.

Suppose for the moment that $n = 1$, and that U is an open interval in $\mathbf{R}$, which may be unbounded. Under these conditions, the previous definition of subharmonicity, suitably interpreted, corresponds to the characterization of convexity in Subsection 9.6.2, as mentioned on p75 of [297].

11.1.1 Averages over balls

Let $a \in U$ be given, and let t be a positive real number such that

$$\overline{B}(a,t) \subseteq U. \tag{11.1.4}$$

If (11.1.2) holds for every positive real number r with $r \le t$, then one can check that

$$u(a) \le \frac{1}{|B(a,t)|} \int_{B(a,t)} u(x)\,dx. \tag{11.1.5}$$

More precisely, this can be seen using the fact that

$$\int_{B(a,t)} u(x)\,dx = \int_0^t \Big(\int_{\partial B(a,r)} u(y')\,dy'\Big)\,dr, \tag{11.1.6}$$

which corresponds to using polar coordinates centered at a, as in Section 6.3.

In particular, the first version of subharmonicity mentioned earlier implies that (11.1.5) holds when (11.1.4) holds. The second version of subharmonicity mentioned earlier implies that

$$\text{(11.1.5) holds for all sufficiently small } t > 0, \text{ depending on } a. \tag{11.1.7}$$

As another version of subharmonicity, we shall consider the condition that

$$\text{(11.1.5) hold for some arbitrarily small } t > 0, \text{ depending on } a. \tag{11.1.8}$$

Note that u satisfies the mean-value property on U, as in Section 6.3, if and only if

$$u \text{ and } -u \text{ are subharmonic on } U, \tag{11.1.9}$$

using the first version of subharmonicity mentioned earlier.

11.2 More on subharmonicity conditions

Let $n \ge 2$ be an integer, let U be a nonempty open subset of $\mathbf{R}^n$, and suppose that u is twice continuously differentiable on U. In Section 6.14, subharmonicity of u was defined to mean that

$$\Delta u \ge 0 \tag{11.2.1}$$

on U. This implies (11.1.2) when (11.1.1) holds, as before, and as in Theorem 4.3 on p76 of [297].

Suppose for the moment that

$$\Delta u(a) > 0 \tag{11.2.2}$$

for some $a \in U$. In this case, one can check that

$$u(a) < \frac{1}{|\partial B(0,1)|} \int_{\partial B(a,r)} u(y')\,dy' \tag{11.2.3}$$

when $r > 0$ is sufficiently small. This can be obtained in essentially the same way as in Section 6.14, or using the Taylor approximation to u at a of degree 2, as in Subsection 6.3.2. We also have that

$$u(a) < \frac{1}{|B(a,t)|} \int_{B(a,t)} u(x)\, dx \tag{11.2.4}$$

when $t > 0$ is sufficiently small. This can be obtained from (11.2.3), or using the Taylor approximation to u at a of degree 2, as before.

If (11.2.2) holds for every $a \in U$, then it follows that u satisfies the second version of subharmonicity mentioned in the previous section, as in (11.1.3). This corresponds to part of the "if" part of Exercise 8 on p236 of [18].

11.2.1 Getting a nonnegative Laplacian

Suppose for the moment again that

$$\Delta u(a) < 0 \tag{11.2.5}$$

for some $a \in U$. This implies that

$$\frac{1}{|\partial B(0,1)|} \int_{\partial B(a,r)} u(y')\, dy' < u(a) \tag{11.2.6}$$

for all sufficiently small $r > 0$, as before. Similarly,

$$\frac{1}{|B(0,1)|} \int_{B(a,t)} u(x)\, dx < u(a) \tag{11.2.7}$$

for all sufficiently small $t > 0$.

This means that u does not satisfy any of the versions of subharmonicity on U mentioned in the previous section when (11.2.5) holds for some $a \in U$. Thus, if u satisfies any of the versions of subharmonicity on U mentioned in the previous section, and if u is twice continuously differentiable on U, then (11.2.1) holds on U. This corresponds to the "only if" part of Exercise 8 on p236 of [18].

11.3 Some properties of subharmonic functions

Let $n \geq 2$ be an integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let u, v be continuous real-valued functions on U. If u satisfies any of the subharmonicity conditions mentioned in Section 11.1 and a is a nonnegative real number, then it is easy to see that

(11.3.1) $\quad a\, u$ satisfies the same subharmonicity condition on U.

If u and v both satisfy the first version of subharmonicity mentioned in Section 11.1, then

(11.3.2) $\quad u + v$ satisfies the same subharmonicity condition on U.

This also works when u and v both satisfy (11.1.3), and when they both satisfy (11.1.7).

If u and v both satisfy the first version of subharmonicity mentioned in Section 11.1 again, then one can check that

$$\max(u, v) \text{ satisfies the same subharmonicity condition on } U. \tag{11.3.3}$$

This works as well when u and v both satisfy (11.1.3), and when they both satisfy (11.1.7). This corresponds to Proposition 11.4 on 225 of [18], and to part of (4) on p79 of [297].

11.3.1 Subharmonicity and uniform convergence

Let $\{u_j\}_{j=1}^{\infty}$ be a sequence of continuous real-valued functions on U that converges to u uniformly on compact subsets of U, as in Subsection 6.5.2. If u_j satisfies the first version of subharmonicity mentioned in Section 11.1 for each j, then u satisfies this condition too.

11.3.2 Compositions with convex functions

Let I be an open interval in the real line, which may be unbounded, and let f be a convex real-valued function on I. Suppose that u takes values in I on U, so that the composition $f \circ u$ is defined as a real-valued function on U. Note that

$$f \circ u \text{ is continuous on } U, \tag{11.3.4}$$

because f is continuous on I, as in Subsection 9.6.1, and u is continuous on U, by hypothesis.

If u is harmonic on U, then one can check that

$$f \circ u \text{ is subharmonic on } U, \tag{11.3.5}$$

using the integral version of Jensen's inequality, as mentioned in Section 9.9.2. This corresponds to part (c) of Problem 5 in Section 2.5 of [81] when $I = \mathbf{R}$ and f is smooth.

Suppose now that f is also monotonically increasing on I. If u satisfies any of the subharmonicity conditions mentioned in Section 11.1, then one can check that $f \circ u$ satisfies the same condition on U, using the integral version of Jen'ssen inequality again. This corresponds to Execise 3 on p236 of [18], and to (2) on p79 of [297].

11.3.3 Convex functions are subharmonic

Suppose that U is also convex, and that u is convex on U, as in Section 9.12. One can check that u is subharmonic on U.

11.4 More on the maximum principle

Let $n \geq 2$ be an integer, let V be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuous real-valued function on V. Suppose that

$$\text{for every } a \in V \text{ there is a } t > 0 \text{ such that } \overline{B}(a,t) \subseteq V \text{ and (11.1.5) holds.} \tag{11.4.1}$$

If u attains its maximum on V, and if V is connected, then

$$u \text{ is constant on } V, \tag{11.4.2}$$

as in Subsection 6.14.3.

Let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuous real-valued function on U. Suppose that u satisfies (11.1.8) on U. If V is a nonempty open set contained in U, then

$$\text{the restriction of } u \text{ to } V \text{ satisfies (11.4.1).} \tag{11.4.3}$$

11.4.1 A helpful version

Let V be a bounded nonempty open set in $\mathbf{R}^n$, and let u be a continuous real-valued function on $\overline{V}$ that satisfies (11.4.1) on V. Under these conditions, the maximum of u on $\overline{V}$ is attained at a point in ∂V, as in Subsection 6.14.3.

Let v be another continuous real-valued function on $\overline{V}$, and suppose that

$$v \text{ is harmonic on } V. \tag{11.4.4}$$

If

$$u \leq v \text{ on } \partial V, \tag{11.4.5}$$

then

$$u \leq v \text{ on } \overline{V}. \tag{11.4.6}$$

This is a version of Theorem 11.3 on p225 of [18], and of Theorem 4.5 on p78 of [297].

To see this, note that $u - v$ is a continuous real-valued function on $\overline{V}$, and that

$$u - v \leq 0 \text{ on } \partial V, \tag{11.4.7}$$

by (11.4.5). Of course, v satisfies the mean-value property on V, because of (11.4.4), as in Section 6.2. One can use this to get that $u - v$ satisfies the analogue of (11.4.1) on V, because u satisfies (11.4.1) on V, by hypothesis. This implies that the maximum of $u - v$ on $\overline{V}$ is attained as a point in ∂V, as before. It follows that

$$u - v \leq 0 \text{ on } \overline{V}, \tag{11.4.8}$$

because of (11.4.7).

11.5 Using the Poisson integral

Let $n \geq 2$ be an integer, let $a \in \mathbf{R}^n$ be given, and let r be a positive real number. Also let u be a continuous real-valued function on $\overline{B}(a, r)$, and suppose that the restriction of u to

$$V = B(a, r) \tag{11.5.1}$$

satisfies (11.4.1). We would like to show that (11.1.2) holds under these conditions.

We can reduce to the case where $a = 0$ and $r = 1$, using a translation and a dilation, so that

$$V = B(0, 1). \tag{11.5.2}$$

Let v be the real-valued function defined on

$$\overline{V} = \overline{B}(0, 1) \tag{11.5.3}$$

using the Poisson integral of the restriction of u to $\partial B(0, 1)$, as in Section 6.12. Observe that

$$u \leq v \text{ on } \overline{B}(0, 1), \tag{11.5.4}$$

as in (11.4.6). This implies that

$$u(0) \leq v(0) \tag{11.5.5}$$

in particular. This is the same as (11.1.2) in this case.

11.5.1 A monotonicity property

If r_0 is a real number with $0 < r_0 < r$, then

$$\frac{1}{|\partial B(a, r_0)|} \int_{\partial B(a, r_0)} u(z')\, dz' \leq \frac{1}{|\partial B(a, r)|} \int_{\partial B(a, r)} u(y')\, dy'. \tag{11.5.6}$$

This corresponds to the first part of Exercise 5 on p236 of [18].

We can reduce to the case where $a = 0$ and $r = 1$, as before. If v is as before, then (11.5.4) implies that

$$\frac{1}{|\partial B(0, r_0)|} \int_{\partial B(0, r_0)} u(z')\, dz' \leq \frac{1}{|\partial B(0, r_0)|} \int_{\partial B(0, r_0)} v(z')\, dz'. \tag{11.5.7}$$

It follows that

$$\frac{1}{|\partial B(0, r_0)|} \int_{\partial B(0, r_0)} u(z')\, dz' \leq v(0), \tag{11.5.8}$$

because v is harmonic on $B(0, 1)$, and thus satisfies the mean-value property.

11.6 Some related characterizations

Let $n \geq 2$ be an integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuous real-valued function on U. We would like to check that u is subharmonic on U if and only if it has the following property. Let V be a nonempty bounded open set in $\mathbf{R}^n$ such that

$$\overline{V} \subseteq U. \tag{11.6.1}$$

and let v be a continuous real-valued function on $\overline{V}$ that is harmonic on V. If (11.4.5) holds, then this property asks that (11.4.6) hold. This corresponds to Exercise 4 on p236 of [18].

That this property holds when u is subharmonic on U follows from the remarks in Subsection 11.4.1. The converse can be obtained using an argument like the one at the beginning of the previous section.

11.6.1 A variant of this property

Similarly, let us check that u is subharmonic on U if and only if it has the following property. Let V be a nonempty connected open set in $\mathbf{R}^n$ such that

$$V \subseteq U, \tag{11.6.2}$$

and let v be a real-valued harmonic function on V. The property asks that

$$u - v \text{ satisfy the strong maximum principle on } V, \tag{11.6.3}$$

as in Section 6.7.1. This means that if $u - v$ attains its maximum on V, then $u - v$ is constant on V.

One can verify that this property implies the one mentioned at the beginnign of the section. This uses the same argument as in Subsection 6.7.2. Conversely, if u s subharmonic on U, then this property may be obtained from the version of the strong maximum principle mentioned at the beginning of Section 11.4.

This characterization of subharmonicity corresponds to Definition 1 on p245 of [7], at least when $n = 2$ and U is connected.

11.6.2 Using milder differentiability conditions

Suppose that the first and second derivatives lf u in each variable exist at every point in U, as in Subsection 6.15.4. If the Laplacian of u is defined on U as before, and if

$$\Delta u \geq 0 \text{ on } U, \tag{11.6.4}$$

then u is subharmonic on U.

To see this, we can use the characterization of subharmonicity mentioned at the beginning of the section. Let V and v be as before, and note that the first and second derivatives of $u - v$ in each variable exist at evry point in V, because v is smooth on V. We also have that

$$\Delta(u - v) = \Delta u \geq 0 \text{ on } V, \tag{11.6.5}$$

because v is harmonic on V. This implies that the maximum of $u - v$ on $\overline{V}$ is attained on ∂V, as in Subsections 6.15.3 and 6.15.4. This means that (11.4.8) holds when (11.4.7) holds, so that u has the property mentioned at the beginning of the section.

11.7 Subharmonicity at a point

Let $n \geq 2$ be an integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuous real-valued function on U. Let us say that u is *subharmonic at a point* $x_0 \in U$ if there is an open set U_0 in $\mathbf{R}^n$ such that $x_0 \in U_0$, $U_0 \subseteq U$, and

$$\text{the restriction of } u \text{ to } U_0 \text{ is subharmonic on } U_0, \tag{11.7.1}$$

as on p245 of [7]. Of course, this depends on the definition of suharmonicity that is being used.

If one uses the first version of subharmonicity in Section 11.1, then it is clear that subharmonicity on U implies subharmonicity at every point in U. If one uses the second version of subharmonicity in Section 11.1, then subharmonicity on U is clearly equivalent to subharmonicity at every point in U. This also works for the two other versions of subharmonicity mentioned in Section 11.1. If u is subharmonic at every point in U with respect to the first version of subharmonicity in Section 11.1, then u is subharmonic on U with respect to the second version of subharmonicity considered there. These subharmonicity conditions are in fact equivalent to each other, as in Section 11.5.

11.7.1 Using Ahlfors' definition of subharmonicity

If u satisfies the subharmonicity property described in Subsection 11.6.1, then it clear that u satisfies this property at every point in U, in the sense mentioned before. It is not too difficult to get the converse directly, as mentioned on p245 of [7].

Let V be a nonempty connected open subset of $\mathbf{R}^n$ that is contained in U, and let v be a real-valued harmonic function on V. Suppose that $u - v$ attains its maximum on V, and that u satisfies the subharmonicity property described in Subsection 11.6.1 at every point in U. One can use this to obtain that the subset of V on which $u - v$ attains its maximum is an open set. This set is also relatively closed in V, because $u - v$ is continuous on V. It follows that this set is equal to V, because V is connected, by hypothesis.

11.7.2 A variant of Ahlfors' definition

Here is another subharmonicity property for u on U. Let W be a nonempty open set in $\mathbf{R}^n$ that is contained in U, and let v be a real-valued harmonic function on W. If

$$u - v \text{ has a local maximum at point } y \in W, \tag{11.7.2}$$

then the property asks that

$$u - v \text{ be constant near } y. \tag{11.7.3}$$

This means that

$$u(x) - v(x) = u(y) - v(y) \tag{11.7.4}$$

for all $x \in W$ sufficiently close to y.

If u satisfies Ahlfors' subharmonicity property at every point in U, then it is easy to see that u satisfies this property. More precisely, this follows by restricting v to a connected open set V contained in W such that $y \in V$ and $u - v$ attains its maximum on V at y. If u satisfies the subharmonicity property described in the preceding paragraph at every point in U, then u clearly satisfies this subharmonicity property on U.

If u satisfies the subharmonicity condition considered in this subsection, then one can check directly that u satisfies Ahlfors' subharmonicity property. This uses the same argument as in the previous subsection.

11.8 Poisson modifications

Let $n \geq 2$ be an integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a continuous real-valued function on U that is subharmonic on U. Also let $a \in U$ be given, and let R be a positive real number such that

$$\overline{B}(a, R) \subseteq U. \tag{11.8.1}$$

Consider the continuous real-valued function v on $\overline{B}(a, R)$ that is harmonic on $B(a, R)$ and equal to u on $\partial B(a, R)$. This may be obtained using the Poisson integral when $a = 0$ and $R = 1$, as in Section 6.12, and otherwise one can reduce to that case using translations and dilations.

Put

$$\begin{aligned} w &= u \quad \text{on } U \setminus B(a, R) \\ &= v \quad \text{on } B(a, R), \end{aligned} \tag{11.8.2}$$

so that w is a real-valued function on U. This may be called the *Poisson modification* of u associated to $B(a, R)$ on U, as on p225 of [18]. One can check that

$$w \text{ is continuous on } U \tag{11.8.3}$$

under these conditions. In fact,

$$w \text{ is subharmonic on } U. \tag{11.8.4}$$

This corresponds to property 4 on p247 of [7] when $n = 2$, and to Theorem 11.5 on p225 of [18].

Note that

$$u \leq v \text{ on } \overline{B}(a, R), \tag{11.8.5}$$

as in Subsection 11.4.1. This implies that

$$u \le w \text{ on } U. \tag{11.8.6}$$

In order to get (11.8.4), it is enough to verify then w satisfies the sub-mean-value property at every point $x \in U$, for sufficiently small radius, and using averages of w over spheres or balls. If $x \in U \setminus B(a, R)$, so that $w(x) = u(x)$, then this follows from the analogous property of u, because of (11.8.6). If $x \in B(a, R)$, then one can use the fact that v satisfies the mean-value property on $B(a, R)$.

11.8.1 A couple of additional remarks

Let u_1 be another continuous real-valued function on U that is subharmonic on U, and let v_1 and w_1 be as before. If

$$u \le u_1 \text{ on } U, \tag{11.8.7}$$

then one can check that

$$v \le v_1 \text{ on } \overline{B}(a, R). \tag{11.8.8}$$

This implies that

$$w \le w_1 \text{ on } U. \tag{11.8.9}$$

Suppose now that u is a continuous real-valued function on $\overline{U}$ that is subharmonic on U. If w is defined on U as in (11.8.2) and we put $w = u$ on ∂U, then w is continuous on $\overline{U}$ as well.

11.9 The Perron process

Let $n \ge 2$ be an integer, let U be a nonempty bounded open subset of $\mathbf{R}^n$, and let f be a continuous real-valued function on ∂U. Note that

$$f \text{ is bounded on } \partial U, \tag{11.9.1}$$

because ∂U is closed and bounded, and thus compact, as in Section 1.9.

Let

$$P_f \tag{11.9.2}$$

be the collection of continuous real-valued functions u on $\overline{U}$ that are subharmonic on U and satisfy

$$u \le f \text{ on } \partial U. \tag{11.9.3}$$

This may be called the *Perron family* associated to f, as on p226 of [18]. If α is a real number such that

$$\alpha \le f \text{ on } \partial U, \tag{11.9.4}$$

then the constant function on $\overline{U}$ equal to α is an element of P_f.

Let β be a real number such that

$$f \le \beta \text{ on } \partial U. \tag{11.9.5}$$

If $u \in P_f$, then

$$u \le \beta \text{ on } \overline{U}. \tag{11.9.6}$$

This follows from the maximum principle, as mentioned at the beginning of Subsection 11.4.1.

If $x \in \overline{U}$, then put

$$h_f(x) = \sup\{u(x) : u \in P_f\}. \tag{11.9.7}$$

This may be called the *Perron function* on $\overline{U}$ associated to f, as on p226 of [18]. More precisely, if $\beta \in \mathbf{R}$ is as in (11.9.5), then

$$h_f \le \beta \text{ on } \overline{U}, \tag{11.9.8}$$

because of (11.9.6). If α is a real number that satisfies (11.9.4), then

$$\alpha \le h_f \text{ on } \overline{U}. \tag{11.9.9}$$

11.9.1 The Perron function is harmonic

It is well known that

$$h_f \text{ is harmonic on } U, \tag{11.9.10}$$

as in Theorem 11.6 on p226 of [18]. This basically corresponds to Lemma 1 on p248 of [7] as well, at least when $n = 2$. The construction in [7] is a bit more complicated, so that one may consider functions f on ∂U that may not be continuous.

The arguments in [7, 18] are quite similar in the beginning, and somewhat different after that. The first part will be discussed in the rest of this section, and the two arguments for the second part will be discussed in the next two sections.

Let $a \in U$ and $R > 0$ be given, such that (11.8.1) holds. It suffices to show that

$$h_f \text{ is harmonic on } B(a, R). \tag{11.9.11}$$

Let

$$x_0 \in B(a, R) \tag{11.9.12}$$

be given. If j s a positive integer, then let $u_{0,j}$ be an element of P_f such that

$$u_{0,j}(x_0) > h_f(x_0) - 1/j. \tag{11.9.13}$$

In [18], one takes $x_0 = a$.

Put

$$\widetilde{u}_{0,j} = \max(u_{0,1}, \ldots, u_{0,j}) \tag{11.9.14}$$

for each j. Observe that

$$\widetilde{u}_{0,j} \in P_f \tag{11.9.15}$$

and

$$\widetilde{u}_{0,j}(x_0) \geq u_{0,j}(x_0) > h_f(x_0) - 1/j \tag{11.9.16}$$

for each j. We also have that

$$\widetilde{u}_{0,j} \leq \widetilde{u}_{0,j+1} \tag{11.9.17}$$

on $\overline{U}$ for each j.

11.9.2 Using Poisson modifications

Let $w_{0,j}$ be the Poisson modification of $\widetilde{u}_{0,j}$ associated to $B(a,R)$, as in the previous section. It is easy to see that

$$w_{0,j} \in P_j \tag{11.9.18}$$

and

$$w_{0,j}(x_0) \geq \widetilde{u}_{0,j}(x_0) > h_f(x_0) - 1/j \tag{11.9.19}$$

for each j. Using (11.9.17), we get that

$$w_{0,j} \leq w_{0,j+1} \tag{11.9.20}$$

on $\overline{U}$ for each j, as in (11.8.9).

Of course,

$$w_{0,j} \leq h_f \text{ on } \overline{U} \tag{11.9.21}$$

for each j, by construction. This implies that $\{w_{0,j}\}_{j=1}^{\infty}$ converges pointwise on $\overline{U}$, because of (11.9.20). Put

$$w_0(x) = \lim_{j \to \infty} w_{0,j}(x) \tag{11.9.22}$$

for every $x \in \overline{U}$, so that w_0 is a real-valued function on $\overline{U}$. Note that

$$w_0(x) \leq h_f(x) \tag{11.9.23}$$

for every $x \in \overline{U}$. In fact,

$$w_0(x_0) = h_f(x_0), \tag{11.9.24}$$

because of (11.9.19).

11.9.3 Using Harnack's principle

Remember that $w_{0,j}$ is harmonic on $B(a,R)$ for each j, as in the previous section. Harnack's principle implies that

$$\{w_{0,j}\}_{j=1}^{\infty} \text{ converges to } w_0 \text{ uniformly on compact subsets of } B(a,R), \tag{11.9.25}$$

and that

$$w_0 \text{ is harmonic on } B(a,R), \tag{11.9.26}$$

as in Section 10.15.

In order to get (11.9.11), it suffices to show that

$$h_f = w_0 \tag{11.9.27}$$

on $B(a,R)$. We shall discuss two proofs of this in the next two sections, following [7, 18], as before.

11.10 The first argument

Let us continue with the same notation and hypotheses as in the previous section. In order to get (11.9.27) using the argument from [7], let $x_1 \in B(a,R)$ be given, and let us show that

$$h_f(x_1) = w_0(x_1). \tag{11.10.1}$$

If j is a positive integer, then let $u_{1,j}$ be an element of P_j such that

$$u_{1,j}(x_1) > h_f(x_1) - 1/j, \tag{11.10.2}$$

as before. Before continuing with the earlier construction, put

$$u_j = \max(u_{0,j}, u_{1,j}) \tag{11.10.3}$$

for each j. Observe that

$$u_j \in P_f \tag{11.10.4}$$

for each j, with

$$u_j(x_0) \geq u_{0,j}(x_0) > h_f(x_0) - 1/j \tag{11.10.5}$$

and

$$u_j(x_1) \geq u_{1,j}(x_1) > h_f(x_1) - 1/j. \tag{11.10.6}$$

Put

$$\widetilde{u}_j = \max(u_1, \ldots, u_j) \tag{11.10.7}$$

for each j. Note that

$$\widetilde{u}_j \in P_f \tag{11.10.8}$$

for each j, as before. Of course,

$$\widetilde{u}_j(x_0) \ge u_j(x_0) > h_f(x_0) - 1/j \tag{11.10.9}$$

and

$$\widetilde{u}_j(x_1) \ge u_j(x_1) > h_f(x_1) - 1/j \tag{11.10.10}$$

for each j. By construction,

$$u_j \le u_{j+1} \tag{11.10.11}$$

on $\overline{U}$ for each j.

11.10.1 Using Poisson modifications again

Let $w_{1,j}$ be the Poisson modification of u_j associated to $B(a, R)$, so that

$$w_{1,j} \in P_f \tag{11.10.12}$$

for each j, as before. We also have that

$$w_{1,j}(x_0) \ge u_j(x_0) > h_f(x_0) - 1/j \tag{11.10.13}$$

and

$$w_{1,j}(x_1) \ge u_j(x_1) > h_f(x_1) - 1/j \tag{11.10.14}$$

for each j. In addition,

$$w_{1,j} \le w_{1,j+1} \tag{11.10.15}$$

on $\overline{U}$ for each j, because of (11.10.11). Note that

$$w_{1,j} \le h_f \text{ on } \overline{U} \tag{11.10.16}$$

for each j, by construction.

As before, $\{w_{1,j}\}_{j=1}^{\infty}$ converges pointwise on $\overline{U}$, and we put

$$w_1(x) = \lim_{j \to \infty} w_{1,j}(x) \tag{11.10.17}$$

for every $x \in \overline{U}$. This is a real-valued function on $\overline{U}$, with

$$w_1(x) \le h_f(x) \tag{11.10.18}$$

for every $x \in \overline{U}$, because of (11.10.16). Using (11.10.13) and (11.10.14), we get that

$$w_1(x_0) = h_f(x_0) \tag{11.10.19}$$

and

$$w_1(x_1) = h_f(x_1), \tag{11.10.20}$$

respectively.

Harnack's principle implies that $\{w_{1,j}\}_{j=1}^{\infty}$ converges to w_1 uniformly on compact subsets of $B(a, R)$, and that

$$w_1 \text{ is harmonic on } B(a, R), \tag{11.10.21}$$

as before.

11.10.2 Comparing w_0 and w_1

Of course,

$$u_{0,j} \le u_j \tag{11.10.22}$$

on $\overline{U}$ for each j, by construction. This implies that

$$\widetilde{u}_{0,j} \le \widetilde{u}_j \tag{11.10.23}$$

on $\overline{U}$ for each j. It follows that

$$w_{0,j} \le w_{1,j} \tag{11.10.24}$$

on $\overline{U}$ for each j, as in (11.8.9). This means that

$$w_0 \le w_1 \tag{11.10.25}$$

on $\overline{U}$.

Note that

$$w_0(x_0) = w_1(x_0), \tag{11.10.26}$$

by (11.9.24) and (11.10.19). It follows that $w_1 - w_0$ attains its maximum at x_0, because of (11.10.25). Using the strong maximum principle, as in Subsection 6.7.1, we obtain that

$$w_0 \equiv w_1 \text{ on } B(a, R). \tag{11.10.27}$$

This implies (11.10.1), by (11.10.20).

11.11 The second argument

We continue with the same notation and hypotheses from Section 11.9. We would like to show (11.9.27), using the argument from [18], as before. In this argument, we take

$$x_0 = a, \tag{11.11.1}$$

as mentioned in Subsection 11.9.1.

Let v be any element of P_f. We would like to show that

$$v \le w_0 \tag{11.11.2}$$

on $B(a, R)$. This will imply that

$$h_f \le w_0 \tag{11.11.3}$$

on $B(a, R)$. Using this, (11.9.27) will follow from (11.9.23).

11.11.1 Using some more Poisson modifications

Observe that

$$\max(w_{0,j}, v) \tag{11.11.4}$$

is an element of P_f for each j, because of (11.9.18). Let v_j be the Poisson modification of (11.11.4) associated to $B(a, R)$ for each j.

It is easy to see that

$$v_j \in P_f \tag{11.11.5}$$

for each j, as before. This implies that

$$v_j(a) \leq w_0(a) \tag{11.11.6}$$

for each j, because of (11.9.24) and (11.11.1).

Of course, v_j is harmonic on $B(a, R)$ for each j, by construction, and

$$\max(w_{0,j}, v) \leq v_j \tag{11.11.7}$$

on $\overline{U}$ for each j, as in Section 11.8. If r is a positive real number strictly less than R, then

$$w_0(a) \geq v_j(a) = \frac{1}{|\partial B(a,r)|} \int_{\partial B(a,r)} v_j(y')\, dy' \tag{11.11.8}$$

for each j, using the mean-value property in the second step. This implies that

$$w_0(a) \geq \frac{1}{|\partial B(a,r)|} \int_{\partial B(a,r)} \max(w_{0,j}(y'), v(y'))\, dy' \tag{11.11.9}$$

for each j.

One can check that

$$\begin{aligned} &\{\max(w_{0,j}, v)\}_{j=1}^{\infty} \text{ converges to } \max(w_0, v) \\ &\text{uniformly on compact subsets of } B(a, R), \end{aligned} \tag{11.11.10}$$

using (11.9.25). One can use this and (11.11.9) to get that

$$w_0(a) \geq \frac{1}{|\partial B(a,r)|} \int_{\partial B(a,r)} \max(w_0(y'), v(y'))\, dy'. \tag{11.11.11}$$

This means that

$$\begin{aligned} &\frac{1}{|\partial B(a,r)|} \int_{\partial B(a,r)} w_0(y')\, dy' \\ &\quad \geq \frac{1}{|\partial B(a,r)|} \int_{\partial B(a,r)} \max(w_0(y'), v(y'))\, dy', \end{aligned} \tag{11.11.12}$$

because of (11.9.26). It follows that (11.11.2) holds on $\partial B(a, r)$. This implies that (11.11.2) holds on $B(a, R)$, because $0 < r < R$ is arbitrary.

11.12 More on the Dirichlet problem

Let us continue with the same notation and hypotheses as in Section 11.9 again. Suppose for the moment that there is a continuous real-valued function h on $\overline{U}$ such that

$$h \text{ is harmonic on } U \tag{11.12.1}$$

and

$$h = f \text{ on } \partial U. \tag{11.12.2}$$

Let us check that

$$h = h_f \text{ on } \overline{U}. \tag{11.12.3}$$

This corresponds to a remark on p249 of [7].

Clearly

$$h \in P_f, \tag{11.12.4}$$

so that

$$h \le h_f \text{ on } \overline{U}, \tag{11.12.5}$$

by the definition of h_f. If $u \in P_f$, then

$$u \le h \tag{11.12.6}$$

on $\overline{U}$, as in Subsection 11.4.1. This implies that

$$h_f \le h \text{ on } \overline{U}, \tag{11.12.7}$$

by the definition of h_f.

We would like to have conditions under which

$$h_f \text{ is continuous on } \overline{U} \tag{11.12.8}$$

and

$$h_f = f \text{ on } \partial U. \tag{11.12.9}$$

More precisely, we already know that h_f is continuous on U, because of (11.9.10). In order to get (11.12.8), it suffices to show that h_f is continuous at every point in ∂U, as a function on $\overline{U}$. This will be discussed further in the next subsection.

11.12.1 Barriers

Let $\zeta_0 \in \partial U$ be given. A continuous real-valued function u_0 on $\overline{U}$ is said to be a *barrier* for U at ζ_0 if it satisfies the following three conditions. First,

$$u_0 \text{ is subharmonic on } U. \tag{11.12.10}$$

Second,

$$u_0 < 0 \text{ on } \overline{U} \setminus \{\zeta_0\}. \tag{11.12.11}$$

Third,

$$u_0(\zeta_0) = 0. \tag{11.12.12}$$

This is mentioned on p227 of [18]. A variant of this is mentioned on p250 of [7], with u_0 asked to be harmonic on U, and we shall see a bit more about that in the next section.

If there is a barrier u_0 for U at ζ_0, then

$$h_f(\zeta_0) = f(\zeta_0) \tag{11.12.13}$$

and

$$h_f \text{ is continuous at } \zeta_0, \text{ as a function on } \overline{U}. \tag{11.12.14}$$

This corresponds to Lemma 2 on p250 of [7] and Theorem 11.7 on p228 of [18], although the former is stated somewhat more broadly, as before.

To see this, let $\epsilon > 0$ be given, and let δ be a positive real number such that

$$f(\zeta_0) - \epsilon < f < f(\zeta_0) + \epsilon \tag{11.12.15}$$

on $\partial U \cap B(\zeta_0, \delta)$. This uses the continuity of f at ζ_0, as a function on ∂U, by hypothesis.

Note that

$$\overline{U} \setminus B(\zeta_0, \delta) \text{ is compact,} \tag{11.12.16}$$

because it is closed and bounded. Put

$$\begin{aligned} c_0 &= \min\{-u_0(x) : x \in \overline{U} \setminus B(\zeta_0, \delta)\} \\ &= -\max\{u_0(x) : x \in \overline{U} \setminus B(\zeta_0, \delta)\}, \end{aligned} \tag{11.12.17}$$

where the minimum or maximum is attained, by the extreme value theorem. We also have that

$$c_0 > 0, \tag{11.12.18}$$

by (11.12.11).

Because f is bounded on ∂U, one can use (11.12.18) to get that there is a positive real number C such that

$$f(\zeta_0) - \epsilon + C\,u_0 < f < f(\zeta_0) + \epsilon - C\,u_0 \tag{11.12.19}$$

on $\overline{U} \setminus B(\zeta_0, \delta)$. More precisely, it suffices to choose C so that

$$|f - f(\zeta_0)| \le C\,c_0 \tag{11.12.20}$$

on $\overline{U} \setminus B(\zeta_o, \delta)$. Observe that (11.12.19) holds on $B(\zeta_0, \delta)$ as well, because of (11.12.15), and because $u_0 \le 0$ on $\overline{U}$, as in (11.12.11) and (11.12.12). This means that (11.12.19) holds on $\overline{U}$.

11.12.2 Estimating h_f near ζ_0

We would like to check that

$$f(\zeta_0) - \epsilon + C\,u_0 \le h_f \le f(\zeta_0) + \epsilon - C\,u_0 \tag{11.12.21}$$

on $\overline{U}$. Observe that

$$f(\zeta_0) - \epsilon + C\, u_0 \in P_f, \tag{11.12.22}$$

because of (11.12.10), (11.12.19), and the definition of P_f. This implies the first inequality in (11.12.21), because of the way that h_f is defined.

To get the second inequality in (11.12.21), let $v \in P_f$ be given. Thus $v \le f$ on ∂U, by definition of P_f, so that

$$v + C\, u_0 \le f(\zeta_0) + \epsilon \tag{11.12.23}$$

on ∂U, by (11.12.19). It follows that (11.12.23) holds on $\overline{U}$, by the maximum principle, as in Subsection 11.4.1, because

$$v + C\, u_0 \tag{11.12.24}$$

is continuous on $\overline{U}$ and subharmonic on U. In fact, the inequality in (11.12.23) is strict, although we do not need that here. This implies the second inequality in (11.12.21).

In particular, we can use (11.12.21) to get that

$$f(\zeta_0) - \epsilon \le h_f(\zeta_0) \le f(\zeta_0) + \epsilon, \tag{11.12.25}$$

because of (11.12.12). This implies (11.12.13), because $\epsilon > 0$ is arbitrary. One can use this and (11.12.21) to get (11.12.14), because u_0 is continuous at ζ_0, by hypothesis.

11.13 More on barriers

Let us continue with the same notation and hypotheses as in the previous section. If

$$\text{there is a barrier for } U \text{ at every } \zeta_0 \in \partial U, \tag{11.13.1}$$

then it follows that (11.12.8) and (11.12.9) hold. This corresponds to a remark on p250 of [7], and to the "if" part of Theorem 11.1 on p228 of [18].

Conversely, suppose that the Dirichlet problem for U is solvable, in the sense that for every continuous real-valued function f on ∂U there is a continuous real-valued function on $\overline{U}$ that is harmonic on U and equal to f on ∂U. If $\zeta_0 \in \partial U$, then

$$f_0(x) = -|x - \zeta_0| \tag{11.13.2}$$

is a continuous real-valued function on ∂U. If u_0 is the corresponding solution to the Dirichlet problem on U, then one can check that

$$u_0 \text{ is a barrier for } U \text{ at } \zeta_0. \tag{11.13.3}$$

More precisely, the maximum principle implies that

$$u_0 \le 0 \text{ on } \overline{U}. \tag{11.13.4}$$

One can use the strong maximum principle to get that

$$u_0 < 0 \tag{11.13.5}$$

on U. It follows that this holds on $\overline{U} \setminus \{\zeta_0\}$, by (11.13.2). This corresponds to the "only if" part of Theorem 11.10 on p228 of [18].

11.13.1 Some simple barriers

Let $\zeta_0 \in \partial U$ be given again. Suppose for the moment that there is a linear functional λ_0 on $\mathbf{R}^n$ such that

$$\lambda_0(x) < \lambda_0(\zeta_0) \tag{11.13.6}$$

for every $x \in \overline{U}$ with $x \neq \zeta_0$. We may say that U satisfies the *exterior half-space condition* at ζ_0 in this case. It is easy to see that this implies that

$$\lambda_0(\zeta_0) - \lambda_0 \text{ is a barrier for } U \text{ at } \zeta_0. \tag{11.13.7}$$

This corresponds to a remark at the top of p251 of [7].

Let us say that U satisfies the *exterior ball condition* at ζ_0 if there is a point $a_0 \in \mathbf{R}^n$ and a radius $r_0 > 0$ such that

$$|\zeta_0 - a_0| = r_0 \tag{11.13.8}$$

and

$$\overline{B}(a_0, r_0) \cap \overline{U} = \{\zeta_0\}. \tag{11.13.9}$$

Let u_{a_0,r_0} be the real-valued function defined on $\mathbf{R}^n \setminus \{a_0\}$ by

$$\begin{aligned} u_{a_0,r_0}(x) &= \log r_0 - \log |x - a_0| \quad \text{when } n = 2 \\ &= |x - a_0|^{2-n} - r_0^{2-n} \quad \text{when } n \geq 3. \end{aligned} \tag{11.13.10}$$

Note that u_0 is harmonic on $\mathbf{R}^n \setminus \{a_0\}$, as in Section 6.1. It is easy to see that

$$\text{the restriction of } u_{a_0,r_0} \text{ to } \overline{U} \text{ is a barrier for } U \text{ at } \zeta_0 \tag{11.13.11}$$

under these conditions, as in Theorem 11.11 on p229 of [18].

11.13.2 Some related geometric conditions

Let us say that U satisfies the *weak exterior ball condition* at ζ_0 if there is a point $a_1 \in \mathbf{R}^n$ and a radius $r_1 > 0$ such that

$$|\zeta_0 - a_1| = r_1 \tag{11.13.12}$$

and

$$\overline{U} \cap B(a_1, r_1) = \emptyset. \tag{11.13.13}$$

Let t_0 be a real number with

$$0 < t_0 < 1, \tag{11.13.14}$$

and put

$$a_0 = (1 - t_0)\,\zeta_0 + t_0\,a_1, \;\; r_0 = t_0\,r_1. \tag{11.13.15}$$

Observe that

$$|\zeta_0 - a_0| = t_0\,|\zeta_0 - a_1| = r_0 \tag{11.13.16}$$

and

$$|a_0 - a_1| = (1 - t_0)\,|\zeta_0 - a_1| = r_1 - r_0. \tag{11.13.17}$$

One can check that

$$\overline{B}(a_0, r_0) \subseteq \overline{B}(a_1, r_1), \tag{11.13.18}$$

with

$$\partial B(a_0, r_0) \cap \partial B(a_1, r_1) = \{\zeta_0\}. \tag{11.13.19}$$

This implies that

(11.13.20) $\quad U$ satisfies the exterior ball condition at ζ_0.

Suppose that $v_0 \in \mathbf{R}^n$ satisfies $|v_0| = 1$, and that

$$x \cdot v_0 \le \zeta_0 \cdot v_0 \tag{11.13.21}$$

for every $x \in \overline{U}$. We may say that U satisfies the *weak exterior half-space condition* in this case. Let $r_0 > 0$ be given, and put

$$a_0 = \zeta_0 + r_0\,v_0. \tag{11.13.22}$$

Note that

$$|\zeta_0 - a_0| = r_0\,|v_0| = r_0. \tag{11.13.23}$$

If $x \in \mathbf{R}^n$, then

$$\begin{aligned} |x - a_0|^2 &= (x - a_0) \cdot (x - a_0) \\ &= (x - \zeta_0) \cdot (x - \zeta_0) - 2\,(x - \zeta_0) \cdot v_0 + r_0^2\,v_0 \cdot v_0 \\ &= |x - \zeta_0|^2 - 2\,(x - \zeta_0) \cdot v_0 + r_0^2. \end{aligned} \tag{11.13.24}$$

If (11.13.21) holds, so that $(x - \zeta_0) \cdot v_0 \le 0$, then it follows that

$$|x - a_0|^2 \ge |x - \zeta_0|^2 + r_0^2. \tag{11.13.25}$$

This implies that (11.13.9) holds, so that (11.13.20) holds.

If U is convex, then it is well known that for each $\zeta_0 \in \partial U$ there is a $v_0 \in \mathbf{R}^n$ such that $|v_0| = 1$ and (11.13.21) holds. A proof of this may be found in Section A.13. This is related to a remark on p229 of [18], and to Corollary 11.12 on p230 of [18]. If the boundary of U is twice continuously differentiable or C^2 in a suitable sense near $\zeta_0 \in \partial U$, then it is well known that U satisfies the exterior ball condition at ζ_0. This is also discussed on p230 of [18].

11.14 Some more barriers using cones

We would like to discuss some more examples of barriers, as in Lemma 11.15 on p231 of [18]. Before getting to that, we consider a refinement of the maximum principle, which is related to Lemma 11.14 on p230 of [18].

Let $n \ge 2$ be an integer, let V be a nonempty bounded open subset of $\mathbf{R}^n$, and let $a \in \partial V$ be given. Also let u be a bounded continuous real-valued function on

$$\overline{V} \setminus \{a\} \tag{11.14.1}$$

that is subharmonic on V. Suppose that there is a real number A such that

$$u \le A \tag{11.14.2}$$

on $\partial V \setminus \{a\}$. Under these conditions, we would like to show that (11.14.2) holds on V as well.

Lemma 11.14 on p230 of [18] deals with a particular point in the boundary of a particular family of bounded open subsets of $\mathbf{R}^n$, and the same argument works here. The same type of argument was used in Section 10.1.

If t is a positive real number, then put

$$V_t = V \setminus \overline{B}(a,t), \tag{11.14.3}$$

which is a bounded open set in $\mathbf{R}^n$ contained in V. We shall be interested in taking t small here, and in particular we may as well suppose that t is small enough so that $V_t \ne \emptyset$. Observe that

$$\begin{aligned}(\overline{V} \setminus \overline{B}(a,t)) \cup (V \cap \partial B(a,t)) &\subseteq \overline{V_t} \subseteq \overline{V} \setminus B(a,t) \\ &= (V \setminus B(a,t)) \cup (\partial V \setminus B(a,t)).\end{aligned} \tag{11.14.4}$$

This implies that

$$\begin{aligned}\partial V_t = \overline{V_t} \setminus V_t &\subseteq ((V \setminus B(a,t)) \setminus V_t) \cup ((\partial V \setminus B(a,t)) \setminus V_t) \\ &= (V \cap \partial B(a,t)) \cup (\partial V \setminus B(a,t)).\end{aligned} \tag{11.14.5}$$

There is a positive real number R such that

$$V \subseteq \overline{B}(a,R), \tag{11.14.6}$$

because V is bounded, by hypothesis. Let $\epsilon > 0$ be given, and consider the real-valued function v_ϵ defined on (11.14.1) by

$$\begin{aligned} v_\epsilon(x) &= u(x) - \epsilon\,|x-a|^{2-n} && \text{when } n \ge 3 \\ &= u(x) + \epsilon\,\log(|x-a|/R) && \text{when } n = 2.\end{aligned} \tag{11.14.7}$$

It is easy to see that v_ϵ is continuous on (11.14.1), with

$$v_\epsilon \le u \tag{11.14.8}$$

on (11.14.1). We also have that v_ϵ is subharmonic on V, because u is subharmonic on V, and using the remarks in Section 6.1.

If t is small enough, then one can verify that

$$v_\epsilon \le A \tag{11.14.9}$$

on ∂V_t, using (11.14.5), and the hypotheses that u be bounded on (11.14.1), and that (11.14.2) hold on $\partial V \setminus \{a\}$. This implies that (11.14.9) holds on $\overline{V_t}$, by the maximum principle, as in Subsection 11.4.1. It follows that (11.14.9) holds on (11.14.1), because we can take t to be arbitrarily small.

One can use this to get that (11.14.2) holds on (11.14.1), because $\epsilon > 0$ is arbitrary.

11.14.1 Some cones

It is convenient here to identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ again, so that $x \in \mathbf{R}^n$ is identified with (x', x_n), where

$$x' = (x_1, \ldots, x_{n-1}). \tag{11.14.10}$$

We shall also use

$$|x'| = \Big(\sum_{j=1}^{n-1} x_j^2\Big)^{1/2} \tag{11.14.11}$$

for the standard Euclidean norm of x', as an element of $\mathbf{R}^{n-1}$. Thus

$$|x| = (|x'|^2 + x_n^2)^{1/2} \tag{11.14.12}$$

is the same as the standard Euclidean norm of x, as an element of $\mathbf{R}^n$.

Let a positive real number α be given, and consider

$$C_\alpha = \{y \in \mathbf{R}^n : |y'| < \alpha\, y_n\}. \tag{11.14.13}$$

One can check that this is an open set in $\mathbf{R}^n$, with closure

$$\overline{C_\alpha} = \{y \in \mathbf{R}^n : |y'| \le \alpha\, y_n\}. \tag{11.14.14}$$

Thus

$$\partial C_\alpha = \overline{C_\alpha} \setminus C_\alpha = \{y \in \mathbf{R}^n : |y'| = \alpha\, y_n\}. \tag{11.14.15}$$

Note that

$$y_n > 0 \tag{11.14.16}$$

when $y \in C_\alpha$, and

$$y_n \ge 0 \tag{11.14.17}$$

when $y \in \overline{C_\alpha}$.

Let $z \in \mathbf{R}^n$ be given, with $z_j = 0$ for $j = 1, \ldots, n-1$, and

$$z_n > 0. \tag{11.14.18}$$

Suppose that $y \in \mathbf{R}^n$ satisfies

$$y_n = (1 + \alpha^2)^{-1} \, z_n, \tag{11.14.19}$$

so that (11.14.16) holds in particular. Equivalently, this means that

$$z_n = y_n + \alpha^2 \, y_n. \tag{11.14.20}$$

In this case, it is easy to see that $y \in \partial C_\alpha$ if and only if

$$(z - y) \cdot y = - \sum_{j=1}^{n-1} y_j^2 + (z_n - y_n) \, y_n = 0. \tag{11.14.21}$$

Similarly, if $y \in \partial C_\alpha$, then (11.14.21) implies

$$z_n = y_n + y_n^{-1} \sum_{j=1}^{n-1} y_j^2 = y_n + \alpha^2 \, y_n. \tag{11.14.22}$$

If $y \in \partial C_\alpha$, then

$$\begin{aligned} |z - y|^2 &= \sum_{j=1}^{n-1} y_j^2 + (z_n - y_n)^2 = \alpha^2 \, y_n^2 + \alpha^4 \, y_n^2 \\ &= \alpha^2 \, (1 + \alpha^2) \, y_n^2 = \alpha^2 \, (1 + \alpha^2)^{-1} \, z_n^2. \end{aligned} \tag{11.14.23}$$

We would like to have that

$$\overline{B}(z, \alpha \, (1 + \alpha^2)^{-1/2} \, z_n) \subseteq \overline{C_\alpha}. \tag{11.14.24}$$

Suppose that $w \in \mathbf{R}^n$ satisfies

$$|z - w| \le \alpha \, (1 + \alpha^2)^{-1/2} \, z_n < z_n. \tag{11.14.25}$$

In particular, this implies that $|z_n - w_n| < z_n$, so that

$$w_n > 0. \tag{11.14.26}$$

Of course, if $w' = 0$, then $w \in C_\alpha$, and so we may as well suppose that $w' \ne 0$.

Let P be the two-dimensional linear subspace of $\mathbf{R}^n$ spanned by w and z. Equivalently, $x \in P$ if and only if

$$x' = t \, w' \tag{11.14.27}$$

for some $t \in \mathbf{R}$. There are exactly two elements y of $\partial C_\alpha \cap P$ that satisfy (11.14.19), with y' a positive and negative multiple of w'. In fact, $\partial C_\alpha \cap P$ consists of the two rays going through these two particular elements. Using some of the previous remarks, we have that the disk

$$\overline{B}(z, \alpha \, (1 + \alpha^2)^{-1/2} \, z_n) \cap P \tag{11.14.28}$$

is tangent to these two rays at these two particular points.

11.14.2 Some related open sets

If r is a positive real number, then put

$$W_{\alpha,r} = B(0,r) \setminus \overline{C_\alpha}. \tag{11.14.29}$$

This is a bounded open set in $\mathbf{R}^n$, with

$$\overline{W_{\alpha,r}} = \overline{B}(0,r) \setminus C_\alpha. \tag{11.14.30}$$

We also have that

$$\partial W_{\alpha,r} = (\partial B(0,r) \setminus C_\alpha) \cup (B(0,r) \cap \partial C_\alpha). \tag{11.14.31}$$

In particular,

$$0 \in \partial W_{\alpha,r}. \tag{11.14.32}$$

This corresponds to a domain in $\mathbf{R}^n$ mentioned on p230 of [18] when $r = 1$.

If $\zeta_0 \in \partial W_{\alpha,r}$ and $\zeta_0 \neq 0$, then

$$W_{\alpha,r} \text{ satisfies the exterior ball condition at } \zeta_0, \tag{11.14.33}$$

as mentioned in the proof of Lemma 11.15 on p231 of [18]. More precisely, if $\zeta_0 \in \partial B(0,r)$, then $B(0,r)$ satifies the exterior half-space condition at ζ_0, as in Subsection 11.13.1. This implies in particular that $W_{\alpha,r}$ satisfies the exterior half-space condition at ζ_0, and thus the exterior ball condition at ζ_0. If $\zeta_0 \in \partial C_\alpha$, then $W_{\alpha,r}$ satisfies the weak exterior ball condition at ζ_0, as in the previous subsection. This implies that $W_{\alpha,r}$ satisfies the exterior ball condition at ζ_0, as in Subsection 11.13.2.

If $x \in \partial W_{\alpha,r}$, then put

$$f_{\alpha,r}(x) = |x|. \tag{11.14.34}$$

This is a continuous real-valued function on $\partial W_{\alpha,r}$, with

$$0 \le f_{\alpha,r} \le r \text{ on } \partial W_{\alpha,r}. \tag{11.14.35}$$

Let

$$u_{\alpha,r} = h_{f_{\alpha,r}} \tag{11.14.36}$$

be the Perron function on $\overline{W_{\alpha,r}}$ associated to $f_{\alpha,r}$, as in Section 11.9. Note that

$$0 \le u_{\alpha,r} \le r \text{ on } \overline{W_{\alpha,r}}, \tag{11.14.37}$$

because of (11.14.35), as before. We also have that

$$u_{\alpha,r} \text{ is harmonic on } W_{\alpha,r}, \tag{11.14.38}$$

as before.

If $\zeta_0 \in \partial W_{\alpha,r}$ and $\zeta_0 \neq 0$, then

$$u_{\alpha,r}(\zeta_0) = f_{\alpha,r}(\zeta_0) = |\zeta_0| \tag{11.14.39}$$

and

$$u_{\alpha,r} \text{ is continuous at } \zeta_0, \text{ as a function on } \overline{W_{\alpha,r}}, \tag{11.14.40}$$

as in Subsection 11.12.1. This uses (11.14.33) to get a barrier for $W_{\alpha,r}$ at ζ_0, as in Subsection 11.13.1. It is easy to see that (11.14.39) holds when $\zeta_0 = 0$ too, using the first inequality in (11.14.37), and the fact that $u_{\alpha,r} \le f_{\alpha,r}$ on $\partial W_{\alpha,r}$, by construction. We would like to show that (11.14.40) holds when $\zeta_0 = 0$ as well, as in the proof of Lemma 11.15 on p231 of [18].

Observe that

$$0 < u_{\alpha,r} < r \text{ on } W_{\alpha,r}, \tag{11.14.41}$$

because of the strong maximum principle, as in Subsection 6.7.1. If r_0 is a positive real number with $r_0 < r$, then put

$$c_0 = c_0(r_0) = \max\{u(x) : x \in \partial B(0, r_0) \setminus C_\alpha\}, \tag{11.14.42}$$

where the maximum is attained, by the extreme value theorem. Clearly

$$c_0(r_0) < r, \tag{11.14.43}$$

because of (11.14.39) and (11.14.41). We also have that

$$r_0 \le c_0(r_0), \tag{11.14.44}$$

because of (11.14.39).

Note that

$$W_{\alpha,r_0} = (r_0/r)\, W_{\alpha,r} = \{(r_0/r)\, x : x \in W_{\alpha,r}\} \tag{11.14.45}$$

using the same notation as in Section A.11. Of course, we can define f_{α,r_0}, u_{α,r_0} in the same way as before. It is easy to see that

$$f_{\alpha,r_0}(x) = (r_0/r)\, f_{\alpha,r}((r/r_0)\, x) \tag{11.14.46}$$

on $\partial W_{\alpha,r_0}$, by construction. This implies that

$$u_{\alpha,r_0}(x) = (r_0/r)\, u_{\alpha,r}((r/r_0)\, x) \tag{11.14.47}$$

on $\overline{W_{\alpha,r_0}}$.

If $x \in \overline{W_{\alpha,r_0}}$, then put

$$\begin{aligned} v_{\alpha,r_0,r}(x) &= u_{\alpha,r}(x) - (c_0(r_0)/r_0)\, u_{\alpha,r_0}(x) \\ &= u_{\alpha,r}(x) - (c_0(r_0)/r)\, u_{\alpha,r}((r/r_0)\, x). \end{aligned} \tag{11.14.48}$$

This function is bounded on $\overline{W_{\alpha,r_0}}$, continuous on $\overline{W_{\alpha,r_0}} \setminus \{0\}$, and harmonic on W_{α,r_0}. Observe that

$$v_{\alpha,r_0,r}(x) \le 0 \tag{11.14.49}$$

when $|x| = r_0$, by (11.14.39) and the definition of $c_0(r_0)$. This also holds when $x \in \partial C_\alpha$, because of (11.14.39) and (11.14.44). It follows that this holds on $\overline{W_{\alpha,r_0}}$, by the remarks at the beginning of the section.

This means that

$$u_{\alpha,r}(x) \le (c_0(r_0)/r)\, u_{\alpha,r}((r/r_0)\, x) \tag{11.14.50}$$

for all $x \in \overline{W_{\alpha,r_0}}$. If $x \in \overline{W_{\alpha,r}}$ and

$$|x| \le (r_0/r)^l\, r \tag{11.14.51}$$

for some nonnegative integer l, then we can repeat the precess to get that

$$u_{\alpha,r}(x) \le (c(r_0)/r)^l\, u_{\alpha,r}((r/r_0)^l\, x). \tag{11.14.52}$$

This implies that $u_{\alpha,r}$ is continuous at 0 as a function on $\overline{W_{\alpha,r}}$, because of (11.14.43).

It follows that

$$u_{\alpha,r} \text{ is a barrier for } W_{\alpha,r} \text{ at } 0, \tag{11.14.53}$$

as in Lemma 11.15 on p231 of [18]. This also uses (11.14.39) and (11.14.41), to get that $-u_{\alpha,r} < 0$ on $\overline{W_{\alpha,r}} \setminus \{0\}$.

11.15 Some additional geometric conditions

Let us continue with the same notation and hypotheses as in the previous section. Put

$$V_{\alpha,r} = W_{\alpha,r} \cup (\mathbf{R}^n \setminus \overline{B}(0,r)) = \mathbf{R}^n \setminus (\overline{B}(0,r) \cap \overline{C_\alpha}). \tag{11.15.1}$$

This is an open set in $\mathbf{R}$, with

$$\overline{V_{\alpha,r}} = \overline{W_{\alpha,r}} \cup (\mathbf{R}^n \setminus B(0,r)) = \mathbf{R}^n \setminus (B(0,r) \cap C_\alpha). \tag{11.15.2}$$

Similarly,

$$\partial V_{\alpha,r} = (\partial B(0,r) \cap \overline{C_\alpha}) \cup (B(0,r) \cap \partial C_\alpha). \tag{11.15.3}$$

Note that

$$0 \in \partial V_{\alpha,r}. \tag{11.15.4}$$

Let us extend $u_{\alpha,r}$ to a real-valued function on $\overline{V_{\alpha,r}}$ by putting

$$u_{\alpha,r} = r \text{ on } \mathbf{R}^n \setminus B(0,r). \tag{11.15.5}$$

This is consistent with the previous definition of $u_{\alpha,r}$ on $\overline{W_{\alpha,r}} \cap \partial B(0,r)$, because of (11.14.39). It follows that

$$\text{this extension is continuous on } \overline{V_{\alpha,r}}. \tag{11.15.6}$$

Using this extension, we also have that

$$-u_{\alpha,r} \text{ is subharmonic on } V_{\alpha,r}. \tag{11.15.7}$$

This uses (11.14.38), and the remarks in Section 11.5.

11.15.1 Using these barriers

Let U be a nonempty bounded open subset of $\mathbf{R}^n$ again, and let $\zeta_0 \in \partial U$ be given. We say that U satisifes the *exterior cone condition* at ζ_0 if there are positive real numbers α, r and an orthogonal transformation T on $\mathbf{R}^n$ such that

$$U \cap (\zeta_0 + T(B(0,r) \cap C_\alpha)) = \emptyset. \tag{11.15.8}$$

This is the same as saying that

$$U \cap (\zeta_0 + T(\overline{B}(0,r) \cap \overline{C_\alpha})) = \emptyset. \tag{11.15.9}$$

Similarly, (11.15.8) is equivalent to asking that

$$\overline{U} \cap (\zeta_0 + T(B(0,r) \cap C_\alpha)) = \emptyset. \tag{11.15.10}$$

In this case,

$$U \text{ has a barrier at } \zeta_0, \tag{11.15.11}$$

as in 11.16 on p232 of [18]. To see this, one may as well suppose that $\zeta_0 = 0$, and that T is the identity mapping on $\mathbf{R}^n$. Thus (11.15.10) is the same as saying that

$$\overline{U} \subseteq \overline{V_{\alpha,r}}. \tag{11.15.12}$$

Under these conditions, the restriction of $-u_{\alpha,r}$ to $\overline{U}$ is a barrier for U at 0. If $n = 2$, then barriers for broader classes of open sets are discussed on p251 of [7].

Chapter 12

Some distribution theory

12.1 Fundamental solutions

Let n be a positive integer, and let u, v be complex-valued functions on $\mathbf{R}^n$, at least one of which has compact support in $\mathbf{R}^n$. If α is a multi-index, and if u, v are $|\alpha|$-times continuously differentiable on $\mathbf{R}^n$, then

$$\int_{\mathbf{R}^n} (\partial^\alpha u)(x)\, v(x)\, dx = (-1)^{|\alpha|} \int_{\mathbf{R}^n} u(x)\, (\partial^\alpha v)(x)\, dx, \tag{12.1.1}$$

by integration by parts.

Let N be a nonnegative integer, and let

$$p(w) = \sum_{|\alpha| \le N} a_\alpha\, w^\alpha \tag{12.1.2}$$

be a polynomial in the n variables $w_1, \ldots, w_n$ of degree less than or equal to N, and with complex coefficients a_α. Put

$$\widetilde{p}(w) = p(-w) = \sum_{|\alpha| \le N} (-1)^{|\alpha|}\, a_\alpha\, w^\alpha, \tag{12.1.3}$$

which is another polynomial in $w_1, \ldots, w_n$ of degree less than or equal to n with complex coefficients, as appropriate. Using these polynomials, we get corresponding differential operators $p(\partial)$ and $\widetilde{p}(\partial)$, as in Section 1.7. If u, v are N-times continuously differentiable on $\mathbf{R}^n$, then

$$\int_{\mathbf{R}^n} (p(\partial) u)(x)\, v(x)\, dx = \int_{\mathbf{R}^n} u(x)\, (\widetilde{p}(\partial) v)(x)\, dx, \tag{12.1.4}$$

because of (12.1.1).

12.1.1 Fundamental solutions of $p(\partial)$

A complex-valued function E on $\mathbf{R}^n$ is said to be a *fundamental solution* of $p(\partial)$ if

$$\int_{\mathbf{R}^n} E(x)\,(\widetilde{p}(\partial)v)(x)\,dx = v(0) \tag{12.1.5}$$

for every smooth function v on $\mathbf{R}^n$ with compact support. This is interpreted as meaning that $(p(\partial))(E)$ is the Dirac delta function δ_0 associated to 0, in the sense of distributions. More precisely, this can be extended to distributions E on $\mathbf{R}^n$. If $p \neq 0$, then a famous theorem of Ehrenpreis and Malgrange states that $p(\partial)$ has a fundamental solution on $\mathbf{R}^n$, which may be a distribution. See [74, 87, 95, 241, 268, 276, 318] for more information.

A fundamental solution for the Laplacian is given by the function N defined in Section 6.8. A fundamental solution for the heat operator

$$\frac{\partial}{\partial t} - \Delta \tag{12.1.6}$$

is given by the heat kernel. Once one has a fundamental solution E for $p(\partial)$, one can solve

$$(p(\partial))(u) = f \tag{12.1.7}$$

by convolving E with f under suitable conditions, in the sense of distributions. If f is a smooth function with compact support on $\mathbf{R}^n$, then this gives a smooth solution u of (12.1.7).

Some basic aspects of distribution theory will be discussed in the next sections, and some additional references about this include [23, 90, 91, 93, 116, 150, 172, 204, 223, 296, 301, 310, 351].

12.2 Spaces of test functions

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Remember that a real or complex-valued function f on U is said to have compact support in U if there is a compact set $E \subseteq \mathbf{R}^n$ such that $E \subseteq U$ and $f = 0$ on $U \setminus E$, as in Subsection 1.9.2. Note that the union of two compact subsets of $\mathbf{R}^n$ is compact too. Using this, it is easy to see that the sum of two real or complex-valued functions on U with compact support in U has compact support in U as well.

Let $C_{com}(U, \mathbf{R})$, $C_{com}(U, \mathbf{C})$ be the spaces of continuous real and complex-valued functions on U with compact support, respectively. These are linear subspaces of the spaces $C(U, \mathbf{R})$, $C(U, \mathbf{C})$ of all continuous real or complex-valued functions on U, as vector spaces over the real and complex numbers, respectively. Similarly, if k is a positive integer, then let $C^k_{com}(U, \mathbf{R})$, $C^k_{com}(U, \mathbf{C})$ be the spaces of k-times continuously-differentiable real and complex-valued functions on U with compact support. It is sometimes convenient to use the same notation with $k = 0$ for the analogous spaces of continuous functions, as before. The spaces of smooth real or complex-valued functions on U with compact support are denoted $C^\infty_{com}(U, \mathbf{R})$, $C^\infty_{com}(U, \mathbf{C})$.

Equivalently,

$$C^k_{com}(U, \mathbf{R}) = C^k(U, \mathbf{R}) \cap C_{com}(U, \mathbf{R}), \tag{12.2.1}$$

$$C^k_{com}(U, \mathbf{C}) = C^k(U, \mathbf{C}) \cap C_{com}(U, \mathbf{C}) \tag{12.2.2}$$

for every $k \ge 1$. These are linear subspaces of $C^k(U, \mathbf{R})$ and $C^k(U, \mathbf{C})$, respectively. Similarly,

$$C^\infty_{com}(U, \mathbf{R}) = C^\infty(U, \mathbf{R}) \cap C_{com}(U, \mathbf{R}), \tag{12.2.3}$$

$$C^\infty_{com}(U, \mathbf{C}) = C^\infty(U, \mathbf{C}) \cap C_{com}(U, \mathbf{C}), \tag{12.2.4}$$

which are linear subspaces of $C^\infty(U, \mathbf{R})$ and $C^\infty(U, \mathbf{C})$, respectively.

If f is a real or complex-valued function on U, then we can extend f to a function on $\mathbf{R}^n$, simply by putting $f = 0$ on $\mathbf{R}^n \backslash U$. If f is a continuous function on U with compact support in U, then this extension of f to $\mathbf{R}^n$ is continuous too. Similarly, if f is k-times continuously differentiable on U for some $k \ge 1$, or if f is smooth on U, and f has compact support in U, then this extension has the analogous property on $\mathbf{R}^n$.

If f is a continuously-differentiable real or complex-valued function on U with compact support in U, then it is easy to see that the partial derivatives of f have compact support in U as well. If α is a multi-index and $|\alpha| \le k$, then ∂^α defines linear mappings from $C^k_{com}(U, \mathbf{R})$, $C^k_{com}(U, \mathbf{C})$ into $C^{k-|\alpha|}_{com}(U, \mathbf{R})$, $C^{k-|\alpha|}_{com}(U, \mathbf{C})$, respectively. Similarly, ∂^α defines linear mappings from $C^\infty_{com}(U, \mathbf{R})$, $C^\infty_{com}(U, \mathbf{C})$ into themselves.

If f, g are continuous real or complex-valued functions on U, then it is well known that their product $f\,g$ is continuous on U. If f and g are both k-times continuously differentiable on U, or both smooth on U, then $f\,g$ has the same property. If either f or g has compact support in U, then $f\,g$ has compact support in U.

Smooth functions on U with compact support in U are also known as *test functions* on U.

12.3 Distributions

A *linear functional* on a vector space V over the real or complex numbers is a linear mapping from V into $\mathbf{R}$ or $\mathbf{C}$, as appropriate. Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. A *distribution* on U is a linear functional on the space $C^\infty_{com}(U, \mathbf{C})$ of complex-valued test functions on U that is continuous in a certain sense. Before describing the continuity condition, let us mention some basic examples.

Let f be a continuous complex-valued function on U. If ϕ is a test function on U, then put

$$\lambda_f(\phi) = \int_U f(x)\,\phi(x)\,dx. \tag{12.3.1}$$

The right side may be interpreted as a Riemann integral over any suitable region that contains the support of ϕ. This defines a linear functional on $C^\infty_{com}(U, \mathbf{C})$. One can check that $\lambda_f(\phi) = 0$ for every test function ϕ on U only when $f \equiv 0$ on U. This implies that f is uniquely determined by λ_f on $C^\infty_{com}(U, \mathbf{C})$.

Similarly, if f is a complex-valued function on U that is locally integrable on U with respect to Lebesgue measure, then the right side of (12.3.1) may be interpreted as a Lebesgue integral. In this case, one can show that f is determined almost everywhere on U with respect to Lebesgue measure by λ_f on $C^\infty_{com}(U, \mathbf{C})$.

If $x \in U$, then

$$\delta_x(\phi) = \phi(x) \tag{12.3.2}$$

defines a linear functional on $C^\infty_{com}(U, \mathbf{C})$. This is the *Dirac distribution* on U associated to x.

12.3.1 Convergent sequences of test functions

The continuity condition used to define distributions can be described in terms of a suitable notion of convergent sequences of test functions. Let $\{\phi_j\}_{j=1}^\infty$ be a sequence of test functions on U, and let ϕ be another test function on U. We say that

$$\{\phi_j\}_{j=1}^\infty \textit{ converges} \text{ to } \phi \text{ in } C^\infty_{com}(U, \mathbf{C}) \tag{12.3.3}$$

if the following two conditions hold. First, there is a compact set $E \subseteq \mathbf{R}^n$ such that $E \subseteq U$ and

$$\phi_j = 0 \text{ on } U \setminus E \tag{12.3.4}$$

for every $j \geq 1$. Second, for every multi-index α, we have that

$$\{\partial^\alpha \phi_j\}_{j=1}^\infty \text{ converges to } \partial^\alpha \phi \text{ uniformly on } U. \tag{12.3.5}$$

In particular, we can take $\alpha = 0$, to get that $\{\phi_j\}_{j=1}^\infty$ converges to ϕ uniformly on U. It follows that

$$\phi = 0 \text{ on } U \setminus E, \tag{12.3.6}$$

because of (12.3.4).

12.3.2 The continuity condition

A linear functional λ on $C^\infty_{com}(U, \mathbf{C})$ is said to be a distribution on U if for every sequence $\{\phi_j\}_{j=1}^\infty$ of test functions on U that converges to a test function ϕ on U, in the sense described in the preceding paragraph, we have that

$$\lim_{j\to\infty} \lambda(\phi_j) = \lambda(\phi). \tag{12.3.7}$$

Alternatively, there is a standard topology defined on $C^\infty_{com}(U, \mathbf{C})$, and it is well known that a linear functional on $C^\infty_{com}(U, \mathbf{C})$ is continuous with respect to this topology if and only if it satisfies this continuity condition in terms of convergent sequences. More precisely, one can show that convergence of sequences in $C^\infty_{com}(U, \mathbf{C})$ with respect to this topology is equivalent to the notion of convergence mentioned in the preceding paragraph. In particular, this implies that continuity with respect to this topology on $C^\infty_{com}(\mathbf{R}^n, \mathbf{C})$ automatically implies the sequential continuity condition (12.3.7). It is well known that the converse holds for linear functionals, but this is more complicated in this case than for metric spaces, for instance.

The space of distributions on U may be denoted

$$C^\infty_{com}(U, \mathbf{C})'. \tag{12.3.8}$$

This is a vector space over the complex numbers, with respect to pointwise addition and scalar multiplication of linear functionals on $C^\infty_{com}(U, \mathbf{C})$.

If f is a continuous complex-valued function on U, or a locally integrable function on U with respect to Lebesgue measure, then it is easy to see that (12.3.1) defines a distribution on U. In this case, it is enough to take $\alpha = 0$ in (12.3.5). We also have that

$$f \mapsto \lambda_f \tag{12.3.9}$$

is a linear mapping from $C(U, \mathbf{C})$ into $C^\infty_{com}(U, \mathbf{C})'$, or from the space of locally integrable functions f on U into $C^\infty_{com}(U, \mathbf{C})'$. It is very easy to see that the Dirac distribution associated to $x \in U$ is indeed a distribution on U.

12.4 Some basic properties of distributions

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Suppose that $\{\phi_j\}_{j=1}^\infty$ is a sequence of test functions on U that converges to a test function ϕ on U, in the sense described in the previous section. This implies that

$$\{\partial_l \phi_j\}_{j=1}^\infty \text{ converges to } \partial_l \phi \tag{12.4.1}$$

in the same sense for each $l = 1, \ldots, n$. Similarly, if a is a smooth complex-valued function on U, then one can check that

$$\{a\, \phi_j\}_{j=1}^\infty \text{ converges to } a\, \phi \tag{12.4.2}$$

in this sense as well. This uses the fact that a and its derivatives of any order are bounded on any compact subset of $\mathbf{R}^n$ that is contained in U.

12.4.1 Differentiating distributions

Let λ be a distribution on U, and for each $l = 1, \ldots, n$, put

$$(\partial_l \lambda)(\phi) = -\lambda(\partial_l \phi) \tag{12.4.3}$$

for every test function ϕ on U. It is easy to see that this defines a distribution on U, which is considered as the partial derivative of λ in the lth variable. If

f is a continuously-differentiable complex-valued function on U, and λ_f is the distribution associated to f as in (12.3.1), then

$$\partial_l \lambda_f = \lambda_{\partial_l f} \tag{12.4.4}$$

is the distribution associated to $\partial_l f$ on U. This basically corresponds to integration by parts. More precisely, this works when f is a continuous function on U such that the partial derivative $\partial f/\partial x_l$ in the lth variable exists at every point in U, and is continuous on U.

It is easy to see that

$$\partial_j(\partial_l \lambda) = \partial_l(\partial_j \lambda) \tag{12.4.5}$$

for each $j, l = 1, \ldots, n$, using the analogous statement for smooth functions. If α is any multi-index, then one can differentiate λ repeatedly, to get that

$$(\partial^\alpha \lambda)(\phi) = (-1)^{|\alpha|}\, \lambda(\partial^\alpha \phi) \tag{12.4.6}$$

for all test functions ϕ on U. If f is an $|\alpha|$-times continuously-differentiable complex-valued function on U, and λ_f is the distribution on U associated to f as before, then

$$\partial^\alpha \lambda_f = \lambda_{\partial^\alpha f} \tag{12.4.7}$$

is the distribution associated to $\partial^\alpha f$ on U.

12.4.2 Multiplying distributions by smooth functions

If a is a smooth complex-valued function on U, then put

$$(a\,\lambda)(\phi) = \lambda(a\,\phi) \tag{12.4.8}$$

for every test function ϕ on U. This defines a distribution on U, which is considered as the product of a and λ. If f is a continuous or simply locally-integrable complex-valued function on U, then

$$a\,\lambda_f = \lambda_{a\,f} \tag{12.4.9}$$

is the distribution on U associated to the usual product $a\,f$ of a and f on U. One can check that

$$\partial_l(a\,\lambda) = (\partial_l a)\,\lambda + a\,(\partial_l \lambda), \tag{12.4.10}$$

as distributions on U, using the usual product rule for partial derivatives of smooth functions on U.

12.4.3 Real-valued distributions

One may consider λ to be real-valued as a distribution on U if

$$\lambda(\phi) \in \mathbf{R} \tag{12.4.11}$$

for every real-valued test function ϕ on U. In this case, one may say that λ is nonnegative as a distribution on U, or

$$\lambda \geq 0, \tag{12.4.12}$$

if

$$\lambda(\phi) \geq 0 \tag{12.4.13}$$

for every nonnegative real-valued test function ϕ on U. If λ is the distribution associated to a continuous function f on U, then these conditions correspond to their usual versions for f. If f is locally integrable with respect to Lebesgue measure, and not necessarily continuous, then the analogous conditions on f should be interpreted as holding almost everywhere with respect to Lebesgue measure, as usual.

12.5 Using a fixed compact set

Let n be a positive integer, and let K be a nonempty compact subset of $\mathbf{R}^n$. Consider the space $C^\infty_K(\mathbf{R}^n, \mathbf{C})$ of smooth complex-valued functions ϕ on $\mathbf{R}^n$ such that

$$\operatorname{supp} \phi \subseteq K. \tag{12.5.1}$$

Equivalently, this means that

$$\phi = 0 \text{ on } \mathbf{R}^n \setminus K. \tag{12.5.2}$$

Note that $C^\infty_K(\mathbf{R}^n, \mathbf{C})$ is a linear subspace of the space $C^\infty_{com}(\mathbf{R}^n, \mathbf{C})$ of all smooth complex-valued functions on $\mathbf{R}^n$ with compact support, as a vector space over the complex numbers.

Let $\{\phi_j\}_{j=1}^\infty$ be a sequence of elements of $C^\infty_K(\mathbf{R}^n, \mathbf{C})$, and let ϕ be another element of $C^\infty_K(\mathbf{R}^n, \mathbf{C})$. Let us say that $\{\phi_j\}_{j=1}^\infty$ *converges* to ϕ in $C^\infty_K(\mathbf{R}^n, \mathbf{C})$ if for every multi-index α,

$$\{\partial^\alpha \phi_j\}_{j=1}^\infty \text{ converges to } \partial^\alpha \phi \text{ uniformly on } K. \tag{12.5.3}$$

Let λ be a linear functional on $C^\infty_K(\mathbf{R}^n, \mathbf{C})$. We can use the notion of convergent sequences in $C^\infty_K(\mathbf{R}^n, \mathbf{C})$ described in the preceding paragraph to define a natural continuity condition for λ. This condition asks that

$$\lim_{j \to \infty} \lambda(\phi_j) = \lambda(\phi) \tag{12.5.4}$$

for every sequence $\{\phi_j\}_{j=1}^\infty$ of elements of $C^\infty_K(\mathbf{R}^n, \mathbf{C})$ that converges to an element ϕ of $C^\infty_K(\mathbf{R}^n, \mathbf{C})$ in this sense.

12.5.1 Another characterization of continuity

Alternatively, there is a standard topology defined on $C_K^\infty(\mathbf{R}^n, \mathbf{C})$, for which convergence of sequences is equivalent to the notion of convergence mentioned before. In this case, one can get the equivalence between continuity and sequential continuity more directly. In particular, the continuity condition for a linear functional mentioned in the preceding paragraph is equivalent to continuity with respect to this topology.

One can show that λ is continuous on $C_K^\infty(\mathbf{R}^n, \mathbf{C})$ with respect to this topology if and only if there are a nonnegative real number C and a nonnegative integer N such that

$$|\lambda(\phi)| \le C \sum_{|\alpha| \le N} \Big(\max_{x \in K} |(\partial^\alpha \phi)(x)| \Big) \tag{12.5.5}$$

for every $\phi \in C_K^\infty(\mathbf{R}^n, \mathbf{C})$. The sum on the right is taken over all multi-indices α with $|\alpha| \le N$, as usual. This condition implies that (12.5.4) holds whenever (12.5.3) holds for all such multi-indices.

12.6 Compact sets in open sets

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Every element of $C_{com}^\infty(U, \mathbf{C})$ can be extended to an element of $C^\infty(\mathbf{R}^n, \mathbf{C})$, by putting it equal to 0 on $\mathbf{R}^n \backslash U$, as in Section 12.2. Using this, we can identify $C_{com}^\infty(U, \mathbf{C})$ with a linear subspace of $C_{com}^\infty(\mathbf{R}^n, \mathbf{C})$. With this identification, $C_{com}^\infty(U, \mathbf{C})$ corresponds to the union of $C_K^\infty(\mathbf{R}^n, \mathbf{C})$ over all nonempty compact subsets K of $\mathbf{R}^n$ such that $K \subseteq U$.

If K is a nonempty compact subset of $\mathbf{R}^n$ that is contained in U, then a convergent sequence in $C_K^\infty(\mathbf{R}^n, \mathbf{C})$, in the sense described in the previous section, may be considered as a convergent sequence in $C_{com}^\infty(U, \mathbf{C})$, in the sense of Subsection 12.3.1. Conversely, any convergent sequence in $C_{com}^\infty(U, \mathbf{C})$, in the sense of Subsection 12.3.1, corresponds to a convergent sequence in $C_K^\infty(\mathbf{R}^n, \mathbf{C})$ for some nonempty compact subset K of $\mathbf{R}^n$ that is contained in U.

Let λ be a linear functional on $C_{com}^\infty(U, \mathbf{C})$. If K is a nonempty compact subset of $\mathbf{R}^n$ that is contained in U, then the restriction of λ to $C_K^\infty(\mathbf{R}^n, \mathbf{C})$ defines a linear functional on that vector space. Observe that λ satisfies the continuity condition on $C_{com}^\infty(U, \mathbf{C})$ described in Subsection 12.3.2 if and only if the restriction of λ to $C_K^\infty(\mathbf{R}^n, \mathbf{C})$ satisfies the continuity condition described in the previous section for every nonempty compact subset K of $\mathbf{R}^n$ that is contained in U. This follows from the remarks about convergent sequences in the preceding paragraph.

12.6.1 Some sequences of compact sets

It is not too difficult to show that there are sequences $\{K_j\}_{j=1}^\infty$ of nonempty compact subsets of $\mathbf{R}^n$ such that

$$\bigcup_{j=1}^{\infty} K_j = U \tag{12.6.1}$$

and K_j is contained in the interior of K_{j+1} for each j. If K is any compact subset of $\mathbf{R}^n$ that is contained in U, then it follows that

$$K \subseteq K_j \tag{12.6.2}$$

for some j. This implies that $C^\infty_{com}(U, \mathbf{C})$ corresponds to

$$\bigcup_{j=1}^{\infty} C^\infty_{K_j}(\mathbf{R}^n, \mathbf{C}), \tag{12.6.3}$$

as a linear subspace of $C^\infty_{com}(\mathbf{R}^n, \mathbf{C})$.

12.7 The Schwartz class

Let n be a positive integer. The *Schwartz class* $\mathcal{S}(\mathbf{R}^n)$ is the space of smooth complex-valued functions f on $\mathbf{R}^n$ such that

$$x^\alpha \, (\partial^\beta f)(x) \tag{12.7.1}$$

is bounded on $\mathbf{R}^n$ for all multi-indices α, β. Equivalently, this means that

$$(1 + |x|^2)^k \, |(\partial^\beta f)(x)| \tag{12.7.2}$$

is bounded on $\mathbf{R}^n$ for every nonnegative integer k and multi-index β. It is easy to see that $\mathcal{S}(\mathbf{R}^n)$ is a linear subspace of the space $C^\infty(\mathbf{R}^n, \mathbf{C})$ of all complex-valued smooth functions on $\mathbf{R}^n$, as a vector space over the complex numbers.

12.7.1 Some basic properties of $\mathcal{S}(\mathbf{R}^n)$

Clearly

$$C^\infty_{com}(\mathbf{R}^n, \mathbf{C}) \subseteq \mathcal{S}(\mathbf{R}^n). \tag{12.7.3}$$

If a is a positive real number and $b \in \mathbf{C}^n$, then one can check that

$$\exp(-a\,|x|^2 + b \cdot x) \in \mathcal{S}(\mathbf{R}^n). \tag{12.7.4}$$

If $f \in \mathcal{S}(\mathbf{R}^n)$ and $c \in \mathbf{R}^n$, then one can verify that

$$f(x + c) \in \mathcal{S}(\mathbf{R}^n). \tag{12.7.5}$$

It is easy to see that

$$\partial^\gamma f \in \mathcal{S}(\mathbf{R}^n) \tag{12.7.6}$$

for every multi-index γ in this case too.

12.7.2 Multilication by polynomials

If $f \in \mathcal{S}(\mathbf{R}^n)$ and p is a polynomial on $\mathbf{R}^n$ with complex coefficients, then one can check that

$$p\,f \in \mathcal{S}(\mathbf{R}^n). \tag{12.7.7}$$

More precisely, this holds for smooth complex-valued functions p on $\mathbf{R}^n$ with the following property. If γ is any multi-index, then we ask that $\partial^\gamma p$ has at most polynomial growth at infinity on $\mathbf{R}^n$. This means that for every such γ, there are a nonnegative real number $C(\gamma)$ and a nonnegative integer $N(\gamma)$ such that

$$|(\partial^\gamma p)(x)| \le C(\gamma)\,(1 + |x|^2)^{N(\gamma)} \tag{12.7.8}$$

for every $x \in \mathbf{R}^n$. Of course, polynomials on $\mathbf{R}^n$ satisfy these conditions.

12.7.3 Convergence of sequences in $\mathcal{S}(\mathbf{R}^n)$

Let $\{f_j\}_{j=1}^\infty$ be a sequence of elements of $\mathcal{S}(\mathbf{R}^n)$, and let f be another element of $\mathcal{S}(\mathbf{R}^n)$. We say that $\{f_j\}_{j=1}^\infty$ *converges* to f in $\mathcal{S}(\mathbf{R}^n)$ if for every pair of multi-indices α, β,

$$x^\alpha\,(\partial^\beta f_j)(x) \to x^\alpha\,(\partial^\beta f)(x) \text{ as } j \to \infty, \tag{12.7.9}$$

uniformly on $\mathbf{R}^n$. This is the same as saying that for every nonnegative integer N and multi-index β,

$$(1 + |x|^2)^N\,|(\partial^\beta f_j)(x) - (\partial^\beta f)(x)| \to 0 \text{ as } j \to \infty, \tag{12.7.10}$$

uniformly on $\mathbf{R}^n$. This is also equivalent to the convergence of $\{f_j\}_{j=1}^\infty$ to f with respect to a standard topology on $\mathcal{S}(\mathbf{R}^n)$. Note that a convergent sequence in $C^\infty_{com}(\mathbf{R}^n, \mathbf{C})$, in the sense described in Subsection 12.3.1, converges as a sequence in $\mathcal{S}(\mathbf{R}^n)$.

If $\{f_j\}_{j=1}^\infty$ converges to f in $\mathcal{S}(\mathbf{R}^n)$, then it is easy to see that $\{\partial^\gamma f_j\}_{j=1}^\infty$ converges to $\partial^\gamma f$ in $\mathcal{S}(\mathbf{R}^n)$ for every multi-index γ. If p is a smooth complex-valued function on $\mathbf{R}^n$ whose derivatives of all orders grow at most polynomially on $\mathbf{R}^n$, as before, then one can check that $\{p\,f_j\}_{j=1}^\infty$ converges to $p\,f$ in $\mathcal{S}(\mathbf{R}^n)$. In particular, this holds when p is a polynomial on $\mathbf{R}^n$.

12.8 Tempered distributions

Let n be a positive integer, and let λ be a linear functional on $\mathcal{S}(\mathbf{R}^n)$. Let us say that λ is continuous on $\mathcal{S}(\mathbf{R}^n)$ if for every sequence $\{\phi_j\}_{j=1}^\infty$ of elements of $\mathcal{S}(\mathbf{R}^n)$ that converges to an element ϕ of $\mathcal{S}(\mathbf{R}^n)$, in the sense described in the previous section, we have that

$$\lim_{j \to \infty} \lambda(\phi_j) = \lambda(\phi). \tag{12.8.1}$$

This is equivalent to the continuity of ϕ with respect to the standard topology on $\mathcal{S}(\mathbf{R}^n)$, which was mentioned in the previous section. Under these conditions, λ is said to be a *tempered distribution* on $\mathbf{R}^n$. The space

$$\mathcal{S}(\mathbf{R}^n)' \tag{12.8.2}$$

of tempered distributions on $\mathbf{R}^n$ is a vector space over the complex numbers, with respect to pointwise addition and scalar multiplication of linear functionals on $\mathcal{S}(\mathbf{R}^n)$.

12.8.1 Some examples of tempered distributions

Let f be a continuous complex-valued function on $\mathbf{R}^n$, and suppose that f grows at most polynomially on $\mathbf{R}^n$, so that

$$|f(x)| \le C\,(1 + |x|^2)^k \tag{12.8.3}$$

for some nonnegative real number C, nonnegative integer k, and every $x \in \mathbf{R}^n$. If $\phi \in \mathcal{S}(\mathbf{R}^n)$, then put

$$\lambda_f(\phi) = \int_{\mathbf{R}^n} f(x)\,\phi(x)\,dx, \tag{12.8.4}$$

where the right side may be defined as in Subsection 7.2.3. One can check that this defines a tempered distribution on $\mathbf{R}^n$. More precisely, this works when

$$f(x)\,(1 + |x|^2)^{-l} \tag{12.8.5}$$

is integrable on $\mathbf{R}^n$ for some nonnegative integer l. This also works when f is a locally integrable function on $\mathbf{R}^n$ with respect to Lebesgue measure such that (12.8.5) is integrable, in which case (12.8.4) should be interpreted as a Lebesgue integral.

12.8.2 Derivatives and some products

One can define derivatives of tempered distributions in the same way as in Section 12.4. Let p be a smooth complex-valued function on $\mathbf{R}^n$ whose derivatives of all orders grow at most polynomially, as in the previous section. If λ is any tempered distribution on $\mathbf{R}^n$, then

$$(p\,\lambda)(\phi) = \lambda(p\,\phi) \tag{12.8.6}$$

defines another tempered distribution on $\mathbf{R}^n$. If f is as in the preceding paragraph, then $p\,f$ satisfies an analogous condition, so that $\lambda_{p\,f}$ is defined as a tempered distribution on $\mathbf{R}^n$ too. Of course,

$$p\,\lambda_f = \lambda_{p\,f}, \tag{12.8.7}$$

as tempered distributions on $\mathbf{R}^n$.

12.8.3 Comparison with ordinary distributions

If λ is a tempered distribution on $\mathbf{R}^n$, then it is easy to see that

(12.8.8) the restriction of λ to $C^\infty_{com}(\mathbf{R}^n, \mathbf{C})$ defines a distribution on $\mathbf{R}^n$.

It is well known that

(12.8.9) λ is uniquely determined by its restriction to $C^\infty_{com}(\mathbf{R}^n, \mathbf{C})$.

More precisely,

$$C^\infty_{com}(\mathbf{R}^n, \mathbf{C}) \text{ is dense in } \mathcal{S}(\mathbf{R}^n), \tag{12.8.10}$$

with respect to the standard topology on $\mathbf{S}(\mathbf{R}^n)$. Equivalently, this means that if $\phi \in \mathcal{S}(\mathbf{R}^n)$, then there is a sequence $\{\phi_j\}_{j=1}^\infty$ of elements of $C^\infty_{com}(\mathbf{R}^n)$ that converges to ϕ in $\mathcal{S}(\mathbf{R}^n)$. The ϕ_j's can be obtained by multiplying ϕ by suitable smooth functions on $\mathbf{R}^n$ with compact support, which are equal to 1 on large bounded subsets of $\mathbf{R}^n$.

Of course, if $x \in \mathbf{R}^n$, then $\delta_x(\phi) = \phi(x)$ defines a tempered distribution on $\mathbf{R}^n$, which is another version of the Dirac distribution associated to x. See [196, 293, 297] for more information about the Schwartz class and tempered distributions, in addition to the references about distributions mentioned in Section 12.1.

12.9 More on $\mathcal{S}(\mathbf{R}^n)$, $\mathcal{S}(\mathbf{R}^n)'$

Let n be a positive integer, let f be an element of the Schwartz class $\mathcal{S}(\mathbf{R}^n)$, and let $c \in \mathbf{R}^n$ be given. One can check that $f(x + c) \in \mathcal{S}(\mathbf{R}^n)$, as a function of $x \in \mathbf{R}^n$, as mentioned in Subsection 12.7.1. More precisely, if α and β are multi-indices, then the boundedness of

$$x^\alpha\, (\partial^\beta f)(x + c) \tag{12.9.1}$$

on $\mathbf{R}^n$ is equivalent to the boundedness of

$$(x - c)^\alpha\, (\partial^\beta f)(x) \tag{12.9.2}$$

on $\mathbf{R}^n$. This can be obtained from the boundedness of

$$x^\gamma\, (\partial^\beta f)(x) \tag{12.9.3}$$

on $\mathbf{R}^n$, for multi-indices γ with $\gamma_j \le \alpha_j$ for each $j = 1, \ldots, n$. This argument also shows that one can get a bound for the absolute value of (12.9.1) on $\mathbf{R}^n$ that grows at most polynomially in $|c|$.

12.9.1 A boundedness condition

Let λ be a linear functional on $\mathcal{S}(\mathbf{R}^n)$. Suppose that there are a nonnegative real number C and nonnegative integers N_1, N_2 such that

$$|\lambda(\phi)| \le C \sum_{|\alpha| \le N_1} \sum_{|\beta| \le N_2} \Big(\sup_{x \in \mathbf{R}^n} |x^\alpha \, (\partial^\beta \phi)(x)| \Big) \tag{12.9.4}$$

for every $\phi \in \mathcal{S}(\mathbf{R}^n)$. Here the first sum is taken over all multi-indices α with $|\alpha| \le N_1$, and the second sum is taken over all multi-indices β with $|\beta| \le N_2$, as usual. Under these conditions, one can check that λ is a tempered distribution on $\mathbf{R}^n$.

In fact, let $\{\phi_j\}_{j=1}^\infty$ be a sequence of elements of $\mathcal{S}(\mathbf{R}^n)$, and let ϕ be another element of $\mathcal{S}(\mathbf{R}^n)$. Suppose that

$$x^\alpha \, (\partial^\beta \phi)(x) \to x^\alpha \, (\partial^\beta \phi)(x) \text{ as } j \to \infty, \tag{12.9.5}$$

uniformly on $\mathbf{R}^n$, for all multi-indices α, β with $|\alpha| \le N_1$ and $|\beta| \le N_2$. If (12.9.4) holds, then it is easy to see that $\lambda(\phi_j) \to \phi$ as $j \to \infty$.

Conversely, if λ is a tempered distribution on $\mathbf{R}^n$, then it is well known that (12.9.4) holds for some $C, N_1, N_2 \ge 0$. This can be obtained from the continuity of λ at 0, with respect to the standard topology on $\mathcal{S}(\mathbf{R}^n)$.

12.9.2 Covergence of sequences of translates

Let f be a continuous complex-valued function on $\mathbf{R}^n$, and let $\{c_j\}_{j=1}^\infty$ be a sequence of elements of $\mathbf{R}^n$ that converges to 0. Put

$$f_j(x) = f(x + c_j) \tag{12.9.6}$$

for each $x \in \mathbf{R}^n$ and $j \ge 1$. Note that

$$f_j \to f \text{ as } j \to \infty \tag{12.9.7}$$

pointwise on $\mathbf{R}^n$, because f is continuous on $\mathbf{R}^n$. More precisely, one can check that (12.9.7) holds uniformly on compact subsets of $\mathbf{R}^n$, because continuous functions are uniformly continuous on compact sets. If f is uniformly continuous on $\mathbf{R}^n$, then (12.9.7) holds uniformly on $\mathbf{R}^n$.

If f is smooth on $\mathbf{R}^n$, then for each multi-index α,

$$\partial^\alpha f_j \to \partial^\alpha f \text{ as } j \to \infty, \tag{12.9.8}$$

uniformly on compact subsets of $\mathbf{R}^n$. If f has compact support in $\mathbf{R}^n$, then one can verify that there is a compact subset of $\mathbf{R}^n$ that contains the supports of f and f_j for each j.

If $f \in \mathcal{S}(\mathbf{R}^n)$, then $f_j \in \mathcal{S}(\mathbf{R}^n)$ for each j, as before. In this case, it is not too difficult to show that (12.9.7) holds in $\mathcal{S}(\mathbf{R}^n)$, in the sense defined in Subsection 12.7.3.

12.10 Some convolutions

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. If $a \in \mathbf{R}^n$ and $E \subseteq \mathbf{R}^n$, then put

$$a + E = \{a + x : x \in E\} \tag{12.10.1}$$

and

$$-E = \{-x : x \in E\}. \tag{12.10.2}$$

Similarly, we put $a - E = a + (-E)$.

Let K be a nonempty compact subset of $\mathbf{R}^n$, and put

$$V = \{a \in \mathbf{R}^n : a - K \subseteq U\}. \tag{12.10.3}$$

One can check that this is an open subset of $\mathbf{R}^n$, because U is an open set.

If ϕ is a complex-valued function on $\mathbf{R}^n$, then put

$$\widetilde{\phi}(y) = \phi(-y) \tag{12.10.4}$$

for every $y \in \mathbf{R}^n$. Also let $\tau_a(\phi)$ be the complex-valued function on $\mathbf{R}^n$ defined by

$$(\tau_a(\phi))(y) = \phi(y - a) \tag{12.10.5}$$

for every $y \in \mathbf{R}^n$. Thus

$$(\tau_a(\widetilde{\phi}))(y) = \phi(a - y) \tag{12.10.6}$$

for every $y \in \mathbf{R}^n$.

Suppose now that ϕ is smooth on $\mathbf{R}^n$, with

$$\operatorname{supp} \phi \subseteq K, \tag{12.10.7}$$

and let λ be a distribution on U. Observe that

$$\operatorname{supp} \tau_a(\widetilde{\phi}) = a - \operatorname{supp} \phi \tag{12.10.8}$$

for every $a \in \mathbf{R}^n$. In particular, if $a \in V$, then

$$\operatorname{supp} \tau_a(\widetilde{\phi}) \subseteq U. \tag{12.10.9}$$

Under these conditions, the *convolution* of λ and ϕ is the complex-valued function $\lambda * \phi$ defined on V by

$$(\lambda * \phi)(a) = \lambda(\tau_a(\widetilde{\phi})). \tag{12.10.10}$$

If $b \in \mathbf{R}^n$, then

$$(\delta_b * \phi)(a) = \phi(a - b) \tag{12.10.11}$$

for every $a \in \mathbf{R}^n$, by (12.10.6).

12.10.1 Some properties of these convolutions

If $\{a_j\}_{j=1}^{\infty}$ is a sequence of elements of V that converges to $a \in V$, then one can check that

$$\lim_{j \to \infty} (\lambda * \phi)(a_j) = (\lambda * \phi)(a). \tag{12.10.12}$$

Equivalently, this means that

$$\lim_{j \to \infty} \lambda(\tau_{a_j}(\widetilde{\phi})) = \lambda(\tau_a(\widetilde{\phi})). \tag{12.10.13}$$

This implies that $\lambda * \phi$ is continuous on V.

It is well known and not too difficult to show that the first partial derivatives of $\lambda * \phi$ exist on V, with

$$\partial_l(\lambda * \phi) = \lambda * (\partial_l \phi) \tag{12.10.14}$$

for each $l = 1, \ldots, n$. One can use this repeatedly, to get that $\lambda * \phi$ is smooth on V, with

$$\partial^\alpha(\lambda * \phi) = \lambda * (\partial^\alpha \phi) \tag{12.10.15}$$

on V for each multi-index α.

One can verify that

$$\lambda * (\partial^\alpha \phi) = (\partial^\alpha \lambda) * \phi \tag{12.10.16}$$

on V for every multi-index α. It follows that

$$\partial^\alpha(\lambda * \phi) = (\partial^\alpha \lambda) * \phi \tag{12.10.17}$$

on V, by (12.10.15).

12.10.2 Some convolutions with tempered distributions

If $\phi \in \mathcal{S}(\mathbf{R}^n)$ and $\lambda \in \mathcal{S}(\mathbf{R}^n)'$, then $\lambda * \phi$ can be defined on $\mathbf{R}^n$ as in (12.10.10). It is well known that this satisfies the same type of properties as before.

In this case, one can also show that

(12.10.18) $\lambda * \phi$ grows at most polynomially on $\mathbf{R}^n$,

using the remarks in the previous section. More precisely,

(12.10.19) the derivatives of $\lambda * \phi$ of all orders grow at most polynomially

too.

12.11 Local solvability

Let n be a positive integer, and let $p(w)$ be a nonzero polynomial on $\mathbf{R}^n$ with complex coefficients. As in Subsection 12.1.1, a theorem of Ehrenpreis and

Malgrange states that there is a distribution E on $\mathbf{R}^n$ that is a fundamental solution of $p(\partial)$, in the sense that

$$p(\partial)(E) = \delta_0. \tag{12.11.1}$$

Let $f \in C^\infty_{com}(\mathbf{R}^n, \mathbf{C})$ be given, and put

$$u = E * f, \tag{12.11.2}$$

which is a smooth complex-valued function on $\mathbf{R}^n$, as in the previous section. Under these conditions, we have that

$$(p(\partial))(u) = (p(\partial))(E) * f = \delta_0 * f = f \tag{12.11.3}$$

on $\mathbf{R}^n$, as mentioned earlier.

Let

$$L = \sum_{|\alpha| \le N} a_\alpha(x)\, \partial^\alpha \tag{12.11.4}$$

be a differential operator whose coefficients $a_\alpha(x)$ are smooth complex-valued functions on $\mathbf{R}^n$. Here N is a nonnegative integer, and the sum is taken over all multi-indices α with $|\alpha| \le N$, as usual. We say that L is *locally solvable* at a point $x_0 \in \mathbf{R}^n$ if for any smooth complex-valued function f on $\mathbf{R}^n$ there is a function (or distribution) u on a neighborhood of x_0 in $\mathbf{R}^n$ that satisfies

$$L(u) = f \tag{12.11.5}$$

on that neighborhood, as in Section F of Chapter 1 of [87]. We may as well take f to have compact support in $\mathbf{R}^n$, as in [87], since otherwise we can multiply f by a smooth function on $\mathbf{R}^n$ with compact support that is equal to 1 on a neighborhood of x_0. Similarly, we could start with any smooth complex-valued function f_0 defined on a neighborhood of x_0 in $\mathbf{R}^n$, and get a smooth function on $\mathbf{R}^n$ with compact support that is equal to f_0 on neighborhood of x_0.

If the coefficients of L are constants, not all equal to 0, then the theorem of Ehrenpreis and Malgrange implies that L is locally solvable at every point in $\mathbf{R}^n$. If

(12.11.6) f and the coefficients a_α are real-analytic near x_0,

and if

$$a_\alpha(x_0) \ne 0 \tag{12.11.7}$$

for some multi-index α with $|\alpha| = N$, then one can get real-analytic solutions to (12.11.5) near x_0 using a famous theorem of Cauchy and Kovalevskaya, as mentioned near the beginning of Section E of Chapter 1 of [87]. This theorem is discussed in Section 4.6.3 of [81], Section D of Chapter 1 of [87], and Section 2.8 of [190].

There is a famous example of H. Lewy of a first-order differential operator on $\mathbf{R}^3$ whose coefficients are constants or linear functions, and for which local solvability does not hold. See Section E of Chapter 1 of [87] for more information.

12.12 Sequences of distributions

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. A sequence $\{\lambda_j\}_{j=1}^\infty$ of distributions on U is said to *converge* to a distribution λ on U if

$$\lim_{j \to \infty} \lambda_j(\phi) = \lambda(\phi) \tag{12.12.1}$$

for every $\phi \in C^\infty_{com}(U, \mathbf{C})$. More precisely, this is the same as convergence with respect to the "weak* topology" on $C^\infty_{com}(\mathbf{R}^n, \mathbf{C})'$.

12.12.1 Convergence of derivatives and products

In this case, it is easy to see that

$$\{\partial^\alpha \lambda_j\}_{j=1}^\infty \text{ converges to } \partial^\alpha \lambda \tag{12.12.2}$$

in the same sense for every multi-index α. Similarly, if a is a smooth complex-valued function on U, then

$$\{a\,\lambda_j\}_{j=1}^\infty \text{ converges to } a\,\lambda \tag{12.12.3}$$

in this sense.

12.12.2 Some examples of convergent sequences

Let $\{f_j\}_{j=1}^\infty$ be a sequence of continuous complex-valued functions on U that converges to a complex-valued function f uniformly on compact sets contained in U. Under these conditions, it is easy to see that the corresponding sequence of distributions $\{\lambda_{f_j}\}_{j=1}^\infty$, as in Section 12.3, converges to the distribution λ_f corresponding to f, in the sense considered here. This also works when $\{f_j\}_{j=1}^\infty$ is a sequence of locally-integrable functions on U that converges to a locally-integrable function f on U with respect to the L^1 metric on any compact set contained in U.

12.12.3 A boundedness condition for sequences

Let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of distributions on U again, and suppose for the moment that for each $\phi \in C^\infty_{com}(U, \mathbf{C})$,

$$\{\lambda_j(\phi)\}_{j=1}^\infty \text{ is a bounded sequence in } \mathbf{C}. \tag{12.12.4}$$

Let K be a nonempty compact subset of $\mathbf{R}^n$ that is contained in U. A famous theorem of Banach and Steinhaus implies that there are a nonnegative real number $C(K)$ and a nonnegative integer $N(K)$ such that

$$|\lambda_j(\phi)| \le C(K) \sum_{|\alpha| \le N(K)} \Big(\max_{x \in K} |(\partial^\alpha \phi)(x)|\Big) \tag{12.12.5}$$

for each $j \ge 1$ and $\phi \in C^\infty_K(\mathbf{R}^n, \mathbf{C})$. This uses the well-known fact that $C^\infty_K(\mathbf{R}^n, \mathbf{C})$ is a "Fréchet space".

12.12.4 Limits of sequences of distributions

Suppose now that for each $\phi \in C^\infty_{com}(U, \mathbf{C})$,

$$\{\lambda_j(\phi)\}_{j=1}^\infty \text{ converges in } \mathbf{C}. \tag{12.12.6}$$

This implies (12.12.4), because convergent sequences are bounded. If λ is defined on $C^\infty_{com}(U, \mathbf{C})$ as in (12.12.1), then it is easy to see that λ is a linear functional on $C^\infty_{com}(U, \mathbf{C})$. We also have that

$$|\lambda(\phi)| \le C(K) \sum_{|\alpha| \le N(K)} \Big(\max_{x \in K} |(\partial^\alpha \phi)(x)|\Big) \tag{12.12.7}$$

for every nonempty compact set K contained in U and $\phi \in C^\infty_K(\mathbf{R}^n, \mathbf{C})$, where $C(K)$ and $N(K)$ are as in the preceding paragraph. This implies that

$$\lambda \text{ is a distribution on } U, \tag{12.12.8}$$

as in Section 12.6. Thus $\{\lambda_j\}_{j=1}^\infty$ converges to λ in the sense described at the beginning of the section. This corresponds to Theorem 6.17 on p146 of [276].

12.12.5 Convergent sequences of tempered distributions

Similarly, a sequence $\{\lambda_j\}_{j=1}^\infty$ of tempered distributions on $\mathbf{R}^n$ is said to *converge* to a tempered distribution λ on $\mathbf{R}^n$ if (12.12.1) holds for every ϕ in $\mathcal{S}(\mathbf{R}^n)$. This is the same as convergence with respect to the weak* topology on $\mathcal{S}(\mathbf{R}^n)'$, as before.

If α is a multi-index, then it follows that $\{\partial^\alpha \lambda_j\}_{j=1}^n$ converges to $\partial^\alpha \lambda$ in the same sense. If a is a smooth complex-valued function on $\mathbf{R}^n$ such that a and all of its derivatives grow at most polynomially on $\mathbf{R}^n$, then $\{a\,\lambda_j\}_{j=1}^\infty$ converges to $a\,\lambda$ in this sense too.

12.12.6 Some sequences of tempered distributions

Let $\{f_j\}_{j=1}^\infty$ be a sequence of complex-valued functions on $\mathbf{R}^n$, let f be another complex-valued function on $\mathbf{R}^n$, and let l be a nonnegative integer. Suppose that the f_j's and f are continuous on $\mathbf{R}^n$, or at least locally integrable, and that the products $f_j(x)\,(1 + |x|^2)^{-l}$ and $f(x)\,(1 + |x|^2)^{-l}$ are integrable on $\mathbf{R}^n$. Thus we get tempered distributions λ_{f_j}, λ_f on $\mathbf{R}^n$, as in Subsection 12.8.1. If

$$\lim_{j \to \infty} \int_{\mathbf{R}^n} |f_j(x) - f(x)|\,(1 + |x|^2)^{-l}\,dx = 0, \tag{12.12.9}$$

then it is easy to see that $\{\lambda_{f_j}\}_{j=1}^\infty$ converges to λ_f, as tempered distributions on $\mathbf{R}^n$.

12.12.7 Bounded sequences of tempered distributions

Suppose that $\{\lambda_j\}_{j=1}^{\infty}$ is a sequence of tempered distributions on $\mathbf{R}^n$ that satisfies (12.12.4) for every $\phi \in \mathcal{S}(\mathbf{R}^n)$. One can use the Banach–Steinhaus theorem to get that there are a nonnegative real number C and nonnegative integers N_1, N_2 such that

$$|\lambda_j(\phi)| \le C \sum_{|\alpha| \le N_1} \sum_{|\beta| \le N_2} \Big(\sup_{x \in \mathbf{R}^n} |x^\alpha \, (\partial^\beta \phi)(x)| \Big) \tag{12.12.10}$$

for each $j \ge 1$ and $\phi \in \mathcal{S}(\mathbf{R}^n)$. This uses the fact that $\mathcal{S}(\mathbf{R}^n)$ is a Fréchet space too.

Suppose that (12.12.6) holds for every $\phi \in \mathcal{S}(\mathbf{R}^n)$, so that (12.12.4) holds in particular, as before. This permits us to define λ as a linear functional on $\mathcal{S}(\mathbf{R}^n)$ by (12.12.1). Note that

$$|\lambda(\phi)| \le C \sum_{|\alpha| \le N_1} \sum_{|\beta| \le N_2} \Big(\sup_{x \in \mathbf{R}^n} |x^\alpha \, (\partial^\beta \phi)(x)| \Big) \tag{12.12.11}$$

for every $\phi \in \mathcal{S}(\mathbf{R}^n)$, by (12.12.10). This implies that λ is a tempered distribution on $\mathbf{R}^n$, as in Subsection 12.9.1.

Chapter 13

Vector-valued functions and systems

13.1 Vector-valued functions

Let n and l be positive integers, and let U be a nonempty open subset of $\mathbf{R}^n$. If $f_1, \ldots, f_l$ are real or complex-valued functions on U, then

$$f(x) = (f_1(x), \ldots, f_l(x)) \tag{13.1.1}$$

defines a mapping from U into $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate. The continuity of f on U can be defined in the usual way, using the standard Euclidean metrics on $\mathbf{R}^n$ and on $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate. It is well known and not difficult to show that this is equivalent to the continuity of $f_1, \ldots, f_l$ as real or complex-valued functions on U, as appropriate.

The spaces of continuous functions on U with values in $\mathbf{R}^l$ and $\mathbf{C}^l$ may be denoted

$$C(U, \mathbf{R}^l) \text{ and } C(U, \mathbf{C}^l), \tag{13.1.2}$$

respectively. These are vector spaces over the real and complex numbers, respectively, with respect to pointwise addition and scalar multiplication of functions. These spaces may be identified with the spaces

$$C(U, \mathbf{R})^l \text{ and } C(U, \mathbf{C})^l \tag{13.1.3}$$

of l-tuples of elements of $C(U, \mathbf{R})$ and $C(U, \mathbf{C})$, respectively.

13.1.1 Differentiability of vector-valued functions

One can define partial derivatives of f, when they exist, in the usual way, using the standard Euclidean metric on $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate. This is equivalent to the existence of the corresponding partial derivative of f_j for each $j = 1, \ldots, l$, in which case the jth component of the partial derivative of f is equal to the corresponding partial derivative of f_j. Similarly, the continuous differentiability of f on U can be defined directly, and is equivalent to the continuous differentiability of f_j on U for each $j = 1, \ldots, l$. If k is any positive integer, then the k-times continuous differentiability of f on U can be defined directly as well, and is equivalent to the k-times continuous differentiability of f_j on U for each $j = 1, \ldots, l$. If f is k-times continuously differentiable on U for every $k \geq 1$, then f is said to be infinitely differentiable or smooth on U, as before.

13.1.2 Spaces of vector-valued functions

Let

$$C^k(U, \mathbf{R}^l) \text{ and } C^k(U, \mathbf{C}^l) \tag{13.1.4}$$

be the spaces of k-times continuously differentiable functions on U with values in $\mathbf{R}^l$ and $\mathbf{C}^l$, respectively, for each $k \geq 1$. We may use the same notation with $k = 0$ for the corresponding spaces of continuous functions, as before. Note that these are linear subspaces of $C(U, \mathbf{R}^l)$ and $C(U, \mathbf{C}^l)$, respectively, as vector spaces over the real and complex numbers, as appropriate, for each k. We may identify these space with the spaces

$$C^k(U, \mathbf{R})^l \text{ and } C^k(U, \mathbf{C})^l \tag{13.1.5}$$

of l-tuples of elements of $C^k(U, \mathbf{R})$ and $C^k(U, \mathbf{C})$, respectively, as usual.

Similarly,

$$C^\infty(U, \mathbf{R}^l) \text{ and } C^\infty(U, \mathbf{C}^l) \tag{13.1.6}$$

denote the spaces of smooth functions on U with values in $\mathbf{R}^l$ and $\mathbf{C}^l$, respectively. These are linear subspaces of $C^k(U, \mathbf{R}^l)$ and $C^k(U, \mathbf{C}^l)$, respectively, for each k. We may identify these spaces with the spaces

$$C^\infty(U, \mathbf{R})^l \text{ and } C^\infty(U, \mathbf{C})^l \tag{13.1.7}$$

of l-tuples of elements of $C^\infty(U, \mathbf{R})$ and $C^\infty(U, \mathbf{C})$, respectively, as before.

13.2 Matrix-valued functions

Let l_1, l_2 be positive integers, and let

$$\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2}),\ \mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2}) \tag{13.2.1}$$

be the spaces of linear mappings from $\mathbf{R}^{l_1}$, $\mathbf{C}^{l_1}$ into $\mathbf{R}^{l_2}$, $\mathbf{C}^{l_2}$, respectively, as vector spaces over the real and complex numbers. Note that these are vector spaces over the real and complex numbers, respectively, with respect to pointwise addition and scalar multiplication of linear mappings. Of course, these linear mappings can be represented in terms of matrices of real or complex numbers, as appropriate, in the usual way. One can use this to identify these spaces with

$$\mathbf{R}^{l_1\, l_2},\ \mathbf{C}^{l_1\, l_2}, \tag{13.2.2}$$

respectively.

13.2.1 Continuity and differentiability properties

Let n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Suppose that $a(x)$ is a function of $x \in U$ with values in $\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})$. This can be identified with a function on U with values in $\mathbf{R}^{l_1\, l_2}$ or $\mathbf{C}^{l_1\, l_2}$, as appropriate, as before. In particular, this can be used to define the usual continuity and differentiability properties of $a(x)$ on U, using the standard Euclidean metric on $\mathbf{R}^{l_1\, l_2}$ or $\mathbf{C}^{l_1\, l_2}$, as appropriate. This is equivalent to the analogous continuity or differentiability properties of the $l_1 \cdot l_2$ real or complex-valued functions on U corresponding to the matrix entries of $a(x)$.

A version of this was mentioned in Section 5.15, for functions defined on an interval in the real line. Similarly, if $v \in \mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate, then

$$(a(x))(v) \tag{13.2.3}$$

defines a function of $x \in U$ with values in $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate. Continuity or differentiability properties of $a(x)$ on U are also equivalent to the analogous properties of (13.2.3) holding for every $v \in \mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate, as a function of $x \in U$ with values in $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate.

13.2.2 Using vector-valued functions $v(x)$ on U

Suppose that $v(x)$ is a function on U with values in $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate, so that

$$(a(x))(v(x)) \tag{13.2.4}$$

is a function on U with values in $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate. If $a(x)$ and $v(x)$ satisfy suitable continuity or differentiability properties on U, then (13.2.4) satisfies the same property on U, as in Section 5.15. In particular,

$$\frac{\partial}{\partial x_j}((a(x))(v(x))) = \Big(\frac{\partial a}{\partial x_j}(x)\Big)(v(x)) + (a(x))\Big(\frac{\partial v}{\partial x_j}(x)\Big) \tag{13.2.5}$$

when the partial derivatives of $a(x)$ and $v(x)$ exist, as before.

13.2.3 Products of matrix-valued functions

Let l_0 be another positive integer, and let $b(x)$ be a function of $x \in U$ with values in $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_1})$ or $\mathcal{L}(\mathbf{C}^{l_0}, \mathbf{C}^{l_1})$, as appropriate. If $x \in U$, then let

$$a(x)\, b(x) \tag{13.2.6}$$

be the composition of $b(x)$ with $a(x)$ as linear mappings, which defines an element of $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_0}, \mathbf{C}^{l_2})$, as appropriate. Of course, this corresponds to multiplication of the matrices associated to $b(x)$ and $a(x)$. If $a(x)$ and $b(x)$ satisfy suitable continuity or differentiability properties on U, then (13.2.6) satisfies the same property on U, as before. In particular,

$$\frac{\partial}{\partial x_j}(a(x)\, b(x)) = \Big(\frac{\partial a}{\partial x_j}(x)\Big)\, b(x) + a(x)\, \Big(\frac{\partial b}{\partial x_j}(x)\Big), \tag{13.2.7}$$

when the partial derivatives of $a(x)$ and $b(x)$ in x_j exist.

13.3 Matrix-valued coefficients

Let l_1, l_2, and n be a positive integer, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let N be a nonnegative integer, and for each multi-index α with order $|\alpha| \le N$, let a_α be a function on U with values in $\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})$. If u is an N-times continuously-differentiable function on U with values in $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate, then let $L(u)$ be the function on U with values in $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate, defined by

$$(L(u))(x) = \sum_{|\alpha| \le N} (a_\alpha(x))((\partial^\alpha u)(x)) \tag{13.3.1}$$

for every $x \in U$. More precisely, if $x \in U$ and α is a multi-index with $|\alpha| \le N$, then

$$(\partial^\alpha u)(x) \tag{13.3.2}$$

is an element of $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$,

$$a_\alpha(x) \tag{13.3.3}$$

is an element of $\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})$, and

$$(a_\alpha(x))((\partial^\alpha u)(x)) \tag{13.3.4}$$

is an element of $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate.

Suppose that

$$a_\alpha \text{ is } r\text{-times continuously differentiable on } U \tag{13.3.5}$$

for some nonnegative integer r, and each multi-index α with $|\alpha| \le N$. If

$$u \text{ is } (N+r)\text{-times continuously differentiable on } U, \tag{13.3.6}$$

then

$$L(u) \text{ is } r\text{-times continuously differentiable on } U, \tag{13.3.7}$$

as in Section 2.4. Under these conditions, L defines a linear mapping from $C^{N+r}(U, \mathbf{R}^{l_1})$ into $C^r(U, \mathbf{R}^{l_2})$, or from $C^{N+r}(U, \mathbf{C}^{l_1})$ into $C^r(U, \mathbf{C}^{l_2})$, as aprorpiate.

Similarly, if

$$a_\alpha \text{ is smooth on } U \tag{13.3.8}$$

for every multi-index α with $|\alpha| \le N$, and

$$u \text{ is smooth on } U, \tag{13.3.9}$$

then

$$L(u) \text{ is smooth on } U \tag{13.3.10}$$

as well. Under these conditions, L defines a linear mapping from $C^\infty(U, \mathbf{R}^{l_1})$ into $C^\infty(U, \mathbf{R}^{l_2})$, or from $C^\infty(U, \mathbf{C}^{l_1})$ into $C^\infty(U, \mathbf{C}^{l_2})$, as appropriate.

Polynomials on $\mathbf{R}^n$ with values in $\mathbf{R}^l$ or $\mathbf{C}^l$ for some positive integer l will be discussed in the next section. One can check that the a_α's are uniquely determined by $L(u)$ for polynomials u with values in $\mathbf{R}^{l_1}$ of degree less than or equal to N, as in Section 2.4.

13.3.1 Compositions of differential operators

Let l_0 be another positive integer, and let $\widetilde{N}$ be another nonnegative integer. Suppose that for each multi-index β with $|\beta| \le \widetilde{N}$, b_β is a function on U with values in $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_1})$ or $\mathcal{L}(\mathbf{C}^{l_0}, \mathbf{C}^{l_1})$. If u is an $\widetilde{N}$-times continuously-differentiable function on U with values in $\mathbf{R}^{l_0}$ or $\mathbf{C}^{l_0}$, as appropriate, then

$$(\widetilde{L}(u))(x) = \sum_{|\beta| \le \widetilde{N}} (b_\beta(x))((\partial^\beta u)(x)) \tag{13.3.11}$$

defines a function on U with values in $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate. If b_β is N-times continuously-differentiable on U for every multi-index β with $|\beta| \le \widetilde{N}$, and u is $(N + \widetilde{N})$-times continuously differentiable on U, then $\widetilde{L}(u)$ is N-times continuously differentiable on U. This implies that

$$L(\widetilde{L}(u)) \tag{13.3.12}$$

is defined as a function on U with values in $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate.

As in Subsection 2.4.1, (13.3.12) may be expressed as

$$(\widehat{L}(u))(x) = \sum_{|\gamma| \le N + \widetilde{N}} (c_\gamma(x))((\partial^\gamma u)(x)). \tag{13.3.13}$$

Here c_γ is a function on U with values in $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_0}, \mathbf{C}^{l_2})$ for each multi-index γ with $|\gamma| \le N + \widetilde{N}$. These functions can be expressed as sums of products of the a_α's with the b_β's and their derivatives of order less than or equal to N, as before. More precisely, these products correspond to compositions of linear mappings from $\mathbf{R}^{l_0}$ or $\mathbf{C}^{l_0}$ into $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$ with linear mappings from $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$ into $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$ to get linear mappings from $\mathbf{R}^{l_0}$ or $\mathbf{C}^{l_0}$ into $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate.

If a_α is r-times continuously differentiable on U for some $r \ge 0$ and every α with $|\alpha| \le N$, and if b_β is $(N + r)$-times continuously differentiable on U for every β with $|\beta| \le \widetilde{N}$, then c_γ is r-times continuously differentiable on U for every γ with $|\gamma| \le N + \widetilde{N}$. If u is also $(N + \widetilde{N} + r)$-times continuously differentiable on U, then $\widetilde{L}(u)$ is $(N+r)$-times continuously differentiable on U, and $\widehat{L}(u)$ is r-times continuously differentiable on U, as before. In particular, if the a_α's and b_β's are smooth on U, then the c_γ's are smooth on U. In this case, if u is smooth on U, then $\widetilde{L}(u)$ and $\widehat{L}(u)$ are smooth on U as well.

13.4 Vector-valued polynomials

Let n and l be positive integers again, and let

$$\mathcal{P}(\mathbf{R}^n, \mathbf{R}^l) \text{ and } \mathcal{P}(\mathbf{R}^n, \mathbf{C}^l) \tag{13.4.1}$$

be the spaces of polynomials on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^l$ and $\mathbf{C}^l$, respectively. These spaces can be identified with the spaces

$$\mathcal{P}(\mathbf{R}^n, \mathbf{R})^l \text{ and } \mathcal{P}(\mathbf{R}^n, \mathbf{C})^l \tag{13.4.2}$$

of l-tuples of polynomials on $\mathbf{R}^n$ with real or complex coefficients, as appropriate. These are also linear subspaces of $C^\infty(\mathbf{R}^n, \mathbf{R}^l)$ and $C^\infty(\mathbf{R}^n, \mathbf{C}^l)$, respectively, as vector spaces over the real or complex numbers, as appropriate.

If k is a nonnegative integer, then let

$$\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l) \text{ and } \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l) \tag{13.4.3}$$

be the spaces of polynomials on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^l$ and $\mathbf{C}^l$, respectively, and degree less than or equal to k. These are linear subspaces of $\mathcal{P}(\mathbf{R}^n, \mathbf{R}^l)$ and $\mathcal{P}(\mathbf{R}^n, \mathbf{C}^l)$, as vector spaces over the real or complex numbers, as appropriate. We can identify these spaces with the spaces

$$\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})^l \text{ and } \mathcal{P}^k(\mathbf{R}^n, \mathbf{C})^l \tag{13.4.4}$$

of l-tuples of polynomials on $\mathbf{R}^n$ with real and complex coefficients, respectively, of degree less than or equal to k.

Note that $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ and $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$ have the same finite dimension, as vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively. This is equal to l times the dimension of $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R})$ and $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C})$, which is the same as the number of multi-indices β with order $|\beta| \le k$, as in Section 5.11.

Let l_1, l_2 be positive integers, and let

$$\mathcal{P}(\mathbf{R}^n, \mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})), \quad \mathcal{P}(\mathbf{R}^n, \mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})) \tag{13.4.5}$$

be the spaces of polynomials on $\mathbf{R}^n$ with coefficients in $\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})$, $\mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})$, respectively. These spaces may be identified with

$$\mathcal{P}(\mathbf{R}^n, \mathbf{R}^{l_1\, l_2}),\ \mathcal{P}(\mathbf{R}^n, \mathbf{C}^{l_1\, l_2}), \tag{13.4.6}$$

respectively, as in Section 13.2.

If k is a nonnegative integer, then let

$$\mathcal{P}^k(\mathbf{R}^n, \mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})), \quad \mathcal{P}^k(\mathbf{R}^n, \mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})) \tag{13.4.7}$$

be the spaces of polynomials on $\mathbf{R}^n$ with coefficients in $\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})$, $\mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})$, respectively, and degree less than or equal to k. These may be identified with the spaces

$$\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^{l_1\, l_2}),\ \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^{l_1\, l_2}), \tag{13.4.8}$$

respectively, as before.

13.4.1 Some remarks about degrees

Suppose that $a(x)$ be a polynomial on $\mathbf{R}^n$ with coefficients in $\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})$, and let $v(x)$ be a polynomial on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate. Observe that

(13.4.9) $a(x)(v(x))$ is a polynomial on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$,

as appropriate, and that

$$\deg(a(x)(v(x))) \le \deg a(x) + \deg v(x). \tag{13.4.10}$$

Let l_0 be another positive integer, and let $b(x)$ be a polynomial on $\mathbf{R}^n$ with coefficients in $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_1})$ or $\mathcal{L}(\mathbf{C}^{l_0}, \mathbf{C}^{l_1})$, as appropriate. The product

$$a(x)\, b(x) \tag{13.4.11}$$

may be defined as in Subsection 13.2.3, and is a polynomial on $\mathbf{R}^n$ with coefficients in $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_0}.\mathbf{C}^{l_2})$, as appropriate. We also have that

$$\deg(a(x)\, b(x)) \le \deg a(x) + \deg b(x). \tag{13.4.12}$$

Let

$$\mathcal{L}(\mathbf{R}^l) = \mathcal{L}(\mathbf{R}^l, \mathbf{R}^l) \text{ and } \mathcal{L}(\mathbf{C}^l) = \mathcal{L}(\mathbf{C}^l, \mathbf{C}^l) \tag{13.4.13}$$

be the spaces of linear mappings from $\mathbf{R}^l$ and $\mathbf{C}^l$ into themselves, respectively, as in Section 5.15. The spaces of polynomials on $\mathbf{R}^n$ with coefficients in $\mathcal{L}(\mathbf{R}^l)$ and $\mathcal{L}(\mathbf{C}^l)$ may be denoted

$$\mathcal{P}(\mathbf{R}^n, \mathcal{L}(\mathbf{R}^l)) \text{ and } \mathcal{P}(\mathbf{R}^n, \mathcal{L}(\mathbf{C}^l)), \tag{13.4.14}$$

respectively. Similarly, the spaces of polynomials on $\mathbf{R}^n$ with coefficients in $\mathcal{L}(\mathbf{R}^l)$ and $\mathcal{L}(\mathbf{C}^l)$ and degree less than or equal to k may be denoted

$$\mathcal{P}^k(\mathbf{R}^n, \mathcal{L}(\mathbf{R}^l)) \text{ and } \mathcal{P}^k(\mathbf{R}^n, \mathcal{L}(\mathbf{C}^l)), \tag{13.4.15}$$

respectively.

13.5 Matrix-valued polynomials

Let n, l_1, and l_2 be positive integers, and let N be a nonnegative integer. Also let

$$p(w) = \sum_{|\alpha| \le N} a_\alpha\, w^\alpha \tag{13.5.1}$$

be a polynomial in the n variables $w_1, \ldots, w_n$ with coefficients in $\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})$ of degree less than or equal to N. Thus, for each multi-index α with order $|\alpha| \le N$, a_α is a linear mapping from $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$ into $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate.

Using p, we get a differential operator

$$p(\partial) = \sum_{|\alpha| \le N} a_\alpha \, \partial^\alpha, \tag{13.5.2}$$

as in Section 1.7. More precisely, this is a differential operator with coefficients in $\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})$, as appropriate, as in Section 13.3.

Let b be an element of $\mathbf{R}^n$ or $\mathbf{C}^n$, as appropriate, so that $\exp(b \cdot x)$ defines a smooth real or complex-valued function of $x \in \mathbf{R}^n$. If $v \in \mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate, then

$$(\exp(b \cdot x)) \, v \tag{13.5.3}$$

defines a smooth function of $x \in \mathbf{R}^n$ with values in $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate. It is easy to see that

$$(p(\partial))((\exp(b \cdot x)) \, v) = (\exp(b \cdot x)) \, (p(b))(v), \tag{13.5.4}$$

which is a function of $x \in \mathbf{R}^n$ with values in $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate. More precisely,

$$p(b) \tag{13.5.5}$$

is defined as a linear mapping from $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$ into $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate, which sends v to an element of $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate. In particular,

$$(p(\partial))((\exp(b \cdot x)) \, v) = 0 \tag{13.5.6}$$

if and only if

$$(p(b))(v) = 0. \tag{13.5.7}$$

13.5.1 Products and compositions

Let l_0 be another positive integer, let N_0 be another nonnegative integer, and let $p_0(w)$ be a polynomial in $w_1, \ldots, w_n$ with coefficients in $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_1})$ or $\mathcal{L}(\mathbf{C}^{l_0}, \mathbf{C}^{l_1})$, as appropriate. Thus

$$p_0(\partial) \tag{13.5.8}$$

is a differential operator with coefficients in $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_1})$ or $\mathcal{L}(\mathbf{C}^{l_0}, \mathbf{C}^{l_1})$, as appropriate. The product

$$p(w) \, p_0(w) \tag{13.5.9}$$

is a polynomial in $w_1, \ldots, w_n$ with coefficients in $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_0}, \mathbf{C}^{l_2})$, as appropriate, of degree less than or equal to $N_0 + N$, as in the previous section. This leads to a differential operator

$$(p \, p_0)(\partial) \tag{13.5.10}$$

with coefficients in $\mathcal{L}(\mathbf{R}^{l_0}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_0}, \mathbf{C}^{l_2})$, as appropriate. One can check that

$$(p \, p_0)(\partial) = p(\partial) \, p_0(\partial), \tag{13.5.11}$$

as in Subsection 1.7.1.

Let U be a nonempty open subset of $\mathbf{R}^n$, and let u be a function on U with values in $\mathbf{R}^{l_0}$ or $\mathbf{C}^{l_0}$, as appropriate, that is $(N_0 + N)$-times continuously differentiable on U. Thus

$$(p_0(\partial))(u) \tag{13.5.12}$$

is a function on U with values in $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate, that is N-times continuously differentiable on U. Under these conditions, we have that

$$((p\, p_0)(\partial))(u) = (p(\partial))((p_0(\partial))(u)) \tag{13.5.13}$$

on U, as in (13.5.11).

Let l be a positive integer, and let us now take $l_1 = l_2 = l$. Let b be an element of $\mathbf{R}^n$ or $\mathbf{C}^n$ again, as appropriate, so that $p(b)$ is a linear mapping from $\mathbf{R}^l$ or $\mathbf{C}^l$ into itself, as appropriate. Suppose that v is an element of $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate, that is an eigenvector of $p(b)$ with eigenvalue λ in $\mathbf{R}$ or $\mathbf{C}$, as appropriate. This implies that

$$(p(\partial))((\exp(b \cdot x))\, v) = \lambda\, (\exp(b \cdot x))\, v, \tag{13.5.14}$$

as in (13.5.4).

Note that

$$\det p(w) \tag{13.5.15}$$

is a polynomial in $w_1, \ldots, w_n$ of degree less than or equal to $N \cdot l$ with real or complex coefficients, as appropriate.

13.6 Polynomials, vectors, and operators

Let n, l_1, and l_2 be positive integers, and let N be a nonnegative integer. Suppose that for each multi-index α with $|\alpha| \le N$, a_α is a polynomial on $\mathbf{R}^n$ with coefficients in $\mathcal{L}(\mathbf{R}^{l_1}, \mathbf{R}^{l_2})$ or $\mathcal{L}(\mathbf{C}^{l_1}, \mathbf{C}^{l_2})$. This can be identified with a polynomial with coefficients in $\mathbf{R}^{l_1\, l_2}$ or $\mathbf{C}^{l_1\, l_2}$, as in Section 13.2.

Using the a_α's, we can define a differential operator L acting on N-times continuously differentiable functions u on $\mathbf{R}^n$ with values in $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate, as in (13.3.1). In this case, L maps polynomials on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$ to polynomials on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^{l_2}$ or $\mathbf{C}^{l_2}$, as appropriate.

Let c be an integer, and suppose that

$$\deg a_\alpha \le |\alpha| - c \tag{13.6.1}$$

for each α, $|\alpha| \le N$, which is interpreted as meaning that $a_\alpha = 0$ when $|\alpha| < c$, as usual. If p is a polynomial on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^{l_1}$ or $\mathbf{C}^{l_1}$, as appropriate, then

$$\deg L(p) \le \deg p - c, \tag{13.6.2}$$

which means that $L(p) = 0$ when $\deg p < c$, as before.

13.6.1 The case where $l_1 = l_2 = l$

Suppose now that $l_1 = l_2 = l$, so that L maps $\mathcal{P}(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}(\mathbf{R}^n, \mathbf{C}^l)$ into itself, as appropriate. If p is a polynomial on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate, then

$$\deg L^j(p) \le \deg p - c\, j \tag{13.6.3}$$

for each $j \ge 1$, by (13.6.2). This means that $L^j(p) = 0$ when $\deg p < c\, j$, as before.

Suppose that $c \ge 0$, and let k be a nonnegative integer. Thus L maps $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$ into itself, as appropriate. Let L_k be the restriction of L to $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$, as appropriate. If $c \ge 1$ and

$$k < c\, j, \tag{13.6.4}$$

then

$$L_k^j = 0, \tag{13.6.5}$$

by (13.6.3). In particular, this means that L_k is nilpotent when $c \ge 1$.

13.6.2 The exponential of $t\, L_k$

Let $m(k)$ be the number of multi-indices β with order $|\beta| \le k$, so that

$$\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l) \text{ and } \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l) \text{ have dimension } l \cdot m(k) \tag{13.6.6}$$

as vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively, as in Section 13.4. This permits us to identify L_k with a linear mapping from $\mathbf{R}^{l\, m(k)}$ or $\mathbf{C}^{l\, m(k)}$ into itself, as appropriate. If $t \in \mathbf{R}$, then we can define the exponential of $t\, L_k$ as a linear mapping on $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$, as appropriate, as in Sections 5.4 and 5.8.

Let q be a polynomial on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate, and of degree less than or equal to k. Note that

$$(\exp(t\, L_k))(q) \tag{13.6.7}$$

is another polynomial on $\mathbf{R}^n$ with coefficients in $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate, and degree less than or equal to k. The coefficients of this polynomial depend on t, and are smooth functions of t. This implies that

$$u(x,t) = ((\exp(t\, L_k))(q))(x) \tag{13.6.8}$$

is smooth as a function of $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ with values in $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate.

If $c \ge 1$, then

$$\exp(t\, L_k) \text{ is a polynomial in } t \tag{13.6.9}$$

whose coefficients are linear mappings on $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$, as appropriate, as in Subsection 5.10.1. It follows that

$$u(x,t) \text{ is a polynomial in } x \text{ and } t \tag{13.6.10}$$

with coefficients in $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate, in this case.

Of course,

$$u(x,0) = q(x) \tag{13.6.11}$$

for every $x \in \mathbf{R}^n$, and

$$\frac{\partial}{\partial t}((\exp(t\,L_k))(q)) = L_k((\exp(t\,L_k))(q)), \tag{13.6.12}$$

as in Sections 5.6 and 5.10. This implies that

$$\frac{\partial u}{\partial t} = L(u) \tag{13.6.13}$$

on $\mathbf{R}^n \times \mathbf{R}$, as in Section 5.12.

A standard approach to dealing with equations with higher-order derivatives in t is to reduce to the case of systems of equations with only first-order derivatives in t. A basic version of this was discussed in Section 5.13. That is much easier to do here, since we are already working with systems of equations.

13.7 Some more products with $\exp(b \cdot x)$

Let n and l be positive integers, also let N be a nonnegative integer. Also let $p(w)$ be a polynomial in the n variables $w_1, \ldots, w_n$ with coefficients in $\mathcal{L}(\mathbf{R}^l)$ or $\mathcal{L}(\mathbf{C}^l)$ of degree less than or equal to N, as in Section 13.5. This leads to a differential operator $p(\partial)$, as before.

Let $b \in \mathbf{R}^n$ or $\mathbf{C}^n$ be given, as appropriate. Observe that

$$p_b(w) = p(w+b) \tag{13.7.1}$$

can be expressed as a polynomial in $w_1, \ldots, w_n$ with coefficients in $\mathcal{L}(\mathbf{R}^l)$ or $\mathcal{L}(\mathbf{C}^l)$, as appropriate, of degree less than or equal to N, as in Section 2.5.

Let $p_b(\partial)$ be the differential operator associated to $p_b(w)$, and let f be an N-times continuously-differentiable function on $\mathbf{R}^n$ with values in $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate. Under these conditions,

$$p(\partial)((\exp(b \cdot x))\, f(x)) = (\exp(b \cdot x))\, (p_b(\partial)(f))(x), \tag{13.7.2}$$

as in Section 5.14.

13.7.1 Exponentials and vector-valued polynomials

If $b \in \mathbf{R}^n$, then let

$$(\exp(b \cdot x))\, \mathcal{P}(\mathbf{R}^n, \mathbf{R}^l) \tag{13.7.3}$$

be the space of functions on $\mathbf{R}^n$ with values in $\mathbf{R}^l$ of the form

$$(\exp(b \cdot x))\, q(x), \tag{13.7.4}$$

where $q \in \mathcal{P}(\mathbf{R}^n, \mathbf{R}^l)$. This is a linear subspace of $C^\infty(\mathbf{R}^n, \mathbf{R}^l)$, as a vector space over the real numbers. Similarly, if k is a nonnegative integer, then let

$$(\exp(b \cdot x))\, \mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l) \tag{13.7.5}$$

be the space of functions on $\mathbf{R}^n$ with values in $\mathbf{R}^l$ of the form (13.7.4), with $q \in \mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$. This is a linear subspace of (13.7.3), as a vector space over $\mathbf{R}$. If $p(w)$ has coefficients in $\mathcal{L}(\mathbf{R}^l)$, then $p(\partial)$ maps (13.7.3) and (13.7.5) into themselves, because of (13.7.2), as in Section 5.14.

If $b \in \mathbf{C}^n$, then let

$$(\exp(b \cdot x))\, \mathcal{P}(\mathbf{R}^n, \mathbf{C}^l) \tag{13.7.6}$$

be the space of functions on $\mathbf{R}^n$ with values in $\mathbf{C}^l$ of the form (13.7.4), with $q \in \mathcal{P}(\mathbf{R}^n, \mathbf{C}^l)$. Similarly, if k is a nonnegative integer, then let

$$(\exp(b \cdot x))\, \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l) \tag{13.7.7}$$

be the space of functions on $\mathbf{R}^n$ with values in $\mathbf{C}^l$ of the form (13.7.4), with $q \in \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$. These are linear subspaces of $C^\infty(\mathbf{R}^n, \mathbf{C}^l)$, as a vector space over the complex numbers. If $p(w)$ has coefficients in $\mathcal{L}(\mathbf{C}^l)$, then $p(\partial)$ maps (13.7.6) and (13.7.7) into themselves, because of (13.7.2), as before.

13.8 Some remarks about nilpotency

Let n, l, N, and $p(w)$ be as at the beginning of the previous section. Suppose for the moment that $p(0)$ is nilpotent, so that

$$p(0)^{r+1} = 0 \tag{13.8.1}$$

on $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate, for some nonnegative integer r. If k is a nonnegative integer, then it follows that

$$p(\partial)^{r+1} \text{ is nilpotent on } \mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l) \text{ or } \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l), \tag{13.8.2}$$

as appropriate, as in Subsection 13.6.1. Of course, this means that

$$p(\partial) \text{ is nilpotent on } \mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l) \text{ or } \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l), \tag{13.8.3}$$

as appropriate.

Put

$$L = p(\partial), \tag{13.8.4}$$

and let L_k be the restriction of L to $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$, as appropriate, for each nonnegative integer k. If $t \in \mathbf{R}$, then we can define

$$\exp(t\, L_k) \tag{13.8.5}$$

as a linear mapping on $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$, as appropriate, as in Sections 5.4 and 5.8. If $p(0)$ is nilpotent, so that L_k is nilpotent, then

$$\exp(t\, L_k) \text{ is a polynomial in } t \tag{13.8.6}$$

whose coefficients are linear mappings on $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$, as appropriate, as in Subsection 5.10.1.

Let q be an element of $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$, as appropriate. If $p(0)$ is nilpotent, then

$$(\exp(t\, L_k))(q) \tag{13.8.7}$$

may be considered as a polynomial in t with coefficients in $\mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l)$ or $\mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l)$, as appropriate. In particular,

$$((\exp(t\, L_k))(q))(x) \tag{13.8.8}$$

is a polynomial in x and t with coefficients in $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate.

13.8.1 Nilpotency of $p(b)$

Let $b \in \mathbf{R}^n$ or $\mathbf{C}^n$ be given, as appropriate, and let $p_b(w)$ be as in (13.7.1). Suppose now that

$$p_b(0) = p(b) \text{ is nilpotent} \tag{13.8.9}$$

on $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate. If k is a nonnegative integer, then

$$\begin{gathered}\text{the restriction of } p_b(\partial) \text{ to } \mathcal{P}^k(\mathbf{R}^n, \mathbf{R}^l) \text{ or } \mathcal{P}^k(\mathbf{R}^n, \mathbf{C}^l), \\ \text{as appropriate, is nilpotent,}\end{gathered} \tag{13.8.10}$$

by the remarks at the beginning of the section. This implies that

$$\begin{gathered}\text{the restriction of } p(\partial) \text{ to (13.7.5) or (13.7.7),} \\ \text{as appropriate, is nilpotent,}\end{gathered} \tag{13.8.11}$$

because of (13.7.2).

13.9 The characteristic polynomial

Let l be a positive integer, and let A be a linear mapping from $\mathbf{R}^l$ or $\mathbf{C}^l$ into itself. If t is a real or complex number, then $A - t\, I$ is another linear mapping from $\mathbf{R}^l$ or $\mathbf{C}^l$ into itself, as appropriate, where I is the identity mapping on $\mathbf{R}^l$ or $\mathbf{C}^l$. Thus

$$\mathrm{ch}_A(t) = \det(A - t\, I) \tag{13.9.1}$$

defines a real or complex-valued function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate. More precisely, $\mathrm{ch}_A(t)$ is a polynomial of degree l in t, with real or complex coefficients, as appropriate. This is known as the *characteristic polynomial* of A.

The characteristic polynomial may be expressed as

$$\mathrm{ch}_A(t) = \sum_{j=0}^{l} c_j\, t^j, \tag{13.9.2}$$

with $c_j \in \mathbf{R}$ or $\mathbf{C}$ for each j, as appropriate. Remember that the determinant of an $l \times l$ matrix is given by a homogeneous polynomial of degree l in the l^2 entries of the matrix. This means that c_j is given by a homogeneous polynomial of degree $l - j$ in the l^2 entries of the $l \times l$ matrix corresponding to A. In particular,

$$c_l = (-1)^l, \tag{13.9.3}$$

and $c_0 = \det A$. The zeros of $\mathrm{ch}_A(t)$ in $\mathbf{R}$ or $\mathbf{C}$ are the same as the eigenvalues of A as a linear mapping on $\mathbf{R}^l$ or $\mathbf{C}^l$, as appropriate, by standard arguments.

A polynomial of degree l in t with complex coefficients is equal to the product of the coefficient of t^l and l linear factors, corresponding to the l zeros of the polynomial in $\mathbf{C}$, with their appropriate multiplicities, by the fundamental theorem of algebra. It follows that $\mathrm{ch}_A(t)$ is uniquely determined by the eigenvalues of A in the complex case, because of (13.9.3).

If A is nilpotent, then it is easy to see that 0 is the only eigenvalue of A. This implies that

$$\mathrm{ch}_A(t) = (-1)^l \, t^l \tag{13.9.4}$$

in the complex case, by the remarks in the preceding paragraph. Equivalently, this means that

$$c_j = 0, \ 0 \le j \le l - 1. \tag{13.9.5}$$

One can get the same conclusion in the real case using the unique extension of a linear mapping from $\mathbf{R}^l$ into itself to a linear mapping from $\mathbf{C}^l$ into itself, as a vector space over the complex numbers. Another proof of this will be mentioned in the next section.

13.9.1 The Cayley–Hamilton theorem

If A is any linear mapping from $\mathbf{R}^l$ or $\mathbf{C}^l$ into itself, then the Cayley–Hamilton theorem states that

$$\mathrm{ch}_A(A) = \sum_{j=0}^{l} c_j \, A^j = 0, \tag{13.9.6}$$

where A^0 is interpreted as being equal to I. If (13.9.4) holds, then it follows that

$$A^l = 0. \tag{13.9.7}$$

13.10 More on nilpotent linear mappings

Let l be a positive integer again, and let A be a linear mapping from $\mathbf{R}^l$ or $\mathbf{C}^l$ into itself. If τ is a real or complex number, then $I - \tau\, A$ defines a linear mapping from $\mathbf{R}^l$ or $\mathbf{C}^l$ into itself, as appropriate. Let r be a nonnegative integer, and observe that

$$(I - \tau\, A) \sum_{j=0}^{r} \tau^j \, A^j = \sum_{j=0}^{r} \tau^j \, A^j - \sum_{j=1}^{r+1} \tau^j \, A^j = I - \tau^{r+1} \, A^{r+1}. \tag{13.10.1}$$

Similarly,

$$\Big(\sum_{j=0}^{r} \tau^j\,A^j\Big)\,(I - \tau\,A) = I - \tau^{r+1}\,A^{r+1}. \tag{13.10.2}$$

If

$$A^{r+1} = 0, \tag{13.10.3}$$

then it follows that $I - \tau\,A$ is invertible, with

$$(I - \tau\,A)^{-1} = \sum_{j=0}^{r} \tau^j\,A^j. \tag{13.10.4}$$

Note that

$$\det(I - \tau\,A) \tag{13.10.5}$$

and

$$\det\Big(\sum_{j=0}^{r} \tau^j\,A^j\Big) \tag{13.10.6}$$

are polynomials in τ with real or complex coefficients, as appropriate. Both of these polynomials are equal to 1 at $\tau = 0$. If (13.10.3) holds, then

$$\det(I - \tau\,A)\,\det\Big(\sum_{j=0}^{r} \tau^j\,A^j\Big) = 1 \tag{13.10.7}$$

for all τ in $\mathbf{R}$ or $\mathbf{C}$, as appropriate. One can use this to get that

$$\det(I - \tau\,A) = 1 \tag{13.10.8}$$

and

$$\det\Big(\sum_{j=0}^{r} \tau^j\,A^j\Big) = 1 \tag{13.10.9}$$

for each τ.

If t is a nonzero real or complex number, as appropriate, then it is easy to see that (13.9.4) is equivalent to (13.10.8), with $\tau = 1/t$. Of course, if (13.9.4) holds for all $t \neq 0$, then it holds when $t = 0$ too. One could also obtain (13.9.4) with $t = 0$ more directly from (13.10.3). This is another way to obtain (13.9.4) from (13.10.3), as mentioned in the previous section.

13.10.1 Polynomials of linear mappings

If

$$f(t) = \sum_{j=0}^{m} b_j\,t^j \tag{13.10.10}$$

is any polynomial with real or complex coefficients, then

$$f(A) = \sum_{j=0}^{m} b_j \, A^j \tag{13.10.11}$$

defines a linear mapping from $\mathbf{R}^l$ or $\mathbf{C}^l$ into itself, as appropriate. If A is any linear mapping from $\mathbf{R}^l$ or $\mathbf{C}^l$ into itself, then it is well known that one can find a nonzero polynomial $f(t)$ of degree at most l^2 such that

$$f(A) = 0. \tag{13.10.12}$$

Of course, this follows from the Cayley–Hamilton theorem, and it can also be obtained more directly from the fact that $\mathcal{L}(\mathbf{R}^l)$, $\mathcal{L}(\mathbf{C}^l)$ have dimension l^2, as vector spaces over the real and complex numbers, respectively.

If A is a linear mapping from $\mathbf{C}^l$ into itself, and $A - t\,I$ is invertible for every $t \in \mathbf{C}$ with $t \neq 0$, then (13.10.12) implies that A is nilpotent, because f can be expressed as the product of a nonzero constant and finitely many linear factors. This invertibility condition holds when (13.9.2) holds for every $t \in \mathbf{C}$, or equivalently (13.10.8) holds for every $\tau \in \mathbf{C}$. If A is a linear mapping from $\mathbf{R}^l$ into itself, then A has a unique extension to a linear mapping from $\mathbf{C}^l$ into itself, as a vector space over the complex numbers, that we may denote by A as well. If (13.9.4) holds for every $t \in \mathbf{R}$, or equivalently (13.10.8) holds for every $\tau \in \mathbf{R}$, then these conditions hold for all $t, \tau \in \mathbf{C}$, because the left sides of these equations are polynomials in t, τ, respectively. The argument in the complex case implies that A is nilpotent on $\mathbf{C}^l$, and thus on $\mathbf{R}^l$.

Chapter 14

Power series in several variables

14.1 Sums over multi-indices

Let n be a positive integer, and consider the set

$$(\mathbf{Z}_+ \cup \{0\})^n \tag{14.1.1}$$

of all n tuples of elements of the set $\mathbf{Z}_+ \cup \{0\}$ of nonnegative integers. Equivalently, this is the set of all multi-indices. If f is a real or complex-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$, then we may be interested in a sum of the form

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha). \tag{14.1.2}$$

Of course, this can be reduced to a finite sum when $f(\alpha) = 0$ for all but finitely many multi-indices α. If $n = 1$, then this may be interpreted as an infinite series.

One can try to define (14.1.2) for any n by reducing to an infinite series. One way to do this is to use the fact that $(\mathbf{Z}_+ \cup \{0\})^n$ is countably infinite, so that one can find a sequence

$$\{\alpha(l)\}_{l=0}^{\infty} \tag{14.1.3}$$

of multi-indices in which every multi-index occurs exactly once. Thus one may try to interpret the sum (14.1.2) as being equal to the infinite series

$$\sum_{l=0}^{\infty} f(\alpha(l)). \tag{14.1.4}$$

14.1.1 Some limits of finite sums

Alternatively, let $E_0, E_1, E_2, E_3, \ldots$ be an infinite sequence of nonempty finite subsets of $(\mathbf{Z}_+ \cup \{0\})^n$ such that

$$E_N \subseteq E_{N+1} \tag{14.1.5}$$

for every nonnegative integer N, and

$$\bigcup_{N=0}^{\infty} E_N = (\mathbf{Z}_+ \cup \{0\})^n. \tag{14.1.6}$$

One may wish to interpret the sum (14.1.2) as being equal to

$$\lim_{N \to \infty} \sum_{\alpha \in E_N} f(\alpha), \tag{14.1.7}$$

if the limit exists.

Let $\{\alpha(l)\}_{l=0}^{\infty}$ be an enumeration of $(\mathbf{Z}_+ \cup \{0\})^n$, as before. If we put

$$E_N = \{\alpha(0), \alpha(1), \ldots, \alpha(N)\} \tag{14.1.8}$$

for each nonnegative integer N, then we get a sequence of nonempty finite subsets of $(\mathbf{Z}_+ \cup \{0\})^n$ that satisfies (14.1.5) and (14.1.6). In this case,

$$\sum_{\alpha \in E_N} f(\alpha) = \sum_{l=0}^{N} f(\alpha(l)) \tag{14.1.9}$$

for each $N \geq 0$, so that (14.1.4) is the same as (14.1.7).

As another basic example, one can take E_N to be

$$\{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n : |\alpha| \leq N\} \tag{14.1.10}$$

for each $N \geq 0$, where $|\alpha|$ is the order of α, as usual. Another possibility is to take E_N to be

$$\{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n : \alpha_j \leq N \text{ for each } j = 1, \ldots, N\} \tag{14.1.11}$$

for every $N \geq 0$. These are the same when $n = 1$, in which case (14.1.7) is the same as the usual interpretation of (14.1.2) as an infinite series.

14.1.2 Sums with nonnegative terms

Let f be a nonnegative real-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$. If A is a nonempty finite subset of $(\mathbf{Z}_+ \cup \{0\})^n$, then

$$\sum_{\alpha \in A} f(\alpha) \tag{14.1.12}$$

is a nonnegative real number. Let us say that f is *summable* on $(\mathbf{Z}_+ \cup \{0\})^n$ if the collection of these finite sums has an upper bound in $\mathbf{R}$. Under these conditions, the sum (14.1.2) may be defined as the supremum or least upper bound of the set of these finite sums. Otherwise, it is sometimes convenient to interpret (14.1.2) as being equal to $+\infty$.

If $n = 1$, then the summability of f is equivalent to the convergence of the corresponding infinite series of nonnegative real numbers, with the same value of the sum. If $\{\alpha(l)\}_{l=0}^\infty$ is any enumeration of $(\mathbf{Z}_+ \cup \{0\})^n$, then the summability of f is equivalent to the convergence of (14.1.4), with the same value of the sum.

Let $E_0, E_1, E_2, E_3, \ldots$ be an infinite sequence of nonempty finite subsets of $(\mathbf{Z}_+ \cup \{0\})^n$ that satisfies (14.1.5) and (14.1.6) again. It is easy to see that f is summable on $(\mathbf{Z}_+ \cup \{0\})^n$ if and only if the sums

$$\sum_{\alpha \in E_N} f(\alpha) \tag{14.1.13}$$

are bounded. In this case, the supremum of these sums is the same as the supremum of the set of sums of the form (14.1.12). We also get that the limit in (14.1.7) exists and is equal to this supremum, because the sums (14.1.13) are monotonically increasing in N.

14.1.3 A nice family of functions

Suppose that f can be expressed as

$$f(\alpha) = \prod_{j=1}^n f_j(\alpha_j), \tag{14.1.14}$$

where f_j is a nonnegative real-valued function on the set $\mathbf{Z}_+ \cup \{0\}$ of nonnegative integers for each $j = 1, \ldots, n$. If E_N is as in (14.1.11), then

$$\sum_{\alpha \in E_N} f(\alpha) = \prod_{j=1}^n \Big(\sum_{\alpha_j = 0}^N f_j(\alpha_j) \Big) \tag{14.1.15}$$

for each $N \geq 0$. If f_j is summable on $\mathbf{Z}_+ \cup \{0\}$ for each $j = 1, \ldots, n$, then it follows that f is summable on $(\mathbf{Z}_+ \cup \{0\})^n$, with

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha) = \prod_{j=1}^n \Big(\sum_{\alpha_j = 0}^\infty f_j(\alpha_j) \Big). \tag{14.1.16}$$

Conversely, if f is summable on $(\mathbf{Z}_+ \cup \{0\})^n$, and if none of the f_j's is identically zero on $\mathbf{Z}_+ \cup \{0\}$, then f_j is summable on $\mathbf{Z}_+ \cup \{0\}$ for each j.

14.1.4 Some examples

As a basic family of examples, let r be an element of the set

$$(\mathbf{R}_+ \cup \{0\})^n \tag{14.1.17}$$

of n-tuples of nonnegative real numbers, and put

$$f(\alpha) = r^\alpha \tag{14.1.18}$$

for each multi-index α. This is of the form (14.1.14), with

$$f_j(\alpha_j) = r^{\alpha_j} \tag{14.1.19}$$

for each $j = 1, \ldots, n$. It follows that f is summable on $(\mathbf{Z}_+ \cup \{0\})^n$ if and only if $r_j < 1$ for each $j = 1, \ldots, n$, in which case

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} r^\alpha = \prod_{j=1}^n (1 - r_j)^{-1}. \tag{14.1.20}$$

14.2 Real and complex-valued functions

Let n be a positive integer, and let f be a real or complex-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$. Let us say that f is *summable* on $(\mathbf{Z}_+ \cup \{0\})^n$ if $|f(\alpha)|$ is summable as a nonnegative real-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$. If f is real valued, then this is equivalent to the summability of $f_+(\alpha) = \max(f(\alpha), 0)$ and $f_-(\alpha) = \max(-f(\alpha), 0)$ on $(\mathbf{Z}_+ \cup \{0\})^n$. If f is complex valued, then summability of f is equivalent to the summability of the real and imaginary parts of f.

If f is a summable real or complex-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$, then the sum (14.1.2) may be defined as a real or complex number, as appropriate, by reducing to the case of nonnegative real-valued summable functions. More precisely, if f is real valued, then the sum may be defined as the difference of the analogous sums for f_+ and f_-. If f is complex valued, then the real and imaginary parts of the sum may be defined as the corresponding sums of the real and imaginary parts of f. In both cases, the sum (14.1.2) may be described equivalently as in (14.1.4) or (14.1.7), because of the analogous statements for nonnegative real-valued summable functions.

14.2.1 A basic inequality

We also have that

$$\left| \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha) \right| \le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} |f(\alpha)| \tag{14.2.1}$$

in both cases. If f is real valued, then this follows directly from the definition of the sum (14.1.2) mentioned in the preceding paragraph. If f is complex valued, and one tries to consider the real and imaginary parts of the sum directly, then one gets an extra factor of 2 on the right side, or $\sqrt{2}$ with a bit more effort. Of course, if $f(\alpha) = 0$ for all but finitely many $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, then (14.2.1) follows from the triangle inequality for the absolute value of a complex number.

One can use this to get (14.2.1), by expressing the sum (14.1.2) as in (14.1.4) or (14.1.7).

One can also reduce the complex case to the real case, by considering $\mathrm{Re}(a\, f)$ with $a \in \mathbf{C}$ and $|a| = 1$. More precisely, one can choose a so that

$$\begin{aligned} (14.2.2) \qquad \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} \mathrm{Re}(a\, f(\alpha)) &= \mathrm{Re}\Big(a \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha)\Big) \\ &= \Big|\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha)\Big|. \end{aligned}$$

14.2.2 More on summable functions

It is easy to see that the spaces of real and complex-valued summable functions on $(\mathbf{Z}_+ \cup \{0\})^n$ are linear subspaces of the spaces of all real and complex-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$, as vector spaces over $\mathbf{R}$ and $\mathbf{C}$, respectively. The linearity of the sum (14.1.2) in f can be obtained from the descriptions of the sum as in (14.1.4) or (14.1.7).

If $n = 1$, then the summability of f is equivalent to the absolute convergence of the corresponding infinite series. Similarly, if $\{\alpha(l)\}_{l=0}^\infty$ is any enumeration of $(\mathbf{Z}_+ \cup \{0\})^n$, then the summablility of f is equivalent to the absolute convergence of (14.1.4).

14.2.3 More on examples

Suppose that f is as in (14.1.14), where f_j is a real or complex-valued summable function on $\mathbf{Z}_+ \cup \{0\}$ for each $j = 1, \ldots, n$. This implies that f is summable on $(\mathbf{Z}_+ \cup \{0\})^n$, as in the previous section. One can check that (14.1.16) holds under these conditions, using the same type of argument as before, or by reducing to the previous case.

Let $z \in \mathbf{C}^n$ be given, and put

$$(14.2.3) \qquad f(\alpha) = z^\alpha$$

for each multi-index α. If $|z_j| < 1$ for each $j = 1, \ldots, n$, then f is summable on $(\mathbf{Z}_+ \cup \{0\})^n$, as in the previous section. In this case,

$$(14.2.4) \qquad \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} z^\alpha = \prod_{j=1}^n (1 - z_j)^{-1},$$

by (14.1.16).

14.3 Cauchy products

Let n be a positive integer, and let f, g be real or complex-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$. If γ is a multi-index, then put

$$h(\gamma) = \sum_{\alpha+\beta=\gamma} f(\alpha)\, g(\beta). \tag{14.3.1}$$

More precisely, the sum on the right is taken over all multi-indices α, β such that $\alpha + \beta = \gamma$. Note that there are only finitely many such multi-indices α, β.

Suppose for the moment that $f(\alpha) = 0$ for all but finitely many multi-indices α, and that $g(\beta) = 0$ for all but finitely many multi-indices β. This implies that $h(\gamma) = 0$ for all but finitely many multi-indices γ. Under these conditions, one can verify that

$$\sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} h(\gamma) = \Big(\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha) \Big) \Big(\sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} g(\beta) \Big). \tag{14.3.2}$$

In fact, both sides of the equation are the same as the sum of $f(\alpha)\, g(\beta)$ over all multi-indices α, β. The sum on the left may be desacribed as the *Cauchy product* of the two sums on the right.

14.3.1 Nonnegative real-valued functions

Suppose now that f, g are nonnegative real-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$, so that h is nonnegative as well. If N is a nonnegative integer, then let E_N be the set of multi-indices α with order $|\alpha| \le N$, as in (14.1.10). Observe that

$$\sum_{\gamma \in E_N} h(\gamma) \le \Big(\sum_{\alpha \in E_N} f(\alpha) \Big) \Big(\sum_{\beta \in E_N} g(\beta) \Big) \tag{14.3.3}$$

and

$$\Big(\sum_{\alpha \in E_N} f(\alpha) \Big) \Big(\sum_{\beta \in E_N} g(\beta) \Big) \le \sum_{\gamma \in E_{2N}} h(\gamma) \tag{14.3.4}$$

for each $N \ge 0$. If f and g are summable on $(\mathbf{Z}_+ \cup \{0\})^n$, then it follows that h is summable too, and that (14.3.2) holds.

14.3.2 Arbitrary functions

If f and g are any real or complex-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$, then

$$|h(\gamma)| \le \sum_{\alpha+\beta=\gamma} |f(\alpha)|\, |g(\beta)| \tag{14.3.5}$$

for every multi-index γ. Suppose that f and g are summable on $(\mathbf{Z}_+ \cup \{0\})^n$, which implies that the right side of (14.3.5) is summable as a function of γ, as in the preceding paragraph. It follows that h is summable as well, with

$$\sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} |h(\gamma)| \le \Big(\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} |f(\alpha)| \Big) \Big(\sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} |g(\beta)| \Big). \tag{14.3.6}$$

One can check that (14.3.2) holds too, by reducing to the case of nonnegative real-valued summable functions.

14.3.3 A family of examples

Let $z \in \mathbf{C}^n$ be given, and suppose that

$$f(\alpha) = a_\alpha \, z^\alpha, \;\; g(\beta) = b_\beta \, z^\beta \tag{14.3.7}$$

for all multi-indices α, β, where a_α, b_β are complex numbers. If we put

$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha \, b_\beta \tag{14.3.8}$$

for each multi-index γ, then we get that

$$h(\gamma) = c_\gamma \, z^\gamma. \tag{14.3.9}$$

14.4 Power series on closed polydisks

Let n be a positive integer, and let $z_0 = (z_{0,1}, \ldots, z_{0,n}) \in \mathbf{C}^n$ be given. Also let a_α be a complex number for each multi-index α, and consider the *power series*

$$f(z) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} a_\alpha \, (z - z_0)^\alpha \tag{14.4.1}$$

in $z_1, \ldots, z_n$, centered at z_0. More precisely, the sum on the right is defined as a complex number for each $z \in \mathbf{C}^n$ such that

$$a_\alpha \, (z - z_0)^\alpha \tag{14.4.2}$$

is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$.

14.4.1 A basic criterion for summability

Let $r \in (\mathbf{R}_+ \cup \{0\})^n$ be given, and suppose for the moment that

$$|a_\alpha| \, r^\alpha \tag{14.4.3}$$

is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$. This implies that (14.4.2) is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$ when

$$|z_j - z_{0,j}| \le r_j, \;\; 1 \le j \le n. \tag{14.4.4}$$

This means that (14.4.1) defines a complex-valued function on the *closed polydisk*

$$\{z \in \mathbf{C}^n : |z_j - z_{0,j}| \le r_j, \, 1 \le j \le n\}, \tag{14.4.5}$$

which is a closed set in $\mathbf{C}^n$, with respect to the standard Euclidean metric.

14.4.2 Approximation by finite sums

Let $\epsilon > 0$ be given, and let $A(\epsilon)$ be a nonempty finite subset of $(\mathbf{Z}_+ \cup \{0\})^n$ such that

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} |a_\alpha| \, r^\alpha < \Big(\sum_{\alpha \in A(\epsilon)} |a_\alpha| \, r^\alpha \Big) + \epsilon. \tag{14.4.6}$$

The existence of such a set follows from the definition of the sum on the left, as the supremum of the corresponding sums over nonempty finite subsets of $(\mathbf{Z}_+ \cup \{0\})^n$, as in Subsection 14.1.2. Using this, we get that

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A(\epsilon)} |a_\alpha| \, r^\alpha < \epsilon, \tag{14.4.7}$$

because of the linearity of the sum.

Let A be a nonempty finite subset of $(\mathbf{Z}_+ \cup \{0\})^n$ such that

$$A(\epsilon) \subseteq A. \tag{14.4.8}$$

If $z \in \mathbf{C}^n$ satisfies (14.4.4), then

$$\begin{aligned} \Big| f(z) - \sum_{\alpha \in A} a_\alpha \, (z - z_0)^\alpha \Big| &= \Big| \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} a_\alpha \, (z - z_0)^\alpha \Big| \\ &\le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} |a_\alpha| \, r^\alpha \\ &\le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A(\epsilon)} |a_\alpha| \, r^\alpha < \epsilon. \end{aligned} \tag{14.4.9}$$

14.4.3 A uniform convergence property

Let $E_0, E_1, E_2, E_3, \ldots$ be an infinite sequence of nonempty finite subsets of $(\mathbf{Z}_+ \cup \{0\})^n$ that satisfy (14.1.5) and (14.1.6). If $z \in \mathbf{C}^n$ satisfies (14.4.4), then

$$\lim_{N \to \infty} \sum_{\alpha \in E_N} a_\alpha \, (z - z_0)^\alpha = f(z), \tag{14.4.10}$$

as in Sections 14.1 and 14.2. In fact, the convergence is uniform over (14.4.5), as in (14.4.9). This corresponds to a classical criterion for uniform convergence of Weierstrass. It follows that f is continuous on (14.4.5), because polynomials are continuous on $\mathbf{C}^n$.

14.5 Power series on open polydisks

Let n be a positive integer, let $z_0 \in \mathbf{C}^n$ be given, and let a_α be a complex number for each multi-index α. Also let t be an element of the set $(\mathbf{R}_+ \cup \{+\infty\})^n$ of positive extended real numbers. Suppose that if $r \in (\mathbf{R}_+ \cup \{0\})^n$ satisfies

$$r_j < t_j, \ 1 \le j \le n, \tag{14.5.1}$$

then (14.4.3) is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$. If $z \in \mathbf{C}^n$ satisfies

$$|z_j - z_{0,j}| < t_j, \ 1 \le j \le n, \tag{14.5.2}$$

then it follows that (14.4.2) is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$.

This implies that (14.4.1) defines a complex-valued function on

$$\{z \in \mathbf{C}^n : |z_j - z_{0,j}| < t_j, \, 1 \le j \le n\}, \tag{14.5.3}$$

which is an open set in $\mathbf{C}^n$, with respect to the standard Euclidean metric. This set may be described as an *open polydisk* in $\mathbf{C}^n$, at least when $t_1, \ldots, t_n$ are finite. One can check that f is continuous on (14.5.3), because its restriction to any closed polydisk (14.4.5) is continuous when (14.5.1) holds, as in the previous section.

14.5.1 Another criterion for summability

Suppose for the moment that $t_1, \ldots, t_n$ are finite, and that

$$|a_\alpha| \, t^\alpha \tag{14.5.4}$$

is bounded as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$. If $r \in (\mathbf{R}_+ \cup \{0\})^n$ satisfies (14.5.1), then

$$r^\alpha \, t^{-\alpha} \tag{14.5.5}$$

is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$, as in Subsection 14.1.4. This implies that (14.4.3) is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$.

14.5.2 Some related summability properties

Let β be a multi-index. If α is another multi-index, then α^β can be defined as a nonnegative integer in the usual way. If $r \in (\mathbf{R}_+ \cup \{0\})^n$ satisfies (14.5.1), then

$$\alpha^\beta \, |a_\alpha| \, r^\alpha \tag{14.5.6}$$

is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$. To see this, one can use an n-tuple $r_0 = (r_{0,1}, \ldots, r_{0,n})$ of positive real numbers such that

$$r_j < r_{0,j} < t_j, \ 1 \le j \le n. \tag{14.5.7}$$

Under these conditions,

$$|a_\alpha| \, r_0^\alpha \tag{14.5.8}$$

is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$, by hypothesis, and

$$\alpha^\beta \, r^\alpha \, r_0^{-\alpha} \tag{14.5.9}$$

is bounded as a function of α on $(\mathbf{Z}_+ \cup \{0\})^n$, by well-known results.

14.5.3 Differentiating power series

If one differentiates the right side of (14.4.1) term-by-term, then one gets a power series of the same type, with suitable coefficients. The remarks in the preceding paragraph imply that this power series has the same summability properties as those considered for $f(z)$ in this section. It is well known that $f(z)$ is smooth on (14.5.3), with derivatives given by differentiating the power series termwise.

More precisely, $f(z)$ is holomorphic on (14.5.3), because polynomials in $z_1, \ldots, z_n$ are holomorphic on $\mathbf{C}^n$. If β is any multi-index, then

$$\frac{\partial^{|\beta|} f}{\partial z^\beta}(z_0) = \beta!\, a_\beta. \tag{14.5.10}$$

Conversely, it is well known that any holomorphic function on (14.5.3) can be expressed as a power series with these summability properties.

14.6 Double sums

Let m and n be positive integers, and let us refer to multi-indices associated to n as *n-multi-indices*, so that we may also consider m-multi-indices and $(m+n)$-multi-indices. Let us identify the set $(\mathbf{Z}_+ \cup \{0\})^{m+n}$ of all $(m+n)$-multi-indices with the set

$$(\mathbf{Z}_+ \cup \{0\})^m \times (\mathbf{Z}_+ \cup \{0\})^n \tag{14.6.1}$$

of ordered pairs (α, β), where α is an m-multi-index, and β is an n-multi-index.

14.6.1 Summable double sum

Let $f(\alpha, \beta)$ be a nonnegative real-valued function on $(\mathbf{Z}_+ \cup \{0\})^{m+n}$, identified with (14.6.1). If

$$f(\alpha, \beta) \text{ is summable on } (\mathbf{Z}_+ \cup \{0\})^{m+n}, \tag{14.6.2}$$

then it is easy to see that for each $\alpha \in (\mathbf{Z}_+ \cup \{0\})^m$,

$$f(\alpha, \beta) \text{ is summable as a function of } \beta \text{ on } (\mathbf{Z}_+ \cup \{0\})^n. \tag{14.6.3}$$

If A is a nonempty finite subset of $(\mathbf{Z}_+ \cup \{0\})^m$, then one can check that

$$\sum_{\alpha \in A} \Big(\sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha, \beta) \Big) \le \sum_{(\alpha,\beta) \in (\mathbf{Z}_+ \cup \{0\})^{m+n}} f(\alpha, \beta). \tag{14.6.4}$$

This implies that

$$\sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha, \beta) \tag{14.6.5}$$

is summable as a nonnegative real-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^m$, with

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^m} \Big(\sum_{\beta \in \mathbf{Z}_+ \cup \{0\})^n} f(\alpha, \beta) \Big) \le \sum_{(\alpha,\beta) \in (\mathbf{Z}_+ \cup \{0\})^{m+n}} f(\alpha, \beta). \tag{14.6.6}$$

14.6.2 Summable iterated sum

Conversely, suppose that (14.6.3) holds for each $\alpha \in (\mathbf{Z}_+ \cup \{0\})^m$, and that

(14.6.7) (14.6.5) is summable as a nonnegative real-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^m$.

One can check that (14.6.2) holds under these conditions, with

$$\sum_{(\alpha,\beta)\in(\mathbf{Z}_+\cup\{0\})^{m+n}} f(\alpha,\beta) \le \sum_{\alpha\in(\mathbf{Z}_+\cup\{0\})^m} \Big(\sum_{\beta\in(\mathbf{Z}_+\cup\{0\})^n} f(\alpha,\beta)\Big). \tag{14.6.8}$$

This means that

$$\sum_{(\alpha,\beta)\in(\mathbf{Z}_+\cup\{0\})^{m+n}} f(\alpha,\beta) = \sum_{\alpha\in(\mathbf{Z}_+\cup\{0\})^n} \Big(\sum_{\beta\in(\mathbf{Z}_+\cup\{0\})^n} f(\alpha,\beta)\Big) \tag{14.6.9}$$

in both cases. Of course, there are analogous statements for summing over α first.

14.6.3 More on double sums

Suppose now that $f(\alpha, \beta)$ is a summable real or complex-valued function on $(\mathbf{Z}_+\cup\{0\})^{m+n}$. This implies that for each $\alpha \in (\mathbf{Z}_+\cup\{0\})^m$, $f(\alpha, \beta)$ is summable as a function of β on $(\mathbf{Z}_+ \cup \{0\})^n$, as before. We also have that

$$\Big|\sum_{\beta\in(\mathbf{Z}_+\cup\{0\})^n} f(\alpha,\beta)\Big| \le \sum_{\beta\in(\mathbf{Z}_+\cup\{0\})^n} |f(\alpha,\beta)| \tag{14.6.10}$$

for every $\alpha \in (\mathbf{Z}_+\cup\{0\})^m$, as in Subsection 14.2.1. The right side is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^m$, as before. It follows that (14.6.5) is summable as a function of α on $(\mathbf{Z}_+ \cup \{0\})^m$. One can check that (14.6.9) holds here too, by reducing to the case of summable nonnegative real-valued functions on $(\mathbf{Z}_+ \cup \{0\})^{m+n}$. There are analogous statements for summing over α first, as before.

14.6.4 A particular case

Now let $f(\alpha)$, $g(\beta)$ be summable real or complex-valued functions of α, β on $(\mathbf{Z}_+ \cup \{0\})^m$, $(\mathbf{Z}_+ \cup \{0\})^n$, respectively. One can check that

$$\phi(\alpha,\beta) = f(\alpha)\, g(\beta) \tag{14.6.11}$$

is summable on $(\mathbf{Z}_+ \cup \{0\})^{m+n}$, by summing

$$|\phi(\alpha,\beta)| = |f(\alpha)|\,|g(\beta)| \tag{14.6.12}$$

one variable at a time. Similarly,

$$\begin{aligned} &\sum_{(\alpha,\beta)\in(\mathbf{Z}_+\cup\{0\})^{m+n}} f(\alpha)\, g(\beta) \\ &\quad = \Big(\sum_{\alpha\in\mathbf{Z}_+\cup\{0\})^m} f(\alpha)\Big)\Big(\sum_{\beta\in(\mathbf{Z}_+\cup\{0\})^n} g(\beta)\Big). \end{aligned} \tag{14.6.13}$$

14.7 Some more rearrangements

Let m and n be positive integers, and let $A(\gamma)$ be a finite set of n-multi-indices for each m-multi-index γ. Suppose that the $A(\gamma)$'s are pairwise disjoint, so that

$$A(\gamma) \cap A(\gamma') = \emptyset \tag{14.7.1}$$

for all m-multi-indices γ, γ' with $\gamma \neq \gamma'$, and that

$$\bigcup_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^m} A(\gamma) = (\mathbf{Z}_+ \cup \{0\})^n. \tag{14.7.2}$$

Let ϕ be a real or complex-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$, and put

$$\psi(\gamma) = \sum_{\alpha \in A(\gamma)} \phi(\alpha) \tag{14.7.3}$$

for every m-multi-index γ. This is interpreted as being equal to 0 when $A(\gamma) = \emptyset$. Note that

$$|\psi(\gamma)| \leq \sum_{\alpha \in A(\gamma)} |\phi(\alpha)| \tag{14.7.4}$$

for every n-multi-index γ.

14.7.1 Nonnegative ϕ

Suppose for the moment that ϕ is a nonnegative real-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$, so that ψ is a nonnegative real-valued function on $(\mathbf{Z}_+ \cup \{0\})^m$. If ϕ is summable on $(\mathbf{Z}_+ \cup \{0\})^n$ then one can check that ψ is summable on $(\mathbf{Z}_+ \cup \{0\})^m$, with

$$\sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^m} \psi(\gamma) \leq \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} \phi(\alpha). \tag{14.7.5}$$

Similarly, if ψ is summable on $(\mathbf{Z}_+ \cup \{0\})^m$, then one can verify that ϕ is summable on $(\mathbf{Z}_+ \cup \{0\})^n$, with

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} \phi(\alpha) \leq \sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^m} \psi(\gamma). \tag{14.7.6}$$

It follows that

$$\sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^m} \psi(\gamma) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} \phi(\alpha) \tag{14.7.7}$$

in both cases.

14.7.2 Arbitrary summable ϕ

If ϕ is a summable real or complex-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$, then one can use (14.7.4) and the remarks in the preceding paragraph to get that ψ is summable on $(\mathbf{Z}_+ \cup \{0\})^m$. One can also check that (14.7.7) holds under these conditions, by reducing to the case of summable nonnegative real-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$.

14.7.3 Examples related to Cauchy products

As a basic class of examples, let us take $n = 2\,m$, and identify the set $(\mathbf{Z}_+ \cup \{0\})^{2m}$ of all $(2\,m)$-multi-indices with the set

$$(\mathbf{Z}_+ \cup \{0\})^m \times (\mathbf{Z}_+ \cup \{0\})^m \tag{14.7.8}$$

of all ordered pairs (α, β) of m-multi-indices, as in the previous section. If γ is an m-multi-index, then put

$$A(\gamma) = \{(\alpha, \beta) \in (\mathbf{Z}_+ \cup \{0\})^{2m} : \alpha + \beta = \gamma\}. \tag{14.7.9}$$

These are pairwise-disjoint nonempty finite subsets of $(\mathbf{Z}_+ \cup \{0\})^{2m}$, whose union is all of $(\mathbf{Z}_+ \cup \{0\})^{2m}$.

Let f, g be real or complex-valued functions on $(\mathbf{Z}_+ \cup \{0\})^m$, and let $\phi(\alpha, \beta)$ be defined on $(\mathbf{Z}_+ \cup \{0\})^{2m}$ as in (14.6.11). In this case,

$$\psi(\gamma) = \sum_{\alpha+\beta=\gamma} f(\alpha)\, g(\beta) \tag{14.7.10}$$

is the same as $h(\gamma)$ in Section 14.3, and the earlier properties of $h(\gamma)$ could also be obtained from the remarks in this and the previous section.

Appendix A

Linear mappings, norms, and differentials

A.1 Invertible linear mappings

Let V be a vector space over the real or complex numbers. A one-to-one linear mapping T from V onto itself is said to be *invertible*, and the corresponding inverse mapping is denoted T^{-1}, as usual. In this case, T^{-1} is invertible on V as well, with

$$(T^{-1})^{-1} = T. \tag{A.1.1}$$

If T_1 and T_2 are invertible linear mappings on V, then it is easy to see that their composition $T_2 \circ T_1$ is invertible on V, with

$$(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}. \tag{A.1.2}$$

The set of invertible linear mappings on V is denoted $GL(V)$. This is called the *general linear group* associated to V. This is a group with respect to composition of mappings on V.

A.1.1 Invertibility and determinants

Suppose now that V has positive finite dimension n, as a vector space over the real or complex numbers, as appropriate. If one chooses a basis for V, then every linear mapping T from V into itself corresponds to an $n \times n$ matrix of real or complex numbers, as appropriate, in a standard way. The *determinant* $\det T$ of T is defined as the determinant of the corresponding matrix. It is well known that this does not depend on the particular choice of basis for V. The identity mapping $I = I_V$ on V corresponds to the usual identity matrix in this way, which has determinant equal to 1, so that

$$\det I = 1. \tag{A.1.3}$$

If T_1, T_2 are linear mappings from V into itself, then

$$\det(T_2 \circ T_1) = (\det T_2)\,(\det T_1), \tag{A.1.4}$$

because of the corresponding property of determinants of products of $n \times n$ matrices. A linear mapping T from V into itself is invertible if and only if

$$\det T \neq 0 \tag{A.1.5}$$

because of the analogous characterization of invertibility of $n \times n$ matrices. In this case, we have that

$$\det(T^{-1}) = (\det T)^{-1}, \tag{A.1.6}$$

because of (A.1.4).

The *special linear group* of V is defined to be the collection $SL(V)$ of linear mappings T from V into itself such that

$$\det T = 1. \tag{A.1.7}$$

This is a subgroup of $GL(V)$, because of (A.1.4).

A.1.2 Invertible matrices

Let $GL(n, \mathbf{R})$ be the set of $n \times n$ matrices of real numbers that are invertible with respect to matrix multiplication, or equivalently that have nonzero determinant. This is called the *general linear group* of these $n \times n$ matrices of real numbers, and it is a group with respect to matrix multiplication. The set of all $n \times n$ matrices of real numbers can be identified with $\mathbf{R}^{n^2}$, by listing the entries of an $n \times n$ matrix in a sequence with n^2 terms. It is well known that $GL(n, \mathbf{R})$ corresponds to an open set in $\mathbf{R}^{n^2}$ in this way. This follows from the fact that the determinant of an $n \times n$ matrix is a polynomial in the entries of the matrix, which defines a continuous function on $\mathbf{R}^{n^2}$.

Similarly, let $SL(n, \mathbf{R})$ be the set of $n \times n$ matrices of real numbers with determinant equal to 1. This is called the *special linear group* of these $n \times n$ matrices of real numbers, and it is a subgroup of $GL(n, \mathbf{R})$. It is easy to see that $SL(n, \mathbf{R})$ corresponds to a closed set in $\mathbf{R}^{n^2}$, because the determinant of an $n \times n$ matrix of real numbers corresponds to a continuous function on $\mathbf{R}^{n^2}$, as in the preceding paragraph. It is well known that $SL(n, \mathbf{R})$ is a smooth submanifold of $GL(n, \mathbf{R})$ of dimension $n^2 - 1$, in the sense that it corresponds to a smooth submanifold of this dimension of the open subset of $\mathbf{R}^{n^2}$ that corresponds to $GL(n, \mathbf{R})$.

A.2 Eigenvalues and eigenvectors

Let V be a vector space over the real or complex numbers, and let T be a linear mapping from V into itself. Also let λ be a real or complex number, as appropriate. An element v of V is said to be an *eigenvector* of T with *eigenvalue* λ if

(A.2.1) $$T(v) = \lambda\, v.$$

The set $E_T(\lambda)$ of these $v \in V$ is called the *eigenspace* of T in V associated to λ. It is easy to see that

(A.2.2) $$E_T(\lambda) \text{ is a linear subspace of } V.$$

More precisely, λ is normally considered to be an *eigenvalue* of T if there is a nonzero $v \in V$ that is an eigenvector of T with eigenvalue λ, so that

(A.2.3) $$E_T(\lambda) \neq \{0\}.$$

However, $v = 0$ is considered to be an element of $E_T(\lambda)$ for every λ. Note that (A.2.1) is the same as saying that

(A.2.4) $$(T - \lambda\, I)(v) = 0,$$

where $I = I_V$ is the identity mapping on V. It follows that λ is an eigenvalue of T if and only if

(A.2.5) $$T - \lambda\, I \text{ is not one-to-one on } V.$$

A.2.1 Commuting linear mappings

Let R be another linear mapping from V into itself. Suppose that R and T *commute* with each other on V, which is to say that

(A.2.6) $$R \circ T = T \circ R.$$

If $v \in V$ is an eigenvector of T with eigenvalue λ, then

(A.2.7) $$T(R(v)) = R(T(v)) = R(\lambda\, v) = \lambda\, R(v).$$

This means that $R(v)$ is also an eigenvector of T with eigenvalue λ, so that

(A.2.8) $$R(v) \in E_T(\lambda).$$

Equivalently, we have that

(A.2.9) $$R(E_T(\lambda)) \subseteq E_T(\lambda).$$

A.2.2 Finite-dimensional vector spaces

Suppose now that V has finite dimension, as a vector space over $\mathbf{R}$ or $\mathbf{C}$. It is well known that a linear mapping from V into itself is one-to-one if and only if it maps V onto itself. In this case, (A.2.5) is the same as saying that

(A.2.10) $$T - \lambda\, I \text{ is not invertible on } V.$$

This means that λ is an eigenvalue of T if and only if

$$\det(T - \lambda I) = 0. \tag{A.2.11}$$

The left side is a polynomial in λ, with coefficients in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, of degree equal to the dimension of V.

If V is a vector space over the complex numbers of positive finite dimension, then it follows that every linear mapping from V into itself has an eigenvalue, because every polynomial of positive degree with complex coefficients has a root in $\mathbf{C}$.

A.3 Linear mappings on $\mathbf{R}^n$

Let n be a positive integer, and remember that the dot product on $\mathbf{R}^n$ is defined as in Section 1.15, and is also known as the standard inner product on $\mathbf{R}^n$. If T is a linear mapping from $\mathbf{R}^n$ into itself, then there is a unique linear mapping T' from $\mathbf{R}^n$ into itself such that

$$T(x) \cdot y = x \cdot T'(y) \tag{A.3.1}$$

for every $x, y \in \mathbf{R}^n$, as in Subsection 1.15.2. This may be called the *adjoint* of T. It is easy to see that

$$(T')' = T, \tag{A.3.2}$$

using (A.3.1), or the fact that the matrix associated to T' in the usual way is the same as the transpose of the matrix associated to T, as before. Note that

$$I' = I, \tag{A.3.3}$$

where $I = I_{\mathbf{R}^n}$ is the identity mapping on $\mathbf{R}^n$.

The space $\mathcal{L}(\mathbf{R}^n)$ of linear mappings from $\mathbf{R}^n$ into itself may be considered as a vector space over the real numbers with respect to pointwise addition and scalar multiplication of linear mappings. This corresponds to the space of $n \times n$ matrices of real numbers, as a vector space over $\mathbf{R}$ with respect to entrywise addition and scalar multiplication of matrices, using the matrices associated to linear mappings in the usual way. Note that

$$T \mapsto T' \tag{A.3.4}$$

is linear, as a mapping from $\mathcal{L}(\mathbf{R}^n)$ into itself. This can be verified using the characterization of T' in (A.3.1), or the fact that the matrix associated to T' is the same as the transpose of the matrix associated to T.

If

$$T' = T, \tag{A.3.5}$$

then we may say that T is *self-adjoint* or *symmetric* with respect to the standard inner product on $\mathbf{R}^n$. Equivalently, this means that

$$T(x) \cdot y = x \cdot T(y) \tag{A.3.6}$$

for all $x, y \in \mathbf{R}^n$. An $n \times n$ matrix of real numbers is said to be *symmetric* if it is equal to its transpose. Thus T is symmetric if and only if the associated matrix is symmetric. If T is symmetric, then it is well known that there is an orthonormal basis for $\mathbf{R}^n$, with respect to the standard inner product, consisting of eigenvectors of T.

Similarly, if

$$T' = -T, \tag{A.3.7}$$

then T is said to be *anti-self-adjoint* or *antisymmetric* with respect to the standard inner product on $\mathbf{R}^n$. This is the same as saying that

$$T(x) \cdot y = -x \cdot T(y) \tag{A.3.8}$$

for all $x, y \in \mathbf{R}^n$. An $n \times n$ matrix of real numbers is said to be *antisymmetric* if it is equal to -1 times its transpose. It follows that T is antisymmetric if and only if the associated matrix is antisymmetric.

A.3.1 More on adjoints

If T_1, T_2 are any linear mappings from $\mathbf{R}^n$ into itself, then

$$(T_1 \circ T_2)' = T_2' \circ T_1', \tag{A.3.9}$$

as in Subsection 5.5.2. This corresponds to a well-known fact about the transpose of a product of matrices. If T is antisymmetric, then one can use this to get that

$$(T \circ T)' = T \circ T, \tag{A.3.10}$$

so that $T^2 = T \circ T$ is self-adjoint.

If T is any linear mapping from $\mathbf{R}^n$ into itself, then

$$(1/2)\,(T + T') \tag{A.3.11}$$

is symmetric,

$$(1/2)(T - T') \tag{A.3.12}$$

is antisymmetric, and T is equal to the sum of (A.3.11) and (A.3.12). One can check that this is the only way in which T can be expressed as a sum of symmetric and antisymmetric linear mappings on $\mathbf{R}^n$.

If

$$T \circ T' = T' \circ T, \tag{A.3.13}$$

then one may say that T is *normal*, although this term is perhaps more commonly used for an analogous property in the complex case. It is easy to see that T is normal if and only if (A.3.11) and (A.3.12) commute with each other.

If T is any linear mapping from $\mathbf{R}^n$ into itself, then

$$\det T' = \det T. \tag{A.3.14}$$

This follows from the fact that the determinant of an $n \times n$ matrix is equal to the determinant of its transpose.

A.4 More on $\mathcal{L}(\mathbf{R}^n)$

Let n be a positive integer, and note that the sets of linear mappings from $\mathbf{R}^n$ into itself that are self-adjoint or anti-self-adjoint are linear subspaces of $\mathcal{L}(\mathbf{R}^n)$. Of course, if a linear mapping T from $\mathbf{R}^n$ into itself is both self-adjoint and anti-self-adjoint, then $T = 0$. Clearly

$$\dim \mathcal{L}(\mathbf{R}^n) = n^2, \tag{A.4.1}$$

because the dimension of the space of $n \times n$ matrices of real numbers is equal to n^2, as a vector space over $\mathbf{R}$. The dimension of the space of self-adjoint linear mappings from $\mathbf{R}^n$ into itself is the same as the dimension of the space of symmetric $n \times n$ matrices of real numbers, which is

$$\frac{n^2 + n}{2}. \tag{A.4.2}$$

Similarly, the dimension of the space of anti-self-adjoint linear mappings from $\mathbf{R}^n$ into itself is the same as the dimension of the space of antisymmetric $n \times n$ matrices of real numbers, namely,

$$\frac{n^2 - n}{2}. \tag{A.4.3}$$

A.4.1 Some properties of $T' \circ T$

If T is any linear mapping from $\mathbf{R}^n$ into itself, then

$$(T' \circ T)' = T' \circ (T')' = T' \circ T, \tag{A.4.4}$$

so that $T' \circ T$ is self-adjoint. Observe that

$$(T' \circ T)(x) \cdot y = T'(T(x)) \cdot y = T(x) \cdot T(y) \tag{A.4.5}$$

for every $x, y \in \mathbf{R}^n$. This implies that

$$(T' \circ T)(x) \cdot x = |T(x)|^2 \geq 0 \tag{A.4.6}$$

for every $x \in \mathbf{R}^n$. It follows that the eigenvalues of $T' \circ T$ are nonnegative.

Of course, if

$$T(x) = 0 \tag{A.4.7}$$

for some $x \in \mathbf{R}^n$, then

$$(T' \circ T)(x) = 0. \tag{A.4.8}$$

Conversely, (A.4.8) implies (A.4.7), because of (A.4.6). This means that

$$\ker(T' \circ T) = \ker T. \tag{A.4.9}$$

A.4.2 Invertibility and adjoints

If T is a one-to-one linear mapping from $\mathbf{R}^n$ onto itself, then one can check that T' is invertible on $\mathbf{R}^n$ as well, with

$$(T')^{-1} = (T^{-1})', \tag{A.4.10}$$

using (A.3.9). In particular, this implies that $T' \circ T$ is invertible on $\mathbf{R}^n$.

Remember that a one-to-one linear mapping T from $\mathbf{R}^n$ onto itself is said to be an orthogonal transformation if it preserves the standard inner product, as in Subsection 1.15.1. This is equivalent to preserving the standard Euclidean norm on $\mathbf{R}^n$, as before.

The condition that T preserve the standard inner product on $\mathbf{R}^n$ is the same as saying that

$$(T' \circ T)(x) \cdot y = x \cdot y \tag{A.4.11}$$

for all $x, y \in \mathbf{R}^n$, because of (A.4.5). Clearly this holds when

$$T' \circ T = I. \tag{A.4.12}$$

Conversely, if (A.4.11) holds for every $x, y \in \mathbf{R}^n$, then one can check that (A.4.12) holds.

Note that (A.4.12) implies that T is one-to-one on $\mathbf{R}^n$, and thus that T maps $\mathbf{R}^n$ onto itself. This means that T is invertible on $\mathbf{R}^n$, so that (A.4.12) is the same as saying that

$$T' = T^{-1}. \tag{A.4.13}$$

Of course, (A.4.13) implies that T and T' commute with each other. This means that (A.3.11) and (A.3.12) commute with each other, as before.

A.5 More on orthogonal transformations

Let n be a positive integer, and let $O(n)$ be the set of all orthogonal transformations on $\mathbf{R}^n$. If T is an orthogonal transformation on $\mathbf{R}^n$, then it is easy to see that T^{-1} is an orthogonal transformation on $\mathbf{R}^n$ as well. One can also verify that the composition of two orthogonal transformations on $\mathbf{R}^n$ is an orthogonal transformation on $\mathbf{R}^n$. This means that $O(n)$ is a subgroup of the group $GL(\mathbf{R}^n)$ of all invertible linear mappings on $\mathbf{R}^n$. This is called the *orthogonal group* associated to $\mathbf{R}^n$, and its standard inner product.

If $T \in O(n)$, then one can check that

$$\det T = \pm 1, \tag{A.5.1}$$

using (A.3.14). Put

$$SO(n) = \{T \in O(n) : \det T = 1\} = O(n) \cap SL(\mathbf{R}^n). \tag{A.5.2}$$

This is a subgroup of $O(n)$ and of $SL(\mathbf{R}^n)$, called the *special orthogonal group*. The elements of $SO(n)$ are known as *rotations* on $\mathbf{R}^n$.

An $n \times n$ matrix of real numbers is said to be an *orthogonal matrix* if it is invertible, with inverse equal to its transpose. These are the same as the matrices that correspond to elements of $O(n)$. The set of these orthogonal matrices is denoted $O(n, \mathbf{R})$. This is a subgroup of the group $GL(n, \mathbf{R})$ of all invertible $n \times n$ matrices of real numbers. This is called the *orthogonal group* of these $n \times n$ matrices of real numbers.

The elements of $O(n, \mathbf{R})$ have determinant ± 1, as before. The set of matrices in $O(n, \mathbf{R})$ with determinant equal to 1 is denoted $SO(n, \mathbf{R})$, so that

$$SO(n, \mathbf{R}) = O(n, \mathbf{R}) \cap SL(n, \mathbf{R}). \tag{A.5.3}$$

These are the matrices that correspond to elements of $SO(n)$, and $SO(n, \mathbf{R})$ is a subgroup of $O(n, \mathbf{R})$ and $SL(n, \mathbf{R})$. This is the *special orthogonal group* of these $n \times n$ matrices of real numbers.

A.5.1 More on $O(n)$

Note that

$$O(n) = \{T \in \mathcal{L}(\mathbf{R}^n) : T' \circ T = I\}, \tag{A.5.4}$$

as in the previous section. Of course, there is an analogous description of $O(n, \mathbf{R})$. Let us identify the space of $n \times n$ matrices of real numbers with $\mathbf{R}^{n^2}$, by listing the entries of such a matrix in a sequence with n^2 terms, as in Subsection A.1.2. It is well known that $O(n, \mathbf{R})$ corresponds to a compact smooth submanifold of $\mathbf{R}^{n^2}$, with dimension equal to (A.4.3). This can be obtained using the implicit function theorem.

More precisely, $O(1)$ consists of ± 1 times the identity mapping on $\mathbf{R}$, and $SO(1)$ contains only the identity mapping on $\mathbf{R}$. Similarly, $O(1, \mathbf{R})$ corresponds to $\{1, -1\}$, considered as a 0-dimensional submanifold of $\mathbf{R}$. If $n \geq 2$, then $n^2 > n$, and (A.4.3) is positive.

Because $O(n, \mathbf{R})$ corresponds to a compact smooth submanifold of $\mathbf{R}^{n^2}$, we can define integrals over it in a standard way, with respect to the element of volume of dimension (A.4.3). If $n = 1$, then this can be interpreted as a sum with two terms in a simple way. We can also take averages over $O(n, \mathbf{R})$, by dividing the integral by the volume of $O(n, \mathbf{R})$ of dimension (A.4.3). We can use this to define integrals and averages over $O(n)$.

A.6 Norms on $\mathbf{R}^n$

Let n be a positive real number, and let N be a nonnegative real-valued function on $\mathbf{R}^n$. We say that N is a *norm* on $\mathbf{R}^n$ if it satisfies the following three conditions. First,

$$N(x) = 0 \text{ if and only if } x = 0. \tag{A.6.1}$$

Second,

$$N(t\, x) = |t|\, N(x) \tag{A.6.2}$$

for every $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. Third,

$$N(x + y) \le N(x) + N(y) \tag{A.6.3}$$

for every $x, y \in \mathbf{R}^n$, which is the *triangle inequality* for N.

It is easy to see that the standard Euclidean norm on $\mathbf{R}^n$ is a norm in this sense, using the properties mentioned in Subsection 1.1.1. If N is any norm on $\mathbf{R}^n$, then

$$N \text{ is convex as a real-valued function on } \mathbf{R}^n, \tag{A.6.4}$$

as mentioned in Section 9.12.

A.6.1 Some basic examples of norms

One can check that

$$\|x\|_1 = \sum_{j=1}^{n} |x_j| \tag{A.6.5}$$

defines a norm on $\mathbf{R}^n$ as well. One can also verify that

$$\|x\|_\infty = \max_{1 \le j \le n} |x_j| \tag{A.6.6}$$

defines a norm on $\mathbf{R}^n$. The standard Euclidean norm on $\mathbf{R}^n$ is sometimes denoted

$$\|x\|_2. \tag{A.6.7}$$

Note that all of these norms are the same as the usual absolute value function on $\mathbf{R}$ when $n = 1$.

A.6.2 Metrics associated to norms

If N is a norm on $\mathbf{R}^n$, then put

$$d_N(x, y) = N(x - y) \tag{A.6.8}$$

for every $x, y \in \mathbf{R}^n$. This is the *metric* on $\mathbf{R}^n$ associated to N. The standard Euclidean metric on $\mathbf{R}^n$ is the same as the metric associated to the standard Euclidean norm, as in Subsection 1.1.2.

Observe that

$$d_N(x, y) = 0 \text{ if and only if } x = y, \tag{A.6.9}$$

because of (A.6.1). It is easy to see that

$$d_N(x, y) = d_N(y, x) \tag{A.6.10}$$

for all $x, y \in \mathbf{R}^n$, using (A.6.2), with $t = -1$. We also have that

$$d_N(x, z) \le d_N(x, y) + d_N(y, z) \tag{A.6.11}$$

for all $x, y, z \in \mathbf{R}^n$, because of (A.6.3). This implies that $\mathbf{R}^n$ is a *metric space* with respect to the metric $d_N(\cdot, \cdot)$. Note that (A.6.11) is called the *triangle inequality* for $d_N(\cdot, \cdot)$, as a metric on $\mathbf{R}^n$.

Clearly d_N is *invariant under translations* on $\mathbf{R}^n$, in the sense that

$$d_N(x + a, y + a) = N((x + a) - (y + a)) = N(x - y) = d_N(x, y) \tag{A.6.12}$$

for every $a, x, y \in \mathbf{R}^n$, as in Subsection 1.1.2.

A.6.3 Open and closed balls

If $x \in \mathbf{R}^n$ and r is a positive real number, then the *open ball* in $\mathbf{R}^n$ centered at x with radius r with respect to N is defined by

$$B_N(x, r) = \{y \in \mathbf{R}^n : N(x - y) < r\}. \tag{A.6.13}$$

The *closed ball* in $\mathbf{R}^n$ centered at x with radius r with respect to N is defined by

$$\overline{B}_N(x, r) = \{y \in \mathbf{R}^n : N(x - y) \le r\}. \tag{A.6.14}$$

These are the same as in Subsection 1.1.3 when N is the standard Euclidean norm on $\mathbf{R}^n$.

One can check that

$$B_N(x, r) \text{ and } \overline{B}_N(x, r) \text{ are convex sets in } \mathbf{R}^n, \tag{A.6.15}$$

as mentioned in Section 1.8. This is related to the remark in Subsection A.7.3.

A.6.4 Norms on $\mathbf{C}^n$

One can define the notion of a *norm* on $\mathbf{C}^n$ in essentially the same way as before. In this case, (A.6.2) should hold for all $x \in \mathbf{C}^n$ and $t \in \mathbf{C}$, where $|t|$ is the absolute value or modulus of t, as in Section 1.4. The standard Euclidean norm on $\mathbf{C}^n$ is a norm in this sense, as in Section 2.6, which may be denoted as in (A.6.7). We get norms on $\mathbf{C}^n$ as in (A.6.5) and (A.6.6) too.

One can identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, as in Section 2.6. Using this, every norm on $\mathbf{C}^n$ may be considered as a norm on $\mathbf{R}^{2n}$. However, not every norm on $\mathbf{R}^{2n}$ corresponds to a norm on $\mathbf{C}^n$ in this way, because norms on $\mathbf{C}^n$ are supposed to satisfy (A.6.2) for all $t \in \mathbf{C}$.

Similarly, one can define the notion of a norm on any vector space over the real or complex numbers.

A.7 Seminorms

Let n be a positive integer, and let N be a nonnegative real-valued function on $\mathbf{R}^n$ again. We say that N defines a *seminorm* on $\mathbf{R}^n$ if it satisfies the second and third conditions in the definition of a norm, which is to say (A.6.2) and

(A.6.3). More precisely, N satisfies the "if" part of the first condition (A.6.1), because of (A.6.2), with $t = 0$. In this case, one can check that the set of $x \in \mathbf{R}^n$ such that

$$N(x) = 0 \tag{A.7.1}$$

is a linear subspace of $\mathbf{R}^n$. Note that seminorms are also sometimes called *pseudonorms*.

If N is a seminorm on $\mathbf{R}^n$, then one can check that N is convex as a real-valued function on $\mathbf{R}^n$. One can also define open and closed balls in $\mathbf{R}^n$ with respect to N as in (A.6.13) and (A.6.14), respectively, and verify that they are convex sets in $\mathbf{R}^n$. This is related to the remark in Subsection A.7.3, as before.

One can define $d_N(x, y)$ for $x, y \in \mathbf{R}^n$ as in (A.6.8), and it satisfies (A.6.10), (A.6.11), and the "if" part of (A.6.9). This may be described as the semimetric or pseudometric on $\mathbf{R}^n$ associated to N.

If $x, y \in \mathbf{R}^n$, then

$$N(x) \le N(y) + N(x - y) \tag{A.7.2}$$

and

$$N(y) \le N(x) + N(x - y). \tag{A.7.3}$$

This implies that

$$|N(x) - N(y)| = \max(N(x) - N(y), N(y) - N(x)) \le N(x - y). \tag{A.7.4}$$

A.7.1 A simple estimate for N

Let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbf{R}^n$, so that the lth coordinate of e_j is equal to one when $j = l$, and to zero otherwise. If $x \in \mathbf{R}^n$, then it follows that

$$x = \sum_{j=1}^{n} x_j \, e_j. \tag{A.7.5}$$

One can use this to get that

$$N(x) \le \sum_{j=1}^{n} |x_j| \, N(e_j). \tag{A.7.6}$$

This implies that there is a nonnegative real number $C(N)$ such that

$$N(x) \le C(N) \, \|x\|_2 \tag{A.7.7}$$

for every $x \in \mathbf{R}^n$, where $\|x\|_2$ is the standard Euclidean norm of x, as before. In fact, one can take

$$C(N) = \Big(\sum_{j=1}^{n} N(e_j)^2\Big)^{1/2}, \tag{A.7.8}$$

using the Cauchy–Schwarz inequality.

Combining this with (A.7.4), we get that

(A.7.9) $$|N(x) - N(y)| \le C(N)\,\|x - y\|_2$$

for all $x, y \in \mathbf{R}^n$. This means that N is Lipschitz of order one with constant $C(N)$ with respect to the standard Euclidean metric on $\mathbf{R}^n$, as in Section 9.3.

A.7.2 Seminorms on other vector spaces

One can define the notion of a seminorm on $\mathbf{C}^n$ in essentially the same way as before, as in Subsection A.6.4. A seminorm on $\mathbf{C}^n$ may be considered as a seminorm on $\mathbf{R}^{2n}$, as before.

In fact, one can define the notion of a seminorm on any vector space over the real or complex numbers, as before.

A.7.3 A remark about convex functions

Let U be a nonempty convex open set in $\mathbf{R}^n$, and let f be a convex function on U, as in Section 9.12. If b is a real number, then it is easy to see that

(A.7.10) $$\{y \in U : f(y) < b\}$$

and

(A.7.11) $$\{y \in U : f(y) \le b\}$$

are convex sets.

A.8 Sublinear functions

Let n be a positive integer, and let p be a real-valued function on $\mathbf{R}^n$. We say that p is *sublinear* on $\mathbf{R}^n$ if it satisfies the following two conditions. First,

(A.8.1) $$p(t\,x) = t\,p(x)$$

for every positive real number t and $x \in \mathbf{R}^n$. Second,

(A.8.2) $$p(x + y) \le p(x) + p(y)$$

for every $x, y \in \mathbf{R}^n$. The same definition can also be used on any vector space over the real numbers, as before.

It is easy to see that (A.8.1) implies that

(A.8.3) $$p(0) = 0.$$

Of course, for $x \ne 0$, (A.8.1) is the same as saying that p is homogeneous of degree one on $\mathbf{R}^n \setminus \{0\}$, as in Subsection 2.8.1.

Let us say that p is *symmetric about* 0 on $\mathbf{R}^n$ if

(A.8.4) $$p(-x) = p(x)$$

for every $x \in \mathbf{R}^n$. If p is sublinear and symmetric about 0 on $\mathbf{R}^n$, then it is easy to see that

$$p(x) \geq 0 \tag{A.8.5}$$

for every $x \in \mathbf{R}^n$. This means that p is a seminorm on $\mathbf{R}^n$ under these conditions.

Note that p is a linear functional on $\mathbf{R}^n$ if and only if p satisfies (A.8.1), and equality holds in (A.8.2) for every $x, y \in \mathbf{R}^n$. This is the same as saying that p and $-p$ are both sublinear on $\mathbf{R}^n$.

A.8.1 Some properties of sublinear functions

If p is sublinear on $\mathbf{R}^n$, then it is easy to see that

$$p \text{ is convex as a real-valued function on } \mathbf{R}^n, \tag{A.8.6}$$

as in Section 9.12. If $x \in \mathbf{R}^n$ and $b \in \mathbf{R}$, then it follows that

$$\{y \in \mathbf{R}^n : p(y-x) < b\} \tag{A.8.7}$$

and

$$\{y \in \mathbf{R}^n : p(y-x) \leq b\} \tag{A.8.8}$$

are convex sets, as in Subsection A.7.3.

If a is a nonnegative real number, then

$$a\,p \text{ is sublinear on } \mathbf{R}^n \tag{A.8.9}$$

too. If p_1, p_2 are sublinear on $\mathbf{R}^n$, then

$$p_1 + p_2 \text{ is sublinear on } \mathbf{R}^n. \tag{A.8.10}$$

One can also check that

$$\max(p_1, p_2) \text{ is sublinear on } \mathbf{R}^n \tag{A.8.11}$$

in this case.

Observe that

$$p(-x) \text{ is sublinear on } \mathbf{R}^n. \tag{A.8.12}$$

This implies that

$$p(x) + p(-x) \tag{A.8.13}$$

and

$$N_p(x) = \max(p(x), p(-x)) \tag{A.8.14}$$

are sublinear on $\mathbf{R}^n$, as in the preceding paragraph. Clearly (A.8.13) and (A.8.14) are symmetric about 0 on $\mathbf{R}^n$, by construction. It follows that (A.8.13) and (A.8.14) are seminorms on $\mathbf{R}^n$, as before.

A.8.2 Lipschitz conditions for sublinear functions

If p is sublinear on $\mathbf{R}^n$, then

$$p(x) \le p(y) + p(x - y) \tag{A.8.15}$$

and

$$p(y) \le p(x) + p(y - x) \tag{A.8.16}$$

for every $x, y \in \mathbf{R}^n$. This implies that

$$\begin{aligned} |p(x) - p(y)| &= \max(p(x) - p(y), p(y) - p(x)) \\ &\le \max(p(x - y), p(y - x)) = N_p(x - y) \end{aligned} \tag{A.8.17}$$

for every $x, y \in \mathbf{R}^n$, where N_p is as in (A.8.14).

Let $C(N_p)$ be a nonnegative real number such that

$$N_p(x) \le C(N_p)\,\|x\|_2 \tag{A.8.18}$$

for every $x \in \mathbf{R}^n$, as in (A.7.7). Using this, we get that

$$|p(x) - p(y)| \le C(N_p)\,\|x - y\|_2 \tag{A.8.19}$$

for every $x, y \in \mathbf{R}^n$.

A.9 Some remarks about directional derivatives

Let n be a positive integer, let U be a nonempty open subset of $\mathbf{R}^n$, and let f be a real-valued function on U. If $x \in U$ and $v \in \mathbf{R}^n$, then put

$$U_{x,v} = \{t \in \mathbf{R} : x + t\,v \in U\}. \tag{A.9.1}$$

It is easy to see that this is an open set in the real line that contains 0.

Put

$$F_{x,v}(t) = f(x + t\,v) \tag{A.9.2}$$

for each $t \in U_{x,v}$. If $F_{x,v}$ is differentiable at 0, then put

$$(D_v f)(x) = F'_{x,v}(0). \tag{A.9.3}$$

This is the usual *directional derivative* of f at x in the direction v, as mentioned in Subsection 1.3.2. Note that

$$(D_v f)(x) = 0 \tag{A.9.4}$$

automatically when $v = 0$.

If a is a real number, then

$$U_{x,a\,v} = \{t \in \mathbf{R} : a\,t \in U_{x,v}\} \tag{A.9.5}$$

and

$$F_{x,a\,v}(t) = F_{x,v}(a\,t) \tag{A.9.6}$$

for every $t \in U_{x,v}$. If the directional derivative of f at x in the direction v exists, then the directional derivative in the direction $a\,v$ exists, and is equal to

$$(D_{a\,v} f)(x) = F'_{x,a\,v}(0) = a\,F'_{x,v}(0) = a\,(D_v f)(x). \tag{A.9.7}$$

A.9.1 One-sided directional derivatives

Remember that the one-sided derivatives of a real-valued function on an open set in the real line are defined as in Section 9.7, when they exist. If the one-sided derivative of $F_{x,v}$ at 0 from the right exists, then the *one-sided directional derivative* at x in the direction v may be defined by

$$(D_v^+ f)(x) = (F_{x,v})'_+(0). \tag{A.9.8}$$

Similarly, if the one-sided derivative of $F_{x,v}$ at 0 from the left exists, then we put

$$(D_v^- f)(x) = (F_{x,v})'_-(0). \tag{A.9.9}$$

The directional derivative of f at x in the direction v exists if and only if these two one-sided directional derivatives exist and are equal, in which case

$$(D_v f)(x) = (D_v^+ f)(x) = (D_v^- f)(x), \tag{A.9.10}$$

as before.

If a is a positive real number, then

$$(D_{a\,v}^+ f)(x) = a\,(D_v^+ f)(x) \tag{A.9.11}$$

when $(D_v^+ f)(x)$ exists, and

$$(D_{a\,v}^- f)(x) = a\,(D_v^- f)(x) \tag{A.9.12}$$

when $(D_v^- f)(x)$ exists. If $a < 0$, then

$$(D_{a,v}^- f)(x) = a\,(D_v^+ f)(x) \tag{A.9.13}$$

when $(D_v f)(x)$ exists, and

$$(D_{a\,v}^+ f)(x) = a\,(D_v^- f)(x) \tag{A.9.14}$$

when $(D_v^- f)(x)$ exists.

A.9.2 Directional derivatives of convex functions

Suppose now that U is a convex open set in $\mathbf{R}^n$, and that f is convex on U, as in Section 9.12. This implies that $U_{x,v}$ is an open interval in the real line, which may be unbounded, and that $F_{x,v}$ is convex on $U_{x,v}$, as before. It follows that (A.9.8) and (A.9.9) exist, as in Subsection 9.7.2. More precisely,

$$\begin{aligned}(D_v^+ f)(x) &= \inf\{t^{-1}\,(F_{x,v}(t) - F_{x,v}(0)) : t \in U_{x,v},\ t > 0\}\\ &= \inf\{t^{-1}\,(f(x + t\,v) - f(x)) : t \in U_{x,v},\ t > 0\}\end{aligned} \tag{A.9.15}$$

and

$$\begin{aligned}(D_v^- f)(x) &= \sup\{t^{-1}\,(F_{x,v}(0) - F_{x,v}(-t)) : -t \in U_{x,v},\ t > 0\}\\ &= \sup\{t^{-1}\,(f(0) - f(x - t\,v)) : -t \in U_{x,v},\ t > 0\},\end{aligned} \tag{A.9.16}$$

as before.

Let w be another element of $\mathbf{R}^n$, and suppose that $t > 0$ is small enough so that

$$x + 2\,t\,v,\ x + 2\,t\,w \in U. \tag{A.9.17}$$

This implies that

$$x + t\,v + t\,w = (1/2)\,(x + 2\,t\,v) + (1/2)\,(x + 2\,t\,w) \in U, \tag{A.9.18}$$

which holds for sufficiently small t anyway. It follows that

$$f(x + t\,v + t\,w) \le (1/2)\,f(x + 2\,t\,v) + (1/2)\,f(x + 2\,t\,w), \tag{A.9.19}$$

because f is convex on U. Equivalently, this means that

$$\begin{aligned} t^{-1}\,(f(x + t\,(v + w)) - f(x)) \quad \le \quad & (2\,t)^{-1}\,(f(x + 2\,t\,v) - f(x)) \\ & +(2\,t)^{-1}\,(f(x + 2\,t\,w) - f(x)). \end{aligned} \tag{A.9.20}$$

Taking the limit as $t \to 0+$, we get that

$$(D^+_{v+w} f)(x) \le (D^+_v f)(x) + (D^+_w f)(w). \tag{A.9.21}$$

This shows that

$$p_x(v) = (D^+_v f)(x) \tag{A.9.22}$$

is a sublinear function of $v \in \mathbf{R}^n$, as in the previous section, using also (A.9.11). This corresponds to part of Theorem 23.1 on p213 of [267].

A.10 Linear functionals on $\mathbf{R}^n$

A *linear functional* on a vector space V over the real numbers is a linear mapping from V into $\mathbf{R}$. Let n be a positive integer, and note that

$$\lambda_u(v) = u \cdot v = \sum_{j=1}^n u_j\,v_j \tag{A.10.1}$$

is a linear functional on $\mathbf{R}^n$ for each $u \in \mathbf{R}^n$. It is easy to see that every linear functional on $\mathbf{R}^n$ is of this form, for a unique $u \in \mathbf{R}^n$.

A.10.1 The Hahn–Banach theorem

Let p be a sublinear function on $\mathbf{R}^n$, as in Section A.8, and let L be a linear subspace of $\mathbf{R}^n$. Also let λ be a linear functional on L, and suppose that

$$\lambda(v) \le p(v) \tag{A.10.2}$$

for every $v \in L$. Under these conditions, then *Hahn–Banach theorem* implies that there is an extension of λ to a linear functional on $\mathbf{R}^n$ that satisfies (A.10.2) for every $v \in \mathbf{R}^n$.

It follows in particular that there is a linear function λ on $\mathbf{R}^n$ that satisfies (A.10.2) for every $v \in \mathbf{R}^n$, by taking $L = \{0\}$. As a more precise version of this, let $w \in \mathbf{R}^n$ with $w \neq 0$ be given, and let

$$L(w) = \{t\,w : t \in \mathbf{R}\} \tag{A.10.3}$$

be the linear subspace of $\mathbf{R}^n$ spanned by w. Consider the linear functional λ on $L(w)$ defined by

$$\lambda(t\,w) = t\,p(w) \tag{A.10.4}$$

for each $t \in \mathbf{R}$. Let us check that λ satisfies (A.10.2) on $L(w)$.

If $t \geq 0$, then

$$\lambda(t\,w) = t\,p(w) = p(t\,w). \tag{A.10.5}$$

If $t < 0$, then

$$\lambda(t\,w) = t\,p(w) = -p(-t\,w). \tag{A.10.6}$$

Observe that

$$0 = p(0) \leq p(t\,w) + p(-t\,w), \tag{A.10.7}$$

so that

$$-p(-t\,w) \leq p(t\,w). \tag{A.10.8}$$

Combining this with (A.10.6), we get that

$$\lambda(t\,w) = -p(-t\,w) \leq p(t\,w) \tag{A.10.9}$$

when $t < 0$.

If p is a linear functional on $\mathbf{R}$, and λ is a linear functional on $\mathbf{R}^n$ that satisfies (A.10.2) for every $v \in \mathbf{R}^n$, then

$$\lambda = p \tag{A.10.10}$$

on $\mathbf{R}^n$. Indeed, if $v \in \mathbf{R}^n$, then we get that

$$\lambda(-v) \leq p(-v), \tag{A.10.11}$$

which implies that

$$p(v) \leq \lambda(v) \tag{A.10.12}$$

in this case.

A.10.2 Convex functions and linear functionals

Let U be a nonempty convex open set in $\mathbf{R}^n$, and let f be a convex real-valued function on U, as in Section 9.12. Also let $x \in U$ be given, and let p_x be the sublinear function on $\mathbf{R}^n$ in (A.9.22). If $v \in \mathbf{R}^n$ and

$$x + v \in U, \tag{A.10.13}$$

then

$$f(x+v) - f(x) \geq (D_v^+ f)(x) = p_x(v). \tag{A.10.14}$$

More precisely, (A.10.13) is the same as saying that

$$1 \in U_{x,v}, \tag{A.10.15}$$

where $U_{x,v}$ is as in (A.9.1). Using this, the first step in (A.10.14) follows from (A.9.15).

Suppose that λ is a linear functional on $\mathbf{R}^n$ such that

$$\lambda(v) \le p_x(v) \tag{A.10.16}$$

for every $v \in \mathbf{R}^n$. In this case, we get that

$$f(x) + \lambda(v) \le f(x+v) \tag{A.10.17}$$

for every $v \in \mathbf{R}^n$ such that (A.10.13) holds.

The existence of a linear functional λ on $\mathbf{R}^n$ such that (A.10.17) holds corresponds to part of Theorem 23.4 on p217 of [267]. Of course, if f is differentiable in an appropriate sense at x, then p_x is a linear functional on $\mathbf{R}^n$, so that one can simply take $\lambda = p_x$. Another approach to the existence of such a linear functional will be mentioned in Subsection A.14.1.

A.11 Nonnegative sublinear functions

Let n be a positive integer, and let N be a nonnegative real-valued function on $\mathbf{R}^n$. Suppose that

$$N(t\,x) = t\,N(x) \tag{A.11.1}$$

for every positive real number t and $x \in \mathbf{R}^n$, so that $N(0) = 0$ in particular, as in Section A.8. If r is a positive real number, then put

$$B_N(r) = \{y \in \mathbf{R}^n : N(y) < r\}. \tag{A.11.2}$$

Similarly, if r is a nonnegative real number, then put

$$\overline{B}_N(r) = \{y \in \mathbf{R}^n : N(y) \le r\}. \tag{A.11.3}$$

If E is a subset of $\mathbf{R}^n$ and $t \in \mathbf{R}$, then put

$$t\,E = \{t\,y : y \in E\}. \tag{A.11.4}$$

Observe that

$$t\,B_N(r) = B_N(t\,r) \tag{A.11.5}$$

for every $r.t > 0$, and that

$$t\,\overline{B}_N(r) = \overline{B}_N(t\,r) \tag{A.11.6}$$

for every $r \ge 0$ and $t > 0$.

Put

$$-E = (-1)\,E = \{-y : y \in E\}. \tag{A.11.7}$$

If

$$-E = E, \tag{A.11.8}$$

then E is said to be *symmetric about* 0. If N is symmetric about 0, as in Section A.8, then $B_N(r)$ is symmetric about 0 for every $r > 0$, and $\overline{B}_N(r)$ is symmetric about 0 for every $r \geq 0$.

A.11.1 Sublinearity and convexity

Suppose for the moment that

$$N(x + y) \leq N(x) + N(y) \tag{A.11.9}$$

for every $x, y \in \mathbf{R}^n$, so that N is sublinear on $\mathbf{R}^n$, as in Section A.8. This implies that $B_N(r)$ is a convex set for every $r > 0$, and that $\overline{B}_N(r)$ is convex for every $r \geq 0$, as in Subsection A.8.1.

Coversely, we would like to show that (A.11.9) holds if either $B_N(1)$ or $\overline{B}_N(1)$ is convex. More precisely, suppose that for every $u, v \in \mathbf{R}^n$ with

$$N(u),\ N(v) < 1 \tag{A.11.10}$$

and every $\tau \in \mathbf{R}$ with $0 < \tau < 1$ we have that

$$N(\tau\, u + (1 - \tau)\, v) \leq 1. \tag{A.11.11}$$

Note that this condition holds when either $B_N(1)$ or $\overline{B}_N(1)$ is convex. We would like to show that (A.11.9) holds in this case.

To do this, let $x, y \in \mathbf{R}^n$ be given, and let t_x, t_y be real numbers with

$$N(x) < t_x,\ N(y) < t_y. \tag{A.11.12}$$

Put

$$u = t_x^{-1}\, x,\ v = t_y^{-1}\, y, \tag{A.11.13}$$

and observe that u, v satisfy (A.11.10), because of (A.11.1). Also put

$$\tau = \frac{t_x}{t_x + t_y}, \tag{A.11.14}$$

so that $0 < \tau < 1$ and

$$1 - \tau = \frac{t_y}{t_x + t_y}. \tag{A.11.15}$$

Clearly

$$\tau\, u + (1 - \tau)\, v = (t_x + t_y)^{-1}\, (x + y). \tag{A.11.16}$$

It follows that

$$N((t_x + t_y)^{-1}\, (x + y)) \leq 1, \tag{A.11.17}$$

because of (A.11.11). This means that

$$N(x + y) < t_x + t_y, \tag{A.11.18}$$

because of (A.11.1). One can use this to get (A.11.9), because t_x and t_y are arbitrary real numbers that satisfy (A.11.12).

A.11.2 An inequality of Minkoswki

Let p be a positive real number, and put

$$\|x\|_p = \Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p} \tag{A.11.19}$$

for every $x \in \mathbf{R}^n$. This is the same as the standard Euclidean norm on $\mathbf{R}^n$ when $p = 2$, and this was mentioned in Subsection A.6.1 when $p = 1$. Of course, this is a nonnegative real-valued function on $\mathbf{R}^n$ for every $p > 0$, and

$$\|x\|_p = 0 \text{ if and only if } x = 0. \tag{A.11.20}$$

It is easy to see that (A.11.19) satisfies (A.11.1) for every $p > 0$ as well.

If $p \geq 1$, then it is well known that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \tag{A.11.21}$$

for every $x, y \in \mathbf{R}^n$. This is *Minkowski's inequality* for finite sums. There are analogous statements for inetgrals, which can be obtained by approximating integrals by finite sums, or using arguments like those for finite sums. Of course, this implies that (A.11.19) is a norm on $\mathbf{R}^n$ when $p \geq 1$. One can use this to get a norm on $\mathbf{C}^n$ as well, as in Subsection A.6.4.

The closed unit ball in $\mathbf{R}^n$ associatied to (A.11.19) is

$$\{y \in \mathbf{R}^n : \|y\|_p \leq 1\} = \Big\{y \in \mathbf{R}^n : \sum_{j=1}^n |y_j|^p \leq 1\Big\}. \tag{A.11.22}$$

In order to get (A.11.19), it suffices to check that (A.11.22) is a convex set in $\mathbf{R}^n$, as in the previous subsection.

If $p \geq 1$, then

$$f_p(w) = |w|^p \tag{A.11.23}$$

is a convex function on the real line, as in Section 9.9. One can use this to verify that (A.11.22) is a convex set in $\mathbf{R}^n$ when $p \geq 1$.

If $n = 1$, then (A.11.19) is the same as the absolute value of a real number. If $n \geq 2$, then one can check that (A.11.19) does not necessarily hold when $p < 1$. Equivalently, (A.11.22) is not convex when $n \geq 2$ and $p < 1$.

If $p \leq 1$, then one can check that

$$\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p \tag{A.11.24}$$

for every $x, y \in \mathbf{R}^n$, using an inequality mentioned in Subsection 9.10.2. One can use this to get that

$$\|x - y\|_p^p \tag{A.11.25}$$

defines a metric on $\mathbf{R}^n$ when $p \leq 1$.

It is easy to see that

$$\|x\|_\infty \leq \|x\|_p \tag{A.11.26}$$

for every $x \in \mathbf{R}^n$ and $p > 0$, where $\|x\|_\infty$ is as in Subsection A.6.1. We also have that

$$\|x\|_p \le n^{1/p}\,\|x\|_\infty \tag{A.11.27}$$

for every $x \in \mathbf{R}^n$ and $p > 0$. One can use this to get that

$$\|x\|_p \to \|x\|_\infty \text{ as } p \to \infty \tag{A.11.28}$$

for every $x \in \mathbf{R}^n$.

A.12 Differentiable mappings

Let m and n be positive integers, and let U be a nonempty open set in $\mathbf{R}^n$. Also let f be a mapping from U into $\mathbf{R}^m$, which is the same as a function on U with values in $\mathbf{R}^m$. Equivalently, if $x \in U$, then

$$f(x) = (f_1(x), \ldots, f_m(x)), \tag{A.12.1}$$

where $f_1, \ldots, f_m$ are real-valued functions on U, as in Section 13.1.

We say that f is *differentiable* at a point $x \in U$ if there is a linear mapping A from $\mathbf{R}^n$ into $\mathbf{R}^m$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A(h)\|_{2,\mathbf{R}^m}}{\|h\|_{2,\mathbf{R}^n}} = 0. \tag{A.12.2}$$

Here $\|\cdot\|_{2,\mathbf{R}^n}$ and $\|\cdot\|_{2,\mathbf{R}^m}$ are the standard Euclidean norms on $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively.

One can check directly that A is unique when it exists, and another way to see this will be mentioned soon. If f is differentiable at x, then we may put

$$f'(x) = A. \tag{A.12.3}$$

This may be called the *differential* of f at x.

If $n = 1$, then this reduces to the usual definition of the derivative of a function of one variable, in the following way. Indeed, a linear mapping from $\mathbf{R}$ into $\mathbf{R}^m$ corresponds exactly to multiplying a real number by a fixed element of $\mathbf{R}^m$. The differential of f at x is the linear mapping that corresponds to multiplying a real number by the usual derivative of f at x, as an element of $\mathbf{R}^m$.

Of course, this is all somewhat simpler when $m = 1$, so that we are dealing with real-valued functions. One can always reduce to that case, by considering the components $f_1, \ldots, f_m$ of f. In particular, one can check that f is differentiable at x if and only if $f_1, \ldots, f_m$ are all differentiable at x, as real-valued functions on U. In this case, the components of the differential of f at x correspond to the differentials of $f_1, \ldots, f_m$ at x.

If f is differentiable at x, then one can check that f is continuous at x. One can also verify that the directional derivative of f at x in any direction $v \in \mathbf{R}^n$ exists in this case, with

$$D_v f(x) = (f'(x))(v). \tag{A.12.4}$$

In particular, the partial derivative of f at x in the kth variable exists for each $k = 1, \ldots, n$, with

(A.12.5) $$\frac{\partial f}{\partial x_k}(x) = (f'(x))(e_k),$$

where $e_1, \ldots, e_n$ are the standard basis vectors for $\mathbf{R}^n$, as in Subsection A.7.1. This can be used to get the uniqueness of the differential when it exists, as mentioned earlier.

Suppose for the moment that f is continuously differentiable on U, in the sense that each of its components is continuously differentiable as a real-valued function on U, as in Subsection 13.1.1. Under these conditions, it is well known that f is differentiable in the sense discussed here at every point in U. The differential of f at $x \in U$ is determined by the partial derivatives of f at x as in (A.12.5). This corresponds to an $m \times n$ matrix in a standard way, whose entries are given by the partial derivatives of the components of f at x.

A.12.1 The chain rule

Let V be a nonempty open set in $\mathbf{R}^m$, and suppose that

(A.12.6) $$f(U) = \{f(x) : x \in U\} \subseteq V.$$

Also let p be another positive integer, and let g be a mapping from V into $\mathbf{R}^p$. Thus the composition $g \circ f$ of f and g may be defined as a mapping from U into $\mathbf{R}^p$, with

(A.12.7) $$(g \circ f)(x) = g(f(x))$$

for every $x \in U$. If f is continuous on U, and g is continuous on V, then it is well known and not difficult to show that $g \circ f$ is continous on U as well.

If f is differentiable at $x \in U$, and g is differentiable at the point $f(x)$ in V, then it is well known that

(A.12.8) $$g \circ f \text{ is differentiable at } x.$$

The differential of $g \circ f$ at x is given by

(A.12.9) $$(g \circ f)'(x) = g'(f(x)) \circ f'(x),$$

as a linear mapping from $\mathbf{R}^n$ into $\mathbf{R}^p$. Equivalently, this is the composition of the differential of f at x, as a linear mapping from $\mathbf{R}^n$ into $\mathbf{R}^m$, with the differential of g at $f(x)$, as a linear mapping from $\mathbf{R}^m$ into $\mathbf{R}^p$. This is the analogue of the *chain rule* here.

A.13 Some properties of convex sets

Let n be a positive integer, and let E be a nonempty closed set in $\mathbf{R}^n$. If $x \in \mathbf{R}^n$, then there is a point $y \in E$ such that

(A.13.1) $$|x - y| \le |x - z|$$

for all $z \in E$. This follows from the extreme value theorem when E is also bounded, and thus compact.

Otherwise, let R be a sufficiently large positive real number, so that

$$E \cap \overline{B}(x, R) \tag{A.13.2}$$

is nonempty. This set is closed and bounded, and thus compact, by construction. It follows that there is an element y of this set such that (A.13.1) for all z in this set, as before. It is easy to see that (A.13.1) holds for all $z \in E$ under these conditions.

Of course, (A.13.1) is the same as saying that

$$|x - y|^2 \le |x - z|^2 \tag{A.13.3}$$

for all $z \in E$. Note that

$$\begin{aligned} |x - z|^2 &= |(x - y) + (y - z)|^2 \\ &= |x - y|^2 + 2\,(x - y)\,(y - z) + |y - z|^2 \end{aligned} \tag{A.13.4}$$

for all $z \in \mathbf{R}^n$. If $z \in E$, then

$$0 \le 2\,(x - y) \cdot (y - z) + |y - z|^2, \tag{A.13.5}$$

by (A.13.3).

A.13.1 Closed convex sets

Suppose now that E is convex as well. If $z \in E$, $t \in \mathbf{R}$, and $0 \le t \le 1$, then

$$(1 - t)\,y + t\,z \tag{A.13.6}$$

is an element of E too. Note that

$$y - ((1 - t)\,y + t\,z) = t\,(y - z). \tag{A.13.7}$$

If we replace z with (A.13.6) in (A.13.5), then we get that

$$0 \le 2\,t\,(z - y) \cdot (y - z) + t^2\,|y - z|^2. \tag{A.13.8}$$

This implies that

$$0 \le 2\,(x - y) \cdot (y - z) + t\,|y - z|^2 \tag{A.13.9}$$

when $0 < t \le 1$.

It follows that

$$0 \le (x - y) \cdot (y - z), \tag{A.13.10}$$

by taking the limit as $t \to 0+$. Equivalently, this means that

$$(x - y) \cdot z \le (x - y) \cdot y. \tag{A.13.11}$$

Of course, this is trivial when $x \in E$, so that $y = x$.

If

$$|x - y| = |x - z|, \tag{A.13.12}$$

then

$$0 = 2\,(x - y) \cdot (y - z) + |y - z|^2, \tag{A.13.13}$$

because of (A.13.4). This implies that

$$y = z, \tag{A.13.14}$$

because of (A.13.10).

A.13.2 Points in ∂E

Suppose now that

$$w \in \partial E = E \cap \overline{(\mathbf{R}^n \setminus E)}. \tag{A.13.15}$$

This implies that there is a sequence $\{x_j\}_{j=1}^\infty$ of elements of $\mathbf{R}^n \setminus E$ such that

$$\{x_j\}_{j=1}^\infty \text{ converges to } w. \tag{A.13.16}$$

If j is a positive integer, then let y_j be the element of E such that

$$|x_j - y_j| \le |x_j - z| \tag{A.13.17}$$

for every $z \in E$, as before. In particular, this means that

$$|x_j - y_j| \le |x_j - w| \tag{A.13.18}$$

for each j, because $w \in E$. It follows that

$$\{y_j\}_{j=1}^\infty \text{ converges to } w \tag{A.13.19}$$

as well.

If $z \in E$, then

$$(x_j - y_j) \cdot (y_j - z) \ge 0 \tag{A.13.20}$$

for each j, as in (A.13.10). Note that $x_j \ne y_j$ for each j, and put

$$u_j = \frac{x_j - y_j}{|x_j - y_j|}. \tag{A.13.21}$$

Clearly

$$u_j \cdot (y_j - z) \ge 0 \tag{A.13.22}$$

for each j and $z \in E$, by (A.13.20).

The unit sphere in $\mathbf{R}^n$ is closed and bounded, and thus compact. This implies that there is a subsequence $\{u_{j_l}\}_{l=1}^\infty$ of $\{u_j\}_{j=1}^\infty$ that converges to a point $u \in \mathbf{R}^n$ with

$$|u| = 1, \tag{A.13.23}$$

as in Subsection 1.9.3. Using the same sequence of indices $\{j_l\}_{l=1}^\infty$, we get subsequences $\{x_{j_l}\}_{l=1}^\infty$ and $\{y_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$, respectively. These two subsequences converge to w, because of (A.13.16) and (A.13.19).

If $z \in E$, then

$$u_{j_l} \cdot (y_{j_l} - z) \geq 0 \tag{A.13.24}$$

for each l, as in (A.13.22). Taking the limit as $l \to \infty$, we get that

$$u \cdot (w - z) \geq 0. \tag{A.13.25}$$

A.14 Some more remarks about convexity

Let n be a positive integer, and let E be a nonempty subset of $\mathbf{R}^n$. If $x \in \mathbf{R}^n$, then we might like to find a point $y \in E$ that minimizes this distahnce to E, as in (A.13.1). If there is a positive real number such that (A.13.2) is nonempty and a closed set, then we can use the same argument as before. If E is also convex, then we get the same conclusions as in Subsection A.13.1.

Suppose now that

$$w \in E \cap \overline{(\mathbf{R}^n \setminus E)} \subseteq \partial E. \tag{A.14.1}$$

Suppose also that there is a positive real number r such that

$$\overline{B}(w, r) \cap E \text{ is a closed set.} \tag{A.14.2}$$

This condition holds automatically when E is a relatively closed set in an open set in $\mathbf{R}^n$.

As before, there is a sequence $\{x_j\}_{j=1}^\infty$ of elements of $\mathbf{R}^n \setminus E$ that converges to w. We may as well ask that

$$|x_j - w| \leq r/2 \tag{A.14.3}$$

for each j. This implies that

$$w \in \overline{B}(x_j, r/2) \subseteq \overline{B}(w, r) \tag{A.14.4}$$

for each j, using the triangle inequality in the second step. It follows that

$$\overline{B}(x_j, r/2) \cap E = \overline{B}(x_j, r/2) \cap (\overline{B}(w, r) \cap E) \tag{A.14.5}$$

is a nonempty closed set for each j. Using this and the remark at the beginning of the section, we can get the same conclusions as in Subsection A.13.2 when E is convex.

If E is convex, then it is well known and not difficult to show that

$$\overline{E} \text{ is convex.} \tag{A.14.6}$$

Of course,

$$\overline{B}(w, r) \cap \overline{E} \tag{A.14.7}$$

is a closed set for every $r > 0$. If (A.14.1) holds, then it is not too difficult to see that

(A.14.8) $$w \in \overline{(\mathbf{R}^n \setminus \overline{E})},$$

so that w is an element of the boundary of $\overline{E}$. Otherwise, w would be an element of the interior of $\overline{E}$, and one could use this to get that w is an element of the interior of E, because E is convex. A more precise version of this may be found in Theorem 6.3 on p46 of [267]. This means that we can replace E with $\overline{E}$ in Subsection A.13.2. This corresponds to Corollary 11.6.1 on p100 of [267].

A.14.1 Epigraphs of convex functions

Let U be a nonempty convex open subset of $\mathbf{R}^n$, let f be a real-valued function on U, and let us identify $\mathbf{R}^{n+1}$ with $\mathbf{R}^n \times \mathbf{R}$ in the usual way. The *epigraph* is the set

(A.14.9) $$E(f) = \{(x, t) \in \mathbf{R}^{n+1} : x \in U,\, t \geq f(x)\},$$

as on p23 of [267]. One can check that f is convex on U if and only if $E(f)$ is a convex set in $\mathbf{R}^{n+1}$. In fact, convexity of a function is defined in terms of convexity of the epigraph in [267].

If $x \in U$, then it is easy to see that

(A.14.10) $$(x, f(x)) \in \partial E(f).$$

Note that

(A.14.11) $$\{(x, t) \in \mathbf{R}^{n+1} : x \in U\}$$

is an open set in $\mathbf{R}^{n+1}$, because U is an open set in $\mathbf{R}^n$. If f is continuous on U, then one can check that (A.14.10) is relatively closed in (A.14.11). More precisely, this works when f is lower semicontinuous on U.

If f is convex on U and $x \in U$, then one can get another approach to the existence of a linear functional on $\mathbf{R}^n$ as in Subsection A.10.2 using the convexity of $E(f)$ and the remarks in Subsection A.13.2. This corresponds to a remark on p217 of [267].

A.15 Conformal mappings

Let n be a positive integer, and remember that a linear mapping T from $\mathbf{R}^n$ onto itself is said to be an orthogonal transformation if it preserves the standard inner product on $\mathbf{R}^n$, as mentioned in Subsection 1.15.1. This is equivalent to asking that T preserve the standard Euclidean norm on $\mathbf{R}^n$, as before.

If T is any linear mapping from $\mathbf{R}^n$ into itself, then the adjoint T' of T with respect to the standard inner product on $\mathbf{R}^n$ may be defined as a linear mapping from $\mathbf{R}^n$ into itself as in Subsection 1.15.2. Remember that the space $\mathcal{L}(\mathbf{R}^n)$ of linear mappings from $\mathbf{R}^n$ into itself may be considered as a vector space over the real numbers, and that $T \mapsto T'$ is a linear mapping from $\mathcal{L}(\mathbf{R}^n)$ into itself, as in Section A.3. We also have that T is an orthogonal transformation on $\mathbf{R}^n$ if and only if

$$T' \circ T = I, \tag{A.15.1}$$

where I is the identity mapping on $\mathbf{R}^n$, as in Subsection A.4.2.

Let us say that a linear mapping A from $\mathbf{R}^n$ into itself is *conformal* if

$$A = a\,T \tag{A.15.2}$$

for some real number $a \neq 0$ and orthogonal transformation T. This implies that

$$A' \circ A = a^2\,T' \circ T = a^2\,I. \tag{A.15.3}$$

Conversely, if

$$A' \circ A = b\,I \tag{A.15.4}$$

for some positive real number b and $a^2 = b$, then

$$(a^{-1}\,A)' \circ (a^{-1}\,A) = b^{-1}\,A' \circ A = I. \tag{A.15.5}$$

This means that $a^{-1}\,A$ is an orthogonal transformation on $\mathbf{R}^n$, so that A is conformal. If (A.15.4) holds for some real number b, then $b \geq 0$, as in Subsection A.4.1.

If A is any linear mapping from $\mathbf{R}^n$ into itself, then $\det A' = \det A$, as mentioned in Subsection A.3.1, so that

$$\det(A' \circ A) = (\det A')\,(\det A) = (\det A)^2. \tag{A.15.6}$$

It is well known that

$$\det(c\,I) = c^n \tag{A.15.7}$$

for any real number c. If (A.15.4) holds, then we get that

$$(\det A)^2 = b^n. \tag{A.15.8}$$

One can use this to get that A is conformal if and only if A is invertible on $\mathbf{R}^n$, so that $\det A \neq 0$, and

$$A' \circ A = |\det A|^{2/n}\,I. \tag{A.15.9}$$

If $\det A = 0$, then (A.15.9) implies that $A' \circ A = 0$, so that $A = 0$, as in Subsection A.4.1.

A.15.1 Conformal differentiable mappings

Let U be a nonempty open subset of $\mathbf{R}^n$, and let f be a mapping from U into $\mathbf{R}^n$. Suppose that f is differentiable at each $x \in U$, as in Section A.12. It is convenient here to use the notation df_x for the differential of f at $x \in U$, which is a linear mapping from $\mathbf{R}^n$ into itself. We say that f is a *conformal mapping* on U if for every $x \in U$,

(A.15.10) df_x is a conformal linear mapping from $\mathbf{R}^n$ onto itself.

If f is a linear mapping from $\mathbf{R}^n$ into itself, then df_x is the same as f for each x. In this case, f is a conformal mapping if and only if f is a conformal linear mapping. If f is defined by a translation on $\mathbf{R}^n$, then $df_x = I$ for each x, so that f is conformal. It is well known and not too difficult to show that

$$x \mapsto x/|x|^2 \tag{A.15.11}$$

defines a conformal mapping from $\mathbf{R}^n \setminus \{0\}$ onto itself.

It is easy to see that the composition of two conformal linear mappings from $\mathbf{R}^n$ onto itself is another conformal linear mapping. One can use this and the chain rule to get that the composition of two conformal differentiable mappings is conformal as well, on the appropriate domain of the composition.

In particular, we can get conformal mappings by composing conformal linear mappings, translations, and the mapping (A.15.11). Conformal mappings of this type are called *Möbius transformations.*

If $n = 1$, then any nonzero linear mapping is conformal, and differentiable functions with nonzero derivative are conformal.

If $n = 2$, then holomorphic functions with nonzero derivative are conformal. The complex-conjugate of such a mapping is conformal as well.

If $n \geq 3$, then a famous theorem of Liouville says that a conformal mapping with sufficient smoothness on a nonempty connected open set is a Möbius transformation. See [103, 195, 263] for more information.

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Index