Abstract

These informal notes are for Math 443 / 538, General Topology, at Rice University in the spring of 2020, starting after spring break. The reader is expected to be familiar with the material covered before spring break, although some topics may be reviewed, for the sake of completeness.

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Part I
Some topology

1 Compact sets

Let \( X \) be a topological space, and let \( K \) be a subset of \( X \). If \( A \) is a nonempty set, \( U_\alpha \) is an open subset of \( X \) for each \( \alpha \in A \), and

\[
K \subseteq \bigcup_{\alpha \in A} U_\alpha,
\]

then \( \{U_\alpha\}_{\alpha \in A} \) is said to be an open covering of \( K \) in \( X \). We say that \( K \) is compact in \( X \) if every open covering of \( K \) can be reduced to a finite subcovering. This means that for every open covering \( \{U_\alpha\}_{\alpha \in A} \) of \( K \) in \( X \) there are finitely many indices \( \alpha_1, \ldots, \alpha_n \in A \) such that

\[
K \subseteq \bigcup_{j=1}^n U_{\alpha_j}.
\]

If \( K \) has only finitely many elements, then it is easy to see that \( K \) is compact. If \( X \) is equipped with the discrete topology and \( K \subseteq X \) is compact, then \( K \) has only finitely many elements. This follows by covering \( K \) with subsets of \( X \) with only one element, which are open sets in this case. If \( a, b \) are real numbers with \( a \leq b \), then it is well known that the closed interval \([a, b]\) in the real line is compact, with respect to the standard topology on \( \mathbb{R} \). If \( X \) is any set equipped with the indiscrete topology, then every subset of \( X \) is compact.

Similarly, let \( X \) be any set equipped with the cofinite topology. This means that \( U \subseteq X \) is an open set when \( U = \emptyset \) or \( X \setminus U \) has only finitely many elements. We also have that every subset \( K \) of \( X \) is compact in this situation. More precisely, let \( \{U_\alpha\}_{\alpha \in A} \) be any open covering of \( K \) in \( X \). We may as well suppose that \( K \neq \emptyset \), which implies that there is an \( \alpha_0 \in A \) such that \( U_{\alpha_0} \neq \emptyset \). It follows that \( U_{\alpha_0} \) contains all but finitely many elements of \( X \), and in particular that \( U_{\alpha_0} \) contains all but finitely many elements of \( K \). The finitely many elements of \( K \setminus U_{\alpha_0} \) can easily be covered by finitely many \( U_\alpha \)'s, so that \( K \) can be covered by finitely many \( U_\alpha \)'s, as desired.

Let \( X \) be a topological space, and let \( Y \) be a subset of \( X \), equipped with the induced topology. If \( K \subseteq Y \), then it is well known that \( K \) is compact as a subset of \( X \) if and only if \( K \) is compact as a subset of \( Y \), with respect to the induced topology. This can be verified directly from the definitions, in essentially the same way as for the analogous statement for metric spaces. The latter is normally formulated a bit differently, and one uses the fact that the relevant topology on \( Y \) is the induced topology.
Let $X$ be a topological space again, and suppose that $K \subseteq X$ is compact, and that $E \subseteq X$ is a closed set. Under these conditions, $K \cap E$ is compact in $X$ as well. To see this, let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of $K \cap E$ in $X$. Note that $X \setminus E$ is an open set in $X$, because $E$ is a closed set. We also have that
\begin{equation}
K = (K \cap E) \cup (K \setminus E) \subseteq \left( \bigcup_{\alpha \in A} U_\alpha \right) \cup (X \setminus E).
\end{equation}
Thus the collection of $U_\alpha$’s, $\alpha \in A$, together with $X \setminus E$, forms an open covering of $K$ in $X$. Because $K$ is compact in $X$, there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that
\begin{equation}
K \subseteq \left( \bigcup_{j=1}^{n} U_{\alpha_j} \right) \cup (X \setminus E).
\end{equation}
This implies that
\begin{equation}
K \cap E \subseteq \bigcup_{j=1}^{n} U_{\alpha_j},
\end{equation}
as desired.

2 Compactness in Hausdorff spaces

Let $X$ be a Hausdorff topological space, let $K$ be a compact subset of $X$, and let $x$ be an element of $X$ not in $K$. If $y \in K$, then $x \neq y$, and hence there are disjoint open sets $U(y), V(y) \subseteq X$ such that $x \in U(y)$ and $y \in V(y)$. The collection of open sets $V(y)$ of this type, with $y \in K$, forms an open covering of $K$ in $X$. Because $K$ is compact, there are finitely many elements $y_1, \ldots, y_n$ of $K$ such that
\begin{equation}
K \subseteq \bigcup_{j=1}^{n} V(y_j).
\end{equation}
Put
\begin{equation}
U = \bigcap_{j=1}^{n} U(y_j), \quad V = \bigcup_{j=1}^{n} V(y_j),
\end{equation}
which are open subsets of $X$. By construction, $x \in U$ and $K \subseteq V$. It is easy to see that
\begin{equation}
U \cap V = \emptyset,
\end{equation}
because $U(y_j) \cap V(y_j) = \emptyset$ for each $j = 1, \ldots, n$. In particular,
\begin{equation}
U \subseteq X \setminus K,
\end{equation}
because $K \subseteq V$.
This shows that the interior of $X \setminus K$ in $X$ is the same as $X \setminus K$, so that $X \setminus K$ is an open set in $X$. Equivalently, $X \setminus K$ can be expressed as the union
of open subsets of $X$, because every element of $X \setminus K$ is contained in an open subset of $X$ that is contained in $X \setminus K$. This means that $K$ is a closed set in $X$.

Let $H$ be another compact subset of $X$, and suppose that

\begin{equation}
H \cap K = \emptyset.
\end{equation}

If $x \in H$, then $x \in X \setminus K$, and so there are disjoint open sets $U_1(x), V_1(x) \subseteq X$ such that $x \in U_1(x)$ and $K \subseteq V_1(x)$, as before. The collection of open sets $U_1(x)$ of this type, with $x \in H$, forms an open covering of $H$ in $X$. Thus there are finitely many elements $x_1, \ldots, x_m$ of $H$ such that

\begin{equation}
H \subseteq \bigcup_{l=1}^{m} U_1(x_l),
\end{equation}

because $H$ is compact. Put

\begin{equation}
U_1 = \bigcup_{l=1}^{m} U_1(x_l), \quad V_1 = \bigcap_{l=1}^{n} V_1(x_l),
\end{equation}

which are open subsets of $X$. Observe that $H \subseteq U_1$ and $K \subseteq V_1$, by construction. One can check that

\begin{equation}
U_1 \cap V_1 = \emptyset,
\end{equation}

because $U_1(x_l) \cap V_1(x_l) = \emptyset$ for every $l = 1, \ldots, m$.

Suppose for the moment that $X$ is compact as a subset of itself. If $E \subseteq X$ is a closed set, then $E$ is compact in $X$. This follows from remarks in the previous section, by considering $E$ as the intersection of itself with $X$. In this situation, the remarks in the preceding paragraph imply that $X$ is a normal topological space.

Suppose now that $X$ is a regular topological space. If $K \subseteq X$ is compact, $E \subseteq X$ is a closed set, and $K \cap E = \emptyset$, then one can show that there are disjoint open sets $U_0, V_0 \subseteq X$ such that $K \subseteq U_0$ and $E \subseteq V_0$, using the same type of arguments as before.

### 3 Continuity and compactness

Let $X$ and $Y$ be topological spaces, and let $f$ be a continuous mapping from $X$ into $Y$. If $K$ is a compact subset of $X$, then

\begin{equation}
f(K) = \{ f(x) : x \in K \}
\end{equation}

is a compact subset of $Y$. This can be shown in the same way as for metric spaces. More precisely, let $\{V_{\alpha}\}_{\alpha \in A}$ be an arbitrary open covering of $f(K)$ in $Y$.

Note that $f^{-1}(V_{\alpha})$ is an open set in $X$ for each $\alpha \in A$, because $f$ is continuous. It is easy to see that

\begin{equation}
K \subseteq \bigcup_{\alpha \in A} f^{-1}(V_{\alpha}),
\end{equation}

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because \( f(K) \subseteq \bigcup_{\alpha \in A} V_{\alpha} \), by hypothesis. Thus \( \{f^{-1}(V_{\alpha})\}_{\alpha \in A} \) is an open covering of \( K \) in \( X \). If \( K \) is compact, then there are finitely many indices \( \alpha_1, \ldots, \alpha_n \in A \) such that

\[
K \subseteq \bigcup_{j=1}^{n} f^{-1}(V_{\alpha_j}).
\]

This implies that

\[
f(K) \subseteq \bigcup_{j=1}^{n} V_{\alpha_j},
\]

as desired.

Suppose for the moment that \( Y \) is the real line with the standard topology. In this case, \( f(K) \) is a closed set in \( \mathbb{R} \), as in the previous section, because the real line is Hausdorff with respect to the standard topology. It is well known and not difficult to show that compact subsets of the real line are bounded too. If \( K \neq \emptyset \), then \( f(K) \neq \emptyset \), and one can use the previous statements to show that \( f(K) \) contains its supremum and infimum in \( \mathbb{R} \). This means that \( f \) attains its maximum and minimum on \( K \), which is a version of the extreme value theorem.

Let \( Y \) be any topological space again, and suppose that \( f \) is a one-to-one continuous mapping from \( X \) onto \( Y \). Let \( g = f^{-1} \) be the inverse of \( f \), as a mapping from \( Y \) onto \( X \). If \( g \) is continuous, then \( f \) is a homeomorphism. We have seen previously that \( g \) is continuous if and only if for every closed set \( E \subseteq X \), \( g^{-1}(E) \) is a closed set in \( Y \). In this situation, this means that for every closed set \( E \subseteq X \), \( f(E) \) is a closed set in \( Y \).

Suppose that \( X \) is compact, as a subset of itself. If \( E \subseteq X \) is a closed set, then it follows that \( E \) is compact, as in Section 1. In this case, we get that \( f(E) \) is compact in \( Y \), as before. If \( Y \) is Hausdorff, then \( f(E) \) is a closed set in \( Y \), as in the previous section. This implies that \( g = f^{-1} \) is continuous in this situation, as in the preceding paragraph.

4 The limit point property

Let \( X \) be a topological space. Remember that a point \( x \in X \) is said to be a limit point of a set \( E \subseteq X \) if for every open set \( U \subseteq X \) with \( x \in U \), there is a \( y \in E \cap U \) such that \( y \neq x \). Let us say that \( x \) is a strong limit point of \( E \) in \( X \) if for every open set \( U \subseteq X \) with \( x \in U \), there are infinitely many elements of \( E \) in \( U \). Thus a strong limit point of \( E \) in \( X \) is automatically a limit point of \( E \) in \( X \). If \( X \) satisfies the first separation condition, and \( x \in X \) is a limit point of \( E \subseteq X \), then one can check that \( x \) is a strong limit point of \( E \) in \( X \).

A subset \( K \) of \( X \) is said to have the limit point property if for every subset \( L \) of \( K \) such that \( L \) has infinitely many elements, there is an element \( x \) of \( K \) that is a limit point of \( L \) in \( X \). Similarly, let us say that \( K \) has the strong limit point property if for every infinite subset \( L \) of \( K \) there is an \( x \in K \) such that \( x \) is a strong limit point of \( L \) in \( X \). If \( K \) has the strong limit point property, then \( K \) automatically has the limit point property. If \( K \) has the limit point
property, and if \( X \) satisfies the first separation condition, then \( K \) has the strong limit point property.

If \( K \subseteq X \) is compact, then \( K \) has the strong limit point property. To see this, let \( L \) be an infinite subset of \( K \), and suppose that \( L \) does not have a strong limit point in \( K \). This means that for each \( x \in K \) there is an open set \( U(x) \subseteq X \) such that \( x \in U(x) \) and \( U(x) \cap L \) has only finitely many elements. Thus \( K \) can be covered by open sets of this type. If \( K \) is compact, then \( K \) can be covered by finitely many open sets of this type. This implies that \( L \) has only finitely many elements, because \( L \subseteq K \). This contradicts the hypothesis that \( L \) have infinitely many elements, as desired.

Suppose now that a subset \( K \) of \( X \) has the strong limit point property. Let \( V_1, V_2, V_3, \ldots \) be an infinite sequence of open subsets of \( X \) such that

\[
K \subseteq \bigcup_{j=1}^{\infty} V_j. \tag{4.1}
\]

We would like to show that there is a positive integer \( n \) such that

\[
K \subseteq \bigcup_{j=1}^{n} V_j. \tag{4.2}
\]

Suppose for the sake of a contradiction that for each positive integer \( n \),

\[
K \nsubseteq \bigcup_{j=1}^{n} V_j. \tag{4.3}
\]

Thus, for each \( n \) in the set \( \mathbb{Z}_+ \) of positive integers, we can choose a point

\[
x_n \in K \setminus \left( \bigcup_{j=1}^{n} V_j \right). \tag{4.4}
\]

Let \( L \) be the set of points \( x_n, n \in \mathbb{Z}_+ \), that have been chosen in this way. Let us check that \( L \) has infinitely many elements. Otherwise, there is an element \( y \) of \( K \) such that \( y = x_n \), for infinitely many \( n \in \mathbb{Z}_+ \). Note that \( y \in V_{j_0} \) for some \( j_0 \in \mathbb{Z}_+ \), because \( y \in K \). This implies that \( x_n \neq y \) when \( n \geq j_0 \), by the way that \( x_n \) was chosen. This contradicts the hypothesis that \( y = x_n \) for infinitely many \( n \), as desired.

Thus \( L \) has infinitely many elements. If \( K \) has the strong limit point property, then it follows that there is an \( x \in K \) such that \( x \) is a strong limit point of \( L \) in \( X \). In particular, \( x \in V_{j_1} \) for some \( j_1 \in \mathbb{Z}_+ \), because \( x \in K \). This implies that \( V_{j_1} \) contains infinitely many elements of \( L \), because \( x \) is a strong limit point of \( L \) in \( X \), and \( V_{j_1} \) is an open set in \( X \). This means that \( x_n \in V_{j_1} \), for infinitely many \( n \in \mathbb{Z}_+ \), by the way that \( L \) was chosen. However, if \( n \geq j_1 \), then \( x_n \notin V_{j_1} \), by construction. This is a contradiction, so that (4.2) holds for some \( n \in \mathbb{Z}_+ \).
5 Sequential compactness

Let \( \{x_j\}_{j=1}^{\infty} \) be a sequence of elements of some set \( X \). Also let \( \{j_l\}_{l=1}^{\infty} \) be a strictly increasing sequence of positive integers, so that \( j_l < j_{l+1} \) for every \( l \in \mathbb{Z}_+ \). Under these conditions, \( \{x_{j_l}\}_{l=1}^{\infty} \) is called a subsequence of \( \{x_j\}_{j=1}^{\infty} \). Now let \( X \) be a topological space. If \( \{x_j\}_{j=1}^{\infty} \) converges to an element \( x \) of \( X \), then it is easy to see that every subsequence \( \{x_{j_l}\}_{l=1}^{\infty} \) of \( \{x_j\}_{j=1}^{\infty} \) converges to \( x \) in \( X \) too.

A subset \( K \) of \( X \) is said to be sequentially compact if for every sequence \( \{x_j\}_{j=1}^{\infty} \) of elements of \( K \) there is a subsequence \( \{x_{j_l}\}_{l=1}^{\infty} \) of \( \{x_j\}_{j=1}^{\infty} \) that converges to an element \( x \) of \( K \). Suppose that \( K \) is sequentially compact, and let us check that \( K \) has the strong limit point property. To do this, let \( L \) be an infinite subset of \( K \). Because \( L \) has infinitely many elements, we can find a sequence \( \{x_j\}_{j=1}^{\infty} \) of distinct elements of \( L \). If \( K \) is sequentially compact, then there is a subsequence \( \{x_{j_l}\}_{l=1}^{\infty} \) of \( \{x_j\}_{j=1}^{\infty} \) that converges to an element \( x \) of \( K \). If \( U \) is an open set in \( X \) that contains \( x \), then it follows that \( x_{j_l} \in U \) for all but finitely many \( l \geq 1 \). This implies that \( x \) is a strong limit point of \( L \) in \( X \), because the terms of the sequence are distinct elements of \( L \).

Let \( K \) be any subset of \( X \), and let \( \{x_j\}_{j=1}^{\infty} \) be a sequence of elements of \( K \). Also let
\[
(5.1) \quad L = \{x_j : j \in \mathbb{Z}_+\}
\]
be the subset of \( K \) consisting of the terms in the sequence. If \( L \) has only finitely many elements, then there is an \( x \in K \) such that \( x_j = x \) for infinitely many \( j \in \mathbb{Z}_+ \). This means that there is a subsequence \( \{x_{j_l}\}_{l=1}^{\infty} \) of \( \{x_j\}_{j=1}^{\infty} \) such that \( x_{j_l} = x \) for every \( l \geq 1 \). Of course, \( \{x_{j_l}\}_{l=1}^{\infty} \) converges to \( x \) in \( X \) in this case.

Suppose now that \( L \) has infinitely many elements. If \( K \) has the strong limit point property, then there is an element \( x \) of \( K \) that is a strong limit point of \( L \) in \( X \). Thus, if \( V \) is an open subset of \( X \) that contains \( x \), then \( V \) contains infinitely many elements of \( L \). This implies that
\[
(5.2) \quad x_j \in V
\]
for infinitely many positive integers \( j \).

Suppose from now on in this section that \( X \) satisfies the first countability condition, so that there is a local base for the topology of \( X \) at each point with only finitely or countably many elements. In particular, there is a local base \( B(x) \) for the topology of \( X \) at the point \( x \) mentioned in the preceding paragraph such that \( B(x) \) has only finitely or countably many elements. Equivalently, there is a sequence \( U_1(x), U_2(x), U_3(x), \ldots \) of open subsets of \( X \) such that \( x \in U_n(x) \) for every positive integer \( n \), and if \( V \subseteq X \) is an open set that contains \( x \), then \( U_n(x) \subseteq V \) for some \( n \). We may also suppose that
\[
(5.3) \quad U_{n+1}(x) \subseteq U_n(x)
\]
for every \( n \geq 1 \), by replacing \( U_n(x) \) with \( U_1(x) \cap \cdots \cap U_n(x) \) for every \( n \).

Using (5.2) with \( V = U_1(x) \), we can get a positive integer \( j_1 \) such that \( x_{j_1} \in U_1(x) \). Suppose that \( j_1 \in \mathbb{Z}_+ \) has been chosen for some positive integer
Using (5.2) with \( V = U_{l+1}(x) \), we get that \( x_j \in U_{l+1}(x) \) for infinitely many \( j \in \mathbb{Z}^+ \). In particular, we can choose \( j_{l+1} \in \mathbb{Z}^+ \) such that \( j_{l+1} > j_l \) and \( x_{j_{l+1}} \in U_{l+1}(x) \).

This leads to a subsequence \( \{x_{j_l}\}_l \) of \( \{x_j\}_j \) such that

\[
(5.4) \quad x_{j_l} \in U_l(x)
\]

for every \( l \in \mathbb{Z}^+ \). It follows that \( \{x_{j_l}\}_l \) converges to \( x \) in \( X \). This shows that \( K \subseteq X \) is sequentially compact when \( K \) has the strong limit point property and \( X \) satisfies the first countability condition.

### 6 Countable compactness

Let \( X \) be a topological space, and let \( K \) be a subset of \( X \). We say that \( K \) is *countably compact* in \( X \) if for every sequence \( U_1, U_2, U_3, \ldots \) of open subsets of \( X \) such that

\[
(6.1) \quad K \subseteq \bigcup_{j=1}^{\infty} U_j,
\]

there is a positive integer \( n \) such that

\[
(6.2) \quad K \subseteq \bigcup_{j=1}^{n} U_j.
\]

We say that \( K \) has the *Lindelöf property* in \( X \) if for every open covering \( \{U_\alpha\}_{\alpha \in A} \) of \( K \) in \( X \) there is a subset \( A_1 \) of \( A \) such that \( A_1 \) has only finitely or countably many elements and

\[
(6.3) \quad K \subseteq \bigcup_{\alpha \in A_1} U_\alpha.
\]

It is easy to see that \( K \) is compact if and only if \( K \) is countably compact and \( K \) has the Lindelöf property. If \( K \) has the strong limit point property, then \( K \) is countably compact, as in Section 4.

Suppose that \( X \) satisfies the *second countability condition*, so that there is a base \( \mathcal{B} \) for the topology of \( X \) such that \( \mathcal{B} \) has only finitely or countably many elements. Let \( A \) be a nonempty set, and let \( U_\alpha \) be an open subset of \( X \) for each \( \alpha \in A \). Under these conditions, a famous theorem of Lindelöf states that there is a subset \( A_1 \) of \( A \) such that \( A_1 \) has only finitely or countably many elements and

\[
(6.4) \quad \bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{\alpha \in A} U_\alpha.
\]

In this case, every subset of \( X \) has the Lindelöf property.

To see this, put

\[
(6.5) \quad \mathcal{B}_\alpha = \{V \in \mathcal{B} : V \subseteq U_\alpha\}
\]

for each \( \alpha \in A \). Observe that

\[
(6.6) \quad U_\alpha = \bigcup \{V : V \in \mathcal{B}_\alpha\}
\]
for every $\alpha \in A$. More precisely, the union on the right is automatically contained in $U_\alpha$, by the definition of $B_\alpha$. In order to get that the union is equal to $U_\alpha$, one uses the hypothesis that $B$ be a base for the topology of $X$.

Put
\[ \tilde{B} = \bigcup_{\alpha \in A} B_\alpha. \]
(6.7)

Of course, $\tilde{B} \subseteq B$, by construction. It follows that $\tilde{B}$ has only finitely or countably many elements, because $B$ has only finitely or countably many elements, by hypothesis. If $V \in \tilde{B}$, then let us choose an element $\alpha(V)$ of $A$ such that $V \in B_\alpha(V)$. Thus
\[ V \subseteq U_{\alpha(V)} \]
(6.8)
for every $V \in \tilde{B}$, by definition of $B_\alpha$.

Let $A_1$ be the set of elements of $A$ of the form $\alpha(V)$ for some $V \in \tilde{B}$ that have been chosen in this way. It is easy to see that $A_1$ has only finitely or countably many elements, because $\tilde{B}$ has only finitely or countably many elements. Observe that
\[ \bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{\alpha \in \tilde{B}} U_\alpha(V) \supseteq \bigcup_{V \in \tilde{B}} V, \]
(6.9)
using (6.8) in the second step. We also have that
\[ \bigcup_{V \in \tilde{B}} V = \bigcup_{\alpha \in A_1} \bigcup_{V \in B_\alpha} V = \bigcup_{\alpha \in A} U_\alpha, \]
(6.10)
This uses the definition (6.7) of $\tilde{B}$ in the first step, and (6.6) in the second step. Combining (6.9) and (6.10), we obtain that
\[ \bigcup_{\alpha \in A_1} U_\alpha \subseteq \bigcup_{\alpha \in A} U_\alpha. \]
(6.11)
This implies (6.4), because $A_1 \subseteq A$, by construction.

### 7 The finite intersection property

Let $X$ be a set, and let $K$ be a subset of $X$. Also let $I$ be a nonempty set, and let $E_j$ be a subset of $X$ for each $j \in I$. We say that $\{E_j\}_{j \in I}$ has the finite intersection property with respect to $K$ if for every finite collection $j_1, \ldots, j_n$ of elements of $I$, we have that
\[ \left( \bigcap_{l=1}^n E_{j_l} \right) \cap K \neq \emptyset. \]
(7.1)
Of course, this holds when
\[ \left( \bigcap_{j \in I} E_j \right) \cap K \neq \emptyset. \]
(7.2)
If \( \{E_j\}_{j \in I} \) has the finite intersection property with respect to \( X \), then we may simply say that \( \{E_j\}_{j \in I} \) has the finite intersection property.

Suppose now that \( X \) is a topological space. It is well known that \( K \subseteq X \) is compact if and only if for every nonempty family \( \{E_j\}_{j \in I} \) of closed subsets of \( X \) with the finite intersection property with respect to \( K \), we have that (7.2) holds. To see this, let a nonempty set \( I \) be given again. If \( E_j \subseteq X \) is a closed set for some \( j \in I \), then

\[
U_j = X \setminus E_j
\]

(7.3)

is an open set in \( X \). Similarly, if \( U_j \subseteq X \) is an open set for some \( j \in I \), then

\[
E_j = X \setminus U_j
\]

(7.4)

is a closed set in \( X \). This defines a simple correspondence between families \( \{E_j\}_{j \in I} \) of closed subsets of \( X \) indexed by \( I \) and families \( \{U_j\}_{j \in I} \) of open subsets of \( X \) indexed by \( I \).

Using this correspondence, we have that (7.1) holds for some finite collection of indices \( j_1, \ldots, j_n \in I \) if and only if

\[
K \not\subseteq \bigcup_{l=1}^{n} U_{j_l}.
\]

(7.5)

Thus \( \{E_j\}_{j \in I} \) has the finite intersection property with respect to \( K \) exactly when \( K \) cannot be covered by finitely many \( U_j \)'s, \( j \in I \). Similarly, (7.2) holds if and only if

\[
K \not\subseteq \bigcup_{j \in I} U_j.
\]

(7.6)

This is the same as saying that \( K \) is not covered by the \( U_j \)'s, \( j \in I \). Using this, it is easy to see that the statement mentioned in the preceding paragraph is equivalent to compactness.

Let us now consider the case where \( I = \mathbb{Z}_+ \). Let \( E_j \) be a subset of \( X \) for each positive integer \( j \). Observe that \( \{E_j\}_{j \in \mathbb{Z}_+} \) has the finite intersection property with respect to \( K \subseteq X \) if and only if

\[
\left( \bigcap_{j=1}^{n} E_j \right) \cap K \neq \emptyset
\]

(7.7)

for every positive integer \( n \). Of course, we could also simply consider the \( E_j \)'s as forming a sequence of subsets of \( X \), and (7.2) is the same as saying that

\[
\left( \bigcap_{j=1}^{\infty} E_j \right) \cap K \neq \emptyset
\]

(7.8)

in this situation. Using the same type of argument as before, we get that \( K \subseteq X \) is countably compact if and only if for every sequence \( E_1, E_2, E_3, \ldots \) of closed subsets of \( X \) that satisfy (7.7) for each positive integer \( n \), we have that (7.8) holds.
8 The strong limit point property

Let $X$ be a topological space, and suppose that $K \subseteq X$ is countably compact. We would like to show that $K$ has the strong limit point property. To do this, let an infinite subset $L$ of $K$ be given. As before, we can find an infinite sequence \( \{x_j\}_{j=1}^\infty \) of distinct elements of $L$, because $L$ has infinitely many elements.

Put
\[
A_l = \{x_j : j \geq l\}
\]
for each positive integer $l$. Note that for each $l$, $A_l \neq \emptyset$ and
\[
A_{l+1} \subseteq A_l.
\]

Let
\[
E_l = \overline{A_l}
\]
be the closure of $A_l$ in $X$ for each $l \geq 1$. Thus
\[
A_l \subseteq E_l
\]
for every $l \geq 1$, which implies in particular that $E_l \neq \emptyset$. It is easy to see that
\[
E_{l+1} \subseteq E_l
\]
for every $l \geq 1$, because of (8.2).

Of course, $E_l$ is a closed set in $X$ for each $l \geq 1$, by construction. Observe that
\[
\bigcap_{l=1}^n E_l = E_n
\]
for every positive integer $n$, by (8.5). It follows that
\[
\left( \bigcap_{l=1}^n E_l \right) \cap K = E_n \cap K
\]
for each $n \geq 1$. We also have that
\[
A_n \subseteq E_n \cap K
\]
for each $n \geq 1$, because of (8.4) and the fact that $A_n \subseteq L \subseteq K$ for every $n$, by construction. This implies that (8.7) is nonempty for every $n$.

Thus $E_1, E_2, E_3, \ldots$ is a sequence of closed sets in $X$ with the finite intersection property with respect to $K$. Because $K$ is countably compact, we get that
\[
\left( \bigcap_{l=1}^\infty E_l \right) \cap K \neq \emptyset,
\]
as in the previous section. Let $x$ be an element of the left side of (8.9). In particular, $x \in K$, and we would like to check that $x$ is a strong limit point of $L$ in $X$. 

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Note that $x$ is adherent to $A_l$ in $X$ for each $l \geq 1$, because $x \in E_l$. Let $U \subseteq X$ be an open set that contains $x$. It follows that

$$A_l \cap U \neq \emptyset$$

for each $l \geq 1$, because $x$ is adherent to $A_l$. This means that for each $l \geq 1$ there is a $j \geq l$ such that

$$x_j \in U,$$

by the definition (8.1) of $A_l$. Thus (8.11) holds for infinitely many positive integers $j$. In particular, $U$ contains infinitely many elements of $L$, because the $x_j$'s are distinct elements of $L$. This implies that $x$ is a strong limit point of $L$ in $X$, as desired.

9 Compactness and bases

Let $X$ be a topological space, and let $B$ be a collection of open subsets of $X$. Let us say that $K \subseteq X$ is compact with respect to $B$ if every open covering of $K$ by elements of $B$ can be reduced to a finite subcovering. More precisely, this means that if \( \{U_\alpha\}_{\alpha \in A} \) is an open covering of $K$ with $U_\alpha \in B$ for every $\alpha \in A$, then there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup_{j=1}^n U_{\alpha_j}$. Of course, if $K$ is a compact subset of $X$, then $K$ is automatically compact with respect to $B$.

Suppose that $B$ is a base for the topology of $X$. If $K \subseteq X$ is compact with respect to $B$, then we would like to check that $K$ is compact in $X$ in the usual sense. To do this, let $\{U_\alpha\}_{\alpha \in A}$ be any open covering of $K$ in $X$. Put

$$B_\alpha = \{V \in B : V \subseteq U_\alpha\}$$

for each $\alpha \in A$. Because $B$ is a base for the topology of $X$,

$$U_\alpha = \bigcup \{V : V \in B_\alpha\}$$

for every $\alpha \in A$, as before. Put

$$\tilde{B} = \bigcup_{\alpha \in A} B_\alpha.$$

Observe that

$$\bigcup_{V \in \tilde{B}} V = \bigcup_{\alpha \in A} \bigcup_{V \in B_\alpha} V = \bigcup_{\alpha \in A} U_\alpha,$$

using the definition of $\tilde{B}$ in the first step, and (9.2) in the second step. It follows that

$$K \subseteq \bigcup_{\alpha \in A} U_\alpha = \bigcup_{V \in \tilde{B}} V.$$
If $K$ is compact with respect to $\mathcal{B}$, then there are finitely many elements $V_1, \ldots, V_n$ of $\mathcal{B}$ such that

$$K \subseteq \bigcup_{j=1}^{n} V_j.$$  

(9.6)

By definition of $\mathcal{B}$, for each $j = 1, \ldots, n$ there is an $\alpha_j \in A$ such that $V \in \mathcal{B}_{\alpha_j}$, which means that

$$V_j \subseteq U_{\alpha_j}.$$  

(9.7)

Combining this with (9.6), we get that

$$K \subseteq \bigcup_{j=1}^{n} U_{\alpha_j},$$  

(9.8)

as desired.

Let $\mathcal{B}$ be any collection of open subsets of $X$. Consider the collection $\mathcal{B}_1$ of all subsets of $X$ that can be expressed as the intersection of finitely many elements of $\mathcal{B}$. Note that the elements of $\mathcal{B}_1$ are all open subsets of $X$. If $\mathcal{B}_1$ is a base for the topology of $X$, then $\mathcal{B}$ is said to be a subbase for the topology of $X$. If $K \subseteq X$ is compact with respect to a subbase $\mathcal{B}$ for the topology of $X$, then Alexander’s subbase theorem says that $K$ is compact in $X$ in the usual sense.

10 Products of two compact sets

Let $X$ and $Y$ be topological spaces, and consider their Cartesian product $X \times Y$, equipped with the corresponding product topology. If $H \subseteq X$ and $K \subseteq Y$ are compact, then a famous theorem of Tychonoff states that $H \times K$ is compact in $X \times Y$. To show this, let $\mathcal{B}$ be the collection of subsets of $X \times Y$ of the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open sets. This is a base for the product topology on $X \times Y$. Thus it suffices to verify that $H \times K$ is compact with respect to $\mathcal{B}$, as in the previous section.

Let $\{U_\alpha \times V_\alpha\}_{\alpha \in A}$ be a covering of $H \times K$ by elements of $\mathcal{B}$, so that $U_\alpha \subseteq X$ and $V_\alpha \subseteq Y$ are open sets for every $\alpha \in A$, and

$$H \times K \subseteq \bigcup_{\alpha \in A} U_\alpha \times V_\alpha.$$  

(10.1)

Let $x \in H$ be given, so that

$$\{x\} \times K \subseteq \bigcup_{\alpha \in A} U_\alpha \times V_\alpha,$$  

(10.2)

because $\{x\} \times K \subseteq H \times K$. Put

$$A(x) = \{\alpha \in A : x \in U_\alpha\}.$$  

(10.3)
Observe that
\begin{equation}
K \subseteq \bigcup_{\alpha \in A(x)} V_\alpha,
\end{equation}
by (10.2). More precisely, if \( y \in K \), then \((x, y) \in \{x\} \times K\), and hence there is an \( \alpha \in A \) such that \((x, y) \in U_\alpha \times V_\alpha\), by (10.2). This means that \( x \in U_\alpha \), so that \( \alpha \in A(x) \), and \( y \in V_\alpha \), as desired. It follows that there is a finite subset \( A_1(x) \) of \( A(x) \) such that
\begin{equation}
K \subseteq \bigcup_{\alpha \in A_1(x)} V_\alpha,
\end{equation}
because \( K \) is compact in \( Y \).

Put
\begin{equation}
U_1(x) = \bigcap_{\alpha \in A_1(x)} U_\alpha.
\end{equation}
This is an open set in \( X \), because \( U_\alpha \) is an open set in \( X \) for every \( \alpha \in A_1(x) \subseteq A \), and \( A_1(x) \) has only finitely many elements. Of course, \( x \in U_1(x) \), because \( x \in U_\alpha \) for every \( \alpha \in A_1(x) \subseteq A(x) \). We can do this for every \( x \in H \), to get an open covering of \( H \) in \( X \). This implies that there are finitely many elements \( x_1, \ldots, x_n \) of \( H \) such that
\begin{equation}
H \subseteq \bigcup_{j=1}^n U_1(x_j),
\end{equation}
because \( H \) is compact in \( X \).

Using (10.7), we get that
\begin{equation}
H \times K \subseteq \left( \bigcup_{j=1}^n U_1(x_j) \right) \times K = \bigcup_{j=1}^n U_1(x_j) \times K.
\end{equation}

We also have that
\begin{equation}
U_1(x_j) \times K \subseteq U_1(x_j) \times \left( \bigcup_{\alpha \in A_1(x)} V_\alpha \right) \subseteq \bigcup_{\alpha \in A_1(x)} U_\alpha \times V_\alpha
\end{equation}
for each \( j = 1, \ldots, n \). This uses (10.5) in the first step, and the fact that \( U_1(x_j) \subseteq U_\alpha \) when \( \alpha \in A_1(x_j) \), by the definition (10.6) of \( U_1(x_j) \), in the second step. Combining (10.8) and (10.9), we get that
\begin{equation}
H \times K \subseteq \bigcup_{j=1}^n \bigcup_{\alpha \in A_1(x_j)} U_\alpha \times V_\alpha.
\end{equation}

Of course, \( \bigcup_{j=1}^n A_1(x_j) \) is a finite subset of \( A \), because \( A_1(x_j) \) is a finite subset of \( A \) for each \( j = 1, \ldots, n \), as desired.
11 Products of more compact sets

Let $X_1, \ldots, X_n$ be finitely many topological spaces for some positive integer $n$, and consider their Cartesian product

$$X = \prod_{j=1}^{n} X_j,$$

equipped with the product topology. Also let $K_j$ be a compact subset of $X_j$ for each $j = 1, \ldots, n$, and put

$$K = \prod_{j=1}^{n} K_j.$$

Under these conditions, $K$ is a compact subset of $X$. This is trivial when $n = 1$, and the $n = 2$ case was discussed in the previous section. Otherwise, one can use induction, as follows.

Suppose that $n \geq 2$, and that the analogous statement holds for $n - 1$. Thus $\prod_{j=1}^{n-1} K_j$ is a compact subset of $\prod_{j=1}^{n-1} X_j$, with respect to the product topology. There is a natural identification of $X$ with

$$\left(\prod_{j=1}^{n-1} X_j\right) \times X_n,$$

where $x = (x_1, \ldots, x_n) \in X$ is identified with $((x_1, \ldots, x_{n-1}), x_n)$, as an element of (11.3). Let us take $\prod_{j=1}^{n-1} X_j$ to be equipped with the product topology, and use the corresponding product topology on (11.3). One can check that this corresponds to the product topology on $X$, using the identification of (11.3) with $X$ just mentioned. As in the previous section,

$$\left(\prod_{j=1}^{n-1} K_j\right) \times K_n$$

is a compact subset of (11.3), with respect to the product topology. Of course, $K$ corresponds to (11.4), with respect to the identification of $X$ with (11.3). It follows that $K$ is a compact subset of $X$, because the identification of $X$ with (11.3) is a homeomorphism with respect to the corresponding product topologies, as before.

Let $n$ be a positive integer again, and $\mathbb{R}^n$ be the usual space of $n$-tuples of real numbers. This is the same as the Cartesian product of $n$ copies of the real line. The standard topology on $\mathbb{R}^n$ may be defined as the product topology corresponding to the standard topology on $\mathbb{R}$. This is the same as the topology determined by the standard Euclidean metric on $\mathbb{R}^n$. Let $a_j$ and $b_j$ be real numbers with $a_j \leq b_j$ for each $j = 1, \ldots, n$. Remember that the closed interval $[a_j, b_j]$ is a compact subset of the real line for each $j = 1, \ldots, n$, with respect to the standard topology. It follows that their Cartesian product

$$\prod_{j=1}^{n} [a_j, b_j]$$

is a compact subset of $\mathbb{R}^n$, with respect to the standard topology.
is compact with respect to the standard topology on \( \mathbb{R}^n \).

Now let \( I \) be a nonempty set, let \( X_j \) be a topological space for each \( j \in I \), and consider the Cartesian product \( X = \prod_{j \in I} X_j \), equipped with the product topology. If \( K_j \subseteq X_j \) is compact for each \( j \in I \), then another famous theorem of Tychonoff states that \( K = \prod_{j \in I} K_j \) is compact in \( X \). Of course, this reduces to the previous statement for \((11.2)\) when \( I \) has only finitely many elements.

## 12 Sequential compactness and finite products

Let \( X_1, \ldots, X_n \) be finitely many topological spaces again, and let

\[
X = \prod_{j=1}^n X_j
\]

be their Cartesian product, equipped with the product topology. Also let \( K_j \) be a sequentially compact subset of \( X_j \) for each \( j = 1, \ldots, n \), and put

\[
K = \prod_{j=1}^n K_j.
\]

We would like to show that \( K \) is sequentially compact in \( X \). To do this, let \( \{x(l)\}_{l=1}^\infty \) be a sequence of elements of \( K \). Thus

\[
x(l) = (x_1(l), \ldots, x_n(l))
\]

for each positive integer \( l \), where \( x_j(l) \in K_j \) for \( j = 1, \ldots, n \). In particular, \( \{x_1(l)\}_{l=1}^\infty \) is a sequence of elements of \( K_1 \). Because \( K_1 \) is sequentially compact, there is a subsequence \( \{x_1(l_m)\}_{m=1}^\infty \) of \( \{x_1(l)\}_{l=1}^\infty \) that converges to an element \( x_1 \) of \( K_1 \) in \( X_1 \). Using the same sequence of indices \( \{l_m\}_{m=1}^\infty \), we get a subsequence \( \{x(l_m)\}_{m=1}^\infty \) of \( \{x(l)\}_{l=1}^\infty \).

Of course, the statement is trivial when \( n = 1 \), and so we may as well suppose that \( n \geq 2 \). As before, \( \{x_2(l_m)\}_{m=1}^\infty \) is a sequence of elements of \( K_2 \), and so there is a subsequence \( \{x_2(l_{m_r})\}_{r=1}^\infty \) of \( \{x_2(l_m)\}_{m=1}^\infty \) that converges to an element \( x_2 \) of \( K_2 \), because \( K_2 \) is sequentially compact. Using the same sequence \( \{m_r\}_{r=1}^\infty \) of indices, we get a subsequence \( \{x(l_{m_r})\}_{r=1}^\infty \) of \( \{x(l_m)\}_{m=1}^\infty \). In particular, \( \{x_1(l_{m_r})\}_{r=1}^\infty \) is a subsequence of \( \{x_1(l_m)\}_{m=1}^\infty \). This implies that \( \{x_1(l_{m_r})\}_{r=1}^\infty \) converges to \( x_1 \) in \( X_1 \), because \( \{x_1(l_{m_r})\}_{r=1}^\infty \) converges to \( x_1 \) in \( X_1 \).

Note that \( \{x(l_{m_r})\}_{r=1}^\infty \) may be considered as a subsequence of the initial sequence \( \{x(l)\}_{l=1}^\infty \) as well. If \( n = 2 \), then we get that \( \{x(l_{m_r})\}_{r=1}^\infty \) converges to

\[
x = (x_1, x_2) \in K_1 \times K_2 = K
\]

in \( X \), as desired. Otherwise, we can repeat the process.

More precisely, suppose that the \( j \)th subsequence of \( \{x(l)\}_{l=1}^\infty \) has been chosen in this way for some positive integer \( j < n \). We would like to choose the
(j + 1)th subsequence of \( \{x(l)\}_{l=1}^{\infty} \) to be a subsequence of the jth subsequence, using the sequential compactness of \( K_{j+1} \), as before. The (j + 1)th subsequence is chosen so that the sequence of (j + 1)th coordinates in \( K_{j+1} \) of the terms of the (j + 1)th subsequence converges to an element \( x_{j+1} \) of \( K_{j+1} \) in \( X_{j+1} \). Observe that the (j + 1)th subsequence is a subsequence of the ith subsequence for each \( i = 1, \ldots, j \). This implies that the sequence of ith coordinates of the terms of the (j + 1)th subsequence converge to \( x_j \) in \( X_j \), because the sequence of ith coordinates of the terms of the ith subsequence converge to \( x_i \) in \( X_i \), by construction.

We can continue in this way until the nth subsequence of \( \{x(l)\}_{l=1}^{\infty} \) is chosen. As before, for each \( i = 1, \ldots, n \), the sequence of ith coordinates of the nth subsequence converges to \( x_i \) in \( X_i \). This implies that the nth subsequence converges to

\[(12.5) \quad x = (x_1, \ldots, x_n) \in K\]

in \( X \). We also have that the nth subsequence is a subsequence of the initial sequence \( \{x(l)\}_{l=1}^{\infty} \), as desired.

### 13 Sequential compactness and countable products

If \( \{l_m\}_{m=1}^{\infty} \) is a strictly increasing sequence of integers, then it is easy to see that

\[(13.1) \quad l_m \geq m \]

for every positive integer \( m \).

Let \( X_1, X_2, X_3, \ldots \) be an infinite sequence of topological spaces, and consider their Cartesian product

\[(13.2) \quad X = \prod_{j=1}^{\infty} X_j,\]

equipped with the product topology. Also let \( K_j \) be a sequentially compact subset of \( X_j \) for each positive integer \( j \), and put

\[(13.3) \quad K = \prod_{j=1}^{\infty} K_j.\]

We want to show that \( K \) is sequentially compact in \( X \).

Let a sequence \( \{x(l)\}_{l=1}^{\infty} \) of elements of \( K \) be given. Note that

\[(13.4) \quad x(l) = \{x_j(l)\}_{j=1}^{\infty}\]

is a sequence for each \( l \), where \( x_j(l) \in K_j \) for every \( j \geq 1 \). In particular, \( \{x_1(l)\}_{l=1}^{\infty} \) is a sequence of elements of \( K_1 \), as before. Hence there is a subsequence \( \{x_1(l_m)\}_{m=1}^{\infty} \) of \( \{x_1(l)\}_{l=1}^{\infty} \) that converges to an element \( x_1 \) of \( K_1 \) in \( X_1 \), because \( K_1 \) is sequentially compact. We can use the same sequence \( \{l_m\}_{m=1}^{\infty} \) of indices to get a subsequence \( \{x(l_m)\}_{m=1}^{\infty} \) of \( \{x(l)\}_{l=1}^{\infty} \).
We can repeat the process, as in the previous section, to get an infinite sequence of subsequences of \( \{x(l)\}_{l=1}^{\infty} \). More precisely, for each positive integer \( j \), we can get a sequence \( \{x(r,j)\}_{r=1}^{\infty} \) of elements of \( K \) with the following two properties. First,

\[
\{x(r,1)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(l)\}_{l=1}^{\infty},
\]

and

\[
\{x(r,j)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(m,j-1)\}_{m=1}^{\infty}
\]

when \( j \geq 2 \). This implies that

\[
\{x(r,j)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(l)\}_{l=1}^{\infty}
\]

for every \( j \geq 1 \), and that

\[
\{x(r,j)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(m,i)\}_{m=1}^{\infty}
\]

when \( 1 \leq i \leq j \). The second property is that for every positive integer \( j \) there is an element \( x_j \) of \( K_j \) such that

\[
x_j \in \{x(r,j)\}_{r=1}^{\infty} \text{ converges to } x_j \text{ in } X_j.
\]

It follows that

\[
\{x_i(r,j)\}_{r=1}^{\infty} \text{ converges to } x_i \text{ in } X_i
\]

when \( 1 \leq i \leq j \), because of (13.8). Put

\[
x = \{x_j\}_{j=1}^{\infty},
\]

which defines an element of \( K \).

We would like to find a subsequence of \( \{x(l)\}_{l=1}^{\infty} \) that converges to \( x \) in \( X \). Put

\[
y(r) = x(r,r)
\]

for each positive integer \( r \), which is the \( r \)th term of the \( r \)th subsequence described in the previous paragraph. One can check that

\[
\{y(r)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(l)\}_{l=1}^{\infty}.
\]

Indeed, for each \( r \geq 1 \), \( y(r) \) is one of the terms of \( \{x(l)\}_{l=1}^{\infty} \), by construction. One can verify that \( y(r+1) \) is chosen among the terms of \( \{x(l)\}_{l=1}^{\infty} \) that occurs after the one corresponding to \( y(r) \), using (13.1). More precisely, \( y(r+1) \) is the \((r+1)\)th term of the \((r+1)\)th subsequence, and the \((r+1)\)th subsequence is a subsequence of the \( r \)th subsequence, by construction. This implies that \( y(r+1) \) occurs in the \( r \)th subsequence after \( y(r) \), as in (13.1). It follows that \( y(r+1) \) occurs after \( y(r) \) in the previous subsequences, including the initial sequence \( \{x(l)\}_{l=1}^{\infty} \).

Similarly, one can check that for each positive integer \( j \),

\[
\{y(r,j)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(m,j)\}_{m=1}^{\infty}.
\]
In particular, if \( r \geq j \), then \( y(r) \) is one of the terms of \( \{ x(m, j) \}_{m=1}^\infty \), because the \( r \)th subsequence is a subsequence of the \( j \)th subsequence, as in (13.8). We also have that \( y(r+1) \) is chosen among the terms of \( \{ x(m, j) \}_{m=1}^\infty \) after the one corresponding to \( y(r) \) when \( r \geq j \), as in the preceding paragraph.

It follows that

\[
\{ y(r) \}_{r=j}^\infty \text{ converges to } x_j \text{ in } X_j
\]

for each positive integer \( j \), because of (13.9) and (13.14). This implies that

\[
\{ y(r) \}_{r=1}^\infty \text{ converges to } x_j \text{ in } X_j
\]

for every \( j \geq 1 \). This means that \( \{ y(r) \}_{r=1}^\infty \) converges to \( x \) with respect to the product topology on \( X \), as desired.

### 14 Bases and finite products

Let \( X_1, \ldots, X_n \) be finitely many topological spaces, and let \( X = \prod_{j=1}^n X_j \) be their Cartesian product, equipped with the product topology. Also let

\[
x = (x_1, \ldots, x_n) \in X
\]

be given. Suppose that for each \( j = 1, \ldots, n \), \( B_j(x_j) \) is a local base for the topology of \( X_j \) at \( x_j \). Put

\[
B(x) = \left\{ \prod_{j=1}^n U_j : U_j \in B_j(x_j) \text{ for each } j = 1, \ldots, n \right\}
\]

It is easy to see that this is a local base for the product topology on \( X \) at \( x \).

Suppose that for each \( j = 1, \ldots, n \), \( B_j(x_j) \) has only finitely or countably many elements. We would like to verify that (14.2) has only finitely or countably many elements as well. In this case,

\[
\prod_{j=1}^n B_j(x_j)
\]

has only finitely or countably many elements, by standard results. There is an obvious mapping from (14.3) onto (14.2), which sends \( (U_1, \ldots, U_n) \) in (14.3) to \( \prod_{j=1}^n U_j \) in (14.2). One can use this to show that (14.2) has only finitely or countably many elements, because of the analogous property for (14.3).

Alternatively, suppose again that for each \( j = 1, \ldots, n \), there is a local base for the topology of \( X_j \) at \( x_j \) with only finitely or countably many elements. This means that for each \( j = 1, \ldots, n \) there is a sequence \( \{ U_{j,l}(x_j) \} \) of open subsets of \( X_j \) such that \( x_j \in U_{j,l}(x_j) \) for every \( l \geq 1 \), and the collection of \( U_{j,l}(x_j) \)'s, \( l \geq 1 \), is a local base for the topology of \( X_j \) at \( x_j \). We may also ask that

\[
U_{j,l+1}(x_j) \subseteq U_{j,l}(x_j)
\]
for each \( j = 1, \ldots, n \) and \( l \geq 1 \), since otherwise we can replace \( U_{j,l}(x_j) \) with \( \bigcap_{k=1}^k U_{j,k}(x_j) \) for every \( j = 1, \ldots, n \) and \( l \geq 1 \). Put

\[
U_l(x) = \prod_{j=1}^n U_{j,l}(x_j)
\]

(14.5)

for each \( l \in \mathbb{Z}_+ \), so that \( U_l(x) \) is an open subset of \( X \) with respect to the product topology, and \( x \in U_l(x) \). One can check that the collection of \( U_l(x) \)'s, \( l \in \mathbb{Z}_+ \), is a local base for the product topology on \( X \) at \( x \).

Similarly, let \( B_j \) be a base for the topology of \( X_j \) for each \( j = 1, \ldots, n \). It is easy to see that

\[
B = \left\{ \prod_{j=1}^n U_j : U_j \in B_j \text{ for each } j = 1, \ldots, n \right\}
\]

(14.6)

is a base for the product topology on \( X \). If \( B_j \) has only finitely or countably many elements for each \( j = 1, \ldots, n \), then (14.6) has only finitely or countably many elements as well. More precisely,

\[
\prod_{j=1}^n B_j
\]

has only finitely or countably many elements in this situation, as before. We can map (14.7) onto (14.6), by sending \((U_1, \ldots, U_n)\) in (14.7) to \( \prod_{j=1}^n U_j \) in (14.6). One can use this and the fact that (14.7) has only finitely or countably many elements to get that (14.6) has only finitely or countably many elements, as before.

15 Bases and countable products

Let \( X_1, X_2, X_3, \ldots \) be an infinite sequence of topological spaces, and let \( X = \prod_{j=1}^\infty X_j \) be their Cartesian product, equipped with the product topology. Also let \( x = (x_j)_{j=1}^\infty \in X \) be given, and let \( B_j(x_j) \) be a local base for the topology of \( X_j \) at \( x_j \) for each \( j \in \mathbb{Z}_+ \). If \( n \) is a positive integer, then put

\[
B^n(x) = \left\{ \prod_{j=1}^n U_j : U_j \in B_j(x_j) \text{ for each } j = 1, \ldots, n, \right. \\
\text{and } U_j = X_j \text{ when } j > n \right\}.
\]

(15.1)

Note that every element of \( B^n(x) \) is an open subset of \( X \) with respect to the product topology. One can check that

\[
B(x) = \bigcup_{n=1}^\infty B^n(x)
\]

(15.2)
is a local base for the product topology on $X$ at $x$.

Suppose that for every $j \in \mathbb{Z}_+$, $\mathcal{B}_j(x_j)$ has only finitely or countably many elements. This implies that
\begin{equation}
\prod_{j=1}^{n} \mathcal{B}_j(x_j)
\end{equation}
has only finitely or countably many elements for each $n \in \mathbb{Z}_+$, by standard arguments, as before. Let $(U_1, \ldots, U_n)$ be an element of (15.3), and put $U_j = X_j$ when $j > n$. Under these conditions, $\prod_{j=1}^\infty U_j$ is an element of (15.1). This defines a mapping from (15.3) onto (15.1). One can use this and the fact that (15.3) has only finitely or countably many elements to obtain that (15.1) has only finitely or countably many elements for each $n \in \mathbb{Z}_+$. It follows that (15.2) has only finitely or countably many elements as well.

Alternatively, suppose again that for each positive integer $j$, there is a local base for the topology of $X_j$ at $x_j$ with only finitely or countably many elements. This means that for each $j \in \mathbb{Z}_+$ there is a sequence $\{U_{j,l}(x_j)\}_{l=1}^\infty$ of open subsets of $X_j$ such that $x_j \in U_{j,l}(x_j)$ for every $l \geq 1$, and the collection of $U_{j,l}(x_j)$'s, $l \geq 1$, is a local base for the topology of $X_j$ at $x_j$. We may also ask that $U_{j,l+1}(x_j) \subseteq U_{j,l}(x_j)$ for every $j, l \geq 1$, as before. If $n \in \mathbb{Z}_+$, then put
\begin{equation}
V_{j,n}(x_j) = U_{j,n}(x_j) \quad \text{for } j = 1, \ldots, n
\end{equation}
\begin{equation}
X_j \quad \text{when } j > n.
\end{equation}
Using this, we put
\begin{equation}
V_n(x) = \prod_{j=1}^{\infty} V_{j,n}(x_j).
\end{equation}
This is an open subset of $X$ with respect to the product topology, and $x \in V_n(x)$. One can check that the collection of $V_n(x)$'s, $n \in \mathbb{Z}_+$, is a local base for the product topology on $X$ at $x$.

Now let $\mathcal{B}_j$ be a base for the topology of $X_j$ for every positive integer $j$. If $n \in \mathbb{Z}_+$, then put
\begin{equation}
\mathcal{B}_n = \left\{ \prod_{j=1}^{\infty} U_j : U_j \in \mathcal{B}_j \text{ for each } j \in \mathbb{Z}_+ight. \\
\left. \quad \text{and } U_j = X_j \text{ when } j > n \right\}.
\end{equation}
Every element of $\mathcal{B}_n$ is an open subset of $X$ with respect to the product topology, by construction. It is not difficult to verify that
\begin{equation}
\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n
\end{equation}
is a base for the product topology on $X$. Suppose that $\mathcal{B}_j$ has only finitely or countably many elements for each $j \in \mathbb{Z}_+$. This implies that $\prod_{j=1}^{n} \mathcal{B}_j$ has only
Finally or countably many elements for each positive integer \(n\), as usual. One can use this to check that (15.6) has only finitely or countably many elements for every \(n \in \mathbb{Z}_+\), as before. This implies that (15.7) has only finitely or countably many elements too.

16 Finite products of metric spaces

Let \((X_1, d_1), \ldots, (X_n, d_n)\) be finitely many metric spaces, and put \(X = \prod_{j=1}^n X_j\). If \(x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X\), then one can check that

\[
d(x, y) = \max_{1 \leq j \leq n} d_j(x_j, y_j)
\]

defines a metric on \(X\). Let

\[
B_j(x_j, r) = \{ w_j \in X_j : d_j(x_j, w_j) < r \}
\]

be the open ball in \(X_j\) centered at \(x_j \in X_j\) with radius \(r > 0\) with respect to \(d_j(\cdot, \cdot)\) for each \(j = 1, \ldots, n\), and let

\[
B(x, r) = \{ w \in X : d(x, w) < r \}
\]

be the open ball in \(X\) centered at \(x \in X\) with radius \(r > 0\) with respect to \(d(\cdot, \cdot)\). It is easy to see that

\[
B(x, r) = \prod_{j=1}^n B_j(x_j, r)
\]

for every \(x \in X\) and \(r > 0\), directly from the definitions. More precisely, \(w \in X\) satisfies \(d(x, w) < r\) if and only if \(d_j(x_j, w_j) < r\) for each \(j = 1, \ldots, n\).

Let us use the term “product topology” to refer to the product topology on \(X\) corresponding to the topologies determined on \(X_1, \ldots, X_n\) by the metrics \(d_1, \ldots, d_n\), respectively. It is well known and not difficult to show that this is the same as the topology determined on \(X\) by the metric (16.1). Indeed, it is well known that an open ball in a metric space is an open set with respect to the topology determined by the metric. This implies that (16.4) is an open set in \(X\) with respect to the product topology for every \(x \in X\) and \(r > 0\). One can use this to check that every open set in \(X\) with respect to the topology determined by the metric (16.1) is also an open set with respect to the product topology. If \(W \subseteq X\) is an open set with respect to the product topology, then one can verify that \(W\) is an open set with respect to the topology determined by the metric (16.1). More precisely, if \(x \in W\), then one can find an \(r > 0\) such that (16.4) is contained in \(W\).

It is easy to see that

\[
\tilde{d}(x, y) = \sum_{j=1}^n d_j(x_j, y_j)
\]
defines a metric on $X$ as well. Observe that
\begin{equation}
(16.6) \quad d(x, y) \leq d(x, y) \leq n d(x, y)
\end{equation}
for every $x, y \in X$. Using this, one can check that the topologies determined on $X$ by (16.1) and (16.5) are the same.

Similarly, put
\begin{equation}
(16.7) \quad d(x, y) = \left( \sum_{j=1}^{n} d_{j}(x_{j}, y_{j})^{2} \right)^{1/2}
\end{equation}
for each $x, y \in X$, using the nonnegative square root on the right side. One can check that this satisfies the triangle inequality on $X$, using the triangle inequality for the standard Euclidean norm on $\mathbb{R}^{n}$. Using this, it is easy to see that (16.7) is a metric on $X$. One can also verify that
\begin{equation}
(16.8) \quad d(x, y) \leq d(x, y) \leq n^{1/2} d(x, y)
\end{equation}
for every $x, y \in X$. This implies that the topologies determined on $X$ by (16.1) and (16.7) are the same, as before.

## 17 Truncating metrics

Let $(X, d(x, y))$ be a metric space, and let $t$ be a positive real number. If $x, y \in X$, then put
\begin{equation}
(17.1) \quad d_{t}(x, y) = \min(d(x, y), t).
\end{equation}
One can check that this defines a metric on $X$.

If $x \in X$ and $r$ is a positive real number, then let
\begin{equation}
(17.2) \quad B_{d}(x, r) = \{ w \in X : d(x, w) < r \}
\end{equation}
be the open ball in $X$ centered at $x$ with radius $r$ with respect to $d(\cdot, \cdot)$. Similarly, let
\begin{equation}
(17.3) \quad B_{d_{t}}(x, r) = \{ w \in X : d_{t}(x, w) < r \}
\end{equation}
be the open ball in $X$ centered at $x$ with radius $r$ with respect to (17.1). Observe that
\begin{equation}
(17.4) \quad B_{d_{t}}(x, r) = \begin{cases} B_{d}(x, r) & \text{when } r \leq t, \\ X & \text{when } r > t. \end{cases}
\end{equation}
Using this, one can check that the topologies determined on $X$ by $d(\cdot, \cdot)$ and (17.1) are the same.
18 Countable products of metric spaces

Let \((X_j, d_j)\) be a metric space for each positive integer \(j\), and put \(X = \prod_{j=1}^{\infty} X_j\).

Put
\[
d'_j(x_j, y_j) = \min(d_j(x_j, y_j), 1/j)
\]
for each \(j \in \mathbb{Z}_+\) and \(x_j, y_j \in X_j\). As in the previous section, for every \(j \in \mathbb{Z}_+\), (18.1) defines a metric on \(X_j\) that determines the same topology on \(X_j\) as \(d_j(\cdot, \cdot)\).

If \(x = \{x_j\}_{j=1}^{\infty}, y = \{y_j\}_{j=1}^{\infty} \in X\), then put
\[
d'(x, y) = \max_{j \in \mathbb{Z}_+} d'_j(x_j, y_j).
\]

More precisely, this is equal to 0 when \(x = y\). If \(x \neq y\), then there is a positive integer \(j_0\) such that \(x_{j_0} \neq y_{j_0}\). This implies that \(d'_{j_0}(x_{j_0}, y_{j_0}) > 0\), so that
\[
d'_j(x_j, y_j) > 0, \quad \text{for all but finitely many } j \in \mathbb{Z}_+.
\]
This means that the right side of (18.2) reduces to the maximum of finitely many terms, and in particular that the maximum is attained.

One can check that (18.2) defines a metric on \(X\). If \(j \in \mathbb{Z}_+, x_j \in X_j\), and \(r\) is a positive real number, then let
\[
B_j(x_j, r) = \{w_j \in X_j : d_j(x_j, w_j) < r\}
\]
and
\[
B'_j(x_j, r) = \{w_j \in X_j : d'_j(x_j, w_j) < r\}
\]
be the open balls in \(X_j\) centered at \(x_j\) with radius \(r\) with respect to \(d_j(\cdot, \cdot)\) and (18.1), respectively. Thus
\[
B'_j(x_j, r) = \begin{cases} B_j(x_j, r) \quad &\text{when } r \leq 1/j \\ X_j \quad &\text{when } r > 1/j \end{cases}
\]
as in (17.4). Also let
\[
B(x, r) = \{w \in X : d(x, w) < r\}
\]
be the open ball in \(X\) centered at \(x \in X\) with radius \(r > 0\) with respect to (18.2).

One can verify that
\[
B(x, r) = \prod_{j=1}^{\infty} B'_j(x_j, r)
\]
for every \(x \in X\) and \(r > 0\). This is the same as saying that \(w \in X\) satisfies \(d(x, w) < r\) if and only if \(d'_j(x_j, w_j) < r\) for every \(j \in \mathbb{Z}_+\). This uses the fact that the maximum is attained on the right side of (18.2).

Let us use the term “product topology” to refer to the product topology on \(X\) corresponding to the topology determined on \(X_j\) by \(d_j(\cdot, \cdot)\) for each \(j \in \mathbb{Z}_+\).
Observe that (18.8) is an open set in $X$ with respect to the product topology for every $x \in X$ and $r > 0$, because of (18.6). One can use this to check that every open set in $X$ with respect to (18.2) is an open set with respect to the product topology. If $W$ is an open subset of $X$ with respect to the product topology and $x \in W$, then it is not too difficult to find an $r > 0$ such that (18.8) is contained in $W$, so that $W$ is an open set with respect to the topology determined by (18.2). Thus the product topology on $X$ is the same as the topology determined on $X$ by (18.2).

19 Continuous real-valued functions

Let $X$ be a nonempty topological space, and let $C(X)$ be the space of continuous real-valued functions on $X$. This uses the standard topology on the real line, as the range of the functions on $X$. Of course, constant functions on $X$ are continuous.

If $f, g \in C(X)$, then $f + g, fg \in C(X)$ too. This can be shown using the same type of arguments as for continuous real-valued functions on the real line, or on a metric space. Alternatively, it is easy to see that

$$x \mapsto (f(x), g(x))$$

is continuous as a mapping from $X$ into $\mathbb{R}^2$, using the product topology on $\mathbb{R}^2$ corresponding to the standard topology on $\mathbb{R}$. One can also show that addition and multiplication on $\mathbb{R}$ are continuous as mappings from $\mathbb{R}^2$ into $\mathbb{R}$, using standard arguments. To get the continuity of $f + g$ and $fg$, one can consider these functions as the compositions of (19.1) with the mappings from $\mathbb{R}^2$ into $\mathbb{R}$ that correspond to addition and multiplication of real numbers, respectively.

Similarly, if $f \in C(X)$ and $f(x) \neq 0$ for each $x \in X$, then $1/f \in C(X)$. This can be obtained using the same type of arguments as for functions on the real line or a metric space, or by considering $1/f$ as the composition of $f$ with the mapping $t \mapsto 1/t$ from $\mathbb{R} \setminus \{0\}$ into itself.

Let us say that $C(X)$ separates points in $X$ if for every $x, y \in X$ with $x \neq y$ there is an $f \in C(X)$ such that

$$f(x) \neq f(y).$$

In this case, $X$ is said to be a Urysohn space. One can check that Urysohn spaces are (completely) Hausdorff, because the real line is (completely) Hausdorff with respect to the standard topology.

Let $(X, d(x, y))$ be a metric space. If $p \in X$, then one can verify that

$$f_p(x) = d(p, x)$$

is continuous on $X$, with respect to the topology determined on $X$ by $d(\cdot, \cdot)$. This implies that $X$ is a Urysohn space.

If $f$ and $g$ are continuous real-valued functions on a topological space $X$ again, then it is not difficult to show that $\max(f(x), g(x))$ and $\min(f(x), g(x))$
are continuous on \( X \) as well, directly from the definitions. Alternatively, one can check that the maximum and minimum of two real numbers define continuous mappings from \( \mathbb{R}^2 \) into \( \mathbb{R} \). The continuity of the maximum and minimum of \( f \) and \( g \) can be obtained from this and the continuity of (19.1) as a mapping from \( X \) into \( \mathbb{R}^2 \), as before. In particular, if \( a \) and \( b \) are real numbers, then \( \max(f(x), a) \) and \( \min(f(x), b) \) are continuous on \( X \). This can also be seen using the continuity of \( \max(t, a) \) and \( \min(t, b) \) as functions of \( t \in \mathbb{R} \), by composing \( f \) with these functions.

Let \( A \) and \( B \) be disjoint subsets of \( X \). A continuous real-valued function \( f \) on \( X \) is said to be a Urysohn function for \( A \) and \( B \) if \( f(x) = 0 \) for every \( x \in A \), \( f(y) = 1 \) for every \( y \in B \), and \( 0 \leq f(w) \leq 1 \) for every \( w \in X \). The third condition can always be arranged by replacing \( f \) with \( \min(\max(f, 0), 1) \), as in the preceding paragraph. If \( f \) is a Urysohn function on \( X \) for \( A \) and \( B \), then \( 1 - f \) is a Urysohn function on \( X \) for \( B \) and \( A \). It is easy to see that \( X \) is a Urysohn space if and only if every pair \( A, B \) of disjoint subsets of \( X \) with only one element each has a Urysohn function.

20 Regularity and normality

Let us say that a topological space \( X \) is regular in the strict sense if for every \( x \in X \) and closed set \( E \subseteq X \) with \( x \notin E \), there are disjoint open subsets \( U, V \) of \( X \) such that \( x \in U \) and \( E \subseteq V \). If \( X \) also satisfies the first (or even zeroth) separation condition, then we say that \( X \) is regular in the strong sense. It is easy to see that \( X \) is Hausdorff in this case. Sometimes one says that \( X \) is regular when \( X \) is regular in the strict sense, and that \( X \) satisfies the third separation condition, or equivalently that \( X \) is a \( T_3 \) space, when \( X \) is regular in the strong sense. However, these terms are sometimes used in the other way.

Similarly, let us say that \( X \) is normal in the strict sense if for every pair \( A, B \) of disjoint closed subsets of \( X \) there are disjoint open subsets \( U, V \) of \( X \) such that \( A \subseteq U \), \( B \subseteq V \). If \( X \) also satisfies the first separation condition, then we say that \( X \) is normal in the strong sense. This implies that \( X \) is regular in the strong sense, and in particular that \( X \) is Hausdorff. As before, normal spaces in the strict sense are often said to be normal, and normal spaces in the strong sense are often said to satisfy the fourth separation condition, or equivalently be \( T_4 \) spaces, but these terms are sometimes used in the other way.

It is well known and not difficult to show that \( X \) is regular in the strict sense if and only if for every \( x \in X \) and open set \( W \subseteq X \) with \( x \in W \) there is an open set \( U \subseteq X \) such that \( x \in U \) and the closure \( \overline{U} \) of \( U \) in \( X \) is contained in \( W \). Similarly, \( X \) is normal in the strict sense if and only if for every closed set \( A \subseteq X \) and open set \( W \subseteq X \) such that \( A \subseteq W \) there is an open set \( U \subseteq X \) such that \( A \subseteq U \) and \( \overline{U} \subseteq W \).

Suppose that \( X \) is normal in the strict sense, and that \( A, B \) are disjoint closed subsets of \( X \). Under these conditions, Urysohn's lemma states that there is a Urysohn function on \( X \) for \( A \) and \( B \). If \( X \) is normal in the strong sense, then it follows that \( X \) is a Urysohn space.
Of course, metric spaces are normal in the strong sense. It is well known that the conclusion of Urysohn’s lemma can be obtained more directly using the metric in this case.

Suppose that $X$ is a Urysohn space, and let $x, y$ be distinct elements of $X$. Thus there is a Urysohn function $f$ on $X$ for $\{x\}$ and $\{y\}$, as in the previous section. Under these conditions,

$$2 \min(f, 1/2)$$

is a Urysohn function on $X$ for $\{x\}$ and

$$\{z \in X : f(z) \geq 1/2\}.$$

Note that

$$\{z \in X : f(z) > 1/2\}$$

is an open subset of $X$ that contains $y$ and is contained in (20.2).

Let $B$ be a compact subset of $X$, and suppose that $x \in X \setminus B$. One can get a Urysohn function on $X$ for $\{x\}$ and $B$, as follows. If $y \in B$, then there is a Urysohn function on $X$ for $\{x\}$ and a neighborhood of $y$ in $X$, as in the preceding paragraph. Because $B$ is compact, $B$ can be covered by finitely many such neighborhoods of its elements. The maximum of the corresponding Urysohn functions for $\{x\}$ and these finitely many neighborhoods of elements of $B$ is a Urysohn function for $\{x\}$ and $B$.

If $A$ and $B$ are disjoint compact subsets of $X$, then one can get a Urysohn function on $X$ for $A$ and $B$ using analogous arguments. More precisely, for each $x \in A$, one can get a Urysohn function for $\{x\}$ and $B$, as before. One can use this to get a Urysohn function on $X$ for a neighborhood of $x$ and $B$. Because $A$ is compact, $A$ can be covered by finitely many such neighborhoods of its elements. The minimum of the corresponding Urysohn functions is a Urysohn function on $X$ for $A$ and $B$.

### 21 Complete regularity

Let us say that a topological space $X$ is completely regular in the strict sense if for every $x \in X$ and closed set $E \subseteq X$ with $x \notin E$ there is an $f \in C(X)$ such that $f(x) \neq 0$ and $f(y) = 0$ for every $y \in E$. In this case, it is easy to modify $f$ a bit, if necessary, to get that $f$ is a Urysohn function for $E$ and $\{x\}$. If $X$ also satisfies the first (or zeroth) separation condition, then we say that $X$ is completely regular in the strong sense. If $X$ is completely regular in the strict sense, then it is easy to see that $X$ is regular in the strict sense, because the real line is Hausdorff with respect to the standard topology. If $X$ is completely regular in the strong sense, then it follows that $X$ is regular in the strong sense. We also have that $X$ is a Urysohn space in this situation. If $X$ is normal in the strong sense, then $X$ is completely regular in the strong sense, by Urysohn’s lemma. As usual, completely regular spaces in the strict sense are sometimes said to be completely regular, and completely regular spaces in the strong sense
may be said to satisfy separation condition three and a half, but these terms may be used the other way.

Let $X$ be a set, and suppose that $\tau_1$ and $\tau_2$ are topologies on $X$, with $\tau_1 \subseteq \tau_2$. If a real-valued function $f$ on $X$ is continuous with respect to $\tau_1$, then it follows that $f$ is continuous with respect to $\tau_2$ as well. If $(X, \tau_1)$ is a Urysohn space, then $(X, \tau_2)$ is a Urysohn space too. This type of argument does not work for complete regularity.

Let $X$ be a topological space again, and let $Y$ be a subset of $X$, equipped with the induced topology. If $f$ is a continuous real-valued function on $X$, then the restriction of $f$ to $Y$ is continuous. If $(X, \tau_1)$ is a Urysohn space, then it is easy to see that $Y$ is a Urysohn space. Similarly, if $X$ is completely regular in the strict sense, then one can check that $Y$ is completely regular in the strict sense too, with respect to the induced topology. In particular, if $X$ is normal in the strong sense, then $X$ is completely regular in the strong sense, by Urysohn’s lemma, and hence $Y$ is completely regular in the strong sense.

Suppose that $X$ is completely regular in the strict sense, and let $x \in X$ and a closed set $E \subseteq X$ be given, with $x \notin E$. Thus there is a Urysohn function on $X$ for $E$ and $\{x\}$, as before. One can use this to get a Urysohn function on $X$ for $E$ and a neighborhood of $x$ in $X$, as in the previous section.

Suppose now that $E \subseteq X$ is a closed set, $K \subseteq X$ is compact, and $E \cap K = \emptyset$. If $x \in K$, then there is a Urysohn function on $X$ for $E$ and a neighborhood of $x$ in $X$, as in the preceding paragraph. Because $K$ is compact, $K$ can be covered by finitely many such neighborhoods of its elements. The maximum of the corresponding Urysohn functions for $E$ and these finitely many neighborhoods of elements of $K$ is a Urysohn function for $E$ and $K$.

## 22 Local compactness and topological manifolds

A topological space $X$ is said to be *locally compact* if for every $x \in X$ there is an open set $U \subseteq X$ and a compact set $K \subseteq X$ such that $x \in U$ and $U \subseteq K$. If $X$ is Hausdorff, then $K$ is a closed set in $X$, and it follows that the closure $\overline{U}$ of $U$ in $X$ is contained in $K$. This implies that $\overline{U}$ is compact in $X$, because it is a closed set contained in a compact set. Sometimes local compactness is defined by asking that $\overline{U}$ be compact. Note that $\mathbb{R}^n$ is locally compact with respect to the standard topology for each positive integer $n$.

Suppose that $X$ is a locally compact Hausdorff space. It is not too difficult to show that $X$ is regular as a topological space. More precisely, one can show that $X$ is completely regular. This can be reduced to Urysohn's lemma, or obtained using similar arguments.

Let $X$ be a topological space, and let $n$ be a positive integer. We say that $X$ is *locally Euclidean of dimension $n$* if for every $x \in X$ there is an open set $U \subseteq X$ such that $x \in U$ and $U$ is homeomorphic to an open subset $W$ of $\mathbb{R}^n$. More precisely, this uses the standard topology on $\mathbb{R}^n$, and the appropriate induced topologies on $U$ and $W$.

Suppose that $X$ is locally Euclidean of dimension $n$, and let us check that $X$
satisfies the first separation condition. Let \( x \) and \( y \) be distinct elements of \( X \), so that we would like to find an open subset of \( X \) that contains \( x \) and not \( y \). By hypothesis, there is an open set \( U \subseteq X \) such that \( x \in U \) and \( U \) is homeomorphic to an open subset of \( \mathbb{R}^n \). If \( y \notin U \), then we can take \( U \) to be the open set that we want. Otherwise, if \( y \in U \), we can use the fact that \( \mathbb{R}^n \) satisfies the first separation condition to find an open set that contains \( x \) and not \( y \).

Let \( x \in X \) be given again, and let \( U \subseteq X \) be an open set such that \( x \in U \) and \( U \) is homeomorphic to an open subset \( W \) of \( \mathbb{R}^n \). If \( K \subseteq \mathbb{R}^n \) is compact and \( K \subseteq W \), then \( K \) is compact as a subset of \( W \), with respect to the induced topology. This means that \( K \) corresponds to a compact subset of \( U \), with respect to the induced topology. It follows that the subset of \( U \) corresponding to \( K \) is compact as a subset of \( X \) as well. In particular, one can use this to check that \( X \) is locally compact.

However, \( X \) may not be Hausdorff, and this is often included as an additional condition. In this case, one can use the regularity of \( \mathbb{R}^n \) to get that \( X \) is regular, a bit more directly than for arbitrary locally compact Hausdorff spaces. Similarly, complete regularity of \( X \) can be obtained more directly from the complete regularity of \( \mathbb{R}^n \).

An \( n \)-dimensional topological manifold is often defined as a Hausdorff topological space \( X \) that is locally Euclidean of dimension \( n \) and satisfies the second countability condition.

### 23 \( \sigma \)-Compactness

Let \( X \) be a topological space. It is easy to see that the union of finitely many compact subsets of \( X \) is compact as well. Similarly, the union of finitely or countably many subsets of \( X \) with the Lindelöf property has the Lindelöf property too. A subset \( E \) of \( X \) is said to be \( \sigma \)-compact if there is a sequence \( K_1, K_2, K_3, \ldots \) of compact subsets of \( X \) such that

\[
E = \bigcup_{j=1}^{\infty} K_j.
\]

This implies that \( E \) has the Lindelöf property, as before.

Suppose that \( X \) is locally compact, so that \( X \) can be covered by open sets that are contained in compact sets. If \( X \) also has the Lindelöf property, then it follows that \( X \) can be covered by finitely or countably many open sets, each of which is contained in a compact set. In particular, this means that \( X \) is \( \sigma \)-compact. Remember that \( X \) has the Lindelöf property when \( X \) satisfies the second countability condition, by Lindelöf’s theorem. It follows that topological manifolds are \( \sigma \)-compact, for instance.

Let \( \{U_\alpha\}_{\alpha \in A} \) be an open covering of \( X \), and suppose that for each \( \alpha \in A \), \( B_\alpha \) is a base for the topology induced on \( U_\alpha \) by the topology on \( X \). Note that the elements of \( B_\alpha \) are open subsets of \( X \) for each \( \alpha \in A \), because the \( U_\alpha \)’s are
open subsets of $X$. Under these conditions, one can check that

$$B = \bigcup_{\alpha \in A} \mathcal{B}_\alpha$$

is a base for the topology of $X$. Of course, if $A$ has only finitely or countably many elements, and if $\mathcal{B}_\alpha$ has only finitely or countably many elements for each $\alpha \in A$, then (23.2) has only finitely or countably many elements. If $X$ has the Lindelöf property, then one can automatically reduce to the case where $A$ has only finitely or countably many elements.

Suppose that $X$ is locally Euclidean of dimension $n$ for some positive integer $n$. Thus $X$ can be covered by open sets that are homeomorphic to open subsets of $\mathbb{R}^n$. If $X$ has the Lindelöf property, then it follows that $X$ can be covered by finitely or countably many open sets that are homeomorphic to open subsets of $\mathbb{R}^n$. Each of these open sets has a countable base for its topology, because of the analogous property for open subsets of $\mathbb{R}^n$. This leads to a countable base for the topology of $X$, as in the preceding paragraph.

One can show that the set $\mathbb{R}^n \setminus \mathbb{Q}$ of irrational numbers is not $\sigma$-compact with respect to the standard topology on $\mathbb{R}$, using the Baire category theorem. Note that every subset of the real line has the Lindelöf property, by Lindelöf's theorem.

### 24 Quotient spaces

Let $X$ be a set. A binary relation $\sim$ on $X$ is said to be an equivalence relation on $X$ if it is reflexive, symmetric, and transitive on $X$, as usual. In this case, if $x \in X$, then

$$[x] = \{y \in X : x \sim y\}$$

is called an equivalence class in $X$. This is a subset of $X$ that contains $x$, by reflexivity. If $x, w \in X$, then $[x] = [w]$ if and only if $x \sim w$, and otherwise $[x]$ and $[w]$ are disjoint.

Conversely, let $\mathcal{P}$ be a partition of $X$, which is to say a collection of pairwise-disjoint nonempty subsets of $X$ whose union is equal to $X$. If $x, y \in X$, then put $x \sim_{\mathcal{P}} y$ when $x$ and $y$ lie in the same element of $\mathcal{P}$. This defines an equivalence relation on $X$, for which the corresponding equivalence classes are the elements of $\mathcal{P}$.

If $\sim$ is any equivalence relation on $X$, then let $X/\sim$ be the corresponding collection of equivalence classes in $X$. In this situation, there is a natural quotient mapping from $X$ onto $X/\sim$, which sends $x \in X$ to $[x] \in X/\sim$.

Let $Y$ be another set, and let $f$ be a mapping from $X$ onto $Y$. If $x, w \in X$, then put $x \sim_f w$ when $f(x) = f(w)$. This defines an equivalence relation on $X$, for which the corresponding equivalence classes are the sets of the form $f^{-1}\{y\}$, $y \in Y$.

Suppose now that $X$ is a topological space. Let us say that $V \subseteq Y$ is an open set exactly when $f^{-1}(V)$ is an open set in $X$. One can check that this
defines a topology on \( Y \), which is called the quotient topology or identification topology associated to \( f \). Equivalently, this means that \( E \subseteq Y \) is a closed set if and only if \( f^{-1}(E) \) is a closed set in \( X \). Note that \( f \) is automatically continuous with respect to the quotient topology on \( Y \).

Remember that \( Y \) satisfies the first separation condition if and only if for each \( y \in Y \), \( \{y\} \) is a closed set in \( Y \). Thus \( Y \) satisfies the first separation condition with respect to the quotient topology associated to \( f \) if and only if for each \( y \in Y \), \( f^{-1}(\{y\}) \) is a closed set in \( X \).

Let \( Z \) be another topological space, and let \( g \) be a mapping from \( Y \) into \( Z \). One can check that \( g \) is continuous with respect to the quotient topology on \( Y \) associated to \( f \) if and only if \( g \circ f \) is continuous as a mapping from \( X \) into \( Z \).

**Part II**

**Some set theory**

**25 Zorn’s lemma**

Let \( (A, \preceq) \) be a partially-ordered set. An element \( b \) of \( A \) is said to be an upper bound of a subset \( E \) of \( A \) if for every \( a \in E \), we have that

\[
\forall a \in E, a \preceq b.
\]

(25.1)

A subset \( C \) of \( A \) is said to be a chain if \( C \) is linearly ordered by the restriction of \( \preceq \) to \( C \). This means that for every \( x, y \in C \), either \( x \preceq y \) or \( y \preceq x \). An element \( x \) of \( A \) is said to be maximal in \( A \) with respect to \( \preceq \) if for every \( y \in A \) with \( x \preceq y \), we have that \( x = y \).

Let us say that \( b \in A \) is a top element of \( A \) if \( b \) is an upper bound for \( A \), so that (25.1) holds for every \( a \in A \). A top element of \( A \) is unique when it exists, and is maximal in \( A \). However, a maximal element of a partially-order set is not necessarily a top element, or unique. A maximal element of a linearly-ordered set is a top element. It is easy to see that a nonempty linearly-ordered set with only finitely many elements has a top element.

If \( A \) is a nonempty partially-ordered set with only finitely many elements, then \( A \) has a maximal element. To see this, let \( a_1 \) be any element of \( A \). If \( a_1 \) is maximal in \( A \), then we can stop. Otherwise, there is an element \( a_2 \) of \( A \) such that \( a_1 \preceq a_2 \) and \( a_1 \neq a_2 \). We can repeat the process a finite number of times to get a maximal element of \( A \), because \( A \) has only finitely many elements.

Let \( A \) be any partially-ordered set again. If every chain in \( A \) has an upper bound in \( A \), then Zorn’s lemma states that \( A \) has a maximal element. Note that the empty set may be considered as a chain in \( A \), so that the hypothesis of Zorn’s lemma implies that \( A \neq \emptyset \). There are well-known arguments for obtaining Zorn’s lemma from the axiom of choice, as in [2]. The converse is much simpler, and we shall return to that later.
Suppose that $A$ is countably infinite, and let \( \{x_j\}_{j=1}^{\infty} \) be a sequence of elements of $A$ in which every element of $A$ occurs exactly once. We can try to find a maximal element of $A$ using a more precise version of the argument for finite sets, as follows. Put $j_1 = 1$, and suppose that $j_l \in \mathbb{Z}_+$ has been chosen for some positive integer $l$. If there is an integer $k > j_l$ such that
\[
x_j \preceq x_k,
\]
then we take $j_{l+1}$ to be the smallest such integer $k$, and otherwise we stop. If this process stops after finitely many steps, then we get a maximal element of $A$. Otherwise, we get an infinite subsequence \( \{x_{j_l}\}_{l=1}^{\infty} \) of \( \{x_j\}_{j=1}^{\infty} \) such that
\[
x_{j_l} \preceq x_{j_{l+1}}
\]
for every $l \geq 1$. In particular, the subset $C$ of $A$ consisting of $x_{j_l}$, $l \in \mathbb{Z}_+$, is a chain in $A$. If $C$ has an upper bound in $A$, then this upper bound is of the form $x_n$ for some positive integer $n$. In this case, there would have to be a positive integer $l_0$ such that $j_{l_0} = n$, because of the way that the $j_l$'s were chosen. This would contradict (25.3), because the $x_{j_l}$'s are supposed to be distinct elements of $A$.

26 Hausdorff’s maximality principle

Let $(A, \preceq)$ be a partially-ordered set again. Hausdorff’s maximality principle states that there is a chain in $A$ that is maximal with respect to inclusion. More precisely, this means that there is a chain $C_0$ in $A$ such that if $C$ is any chain in $A$ with $C_0 \subseteq C$, then $C_0 = C$.

If $A$ has only finitely many elements, then $A$ has only finitely many subsets, and in particular there are only finitely many chains in $A$. In this case, one can find a maximal chain in $A$ as in the previous section. Alternatively, one can keep adding points to a chain in $A$ until it is no longer possible to have a chain in $A$.

Similarly, suppose that $A$ is countably infinite, and let \( \{x_j\}_{j=1}^{\infty} \) be a sequence of elements of $A$ in which every element of $A$ occurs exactly once. Of course, $C_1 = \{x_1\}$ is a chain in $A$. If $x_1 \preceq x_2$ or $x_2 \preceq x_1$, then put $C_2 = \{x_1, x_2\}$, which is a chain in $A$, and otherwise put $C_2 = C_1$. If $C_n \subseteq \{x_1, \ldots, x_n\}$ has been chosen in this way for some positive integer $n$, then we can define $C_{n+1}$ as follows. If $C_n \cup \{x_{n+1}\}$ is a chain in $A$, then we take it to be $C_{n+1}$. Otherwise, put $C_{n+1} = C_n$. One can check that $C_n$ is a maximal chain in $\{x_1, \ldots, x_n\}$ for each positive integer $n$, as in the preceding paragraph. One can also verify that
\[
C = \bigcup_{n=1}^{\infty} C_n
\]
is a maximal chain in $A$.

There are well-known arguments for obtaining Hausdorff’s maximality principle from the axiom of choice, as in [2] again.
Let us see how Zorn’s lemma can be obtained from Hausdorff’s maximality principle. Let \((A, \preceq)\) be a partially-ordered set, and let \(C_0\) be a maximal chain in \(A\), as in Hausdorff’s maximality principle. The hypothesis of Zorn’s lemma implies that there is a \(b \in A\) such that \(b\) is an upper bound for \(C_0\). We would like to check that \(b\) is a maximal element of \(A\) under these conditions. To do this, suppose that \(y \in A\) satisfies \(b \preceq y\). It is easy to see that \(C_0 \cup \{b, y\}\) is a chain in \(A\) too in this situation. This implies that \(C_0 = C_0 \cup \{b, y\}\), because \(C_0\) is a maximal chain in \(A\). This means that \(b, y \in C_0\). It follows that \(y \preceq b\), because \(y \in C_0\) and \(b\) is an upper bound for \(C_0\). Thus \(b = y\), because \(b \preceq y\), by hypothesis, as desired.

27 Maximal chains from Zorn’s lemma

We can also obtain Hausdorff’s maximality principle from Zorn’s lemma. Let \((A, \preceq)\) be a partially-ordered set again, and let \(C\) be the collection of all chains in \(A\). We may consider \(C\) as a partially-ordered set with respect to inclusion, i.e., using \(C_1 \subseteq C_2\) for \(C_1, C_2 \in C\) as the partial ordering on \(C\). Hausdorff’s maximality principle is exactly the statement that \(C\) has a maximal element. Let us check that \(C\) satisfies the hypothesis of Zorn’s lemma.

Let \(E\) be a chain in \(C\). This means that \(E\) is a collection of chains in \(A\) such that for every \(C_1, C_2 \in E\), either \(C_1 \subseteq C_2\) or \(C_2 \subseteq C_1\). We would like to show that \(E\) has an upper bound in \(C\).

Put

\[
C(E) = \bigcup_{C \in E} C.
\]

More precisely, each element \(C\) of \(E\) is a subset of \(A\), so that their union is a subset of \(A\) as well. If \(E = \emptyset\), then \(C(E)\) is interpreted as being the empty set too.

We would like to verify that \(C(E)\) is a chain in \(A\). Let \(x, y \in C(E)\) be given. By definition of \(C(E)\), there are \(C_x, C_y \in E\) such that \(x \in C_x\) and \(y \in C_y\). Because \(E\) is a chain in \(C\), we have that \(C_x \subseteq C_y\) or \(C_y \subseteq C_x\). It follows that \(x\) and \(y\) are both contained in \(C_x\), or that \(x\) and \(y\) are both contained in \(C_y\). In either case, we get that \(x \preceq y\) or \(y \preceq x\), because \(C_x\) and \(C_y\) are chains in \(A\). This shows that \(C(E)\) is a chain in \(A\).

Equivalently, this means that \(C(E) \in C\). Of course, if \(C \in E\), then \(C \subseteq C(E)\), by construction. Thus \(C(E)\) is an upper bound for \(E\) in \(C\). This implies that \(C\) satisfies the hypothesis of Zorn’s lemma. In this situation, the conclusion of Zorn’s lemma is that \(C\) has a maximal element, as desired.

28 The axiom of choice

Let us show how the axiom of choice can be obtained from Zorn’s lemma or Hausdorff’s maximality principle. Let \(I\) be a nonempty set, and let \(X_i\) be a
nonempty set for each \( j \in I \). We would like to show that there is a mapping \( f \) from \( I \) into \( \bigcup_{j \in I} X_j \) such that \( f(j) \in X_j \) for every \( j \in I \).

Let \( A \) be the set of ordered pairs \((I_0, f_0)\), where \( I_0 \) is a subset of \( I \), and \( f_0 \) is a mapping from \( I_0 \) into \( \bigcup_{j \in I_0} X_j \) such that \( f_0(j) \in X_j \) for every \( j \in I_0 \). If \((I_1, f_1), (I_2, f_2) \in A\), then put

\[
(I_1, f_1) \preceq (I_2, f_2)
\]

when \( I_1 \subseteq I_2 \) and \( f_1 = f_2 \) on \( I_1 \). It is easy to see that this defines a partial ordering on \( A \).

Let \( C \) be a chain in \( A \), and let us check that \( C \) has an upper bound in \( A \). Put

\[
I_C = \bigcup_{(I_0, f_0) \in C} I_0,
\]

which is a subset of \( I \). We would like to define a mapping \( f_C \) from \( I_C \) into \( \bigcup_{j \in I_C} X_j \) as follows. If \( j \in I_C \), then there is an element \((I_1, f_1)\) of \( C \) such that \( j \in I_1 \), by definition of \( I_C \). In this case, we would like to put

\[
f_C(j) = f_1(j).
\]

We need to check that this does not depend on the particular choice of \((I_1, f_1)\). Suppose that \((I_2, f_2)\) is another element of \( C \) such that \( j \in I_2 \). Note that \((I_1, f_1) \preceq (I_2, f_2)\) or \((I_2, f_2) \preceq (I_1, f_1)\), because \( C \) is a chain in \( A \). In both cases, we have that \( f_1(j) = f_2(j) \), so that \( f_C(j) \) is well defined. Of course, \( f_C(j) \in X_j \), so that \((I_C, f_C) \in A\). If \((I_0, f_0) \) is an element of \( C \), then \( I_0 \subseteq I_C \) and \( f_0 = f_C \) on \( I_0 \), by construction. This means that \((I_0, f_0) \preceq (I_C, f_C)\), so that \((I_C, f_C) \) is an upper bound for \( C \) in \( A \).

This shows that \( A \) satisfies the hypothesis of Zorn’s lemma, so that Zorn’s lemma implies that \( A \) has a maximal element. Alternatively, Hausdorff’s maximality principle says that \( A \) has a maximal chain. The previous argument implies that such a maximal chain has an upper bound in \( A \), which is a maximal element of \( A \), as before.

If \((I_0, f_0) \) is a maximal element of \( A \), then we would like to show that \( I_0 = I \). Otherwise, there is an element \( j_1 \) of \( I \) not in \( I_0 \), and we put \( I_1 = I_0 \cup \{ j_1 \} \). Let \( x_{j_1} \) be an element of \( X_{j_1} \). Consider the mapping \( f_1 \) from \( I_1 \) into \( \bigcup_{j \in I_1} X_j \) defined by putting \( f_1(j) = f_0(j) \) when \( j \in I_0 \) and \( f_1(j_1) = x_{j_1} \). Thus \( f_1(j) \in X_j \) for every \( j \in I_1 \), so that \((I_1, f_1) \in A \). We also have that \((f_0, I_0) \preceq (I_1, f_1)\) and \((I_0, f_0) \neq (I_1, f_1)\), by construction. This contradicts the maximality of \((I_0, f_0) \) in \( A \). It follows that \( I_0 = I \), as desired.

### 29 Injective mappings

Let \( A \) and \( B \) be sets. We would like to show how Zorn’s lemma or Hausdorff’s maximality principle implies that there is either a one-to-one mapping from \( A \) into \( B \), or a one-to-one mapping from \( B \) into \( A \). Let \( A \) be the collection of
ordered triples \((A_0, B_0, f_0)\), where \(A_0 \subseteq A\), \(B_0 \subseteq B\), and \(f_0\) is a one-to-one mapping from \(A_0\) onto \(B_0\). If \((A_1, B_1, f_1), (A_2, B_2, f_2) \in \mathcal{A}\), then put

\[(A_1, B_1, f_1) \preceq (A_2, B_2, f_2)\]

when \(A_1 \subseteq A_2\), \(B_1 \subseteq B_2\), and \(f_1 = f_2\) on \(A_1\). One can check that this defines a partial ordering on \(\mathcal{A}\).

Let \(\mathcal{C}\) be a chain in \(\mathcal{A}\), and let us verify that \(\mathcal{C}\) has an upper bound in \(\mathcal{A}\). Put

\[A_\mathcal{C} = \bigcup_{(A_0, B_0, f_0) \in \mathcal{C}} A_0\]

and

\[B_\mathcal{C} = \bigcup_{(A_0, B_0, f_0) \in \mathcal{C}} B_0,\]

which are subsets of \(A\) and \(B\), respectively. We would like to define a mapping \(f_\mathcal{C}\) from \(A_\mathcal{C}\) into \(B_\mathcal{C}\) as follows. If \(x \in A_\mathcal{C}\), then there is an element \((A_1, B_1, f_1)\) of \(\mathcal{C}\) such that \(x \in A_1\), and we would like to put

\[f_\mathcal{C}(x) = f_1(x)\].

One can check that this does not depend on the particular choice of \((A_1, B_1, f_1)\), as in the previous section. Note that (29.4) is an element of \(B_1 \subseteq B_\mathcal{C}\), so that \(f_\mathcal{C}\) is a well-defined mapping from \(A_\mathcal{C}\) into \(B_\mathcal{C}\). We would like to verify that \(f_\mathcal{C}\) is a one-to-one mapping from \(A_\mathcal{C}\) onto \(B_\mathcal{C}\), so that \((A_\mathcal{C}, B_\mathcal{C}, f_\mathcal{C}) \in \mathcal{A}\).

Let \(x_1\) and \(x_2\) be distinct elements of \(A_\mathcal{C}\). It follows that there are elements \((A_1, B_1, f_1)\) and \((A_2, B_2, f_2)\) of \(\mathcal{C}\) such that \(x_1 \in A_1\) and \(x_2 \in A_2\), by the definition (29.2) of \(A_\mathcal{C}\). Because \(\mathcal{C}\) is a chain in \(\mathcal{A}\), either \((A_1, B_1, f_1) \preceq (A_2, B_2, f_2)\) or \((A_2, B_2, f_2) \preceq (A_1, B_1, f_1)\). In the first case, we get that \(A_1 \subseteq A_2\), \(x_1, x_2 \in A_2\), and hence \(f_\mathcal{C}(x_1) = f_2(x_1)\), \(f_\mathcal{C}(x_2) = f_2(x_2)\). Of course, \(f_2\) is injective on \(A_2\), by hypothesis, so that \(f_2(x_1) \neq f_2(x_2)\). This means that \(f_\mathcal{C}(x_1) \neq f_\mathcal{C}(x_2)\) in this case, and the other case can be handled in the same way. This shows that \(f_\mathcal{C}\) is injective on \(A_\mathcal{C}\).

Now let \(y \in B_\mathcal{C}\) be given. By the definition (29.3) of \(B_\mathcal{C}\), there is an element \((A_0, B_0, f_0)\) of \(\mathcal{C}\) such that \(y \in B_0\). This implies that there is an \(x \in A_0\) such that \(f_0(x) = y\), because \(f_0\) maps \(A_0\) onto \(B_0\), by hypothesis. It follows that \(x \in A_\mathcal{C}\), and that \(f_\mathcal{C}(x) = y\), by the definitions of \(A_\mathcal{C}\) and \(f_\mathcal{C}\). Thus \(f_\mathcal{C}\) maps \(A_\mathcal{C}\) onto \(B_\mathcal{C}\).

This shows that \((A_\mathcal{C}, B_\mathcal{C}, f_\mathcal{C}) \in \mathcal{A}\). It is easy to see that \((A_\mathcal{C}, B_\mathcal{C}, f_\mathcal{C})\) is an upper bound for \(\mathcal{C}\) in \(\mathcal{A}\), by construction. This means that \(\mathcal{A}\) satisfies the hypothesis of Zorn’s lemma, so that Zorn’s lemma implies that \(\mathcal{A}\) has a maximal element. Alternatively, Hausdorff’s maximality principle says that \(\mathcal{A}\) has a maximal chain, and the upper bound in \(\mathcal{A}\) for such a maximal chain obtained as before is a maximal element of \(\mathcal{A}\).

If \((A_0, B_0, f_0)\) is a maximal element of \(\mathcal{A}\), then we would like to show that \(A_0 = A\) or \(B_0 = B\). Otherwise, there is an element \(a_1\) of \(A\) not in \(A_0\), and an element \(b_1\) of \(B\) not in \(B_0\). Put \(A_1 = A_0 \cup \{a_1\}\) and \(B_1 = B_0 \cup \{b_1\}\), so
A maximality of \((E_a)\) smallest element. More precisely, this means that if \(x \preceq y\) or \(y \preceq x\), then there is a smallest element in \(E\) such that \(x \preceq y\) for every \(y \in E\). Note that the smallest element in \(E\) is automatically unique.

It is easy to see how the axiom of choice can be obtained from the well-ordering principle. Let \((A, \preceq)\) be a nonempty set, and let \(X_j\) be a nonempty set, \(j \in I\). The well-ordering principle implies that \(\bigcup_{j \in I} X_j\) can be well ordered. If \(j \in I\), then let \(f(j)\) be the smallest element of \(X_j\) with respect to this ordering. This defines a mapping \(f\) from \(I\) into \(\bigcup_{j \in I} X_j\) such that \(f(j) \in X_j\) for each \(j \in I\).

Let \((A, \preceq)\) be a partially-ordered set, and let \(B\) be a subset of \(A\). It is easy to see that the restriction of \(\preceq\) to \(B\) is a partial ordering on \(B\). If \(A\) is linearly ordered by \(\preceq\), then \(B\) is linearly ordered by the restriction of \(\preceq\) to \(B\). If \(A\) is well ordered by \(\preceq\), then \(B\) is well ordered by \(\preceq\) too.

Let \((A, \preceq)\) be a partially-ordered set again. A subset \(B\) of \(A\) is said to be an ideal in \(A\) if for every \(x \in A\) and \(y \in B\) with \(x \preceq y\) we have that \(x \in B\). If \(a \in A\), then the segment in \(A\) associated to \(a\) is defined by

\[
S(a) = S_A(a) = \{x \in A : x \preceq a, x \neq a\}.
\] (30.1)

It is easy to see that segments in \(A\) are ideals in \(A\). Of course, \(A\) is an ideal in itself.

Suppose that \((A, \preceq)\) is a linearly-ordered set. If \(a \in A\), then it is easy to see that

\[
A \setminus S(a) = \{x \in A : a \preceq x\}. \tag{30.2}
\]

Suppose now that \((A, \preceq)\) is a well-ordered set, and that \(B \subseteq A\) is an ideal in \(A\). If \(B \neq A\), then there is a smallest element \(a_0\) of \(A \setminus B\). One can check that

\[
B = S(a_0), \tag{30.3}
\]
under these conditions. More precisely, $S(a_0) \subseteq B$ automatically, because $a_0$ is the smallest element of $A \setminus B$. To show that $B \subseteq S(a_0)$, suppose for the sake of a contradiction that $x \in B$ and $x \notin S(a_0)$. This implies that $a_0 \leq x$, because $A$ is linearly ordered by $\leq$, as in (30.2). It follows that $a_0 \in B$, because $B$ is an ideal in $A$. This contradict the fact that $a_0 \in A \setminus B$, as desired.

Let $(A, \preceq)$ be a linearly-ordered set. If every segment in $A$ is well ordered by $\preceq$, then $A$ is well ordered by $\preceq$. To see this, let $E$ be a nonempty subset of $A$, and let us show that $E$ has a smallest element. Let $a$ be an element of $E$. If $a$ is already the smallest element of $E$, then we can stop. Otherwise, $E \setminus S(a) \neq \emptyset$.

In this case, $E \cap S(a)$ has a smallest element, because $S(a)$ is well ordered, by hypothesis. It is easy to see that the smallest element of $E \cap S(a)$ is also the smallest element of $E$, using (30.2), as desired.

### 31 The well-ordering principle

The well-ordering principle can be obtained from Zorn’s lemma or Hausdorff’s maximality principle as follows. Let $A$ be a set, on which we would like to find a well ordering. Let $A$ be the collection of ordered pairs $(A_0, \preceq_0)$, where $A_0$ is a subset of $A$ well ordered by $\preceq_0$. If $(A_1, \preceq_1), (A_2, \preceq_2) \in A$, then put

$$(A_1, \preceq_1) \preceq_A (A_2, \preceq_2)$$

when $A_1 \subseteq A_2$, the restriction of $\preceq_2$ to $A_1$ is the same as $\preceq_1$, and $A_1$ is an ideal in $A_2$ with respect to $\preceq_2$. One can check that this defines a partial ordering on $A$. Let $C$ be a chain in $A$. We would like to show that $C$ has an upper bound in $A$, as usual. Put

$$(31.2) \quad A_C = \bigcup_{(A_0, \preceq_0) \in C} A_0,$$

which is a subset of $A$. We first need to define an ordering $\preceq_C$ on $A_C$.

Let $x, y \in A_C$ be given, so that there are $(A_1, \preceq_1), (A_2, \preceq_2) \in C$ such that $x \in A_1$ and $y \in A_2$. Because $C$ is a chain in $A$, we have that $(A_1, \preceq_1) \preceq_A (A_2, \preceq_2)$ or $(A_2, \preceq_2) \preceq_A (A_1, \preceq_1)$. In particular, this means that $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. It follows that $x, y \in A_1$ or $x, y \in A_2$.

Let $(A_0, \preceq_0)$ be any element of $C$ such that $x, y \in A_0$, the existence of which follows from the remarks in the preceding paragraph. We would like to put

$$(31.3) \quad x \preceq_C y$$

when $x \preceq_0 y$. One can check that this does not depend on the particular element $(A_0, \preceq_0)$ of $C$ with $x, y \in A_0$, because $C$ is a chain in $A$.

It is not difficult to show that $A_C$ is linearly ordered by $\preceq_C$, using the fact that the elements of $C$ are linearly-ordered sets. More precisely, to check that
is transitive on $A_C$, let $x, y, z \in A_C$ be given. One can verify that there is an element $(A_0, \leq_0)$ of $C$ such that $x, y, z \in A_0$, using the fact that $C$ is a chain in $A$, as before. If $x \leq_C y$ and $y \leq_C z$, then one can get that $x \leq_C z$ using transitivity of $\leq_0$ on $A_0$.

Let $(A_1, \leq_1)$ be any element of $C$. By construction, $A_1 \subseteq A_C$, and $\leq_1$ agrees with $\leq_1$ on $A_1$. Let us check that $A_1$ is an ideal in $A_C$ with respect to $\leq_C$. Let $x \in A_C$ and $y \in A_1$ be given, with $x \leq_C y$. By definition of $A_C$, there is an element $(A_2, \leq_2)$ of $C$ such that $x \in A_2$. We also have that $(A_1, \leq_1) \leq_A (A_2, \leq_2)$ or $(A_2, \leq_2) \leq_A (A_1, \leq_1)$, because $C$ is a chain in $A$. In the first case, $x, y \in A_2$ and $x \leq_2 y$, which implies that $x \in A_1$, because $y \in A_1$ and $A_1$ is an ideal in $A_2$. In the second case, $x \in A_2 \subseteq A_1$. Thus $x \in A_1$ in both cases, as desired.

Let us check that $A_C$ is well ordered by $\leq_C$. Let $a \in A_C$ be given, and let us verify that the corresponding segment $S_{A_C}(a)$ in $A_C$ is well ordered by $\leq_C$. By construction, there is an $(A_0, \leq_0) \in C$ such that $a \in A_0$. We also have that

$$S_{A_C}(a) \subseteq A_0,$$

because $A_0$ is an ideal in $A_C$, as in the preceding paragraph. It follows that $S_{A_C}(a)$ is well ordered by $\leq_C$, because $\leq_0$ and $\leq_C$ are the same on $A_0$, and $A_0$ is well ordered by $\leq_0$.

This shows that $(A_C, \leq_C)$ is an element of $A$. If $(A_0, \leq_0) \in C$, then it is easy to see that $(A_0, \leq_0) \leq_A (A_C, \leq_C)$, by the previous remarks. This means that $(A_C, \leq_C)$ is an upper bound for $C$ in $A$. It follows that $A$ has a maximal element, by Zorn’s lemma. Alternatively, Hausdorff’s maximality principle implies that $A$ has a maximal chain, and an upper bound for such a chain is a maximal element of $A$, as usual.

Let $(A_1, \leq_1)$ be a maximal element of $A$. We would like to show that $A_1 = A$. Otherwise, there is an $a_2 \in A$ such that $a_2 \notin A_1$. Put $A_2 = A_1 \cup \{a_2\}$, and let us define $\leq_2$ on $A_2$ as follows. We take $\leq_2$ to be the same as $\leq_1$ on $A_1$, and we put $x \leq_2 a_2$ for every $x \in A_2$. It is easy to see that $A_2$ is well ordered by $\leq_2$, because $A_1$ is well ordered by $\leq_1$. Clearly $A_1$ is an ideal in $A_2$, by construction. Thus $(A_2, \leq_2) \in A$ and $(A_1, \leq_1) \leq_A (A_2, \leq_2)$. This contradicts the maximality of $(A_1, \leq_1)$ in $A$, because $A_1 \neq A_2$.

### 32 Order Isomorphisms

Let $(A_1, \leq_1)$ and $(A_2, \leq_2)$ be partially-ordered sets. A one-to-one mapping $f$ from $A_1$ onto $A_2$ is said to be an **order isomorphism** if for every $x, y \in A_1$,

$$x \leq_1 y \text{ if and only if } f(x) \leq_2 f(y).$$

In this case, the inverse mapping $f^{-1}$ is an order isomorphism from $A_2$ onto $A_1$. Let $(A_3, \leq_3)$ be another partially-ordered set, and suppose that $g$ is an order isomorphism from $A_2$ onto $A_3$. Under these conditions, $g \circ f$ is an order isomorphism from $A_1$ onto $A_3$. 

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An order isomorphism from a partially-ordered set onto itself may be called an \textit{order automorphism}. Of course, the identity mapping on any partially-ordered set is an order automorphism.

Now let \((A_1, \preceq_1)\) and \((A_2, \preceq_2)\) be well-ordered sets, and let \(f\) be an order isomorphism from \(A_1\) onto \(A_2\). If \(E_1\) is a nonempty subset of \(A_1\), then \(f\) maps the smallest element of \(E_1\) to the smallest element of \(f(E_1)\). In particular, if \(A_1 \neq \emptyset\), then \(f\) maps the smallest element of \(A_1\) to the smallest element of \(A_2\).

Let \((A, \preceq)\) be a well-ordered set, and let \(f\) be an order automorphism on \(A\). We would like to check that \(f\) is the identity mapping on \(A\). Otherwise, let \(a\) be the smallest element of \(A\) such that \(f(a) \neq a\). Thus \(f\) is the identity mapping on the segment \(S(a)\), and in particular \(f\) maps \(S(a)\) onto itself. This implies that \(f\) maps the complement of \(S(a)\) in \(A\) onto itself. However, \(a\) is the smallest element of the complement of \(S(a)\) in \(A\). It follows that \(f(a) = a\), which is a contradiction.

Let \((A_1, \preceq_1)\) and \((A_2, \preceq_2)\) be well-ordered sets, and suppose that \(f\) and \(g\) are order isomorphisms from \(A_1\) onto \(A_2\). This implies that \(g^{-1} \circ f\) is an order automorphism on \(A_1\). It follows that \(g^{-1} \circ f\) is the identity mapping on \(A_1\), as in the preceding paragraph. Of course, this means that \(g = f\), so that an order isomorphism between well-ordered sets is unique when it exists.

Let \((A, \preceq)\) be a well-ordered set again, and let \(B\) be an ideal in \(A\). As before, \(B\) is well ordered by the restriction of \(\preceq\) to \(B\). Suppose that \(f\) is an order isomorphism from \(A\) onto \(B\). Under these conditions, \(f\) is the identity mapping on \(A\), so that \(B = f(A) = A\). This is an extension of the earlier statement for order automorphisms on \(A\), which can be shown in essentially the same way. More precisely, if \(f\) is not the identity mapping on \(A\), then there is a smallest element \(a\) of \(A\) such that \(f(a) \neq a\). This means that \(f\) is equal to the identity mapping on \(S(a)\), so that \(f(S(a)) = S(a)\). It follows that \(f(a) \not\in S(a)\), because \(a \not\in S(a)\) and \(f\) is injective, which is to say that \(a \preceq f(a)\). This implies that \(a \in B\), because \(f(a) \in f(A) = B\) and \(B\) is an ideal in \(A\). Note that

\[
(32.2) \quad f(A \setminus S(a)) = f(A) \setminus f(S(a)) = f(A) \setminus S(a).
\]

Of course, \(a\) is the smallest element of \(A \setminus S(a)\), and \(a\) is the smallest element of \(32.2\) too. Thus \(f(a) = a\), which is a contradiction.

Alternatively, one can show more directly that a well-ordered set cannot be order-isomorphic to any of its segments, as in Theorem 20 on p51 of [2].

Let \((A_1, \preceq_1)\) and \((A_2, \preceq_2)\) be partially-ordered sets again, and let \(f\) be an order isomorphism from \(A_1\) onto \(A_2\). If \(B_1\) is an ideal in \(A_1\), then \(f(B_1)\) is an ideal in \(A_2\). If \(a_1 \in A_1\), then \(f\) maps the corresponding segment \(S_{A_1}(a_1)\) in \(A_1\) onto the segment \(S_{A_2}(f(a_1))\) associated to \(f(a_1)\) in \(A_2\).

Suppose that \(A_1\) and \(A_2\) are well ordered by \(\preceq_1\) and \(\preceq_2\), respectively. If \(a_1 \in A_1\), then there is at most one \(a_2 \in A_2\) such that \(S_{A_1}(a_1)\) is order isomorphic to \(S_{A_2}(a_2)\), with respect to the restrictions of \(\preceq_1\) and \(\preceq_2\) to \(S_{A_1}(a_1)\) and \(S_{A_2}(a_2)\), respectively. Otherwise, there are elements \(a_2'\) and \(a_2''\) of \(A_2\) such that \(a_2' \neq a_2''\) and \(S_{A_2}(a_2')\) is order isomorphic to \(S_{A_2}(a_2'')\). We may as well suppose that \(a_2' \preceq_2 a_2''\), because \(A_2\) is linearly ordered by \(\preceq_2\), and otherwise we could interchange
the roles of \( a'_0 \) and \( a''_0 \). This means that \( a'_0 \in S_A(a''_0) \), because \( a'_0 \neq a''_0 \). Thus \( S_A(a'_0) \) may be considered as a segment in \( S_A(a''_0) \). This contradicts the fact that a well-ordered set cannot be order isomorphic to any of its segments, as before.

This can also be used to prove the uniqueness of order isomorphisms between well-ordered sets, by considering the images of segments under the order isomorphisms.

### 33 Order isomorphisms, continued

Let \((A, \preceq_A)\) and \((B, \preceq_B)\) be well-ordered sets. Under these conditions, Theorem 21 on p51 of [2] states that \(A\) is order-isomorphic to an ideal in \(B\), or \(B\) is order-isomorphic to an ideal in \(A\). More precisely, this means that either \(A\) is order-isomorphic to \(B\), \(A\) is order-isomorphic to a segment in \(B\), or \(B\) is order-isomorphic to a segment in \(A\). It is easy to see that only one of these three possibilities can occur, because a well-ordered set cannot be order-isomorphic to any of its segments, as in the previous section.

Let \(I_A\) be the set of \( a \in A \) for which there is a \( b \in B \) such that the segment \( S_A(a) \) corresponding to \( a \) in \( A \) is order-isomorphic to the segment \( S_B(b) \) corresponding to \( b \) in \( B \). One can check that \( b \) is uniquely determined by this property, because a well-ordered set cannot be order-isomorphic to any of its segments. If \( a \in I_A \), then let \( f(a) \) be the element of \( B \) such that \( S_A(a) \) is order-isomorphic to \( S_B(f(a)) \). This defines a mapping from \( I_A \) into \( B \).

Let \( a \in I_A \) be given, so that there is an order isomorphism \( \phi_a \) from \( S_A(a) \) onto \( S_B(f(a)) \). More precisely, this uses the restrictions of \( \preceq_A \) and \( \preceq_B \) to \( S_A(a) \) and \( S_B(f(a)) \), respectively. If \( a_0 \in S_A(a) \), then \( S_A(a_0) \) is the same as the segment corresponding to \( a_0 \) in \( S_A(a) \). Similarly, \( S_B(f(a_0)) \) is the same as the segment corresponding to \( \phi_a(a_0) \) in \( S_B(f(a)) \). It follows that

\[
(33.1) \quad \phi_a(S_A(a_0)) = S_B(f(a_0)),
\]

as in the previous section. Note that the restriction of \( \phi_a \) to \( S_A(a) \) is an order isomorphism onto \( S_B(f(a_0)) \). This means that \( a_0 \in I_A \), with

\[
(33.2) \quad f(a_0) = \phi_a(a_0).
\]

Thus

\[
(33.3) \quad S_A(a) \subseteq I_A,
\]

and

\[
(33.4) \quad f(S_A(a)) = \phi_a(S_A(a)) = S_B(f(a)).
\]

In particular, \( I_A \) is an ideal in \( A \).

Similarly, let \( I_B \) be the set of \( b \in B \) such that \( S_B(b) \) is order-isomorphic to \( S_A(a) \) for some \( a \in A \). This means that \( a \in I_A \) and \( f(a) = b \), and in fact

\[
(33.5) \quad f(I_A) = I_B.
\]
It is easy to see that $I_B$ is an ideal in $B$, using (33.4). One can check that $f$ is an order isomorphism from $I_A$ onto $I_B$, with respect to the restrictions of $\leq_A$ and $\leq_B$ to $I_A$ and $I_B$, respectively.

Suppose for the sake of a contradiction that $I_A \neq A$ and $I_B \neq B$. This implies that there are $x \in A$ and $y \in B$ such that $I_A = S_A(x)$ and $I_B = S_B(y)$, as in Section 30. It follows that $x \in I_A$ and $y \in I_B$, because $f$ is an order isomorphism from $I_A$ onto $I_B$. This is a contradiction, and so we get that $I_A = A$ or $I_B = B$, as desired.

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