Some basic topics in topology and set theory

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Preface

These informal notes are intended to complement more detailed texts, some of which are mentioned in the bibliography. The reader is expected to be familiar with some basic notions in analysis, including metric spaces and countability. The brief introduction in [174] may be of interest in this regard.

Some references concerning related topics are mentioned in the bibliography too. This includes more involved areas connected to topology and set theory, as well as more on some of the sorts of questions discussed here.

In particular, some very nice surveys may be found in [20, 21, 27, 32, 41, 42, 44, 45, 89, 90, 134, 169, 171, 203, 238, 278, 312, 317, 321, 322, 340, 341, 366, 376, 377, 383, 395, 413, 440, 441, 443, 447, 448, 449, 450].

Some additional expositions that one may find to be quite informative include [15, 16, 23, 53, 55, 64, 65, 68, 73, 74, 136, 143, 149, 151, 161, 162, 163, 165, 219, 233, 245, 282, 339, 351, 407, 410, 415, 426].

Some more articles with some helpful background material are [6, 7, 12, 14, 22, 48, 51, 54, 56, 57, 58, 59, 60, 61, 62, 80, 84, 113, 115, 116, 122, 123, 144, 145, 155, 158, 159, 195, 206, 212, 217, 239, 246, 253, 279, 280, 288, 292, 293, 300, 305, 324, 327, 328, 336, 388, 409, 420, 435].

Of course, there are quite a few other papers in the bibliography related to topology and set theory. See [81, 82, 138, 229, 308, 337, 453, 454] for some more ways in which topological notions or properties may be used.

Although various aspects of history may be found in a number of the references, one may be particularly interested in the collections [24, 25, 26, 218].

Some interesting perspectives may be found in [4, 5, 63, 70, 83, 131, 137, 141, 142, 150, 194, 214, 386, 417, 421, 433, 438].

One may also enjoy the contributions [29, 104, 125, 140, 157, 164, 170, 172, 177, 196, 197, 198, 223, 224, 234, 259, 284, 285, 298, 332, 333, 334, 335, 343, 344, 345, 347, 348, 356, 371, 374, 397, 398, 402, 419, 437, 439, 458, 460] to the "Miscellanea" section of the American Mathematical Monthly, for instance. Some suggestions for questions to ask at the end of a lecture in mathematics may be found in [289].

I would like to dedicate these notes to the memory of Juha Heinonen, who was also interested in various connections between analysis, geometry, and topology. This is related to the notions of *Lipschitz* and *bilipschitz* mappings, as in [286, 287], and of *quasisymmetric embeddings*, as in [430, 432]. Some additional related notions may be found in [34]. Of course, there are many other types of connections between these areas as well, some aspects of which may be found in some of the other references included here.

The reader may notice that there are a number of references to articles in the American Mathematical Monthly and Mathematics Magazine in the bibliography. These journals aim to have papers that are accessible to rather wide mathematical audiences, and in particular hopefully fit well with these notes. These journals also happen to have a lot of papers on topics related to those discussed here. However, there are surely many more papers with similar qualities that have not been listed here, in other publications as well.

It is well known that there are many connections between abstract algebra and each of topology and set theory. Although we do not really get into this here, readers familiar with some abstract algebra can find many related topics in the bibliography. A brief introduction to groups, rings, and fields may be found in [18]. Note that one may be interested in algebraic structures related to topological spaces, as in *algebraic topology*, as well as topological structures on algebraic objects, as in *topological algebra*.

Contents

Chapter 1

Some basic notions in topology

1.1 Some definitions and examples

Let X be a set.

Definition 1.1.1 *A collection τ of subsets of X is said to define a* topology *on X if it satisfies the following three conditions. First,*

$$\emptyset,\, X \in \tau. \tag{1.1.2}$$

Second, if $U_1, \ldots, U_n$ are finitely many elements of τ, then

$$\bigcap_{j=1}^{n} U_j \in \tau. \tag{1.1.3}$$

Third, if A is a nonempty set, and $U_\alpha \in \tau$ for every $\alpha \in A$, then

$$\bigcup_{\alpha \in A} U_\alpha \in \tau. \tag{1.1.4}$$

In this case, (X, τ) is said to be a topological space*, and the elements of τ are called* open sets *in X.*

Sometimes we may refer to a topological space X, in which case the topology τ is implicit.

1.1.1 Some basic examples

Definition 1.1.5 *If X is any set, then the* discrete topology *on X is defined by taking τ to be the collection of all subsets of X. The* indiscrete topology *on X is defined by taking τ to be the collection consisting of only the empty set and X.*

It is easy to see that the discrete and indiscrete topologies satisfy the requirements of a topology.

If $d(x, y)$ is a metric on a set X, then the notion of an open subset of X with respect to $d(\cdot, \cdot)$ can be defined in a standard way. An extension of this will be mentioned in the next section. It is well known and not difficult to check that this defines a topology on X.

In particular, the *discrete metric* on a set X is defined by putting $d(x, y) = 1$ when x and y are distinct elements of X, and $d(x, x) = 0$ for every $x \in X$. This defines a metric on X, for which the corresponding topology is the discrete topology.

Let $\mathbf{R}$ be the real line, as usual. If $x \in \mathbf{R}$, then the *absolute value* of x is defined by $|x| = x$ when $x \geq 0$, and $|x| = -x$ when $x \leq 0$. The *standard Euclidean metric on* $\mathbf{R}$ is defined by

$$d(x, y) = |x - y|. \tag{1.1.6}$$

The *standard topology on* $\mathbf{R}$ may be defined as the topology determined on $\mathbf{R}$ by (1.1.6).

1.1.2 Some basic separation conditions

Let (X, τ) be a topological space.

Definition 1.1.7 *We say that (X, τ) satisfies the* first separation condition *if for every pair x, y of distinct elements of X, there is an open subset U of X such that $x \in U$ and $y \notin U$. Equivalently, we say that X is a T_1* space *in this case.*

The first separation condition is supposed to be symmetric in x and y, so that there should also be an open set $V \subseteq X$ such that $y \in V$ and $x \notin V$.

Definition 1.1.8 *We say that (X, τ) satisfies the* zeroth separation condition *if for every pair x, y of distinct elements of X there is an open set $W \subseteq X$ such that either $x \in W$ and $y \notin W$, or $y \in W$ and $x \notin W$. In this case, we may also say that (X, τ) is a T_0* space.

Thus the first separation condition implies the zeroth separation condition.

Definition 1.1.9 *We say that (X, τ) satisfies the* second separation condition *if for every pair x, y of distinct elements of X there are disjoint open subsets U, V of X such that $x \in U$ and $y \in V$. We may also say that (X, τ) is a T_2* space *in this situation, or equivalently that (X, τ) is* Hausdorff.

Hausdorff spaces obviously satisfy the first separation condition. If X is a set with at least two elements equipped with the indiscrete topology, then X does not satisfy the zeroth separation condition. Any set equipped with the discrete topology is Hausdorff.

1.2 Semimetrics and intervals

Let X be a set.

Definition 1.2.1 *A nonnegative real-valued function $d(x, y)$ defined for x, y in X is said to be a* semimetric *or* pseudometric *on X if it satisfies the following three conditions. First,*

$$d(x, x) = 0 \tag{1.2.2}$$

for every $x \in X$. Second, $d(x, y)$ is symmetric in x and y, so that

$$d(x, y) = d(y, x) \tag{1.2.3}$$

for every $x, y \in X$. Third, the triangle inequality

$$d(x, z) \le d(x, y) + d(y, z) \tag{1.2.4}$$

holds for every $x, y, z \in X$. If we also have that

$$d(x, y) > 0 \tag{1.2.5}$$

for every $x, y \in X$ with $x \ne y$, then $d(\cdot, \cdot)$ is said to be a metric *on X.*

1.2.1 Open balls and open sets

Let $d(\cdot, \cdot)$ be a semimetric on X. If $x \in X$ and r is a positive real number, then the *open ball* in X centered at x with radius r with respect to $d(\cdot, \cdot)$ is defined by

$$B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}. \tag{1.2.6}$$

A subset U of X is said to be an *open set* with respect to $d(\cdot, \cdot)$ if for every $x \in U$ there is an $r > 0$ such that

$$B(x, r) \subseteq U. \tag{1.2.7}$$

It is well known and not difficult to show that this defines a topology on X. If $d(x, y) = 0$ for every $x, y \in X$, then $B(x, r) = X$ for every $x \in X$ and $r > 0$, and the corresponding topology on X is the indiscrete topology.

If $w \in X$ and $t > 0$, then one can check that $B(w, t)$ is an open set in X with respect to $d(\cdot, \cdot)$. More precisely, if $x \in B(w, t)$, then $r = t - d(w, x) > 0$, and one can verify that

$$B(x, r) \subseteq B(w, t), \tag{1.2.8}$$

using the triangle inequality.

If $w_1, w_2 \in X$ satisfy $d(w_1, w_2) > 0$, then it is easy to see that

$$B(w_1, d(w_1, w_2)/2) \cap B(w_2, d(w_1, w_2)/2) = \emptyset, \tag{1.2.9}$$

using the triangle inequality again. In particular, if $d(\cdot, \cdot)$ is a metric on X, then X is Hausdorff with respect to the topology determined by $d(\cdot, \cdot)$. If X satisfies the zeroth separation condition with respect to the topology determined by a semimetric $d(\cdot, \cdot)$, then $d(\cdot, \cdot)$ is a metric on X.

1.2.2 Intervals in the set of extended real numbers

Now let X be the set of *extended real numbers*, which consists of the real numbers together with $+\infty$ and $-\infty$, where

$$-\infty < x < +\infty \tag{1.2.10}$$

for every $x \in \mathbf{R}$. If $a, b \in X$ and $a < b$, then the corresponding *open interval* is defined by

$$(a, b) = \{x \in \mathbf{R} : a < x < b\}. \tag{1.2.11}$$

It is easy to see that this is an open set in $\mathbf{R}$ with respect to the standard topology. If $a, b \in \mathbf{R}$, then this is the same as the open ball centered at the midpoint $(a+b)/2$ with radius $(b-a)/2$ with respect to the standard Euclidean metric on $\mathbf{R}$.

If $a, b \in X$ and $a \leq b$, then the corresponding *closed interval* is defined by

$$[a, b] = \{x \in X : a \leq x \leq b\}. \tag{1.2.12}$$

If $a < b$, then we get the half-open, half-closed intervals

$$[a, b) = \{x \in X : a \leq x < b\} \tag{1.2.13}$$

and

$$(a, b] = \{x \in X : a < x \leq b\}. \tag{1.2.14}$$

Let us say that $U \subseteq X$ is an open set if for each $x \in U$, one of the following three conditions holds. If $x \in \mathbf{R}$, then there are $a, b \in \mathbf{R}$ such that $a < x < b$ and

$$(a, b) \subseteq U. \tag{1.2.15}$$

If $x = +\infty$, then there is an $a \in \mathbf{R}$ such that

$$(a, +\infty] \subseteq U. \tag{1.2.16}$$

If $x = -\infty$, then there is a $b \in \mathbf{R}$ such that

$$[-\infty, b) \subseteq U. \tag{1.2.17}$$

One can check that this defines a topology on X, which we may refer to as the standard topology on the set of extended real numbers.

If $U \subseteq \mathbf{R}$, then U is an open set in X if and only if U is an open set in $\mathbf{R}$ with respect to the standard topology. In particular, open intervals in $\mathbf{R}$ are open sets in X with respect to this topology. Similarly, one can verify that $(a, +\infty]$ and $[-\infty, b)$ are open sets in X with respect to this topology for all $a, b \in \mathbf{R}$. Using this, it is easy to see that X is Hausdorff with respect to this topology.

1.3 Dense sets and stronger topologies

Let (X, τ) be a topological space.

Definition 1.3.1 *A subset E of X is said to be* dense *in X with respect to τ if for every $x \in X$ and open set $U \subseteq X$ with $x \in U$ there is a $y \in E$ such that $y \in U$. Equivalently, this means that for every nonempty open set $U \subseteq X$, we have that*

$$E \cap U \neq \emptyset. \tag{1.3.2}$$

Of course, X is dense in itself. If X is any set equipped with the discrete topology, then X is the only dense subset of itself. If X is a nonempty set equipped with the indiscrete topology, then $E \subseteq X$ is dense in X if and only if $E \neq \emptyset$. It is well known that the set $\mathbf{Q}$ of rational numbers is dense in the real line with respect to the standard topology.

1.3.1 Stronger topologies

Proposition 1.3.3 *Let X be a set, and let τ, $\widetilde{\tau}$ be topologies on X such that*

$$\tau \subseteq \widetilde{\tau}. \tag{1.3.4}$$

In this case, we may say that $\widetilde{\tau}$ is at least as strong as τ on X.

(a) If $j \in \{0, 1, 2\}$ and (X, τ) satisfies the jth separation condition, then $(X, \widetilde{\tau})$ satisfies the jth separation condition as well.

(b) If $E \subseteq X$ is dense in X with respect to $\widetilde{\tau}$, then E is dense in X with respect to τ.

This can be verified directly from the definitions.

1.3.2 The topologies τ_+, τ_- on the real line

Let τ_+ be the collection of subsets U of the real line such that for every $x \in U$ there is a $b \in \mathbf{R}$ with $x < b$ and

$$[x, b) \subseteq U. \tag{1.3.5}$$

Similarly, let τ_- be the collection of subsets U of $\mathbf{R}$ such that for every $x \in U$ there is an $a \in \mathbf{R}$ with $a < x$ and

$$(a, x] \subseteq U. \tag{1.3.6}$$

One can check that τ_+ and τ_- define topologies on $\mathbf{R}$. Every open set in $\mathbf{R}$ with respect to the standard topology is an open set with respect to τ_+ and τ_-. More precisely,

$$\tau_+ \cap \tau_- \tag{1.3.7}$$

is the same as the standard topology on $\mathbf{R}$.

1.3.3 Some properties of τ_+, τ_-

If $a \in \mathbf{R}$, $b \in \mathbf{R} \cup \{+\infty\}$, and $a < b$, then it is easy to see that

$$[a, b) \tag{1.3.8}$$

is an open set with respect to τ_+. Similarly, if $a \in \mathbf{R} \cup \{-\infty\}$, $b \in \mathbf{R}$, and $a < b$, then

$$(a, b] \tag{1.3.9}$$

is an open set with respect to τ_-. Note that the real line is Hausdorff with respect to τ_+ and τ_-, by Proposition 1.3.3, and because $\mathbf{R}$ is Hausdorff with respect to the standard topology. If $E \subseteq \mathbf{R}$ is dense in $\mathbf{R}$ with respect to the standard topology, then one can check that E is dense in $\mathbf{R}$ with respect to τ_+ and τ_-. Of course, if E is dense in $\mathbf{R}$ with respect to τ_+ or τ_-, then E is dense in $\mathbf{R}$ with respect to the standard topology, as in Proposition 1.3.3.

1.4 Closed sets and limit points

Let (X, τ) be a topological space.

Definition 1.4.1 *A subset E of X is said to be a* closed set *in X with respect to τ if the complement*

$$X \setminus E = \{x \in X : x \notin E\} \tag{1.4.2}$$

of E in X is an open set in X with respect to τ.

Of course,

$$X \setminus (X \setminus W) = W \tag{1.4.3}$$

for every subset W of X. This implies that the complement of an open subset of X is a closed set.

Note that X and the empty set are closed subsets of X. If $E_1, \ldots, E_n$ are finitely many closed subsets of X, then their union is a closed set as well. More precisely,

$$X \setminus \Big(\bigcup_{j=1}^{n} E_j\Big) = \bigcap_{j=1}^{n} (X \setminus E_j) \tag{1.4.4}$$

is an open set in X, by the definition of a topology.

If A is a nonempty set, and E_α is a closed set in X for every $\alpha \in A$, then $\bigcap_{\alpha \in A} E_\alpha$ is a closed set in X too. Indeed,

$$X \setminus \Big(\bigcap_{\alpha \in A} E_\alpha\Big) = \bigcup_{\alpha \in A} (X \setminus E_\alpha) \tag{1.4.5}$$

is an open set in X in this case, by the definition of a topological space.

1.4.1 Limit points, adherent points, and closures of sets

Definition 1.4.6 *A point $p \in X$ is said to be a* limit point *of $E \subseteq X$ if for every open set $U \subseteq X$ with $p \in U$ there is a $q \in E \cap U$ such that $p \neq q$. We say that $p \in X$ is* adherent *to E if for every open set $U \subseteq X$ with $p \in U$, we have that $E \cap U \neq \emptyset$.*

Thus every element of E and every limit point of E in X is adherent to E. If $p \in X$ is adherent to E and $p \notin E$, then p is a limit point of E in X.

Definition 1.4.7 *The* closure *of a subset E of X is the set $\overline{E}$ consisting of all points in X that are adherent to E. Equivalently, $\overline{E}$ is the set of $p \in X$ such that $p \in E$, p is a limit point of E, or both.*

Note that $E \subseteq X$ is dense in X if and only if $\overline{E} = X$.

Proposition 1.4.8 *If $E \subseteq X$ is a closed set, then $\overline{E} = E$.*

It suffices to check that $\overline{E} \subseteq E$. If E is a closed set and $p \in X \setminus E$, then it is easy to see that p is not adherent to E, because $X \setminus E$ is an open set. Thus $p \notin \overline{E}$, as desired.

1.4.2 Interiors of sets

Definition 1.4.9 *The* interior *of a subset A of X is the set $\operatorname{Int} A$ of $p \in A$ for which there is an open set $U \subseteq X$ such that $p \in U$ and $U \subseteq A$. Equivalently, $\operatorname{Int} A$ is the union of all the open subsets of X that are contained in A.*

Thus $\operatorname{Int} A$ is automatically an open subset of X, and $\operatorname{Int} A = A$ when A is an open set in X.

Proposition 1.4.10 *If E is any subset of X, then*

$$X \setminus \overline{E} = \operatorname{Int}(X \setminus E). \tag{1.4.11}$$

In particular, $\overline{E}$ is a closed set in X.

Indeed, $p \in X$ is not adherent to E exactly when there is an open set $U \subseteq X$ such that $p \in U$ and $U \cap E = \emptyset$. This is the same as saying that $p \in U$ and $U \subseteq X \setminus E$, which means that $p \in \operatorname{Int}(X \setminus E)$.

1.5 Induced topologies

Let (X, τ) be a topological space, and let Y be a subset of X.

Definition 1.5.1 *A subset E of Y is said to be* relatively open *in Y if there is an open set $U \subseteq X$ such that*

$$E = U \cap Y. \tag{1.5.2}$$

Proposition 1.5.3 *The collection of relatively open subsets of Y defines a topology on Y, which is called the* induced topology *on Y.*

Clearly $Y = X \cap Y$ and $\emptyset = \emptyset \cap Y$ are relatively open in Y. If $E_1, \ldots, E_n$ are finitely many relatively open subsets of Y, then there are open subsets $U_1, \ldots, U_n$ of X such that $E_j = U_j \cap Y$ for each $j = 1, \ldots, n$. This implies that

$$\bigcap_{j=1}^{n} E_j = \bigcap_{j=1}^{n} (U_j \cap Y) = \Big(\bigcap_{j=1}^{n} U_j\Big) \cap Y \tag{1.5.4}$$

is relatively open in Y.

Let A be a nonempty set, and suppose that $E_\alpha \subseteq Y$ is relatively open for every $\alpha \in A$. Thus, for each $\alpha \in A$, there is an open subset U_α of X such that $E_\alpha = U_\alpha \cap Y$. It follows that

$$\bigcup_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A} (U_\alpha \cap Y) = \Big(\bigcup_{\alpha \in A} U_\alpha\Big) \cap Y \tag{1.5.5}$$

is relatively open in Y, as desired.

1.5.1 Closed sets with respect to the induced topology

Proposition 1.5.6 *A subset A of Y is a closed set with respect to the induced topology if and only if there is a closed set $A_1 \subseteq X$ such that*

$$A = A_1 \cap Y. \tag{1.5.7}$$

If A_1 is a closed set in X, then $X \setminus A_1$ is an open set in X, and hence $(X \setminus A_1) \cap Y$ is relatively open in Y. One can check that

$$Y \setminus (A_1 \cap Y) = (X \setminus A_1) \cap Y, \tag{1.5.8}$$

so that $A_1 \cap Y$ is a closed set in Y with respect to the induced topology. Conversely, if $A \subseteq Y$ is a closed set with respect to the induced topology, then $Y \setminus A$ is relatively open in Y. This implies that there is an open subset U of X such that

$$Y \setminus A = U \cap Y. \tag{1.5.9}$$

This means that $A_1 = X \setminus U$ is a closed set in X, and it is easy to see that (1.5.7) holds.

1.5.2 Induced topologies and separation conditions

Proposition 1.5.10 *If $j \in \{0, 1, 2\}$ and X satisfies the jth separation condition, then Y satisfies the jth separation condition, with respect to the induced topology.*

This can be verified directly from the definitions.

1.5.3 Induced topologies and semimetrics

Let $d(\cdot,\cdot)$ be a semimetric on X, and suppose now that X is equipped with the topology determined by $d(\cdot,\cdot)$. It is easy to see that the restriction of $d(x,y)$ to $x,y \in Y$ defines a semimetric on Y.

Proposition 1.5.11 *Under the conditions just mentioned, the induced topology on Y is the same as the topology determined on Y by the restriction of $d(\cdot,\cdot)$ to Y.*

If $x \in X$ and $r > 0$, then let $B_X(x,r)$ be the open ball in X centered at x with radius r with respect to $d(\cdot,\cdot)$. Similarly, if $x \in Y$, then let $B_Y(x,r)$ be the open ball in Y centered at x with radius $r > 0$ with respect to the restriction of $d(\cdot,\cdot)$ to Y. It is easy to see that

$$B_Y(x,r) = B_X(x,r) \cap Y \tag{1.5.12}$$

for every $x \in Y$ and $r > 0$.

If E is a relatively open subset of Y, then there is an open subset U of X such that $E = U \cap Y$. One can check directly that E is an open set in Y with respect to the topology determined by the restriction of $d(\cdot,\cdot)$ to Y, using the analogous property of U in X.

To show the converse, observe first that (1.5.12) is relatively open in Y for every $x \in Y$ and $r > 0$, because open balls in X are open sets. If $E \subseteq Y$ is an open set with respect to the topology determined by the restriction of $d(\cdot,\cdot)$ to Y, then E can be expressed as a union of open balls in Y. This implies that E is relatively open in Y, because the union of any family of relatively open subsets of Y is relatively open in Y, as before.

1.6 Convergent sequences

Let (X,τ) be a topological space.

Definition 1.6.1 *A sequence $\{x_j\}_{j=1}^\infty$ of elements of X is said to* converge *to an element x of X if for every open subset U of X with $x \in U$ there is a positive integer L such that*

$$x_j \in U \tag{1.6.2}$$

for every $j \geq L$.

If $x_j = x$ for all but finitely many j, then $\{x_j\}_{j=1}^\infty$ automatically converges to x. If X is equipped with the discrete topology, and $\{x_j\}_{j=1}^\infty$ converges to x, then $x_j = x$ for all but finitely many j. If X is equipped with the indiscrete topology, then every sequence of elements of X converges to every element of X.

1.6.1 Some properties of convergent sequences

Proposition 1.6.3 *If (X, τ) is a Hausdorff topological space, then a sequence of elements of X can converge to at most one element of X.*

This is not difficult to show, and an extension of this will be discussed soon.

Proposition 1.6.4 *Let E be a subset of X, and let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of E. If $\{x_j\}_{j=1}^{\infty}$ converges to an element x of X, then x is adherent to E. If, for each positive integer j, we also have that $x_j \neq x$, then x is a limit point of E in X.*

This can be verified directly from the definitions.

Proposition 1.6.5 *Let Y be a subset of X, let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of Y, and let x be an element of Y. Under these conditions, $\{x_j\}_{j=1}^{\infty}$ converges to x in X if and only if $\{x_j\}_{j=1}^{\infty}$ converges to x in Y, with respect to the induced topology.*

This can also be verified directly from the definitions.

1.6.2 Convergent sequences in R with respect to τ_+, τ_-

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of real numbers, and let x be a real number too. If $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to either of the topologies τ_+ or τ_- defined in Subsection 1.3.2, then $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to the standard topology on $\mathbf{R}$. More precisely, if $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to τ_+, then

$$x_j \geq x \quad \text{for all but finitely many } j. \tag{1.6.6}$$

This can be seen by taking $U = [x, +\infty)$ in (1.6.2). Conversely, if $\{x_j\}_{j=1}^{\infty}$ satisfies (1.6.6), and $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to the standard topology on $\mathbf{R}$, then $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to τ_+.

Similarly, if $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to τ_-, then

$$x_j \leq x \quad \text{for all but finitely many } j. \tag{1.6.7}$$

Conversely, if $\{x_j\}_{j=1}^{\infty}$ satisfies (1.6.7), and if $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to the standard topology on $\mathbf{R}$, then $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to τ_-.

1.7 Local bases

Let (X, τ) be a topological space, and let x be an element of X.

Definition 1.7.1 *A collection $\mathcal{B}(x)$ of open subsets of X is said to be a* local base for the topology of X at x *if it satisfies the following two conditions. First, for every $U \in \mathcal{B}(x)$, we have that $x \in U$. Second, if V is an open subset of X such that $x \in V$, then there is an element U of $\mathcal{B}(x)$ with*

$$U \subseteq V. \tag{1.7.2}$$

The collection of all open subsets U of X with $x \in U$ is a local base for the topology of X at x. If X is equipped with the indiscrete topology, then X is the only element of this collection. If X is equipped with the discrete topology, then the collection consisting only of $\{x\}$ is a local base for the topology of X at x.

1.7.1 Countable local bases

We are often particularly interested in situations where there may be a local base for the topology of X at x with only finitely or countably many elements, which can therefore be listed in a sequence. Equivalently, we may wish to have a sequence $U_1(x), U_2(x), U_3(x), \ldots$ of open subsets of X such that the collection of the $U_j(x)$'s is a local base for the topology of X at x. This means that

$$x \in U_j(x) \quad \text{for every } j, \tag{1.7.3}$$

and that

$$\text{for every open set } V \subseteq X \text{ with } x \in V \\ \text{we have that } U_j(x) \subseteq V \text{ for some } j. \tag{1.7.4}$$

In this situation, it is frequently helpful to also ask that

$$U_{j+1}(x) \subseteq U_j(x) \quad \text{for every } j. \tag{1.7.5}$$

This can always be arranged, by replacing $U_j(x)$ with $\bigcap_{l=1}^{j} U_l(x)$ for each j, if necessary.

If the topology on X is determined by a semimetric $d(\cdot, \cdot)$, then

$$U_j(x) = B(x, 1/j) \tag{1.7.6}$$

satisfies the conditions just mentioned. If X is the real line equipped with the topology τ_+ defined in Subsection 1.3.2, then

$$U_j(x) = [x, x + 1/j) \tag{1.7.7}$$

satisfies these conditions. Similarly, if $X = \mathbf{R}$ equipped with the topology τ_-, then

$$U_j(x) = (x - 1/j, x] \tag{1.7.8}$$

satisfies these conditions.

1.7.2 A criterion for the convergence of a sequence

Let (X, τ) be any topological space again, and let $x \in X$ be given.

Proposition 1.7.9 *Suppose that $\{U_j(x)\}_{j=1}^{\infty}$ is a sequence of open subsets of X that satisfies (1.7.3), (1.7.4), and (1.7.5). Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of X such that*

$$x_j \in U_j(x) \tag{1.7.10}$$

for every j. Under these conditions, $\{x_j\}_{j=1}^{\infty}$ converges to x in X.

This is easy to check, directly from the definitions.

Corollary 1.7.11 *Suppose that there is a local base for the topology of X at x with only finitely or countably many elements. If x is adherent to $E \subseteq X$, then there is a sequence $\{x_j\}_{j=1}^\infty$ of elements of E that converges to x in X. If x is a limit point of E in X, then we may also ask that $x_j \neq x$ for each j.*

The hypothesis of the corollary implies that there is a sequence $\{U_j(x)\}_{j=1}^\infty$ of open sets in X as in the previous proposition. If x is adherent to E, then we can choose

$$x_j \in E \cap U_j(x) \tag{1.7.12}$$

for each j. If x is a limit point of E, then we can also choose x_j to be different from x for every j.

1.7.3 Local bases and induced topologies

Proposition 1.7.13 *Let Y be a subset of X such that $x \in Y$. If $\mathcal{B}(x)$ is a local base for the topology of X at x, then*

$$\mathcal{B}_Y(x) = \{U \cap Y : U \in \mathcal{B}(x)\} \tag{1.7.14}$$

is a local base for the induced topology on Y at x.

This can be verified directly from the definitions. Note that if $\mathcal{B}(x)$ has only finitely or countably many elements, then $\mathcal{B}_Y(x)$ has only finitely or countably many elements.

1.7.4 The first countability condition

Definition 1.7.15 *We say that (X, τ) satisfies the* first countability condition *if for every $x \in X$ there is a local base $\mathcal{B}(x)$ for the topology of X at x with only finitely or countably many elements.*

If (X, τ) satisfies the first countability condition and $Y \subseteq X$, then Y satisfies the first countability condition with respect to the induced topology, by the previous remark.

1.8 Directed systems and nets

Let A be a set.

Definition 1.8.1 *A binary relation $\preceq$ on A is said to be a* partial ordering *if it satisfies the following three conditions. First, $\preceq$ should be* reflexive *on A, in the sense that*

$$a \preceq a \quad \textit{for every } a \in A. \tag{1.8.2}$$

Second,

(1.8.3) *if $a, b \in A$ satisfy $a \preceq b$ and $b \preceq a$, then $a = b$.*

Third, $\preceq$ should be transitive *on A, in the sense that*

(1.8.4) *if $a, b, c \in A$ satisfy $a \preceq b$ and $b \preceq c$, then $a \preceq c$.*

If, in addition,

(1.8.5) *for every $a, b \in A$, we have that $a \preceq b$ or $b \preceq a$,*

then $\preceq$ is said to be a linear ordering *or* total ordering *on A.*

A binary relation on A that is reflexive and transitive may be called a *pre-order* on A.

Definition 1.8.6 *We say that $(A, \preceq)$ is a* directed system *if $\preceq$ is a partial ordering on A, and if*

(1.8.7) *for every $a, b \in A$ there is a $c \in A$ such that $a \preceq c$ and $b \preceq c$.*

Sometimes one also considers pre-orderings that satisfy (1.8.7), which can work just as well for some purposes. Note that linearly-ordered sets are directed systems.

1.8.1 Nets of elements of sets

Suppose that $(A, \preceq)$ is a nonempty directed system, and let X be a set.

Definition 1.8.8 *A* net *$\{x_a\}_{a \in A}$ of elements of X indexed by A assigns to each $a \in A$ an element x_a of X.*

Thus this is a function defined on A with values in X, where A is also equipped with the partial ordering $\preceq$. If A is the set $\mathbf{Z}_+$ of positive integers with the standard ordering, then this is the same as a sequence of elements of X.

1.8.2 Convergence of nets

Now let (X, τ) be a topological space.

Definition 1.8.9 *A net $\{x_a\}_{a \in A}$ of elements of X indexed by A is said to* converge *to an element x of X if for every open subset U of X with $x \in U$ there is a $b \in A$ such that*

(1.8.10) $$x_a \in U$$

for every $a \in A$ with $b \preceq a$.

This reduces to the earlier definition of convergent sequences when $A = \mathbf{Z}_+$ with the standard ordering.

1.8.3 Uniqueness of the limit in Hausdorff spaces

Proposition 1.8.11 *Let $\{x_a\}_{a \in A}$ be a net of elements of X that converges to elements x and x' of X. If (X, τ) is Hausdorff, then $x = x'$.*

Suppose for the sake of a contradiction that $x \neq x'$. This implies that there are disjoint open subsets U and V of X such that $x \in U$ and $x' \in V$, because X is Hausdorff. Using the convergence of $\{x_a\}_{a \in A}$ to x, we get that there is a $b \in A$ such that (1.8.10) holds for every $a \in A$ with $b \preceq a$. Similarly, there is a $c \in A$ such that

$$x_a \in V \tag{1.8.12}$$

for every $a \in A$ with $c \preceq a$, because $\{x_a\}_{a \in A}$ converges to x' in X.

There is an $a \in A$ such that $b \preceq a$ and $c \preceq a$, by the definition of a directed system. In this case, we get that $x_a \in U \cap V$, contradicting the disjointness of U and V.

Conversely, if (X, τ) is not Hausdorff, then one can find a net of elements of X indexed by some nonempty directed system that converges to two different elements of X. If (X, τ) also satisfies the first countability condition, then one can find a sequence of elements of X that converges to two different elements of X.

1.9 More on convergent nets

Let $(A, \preceq)$ be a nonempty directed system, and let (X, τ) be a topological space.

Proposition 1.9.1 *Let E be a subset of X, and let $\{x_a\}_{a \in A}$ be a net of elements of E indexed by A. If $\{x_a\}_{a \in A}$ converges to an element x of X, then x is adherent to E in X. If, for every $a \in A$, we also have that $x_a \neq x$, then x is a limit point of E in X.*

This follows easily from the definitions, as in the case of sequences.

Proposition 1.9.2 *Let Y be a subset of X, let $\{x_a\}_{a \in A}$ be a net of elements of Y indexed by A, and let x be an element of Y. Under these conditions, $\{x_a\}_{a \in A}$ converges to x in X if and only if $\{x_a\}_{a \in A}$ converges to x in Y, with respect to the induced topology.*

This can also be verified directly from the definitions, as before.

1.9.1 Using a local base to get a directed system

Let $x \in X$ be given, and let $\mathcal{B}(x)$ be a local base for the topology of X at x. Let us define a binary relation $\preceq$ on $\mathcal{B}(x)$ by

$$U \preceq V \quad \text{when } U, V \in \mathcal{B}(x) \text{ satisfy } V \subseteq U. \tag{1.9.3}$$

Proposition 1.9.4 *With respect to the binary relation $\preceq$ defined in (1.9.3), $\mathcal{B}(x)$ is a directed system.*

It is easy to see that $\preceq$ is a partial ordering on $\mathcal{B}(x)$. To check that $\mathcal{B}(x)$ is a directed system with respect to $\preceq$, let $U, V \in \mathcal{B}(x)$ be given. Thus U and V are open subsets of X that contain x, so that $U \cap V$ is an open set that contains x too. It follows that there is a $W \in \mathcal{B}(x)$ such that

$$W \subseteq U \cap V, \tag{1.9.5}$$

by definition of a local base. This implies that $U \preceq W$ and $V \preceq W$, as desired.

1.9.2 A criterion for the convergence of a net

Proposition 1.9.6 *If $\{x_U\}_{U \in \mathcal{B}(x)}$ is a net of elements of X indexed by $\mathcal{B}(x)$ such that*

$$x_U \in U \tag{1.9.7}$$

for every $U \in \mathcal{B}(x)$, then $\{x_U\}_{U \in \mathcal{B}(x)}$ converges to x in X.

Let W be an arbitrary open subset of X that contains x. By the definition of a local base, there is a $V \in \mathcal{B}(x)$ such that $V \subseteq W$. If $U \in \mathcal{B}(x)$ and $V \preceq U$, then

$$x_U \in U \subseteq V \subseteq W, \tag{1.9.8}$$

as desired.

Corollary 1.9.9 *If x is adherent to $E \subseteq X$, then there is a net $\{x_U\}_{U \in \mathcal{B}(x)}$ of elements of E indexed by $\mathcal{B}(x)$ that converges to x. If x is a limit point of E in X, then we may also ask that $x_U \neq x$ for every $U \in \mathcal{B}(x)$.*

If x is adherent to E, then we can choose

$$x_U \in E \cap U \tag{1.9.10}$$

for every $U \in \mathcal{B}(x)$. If x is a limit point of E, then we can also choose x_U to be different from x for every $U \in \mathcal{B}(x)$.

1.10 Regular topological spaces

Let (X, τ) be a topological space.

Definition 1.10.1 *We say that (X, τ) is* regular in the strict sense *if for every point $p \in X$ and closed set $E \subseteq X$ with $p \notin E$ there are disjoint open subsets U and V of X such that*

$$p \in U \quad \text{and} \quad E \subseteq V. \tag{1.10.2}$$

If (X, τ) also satisfies the zeroth separation condition, then (X, τ) is said to be regular in the strong sense.

Sometimes one may say that (X, τ) is *regular* when X is regular in the strict sense. In this case, one may say that X satisfies the *third separation condition*, or equivalently that (X, τ) is a T_3 *space*, when X is regular in the strong sense. However, the opposite convention is sometimes used as well. Alternatively, regularity, the third separation condition, and T_3 spaces may be used to refer to regularity in the strong sense, and regularity in the strict sense may be described in other ways.

1.10.1 Regularity in the strong sense implies Hausdorff

Proposition 1.10.3 *If (X, τ) is regular in the strong sense, then (X, τ) is Hausdorff.*

To see this, let x and y be distinct elements of X. Thus there is an open set that contains one of x and y and not the other, because (X, τ) satisfies the zeroth separation condition. Equivalently, this means that there is a closed set in X that contains one of x and y, and not the other. It follows that x and y are contained in disjoint open subsets of X, as desired.

1.10.2 Regularity and semimetrics

Proposition 1.10.4 *If the topology on X is determined by a semimetric $d(\cdot, \cdot)$, then X is regular in the strict sense. If $d(\cdot, \cdot)$ is a metric on X, then X is regular in the strong sense.*

Of course, the second statement follows from the first.

Lemma 1.10.5 *If $p \in X$ and r is a nonnegative real number, then*

$$V(p, r) = \{x \in X : d(p, x) > r\} \tag{1.10.6}$$

is an open set in X.

If $x \in V(p, r)$, then $t = d(p, x) - r > 0$, and one can check that

$$B(x, t) \subseteq V(p, r), \tag{1.10.7}$$

using the triangle inequality.

To prove the first part of Proposition 1.10.4, let $p \in X$ and a closed set $E \subseteq X$ be given, with $p \notin E$. Thus $X \setminus E$ is an open set in X that contains p, so that

$$B(p, r) \subseteq X \setminus E \tag{1.10.8}$$

for some $r > 0$. It is easy to see that $U = B(p, r/2)$ and $V = V(p, r/2)$ are disjoint open subsets of X that contain p and E, respectively.

1.10.3 Regularity of R with respect to τ_+, τ_-

Proposition 1.10.9 *The real line is regular in the strong sense with respect to the topologies τ_+ and τ_- defined in Subsection 1.3.2.*

Of course, we already know that $\mathbf{R}$ is Hausdorff with respect to τ_+ and τ_-, and so we only need to verify regularity in the strict sense. We shall do this for τ_+, the argument for τ_- being analogous. Let $p \in \mathbf{R}$ and a closed set $E \subseteq \mathbf{R}$ with respect to τ_+ be given, with $p \notin E$. This means that $\mathbf{R} \setminus E$ is an open set in $\mathbf{R}$ with respect to τ_+ that contains p, so that there is a $b \in \mathbf{R}$ such that $p < b$ and

$$[p, b) \subseteq \mathbf{R} \setminus E. \tag{1.10.10}$$

Equivalently,

$$E \subseteq \mathbf{R} \setminus [p, b) = (-\infty, p) \cup [b, +\infty). \tag{1.10.11}$$

In this situation, $[p, b)$ and $(-\infty, p) \cup [b, +\infty)$ are disjoint open subsets of $\mathbf{R}$ with respect to τ_+ that contain p and E, as desired.

Let (X, τ) be a topological space again, and let Y be a subset of X.

1.10.4 Regularity and induced topologies

Proposition 1.10.12 *If (X, τ) is regular in the strict sense, then Y is regular in the strict sense, with respect to the induced topology. Similarly, if (X, τ) is regular in the strong sense, then Y is regular in the strong sense, with respect to the induced topology.*

The second statement follows from the first, and the analogous fact for the zeroth separation condition. To prove the first statement, let $p \in Y$ and a closed set $A \subseteq Y$ with respect to the induced topology be given, with $p \notin A$. Remember that there is a closed set $A_1 \subseteq X$ such that $A = A_1 \cap Y$, because A is a closed set with respect to the induced topology. Note that $p \notin A_1$, because $p \in Y \setminus A$. It follows that there are disjoint open subsets U and V of X that contain p and A_1, respectively, because (X, τ) is regular in the strict sense. This implies that $U \cap Y$ and $V \cap Y$ are disjoint relatively open subsets of Y that contain p and A, respectively, as desired.

1.11 Completely Hausdorff spaces

Let (X, τ) be a topological space.

Proposition 1.11.1 *The following condition is equivalent to (X, τ) being regular in the strict sense: for every point $p \in X$ and open set $W \subseteq X$ with $p \in W$, there is an open set $U \subseteq X$ such that $p \in U$ and the closure $\overline{U}$ of U in X is contained in W.*

Let $p \in X$ and an open set $W \subseteq X$ with $p \in W$ be given, so that $E = X \setminus W$ is a closed set that does not contain p. If (X, τ) is regular in the strict sense, then there are disjoint open subsets U and V of X such that $p \in U$ and $E \subseteq V$. It is easy to see that $\overline{U} \cap V = \emptyset$ in this situation, which means that

$$\overline{U} \subseteq X \setminus V \subseteq X \setminus E = W, \tag{1.11.2}$$

as desired.

Conversely, let $p \in X$ and a closed set $E \subseteq X$ with $p \notin E$ be given. Thus $W = X \setminus E$ is an open set that contains p. If (X, τ) satisfies the condition in the statement of the proposition, then there is an open set $U \subseteq X$ such that $p \in U$ and $\overline{U} \subseteq W$. Put $V = X \setminus \overline{U}$, which is an open set in X, because $\overline{U}$ is a closed set. Clearly $U \cap V = \emptyset$, and

$$E = X \setminus W \subseteq X \setminus \overline{U} = V, \tag{1.11.3}$$

as desired.

1.11.1 The completely Hausdorff condition

Definition 1.11.4 *We say that (X, τ) is* completely Hausdorff *if for every pair x, y of distinct elements of X there are open subsets U and V of X such that $x \in U$, $y \in V$, and*

$$\overline{U} \cap \overline{V} = \emptyset. \tag{1.11.5}$$

In this case we may also say that (X, τ) satisfies separation condition number two and a half, *or equivalently that (X, τ) is a $T_{2\frac{1}{2}}$* space.

Of course, completely Hausdorff spaces are Hausdorff in particular.

Proposition 1.11.6 *If (X, τ) is regular in the strong sense, then (X, τ) is completely Hausdorff.*

Let x, y be distinct elements of X. If (X, τ) is regular in the strong sense, then (X, τ) is Hausdorff, and so there are disjoint open subsets U and V of X with $x \in U$ and $y \in V$. Using Proposition 1.11.1, we get open subsets U_1 and V_1 of X such that $x \in U_1$, $\overline{U_1} \subseteq U$, $y \in V_1$, and $\overline{V_1} \subseteq V$. It follows that $\overline{U_1} \cap \overline{V_1} = \emptyset$, as desired.

1.11.2 More on completely Hausdorff spaces

Proposition 1.11.7 *Let $\widetilde{\tau}$ be another topology on X such that $\tau \subseteq \widetilde{\tau}$. If (X, τ) is completely Hausdorff, then $(X, \widetilde{\tau})$ is completely Hausdorff too.*

If E is any subset of X, then let $\overline{E}_\tau$ and $\overline{E}_{\widetilde{\tau}}$ be the closures of E with respect to τ and $\widetilde{\tau}$, respectively. It is easy to see that

$$\overline{E}_{\widetilde{\tau}} \subseteq \overline{E}_\tau, \tag{1.11.8}$$

because $\tau \subseteq \widetilde{\tau}$. Using this, one can verify the proposition directly from the definition of the completely Hausdorff property.

Suppose for the moment that (X, τ) is regular in the strict or strong sense. If $\widetilde{\tau}$ is as before, then it is not so clear if $(X, \widetilde{\tau})$ should have the same property.

Proposition 1.11.9 *If (X, τ) is completely Hausdorff and $Y \subseteq X$, then Y is completely Hausdorff with respect to the induced topology.*

If $E \subseteq Y$ and $p \in Y$, then one can check that

$$p \text{ is adherent to } E \text{ in } X \tag{1.11.10}$$

if and only if

$$p \text{ is adherent to } E \text{ in } Y, \text{ with respect to the induced topology.} \tag{1.11.11}$$

This implies that the closure of E in Y, with respect to the induced topology, is the same as the intersection of Y with the closure of E in X. Using this, the proposition can be verified directly from the definitions.

1.12 Separated sets and connectedness

Let (X, τ) be a topological space.

Definition 1.12.1 *A pair A, B of subsets of X are said to be* separated *in X if*

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset. \tag{1.12.2}$$

Disjoint closed subsets of X are obviously separated in X. It is easy to see that disjoint open subsets of X are separated in X as well.

Proposition 1.12.3 *Let Y be a subset of X, and let A, B be subsets of Y. Under these conditions, A and B are separated in X if and only if A and B are separated in Y, with respect to the induced topology.*

This follows from the fact that the closure of a subset E of Y, with respect to the induced topology on Y, is the same as the intersection of Y with the closure of E in X.

Proposition 1.12.4 *If A and B are separated subsets of X such that $A \cup B = X$, then A and B are both open and closed in X.*

It is easy to see that $\overline{A} = A$ and $\overline{B} = B$ in this case, so that A and B are closed sets. This implies that A and B are open sets in X too, because their complements in X are closed sets.

1.12.1 Connectedness

Definition 1.12.5 *A subset E of X is said to be* connected *in X if E cannot be expressed as the union of two nonempty separated sets in X.*

It is well known that a subset E of the real line is connected with respect to the standard topology on $\mathbf{R}$ if and only if for every $x, y \in E$ with $x < y$, we have that

$$[x, y] \subseteq E. \tag{1.12.6}$$

Proposition 1.12.7 *Let Y be a subset of X, and let E be a subset of Y. Under these conditions, E is connected as a subset of X if and only if E is connected as a subset of Y, with respect to the induced topology.*

This follows from Proposition 1.12.3.

Proposition 1.12.8 *The following three statements are equivalent: (a) X is connected, as a subset of itself; (b) X cannot be expressed as the union of two nonempty disjoint open sets; (c) X cannot be expressed as the union of two nonempty disjoint closed sets.*

This follows from Proposition 1.12.4.

If X is any set equipped with the indiscrete topology, then X is connected. If X is any set with at least two elements equipped with the discrete topology, then X is not connected.

1.13 Normality and complete normality

Let (X, τ) be a topological space.

Definition 1.13.1 *We say that (X, τ) is* normal in the strict sense *if for every pair A, B of disjoint closed subsets of X, there are disjoint open subsets U, V of X such that $A \subseteq U$ and $B \subseteq V$. If (X, τ) also satisfies the first separation condition, then (X, τ) is said to be* normal in the strong sense*.*

Sometimes one may say that (X, τ) is *normal* when X is normal in the strict sense, and that X satisfies the *fourth separation condition*, or equivalently that X is a T_4 *space*, when X is normal in the strong sense. The opposite convention may be used sometimes as well. One may also use normality, the fourth separation condition, and T_4 spaces for normality in the strong sense, and refer to normality in the strict sense in other ways.

1.13.1 Normality in the strong sense

Proposition 1.13.2 *If (X, τ) is normal in the strong sense, then (X, τ) is Hausdorff, and regular in the strong sense.*

One can check that (X, τ) satisfies the first separation condition if and only if for every $p \in X$, $\{p\}$ is a closed set in X. Of course, this is the same as saying that $X \setminus \{p\}$ is an open set in X for every $p \in X$. More precisely, the first separation condition says that every element of $X \setminus \{p\}$ is an element of the interior of $X \setminus \{p\}$. The proposition follows easily from this and the definitions.

1.13.2 Normality in the strict sense

Proposition 1.13.3 *The following condition is equivalent to (X, τ) being normal in the strict sense: if $A \subseteq X$ is a closed set, $W \subseteq X$ is an open set, and $A \subseteq W$, then there is an open set $U \subseteq X$ such that $A \subseteq U$ and $\overline{U} \subseteq W$.*

Let a closed set $A \subseteq X$ and an open set $W \subseteq X$ with $A \subseteq W$ be given, so that $B = X \setminus W$ is a closed set in X that is disjoint from A. If (X, τ) is normal in the strict sense, then there are disjoint open subsets U, V of X such that $A \subseteq U$ and $B \subseteq V$. In this situation, we have that $\overline{U} \cap V = \emptyset$, so that

$$\overline{U} \subseteq X \setminus V \subseteq X \setminus B = W, \tag{1.13.4}$$

as desired.

Conversely, let A and B be disjoint closed subsets of X, so that $W = X \setminus B$ is an open set that contains A. If (X, τ) satisfies the condition in the statement of the proposition, then there is an open set $U \subseteq X$ such that $A \subseteq U$ and $\overline{U} \subseteq W$. Thus $V = X \setminus \overline{U}$ is an open set in X that is disjoint from U. We also have that

$$B = X \setminus W \subseteq X \setminus \overline{U} = V, \tag{1.13.5}$$

as desired.

1.13.3 Complete normality

Definition 1.13.6 *We say that (X, τ) is* completely normal in the strict sense *if for every pair A, B of separated subsets of X there are disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. If (X, τ) also satisfies the first separation condition, then (X, τ) is said to be* completely normal in the strong sense.

If (X, τ) is completely normal in the strict sense, then (X, τ) is normal in the strict sense, because disjoint closed subsets of X are separated in X. Similarly, if (X, τ) is completely normal in the strong sense, then (X, τ) is normal in the strong sense. Sometimes one may say that (X, τ) is *completely normal* when (X, τ) is completely normal in the strict sense, and that (X, τ) satisfies the *fifth separation condition*, or equivalently that (X, τ) is a T_5 *space*, when (X, τ) is completely normal in the strong sense. As before, the opposite convention may sometimes be used too. One may also use complete normality, the fifth separation condition, and T_5 spaces for complete normality in the strong sense, and refer to complete normality in the strict sense in other ways.

Proposition 1.13.7 *If (X, τ) is completely normal in the strict sense and Y is a subset of X, then Y is completely normal in the strict sense, with respect to the induced topology. If (X, τ) is completely normal in the strong sense, then Y is completely normal in the strong sense, with respect to the induced topology.*

The first part can be obtained from the definitions, using Proposition 1.12.3. The second part follows from the first part and the fact that Y satisfies the first separation condition when (X, τ) has this property.

1.14 Examples of completely normal spaces

Proposition 1.14.1 *If X is a set with a semimetric $d(\cdot, \cdot)$, then X is completely normal in the strict sense with respect to the topology determined by $d(\cdot, \cdot)$. If $d(\cdot, \cdot)$ is a metric on X, then X is completely normal in the strong sense with respect to the topology determined by $d(\cdot, \cdot)$.*

The second statement follows immediately from the first. To prove the first statement, let A and B be separated subsets of X. Thus, for each $x \in A$, there is a positive real number $r(x)$ such that

$$B(x, r(x)) \cap B = \emptyset, \tag{1.14.2}$$

because $x \not\in \overline{B}$. Similarly, for every $y \in B$, there is a $t(y) > 0$ such that

$$B(y, t(y)) \cap A = \emptyset, \tag{1.14.3}$$

because $y \not\in \overline{A}$. Put

$$U = \bigcup_{x \in A} B(x, r(x)/2), \quad V = \bigcup_{y \in B} B(y, t(y)/2). \tag{1.14.4}$$

These are open sets in X, because open balls are open sets, and unions of open sets are open sets. We also have that $A \subseteq U$ and $B \subseteq V$, because every $x \in A$ is contained in $B(x, r(x)/2)$, and every $y \in B$ is contained in $B(y, t(y)/2)$.

Suppose for the sake of a contradiction that $U \cap V \neq \emptyset$. This means that there are $x \in A$, $y \in B$, and $w \in X$ such that

$$w \in B(x, r(x)/2) \cap B(y, t(y)/2). \tag{1.14.5}$$

It follows that

$$d(x, y) \leq d(x, w) + d(w, y) < r(x)/2 + t(y)/2. \tag{1.14.6}$$

However,

$$d(x, y) \geq r(x),\ t(y), \tag{1.14.7}$$

by (1.14.2) and (1.14.3). This is a contradiction, as desired.

1.14.1 Complete normality of R with respect to τ_+, τ_-

Proposition 1.14.8 *The real line is completely normal in the strong sense with respect to the topologies τ_+ and τ_- defined in Subsection 1.3.2.*

We already know that the real line is Hausdorff with respect to τ_+ and τ_-, and so we only have to check complete normality in the strict sense. We shall do this for τ_+, the argument for τ_- being analogous. Let A and B be subsets of $\mathbf{R}$ that are separated with respect to τ_+. If $x \in A$, then there is a real number $b_1(x)$ such that $x < b_1(x)$ and

$$[x, b_1(x)) \cap B = \emptyset, \tag{1.14.9}$$

because x is not in the closure of B with respect to τ_+. Similarly, if $y \in B$, then there is a real number $b_2(y)$ such that $y < b_2(y)$ and

$$[y, b_2(y)) \cap A = \emptyset, \tag{1.14.10}$$

because y is not in the closure of A with respect to τ_+. Put

$$U = \bigcup_{x \in A} [x, b_1(x)), \quad V = \bigcup_{y \in B} [y, b_2(y)), \tag{1.14.11}$$

which are open subsets of $\mathbf{R}$ with respect to τ_+, because these intervals are open sets with respect to τ_+. Note that $A \subseteq U$ and $B \subseteq V$, by construction.

If $x \in A$ and $y \in B$, then

$$[x, b_1(x)) \cap [y, b_2(y)) = \emptyset. \tag{1.14.12}$$

More precisely, if $x \le y$, then $b_1(x) \le y$, by (1.14.9). Similarly, if $y \le x$, then $b_2(y) \le x$, by (1.14.10). It is easy to see that (1.14.12) holds in both cases. This implies that $U \cap V = \emptyset$, as desired.

1.15 More on connectedness

Let (X, τ) be a topological space.

Proposition 1.15.1 *If E is a connected subset of X, then the closure $\overline{E}$ of E in X is connected too.*

Suppose for the sake of a contradiction that $\overline{E}$ is not connected, so that there are nonempty separated subsets A and B of X such that $\overline{E} = A \cup B$. Put

$$A_1 = A \cap E, \quad B_1 = B \cap E, \tag{1.15.2}$$

and note that $E = A_1 \cup B_1$. It is easy to see that A_1 and B_1 are separated in X, because $A_1 \subseteq A$ and $B_1 \subseteq B$. We would like to check that A_1 and B_1 are nonempty.

Let x be an element of A. Thus $x \notin \overline{B}$, because $A \cap \overline{B} = \emptyset$, by hypothesis. This means that there is an open subset U of X such that $x \in U$ and $B \cap U = \emptyset$. However, $E \cap U \neq \emptyset$, because $x \in A \subseteq \overline{E}$. It follows that $A_1 \neq \emptyset$, because $E \cap U \subseteq A_1$. Similarly, $B_1 \neq \emptyset$. This implies that E is not connected, as desired.

Proposition 1.15.3 *Let I be a nonempty set, and suppose that E_j is a connected subset of X for every $j \in I$. If*

$$\bigcap_{j \in I} E_j \neq \emptyset, \tag{1.15.4}$$

then $\bigcup_{j \in I} E_j$ is connected in X.

Suppose for the sake of a contradiction that $\bigcup_{j \in I} E_j$ is not connected, so that there are nonempty separated subsets A and B of X such that $\bigcup_{j \in L} E_j = A \cup B$. Let x be an element of $\bigcap_{j \in I} E_j$. We may as well suppose that $x \in A$, by interchanging the roles of A and B, if necessary. Let y be an element of B, and let j_0 be an element of I such that $y \in E_{j_0}$. Put

$$A_0 = A \cap E_{j_0}, \quad B_0 = B \cap E_{j_0}, \tag{1.15.5}$$

and observe that $E_{j_0} = A_0 \cup B_0$. As before, A_0 and B_0 are separated in X, because $A_0 \subseteq A$ and $B_0 \subseteq B$. We also have that $x \in A_0$ and $y \in B_0$, by construction, so that $A_0, B_0 \neq \emptyset$. This implies that E_{j_0} is not connected, which is a contradiction.

1.15.1 A criterion for connectedness

Proposition 1.15.6 *Let E be a subset of X. Suppose that for every pair of elements x, y of E there is a connected subset $E(x, y)$ of X such that*

$$x, y \in E(x, y) \tag{1.15.7}$$

and

$$E(x, y) \subseteq E. \tag{1.15.8}$$

Under these conditions, E is connected in X.

Suppose for the sake of a contradiction that E is not connected in X, so that there are nonempty separated subsets A and B of X such that $E = A \cup B$. Let x be an element of A, let y be an element of B, and let $E(x, y)$ be as in the statement of the proposition. Put

$$A_2 = A \cap E(x, y), \quad B_2 = B \cap E(x, y), \tag{1.15.9}$$

which satisfy $x \in A_2$, $y \in B_2$, and $A_2 \cup B_2 = E(x, y)$. We also have that A_2 and B_2 are separated in X, because $A_2 \subseteq A$ and $B_2 \subseteq B$. This implies that $E(x, y)$ is not connected, which is a contradiction.

Of course, if E is connected, then E satisfies the condition mentioned in Proposition 1.15.6, with $E(x, y) = E$ for every $x, y \in E$.

Alternatively, Proposition 1.15.6 could be obtained from Proposition 1.15.3. More precisely, Proposition 1.15.6 is trivial when $E = \emptyset$. Otherwise, one can fix $x \in E$, and use the fact that E is the union of $E(x, y)$, $y \in E$.

If E_1 and E_2 are connected subsets of X and $E_1 \cap E_2 \neq \emptyset$, then $E_1 \cup E_2$ is connected in X as well, as in Proposition 1.15.3. One could also prove this directly, and use this to obtain Proposition 1.15.3 from Proposition 1.15.6.

Chapter 2

Cardinality and some more topology

2.1 Cardinal equivalence

Let A and B be sets, and let f be a function defined on A with values in B. This is also known as a *mapping* from A into B, which may be expressed by $f : A \to B$. As usual, we say that f is *one-to-one* or *injective* if for every $x, y \in A$ with $x \neq y$, we have that $f(x) \neq f(y)$. We say that f maps A *onto* B, or equivalently that f is *surjective*, if for every $z \in B$ there is an $x \in A$ such that $f(x) = z$. A one-to-one mapping from A onto B is also known as a *bijection*, or a *one-to-one correspondence*.

Let C be another set, and let g be a mapping from B into C. Thus the *composition* $g \circ f$ is the mapping from A into C defined by

$$(g \circ f)(x) = g(f(x)) \tag{2.1.1}$$

for every $x \in A$. If f and g are injections, then it is easy to see that $g \circ f$ is an injection. If f and g are surjections, then $g \circ f$ is a surjection. It follows that if f and g are bijections, then $g \circ f$ is a bijection.

If f is a one-to-one mapping from A onto B, then the inverse mapping is defined on B by $f^{-1}(f(x)) = x$ for every $x \in A$, which is a one-to-one mapping from B onto A. If g is a one-to-one mapping from B onto C, then one can check that

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}. \tag{2.1.2}$$

If there is a one-to-one mapping from A onto B, then let us express this by

$$\#A = \#B. \tag{2.1.3}$$

Note that $\#A = \#A$, because the identity mapping on A is a one-to-one mapping from A onto itself. It is easy to see that $\#A = \#B$ implies $\#B = \#A$, because the inverse of a bijection is a bijection. Similarly, $\#A = \#B$ and

$\#B = \#C$ imply that $\#A = \#C$, because the composition of bijections is a bijection.

To say that A is a finite set with exactly n elements for some positive integer n means that

$$\#A = \#\{1, \ldots, n\}. \tag{2.1.4}$$

By definition, A is *countably infinite* when

$$\#A = \#\mathbf{Z}_+. \tag{2.1.5}$$

2.1.1 The power set

Let $\mathcal{P}(A)$ be the *power set* of A, which is the set of all subsets of A. Suppose that f is a mapping from A into $\mathcal{P}(A)$, and put

$$B = \{a \in A : a \notin f(a)\}, \tag{2.1.6}$$

which is a subset of A. If $x \in A$, then it is easy to see that

$$f(x) \neq B. \tag{2.1.7}$$

More precisely, if $x \in f(x)$, then $x \notin B$, so that (2.1.7) holds. Otherwise, if $x \notin f(x)$, then $x \in B$, and (2.1.7) holds. Thus f cannot map A onto $\mathcal{P}(A)$. In particular,

$$\#A \neq \#\mathcal{P}(A). \tag{2.1.8}$$

A set is said to be *uncountable* if it is neither finite nor countably infinite. Note that $\mathcal{P}(\mathbf{Z}_+)$ is uncountable, by (2.1.8).

Let $\mathcal{B}$ be the set of infinite sequences $x = \{x_j\}_{j=1}^\infty$ such that for each positive integer j, $x_j = 0$ or 1. It is easy to see that

$$\#\mathcal{B} = \#\mathcal{P}(\mathbf{Z}_+), \tag{2.1.9}$$

using the correspondence between elements of $\mathcal{B}$ and the subsets of $\mathbf{Z}_+$ on which they are equal to 1.

2.1.2 The cardinality of $[0,1]$

It is well known that

$$\#\mathcal{B} = \#[0,1]. \tag{2.1.10}$$

More precisely, we can start with the usual mapping that sends $x \in \mathcal{B}$ to $\sum_{j=1}^\infty x_j\, 2^{-j}$, which is an element of $[0,1]$. This mapping is surjective but not injective, and we can modify it on a countable set to get a bijection. If n is a nonnegative integer, then put

$$A_{n,0} = \{x \in \mathcal{B} : x_j = 0 \text{ for every } j > n\} \tag{2.1.11}$$

and

$$A_{n,1} = \{x \in \mathcal{B} : x_j = 1 \text{ for every } j > n\}. \tag{2.1.12}$$

These are finite sets with exactly 2^n elements, and

$$A = \bigcup_{n=0}^{\infty} (A_{n,0} \cup A_{n,1}) \tag{2.1.13}$$

is a countably-infinite subset of $\mathcal{B}$. Also let E be the subset of $[0,1]$ consisting of nonnegative integer multiples of nonnegative integer powers of $1/2$, which is countably infinite as well. The restriction of the mapping from $\mathcal{B}$ onto $[0,1]$ mentioned earlier to $\mathcal{B} \setminus A$ is a one-to-one mapping onto $[0,1] \setminus E$. Because A and E are both countably infinite, one can find a one-to-one mapping from A onto E. One can combine these two mappings to get a one-to-one mapping from $\mathcal{B}$ onto $[0,1]$, as desired.

2.2 Comparing cardinalities

If A and B are sets, and if there is a one-to-one mapping from A into B, then we may express this by

$$\#A \le \#B. \tag{2.2.1}$$

If $\#A = \#B$, then it follows that $\#A \le \#B$ and $\#B \le \#A$. In particular, $\#A \le \#A$ automatically. If A, B, and C are sets such that $\#A \le \#B$ and $\#B \le \#C$, then it is easy to see that $\#A \le \#C$, because the composition of injective mappings is injective.

A famous theorem of Bernstein and Schröder states that if A and B are sets with $\#A \le \#B$ and $\#B \le \#A$, then $\#A = \#B$. This corresponds to Theorem 7 on p33 of [236].

If A is any set, then

$$a \mapsto \{a\} \tag{2.2.2}$$

defines a one-to-one mapping from A into the power set $\mathcal{P}(A)$ of A. This implies that

$$\#A \le \#\mathcal{P}(A). \tag{2.2.3}$$

Let A be a nonempty set, and let f be a one-to-one mapping from A into a set B. Let a_0 be an element of A, and consider the mapping g from B into A defined as follows. We put

$$g(f(a)) = a \tag{2.2.4}$$

for every $a \in A$, and $g(z) = a_0$ for every $z \in B \setminus f(A)$. Here

$$f(A) = \{f(a) : a \in A\} \tag{2.2.5}$$

is the image of A under f, as usual. Clearly g maps B onto A.

2.2.1 The axiom of choice

Let I be a nonempty set, and suppose that X_j is a nonempty set for every $j \in I$. Under these conditions, the *axiom of choice* states that there is a function f defined on I with values in $\bigcup_{j \in I} X_j$ such that

$$f(j) \in X_j \tag{2.2.6}$$

for every $j \in I$.

Let g be a mapping from a set B onto a nonempty set A. Thus, if $a \in A$, then

$$g^{-1}(\{a\}) = \{z \in B : g(z) = a\} \neq \emptyset. \tag{2.2.7}$$

Using the axiom of choice, we get a mapping f from A into B such that $f(a)$ is an element of $g^{-1}(\{a\})$ for every $a \in A$. This is the same as saying that $g(f(a)) = a$ for every $a \in A$. In particular, it follows that f is a one-to-one mapping from A into B.

If A and B are any two sets, then there are well-known arguments using the axiom of choice to get that $\#A \leq \#B$ or $\#B \leq \#A$.

Let A be an infinite set, and let C be a set such that

$$\#C \leq \#A. \tag{2.2.8}$$

There are well-known arguments using the axiom of choice to get that

$$\#A = \#(A \cup C). \tag{2.2.9}$$

Of course, $\#A \leq \#(A \cup C)$, because the obvious inclusion mapping from A into $A \cup C$ is an injection. Note that we can reduce to the case where $A \cap C = \emptyset$, by replacing C with $C \setminus A$.

If C has only finitely or countably many elements, then we can get (2.2.9) more directly, as follows. Let B be a countably-infinite subset of A, so that $B \cup C$ is countably-infinite as well. In particular, there is a one-to-one mapping from B onto $B \cup C$. If $A \cap C = \emptyset$, then we can combine this with the identity mapping on $A \setminus B$ to get a one-to-one mapping from A onto $A \cup C$, as desired.

If a and b are real numbers with $a < b$, then we get that

$$\#(a, b) = \#[a, b) = \#(a, b] = \#[a, b]. \tag{2.2.10}$$

One can use explicit mappings to get that $\#[a, b] = \#[0, 1]$ and $\#\mathbf{R} = \#(-1, 1)$. One can also use the argument in the previous paragraph to get that $\#(\mathbf{R} \setminus \mathbf{Q}) = \#\mathbf{R}$.

2.3 Products and exponentials

The *Cartesian product* of two sets A and B is the set $A \times B$ of all ordered pairs (a, b), with $a \in A$ and $b \in B$. Let $\widetilde{A}$ and $\widetilde{B}$ be sets, let ϕ be a one-to-one mapping from A onto $\widetilde{A}$, and let ψ be a one-to-one mapping from B onto $\widetilde{B}$. Under these conditions,

$$(a, b) \mapsto (\phi(a), \psi(b)) \tag{2.3.1}$$

is a one-to-one mapping from $A \times B$ onto $\widetilde{A} \times \widetilde{B}$. This implies that

$$\#(A \times B) = \#(\widetilde{A} \times \widetilde{B}). \tag{2.3.2}$$

Note that

$$(a, b) \mapsto (b, a) \tag{2.3.3}$$

is a one-to-one mapping from $A \times B$ onto $B \times A$. It follows that

$$\#(A \times B) = \#(B \times A). \tag{2.3.4}$$

If A is an infinite set, then there are well-known arguments using the axiom of choice to get that

$$\#(A \times A) = \#A. \tag{2.3.5}$$

Of course, this can be shown more directly when A is countably infinite.

2.3.1 Exponentials

If A and B are any two sets again, then the space of all mappings from A into B may be denoted B^A. Let $\widetilde{A}$ and $\widetilde{B}$ be sets with one-to-one mappings ϕ and ψ from A onto $\widetilde{A}$ and from B onto $\widetilde{B}$, respectively, again. If f is a mapping from A into B, then $\psi \circ f \circ \phi^{-1}$ is a mapping from $\widetilde{A}$ into $\widetilde{B}$. It is easy to see that

$$f \mapsto \psi \circ f \circ \phi^{-1} \tag{2.3.6}$$

is a one-to-one mapping from B^A onto $\widetilde{B}^{\widetilde{A}}$. Thus

$$\#B^A = \#\widetilde{B}^{\widetilde{A}} \tag{2.3.7}$$

under these conditions.

If $B = \{0, 1\}$, then B^A may be denoted 2^A. There is a simple one-to-one correspondence between 2^A and the power set $\mathcal{P}(A)$ of A, where a subset of A corresponds to the function on A that is equal to 1 on the subset, and to 0 on the complement of the subset in A. This means that

$$\#2^A = \#\mathcal{P}(A). \tag{2.3.8}$$

2.3.2 Exponentials of disjoint unions

If A, B, and C are sets with $A \cap B = \emptyset$, then there is a natural one-to-one correspondence between $C^{A \cup B}$ and $C^A \times C^B$. This is because a mapping from $A \cup B$ into C corresponds exactly to a pair of mappings from A into C and from B into C. It follows that

$$\#C^{A \cup B} = \#(C^A \times C^B). \tag{2.3.9}$$

If A and B are countably infinite, then it is well known that $A \cup B$ is countably infinite. Of course, it is easy to find disjoint countably-infinite sets, to get that

$$\#C^{\mathbf{Z}_+} = \#(C^{\mathbf{Z}_+} \times C^{\mathbf{Z}_+}). \tag{2.3.10}$$

In particular, we can take $C = \{0, 1\}$, to get that

$$\#2^{\mathbf{Z}_+} = \#(2^{\mathbf{Z}_+} \times 2^{\mathbf{Z}_+}). \tag{2.3.11}$$

This implies that $\#([0,1] \times [0,1]) = \#[0,1]$, $\#(\mathbf{R} \times \mathbf{R}) = \#\mathbf{R}$.

2.4 Some more exponentials

Let A, B_1, and B_2 be sets, and suppose that ψ is a one-to-one mapping from B_1 into B_2. If f is a mapping from A into B_1, then $\psi \circ f$ is a mapping from A into B_2. Thus

$$f \mapsto \psi \circ f \tag{2.4.1}$$

defines a mapping from B_1^A into B_2^A. It is easy to see that (2.4.1) is injective, because ψ is injective, by hypothesis. This shows that

$$\#B_1^A \le \#B_2^A \tag{2.4.2}$$

when $\#B_1 \le \#B_2$.

Similarly, let A_1, A_2, and B be sets, and suppose that ρ is a mapping from A_2 onto A_1. If f is a mapping from A_1 into B, then $f \circ \rho$ is a mapping from A_2 into B. This defines a mapping

$$f \mapsto f \circ \rho \tag{2.4.3}$$

from B^{A_1} into B^{A_2}. Observe that (2.4.3) is injective, because $\rho(A_2) = A_1$, by hypothesis. This implies that

$$\#B^{A_1} \le \#B^{A_2} \tag{2.4.4}$$

when $\#A_1 \le \#A_2$.

2.4.1 Exponentials of products

If A, B, and C are sets, then there is a simple one-to-one correspondence between the set $C^{A \times B}$ of mappings from $A \times B$ into C and the set $(C^B)^A$ of all mappings from A into the set C^B of all mappings from B into C. More precisely, let $f(a, b)$ be a mapping from $A \times B$ into C, and put

$$f_a(b) = f(a, b) \tag{2.4.5}$$

for every $a \in A$ and $b \in B$. Thus, for each $a \in A$, this defines f_a as a mapping from B into C. It follows that

$$a \mapsto f_a \tag{2.4.6}$$

defines a mapping from A into C^B. This defines a mapping

$$f \mapsto (2.4.6) \tag{2.4.7}$$

from $C^{A\times B}$ into $(C^B)^A$. Conversely, every mapping from A into C^B leads to a mapping from $A \times B$ into C in this way. In particular, we get that

$$\#C^{A\times B} = \#(C^B)^A. \tag{2.4.8}$$

If A is an infinite set, then we can take $B = A$ in (2.4.8) and use (2.3.5) to obtain that

$$\#(C^A)^A = \#C^{A\times A} = \#C^A. \tag{2.4.9}$$

Of course, this also uses (2.3.7) in the second step. Let E be a set such that

$$\#C \le \#E \le \#C^A. \tag{2.4.10}$$

Using (2.4.2), we get that

$$\#C^A \le \#E^A \le \#(C^A)^A = \#C^A, \tag{2.4.11}$$

so that

$$\#C^A = \#E^A. \tag{2.4.12}$$

In particular, we can take $C = \{0, 1\}$ and $E = A$, to obtain that

$$\#A^A = \#2^A. \tag{2.4.13}$$

2.5 Strong limit points and separability

Let (X, τ) be a topological space.

Definition 2.5.1 *A point $p \in X$ is said to be a* strong limit point *of a subset E of X if for every open subset U of X with $p \in U$, there are infinitely many elements of E in U. Similarly, $p \in X$ is said to be a* condensation point *if for every open set $U \subseteq X$ with $p \in U$, $E \cap U$ is uncountable.*

Thus a strong limit point of E is automatically a limit point of E, and a condensation point of E is a strong limit point of E. Of course, there are analogous notions using other conditions on the cardinality of $E \cap U$ as in Definition 2.5.1.

Proposition 2.5.2 *If $p \in X$ is a limit point of $E \subseteq X$, and if X satisfies the first separation condition, then p is a strong limit point of E in X.*

If p is not a strong limit point of E, then there is an open set $U \subseteq X$ such that $p \in U$ and $E \cap U$ has only finitely many elements. If X satisfies the first separation condition, then one can find a smaller open set $V \subseteq X$ such that $p \in V$ and V does not contain any elements of E, other than p, if $p \in E$. This implies that p is not a limit point of E in X.

2.5.1 Separable topological spaces

Definition 2.5.3 *We say that* (X, τ) *is* separable *if there is a dense subset* E *of* X *such that* E *has only finitely or countably many elements.*

Suppose that (X, τ) is a Hausdorff topological space that satisfies the first countability condition, and that E is a dense subset of X. An infinite sequence of elements of E is the same as a function on the set $\mathbf{Z}_+$ with values in E, which is an element of $E^{\mathbf{Z}_+}$. Let $\mathcal{C}$ be the collection of all sequences of elements of E that converge to an element of X, so that $\mathcal{C}$ may be considered as a subset of $E^{\mathbf{Z}_+}$. Remember that the limit of a convergent sequence is unique in a Hausdorff space. Thus

$$\{x_j\}_{j=1}^\infty \mapsto \lim_{j \to \infty} x_j \tag{2.5.4}$$

defines a function on $\mathcal{C}$ with values in X.

If E is dense in X, and (X, τ) satisfies the first countability condition, then every element of X is the limit of a convergent sequence of elements of E. This means that (2.5.4) maps $\mathcal{C}$ onto X under these conditions. It follows that

$$\#X \le \#\mathcal{C}, \tag{2.5.5}$$

using the axiom of choice, as in Subsection 2.2.1. This implies that

$$\#X \le \#E^{\mathbf{Z}_+}, \tag{2.5.6}$$

because $\mathcal{C} \subseteq E^{\mathbf{Z}_+}$. If E has only finitely or countably many elements, then we obtain that

$$\#X \le \#\mathbf{Z}_+^{\mathbf{Z}_+} = \#2^{\mathbf{Z}_+}. \tag{2.5.7}$$

Note that the real line is separable with respect to the standard topology, because $\mathbf{Q}$ is a countable dense subset of $\mathbf{R}$. We have also seen that $\mathbf{R}$ is Hausdorff and satisfies the first countability condition with respect to the standard topology, and that $\#\mathbf{R} = \#2^{\mathbf{Z}_+}$.

2.6 Bases for topologies

Let (X, τ) be a topological space.

Definition 2.6.1 *A collection* $\mathcal{B}$ *of open subsets of* X *is said to be a* base *for the topology* τ *on* X *if for every* $x \in X$ *and open subset* V *of* X *with* $x \in V$ *there is an element* U *of* $\mathcal{B}$ *such that* $x \in U$ *and* $U \subseteq V$.

If $\mathcal{B}$ is a base for τ and W is an open subset of X, then

$$W = \bigcup \{U : U \in \mathcal{B}, U \subseteq W\}. \tag{2.6.2}$$

More precisely, the right side of (2.6.2) is automatically contained in W. If $\mathcal{B}$ is a base for τ, then every element of W is contained in the union on the right side of (2.6.2). Conversely, if $\mathcal{B}$ is a collection of open subsets of X, and if every open subset of X can be expressed as a union of elements of $\mathcal{B}$, then it is easy to see that $\mathcal{B}$ is a base for τ.

2.6.1 Bases and local bases

If $\mathcal{B}$ is a base for τ and $x \in X$, then

$$\mathcal{B}(x) = \{U \in \mathcal{B} : x \in U\} \tag{2.6.3}$$

is a local base for the topology of X at x. Conversely, if $\mathcal{B}$ is a collection of open subsets of X, and for each $x \in X$, (2.6.3) is a local base for the topology of X at x, then $\mathcal{B}$ is a base for τ.

2.6.2 The second countability condition

Definition 2.6.4 *If there is a base $\mathcal{B}$ for τ such that $\mathcal{B}$ has only finitely or countably many elements, then we say that (X, τ) satisfies the* second countability condition.

Suppose that (X, τ) satisfies the second countability condition, and let $\mathcal{B}$ be a base for τ with only finitely or countably many elements. If $x \in X$, then it follows that (2.6.3) has only finitely or countably many elements as well. This means that (X, τ) satisfies the first countability condition.

2.6.3 Bases and semimetrics

Proposition 2.6.5 *Let $d(\cdot, \cdot)$ be a semimetric on X, and suppose that τ is the topology determined on X by $d(\cdot, \cdot)$. Also let E be a dense subset of X, and let $\mathcal{B}_E$ be the collection of open balls in X centered at elements of E with radius $1/j$ for some positive integer j. Under these conditions, $\mathcal{B}_E$ is a base for τ. If E has only finitely or countably many elements, then $\mathcal{B}_E$ has only finitely or countably many elements too.*

Let $x \in X$ and a positive real number r be given, and let j be a positive integer such that $2/j < r$. Because E is dense in X, there is a $y \in E$ such that

$$d(x, y) < 1/j. \tag{2.6.6}$$

This implies that $x \in B(y, 1/j)$, and one can check that

$$B(y, 1/j) \subseteq B(x, r), \tag{2.6.7}$$

using the triangle inequality. It follows that $\mathcal{B}_E$ is a base for τ, because $B(y, 1/j)$ is an element of $\mathcal{B}_E$, by construction.

If j is a positive integer, then let $\mathcal{B}_{E,j}$ be the collection of open balls in X centered at elements of E with radius $1/j$, so that

$$\mathcal{B}_E = \bigcup_{j=1}^{\infty} \mathcal{B}_{E,j}. \tag{2.6.8}$$

If E has only finitely or countably many elements, then it is easy to see that $\mathcal{B}_{E,j}$ has only finitely or countably many elements for each $j \geq 1$. This implies that (2.6.8) has only finitely or countably many elements, by standard results about countable sets.

2.6.4 Second countability implies separability

Proposition 2.6.9 *If (X, τ) satisfies the second countability condition, then (X, τ) is separable.*

Let $\mathcal{B}$ be a base for τ. If $U \in \mathcal{B}$ and $U \neq \emptyset$, then let us choose a point x_U in U, and let E be the set of points chosen in this way. It is easy to see that E is dense in X with respect to τ. If $\mathcal{B}$ has only finitely or countably many elements, then one can check that E has only finitely or countably many elements as well.

A topological space that satisfies the second countability condition is sometimes said to be *completely separable*.

2.6.5 Bases and induced topologies

Proposition 2.6.10 *Let Y be a subset of X. If $\mathcal{B}$ is a base for τ, then*

$$\mathcal{B}_Y = \{U \cap Y : U \in \mathcal{B}\} \tag{2.6.11}$$

is a base for the induced topology on Y.

In particular, if (X, τ) satisfies the second countability condition, then Y satisfies the second countability condition with respect to the induced topology.

The first part can be verified directly from the definitions. In the second part, if $\mathcal{B}$ has only finitely or countably many elements, then one can verify that $\mathcal{B}_Y$ has only finitely or countably many elements.

2.6.6 Some remarks about bases for τ_+, τ_-

Let τ_+ be the topology defined on the real line as in Subsection 1.3.2, and let $\mathcal{B}_+$ be a base for τ_+. If $x \in \mathbf{R}$, then there is an element $U(x)$ of $\mathcal{B}_+$ such that

$$x \in U(x) \quad \text{and} \quad U(x) \subseteq [x, +\infty), \tag{2.6.12}$$

because $[x, +\infty)$ is an open set with respect to τ_+. If $y \in \mathbf{R}$ and $x \neq y$, then it is easy to see that

$$U(x) \neq U(y). \tag{2.6.13}$$

If $\mathcal{B}_+$ has only finitely or countably many elements, then one could use this to get that the real line has only finitely or countably many elements, which is a contradiction. Thus $\mathcal{B}_+$ is uncountable, which means that $(\mathbf{R}, \tau_+)$ does not satisfy the second countability condition.

Remember that $\mathbf{Q}$ is a countable set which is dense in $\mathbf{R}$ with respect to τ_+, so that $(\mathbf{R}, \tau_+)$ is separable. It follows that there is no metric on $\mathbf{R}$ for which τ_+ is the corresponding topology, by Proposition 2.6.5.

Of course, there are analogous statements for the topology τ_- defined on $\mathbf{R}$ in Subsection 1.3.2.

2.6.7 Some more remarks about bases

Let (X, τ) be any topological space again, and let $\mathcal{B}$ be a base for τ. If V is an open subset of X, then put

$$\phi(V) = \{U \in \mathcal{B} : U \subseteq V\}, \tag{2.6.14}$$

which is a subset of $\mathcal{B}$. This defines ϕ as a mapping from τ into the power set $\mathcal{P}(\mathcal{B})$ of all subsets of $\mathcal{B}$.

If $\mathcal{E}$ is any subset of $\mathcal{B}$, then put

$$\psi(\mathcal{E}) = \bigcup_{U \in \mathcal{E}} U, \tag{2.6.15}$$

which is interpreted as being the empty set when $\mathcal{E} = \emptyset$. Note that (2.6.15) is an open subset of X, because the elements of $\mathcal{B}$ are open subsets of X, by hypothesis. This defines ψ as a mapping from $\mathcal{P}(\mathcal{B})$ into τ.

The composition $\psi \circ \phi$ of ϕ and ψ is the identity mapping on τ, as in (2.6.2). Thus ϕ is an injection, and ψ is a surjection. In particular,

$$\#\tau \le \#\mathcal{P}(\mathcal{B}). \tag{2.6.16}$$

2.7 Continuous mappings

Let X and Y be topological spaces.

Definition 2.7.1 *A mapping f from X into Y is said to be* continuous *at a point $x \in X$ if for every open subset V of Y with $f(x) \in V$ there is an open subset U of X such that $x \in U$ and*

$$f(U) \subseteq V. \tag{2.7.2}$$

It is easy to see that this reduces to the usual definition of continuity for mappings between metric spaces when the topologies are determined by metrics.

2.7.1 Continuity and convergent nets

Proposition 2.7.3 *Let $(A, \preceq)$ be a nonempty directed system, and let $\{x_a\}_{a \in A}$ be a net of elements of X indexed by A that converges to x in X. If a mapping f from X into Y is continuous at x, then $\{f(x_a)\}_{a \in A}$ converges to $f(x)$, as a net of elements of Y indexed by A.*

Let V be an open subset of Y that contains $f(x)$, and let U be an open subset of X that contains x and satisfies (2.7.2). Because $\{x_a\}_{a \in A}$ converges to x in X, there is a $b \in A$ such that $x_a \in U$ for every $a \in A$ with $b \preceq a$. This implies that

$$f(x_a) \in f(U) \subseteq V \tag{2.7.4}$$

for every $a \in A$ with $b \preceq a$, as desired.

2.7.2 The converse

Conversely, let f be a mapping from X into Y, and suppose that f is not continuous at $x \in X$. This means that there is an open set $V \subseteq Y$ such that $f(x) \in V$, and for every open set $U \subseteq X$ with $x \in U$, we have that

$$f(U) \not\subseteq V. \tag{2.7.5}$$

Let $\mathcal{B}(x)$ be a local base for the topology of X at x. If $U \in \mathcal{B}(x)$, then let us choose a point $x_U \in U$ such that

$$f(x_U) \notin V. \tag{2.7.6}$$

Remember that $\mathcal{B}(x)$ is a directed system with respect to the partial ordering defined by putting $U_1 \preceq U_2$ when $U_1, U_2 \in \mathcal{B}(x)$ satisfies $U_2 \subseteq U_1$. In this situation, we have seen that $\{x_U\}_{U \in \mathcal{B}(x)}$ converges to x, as a net of elements of X indexed by $\mathcal{B}(x)$. However, $\{f(x_U)\}_{U \in \mathcal{B}(x)}$ does not converge to $f(x)$ in Y.

2.7.3 Another version of the converse

Suppose for the moment that there is a local base for the topology of X at x with only finitely or countably many elements. This implies that there is a sequence $U_1(x), U_2(x), U_3(x), \ldots$ of open subsets of X such that $x \in U_j(x)$ for every $j \geq 1$, the collection of $U_j(x)$'s, $j \geq 1$, is a local base for the topology of X at x, and $U_{j+1}(x) \subseteq U_j(x)$ for every $j \geq 1$.

If $V \subseteq Y$ is as in the previous subsection, then, for each positive integer j, we can choose $x_j \in U_j(x)$ such that

$$f(x_j) \notin V \tag{2.7.7}$$

It follows that $\{x_j\}_{j=1}^{\infty}$ converges to x in X, while $\{f(x_j)\}_{j=1}^{\infty}$ does not converge to $f(x)$ in Y.

2.7.4 Continuity on X

Definition 2.7.8 *A mapping f from X into Y is said to be* continuous *on X if f is continuous at every point in X.*

Proposition 2.7.9 *A mapping f from X into Y is continuous if and only if for every open set $V \subseteq Y$,*

$$f^{-1}(V) = \{x \in X : f(x) \in V\} \tag{2.7.10}$$

is an open subset of X. This is also equivalent to the condition that for every closed set $E \subseteq Y$, $f^{-1}(E)$ is a closed set in X.

Suppose that f is continuous, and let V be an open subset of Y. If x is an element of $f^{-1}(V)$, then one can use the continuity of f at x to get that x is an element of the interior of $f^{-1}(V)$ in X. This implies that $f^{-1}(V)$ is an open set in X, as desired.

Conversely, let $x \in X$ and an open set $V \subseteq Y$ that contains $f(x)$ be given. Observe that $U = f^{-1}(V)$ contains x and satisfies

$$f(U) = f(f^{-1}(V)) \subseteq V. \tag{2.7.11}$$

If $f^{-1}(V)$ is an open set in X, then the conditions for continuity of f at x are satisfied.

The second part of the proposition can be obtained from the first part, using the fact that

$$f^{-1}(Y \setminus E) = X \setminus f^{-1}(E) \tag{2.7.12}$$

for every $E \subseteq Y$.

2.7.5 Some basic classes of examples

Let us consider a few basic classes of examples of continuous mappings. If X is equipped with the discrete topology, then any mapping f from X into any topological space Y is continuous. Similarly, if Y is equipped with the indiscrete topology, then any mapping f from any topological space X into Y is continuous.

Let X be a set, and let τ, $\widetilde{\tau}$ be topologies on X. It is easy to see that the identity mapping on X is continuous as a mapping from X equipped with $\widetilde{\tau}$ into X equipped with τ if and only if $\tau \subseteq \widetilde{\tau}$.

Let Y be a topological space, and let X be a subset of Y, equipped with the induced topology. One can check that the obvious inclusion mapping from X into Y, which sends every element of X to itself, is continuous as a mapping from X into Y.

Let X be any set equipped with the indiscrete topology, and let Y be a topological space that satisfies the zeroth separation condition. If f is a continuous mapping from X into Y, then f is constant on X.

2.7.6 Locally constant mappings

A mapping f from a topological space X into a set Y is said to be *locally constant* if for every $x \in X$ there is an open subset $U(x)$ of X such that $x \in U(x)$ and f is constant on $U(x)$. It is easy to see that a locally constant mapping from a topological space X into a topological space Y is continuous. If Y is equipped with the discrete topology, then any continuous mapping from a topological space X into Y is locally constant.

2.7.7 Locally constant mappings and connectedness

If f is a locally constant mapping from a connected topological space X into a set Y, then one can check that f is constant on X. Conversely, if X is not connected, then one can find a locally constant mapping on X that is not constant.

2.8 More on continuity

Let X, Y, and Z be topological spaces.

Proposition 2.8.1 *Let f be a mapping from X into Y, and let g be a mapping from Y into Z. If f is continuous at $x \in X$, and g is continuous at $f(x)$, then their composition $g \circ f$ is continuous at x, as a mapping from X into Z. In particular, if f is continuous on X, and g is continuous on Y, then $g \circ f$ is continuous as a mapping from X into Z.*

The first part can be verified directly from the definitions. The second part follows from the first part, and can also be obtained from the characterization of continuity in Proposition 2.7.9.

2.8.1 Homeomorphisms

Definition 2.8.2 *A one-to-one mapping f from X onto Y is said to be a* homeomorphism *if f is continuous as a mapping from X into Y, and the inverse mapping f^{-1} is continuous as a mapping from Y into X.*

The identity mapping on X is a homeomorphism from X onto itself. If f is a homeomorphism from X onto Y, then the inverse mapping f^{-1} is a homeomorphism from Y onto X. In this case, if g is a homeomorphism from Y onto Z, then $g \circ f$ is a homeomorphism from X onto Z.

Consider the mapping f from the real line onto itself defined by $f(x) = -x$ for every $x \in \mathbf{R}$. This is a homeomorphism as a mapping from the real line onto itself with respect to the standard topology. This is also a homeomorphism from the real line equipped with the topology τ_+ defined in Subsection 1.3.2 onto $\mathbf{R}$ equipped with the analogous topology τ_-.

2.8.2 Open mappings

Definition 2.8.3 *A mapping f from X into Y is said to be an* open mapping *if for every open subset U of X, $f(U)$ is an open subset of Y.*

If f is a one-to-one mapping from X onto Y, then f is an open mapping if and only if f^{-1} is continuous as a mapping from Y into X.

2.8.3 Restrictions of continuous mappings

Proposition 2.8.4 *Let f be a continuous mapping from X into Y, and let X_0 be a subset of X. Under these conditions, the restriction of f to X_0 is continuous as a mapping from X_0 into Y, with respect to the topology induced on X_0 by the topology on X.*

This can be verified directly from the definitions. Alternatively, one can use the fact that the restriction of f to X_0 is the same as the composition of f with the inclusion mapping from X_0 into X.

Proposition 2.8.5 *Let Y_0 be a subset of Y, and let f be a mapping from X into Y_0. In this situation, f is continuous as a mapping from X into Y_0, with respect to the topology induced on Y_0 by the topology on Y, if and only if f is continuous as a mapping from X into Y.*

This can be verified directly from the definitions too.

2.8.4 Continuity and connectedness

Proposition 2.8.6 *If f is a continuous mapping from X into Y, and E is a connected subset of X, then $f(E)$ is connected as a subset of Y.*

It is not difficult to show this directly, using the definition of connectedness in terms of separated sets. Alternatively, one can use the previous two propositions to reduce to the case where $E = X$ and $f(X) = Y$. This also uses Proposition 1.12.7, concerning connectedness and induced topologies, in each of X and Y. If Y is not connected, then Y can be expressed as the union of two nonempty disjoint open sets V_1 and V_2. The continuity of f implies that $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are open subsets of X. It is easy to see that $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint subsets of X whose union is X, because of the corresponding properties of V_1 and V_2 in Y. We also have that $f^{-1}(V_1), f^{-1}(V_2) \neq \emptyset$, because $V_1, V_2 \neq \emptyset$, and $f(X) = Y$. This implies that X is not connected, as desired.

2.8.5 Path connectedness

Definition 2.8.7 *A subset E of X is said to be* path connected *in X if for every pair of points $x, w \in E$ there is a continuous mapping p from the closed unit interval $[0, 1]$ into X such that $p(0) = x$, $p(1) = y$, and*

$$p([0, 1]) \subseteq E. \tag{2.8.8}$$

More precisely, this uses the topology induced on $[0, 1]$ by the standard topology on $\mathbf{R}$.

Proposition 2.8.9 *If E is a path-connected subset of X, then E is connected in X.*

Let $x, w \in E$ be given, and let p be as in the definition of path connectedness. Remember that $[0, 1]$ is connected as a subset of the real line, with respect to the standard topology on $\mathbf{R}$. This implies that $[0, 1]$ is connected as a subset of itself, with respect to the induced topology. It follows that $p([0, 1])$ is connected as a subset of X. One can use this to get the connectedness of E, as in Proposition 1.15.6.

Proposition 2.8.10 *If f is a continuous mapping from X into Y, and E is a path-connected subset of X, then $f(E)$ is path connected in Y.*

This follows from the definition of path connectedness, and the fact that compositions of continuous mappings are continuous.

Proposition 2.8.11 *Let X_0 be a subset of X, and let E be a subset of X_0. Under these conditions, E is path connected in X_0, with respect to the topology induced by the topology on X, if and only if E is path connected in X.*

This follows from the definition of path connectedness and Proposition 2.8.5.

2.9 The product topology

Let I be a nonempty set, and let X_j be a set for each $j \in I$. The *Cartesian product* of the X_j's, $j \in I$, is the set

$$\prod_{j \in I} X_j \tag{2.9.1}$$

consisting of all functions f defined on I with values in $\bigcup_{j \in I} X_j$ such that $f(j) \in X_j$ for every $j \in I$.

If $I = \{1, \ldots, n\}$ for some positive integer n, then the Cartesian product of the X_j's may be denoted

$$\prod_{j=1}^{n} X_j, \tag{2.9.2}$$

and its elements identified with n-tuples $x = (x_1, \ldots, x_n)$ such that $x_j \in X_j$ for each $j = 1, \ldots, n$. Similarly, if $I = \mathbf{Z}_+$, then the Cartesian product of the X_j's may be denoted

$$\prod_{j=1}^{\infty} X_j, \tag{2.9.3}$$

and its elements identified with sequences $x = \{x_j\}_{j=1}^{\infty}$ with $x_j \in X_j$ for every $j \geq 1$.

Suppose for the moment that U_j and V_j are subsets of X_j for each $j \in I$. Observe that

$$\Big(\prod_{j \in I} U_j\Big) \cap \Big(\prod_{j \in I} V_j\Big) = \prod_{j \in I} (U_j \cap V_j). \tag{2.9.4}$$

2.9.1 Defining the product topology

Suppose now that X_j is a topological space for every $j \in I$. A subset W of $\prod_{j \in I} X_j$ is said to be an open set with respect to the *product topology* if for every $f \in W$ and $j \in I$ there is an open subset U_j of X_j such that

$$f(j) \in U_j \quad \text{for every } j \in I, \tag{2.9.5}$$

$$\prod_{j \in I} U_j \subseteq W, \tag{2.9.6}$$

and

$$U_j = X_j \quad \text{for all but finitely many } j \in I. \tag{2.9.7}$$

Note that (2.9.5) is the same as saying that

$$f \in \prod_{j \in I} U_j. \tag{2.9.8}$$

Of course, (2.9.7) holds automatically when I has only finitely many elements. One can verify that this defines a topology on $\prod_{j \in I} X_j$.

2.9.2 The strong product topology

Similarly, let us say that a subset W of $\prod_{j \in I} X_j$ is an open set with respect to the *strong product topology* if for every $f \in W$ and $j \in I$ there is an open set $U_j \subseteq X_j$ that satisfies (2.9.5) and (2.9.6). One can check that this defines a topology on $\prod_{j \in I} X_j$ as well. Of course, the strong product topology on $\prod_{j \in I} X_j$ is automatically at least as strong as the product topology. If I has only finitely many elements, then the product topology and the strong product topology on $\prod_{j \in I} X_j$ are the same.

2.9.3 Some basic open sets

If U_j is an open subset of X_j for every $j \in I$, then it is easy to see that

$$U = \prod_{j \in I} U_j \tag{2.9.9}$$

is an open set in $\prod_{j \in I} X_j$ with respect to the strong product topology. The collection of these open sets is a base for the strong product topology on $\prod_{j \in I} X_j$, by construction.

Similarly, if $U_j \subseteq X_j$ is an open set for every $j \in I$, and if $U_j = X_j$ for all but finitely many $j \in I$, then (2.9.9) is an open set with respect to the product topology on $\prod_{j \in I} X_j$. The collection of these open sets is a base for the product topology on $\prod_{j \in I} X_j$, by construction again.

2.9.4 Using the discrete topology on the factors

Suppose for the moment that X_j is equipped with the discrete topology for every $j \in I$. In this case, the strong product topology on $\prod_{j \in I} X_j$ is the same as the discrete topology. Suppose in addition that $X_j \neq \emptyset$ for every $j \in I$, and that X_j has at least two elements for infinitely many $j \in I$. Under these conditions, one can check that the product topology on $\prod_{j \in I} X_j$ is not the discrete topology.

2.9.5 The standard topology on $\mathbf{R}^n$

If n is a positive integer, then the space $\mathbf{R}^n$ of n-tuples of real numbers is the same as the nth Cartesian power of $\mathbf{R}$. The product topology on $\mathbf{R}^n$ corresponding to the standard topology on the real line may be considered as the *standard topology on* $\mathbf{R}^n$.

2.10 Coordinate mappings

Let I be a nonempty set again, let X_j be a set for each $j \in I$, and let

$$X = \prod_{j \in I} X_j \tag{2.10.1}$$

be the corresponding Cartesian product. If $l \in I$, then let p_l be the lth *coordinate mapping* from X into X_l, which is defined by

$$p_l(f) = f(l) \tag{2.10.2}$$

for every $f \in X$.

2.10.1 Some properties of p_l

Let W_l be a subset of X_l, and for each $j \in I$, let $\widetilde{W}_{j,l}$ be the subset of X_j defined by

$$\begin{aligned} \widetilde{W}_{j,l} &= W_l \quad \text{when } j = l \\ &= X_j \quad \text{when } j \neq l. \end{aligned} \tag{2.10.3}$$

Observe that

$$p_l^{-1}(W_l) = \prod_{j \in I} \widetilde{W}_{j,l}. \tag{2.10.4}$$

Similarly, let U_j be a subset of X_j for every $j \in I$, and put $U = \prod_{j \in I} U_j$. If $U_j \neq \emptyset$ for every $j \in I$, then

$$p_l(U) = U_l \tag{2.10.5}$$

for every $l \in I$.

2.10.2 Continuity of the coordinate mappings

Suppose now that X_j is a topological space for each $j \in I$.

Proposition 2.10.6 *If $l \in I$, then p_l is a continuous mapping from X into X_l with respect to the product topology on X, and hence with respect to the strong product topology on X. We also have that p_l is an open mapping from X into X_l with respect to the strong product topology on X, and hence with respect to the product topology.*

If W_l is an open subset of X_l for some $l \in I$, then (2.10.4) is an open set in X with respect to the product topology. This implies the first part of the proposition. The second part can be obtained from (2.10.5).

2.10.3 Products of closed sets

Corollary 2.10.7 *If E_j is a closed set in X_j for each $j \in I$, then $E = \prod_{j\in I} E_j$ is a closed set in X with respect to the product topology, and hence with respect to the strong product topology.*

If $l \in I$, then $p_l^{-1}(E_l)$ is a closed set in X with respect to the product topology, by the previous proposition. This implies that

$$E = \bigcap_{l\in I} p_l^{-1}(E_l) \tag{2.10.8}$$

is a closed set in X with respect to the product topology as well. Alternatively, one can verify that $X \setminus E$ is an open set with respect to the product topology on X.

2.10.4 Convergence of nets and the product topology

Proposition 2.10.9 *Let $(A, \preceq)$ be a nonempty directed system, let $\{f_a\}_{a\in A}$ be a net of elements of X indexed by A, and let f be another element of X. Under these conditions, $\{f_a\}_{a\in A}$ converges to f with respect to the product topology on X if and only if for each $l \in I$, $\{f_a(l)\}_{a\in A}$ converges to $f(l)$ as a net of elements of X_l.*

The "only if" part of the proposition follows from the fact that p_l is continuous with respect to the product topology on X for every $l \in I$. The "if" part can be verified directly from the definitions.

Of course, if $\{f_a\}_{a\in A}$ converges to f with respect to the strong product topology on X, then $\{f_a\}_{a\in A}$ converges to f with respect to the product topology on X.

2.10.5 Closures of products of sets

Proposition 2.10.10 *Let A_j be a subset of X_j for each $j \in I$, and let $\overline{A_j}$ be the closure of A_j in X_j for every $j \in I$, as usual. The closure of $A = \prod_{j\in I} A_j$ in X with respect to the product topology or the strong product topology is equal to $\prod_{j\in I} \overline{A_j}$.*

This can be verified directly from the definitions. More precisely, the closure of A with respect to the strong product topology on X is contained in the closure of A with respect to the product topology on X. One can check that the closure of A with respect to the product topology on X is contained in $\prod_{j\in I} \overline{A_j}$, and that $\prod_{j\in I} \overline{A_J}$ is contained in the closure of A with respect to the strong product topology on X.

2.10.6 Some basic separation conditions for the product

Proposition 2.10.11 *Let $k \in \{0, 1, 2, 2\frac{1}{2}\}$ be given, and suppose that X_j satisfies the kth separation condition for every $j \in I$. Under these conditions, X satisfies the kth separation condition with respect to the product topology, and hence with respect to the strong product topology.*

If f and g are distinct elements of X, then there is an $l \in I$ such that $f(l) \neq g(l)$. In this situation, the kth separation condition for f and g in X can be obtained from the analogous condition for $f(l)$ and $g(l)$ in X_l.

2.10.7 Regularity of the product

Proposition 2.10.12 *If X_j is regular in the strict sense for every $j \in I$, then X is regular in the strict sense with respect to the product topology and the strong product topology.*

Let $f \in X$ be given, and let W be an open subset of X with respect to the product topology or strong product topology, with $f \in W$. It follows that for every $j \in I$ there is an open set $V_j \subseteq X_j$ such that $f(j) \in V_j$,

$$V = \prod_{j \in I} V_j \subseteq W, \tag{2.10.13}$$

and $V_j = X_j$ for all but finitely many $j \in I$ in the case of the product topology. If $j \in I$, there is an open set $U_j \subseteq X_j$ such that

$$f(j) \in U_j \text{ and } \overline{U_j} \subseteq V_j, \tag{2.10.14}$$

because X_j is regular in the strict sense. We may as well take

$$U_j = X_j \text{ when } V_j = X_j, \tag{2.10.15}$$

so that $U_j = X_j$ for all but finitely many $j \in I$ in the case of the product topology. Thus $U = \prod_{j \in I} U_j$ is an open set in X with respect to the product topology or strong product topology, as appropriate. Of course, $f \in U$, by construction. The closure of U with respect to the product topology or strong product topology is equal to $\prod_{j \in I} \overline{U_j}$, as before. This is contained in V, by construction, and hence in W, as desired.

Corollary 2.10.16 *If X_j is regular in the strong sense for every $j \in I$, then X is regular in the strong sense with respect to the product topology and the strong product topology.*

2.11 Bases and finite products

Let $X_1, \ldots, X_n$ be finitely many topological spaces, and let $X = \prod_{j=1}^n X_j$ be their Cartesian product, equipped with the product topology. Also let

$$x = (x_1, \ldots, x_n) \in X \tag{2.11.1}$$

be given. Suppose that for each $j = 1, \ldots, n$, $\mathcal{B}_j(x_j)$ is a local base for the topology of X_j at x_j. Put

$$\mathcal{B}(x) = \biggl\{ \prod_{j=1}^{n} U_j : U_j \in \mathcal{B}_j(x_j) \text{ for each } j = 1, \ldots, n \biggr\}. \tag{2.11.2}$$

It is easy to see that this is a local base for the product topology on X at x.

2.11.1 Countability of $\mathcal{B}(x)$

Suppose that for each $j = 1, \ldots, n$, $\mathcal{B}_j(x_j)$ has only finitely or countably many elements. We would like to verify that (2.11.2) has only finitely or countably many elements as well. In this case,

$$\prod_{j=1}^{n} \mathcal{B}_j(x_j) \tag{2.11.3}$$

has only finitely or countably many elements, by standard results. There is an obvious mapping from (2.11.3) onto (2.11.2), which sends $(U_1, \ldots, U_n)$ in (2.11.3) to $\prod_{j=1}^{n} U_j$ in (2.11.2). One can use this to show that (2.11.2) has only finitely or countably many elements, because of the analogous property for (2.11.3).

2.11.2 Another local base for the product topology at x

Alternatively, suppose again that for each $j = 1, \ldots, n$, there is a local base for the topology of X_j at x_j with only finitely or countably many elements. This means that for each $j = 1, \ldots, n$ there is a sequence $\{U_{j,l}(x_j)\}_{l=1}^{\infty}$ of open subsets of X_j such that $x_j \in U_{j,l}(x_j)$ for every $l \geq 1$, and the collection of $U_{j,l}(x_j)$'s, $l \geq 1$, is a local base for the topology of X_j at x_j.

We may also ask that

$$U_{j,l+1}(x_j) \subseteq U_{j,l}(x_j) \tag{2.11.4}$$

for each $j = 1, \ldots, n$ and $l \geq 1$, since otherwise we can replace $U_{j,l}(x_j)$ with $\bigcap_{k=1}^{l} U_{j,k}(x_j)$ for every $j = 1, \ldots, n$ and $l \geq 1$, as before.

Put

$$U_l(x) = \prod_{j=1}^{n} U_{j,l}(x_j) \tag{2.11.5}$$

for each $l \in \mathbf{Z}_+$, so that $U_l(x)$ is an open subset of X with respect to the product topology, and $x \in U_l(x)$. One can check that the collection of $U_l(x)$'s, $l \in \mathbf{Z}_+$, is a local base for the product topology on X at x.

2.11.3 Some bases for the product topology on X

Similarly, let $\mathcal{B}_j$ be a base for the topology of X_j for each $j = 1, \ldots, n$. It is easy to see that

$$\mathcal{B} = \left\{ \prod_{j=1}^{n} U_j : U_j \in \mathcal{B}_j \text{ for each } j = 1, \ldots, n \right\} \tag{2.11.6}$$

is a base for the product topology on X.

If $\mathcal{B}_j$ has only finitely or countably many elements for each $j = 1, \ldots, n$, then (2.11.6) has only finitely or countably many elements as well. More precisely,

$$\prod_{j=1}^{n} \mathcal{B}_j \tag{2.11.7}$$

has only finitely or countably many elements in this situation, as before. We can map (2.11.7) onto (2.11.6), by sending $(U_1, \ldots, U_n)$ in (2.11.7) to $\prod_{j=1}^{n} U_j$ in (2.11.6). One can use this and the fact that (2.11.7) has only finitely or countably many elements to get that (2.11.6) has only finitely or countably many elements, as before.

2.12 Bases and countable products

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of topological spaces, and let $X = \prod_{j=1}^{\infty} X_j$ be their Cartesian product, equipped with the product topology. Also let $x = \{x_j\}_{j=1}^{\infty} \in X$ be given, and let $\mathcal{B}_j(x_j)$ be a local base for the topology of X_j at x_j for each $j \in \mathbf{Z}_+$.

2.12.1 A local base for the product topology at x

If n is a positive integer, then put

$$\begin{aligned} \mathcal{B}^n(x) \quad = \quad & \left\{ \prod_{j=1}^{\infty} U_j : U_j \in \mathcal{B}_j(x_j) \text{ for each } j = 1, \ldots, n, \right. \\ & \qquad\qquad \left. \text{and } U_j = X_j \text{ when } j > n \right\}. \end{aligned} \tag{2.12.1}$$

Note that every element of $\mathcal{B}^n(x)$ is an open subset of X with respect to the product topology. One can check that

$$\mathcal{B}(x) = \bigcup_{n=1}^{\infty} \mathcal{B}^n(x) \tag{2.12.2}$$

is a local base for the product topology on X at x.

2.12.2 Countability of $\mathcal{B}^n(x)$

Suppose that for every $j \in \mathbf{Z}_+$, $\mathcal{B}_j(x_j)$ has only finitely or countably many elements. This implies that

$$\prod_{j=1}^{n} \mathcal{B}_j(x_j) \tag{2.12.3}$$

has only finitely or countably many elements for each $n \in \mathbf{Z}_+$, by standard arguments, as before.

Let $(U_1, \ldots, U_n)$ be an element of (2.12.3), and put $U_j = X_j$ when j is larger than n. Under these conditions, $\prod_{j=1}^{\infty} U_j$ is an element of (2.12.1). This defines a mapping from (2.12.3) onto (2.12.1). One can use this and the fact that (2.12.3) has only finitely or countably many elements to obtain that (2.12.1) has only finitely or countably many elements for each $n \in \mathbf{Z}_+$. It follows that (2.12.2) has only finitely or countably many elements as well.

2.12.3 Another countable local base for X at x

Alternatively, suppose again that for each positive integer j, there is a local base for the topology of X_j at x_j with only finitely or countably many elements. This means that for each $j \in \mathbf{Z}_+$ there is a sequence $\{U_{j,l}(x_j)\}_{l=1}^{\infty}$ of open subsets of X_j such that $x_j \in U_{j,l}(x_j)$ for every $l \geq 1$, and the collection of $U_{j,l}(x_j)$'s, $l \geq 1$, is a local base for the topology of X_j at x_j. We may also ask that $U_{j,l+1}(x_j) \subseteq U_{j,l}(x_j)$ for every $j, l \geq 1$, as before.

If $n \in \mathbf{Z}_+$, then put

$$\begin{aligned} V_{j,n}(x_j) &= U_{j,n}(x_j) \quad \text{for } j = 1, \ldots, n \\ &= X_j \qquad\quad \text{when } j > n. \end{aligned} \tag{2.12.4}$$

Using this, we put

$$V_n(x) = \prod_{j=1}^{\infty} V_{j,n}(x_j). \tag{2.12.5}$$

This is an open subset of X with respect to the product topology, and $x \in V_n(x)$. One can check that the collection of $V_n(x)$'s, $n \in \mathbf{Z}_+$, is a local base for the product topology on X at x.

2.12.4 Some bases for X

Now let $\mathcal{B}_j$ be a base for the topology of X_j for every positive integer j. If $n \in \mathbf{Z}_+$, then put

$$\begin{aligned} \mathcal{B}^n = \biggl\{ \prod_{j=1}^{\infty} U_j : U_j \in \mathcal{B}_j \text{ for each } j = 1, \ldots, n \\ \text{and } U_j = X_j \text{ when } j > n \biggr\}. \end{aligned} \tag{2.12.6}$$

Every element of $\mathcal{B}^n$ is an open subset of X with respect to the product topology, by construction. It is not difficult to verify that

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}^n \tag{2.12.7}$$

is a base for the product topology on X.

2.12.5 Countable bases for X

Suppose that $\mathcal{B}_j$ has only finitely or countably many elements for each $j \in \mathbf{Z}_+$. This implies that $\prod_{j=1}^{n} \mathcal{B}_j$ has only finitely or countably many elements for each positive integer n, as usual. One can use this to check that (2.12.6) has only finitely or countably many elements for every $n \in \mathbf{Z}_+$, as before. This implies that (2.12.7) has only finitely or countably many elements too.

2.12.6 Some local bases for the strong product topology

Let I be a nonempty set, let X_j be a topological space for each $j \in I$, and let $X = \prod_{j \in I} X_j$ be the Cartesian product of the X_j's. Also let $f \in X$ be given, and let $\mathcal{B}_j(f(j))$ be a local base for the topology of X_j at $f(j)$ for each $j \in I$. Under these conditions, one can check that

$$\mathcal{B}(f) = \left\{ \prod_{j \in I} U_j : U_j \in \mathcal{B}_j(f(j)) \text{ for each } j \in I \right\} \tag{2.12.8}$$

is a local base for the strong product topology on X at f.

If $\mathcal{B}_j(f(j))$ has only finitely many elements for some $j \in I$, then one can reduce to the case where $\mathcal{B}_j(f(j))$ has only one element, because the interesection of finitely many open subsets of X_j is an open set too. Note that

$$\prod_{j \in I} \mathcal{B}_j(f(j)) \tag{2.12.9}$$

is uncountable when $\mathcal{B}_j(f(j))$ has at least two elements for infinitely many $j \in I$.

2.12.7 Some bases for the strong product topology

Similarly, if $\mathcal{B}_j$ is a base for the topology of X_j for each $j \in I$, then one can verify that

$$\mathcal{B} = \left\{ \prod_{j \in I} U_j : U_j \in \mathcal{B}_j \text{ for each } j \in I \right\} \tag{2.12.10}$$

is a base for the strong product topology on X. Suppose that $X_j \neq \emptyset$ for each $j \in I$, so that $\mathcal{B}_j \neq \emptyset$ for each $j \in I$. If $\mathcal{B}_j$ has at least two elements for infinitely many $j \in I$, then

$$\prod_{j \in I} \mathcal{B}_j \tag{2.12.11}$$

is uncountable.

2.13 Finite products and semimetrics

Let n be a positive integer, and let X_j be a set with a semimetric $d_j(\cdot, \cdot)$ for each $j = 1, \ldots, n$. Put $X = \prod_{j=1}^n X_j$. If $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X$, then one can check that

$$d(x, y) = \max_{1 \le j \le n} d_j(x_j, y_j) \tag{2.13.1}$$

defines a semimetric on X. If $d_j(\cdot, \cdot)$ is a metric on X_j for every $j = 1, \ldots, n$, then (2.13.1) is a metric on X.

2.13.1 Open balls in X with respect to d

Let

$$B_j(x_j, r) = \{w_j \in X_j : d_j(x_j, w_j) < r\} \tag{2.13.2}$$

be the open ball in X_j centered at $x_j \in X_j$ with radius $r > 0$ with respect to $d_j(\cdot, \cdot)$ for each $j = 1, \ldots, n$, and let

$$B(x, r) = \{w \in X : d(x, w) < r\} \tag{2.13.3}$$

be the open ball in X centered at $x \in X$ with radius $r > 0$ with respect to $d(\cdot, \cdot)$. It is easy to see that

$$B(x, r) = \prod_{j=1}^n B_j(x_j, r) \tag{2.13.4}$$

for every $x \in X$ and $r > 0$, directly from the definitions. More precisely, $w \in X$ satisfies $d(x, w) < r$ if and only if $d_j(x_j, w_j) < r$ for each $j = 1, \ldots, n$.

2.13.2 Relation with the product topology on X

Let us use the term "product topology" to refer to the product topology on X corresponding to the topologies determined on $X_1, \ldots, X_n$ by the semimetrics $d_1, \ldots, d_n$, respectively. It is well known and not difficult to show that this is the same as the topology determined on X by the semimetric (2.13.1).

Remember that an open ball is an open set with respect to the topology determined by the corresponding semimetric. This implies that (2.13.4) is an open set in X with respect to the product topology for every $x \in X$ and $r > 0$. One can use this to check that every open set in X with respect to the topology determined by the semimetric (2.13.1) is also an open set with respect to the product topology.

If $W \subseteq X$ is an open set with respect to the product topology, then one can verify that W is an open set with respect to the topology determined by the semimetric (2.13.1). More precisely, if $x \in W$, then one can find an $r > 0$ such that (2.13.4) is contained in W.

2.13.3 Another semimetric on X

It is easy to see that

$$\widetilde{d}(x,y) = \sum_{j=1}^{n} d_j(x_j, y_j) \tag{2.13.5}$$

defines a semimetric on X as well. If $d_j(\cdot,\cdot)$ is a metric on X_j for each $j = 1, \ldots, n$, then (2.13.5) is a metric on X, as before. Observe that

$$d(x,y) \le \widetilde{d}(x,y) \le n\, d(x,y) \tag{2.13.6}$$

for every $x, y \in X$. Using this, one can check that the topologies determined on X by (2.13.1) and (2.13.5) are the same.

2.13.4 An additional semimetric on X

Similarly, put

$$\widehat{d}(x,y) = \Big(\sum_{j=1}^{n} d_j(x_j, y_j)^2\Big)^{1/2} \tag{2.13.7}$$

for each $x, y \in X$, using the nonnegative square root on the right side. One can check that this satisfies the triangle inequality on X, using the triangle inequality for the standard Euclidean norm on $\mathbf{R}^n$. Using this, it is easy to see that (2.13.7) is a semimetric on X, which is a metric on X when $d_j(\cdot,\cdot)$ is a metric on X_j for each $j = 1, \ldots, n$.

One can also verify that

$$d(x,y) \le \widehat{d}(x,y) \le n^{1/2}\, d(x,y) \tag{2.13.8}$$

for every $x, y \in X$. This implies that the topologies determined on X by (2.13.1) and (2.13.7) are the same, as before.

2.14 Truncating semimetrics

Let X be a set, let $d(x,y)$ be a semimetric on X, and let t be a positive real number. If $x, y \in X$, then put

$$d_t(x,y) = \min(d(x,y), t). \tag{2.14.1}$$

One can check that this defines a semimetric on X. If $d(x,y)$ is a metric on X, then (2.14.1) is a metric on X too.

2.14.1 Open balls in X with respect to d_t

If $x \in X$ and r is a positive real number, then let

$$B_d(x,r) = \{w \in X : d(x,w) < r\} \tag{2.14.2}$$

be the open ball in X centered at x with radius r with respect to $d(\cdot,\cdot)$. Similarly, let

$$B_{d_t}(x,r) = \{w \in X : d_t(x,w) < r\} \tag{2.14.3}$$

be the open ball in X centered at x with radius r with respect to (2.14.1). Observe that

$$\begin{aligned} B_{d_t}(x,r) &= B_d(x,r) \quad \text{when } r \le t \\ &= X \qquad\qquad \text{when } r > t. \end{aligned} \tag{2.14.4}$$

Using this, one can check that the topologies determined on X by $d(\cdot,\cdot)$ and (2.14.1) are the same.

2.15 Countable products and semimetrics

Let $X_1, X_2, X_3, \ldots$ be a sequence of sets, let $d_j(\cdot,\cdot)$ be a semimetric on X_j for each $j \ge 1$, and put $X = \prod_{j=1}^{\infty} X_j$. Put

$$d_j'(x_j, y_j) = \min(d_j(x_j, y_j), 1/j) \tag{2.15.1}$$

for each $j \in \mathbf{Z}_+$ and $x_j, y_j \in X_j$. As in the previous section, for every $j \in \mathbf{Z}_+$, (2.15.1) defines a semimetric on X_j that determines the same topology on X_j as $d_j(\cdot,\cdot)$. If $d_j(\cdot,\cdot)$ is a metric on X_j, then (2.15.1) is a metric on X_j too, as before.

2.15.1 A semimetric on X

If $x = \{x_j\}_{j=1}^{\infty}, y = \{y_j\}_{j=1}^{\infty} \in X$, then put

$$d(x,y) = \max_{j \in \mathbf{Z}_+} d_j'(x_j, y_j). \tag{2.15.2}$$

Of course, this is equal to 0 when $d_j'(x_j, y_j) = 0$ for every $j \ge 1$, and in particular when $x = y$. Otherwise, suppose that there is a positive integer j_0 such that $d_{j_0}'(x_{j_0}, y_{j_0}) > 0$. In this case,

$$d_j'(x_j, y_j) \le 1/j < d_{j_0}(x_{j_0}, y_{j_0}) \tag{2.15.3}$$

for all but finitely many $j \in \mathbf{Z}_+$. This means that the right side of (2.15.2) reduces to the maximum of finitely many terms, so that the maximum is attained.

One can check that (2.15.2) defines a semimetric on X. If $d_j(\cdot,\cdot)$ is a metric on X_j for each $j \ge 1$, then (2.15.2) is a metric on X as well.

2.15.2 The corresponding open balls

If $j \in \mathbf{Z}_+$, $x_j \in X_j$, and r is a positive real number, then let

$$B_j(x_j, r) = \{w_j \in X_j : d_j(x_j, w_j) < r\} \tag{2.15.4}$$

and

$$B'_j(x_j, r) = \{w_j \in X_j : d'_j(x_j, w_j) < r\} \tag{2.15.5}$$

be the open balls in X_j centered at x_j with radius r with respect to $d_j(\cdot, \cdot)$ and (2.15.1), respectively. Thus

$$\begin{aligned} B'_j(x_j, r) &= B_j(x_j, r) \quad \text{when } r \le 1/j \\ &= X_j \qquad\quad\ \, \text{when } r > 1/j, \end{aligned} \tag{2.15.6}$$

as in (2.14.4). Also let

$$B(x, r) = \{w \in X : d(x, w) < r\} \tag{2.15.7}$$

be the open ball in X centered at $x \in X$ with radius $r > 0$ with respect to (2.15.2).

One can verify that

$$B(x, r) = \prod_{j=1}^{\infty} B'_j(x_j, r) \tag{2.15.8}$$

for every $x \in X$ and $r > 0$. This is the same as saying that $w \in X$ satisfies $d(x, w) < r$ if and only if $d'_j(x_j, w_j) < r$ for every $j \in \mathbf{Z}_+$. This uses the fact that the maximum is attained on the right side of (2.15.2).

2.15.3 Relation between d and the product topology on X

Let us use the term "product topology" to refer to the product topology on X corresponding to the topology determined on X_j by $d_j(\cdot, \cdot)$ for each $j \in \mathbf{Z}_+$. Observe that (2.15.8) is an open set in X with respect to the product topology for every $x \in X$ and $r > 0$, because of (2.15.6). One can use this to check that every open set in X with respect to (2.15.2) is an open set with respect to the product topology.

If W is an open subset of X with respect to the product topology and $x \in W$, then it is not too difficult to find an $r > 0$ such that (2.15.8) is contained in W, so that W is an open set with respect to the topology determined by (2.15.2). Thus the product topology on X is the same as the topology determined on X by (2.15.2).

Chapter 3

Compactness and related topics

3.1 Compact sets

Let X be a topological space, and let K be a subset of X.

Definition 3.1.1 *If A is a nonempty set, U_α is an open subset of X for each $\alpha \in A$, and*

$$K \subseteq \bigcup_{\alpha \in A} U_\alpha, \tag{3.1.2}$$

then $\{U_\alpha\}_{\alpha \in A}$ is said to be an open covering *of K in X. We say that K is* compact *in X if every open covering of K can be reduced to a finite subcovering. This means that for every open covering $\{U_\alpha\}_{\alpha \in A}$ of K in X there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that*

$$K \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}. \tag{3.1.3}$$

3.1.1 Some examples of compact sets

If K has only finitely many elements, then it is easy to see that K is compact.

If X is equipped with the discrete topology and $K \subseteq X$ is compact, then K has only finitely many elements. This follows by covering K with subsets of X with only one element, which are open sets in this case.

If a, b are real numbers with $a \le b$, then it is well known that the closed interval $[a, b]$ in the real line is compact, with respect to the standard topology on $\mathbf{R}$.

If X is any set equipped with the indiscrete topology, then every subset of X is compact.

3.1.2 Another class of examples

Similarly, let X be any set equipped with the cofinite topology. This means that $U \subseteq X$ is an open set when $U = \emptyset$ or $X \setminus U$ has only finitely many elements. We also have that every subset K of X is compact in this situation.

More precisely, let $\{U_\alpha\}_{\alpha \in A}$ be any open covering of K in X. We may as well suppose that $K \neq \emptyset$, which implies that there is an $\alpha_0 \in A$ such that $U_{\alpha_0} \neq \emptyset$. It follows that U_{α_0} contains all but finitely many elements of X, and in particular that U_{α_0} contains all but finitely many elements of K. The finitely many elements of $K \setminus U_{\alpha_0}$ can easily be covered by finitely many U_α's, so that K can be covered by finitely many U_α's, as desired.

3.1.3 Compactness and induced topologies

Let X be any topological space again.

Proposition 3.1.4 *Let Y be a subset of X, and let K be a subset of Y. Under these conditions, K is compact as a subset of X if and only if K is compact as a subset of Y, with respect to the induced topology.*

This can be verified directly from the definitions.

3.1.4 Intersections of compact sets and closed sets

Proposition 3.1.5 *If K is a compact subset of X, and E is a closed set in X, then $K \cap E$ is compact in X as well.*

To see this, let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of $K \cap E$ in X. Note that $X \setminus E$ is an open set in X, because E is a closed set. We also have that

$$K = (K \cap E) \cup (K \setminus E) \subseteq \Big(\bigcup_{\alpha \in A} U_\alpha\Big) \cup (X \setminus E). \tag{3.1.6}$$

Thus the collection of U_α's, $\alpha \in A$, together with $X \setminus E$, forms an open covering of K in X. Because K is compact in X, there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \Big(\bigcup_{j=1}^{n} U_{\alpha_j}\Big) \cup (X \setminus E). \tag{3.1.7}$$

This implies that

$$K \cap E \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}, \tag{3.1.8}$$

as desired.

3.1.5 An additional class of examples

Proposition 3.1.9 *If $\{x_j\}_{j=1}^\infty$ is a sequence of elements of X that converges to an element x of X, then*

$$K = \{x_j : j \in \mathbf{Z}_+\} \cup \{x\} \tag{3.1.10}$$

is a compact set in X.

This can be verified directly from the definitions.

3.2 Compactness in Hausdorff spaces

Let X be a topological space.

Proposition 3.2.1 *If X is Hausdorff, K is a compact subset of X, and x is an element of X not in K, then there are disjoint open subsets U and V of X such that $x \in U$ and $K \subseteq V$.*

If $y \in K$, then $x \neq y$, and hence there are disjoint open sets $U(y), V(y) \subseteq X$ such that $x \in U(y)$ and $y \in V(y)$. The collection of open sets $V(y)$ of this type, with $y \in K$, forms an open covering of K in X. Because K is compact, there are finitely many elements $y_1, \ldots, y_n$ of K such that

$$K \subseteq \bigcup_{j=1}^n V(y_j). \tag{3.2.2}$$

Put

$$U = \bigcap_{j=1}^n U(y_j), \quad V = \bigcup_{j=1}^n V(y_j), \tag{3.2.3}$$

which are open subsets of X. By construction, $x \in U$ and $K \subseteq V$. It is easy to see that

$$U \cap V = \emptyset, \tag{3.2.4}$$

because $U(y_j) \cap V(y_j) = \emptyset$ for each $j = 1, \ldots, n$. In particular,

$$U \subseteq X \setminus K, \tag{3.2.5}$$

because $K \subseteq V$.

Corollary 3.2.6 *If X is Hausdorff, and K is a compact subset of X, then K is a closed set in X.*

Indeed, the previous argument shows that every element of $X \setminus K$ is in the interior of $X \setminus K$.

3.2.1 Disjoint compact sets

Proposition 3.2.7 *If X is Hausdorff, and H, K are disjoint compact subsets of X, then there are disjoint open subsets U_1, V_1 of X such that $H \subseteq U_1$ and $K \subseteq V_1$.*

If $x \in H$, then $x \in X \setminus K$, and so there are disjoint open subsets $U_1(x)$, $V_1(x)$ of X such that $x \in U_1(x)$ and $K \subseteq V_1(x)$, as before. The collection of open sets $U_1(x)$ of this type, with $x \in H$, forms an open covering of H in X. Thus there are finitely many elements $x_1, \ldots, x_m$ of H such that

$$H \subseteq \bigcup_{l=1}^{m} U_1(x_l), \tag{3.2.8}$$

because H is compact. Put

$$U_1 = \bigcup_{l=1}^{m} U_1(x_l), \quad V_1 = \bigcap_{l=1}^{m} V_1(x_l), \tag{3.2.9}$$

which are open subsets of X. Observe that $H \subseteq U_1$ and $K \subseteq V_1$, by construction. One can check that

$$U_1 \cap V_1 = \emptyset, \tag{3.2.10}$$

because $U_1(x_l) \cap V_1(x_l) = \emptyset$ for every $l = 1, \ldots, m$.

Corollary 3.2.11 *If X is Hausdorff and compact as a subset of itself, then X is normal in the strong sense.*

This follows from the previous proposition and the fact that closed subsets of X are compact, by Proposition 3.1.5.

3.2.2 Disjoint compact and closed sets

Proposition 3.2.12 *If X is regular in the strict sense, K is a compact subset of X, E is a closed set in X, and*

$$K \cap E = \emptyset, \tag{3.2.13}$$

then there are disjoint open subsets U_0, V_0 of X such that $K \subseteq U_0$ and $E \subseteq V_0$.

This can be shown using the same type of argument as for Proposition 3.2.7.

Corollary 3.2.14 *If X is regular in the strict sense, and X is compact as a subset of itself, then X is normal in the strict sense.*

This uses the fact that closed subsets of X are compact when X is compact, as before.

3.3 Continuity and compactness

Let X and Y be topological spaces.

Theorem 3.3.1 *If f is a continuous mapping from X into Y, and K is a compact subset of X, then $f(K)$ is a compact subset of Y.*

Let $\{V_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of $f(K)$ in Y. Note that $f^{-1}(V_\alpha)$ is an open set in X for each $\alpha \in A$, because f is continuous. It is easy to see that

$$K \subseteq \bigcup_{\alpha \in A} f^{-1}(V_\alpha), \tag{3.3.2}$$

because $f(K) \subseteq \bigcup_{\alpha \in A} V_\alpha$, by hypothesis. Thus $\{f^{-1}(V_\alpha)\}_{\alpha \in A}$ is an open covering of K in X. If K is compact, then there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \bigcup_{j=1}^{n} f^{-1}(V_{\alpha_j}). \tag{3.3.3}$$

This implies that

$$f(K) \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}, \tag{3.3.4}$$

as desired.

3.3.1 The extreme value theorem

Corollary 3.3.5 (Extreme value theorem) *Suppose that f is a continuous real-valued function on X, with respect to the standard topology on $\mathbf{R}$. If K is a nonempty compact subset of X, then f attains its maximum and minimum on K.*

In this case, $f(K)$ is a closed set in $\mathbf{R}$, as in the previous section, because the real line is Hausdorff with respect to the standard topology. It is well known and not difficult to show that compact subsets of the real line are bounded too. If $K \neq \emptyset$, then $f(K) \neq \emptyset$, and one can use the previous statements to show that $f(K)$ contains its supremum and infimum in $\mathbf{R}$, as desired.

3.3.2 Compactness and homeomorphisms

Proposition 3.3.6 *If X is compact, Y is Hausdorff, and f is a one-to-one continuous mapping from X onto Y, then f is a homeomorphism.*

Let $g = f^{-1}$ be the inverse of f, as a mapping from Y onto X. We have seen previously that g is continuous if and only if for every closed set $E \subseteq X$, $g^{-1}(E)$ is a closed set in Y. In this situation, this means that for every closed set $E \subseteq X$, $f(E)$ is a closed set in Y.

If X is compact, and $E \subseteq X$ is a closed set, then it follows that E is compact, as in Proposition 3.1.5. In this case, we get that $f(E)$ is compact in Y, as before. If Y is Hausdorff, then $f(E)$ is a closed set in Y, as in the previous section. This implies that $g = f^{-1}$ is continuous in this situation, as in the preceding paragraph.

3.4 The limit point property

Let X be a topological space.

Definition 3.4.1 *A subset K of X is said to have the* limit point property *if for every subset L of K such that L has infinitely many elements, there is an element x of K that is a limit point of L in X. Similarly, let us say that K has the* strong limit point property *if for every infinite subset L of K there is an $x \in K$ such that x is a strong limit point of L in X.*

If K has the strong limit point property, then K automatically has the limit point property. If K has the limit point property, and if X satisfies the first separation condition, then K has the strong limit point property.

3.4.1 Compactness implies the strong limit point property

Proposition 3.4.2 *If K is a compact subset of X, then K has the strong limit point property.*

To see this, let an infinite subset L of K be given. Suppose for the sake of a contradiction that L does not have a strong limit point in K. This means that for each $x \in K$ there is an open set $U(x) \subseteq X$ such that $x \in U(x)$ and $U(x) \cap L$ has only finitely many elements. Thus K can be covered by open sets of this type. If K is compact, then K can be covered by finitely many open sets of this type. This implies that L has only finitely many elements, because $L \subseteq K$. This contradicts the hypothesis that L have infinitely many elements, as desired.

3.4.2 A partial converse

Proposition 3.4.3 *Suppose that $K \subseteq X$ has the strong limit point property. If $V_1, V_2, V_3, \ldots$ is an infinite sequence of open subsets of X such that*

$$K \subseteq \bigcup_{j=1}^{\infty} V_j, \tag{3.4.4}$$

then there is a positive integer n such that

$$K \subseteq \bigcup_{j=1}^{n} V_j. \tag{3.4.5}$$

Suppose for the sake of a contradiction that for each positive integer n,

$$K \not\subseteq \bigcup_{j=1}^{n} V_j. \tag{3.4.6}$$

Thus, for every $n \in \mathbf{Z}_+$, we can choose a point

$$x_n \in K \setminus \Big(\bigcup_{j=1}^{n} V_j\Big). \tag{3.4.7}$$

Let L be the set of points x_n, $n \in \mathbf{Z}_+$, that have been chosen in this way. Let us check that L has infinitely many elements. Otherwise, there is an element y of K such that $y = x_n$ for infinitely many $n \in \mathbf{Z}_+$. Note that $y \in V_{j_0}$ for some $j_0 \in \mathbf{Z}_+$, because $y \in K$. This implies that $x_n \neq y$ when $n \geq j_0$, by the way that x_n was chosen. This contradicts the hypothesis that $y = x_n$ for infinitely many n, as desired.

Thus L has infinitely many elements. If K has the strong limit point property, then it follows that there is an $x \in K$ such that x is a strong limit point of L in X. In particular, $x \in V_{j_1}$ for some $j_1 \in \mathbf{Z}_+$, because $x \in K$. This implies that V_{j_1} contains infinitely many elements of L, because x is a strong limit point of L in X, and V_{j_1} is an open set in X. This means that $x_n \in V_{j_1}$ for infinitely many $n \in \mathbf{Z}_+$, by the way that L was chosen. However, if $n \geq j_1$, then $x_n \notin V_{j_1}$, by construction. This is a contradiction, so that (3.4.5) holds for some $n \in \mathbf{Z}_+$.

3.4.3 Some related results

Proposition 3.4.8 *Let K be a subset of a subset Y of X. Under these conditions, K has the limit point property as a subset of X if and only if K has the limit point property as a subset of Y, with respect to the induced topology. Similarly, K has the strong limit point property as a subset of X if and only if K has the strong limit point property in Y, with respect to the induced topology.*

This is easy to see, directly from the definitions.

Proposition 3.4.9 *Let E be a closed set in X. If $K \subseteq X$ has the limit point property, then $K \cap E$ has the limit point property too. If K has the strong limit point property, then $K \cap E$ has the strong limit point property as well.*

This follows from the fact that E contains all of its limit points in X.

Proposition 3.4.10 *Let f be a continuous mapping from X into another topological space Y. If $K \subseteq X$ has the strong limit point property, then $f(K)$ has the strong limit point property in Y.*

Let L be an infinite subset of $f(K)$. If $y \in L$, then choose an element w of K such that $y = f(w)$, and let L_0 be the set of points in K chosen in this way.

Thus L_0 is an infinite subset of K, which has a strong limit point x in K, by hypothesis. One can check that $f(x)$ is a strong limit point of L in Y, using the continuity of f at x. This also uses the fact that f is injective on L_0, by construction.

3.5 Sequential compactness

Let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of some set X. Also let $\{j_l\}_{l=1}^\infty$ be a strictly increasing sequence of positive integers, so that $j_l < j_{l+1}$ for every $l \in \mathbf{Z}_+$. Under these conditions, $\{x_{j_l}\}_{l=1}^\infty$ is called a *subsequence* of $\{x_j\}_{j=1}^\infty$. Now let X be a topological space. If $\{x_j\}_{j=1}^\infty$ converges to an element x of X, then it is easy to see that every subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ converges to x in X too.

Definition 3.5.1 *A subset K of X is said to be* sequentially compact *if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of K there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ that converges to an element x of K.*

3.5.1 A consequence of sequential compactness

Proposition 3.5.2 *If $K \subseteq X$ is sequentially compact, then K has the strong limit point property.*

Let L be an infinite subset of K. Because L has infinitely many elements, we can find a sequence $\{x_j\}_{j=1}^\infty$ of distinct elements of L. If K is sequentially compact, then there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ that converges to an element x of K. If U is an open set in X that contains x, then it follows that $x_{j_l} \in U$ for all but finitely many $l \geq 1$. This implies that x is a strong limit point of L in X, because the terms of the sequence are distinct elements of L.

3.5.2 A criterion for sequential compactness

Proposition 3.5.3 *If $K \subseteq X$ has the strong limit point property, and if X satisfies the first countability condition, then K is sequentially compact.*

Let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of K, and let

$$L = \{x_j : j \in \mathbf{Z}_+\} \tag{3.5.4}$$

be the subset of K consisting of the terms in the sequence. If L has only finitely many elements, then there is an $x \in K$ such that $x_j = x$ for infinitely many $j \in \mathbf{Z}_+$. This means that there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that $x_{j_l} = x$ for every $l \geq 1$. Of course, $\{x_{j_l}\}_{l=1}^\infty$ converges to x in X in this case.

Suppose now that L has infinitely many elements. If K has the strong limit point property, then there is an element x of K that is a strong limit point of

L in X. Thus, if V is an open subset of X that contains x, then V contains infinitely many elements of L. This implies that

$$x_j \in V \tag{3.5.5}$$

for infinitely many positive integers j.

If X satisfies the first countability condition, then there is a local base $\mathcal{B}(x)$ for the topology of X at the point x mentioned in the preceding paragraph such that $\mathcal{B}(x)$ has only finitely or countably many elements. Equivalently, there is a sequence $U_1(x), U_2(x), U_3(x), \ldots$ of open subsets of X that contain x and form a local base for the topology of X at x. We may also suppose that

$$U_{n+1}(x) \subseteq U_n(x) \tag{3.5.6}$$

for every $n \geq 1$, by replacing $U_n(x)$ with $U_1(x) \cap \cdots \cap U_n(x)$ for every n, as usual.

Using (3.5.5) with $V = U_1(x)$, we can get a positive integer j_1 such that $x_{j_1} \in U_1(x)$. Suppose that $j_l \in \mathbf{Z}_+$ has been chosen for some positive integer l. Using (3.5.5) with $V = U_{l+1}(x)$, we get that $x_j \in U_{l+1}(x)$ for infinitely many $j \in \mathbf{Z}_+$. In particular, we can choose $j_{l+1} \in \mathbf{Z}_+$ such that $j_{l+1} > j_l$ and $x_{j_{l+1}} \in U_{l+1}(x)$.

This leads to a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that

$$x_{j_l} \in U_l(x) \tag{3.5.7}$$

for every $l \in \mathbf{Z}_+$. It follows that $\{x_{j_l}\}_{l=1}^\infty$ converges to x in X, as desired.

3.5.3 Sequential compactness and induced topologies

Proposition 3.5.8 *Suppose that $K \subseteq Y \subseteq X$. Under these conditions, K is sequentially compact as a subset of X if and only if K is sequentially compact as a subset of Y, with respect to the induced topology.*

This is easy to verify, using the analogous statement for convergence of sequences in Y.

3.5.4 Sequentially closed sets

Definition 3.5.9 *A subset E of X is said to be* sequentially closed *if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of E that converges to an element x of X, we have that $x \in E$.*

It is easy to see that closed subsets of X are sequentially closed. If X satisfies the first countability condition, then sequentially closed subsets of X are closed sets.

Proposition 3.5.10 *If $K \subseteq X$ is sequentially compact and $E \subseteq X$ is sequentially closed, then $K \cap E$ is sequentially compact in X.*

This can be verified directly from the definitions.

Proposition 3.5.11 *If $E \subseteq X$ is sequentially compact, and X is Hausdorff, then E is sequentially closed in X.*

Indeed, let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of E that converges to an element x of X. Because E is sequentially compact in X, there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ that converges to an element x' of E. Note that $\{x_{j_l}\}_{l=1}^\infty$ converges to x too, as before. If X is Hausdorff, then $x = x'$, so that $x \in E$, as desired.

3.5.5 Sequential continuity

Let Y be another topological space.

Definition 3.5.12 *A mapping f from X into Y is said to be* sequentially continuous at a point $x \in X$ *if for every sequence $\{x_j\}_{j=1}^\infty$ of elements of X that converges to x, $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in Y. If f is sequentially continuous at every point in X, then f is said to be* sequentially continuous *on X.*

If f is continuous at x in the usual sense, then f is sequentially continuous at x. The converse holds when there is a local base for the topology of X at x with only finitely or countably many elements.

Proposition 3.5.13 *If f is a sequentially continuous mapping from X into Y, and $K \subseteq X$ is sequentially compact, then $f(K)$ is sequentially compact in Y.*

Let $\{y_j\}_{j=1}^\infty$ be a sequence of elements of $f(K)$. If j is a positive integer, then let us choose an element x_j of K such that $y_j = f(x_j)$. Because K is sequentially compact, there is a subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ that converges to an element x of K. This implies that $\{f(x_{j_l})\}_{l=1}^\infty$ converges to $f(x)$ in Y, because f is sequentially continuous at x, by hypothesis. Thus $\{y_{j_l}\}_{l=1}^\infty = \{f(x_{j_l})\}_{l=1}^\infty$ is a subsequence of $\{y_j\}_{j=1}^\infty$ that converges to $y = f(x) \in f(K)$ in Y, as desired.

3.6 Countable compactness

Let X be a topological space, and let K be a subset of X.

Definition 3.6.1 *We say that K is* countably compact *in X if for every sequence $U_1, U_2, U_3, \ldots$ of open subsets of X such that*

$$K \subseteq \bigcup_{j=1}^\infty U_j, \tag{3.6.2}$$

there is a positive integer n such that

$$K \subseteq \bigcup_{j=1}^n U_j. \tag{3.6.3}$$

If K has the strong limit point property, then K is countably compact, as in Section 3.4.

3.6.1 The Lindelöf property

Definition 3.6.4 *We say that K has the* Lindelöf property *in X if for every open covering $\{U_\alpha\}_{\alpha \in A}$ of K in X there is a subset A_1 of A such that A_1 has only finitely or countably many elements and*

$$K \subseteq \bigcup_{\alpha \in A_1} U_\alpha. \tag{3.6.5}$$

It is easy to see that K is compact if and only if K is countably compact and K has the Lindelöf property.

3.6.2 Lindelöf's theorem

Theorem 3.6.6 (Lindelöf) *Let A be a nonempty set, and let U_α be an open subset of X for each $\alpha \in A$. If X satisfies the second countability condition, then there is a subset A_1 of A such that A_1 has only finitely or countably many elements and*

$$\bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{\alpha \in A} U_\alpha. \tag{3.6.7}$$

By hypothesis, there is a base $\mathcal{B}$ for the topology of X such that $\mathcal{B}$ has only finitely or countably many elements. If $\alpha \in A$, then put

$$\mathcal{B}_\alpha = \{V \in \mathcal{B} : V \subseteq U_\alpha\}. \tag{3.6.8}$$

Observe that

$$U_\alpha = \bigcup \{V : V \in \mathcal{B}_\alpha\} \tag{3.6.9}$$

for every $\alpha \in A$. More precisely, the union on the right is automatically contained in U_α, by the definition of $\mathcal{B}_\alpha$. In order to get that the union is equal to U_α, one uses the hypothesis that $\mathcal{B}$ be a base for the topology of X.

Put

$$\widetilde{\mathcal{B}} = \bigcup_{\alpha \in A} \mathcal{B}_\alpha. \tag{3.6.10}$$

Of course, $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$, by construction. It follows that $\widetilde{\mathcal{B}}$ has only finitely or countably many elements, because $\mathcal{B}$ has only finitely or countably many elements, by hypothesis. If $V \in \widetilde{\mathcal{B}}$, then let us choose an element $\alpha(V)$ of A such that $V \in \mathcal{B}_{\alpha(V)}$. Thus

$$V \subseteq U_{\alpha(V)} \tag{3.6.11}$$

for every $V \in \widetilde{\mathcal{B}}$, by definition of $\mathcal{B}_\alpha$.

Let A_1 be the set of elements of A of the form $\alpha(V)$ for some $V \in \widetilde{\mathcal{B}}$ that have been chosen in this way. It is easy to see that A_1 has only finitely or countably many elements, because $\widetilde{\mathcal{B}}$ has only finitely or countably many elements. Observe that

$$\bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{V \in \widetilde{\mathcal{B}}} U_{\alpha(V)} \supseteq \bigcup_{V \in \widetilde{\mathcal{B}}} V, \tag{3.6.12}$$

using (3.6.11) in the second step. We also have that

$$\bigcup_{V \in \widetilde{\mathcal{B}}} V = \bigcup_{\alpha \in A} \bigcup_{V \in \mathcal{B}_\alpha} V = \bigcup_{\alpha \in A} U_\alpha. \tag{3.6.13}$$

This uses the definition (3.6.10) of $\widetilde{\mathcal{B}}$ in the first step, and (3.6.9) in the second step. Combining (3.6.12) and (3.6.13), we obtain that

$$\bigcup_{\alpha \in A} U_\alpha \subseteq \bigcup_{\alpha \in A_1} U_\alpha. \tag{3.6.14}$$

This implies (3.6.7), because $A_1 \subseteq A$, by construction.

Corollary 3.6.15 *If X satisfies the second countability condition, then every subset of X has the Lindelöf property.*

3.6.3 Some additional results

Proposition 3.6.16 *Suppose that $K \subseteq Y \subseteq X$. Under these conditions, K is countably compact as a subset of X if and only if K is countably compact as a subset of Y, with respect to the induced topology. Similarly, K has the Lindelöf property as a subset of X if and only if K has the Lindelöf property as a subset of Y, with respect to the induced topology.*

This can be verified directly from the definitions, as in the case of ordinary compactness.

Proposition 3.6.17 *Let E be a closed set in X. If $K \subseteq X$ is countably compact, then $K \cap E$ is countably compact too. If K has the Lindelöf property, then $K \cap E$ has the Lindelöf property as well.*

This can be shown in essentially the same way as for ordinary compactness.

Proposition 3.6.18 *Let f be a continuous mapping from X into another topological space Y. If $K \subseteq X$ is countably compact, then $f(K)$ is countably compact in Y. If K has the Lindelöf property, then $f(K)$ has the Lindelöf property in Y.*

This is similar to the analogous statement for ordinary compactness.

3.7 The finite intersection property

Let X be a set, and let K be a subset of X.

Definition 3.7.1 *Let I be a nonempty set, and let E_j be a subset of X for each $j \in I$. We say that $\{E_j\}_{j \in I}$ has the* finite intersection property *with respect to K if for every finite collection $j_1, \ldots, j_n$ of elements of I, we have that*

$$\Big(\bigcap_{l=1}^{n} E_{j_l}\Big) \cap K \neq \emptyset. \tag{3.7.2}$$

If $\{E_j\}_{j \in I}$ has the finite intersection property with respect to X, then we may simply say that $\{E_j\}_{j \in I}$ has the finite intersection property.

Of course, (3.7.2) holds when

$$\Big(\bigcap_{j \in I} E_j\Big) \cap K \neq \emptyset. \tag{3.7.3}$$

3.7.1 A characterization of compactness

Suppose now that X is a topological space.

Proposition 3.7.4 *A subset K of X is compact if and only if for all nonempty families $\{E_j\}_{j \in I}$ of closed subsets of X with the finite intersection property with respect to K, we have that (3.7.3) holds.*

To see this, let a nonempty set I be given again. If $E_j \subseteq X$ is a closed set for some $j \in I$, then

$$U_j = X \setminus E_j \tag{3.7.5}$$

is an open set in X. Similarly, if $U_j \subseteq X$ is an open set for some $j \in I$, then

$$E_j = X \setminus U_j \tag{3.7.6}$$

is a closed set in X. This defines a simple correspondence between families $\{E_j\}_{j \in I}$ of closed subsets of X indexed by I and families $\{U_j\}_{j \in I}$ of open subsets of X indexed by I.

Using this correspondence, we have that (3.7.2) holds for some finite collection of indices $j_1, \ldots, j_n \in I$ if and only if

$$K \not\subseteq X \setminus \Big(\bigcap_{j=1}^{n} E_j\Big) = \bigcup_{l=1}^{n} U_{j_l}. \tag{3.7.7}$$

Thus $\{E_j\}_{j \in I}$ has the finite intersection property with respect to K exactly when K cannot be covered by finitely many U_j's, $j \in I$. Similarly, (3.7.3) holds if and only if

$$K \not\subseteq X \setminus \Big(\bigcap_{j \in I} E_j\Big) = \bigcup_{j \in I} U_j. \tag{3.7.8}$$

This is the same as saying that K is not covered by the U_j's, $j \in I$. Using this, it is easy to see that the condition mentioned in the proposition is equivalent to compactness.

3.7.2 A characterization of countable compactness

Let us now consider the case where $I = \mathbf{Z}_+$.

Proposition 3.7.9 *A subset K of X is countably compact if and only if for every sequence $E_1, E_2, E_3, \ldots$ of closed subsets of X that satisfies*

$$\Big(\bigcap_{j=1}^{n} E_j\Big) \cap K \neq \emptyset \tag{3.7.10}$$

for each positive integer n, we have that

$$\Big(\bigcap_{j=1}^{\infty} E_j\Big) \cap K \neq \emptyset. \tag{3.7.11}$$

Let E_j be a subset of X for each positive integer j. Observe that $\{E_j\}_{j \in \mathbf{Z}_+}$ has the finite intersection property with respect to $K \subseteq X$ if and only if (3.7.10) holds for every positive integer n. In this situation, (3.7.11) is the same as (3.7.3) with $I = \mathbf{Z}_+$. Thus the proposition follows from the same type of argument as before.

3.8 The strong limit point property

Let X be a topological space.

Proposition 3.8.1 *If $K \subseteq X$ is countably compact, then K has the strong limit point property.*

To show this, let an infinite subset L of K be given. As before, we can find an infinite sequence $\{x_j\}_{j=1}^{\infty}$ of distinct elements of L, because L has infinitely many elements. Put

$$A_l = \{x_j : j \geq l\} \tag{3.8.2}$$

for each positive integer l. Note that for each l, $A_l \neq \emptyset$ and

$$A_{l+1} \subseteq A_l. \tag{3.8.3}$$

Let

$$E_l = \overline{A_l} \tag{3.8.4}$$

be the closure of A_l in X for each $l \geq 1$. Thus

$$A_l \subseteq E_l \tag{3.8.5}$$

for every $l \geq 1$, which implies in particular that $E_l \neq \emptyset$. It is easy to see that

$$E_{l+1} \subseteq E_l \tag{3.8.6}$$

for every $l \geq 1$, because of (3.8.3).

Of course, E_l is a closed set in X for each $l \geq 1$, by construction. Observe that

$$\bigcap_{l=1}^{n} E_l = E_n \tag{3.8.7}$$

for every positive integer n, by (3.8.6). It follows that

$$\Big(\bigcap_{l=1}^{n} E_l\Big) \cap K = E_n \cap K \tag{3.8.8}$$

for each $n \geq 1$. We also have that

$$A_n \subseteq E_n \cap K \tag{3.8.9}$$

for each $n \geq 1$, because of (3.8.5) and the fact that $A_n \subseteq L \subseteq K$ for every n, by construction. This implies that (3.8.8) is nonempty for every n.

Thus $E_1, E_2, E_3, \ldots$ is a sequence of closed sets in X with the finite intersection property with respect to K. Because K is countably compact, we get that

$$\Big(\bigcap_{l=1}^{\infty} E_l\Big) \cap K \neq \emptyset, \tag{3.8.10}$$

as in the previous section. Let x be an element of the left side of (3.8.10). In particular, $x \in K$, and we would like to check that x is a strong limit point of L in X.

Note that x is adherent to A_l in X for each $l \geq 1$, because $x \in E_l$. Let $U \subseteq X$ be an open set that contains x. It follows that

$$A_l \cap U \neq \emptyset \tag{3.8.11}$$

for each $l \geq 1$, because x is adherent to A_l. This means that for each $l \geq 1$ there is a $j \geq l$ such that

$$x_j \in U, \tag{3.8.12}$$

by the definition (3.8.2) of A_l. Thus (3.8.12) holds for infinitely many positive integers j. In particular, U contains infinitely many elements of L, because the x_j's are distinct elements of L. This implies that x is a strong limit point of L in X, as desired.

3.9 Compactness and bases

Let X be a topological space, and let $\mathcal{B}$ be a collection of open subsets of X.

Definition 3.9.1 *We say that $K \subseteq X$ is* compact with respect to $\mathcal{B}$ *if every open covering of K by elements of $\mathcal{B}$ can be reduced to a finite subcovering. More precisely, this means that if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of K with $U_\alpha \in \mathcal{B}$ for every $\alpha \in A$, then there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that $K \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}$.*

Of course, if K is a compact subset of X, then K is automatically compact with respect to $\mathcal{B}$. Countable compactness and the Lindelöf property with respect to $\mathcal{B}$ can be defined analogously.

3.9.1 A criterion for compactness

Proposition 3.9.2 *Suppose that $\mathcal{B}$ is a base for the topology of X. If $K \subseteq X$ is compact with respect to $\mathcal{B}$, then K is compact in X in the usual sense.*

The analogous statement for the Lindelöf property can be shown similarly. Note that Lindelöf's theorem follows from the version of Proposition 3.9.2 for the Lindelöf property. However, the proofs of Proposition 3.9.2 and its version for the Lindelöf property are also very similar to the proof of Lindelöf's theorem.

3.9.2 The proof of the proposition

To prove the proposition, let $\{U_\alpha\}_{\alpha \in A}$ be any open covering of K in X. Put

$$\mathcal{B}_\alpha = \{V \in \mathcal{B} : V \subseteq U_\alpha\} \tag{3.9.3}$$

for each $\alpha \in A$. Because $\mathcal{B}$ is a base for the topology of X,

$$U_\alpha = \bigcup \{V : V \in \mathcal{B}_\alpha\} \tag{3.9.4}$$

for every $\alpha \in A$, as before. Put

$$\widetilde{\mathcal{B}} = \bigcup_{\alpha \in A} \mathcal{B}_\alpha. \tag{3.9.5}$$

Observe that

$$\bigcup_{V \in \widetilde{\mathcal{B}}} V = \bigcup_{\alpha \in A} \bigcup_{V \in \mathcal{B}_\alpha} V = \bigcup_{\alpha \in A} U_\alpha, \tag{3.9.6}$$

using the definition of $\widetilde{\mathcal{B}}$ in the first step, and (3.9.4) in the second step. It follows that

$$K \subseteq \bigcup_{\alpha \in A} U_\alpha = \bigcup_{V \in \widetilde{\mathcal{B}}} V. \tag{3.9.7}$$

If K is compact with respect to $\mathcal{B}$, then there are finitely many elements $V_1, \ldots, V_n$ of $\widetilde{\mathcal{B}}$ such that

$$K \subseteq \bigcup_{j=1}^{n} V_j. \tag{3.9.8}$$

By definition of $\widetilde{\mathcal{B}}$, for each $j = 1, \ldots, n$ there is an $\alpha_j \in A$ such that $V_j \in \mathcal{B}_{\alpha_j}$, which means that

$$V_j \subseteq U_{\alpha_j}. \tag{3.9.9}$$

Combining this with (3.9.8), we get that

$$K \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}, \tag{3.9.10}$$

as desired.

3.9.3 Subbases

Let $\mathcal{B}$ be any collection of open subsets of X. Consider the collection $\mathcal{B}_1$ of all subsets of X that can be expressed as the intersection of finitely many elements of $\mathcal{B}$. Note that the elements of $\mathcal{B}_1$ are all open subsets of X. If $\mathcal{B}_1$ is a base for the topology of X, then $\mathcal{B}$ is said to be a *subbase* for the topology of X.

As an example, let $\mathcal{B}$ be the collection of subsets of the real line that are open half-lines, which is to say sets of the form $(a, +\infty)$ and $(-\infty, b)$ for $a, b \in \mathbf{R}$. In this case, $\mathcal{B}_1$ consists of all open half-lines in $\mathbf{R}$, as well as all open intervals in $\mathbf{R}$. It is easy to see that $\mathcal{B}_1$ is a base for the standard topology on $\mathbf{R}$, so that $\mathcal{B}$ is a subbase for this topology.

3.9.4 Alexander's subbase theorem

If $K \subseteq X$ is compact with respect to a subbase $\mathcal{B}$ for the topology of X, then *Alexander's subbase theorem* says that K is compact in X in the usual sense. The proof of this theorem uses the axiom of choice in a significant way. More precisely, this involves Zorn's lemma or the Hausdorff maximality principle, which will be discussed in the next chapter.

3.10 Products of two compact sets

Let X and Y be topological spaces, and consider their Cartesian product $X \times Y$, equipped with the corresponding product topology.

Theorem 3.10.1 (Tychonoff) *If $H \subseteq X$ and $K \subseteq Y$ are compact, then their product $H \times K$ is compact in $X \times Y$.*

To show this, let $\mathcal{B}$ be the collection of subsets of $X \times Y$ of the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open sets. This is a base for the product topology on $X \times Y$. Thus it suffices to verify that $H \times K$ is compact with respect to $\mathcal{B}$, as in the previous section.

Let $\{U_\alpha \times V_\alpha\}_{\alpha \in A}$ be a covering of $H \times K$ by elements of $\mathcal{B}$, so that $U_\alpha \subseteq X$ and $V_\alpha \subseteq Y$ are open sets for every $\alpha \in A$, and

$$H \times K \subseteq \bigcup_{\alpha \in A} (U_\alpha \times V_\alpha). \tag{3.10.2}$$

Let $x \in H$ be given, so that

$$\{x\} \times K \subseteq \bigcup_{\alpha \in A} (U_\alpha \times V_\alpha). \tag{3.10.3}$$

because $\{x\} \times K \subseteq H \times K$.

3.10.1 Some open coverings of K in Y

Put

$$A(x) = \{\alpha \in A : x \in U_\alpha\}. \tag{3.10.4}$$

Observe that

$$K \subseteq \bigcup_{\alpha \in A(x)} V_\alpha, \tag{3.10.5}$$

by (3.10.3). More precisely, if $y \in K$, then $(x, y) \in \{x\} \times K$, and hence there is an $\alpha \in A$ such that $(x, y) \in U_\alpha \times V_\alpha$, by (3.10.3). This means that $x \in U_\alpha$, so that $\alpha \in A(x)$, and $y \in V_\alpha$, as desired. It follows that there is a finite subset $A_1(x)$ of $A(x)$ such that

$$K \subseteq \bigcup_{\alpha \in A_1(x)} V_\alpha, \tag{3.10.6}$$

because K is compact in Y.

3.10.2 An open covering of H in X

Put

$$U_1(x) = \bigcap_{\alpha \in A_1(x)} U_\alpha. \tag{3.10.7}$$

This is an open set in X, because U_α is an open set in X for every $\alpha \in A_1(x) \subseteq A$, and $A_1(x)$ has only finitely many elements. Of course, $x \in U_1(x)$, because $x \in U_\alpha$ for every $\alpha \in A_1(x) \subseteq A(x)$. We can do this for every $x \in H$, to get an open covering of H in X. This implies that there are finitely many elements $x_1, \ldots, x_n$ of H such that

$$H \subseteq \bigcup_{j=1}^{n} U_1(x_j), \tag{3.10.8}$$

because H is compact in X.

3.10.3 Using these finite subcoverings

Using (3.10.8), we get that

$$H \times K \subseteq \Big(\bigcup_{j=1}^{n} U_1(x_j)\Big) \times K = \bigcup_{j=1}^{n} (U_1(x_j) \times K). \tag{3.10.9}$$

We also have that

$$\begin{aligned} U_1(x_j) \times K \subseteq U_1(x_j) \times \Big(\bigcup_{\alpha \in A_1(x_j)} V_\alpha\Big) &= \bigcup_{\alpha \in A_1(x_j)} (U_1(x_j) \times V_\alpha) \\ &\subseteq \bigcup_{\alpha \in A_1(x_j)} (U_\alpha \times V_\alpha) \end{aligned} \tag{3.10.10}$$

for each $j = 1, \ldots, n$. This uses (3.10.6) in the first step, and the fact that $U_1(x_j) \subseteq U_\alpha$ when $\alpha \in A_1(x_j)$, by the definition (3.10.7) of $U_1(x_j)$, in the third step. Combining (3.10.9) and (3.10.10), we get that

$$H \times K \subseteq \bigcup_{j=1}^{n} \bigcup_{\alpha \in A_1(x_j)} (U_\alpha \times V_\alpha). \tag{3.10.11}$$

Of course, $\bigcup_{j=1}^n A_1(x_j)$ is a finite subset of A, because $A_1(x_j)$ is a finite subset of A for each $j = 1, \ldots, n$, as desired.

3.11 Products of more compact sets

Let $X_1, \ldots, X_n$ be finitely many topological spaces for some positive integer n, and consider their Cartesian product

$$X = \prod_{j=1}^{n} X_j, \tag{3.11.1}$$

equipped with the product topology. Also let K_j be a compact subset of X_j for each $j = 1, \ldots, n$, and put

$$K = \prod_{j=1}^{n} K_j. \tag{3.11.2}$$

Under these conditions, K is a compact subset of X. This is trivial when $n = 1$, and the $n = 2$ case was discussed in the previous section. Otherwise, one can use induction, as follows.

3.11.1 An induction argument

Suppose that $n \geq 2$, and that the analogous statement holds for $n - 1$. Thus $\prod_{j=1}^{n-1} K_j$ is a compact subset of $\prod_{j=1}^{n-1} X_j$, with respect to the product topology. There is a natural identification of X with

$$\Big(\prod_{j=1}^{n-1} X_j\Big) \times X_n, \tag{3.11.3}$$

where $x = (x_1, \ldots, x_n) \in X$ is identified with $((x_1, \ldots, x_{n-1}), x_n)$, as an element of (3.11.3). Let us take $\prod_{j=1}^{n-1} X_j$ to be equipped with the product topology, and use the corresponding product topology on (3.11.3). One can check that this corresponds to the product topology on X, using the identification of (3.11.3) with X just mentioned.

As in the previous section,

$$\Big(\prod_{j=1}^{n-1} K_j\Big) \times K_n \tag{3.11.4}$$

is a compact subset of (3.11.3), with respect to the product topology. Of course, K corresponds to (3.11.4), with respect to the identification of X with (3.11.3). It follows that K is a compact subset of X, because the identification of X with (3.11.3) is a homeomorphism with respect to the corresponding product topologies, as before.

3.11.2 Some compact subsets of $\mathbf{R}^n$

Let n be a positive integer again, and $\mathbf{R}^n$ be the usual space of n-tuples of real numbers. This is the same as the Cartesian product of n copies of the real line. The standard topology on $\mathbf{R}^n$ may be defined as the product topology corresponding to the standard topology on $\mathbf{R}$, as before. This is the same as the topology determined by the standard Euclidean metric on $\mathbf{R}^n$.

Let a_j and b_j be real numbers with $a_j \le b_j$ for each $j = 1, \ldots, n$. Remember that the closed interval $[a_j, b_j]$ is a compact subset of the real line for each $j = 1, \ldots, n$, with respect to the standard topology. It follows that their Cartesian product

$$\prod_{j=1}^{n} [a_j, b_j] \tag{3.11.5}$$

is compact with respect to the standard topology on $\mathbf{R}^n$.

3.11.3 Compactness of arbitrary products of compact sets

Now let I be a nonempty set, let X_j be a topological space for each $j \in I$, and consider the Cartesian product $X = \prod_{j \in I} X_j$, equipped with the product topology. If $K_j \subseteq X_j$ is compact for each $j \in I$, then another famous theorem of Tychonoff states that $K = \prod_{j \in I} K_j$ is compact in X. Of course, this reduces to the previous statement for (3.11.2) when I has only finitely many elements.

Let $\mathcal{B}$ be the collection of subsets of X of the form $U = \prod_{j \in I} U_j$, where $U_j = X_j$ for all but one $j \in I$, and U_j is an open set in X_j for that j. It is easy to see that $\mathcal{B}$ is a subbase for the product topology on X. To show Tychonoff's theorem, it suffices to get that K is compact with respect to $\mathcal{B}$, by Alexander's subbase theorem.

3.12 Sequential compactness and finite products

Let $X_1, \ldots, X_n$ be finitely many topological spaces again, and let

$$X = \prod_{j=1}^{n} X_j \tag{3.12.1}$$

be their Cartesian product, equipped with the product topology.

Proposition 3.12.2 *If K_j is a sequentially compact subset of X_j for each $j = 1, \ldots, n$, then*

$$K = \prod_{j=1}^{n} K_j \tag{3.12.3}$$

is sequentially compact in X.

3.12.1 An initial subsequence

To see this, let $\{x(l)\}_{l=1}^{\infty}$ be a sequence of elements of K. Thus

$$x(l) = (x_1(l), \ldots, x_n(l)) \tag{3.12.4}$$

for each positive integer l, where $x_j(l) \in K_j$ for $j = 1, \ldots, n$. In particular, $\{x_1(l)\}_{l=1}^{\infty}$ is a sequence of elements of K_1. Because K_1 is sequentially compact, there is a subsequence $\{x_1(l_m)\}_{m=1}^{\infty}$ of $\{x_1(l)\}_{l=1}^{\infty}$ that converges to an element x_1 of K_1 in X_1. Using the same sequence of indices $\{l_m\}_{m=1}^{\infty}$, we get a subsequence $\{x(l_m)\}_{m=1}^{\infty}$ of $\{x(l)\}_{l=1}^{\infty}$.

3.12.2 A subsequence of the first subsequence when $n \geq 2$

Of course, the statement is trivial when $n = 1$, and so we may as well suppose that $n \geq 2$. As before, $\{x_2(l_m)\}_{m=1}^{\infty}$ is a sequence of elements of K_2, and so there is a subsequence $\{x_2(l_{m_r})\}_{r=1}^{\infty}$ of $\{x_2(l_m)\}_{m=1}^{\infty}$ that converges to an element x_2 of K_2, because K_2 is sequentially compact. Using the same sequence $\{m_r\}_{r=1}^{\infty}$ of indices, we get a subsequence $\{x(l_{m_r})\}_{r=1}^{\infty}$ of $\{x(l_m)\}_{m=1}^{\infty}$. In particular, $\{x_1(l_{m_r})\}_{r=1}^{\infty}$ is a subsequence of $\{x_1(l_m)\}_{m=1}^{\infty}$. This implies that $\{x_1(l_{m_r})\}_{r=1}^{\infty}$ converges to x_1 in X_1, because $\{x_1(l_m)\}_{m=1}^{\infty}$ converges to x_1 in X_1.

Note that $\{x(l_{m_r})\}_{r=1}^{\infty}$ may be considered as a subsequence of the initial sequence $\{x(l)\}_{l=1}^{\infty}$ as well. If $n = 2$, then we get that $\{x(l_{m_r})\}_{r=1}^{\infty}$ converges to

$$x = (x_1, x_2) \in K_1 \times K_2 = K \tag{3.12.5}$$

in X, as desired. Otherwise, we can repeat the process.

3.12.3 Repeating the process

More precisely, suppose that the jth subsequence of $\{x(l)\}_{l=1}^{\infty}$ has been chosen in this way for some positive integer $j < n$. We would like to choose the $(j+1)$th subsequence of $\{x(l)\}_{l=1}^{\infty}$ to be a subsequence of the jth subsequence, using the sequential compactness of K_{j+1}, as before. The $(j+1)$th subsequence is chosen so that the sequence of $(j+1)$th coordinates in K_{j+1} of the terms of the $(j+1)$th subsequence converges to an element x_{j+1} of K_{j+1} in X_{j+1}. Observe that the $(j+1)$th subsequence is a subsequence of the ith subsequence for each $i = 1, \ldots, j$. This implies that the sequence of ith coordinates of the terms of the $(j+1)$th subsequence converge to x_i in X_i, because the sequence of ith coordinates of the terms of the ith subsequence converge to x_i in X_i, by construction.

We can continue in this way until the nth subsequence of $\{x(l)\}_{l=1}^\infty$ is chosen. As before, for each $i = 1, \ldots, n$, the sequence of ith coodinates of the nth subsequence converges to $x_i \in K_i$ in X_i. This implies that the nth subsequence converges to

$$x = (x_1, \ldots, x_n) \in K \tag{3.12.6}$$

in X. We also have that the nth subsequence is a subsequence of the initial sequence $\{x(l)\}_{l=1}^\infty$, as desired.

3.13 Sequential compactness in countable products

If $\{l_m\}_{m=1}^\infty$ is a strictly increasing sequence of positive integers, then it is easy to see that

$$l_m \geq m \tag{3.13.1}$$

for every positive integer m.

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of topological spaces, and consider their Cartesian product

$$X = \prod_{j=1}^\infty X_j, \tag{3.13.2}$$

equipped with the product topology.

Proposition 3.13.3 *If K_j is a sequentially compact subset of X_j for each positive integer j, then*

$$K = \prod_{j=1}^\infty K_j \tag{3.13.4}$$

is sequentially compact in X.

3.13.1 A sequence of subsequences

To show this, let a sequence $\{x(l)\}_{l=1}^\infty$ of elements of K be given. Note that

$$x(l) = \{x_j(l)\}_{j=1}^\infty \tag{3.13.5}$$

is a sequence for each l, where $x_j(l) \in K_j$ for every $j \geq 1$. In particular, $\{x_1(l)\}_{l=1}^\infty$ is a sequence of elements of K_1, as before. Hence there is a subsequence $\{x_1(l_m)\}_{m=1}^\infty$ of $\{x_1(l)\}_{l=1}^\infty$ that converges to an element x_1 of K_1 in X_1, because K_1 is sequentially compact. We can use the same sequence $\{l_m\}_{m=1}^\infty$ of indices to get a subsequence $\{x(l_m)\}_{m=1}^\infty$ of $\{x(l)\}_{l=1}^\infty$.

We can repeat the process, as in the previous section, to get an infinite sequence of subsequences of $\{x(l)\}_{l=1}^\infty$. More precisely, for each positive integer

j, we can get a sequence $\{x(r,j)\}_{r=1}^{\infty}$ of elements of K with the following two poperties. First,

(3.13.6) $$\{x(r,1)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(l)\}_{l=1}^{\infty},$$

and

(3.13.7) $$\{x(r,j)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(m,j-1)\}_{m=1}^{\infty}$$

when $j \geq 2$. This implies that

(3.13.8) $$\{x(r,j)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(l)\}_{l=1}^{\infty}$$

for every $j \geq 1$, and that

(3.13.9) $$\{x(r,j)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(m,i)\}_{m=1}^{\infty}$$

when $1 \leq i \leq j$. The second property is that for every positive integer j there is an element x_j of K_j such that

(3.13.10) $$\{x_j(r,j)\}_{r=1}^{\infty} \text{ converges to } x_j \text{ in } X_j.$$

It follows that

(3.13.11) $$\{x_i(r,j)\}_{r=1}^{\infty} \text{ converges to } x_i \text{ in } X_i$$

when $1 \leq i \leq j$, because of (3.13.9). Put

(3.13.12) $$x = \{x_j\}_{j=1}^{\infty},$$

which defines an element of K.

3.13.2 Another subsequence

We would like to find a subsequence of $\{x(l)\}_{l=1}^{\infty}$ that converges to x in X. Put

(3.13.13) $$y(r) = x(r,r)$$

for each positive integer r, which is the rth term of the rth subsequence described in the previous paragraph. One can check that

(3.13.14) $$\{y(r)\}_{r=1}^{\infty} \text{ is a subsequence of } \{x(l)\}_{l=1}^{\infty}.$$

Indeed, for each $r \geq 1$, $y(r)$ is one of the terms of $\{x(l)\}_{l=1}^{\infty}$, by construction. One can verify that $y(r+1)$ is chosen among the terms of $\{x(l)\}_{l=1}^{\infty}$ that occurs after the one corresponding to $y(r)$, using (3.13.1). More precisely, $y(r+1)$ is the $(r+1)$th term of the $(r+1)$th subsequence, and the $(r+1)$th subsequence is a subsequence of the rth subsequence, by construction. This implies that $y(r+1)$ occurs in the rth subsequence after $y(r)$, as in (3.13.1). It follows that $y(r+1)$ occurs after $y(r)$ in the previous subsequences, including the initial sequence $\{x(l)\}_{l=1}^{\infty}$.

3.13.3 More on this subsequence

Similarly, one can check that for each positive integer j,

$$\{y(r)\}_{r=j}^{\infty} \text{ is a subsequence of } \{x(m,j)\}_{m=1}^{\infty}. \tag{3.13.15}$$

In particular, if $r \geq j$, then $y(r)$ is one of the terms of $\{x(m,j)\}_{m=1}^{\infty}$, because the rth subsequence is a subsequence of the jth subsequence, as in (3.13.9). We also have that $y(r+1)$ is chosen among the terms of $\{x(m,j)\}_{m=1}^{\infty}$ after the one corresponding to $y(r)$ when $r \geq j$, as in the preceding paragraph.

It follows that

$$\{y_j(r)\}_{r=j}^{\infty} \text{ converges to } x_j \text{ in } X_j \tag{3.13.16}$$

for each positive integer j, because of (3.13.10) and (3.13.15). This implies that

$$\{y_j(r)\}_{r=1}^{\infty} \text{ converges to } x_j \text{ in } X_j \tag{3.13.17}$$

for every $j \geq 1$. This means that $\{y(r)\}_{r=1}^{\infty}$ converges to x with respect to the product topology on X, as desired.

3.14 Compactness and sequential compactness

If a topological space X satisfies the first countability condition and $K \subseteq X$ is compact, then

$$K \text{ is sequentially compact.} \tag{3.14.1}$$

More precisely, compactness implies the strong limit point property, which implies sequential compactness when X satisfies the first countability condition.

Suppose for the moment that X is any set equipped with the cofinite topology, as in Subsection 3.1.2. Remember that every subset of X is compact in this case. One can check that every subset of X is also sequentially compact in this case. However, one can verify that X satisfies the first countability condition only when X has only finitely or countably many elements.

One can get examples of compact Hausdorff topological spaces that do not satisfy the first countability condition using uncountable products of compact Hausdorff spaces and Tychonoff's theorem. One can use uncountable products of finite sets, each of which has at least two elements, and is equipped with the discrete topology, for instance. One could also use uncountable products of closed intervals in the real line, equipped with the topology induced by the standard topology on $\mathbf{R}$.

Let I be a nonempty set, let X_j be a topological space for each $j \in I$, and let $X = \prod_{j \in I} X_j$ be the Cartesian product of the X_j's, equipped with the product topology. Suppose that $K_j \subseteq X_j$ is compact for each $j \in I$, so that $K = \prod_{j \in I} K_j$ is a compact subset of X, by Tychonoff's theorem. If I has only finitely or countably many elements, and X_j satisfies the first countability condition for each $j \in I$, then we have seen that X satisfies the first countability condition with respect to the product topology. In this case, the compactness of K implies that K is sequentially compact, as before. Alternatively, K_j is sequentially compact in X_j for each $j \in I$ under these conditions, which implies that K is sequentially compact, as in the previous two sections.

If K is a sequentially compact subset of a topological space X, then we have seen that K has the strong limit point property, which implies that

$$K \text{ is countably compact.} \tag{3.14.2}$$

If K also satisfies the Lindelöf property, then it follows that K is compact. If X satisfies the second countability condition, then K automatically satisfies the Lindelöf property, by Lindelöf's theorem. If X is the Cartesian product of finitely or countably many topological spaces, each of which satisfies the second countability condition, then we have seen that X satisfies the second countability condition as well, with respect to the product topology.

It is well known that

(3.14.3) sequentially compact subsets of metric spaces are compact.

More precisely, one can reduce to showing that a sequentially compact metric space X is compact. Part of the proof is to show that X is "totally bounded". This implies that X is separable, and thus satisfies the second countability condition.

Suppose that X is the product of finitely or countably many topological spaces, each of which has a metric that determines the topology. We have seen that X has a metric that determines the corresponding product topology in this case. This implies that sequentially compact subsets of X are compact, as in the preceding paragraph.

3.15 Another result of Tychonoff

Let X be a topological space. If X has the Lindelöf property, and X is regular in the strict sense, then a well-known result of Tychonoff states that X is normal in the strict sense.

To see this, let A and B be disjoint closed subsets of X. Note that A and B each have the Lindelöf property in X, because A and B are closed subsets of X and X has the Lindelöf property, as in Proposition 3.6.17. If $x \in A$, then there is an open set $U(x) \subseteq X$ such that

$$x \in U(x) \quad \text{and} \quad \overline{U(x)} \subseteq X \setminus B, \tag{3.15.1}$$

because X is regular in the strict sense, and $X \setminus B$ is an open set that contains x. Similarly, if $y \in B$, then there is an open set $V(y) \subseteq X$ such that

$$y \in V(y) \quad \text{and} \quad \overline{V(y)} \subseteq X \setminus A. \tag{3.15.2}$$

Using the Lindelöf property for A and B in X, we can get sequences $\{U_j\}_{j=1}^\infty$ and $\{V_j\}_{j=1}^\infty$ of open subsets of X such that

$$A \subseteq \bigcup_{j=1}^\infty U_j, \quad B \subseteq \bigcup_{j=1}^\infty V_j, \tag{3.15.3}$$

and

$$\overline{U_j} \subseteq X \setminus B, \quad \overline{V_j} \subseteq X \setminus A \tag{3.15.4}$$

for every $j \geq 1$. Put

$$\widetilde{U}_j = U_j \setminus \bigcup_{l=1}^{j} \overline{V_l}, \quad \widetilde{V}_j = V_j \setminus \bigcup_{l=1}^{j} \overline{U_l} \tag{3.15.5}$$

for each $j \geq 1$. It is easy to see that $\widetilde{U}_j$ and $\widetilde{V}_j$ are open subsets of X for every $j \geq 1$. One can check that

$$A \subseteq \bigcup_{j=1}^{\infty} \widetilde{U}_j, \quad B \subseteq \bigcup_{j=1}^{\infty} \widetilde{V}_j, \tag{3.15.6}$$

using (3.15.3) and (3.15.4).

Thus

$$\widetilde{U} = \bigcup_{j=1}^{\infty} \widetilde{U}_j, \quad \widetilde{V} = \bigcup_{j=1}^{\infty} \widetilde{V}_j \tag{3.15.7}$$

are open subsets of X that contain A and B, respectively. We would like to verify that $\widetilde{U} \cap \widetilde{V} = \emptyset$. To do this, it suffices to check that

$$\widetilde{U}_j \cap \widetilde{V}_l = \emptyset \tag{3.15.8}$$

for every $j, l \geq 1$. If $l \leq j$, then this follows from the definition of $\widetilde{U}_j$, and the fact that $\widetilde{V}_l \subseteq V_l$. The case where $j \leq l$ can be handled analogously.

Chapter 4

Some more set theory

4.1 Zorn's lemma

Let $(A, \preceq)$ be a partially-ordered set. An element b of A is said to be an *upper bound* of a subset E of A if for every $a \in E$, we have that

$$a \preceq b. \tag{4.1.1}$$

A subset C of A is said to be a *chain* if C is linearly ordered by the restriction of $\preceq$ to C. This means that for every $x, y \in C$,

$$\text{either } x \preceq y \text{ or } y \preceq x \text{ (or both, i.e., } x = y). \tag{4.1.2}$$

An element x of A is said to be *maximal* in A with respect to $\preceq$ if

$$\text{for every } y \in A \text{ with } x \preceq y, \text{we have that } x = y. \tag{4.1.3}$$

Let us say that $b \in A$ is a *top element* of A if b is an upper bound for A, so that (4.1.1) holds for every $a \in A$. A top element of A is unique when it exists, and is maximal in A. However, a maximal element of a partially-order set is not necessarily a top element, or unique. A maximal element of a linearly-ordered set is a top element. It is easy to see that a nonempty linearly-ordered set with only finitely many elements has a top element.

If A is a nonempty partially-ordered set with only finitely many elements, then A has a maximal element. To see this, let a_1 be any element of A. If a_1 is maximal in A, then we can stop. Otherwise, there is an element a_2 of A such that

$$a_1 \preceq a_2 \text{ and } a_1 \neq a_2. \tag{4.1.4}$$

We can repeat the process a finite number of times to get a maximal element of A, because A has only finitely many elements.

4.1.1 The statement of Zorn's lemma

Let A be any partially-ordered set again. If

(4.1.5) every chain in A has an upper bound in A,

then *Zorn's lemma* states that

(4.1.6) A has a maximal element.

Note that the empty set may be considered as a chain in A, so that the hypothesis of Zorn's lemma implies that $A \neq \emptyset$. There are well-known arguments for obtaining Zorn's lemma from the axiom of choice, as in [236], for instance. The converse is much simpler, and we shall return to that later.

4.1.2 Countably-infinite sets A

Suppose that A is countably infinite, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of A in which every element of A occurs exactly once. We can try to find a maximal element of A using a more precise version of the argument for finite sets, as follows. Put $j_1 = 1$, and suppose that $j_l \in \mathbf{Z}_+$ has been chosen for some positive integer l. If there is an integer $k > j_l$ such that

$$x_{j_l} \preceq x_k, \tag{4.1.7}$$

then we take j_{l+1} to be the smallest such integer k, and otherwise we stop. If this process stops after finitely many steps, then we get a maximal element of A. Otherwise, we get an infinite subsequence $\{x_{j_l}\}_{l=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ such that

$$x_{j_l} \preceq x_{j_{l+1}} \tag{4.1.8}$$

for every $l \geq 1$. In particular,

$$C = \{x_{j_l} : l \in \mathbf{Z}_+\} \tag{4.1.9}$$

is a chain in A. If C has an upper bound in A, then this upper bound is of the form x_n for some positive integer n. In this case, there would have to be a positive integer l_0 such that $j_{l_0} = n$, because of the way that the j_l's were chosen. This would contradict (4.1.8), because the x_j's are supposed to be distinct elements of A.

4.2 Hausdorff's maximality principle

Let $(A, \preceq)$ be a partially-ordered set again. *Hausdorff's maximality principle* states that

(4.2.1) there is a chain in A that is maximal with respect to inclusion.

More precisely, this means that there is a chain C_0 in A such that

$$\text{if } C \text{ is any chain in } A \text{ with } C_0 \subseteq C, \text{ then } C_0 = C. \tag{4.2.2}$$

If A has only finitely many elements, then A has only finitely many subsets, and in particular there are only finitely many chains in A. In this case, one can find a maximal chain in A as in the previous section. Alternatively, one can keep adding points to a chain in A until it is no longer possible to have a chain in A.

4.2.1 Maximal chains in countable sets

Similarly, suppose that A is countably infinite, and let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of A in which every element of A occurs exactly once. Of course, $C_1 = \{x_1\}$ is a chain in A. If $x_1 \preceq x_2$ or $x_2 \preceq x_1$, then put $C_2 = \{x_1, x_2\}$, which is a chain in A, and otherwise put $C_2 = C_1$. If $C_n \subseteq \{x_1, \ldots, x_n\}$ has been chosen in this way for some positive integer n, then we can define C_{n+1} as follows. If $C_n \cup \{x_{n+1}\}$ is a chain in A, then we take it to be C_{n+1}. Otherwise, put $C_{n+1} = C_n$. One can check that C_n is a maximal chain in $\{x_1, \ldots, x_n\}$ for each positive integer n, as in the preceding paragraph. One can also verify that

$$C = \bigcup_{n=1}^{\infty} C_n \tag{4.2.3}$$

is a maximal chain in A.

There are well-known arguments for obtaining Hausdorff's maximality principle from the axiom of choice, as in [236] again.

4.2.2 Zorn's lemma from maximal chains

Let us see how Zorn's lemma can be obtained from Hausdorff's maximality principle. Let $(A, \preceq)$ be a partially-ordered set, and let C_0 be a maximal chain in A, as in Hausdorff's maximality principle. The hypothesis of Zorn's lemma implies that there is a $b \in A$ such that

$$b \text{ is an upper bound for } C_0. \tag{4.2.4}$$

We would like to check that

$$b \text{ is a maximal element of } A \tag{4.2.5}$$

under these conditions. To do this, suppose that $y \in A$ satisfies $b \preceq y$. It is easy to see that

$$C_0 \cup \{b, y\} \text{ is a chain in } A \tag{4.2.6}$$

too in this situation. This implies that

$$C_0 = C_0 \cup \{b, y\}, \tag{4.2.7}$$

because C_0 is a maximal chain in A. This means that $b, y \in C_0$. It follows that $y \preceq b$, because $y \in C_0$ and b is an upper bound for C_0. Thus $b = y$, because $b \preceq y$, by hypothesis, as desired.

4.3 Maximal chains from Zorn's lemma

We can also obtain Hausdorff's maximality principle from Zorn's lemma. Let $(A, \preceq)$ be a partially-ordered set again, and let $\mathcal{C}$ be the collection of all chains in A. We may consider $\mathcal{C}$ as a partially-ordered set with respect to inclusion, i.e., using $C_1 \subseteq C_2$ for $C_1, C_2 \in \mathcal{C}$ as the partial ordering on $\mathcal{C}$. Hausdorff's maximality principle is exactly the statement that

$$\mathcal{C} \text{ has a maximal element.} \tag{4.3.1}$$

Let us check that $\mathcal{C}$ satisfies the hypothesis of Zorn's lemma.

Let $\mathcal{E}$ be a chain in $\mathcal{C}$. This means that $\mathcal{E}$ is a collection of chains in A such that for every $C_1, C_2 \in \mathcal{E}$, either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. We would like to show that $\mathcal{E}$ has an upper bound in $\mathcal{C}$.

4.3.1 An upper bound for $\mathcal{E}$ in $\mathcal{C}$

Put

$$C(\mathcal{E}) = \bigcup_{C \in \mathcal{E}} C. \tag{4.3.2}$$

More precisely, each element C of $\mathcal{E}$ is a subset of A, so that their union is a subset of A as well. If $\mathcal{E} = \emptyset$, then $C(\mathcal{E})$ is interpreted as being the empty set too.

We would like to verify that $C(\mathcal{E})$ is a chain in A. Let $x, y \in C(\mathcal{E})$ be given. By definition of $C(\mathcal{E})$, there are $C_x, C_y \in \mathcal{E}$ such that

$$x \in C_x \text{ and } y \in C_y. \tag{4.3.3}$$

Because $\mathcal{E}$ is a chain in $\mathcal{C}$, we have that $C_x \subseteq C_y$ or $C_y \subseteq C_x$. It follows that x and y are both contained in C_x, or that x and y are both contained in C_y. In either case, we get that $x \preceq y$ or $y \preceq x$, because C_x and C_y are chains in A. This shows that $C(\mathcal{E})$ is a chain in A.

Equivalently, this means that $C(\mathcal{E}) \in \mathcal{C}$. Of course, if $C \in \mathcal{E}$, then

$$C \subseteq C(\mathcal{E}), \tag{4.3.4}$$

by construction. Thus $C(\mathcal{E})$ is an upper bound for $\mathcal{E}$ in $\mathcal{C}$. This implies that $\mathcal{C}$ satisfies the hypothesis of Zorn's lemma. In this situation, the conclusion of Zorn's lemma is that $\mathcal{C}$ has a maximal element, as desired.

4.4 The axiom of choice

Let us show how the axiom of choice can be obtained from Zorn's lemma or Hausdorff's maximality principle. Let I be a nonempty set, and let X_j be a nonempty set for each $j \in I$. We would like to show that there is a mapping f from I into $\bigcup_{j \in I} X_j$ such that

$$f(j) \in X_j \tag{4.4.1}$$

for every $j \in I$.

4.4.1 A partially-ordered set A

Let A be the set of ordered pairs (I_0, f_0), where I_0 is a subset of I, and f_0 is a mapping from I_0 into $\bigcup_{j \in I_0} X_j$ such that $f_0(j) \in X_j$ for every $j \in I_0$. If $(I_1, f_1), (I_2, f_2) \in A$, then put

$$(I_1, f_1) \preceq (I_2, f_2) \tag{4.4.2}$$

when $I_1 \subseteq I_2$ and $f_1 = f_2$ on I_1. It is easy to see that this defines a partial ordering on A.

4.4.2 Upper bounds for arbitrary chains in A

Let C be a chain in A, and let us check that C has an upper bound in A. Put

$$I_{\mathcal{C}} = \bigcup_{(I_0, f_0) \in \mathcal{C}} I_0, \tag{4.4.3}$$

which is a subset of I. We would like to define a mapping $f_{\mathcal{C}}$ from $I_{\mathcal{C}}$ into $\bigcup_{j \in I_{\mathcal{C}}} X_j$ as follows. If $j \in I_{\mathcal{C}}$, then there is an element (I_1, f_1) of $\mathcal{C}$ such that $j \in I_1$, by definition of $I_{\mathcal{C}}$. In this case, we would like to put

$$f_{\mathcal{C}}(j) = f_1(j). \tag{4.4.4}$$

We need to check that this does not depend on the particular choice of (I_1, f_1). Suppose that (I_2, f_2) is another element of $\mathcal{C}$ such that $j \in I_2$. Note that

$$(I_1, f_1) \preceq (I_2, f_2) \text{ or } (I_2, f_2) \preceq (I_1, f_1), \tag{4.4.5}$$

because $\mathcal{C}$ is a chain in A. In both cases, we have that $f_1(j) = f_2(j)$, so that $f_{\mathcal{C}}(j)$ is well defined. Of course, $f_{\mathcal{C}}(j) = f_1(j) \in X_j$, so that $(I_{\mathcal{C}}, f_{\mathcal{C}}) \in A$. If (I_0, f_0) is an element of $\mathcal{C}$, then $I_0 \subseteq I_{\mathcal{C}}$ and $f_0 = f_{\mathcal{C}}$ on I_0, by construction. This means that

$$(I_0, f_0) \preceq (I_{\mathcal{C}}, f_{\mathcal{C}}), \tag{4.4.6}$$

so that $(I_{\mathcal{C}}, f_{\mathcal{C}})$ is an upper bound for $\mathcal{C}$ in A.

This shows that A satisfies the hypothesis of Zorn's lemma, so that Zorn's lemma implies that A has a maximal element. Alternatively, Hausdorff's maximality principle says that A has a maximal chain. The previous argument implies that such a maximal chain has an upper bound in A, which is a maximal element of A, as before.

If (I_0, f_0) is a maximal element of A, then we would like to show that

$$I_0 = I. \tag{4.4.7}$$

Otherwise, there is an element j_1 of I not in I_0, and we put $I_1 = I_0 \cup \{j_1\}$. Let x_{j_1} be an element of X_{j_1}. Consider the mapping f_1 from I_1 into $\bigcup_{j \in I_1} X_j$ defined by putting $f_1(j) = f_0(j)$ when $j \in I_0$ and $f_1(j_1) = x_{j_1}$. Thus $f_1(j) \in X_j$ for every $j \in I_1$, so that $(I_1, f_1) \in A$. We also have that $(I_0, f_0) \preceq (I_1, f_1)$ and $(I_0, f_0) \neq (I_1, f_1)$, by construction. This contradicts the maximality of (I_0, f_0) in A. It follows that $I_0 = I$, as desired.

4.5 Injective mappings

Let A and B be sets. We would like to show how Zorn's lemma or Hausdorff's maximality principle implies that there is either a one-to-one mapping from A into B, or a one-to-one mapping from B into A.

4.5.1 A partially-ordered set $\mathcal{A}$

Let $\mathcal{A}$ be the collection of ordered triples (A_0, B_0, f_0), where $A_0 \subseteq A$, $B_0 \subseteq B$, and f_0 is a one-to-one mapping from A_0 onto B_0. If $(A_1, B_1, f_1), (A_2, B_2, f_2) \in \mathcal{A}$, then put

$$(A_1, B_1, f_1) \preceq (A_2, B_2, f_2) \tag{4.5.1}$$

when $A_1 \subseteq A_2$, $B_1 \subseteq B_2$, and $f_1 = f_2$ on A_1. One can check that this defines a partial ordering on $\mathcal{A}$.

4.5.2 Upper bounds for arbitrary chains in $\mathcal{A}$

Let $\mathcal{C}$ be a chain in $\mathcal{A}$, and let us verify that $\mathcal{C}$ has an upper bound in $\mathcal{A}$. Put

$$A_{\mathcal{C}} = \bigcup_{(A_0, B_0, f_0) \in \mathcal{C}} A_0 \tag{4.5.2}$$

and

$$B_{\mathcal{C}} = \bigcup_{(A_0, B_0, f_0) \in \mathcal{C}} B_0, \tag{4.5.3}$$

which are subsets of A and B, respectively. We would like to define a mapping $f_{\mathcal{C}}$ from $A_{\mathcal{C}}$ into $B_{\mathcal{C}}$ as follows. If $x \in A_{\mathcal{C}}$, then there is an element (A_1, B_1, f_1) of $\mathcal{C}$ such that $x \in A_1$, and we would like to put

$$f_{\mathcal{C}}(x) = f_1(x). \tag{4.5.4}$$

One can check that this does not depend on the particular choice of (A_1, B_1, f_1), as in the previous section. Note that (4.5.4) is an element of $B_1 \subseteq B_{\mathcal{C}}$, so that $f_{\mathcal{C}}$ is a well-defined mapping from $A_{\mathcal{C}}$ into $B_{\mathcal{C}}$. We would like to verify that $f_{\mathcal{C}}$ is a one-to-one mapping from $A_{\mathcal{C}}$ onto $B_{\mathcal{C}}$, so that $(A_{\mathcal{C}}, B_{\mathcal{C}}, f_{\mathcal{C}}) \in \mathcal{A}$.

Let x_1 and x_2 be distinct elements of $A_{\mathcal{C}}$. It follows that there are elements (A_1, B_1, f_1) and (A_2, B_2, f_2) of $\mathcal{C}$ such that $x_1 \in A_1$ and $x_2 \in A_2$, by the definition (4.5.2) of $A_{\mathcal{C}}$. Because $\mathcal{C}$ is a chain in $\mathcal{A}$, we have that

$$\text{either } (A_1, B_1, f_1) \preceq (A_2, B_2, f_2) \text{ or } (A_2, B_2, f_2) \preceq (A_1, B_1, f_1). \tag{4.5.5}$$

In the first case, we get that $A_1 \subseteq A_2$, $x_1, x_2 \in A_2$, and hence $f_{\mathcal{C}}(x_1) = f_2(x_1)$, $f_{\mathcal{C}}(x_2) = f_2(x_2)$. Of course, f_2 is injective on A_2, by hypothesis, so that

$$f_2(x_1) \neq f_2(x_2) \tag{4.5.6}$$

This means that

$$f_{\mathcal{C}}(x_1) \neq f_{\mathcal{C}}(x_2) \tag{4.5.7}$$

in this case, and and the other case can be handled in the same way. This shows that $f_{\mathcal{C}}$ is injective on $A_{\mathcal{C}}$.

Now let $y \in \mathcal{B}_{\mathcal{C}}$ be given. By the definition (4.5.3) of $B_{\mathcal{C}}$, there is an element (A_0, B_0, f_0) of $\mathcal{C}$ such that $y \in B_0$. This implies that there is an $x \in A_0$ such that

(4.5.8) $$f_0(x) = y,$$

because f_0 maps A_0 onto B_0, by hypothesis. It follows that $x \in A_{\mathcal{C}}$, and that

(4.5.9) $$f_{\mathcal{C}}(x) = y,$$

by the definitions of $A_{\mathcal{C}}$ and $f_{\mathcal{C}}$. Thus $f_{\mathcal{C}}$ maps $A_{\mathcal{C}}$ onto $B_{\mathcal{C}}$.

This shows that $(A_{\mathcal{C}}, B_{\mathcal{C}}, f_{\mathcal{C}}) \in \mathcal{A}$. It is easy to see that $(A_{\mathcal{C}}, B_{\mathcal{C}}, f_{\mathcal{C}})$ is an upper bound for $\mathcal{C}$ in $\mathcal{A}$, by construction. This means that $\mathcal{A}$ satisfies the hypothesis of Zorn's lemma, so that Zorn's lemma implies that $\mathcal{A}$ has a maximal element. Alternatively, Hausdorff's maximality principle says that $\mathcal{A}$ has a maximal chain, and the upper bound in $\mathcal{A}$ for such a maximal chain obtained as before is a maximal element of $\mathcal{A}$.

If (A_0, B_0, f_0) is a maximal element of $\mathcal{A}$, then we would like to show that

(4.5.10) $$A_0 = A \text{ or } B_0 = B.$$

Otherwise, there is an element a_1 of A not in A_0, and an element b_1 of B not in B_0. Put $A_1 = A_0 \cup \{a_1\}$ and $B_1 = B_0 \cup \{b_1\}$, so that $A_1 \subseteq A$ and $B_1 \subseteq B$. Let f_1 be the mapping from A_1 into B_1 defined by putting $f_1(x) = f_0(x)$ when $x \in A_0$ and $f_1(a_1) = b_1$. Thus f_1 is a one-to-one mapping from A_1 onto B_1, so that $(A_1, B_1, f_1) \in \mathcal{A}$. By construction, $(A_0, B_0, f_0) \preceq (A_1, B_1, f_1)$ and $(A_0, B_0, f_0) \neq (A_1, B_1, f_1)$, contradicting the maximality of (A_0, B_0, f_0) in $\mathcal{A}$. This shows that $A_0 = A$ or $B_0 = B$.

If $A_0 = A$, then f_0 is a one-to-one mapping from A into B. If $B_0 = B$, then the inverse of f_0 is a one-to-one mapping from B into A.

4.6 Well-ordered sets

Let $(A, \preceq)$ be a linearly-ordered set. Thus, for each $x, y \in A$, we have that $x \preceq y$ or $y \preceq x$. We say that A is *well ordered* by $\preceq$ if

(4.6.1) every nonempty subset of A has a smallest element.

More precisely, this means that if E is a nonempty subset of A, then

(4.6.2) there is an $x \in E$ such that $x \preceq y$ for every $y \in E$.

Note that the smallest element in E is automatically unique.

It is easy to see that a linearly-ordered set with only finitely many elements is well ordered. The set $\mathbf{Z}_+$ of positive integers is well ordered by its standard ordering.

4.6.1 The well-ordering principle

Zermelo's *well-ordering principle* states that every set can be well ordered. This can be obtained from the axiom of choice, as discussed in [236], for instance.

It is easy to see how the axiom of choice can be obtained from the well-ordering principle. Let I be a nonempty set, and suppose that X_j is a nonempty set for each $j \in I$. The well-ordering principle implies that $\bigcup_{j \in I} X_j$ can be well ordered. If $j \in I$, then let $f(j)$ be the smallest element of X_j with respect to this ordering. This defines a mapping f from I into $\bigcup_{j \in I} X_j$ such that $f(j) \in X_j$ for each $j \in I$.

4.6.2 Ideals and segments in partially-ordered sets

Let $(A, \preceq)$ be a partially-ordered set, and let B be a subset of A. It is easy to see that the restriction of $\preceq$ to B is a partial ordering on B. If A is linearly ordered by $\preceq$, then B is linearly ordered by the restriction of $\preceq$ to B. If A is well ordered by $\preceq$, then B is well ordered by $\preceq$ too.

Let $(A, \preceq)$ be a partially-ordered set again. A subset B of A is said to be an *ideal* in A if

(4.6.3) for every $x \in A$ and $y \in B$ with $x \preceq y$ we have that $x \in B$.

If $a \in A$, then the *segment* in A associated to a is defined by

$$S(a) = S_A(a) = \{x \in A : x \preceq a,\ x \neq a\}. \tag{4.6.4}$$

It is easy to see that segments in A are ideals in A. Of course, A is an ideal in itself.

Suppose that $(A, \preceq)$ is a linearly-ordered set. If $a \in A$, then it is easy to see that

$$A \setminus S(a) = \{x \in A : a \preceq x\}. \tag{4.6.5}$$

4.6.3 Ideals in well-ordered sets

Suppose now that $(A, \preceq)$ is a well-ordered set, and that $B \subseteq A$ is an ideal in A. If $B \neq A$, then there is a smallest element a_0 of $A \setminus B$. One can check that

$$B = S(a_0) \tag{4.6.6}$$

under these conditions. More precisely,

$$S(a_0) \subseteq B \tag{4.6.7}$$

automatically, because a_0 is the smallest element of $A \setminus B$. To show that

$$B \subseteq S(a_0), \tag{4.6.8}$$

suppose for the sake of a contradiction that $x \in B$ and $x \notin S(a_0)$. This implies that $a_0 \preceq x$, because A is linearly ordered by $\preceq$, as in (4.6.5). It follows that $a_0 \in B$, because B is an ideal in A. This contradicts the fact that $a_0 \in A \setminus B$, as desired.

4.6.4 Checking that a linear ordering is a well ordering

Let $(A, \preceq)$ be a linearly-ordered set. If

$$\text{every segment in } A \text{ is well ordered by } \preceq, \tag{4.6.9}$$

then A is well ordered by $\preceq$. To see this, let E be a nonempty subset of A, and let us show that E has a smallest element. Let a be an element of E. If a is already the smallest element of E, then we can stop. Otherwise,

$$E \cap S(a) \neq \emptyset. \tag{4.6.10}$$

In this case, $E \cap S(a)$ has a smallest element, because $S(a)$ is well ordered, by hypothesis. It is easy to see that the smallest element of $E \cap S(a)$ is also the smallest element of E, using (4.6.5), as desired.

4.7 The well-ordering principle

The well-ordering principle can be obtained from Zorn's lemma or Hausdorff's maximality principle as follows. Let A be a set, on which we would like to find a well ordering.

4.7.1 The collection of well-ordered subsets of A

Let $\mathcal{A}$ be the collection of ordered pairs $(A_0, \preceq_0)$, where A_0 is a subset of A well ordered by $\preceq_0$. If $(A_1, \preceq_1), (A_2, \preceq_2) \in \mathcal{A}$, then put

$$(A_1, \preceq_1) \preceq_{\mathcal{A}} (A_2, \preceq_2) \tag{4.7.1}$$

when $A_1 \subseteq A_2$, the restriction of $\preceq_2$ to A_1 is the same as $\preceq_1$, and A_1 is an ideal in A_2 with respect to $\preceq_2$. One can check that this defines a partial ordering on $\mathcal{A}$.

4.7.2 Upper bounded for chains of well-ordered subsets

Let $\mathcal{C}$ be a chain in $\mathcal{A}$. We would like to show that $\mathcal{C}$ has an upper bound in $\mathcal{A}$, as usual. Put

$$A_{\mathcal{C}} = \bigcup_{(A_0, \preceq_0) \in \mathcal{C}} A_0, \tag{4.7.2}$$

which is a subset of A. We first need to define an ordering $\preceq_{\mathcal{C}}$ on $A_{\mathcal{C}}$.

Let $x, y \in A_{\mathcal{C}}$ be given, so that there are $(A_1, \preceq_1), (A_2, \preceq_2) \in \mathcal{C}$ such that $x \in A_1$ and $y \in A_2$. Because $\mathcal{C}$ is a chain in $\mathcal{A}$, we have that

$$(A_1, \preceq_1) \preceq_{\mathcal{A}} (A_2, \preceq_2) \text{ or } (A_2, \preceq_2) \preceq_{\mathcal{A}} (A_1, \preceq_1). \tag{4.7.3}$$

In particular, this means that $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. It follows that $x, y \in A_1$ or $x, y \in A_2$.

Let $(A_0, \preceq_0)$ be any element of $\mathcal{C}$ such that $x, y \in A_0$, the existence of which follows from the remarks in the preceding paragraph. We would like to put

$$x \preceq_{\mathcal{C}} y \tag{4.7.4}$$

when $x \preceq_0 y$. One can check that this does not depend on the particular element $(A_0, \preceq_0)$ of $\mathcal{C}$ with $x, y \in A_0$, because $\mathcal{C}$ is a chain in $\mathcal{A}$.

It is not difficult to show that $A_{\mathcal{C}}$ is linearly ordered by $\preceq_{\mathcal{C}}$, using the fact that the elements of $\mathcal{C}$ are linearly-ordered sets. More precisely, to check that $\preceq_{\mathcal{C}}$ is transitive on $A_{\mathcal{C}}$, let $x, y, z \in A_{\mathcal{C}}$ be given. One can verify that there is an element $(A_0, \prec_0)$ of $\mathcal{C}$ such that $x, y, z \in A_0$, using the fact that $\mathcal{C}$ is a chain in $\mathcal{A}$, as before. If $x \preceq_{\mathcal{C}} y$ and $y \preceq_{\mathcal{C}} z$, then one can get that $x \preceq_{\mathcal{C}} z$ using transitivity of $\preceq_0$ on A_0.

4.7.3 Elements of $\mathcal{C}$ correspond to ideals in $A_{\mathcal{C}}$

Let $(A_1, \preceq_1)$ be any element of $\mathcal{C}$. By construction, $A_1 \subseteq A_{\mathcal{C}}$, and $\preceq_{\mathcal{C}}$ agrees with $\preceq_1$ on A_1. Let us check that A_1 is an ideal in $A_{\mathcal{C}}$ with respect to $\preceq_{\mathcal{C}}$. Let $x \in A_{\mathcal{C}}$ and $y \in A_1$ be given, with $x \preceq_{\mathcal{C}} y$. By definition of $A_{\mathcal{C}}$, there is an element $(A_2, \preceq_2)$ of $\mathcal{C}$ such that $x \in A_2$. We also have that (4.7.3) holds, because $\mathcal{C}$ is a chain in $\mathcal{A}$. In the first case, $x, y \in A_2$ and $x \preceq_2 y$, which implies that $x \in A_1$, because $y \in A_1$ and A_1 is an ideal in A_2. In the second case, $x \in A_2 \subseteq A_1$. Thus $x \in A_1$ in both cases, as desired.

4.7.4 Checking that $A_{\mathcal{C}}$ is well ordered by $\preceq_{\mathcal{C}}$

Let us check that $A_{\mathcal{C}}$ is well ordered by $\preceq_{\mathcal{C}}$. Let $a \in A_{\mathcal{C}}$ be given, and let us verify that the corresponding segment $S_{A_{\mathcal{C}}}(a)$ in $A_{\mathcal{C}}$ is well ordered by $\preceq_{\mathcal{C}}$. By construction, there is an $(A_0, \preceq_0) \in \mathcal{C}$ such that $a \in A_0$. We also have that

$$S_{A_{\mathcal{C}}}(a) \subseteq A_0, \tag{4.7.5}$$

because A_0 is an ideal in $A_{\mathcal{C}}$, as in the preceding paragraph. It follows that $S_{A_{\mathcal{C}}}(a)$ is well ordered by $\preceq_{\mathcal{C}}$, because $\preceq_0$ and $\preceq_{\mathcal{C}}$ are the same on A_0, and A_0 is well ordered by $\preceq_0$.

This shows that $(A_{\mathcal{C}}, \preceq_{\mathcal{C}})$ is an element of $\mathcal{A}$. If $(A_0, \preceq_0) \in \mathcal{C}$, then it is easy to see that $(A_0, \preceq_0) \preceq_{\mathcal{A}} (A_{\mathcal{C}}, \preceq_{\mathcal{C}})$, by the previous remarks. This means that $(A_{\mathcal{C}}, \preceq_{\mathcal{C}})$ is an upper bound for $\mathcal{C}$ in $\mathcal{A}$. It follows that $\mathcal{A}$ has a maximal element, by Zorn's lemma. Alternatively, Hausdorff's maximality principle implies that $\mathcal{A}$ has a maximal chain, and an upper bound for such a chain is a maximal element of $\mathcal{A}$, as usual.

Let $(A_1, \preceq_1)$ be a maximal element of $\mathcal{A}$. We would like to show that $A_1 = A$. Otherwise, there is an $a_2 \in A$ such that $a_2 \notin A_1$. Put $A_2 = A_1 \cup \{a_2\}$, and let us define $\preceq_2$ on A_2 as follows. We take $\preceq_2$ to be the same as $\preceq_1$ on A_1, and we put $x \preceq_2 a_2$ for every $x \in A_2$. It is easy to see that A_2 is well ordered by $\preceq_2$, because A_1 is well ordered by $\preceq_1$. Clearly A_1 is an ideal in A_2, by construction. Thus $(A_2, \preceq_2) \in \mathcal{A}$ and $(A_1, \preceq_1) \preceq_{\mathcal{A}} (A_2, \preceq_2)$. This contradicts the maximality of $(A_1, \preceq_1)$ in $\mathcal{A}$, because $A_1 \neq A_2$.

4.8 Order isomorphisms

Let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be partially-ordered sets. A one-to-one mapping f from A_1 onto A_2 is said to be an *order isomorphism* if for every $x, y \in A_1$,

$$x \preceq_1 y \quad \text{if and only if} \quad f(x) \preceq_2 f(y). \tag{4.8.1}$$

In this case, the inverse mapping f^{-1} is an order isomorphism from A_2 onto A_1. Let $(A_3, \preceq_3)$ be another partially-ordered set, and suppose that g is an order isomorphism from A_2 onto A_3. Under these conditions, $g \circ f$ is an order isomorphism from A_1 onto A_3.

An order isomorphism from a partially-ordered set onto itself may be called an *order automorphism*. Of course, the identity mapping on any partially-ordered set is an order automorphism.

Now let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be well-ordered sets, and let f be an order isomorphism from A_1 onto A_2. If E_1 is a nonempty subset of A_1, then f maps the smallest element of E_1 to the smallest element of $f(E_1)$. In particular, if $A_1 \neq \emptyset$, then f maps the smallest element of A_1 to the smallest element of A_2.

4.8.1 Automorphisms of well-ordered sets

Let $(A, \preceq)$ be a well-ordered set, and let f be an order automorphism on A. We would like to check that

$$f \text{ is the identity mapping on } A. \tag{4.8.2}$$

Otherwise, let a be the smallest element of A such that $f(a) \neq a$. Thus f is the identity mapping on the segment $S(a)$, and in particular f maps $S(a)$ onto itself. This implies that f maps the complement of $S(a)$ in A onto itself. However, a is the smallest element of the complement of $S(a)$ in A. It follows that $f(a) = a$, which is a contradiction.

4.8.2 Uniqueness of isomorphisms between well-ordered sets

Let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be well-ordered sets, and suppose that f and g are order isomorphisms from A_1 onto A_2. This implies that $g^{-1} \circ f$ is an order automorphism on A_1. It follows that $g^{-1} \circ f$ is the identity mapping on A_1, as in the preceding paragraph. Of course, this means that

$$g = f, \tag{4.8.3}$$

so that an order isomorphism between well-ordered sets is unique when it exists.

4.8.3 Isomorphisms onto ideals in well-ordered sets

Let $(A, \preceq)$ be a well-ordered set again, and let B be an ideal in A. As before, B is well ordered by the restriction of $\preceq$ to B. Suppose that f is an order isomorphism from A onto B. Under these conditions, f is the identity mapping on A, so that

$$B = f(A) = A. \tag{4.8.4}$$

This is an extension of the earlier statement for order automorphisms on A, which can be shown in essentially the same way. More precisely, if f is not the identity mapping on A, then there is a smallest element a of A such that $f(a) \neq a$. This means that f is equal to the identity mapping on $S(a)$, so that $f(S(a)) = S(a)$. It follows that $f(a) \notin S(a)$, because $a \notin S(a)$ and f is injective, which is to say that $a \preceq f(a)$. This implies that $a \in B$, because $f(a) \in f(A) = B$ and B is an ideal in A. Note that

$$f(A \setminus S(a)) = f(A) \setminus f(S(a)) = f(A) \setminus S(a). \tag{4.8.5}$$

Of course, a is the smallest element of $A \setminus S(a)$, and a is the smallest element of (4.8.5) too. Thus $f(a) = a$, which is a contradiction.

Alternatively, one can show more directly that a well-ordered set cannot be order-isomorphic to any of its segments, as in Theorem 20 on p51 of [236].

4.8.4 Uniqueness of isomorphic segments in well-ordered sets

Let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be partially-ordered sets again, and let f be an order isomorphism from A_1 onto A_2. If B_1 is an ideal in A_1, then $f(B_1)$ is an ideal in A_2. If $a_1 \in A_1$, then f maps the corresponding segment $S_{A_1}(a_1)$ in A_1 onto the segment $S_{A_2}(f(a_1))$ associated to $f(a_1)$ in A_2.

Suppose that A_1 and A_2 are well ordered by $\preceq_1$ and $\preceq_2$, respectively. If $a_1 \in A_1$, then there is at most one $a_2 \in A_2$ such that $S_{A_1}(a_1)$ is order isomorphic to $S_{A_2}(a_2)$, with respect to the restrictions of $\preceq_1$ and $\preceq_2$ to $S_{A_1}(a_1)$ and $S_{A_2}(a_2)$, respectively. Otherwise, there are elements a_2' and a_2'' of A_2 such that $a_2' \neq a_2''$ and $S_{A_2}(a_2')$ is order isomorphic to $S_{A_2}(a_2'')$. We may as well suppose that $a_2' \preceq_2 a_2''$, because A_2 is linearly ordered by $\preceq_2$, and otherwise we could interchange the roles of a_2' and a_2''. This means that $a_2' \in S_{A_2}(a_2'')$, because $a_2' \neq a_2''$. Thus $S_{A_2}(a_2')$ may be considered as a segment in $S_{A_2}(a_2'')$. This contradicts the fact that a well-ordered set cannot be order isomorphic to any of its segments, as before.

This can also be used to prove the uniqueness of order isomorphisms between well-ordered sets, by considering the images of segments under the order isomorphisms.

4.9 Order isomorphisms, continued

Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be well-ordered sets. Under these conditions, Theorem 21 on p51 of [236] states that

$$\begin{aligned} &A \text{ is order-isomorphic to an ideal in } B, \\ &\text{or } B \text{ is order-isomorphic to an ideal in } A. \end{aligned} \tag{4.9.1}$$

More precisely, this means that

$$\text{(4.9.2)} \qquad \begin{aligned} &\text{either } A \text{ is order-isomorphic to } B,\\ &A \text{ is order-isomorphic to a segment in } B,\\ &\text{or } B \text{ is order-isomorphic to a segment in } A. \end{aligned}$$

It is easy to see that only one of these three possibilities can occur, because a well-ordered set cannot be order-isomorphic to any of its segments, as in the previous section.

4.9.1 A subset I_A of A

Let I_A be the set of $a \in A$ for which there is a $b \in B$ such that the segment $S_A(a)$ corresponding to a in A is order-isomorphic to the segment $S_B(b)$ corresponding to b in B. One can check that b is uniquely determined by this property, because a well-ordered set cannot be order-isomorphic to any of its segments. If $a \in I_A$, then let $f(a)$ be the element of B such that $S_A(a)$ is order-isomorphic to $S_B(f(a))$. This defines a mapping from I_A into B.

4.9.2 More on I_A

Let $a \in I_A$ be given, so that there is an order isomorphism ϕ_a from $S_A(a)$ onto $S_B(f(a))$. More precisely, this uses the restrictions of $\preceq_A$ and $\preceq_B$ to $S_A(a)$ and $S_B(f(a))$, respectively.

If $a_0 \in S_A(a)$, then $S_A(a_0)$ is the same as the segment corresponding to a_0 in $S_A(a)$. Similarly, $\phi_a(a_0) \in S_B(f(a))$, and $S_B(\phi_a(a_0))$ is the same as the segment corresponding to $\phi_a(a_0)$ in $S_B(f(a))$. It follows that

$$\phi_a(S_A(a_0)) = S_B(\phi_a(a_0)), \tag{4.9.3}$$

as in the previous section.

Note that the restriction of ϕ_a to $S_A(a_0)$ is an order isomorphism onto $S_B(\phi_a(a_0))$. This means that $a_0 \in I_A$, with

$$f(a_0) = \phi_a(a_0). \tag{4.9.4}$$

Thus

$$S_A(a) \subseteq I_A, \tag{4.9.5}$$

and

$$f(S_A(a)) = \phi_a(S_A(a)) = S_B(f(a)). \tag{4.9.6}$$

In particular, I_A is an ideal in A.

4.9.3 A subset I_B of B

Similarly, let I_B be the set of $b \in B$ such that $S_B(b)$ is order-isomorphic to $S_A(a)$ for some $a \in A$. This means that $a \in I_A$ and $f(a) = b$, and in fact

$$f(I_A) = I_B. \tag{4.9.7}$$

It is easy to see that I_B is an ideal in B, using (4.9.6). One can check that f is an order isomorphism from I_A onto I_B, with respect to the restrictions of $\preceq_A$ and $\preceq_B$ to I_A and I_B, respectively.

4.9.4 Either $I_A = A$ or $I_B = B$

Suppose for the sake of a contradiction that $I_A \neq A$ and $I_B \neq B$. This implies that there are $x \in A$ and $y \in B$ such that $I_A = S_A(x)$ and $I_B = S_B(y)$, as in Subsection 4.6.3. It follows that $x \in I_A$ and $y \in I_B$, because f is an order isomorphism from I_A onto I_B. This is a contradiction, and so we get that $I_A = A$ or $I_B = B$, as desired.

Chapter 5

Some additional notions in topology

5.1 Continuous real-valued functions

Let X be a nonempty topological space, and let $C(X)$ be the space of continuous real-valued functions on X. This uses the standard topology on the real line, as the range of the functions on X. Of course, constant functions on X are continuous.

5.1.1 Sums and products

If $f, g \in C(X)$, then

$$f + g, f\,g \in C(X) \tag{5.1.1}$$

too. This can be shown using the same type of arguments as for continuous real-valued functions on the real line, or on a metric space.

Alternatively, it is easy to see that

$$x \mapsto (f(x), g(x)) \tag{5.1.2}$$

is continuous as a mapping from X into $\mathbf{R}^2$, using the product topology on $\mathbf{R}^2$ corresponding to the standard topology on $\mathbf{R}$. One can also show that

(5.1.3) addition and multiplication on $\mathbf{R}$ are continuous as mappings from $\mathbf{R}^2$ into $\mathbf{R}$,

using standard arguments. To get the continuity of $f + g$ and $f\,g$, one can consider these functions as the compositions of (5.1.2) with the mappings from $\mathbf{R}^2$ into $\mathbf{R}$ that correspond to addition and multiplication of real numbers, respectively.

5.1.2 Reciprocals of continuous functions

Similarly, if $f \in C(X)$ and $f(x) \neq 0$ for each $x \in X$, then

$$1/f \in C(X). \tag{5.1.4}$$

This can be obtained using the same type of arguments as for functions on the real line or a metric space, or by considering $1/f$ as the composition of f with the mapping $t \mapsto 1/t$ from $\mathbf{R} \setminus \{0\}$ into itself.

5.1.3 Urysohn spaces

Let us say that $C(X)$ *separates points* in X if for every $x, y \in X$ with $x \neq y$ there is an $f \in C(X)$ such that

$$f(x) \neq f(y). \tag{5.1.5}$$

In this case, X is said to be a *Urysohn space*. One can check that

(5.1.6) Urysohn spaces are completely Hausdorff.

This uses the fact that the real line is completely Hausdorff with respect to the standard topology.

Let X be a set, and suppose that τ and $\widetilde{\tau}$ are topologies on X, with $\tau \subseteq \widetilde{\tau}$. If a real-valued function f on X is continuous with respect to τ, then

(5.1.7) f is continuous with respect to $\widetilde{\tau}$

too. If (X, τ) is a Urysohn space, then it follows that

(5.1.8) $(X, \widetilde{\tau})$ is a Urysohn space

as well.

5.1.4 Continuous functions from semimetrics

Now let X be a set with a semimetric $d(x, y)$. If $p \in X$, then one can verify that

$$f_p(x) = d(p, x) \tag{5.1.9}$$

is continuous on X, with respect to the topology determined by $d(\cdot, \cdot)$. If $d(\cdot, \cdot)$ is a metric on X, then it follows that X is a Urysohn space with respect to this topology.

5.1.5 Maxima and minima of continuous real-valued functions

If f and g are continuous real-valued functions on a topological space X again, then it is not difficult to show that

$$\max(f(x), g(x)) \quad \text{and} \quad \min(f(x), g(x)) \tag{5.1.10}$$

are continuous on X as well, directly from the definitions.

Alternatively, one can check that

(5.1.11) the maximum and minimum of two real numbers define continuous mappings from $\mathbf{R}^2$ into $\mathbf{R}$.

The continuity of the maximum and minimum of f and g can be obtained from this and the continuity of (5.1.2) as a mapping from X into $\mathbf{R}^2$, as before.

In particular, if a and b are real numbers, then

$$\max(f(x), a) \text{ and } \min(f(x), b) \tag{5.1.12}$$

are continuous on X. This can also be seen using the continuity of

$$\max(t, a) \text{ and } \min(t, b) \tag{5.1.13}$$

as functions of $t \in \mathbf{R}$, by composing f with these functions.

5.2 Urysohn functions

Let X be a topological space, and let A and B be disjoint subsets of X. A continuous real-valued function f on X is said to be a *Urysohn function* for A and B if

$$f(x) = 0 \quad \text{for every } x \in A, \tag{5.2.1}$$

$$f(y) = 1 \quad \text{for every } y \in B, \tag{5.2.2}$$

and

$$0 \le f(w) \le 1 \quad \text{for every } w \in X. \tag{5.2.3}$$

The third condition can always be arranged by replacing f with

$$\min(\max(f, 0), 1), \tag{5.2.4}$$

as in the previous section.

If f is a Urysohn function on X for A and B, then

(5.2.5) $1 - f$ is a Urysohn function on X for B and A.

It is easy to see that X is a Urysohn space if and only if every pair A, B of disjoint subsets of X with only one element each has a Urysohn function.

5.2.1 Urysohn's lemma

If X is normal in the strict sense, and A, B are disjoint closed subsets of X, then *Urysohn's lemma* states that

(5.2.6) there is a Urysohn function on X for A and B.

One can check that this property implies that X is normal in the strict sense. If X is normal in the strong sense, then it follows that X is a Urysohn space.

5.2.2 Urysohn's metrization theorem

Urysohn's metrization theorem states that if X is second countable and normal in the strong sense, then there is a metric on X that determines the same topology. Tychonoff extended this to second countable spaces that are regular in the strong sense.

5.2.3 Some properties of Urysohn spaces

Suppose that X is a Urysohn space, and let x, y be distinct elements of X. Thus there is a Urysohn function f on X for $\{x\}$ and $\{y\}$, as before. Under these conditions,

$$2\ \min(f, 1/2) \tag{5.2.7}$$

is a Urysohn function on X for $\{x\}$ and

$$\{z \in X : f(z) \geq 1/2\}. \tag{5.2.8}$$

Note that

$$\{z \in X : f(z) > 1/2\} \tag{5.2.9}$$

is an open subset of X that contains y and is contained in (5.2.8).

Let B be a compact subset of X, and suppose that $x \in X \setminus B$. One can get

$$\text{a Urysohn function on } X \text{ for } \{x\} \text{ and } B, \tag{5.2.10}$$

as follows. If $y \in B$, then there is a Urysohn function on X for $\{x\}$ and a neighborhood of y in X, as in the preceding paragraph. Because B is compact, B can be covered by finitely many such neighborhoods of its elements. The maximum of the corresponding Urysohn functions for $\{x\}$ and these finitely many neighborhoods of elements of B is a Urysohn function for $\{x\}$ and B.

If A and B are disjoint compact subsets of X, then one can get

$$\text{a Urysohn function on } X \text{ for } A \text{ and } B \tag{5.2.11}$$

using analogous arguments. More precisely, for each $x \in A$, one can get a Urysohn function for $\{x\}$ and B, as before. One can use this to get a Urysohn function on X for a neighborhood of x and B. Because A is compact, A can be covered by finitely many such neighborhoods of its elements. The minimum of the corresponding Urysohn functions is a Urysohn function on X for A and B.

5.3 Complete regularity

Let us say that a topological space X is *completely regular in the strict sense* if for every $x \in X$ and closed set $E \subseteq X$ with $x \notin E$ there is an $f \in C(X)$ such that $f(x) \neq 0$ and

$$f(y) = 0 \quad \text{for every } y \in E. \tag{5.3.1}$$

In this case, it is easy to modify f a bit, if necessary, to get that

$$f \text{ is a Urysohn function for } E \text{ and } \{x\}. \tag{5.3.2}$$

If X also satisfies the zeroth separation condition, then we say that X is *completely regular in the strong sense*. In this case, X is said to be a *Tychonoff space* as well.

As usual, completely regular spaces in the strict sense are sometimes said to be *completely regular*, and completely regular spaces in the strong sense may be said to satisfy *separation condition number three and a half*, or equivalently be $T_{3\frac{1}{2}}$ *spaces*, but these terms may be used the other way. Alternatively, complete regularity, separation condition number three and a half, and $T_{3\frac{1}{2}}$ spaces may be used for complete regularity in the strong sense, and one may refer to complete regularity in the strict sense in other ways.

5.3.1 Relation with other separation conditions

If X is completely regular in the strict sense, then it is easy to see that

$$X \text{ is regular in the strict sense,} \tag{5.3.3}$$

because the real line is Hausdorff with respect to the standard topology.

If X is completely regular in the strong sense, then it follows that

$$X \text{ is regular in the strong sense.} \tag{5.3.4}$$

We also have that

$$X \text{ is a Urysohn space} \tag{5.3.5}$$

in this situation. If X is normal in the strong sense, then

$$X \text{ is completely regular in the strong sense,} \tag{5.3.6}$$

by Urysohn's lemma.

If X is regular in the strict sense, and X is normal in the strict sense, then

$$X \text{ is completely regular in the strict sense.} \tag{5.3.7}$$

To see this, let $x \in X$ and a closed set $E \subseteq X$ be given, with $x \notin E$. One can use regularity in the strict sense to get an open set $U \subseteq X$ such that $x \in U$ and $\overline{U} \subseteq X \setminus E$. In particular, E and $\overline{U}$ are disjoint closed subsets of X, and $x \in \overline{U}$. Thus one can use Urysohn's lemma to get a suitable element of $C(X)$.

5.3.2 Relation with induced topologies

Let X be a topological space again, and let Y be a subset of X, equipped with the induced topology. If f is a continuous real-valued function on X, then

$$\text{the restriction of } f \text{ to } Y \text{ is continuous.} \tag{5.3.8}$$

If X is a Urysohn space, then it is easy to see that

$$Y \text{ is a Urysohn space.} \tag{5.3.9}$$

Similarly, if X is completely regular in the strict sense, then one can check that

$$Y \text{ is completely regular in the strict sense} \tag{5.3.10}$$

too, with respect to the induced topology. In particular, if X is normal in the strong sense, then X is completely regular in the strong sense, by Urysohn's lemma, and hence

$$Y \text{ is completely regular in the strong sense.} \tag{5.3.11}$$

5.3.3 More on completely regular spaces

Suppose that X is completely regular in the strict sense, and let $x \in X$ and a closed set $E \subseteq X$ be given, with $x \notin E$. Thus there is a Urysohn function on X for E and $\{x\}$, as before. One can use this to get

$$\text{a Urysohn function on } X \text{ for } E \text{ and a neighborhood of } x \text{ in } X, \tag{5.3.12}$$

as in the previous section.

Suppose now that $E \subseteq X$ is a closed set, $K \subseteq X$ is compact, and $E \cap K = \emptyset$. If $x \in K$, then there is a Urysohn function on X for E and a neighborhood of x in X, as in the preceding paragraph. Because K is compact, K can be covered by finitely many such neighborhoods of its elements. The maximum of the corresponding Urysohn functions for E and these finitely many neighborhoods of elements of K

$$\text{is a Urysohn function for } E \text{ and } K. \tag{5.3.13}$$

5.4 Local compactness and manifolds

A topological space X is said to be *locally compact* if for every $x \in X$ there is an open set $U \subseteq X$ and a compact set $K \subseteq X$ such that

$$x \in U \text{ and } U \subseteq K. \tag{5.4.1}$$

If X is Hausdorff, then K is a closed set in X, and it follows that the closure $\overline{U}$ of U in X is contained in K. This implies that

$$\overline{U} \text{ is compact in } X, \tag{5.4.2}$$

because $\overline{U}$ is a closed set contained in a compact set. Sometimes local compactness is defined by asking that $\overline{U}$ be compact.

Note that $\mathbf{R}^n$ is locally compact with respect to the standard topology for each positive integer n.

5.4.1 Locally compact Hausdorff spaces

Suppose that X is a locally compact Hausdorff space. It is not too difficult to show that

(5.4.3) $$X \text{ is regular as a topological space.}$$

More precisely, one can show that

(5.4.4) $$X \text{ is completely regular.}$$

This can be reduced to Urysohn's lemma, or obtained using similar arguments.

5.4.2 Locally Euclidean of dimension n

Let X be a topological space, and let n be a positive integer. We say that X is *locally Euclidean of dimension* n if for every $x \in X$ there is an open set $U \subseteq X$ such that $x \in U$ and

(5.4.5) $$U \text{ is homeomorphic to an open subset } W \text{ of } \mathbf{R}^n.$$

More precisely, this uses the standard topology on $\mathbf{R}^n$, and the appropriate induced topologies on U and W.

Suppose that X is locally Euclidean of dimension n, and let us check that

(5.4.6) $$X \text{ satisfies the first separation condition.}$$

Let x and y be distinct elements of X, so that we would like to find an open subset of X that contains x and not y. By hypothesis, there is an open set $U \subseteq X$ such that $x \in U$ and U is homeomorphic to an open subset of $\mathbf{R}^n$. If $y \notin U$, then we can take U to be the open set that we want. Otherwise, if $y \in U$, we can use the fact that $\mathbf{R}^n$ satisfies the first separation condition to find an open set that contains x and not y.

Note that

(5.4.7) $$X \text{ satisfies the first countability condition,}$$

because of the analogous property of $\mathbf{R}^n$.

Let $x \in X$ be given again, and let $U \subseteq X$ be an open set such that $x \in U$ and U is homeomorphic to an open subset W of $\mathbf{R}^n$. If $K \subseteq \mathbf{R}^n$ is compact and $K \subseteq W$, then

(5.4.8) $$K \text{ is compact as a subset of W,}$$

with respect to the induced topology. This means that

(5.4.9) $$K \text{ corresponds to a compact subset of } U,$$

with respect to the induced topology. It follows that

(5.4.10) $$\text{the subset of } U \text{ corresponding to } K \text{ is compact as a subset of } X$$

as well. In particular, one can use this to check that X is locally compact.

However, X may not be Hausdorff, and this is often included as an additional condition. In this case, one can use the regularity of $\mathbf{R}^n$ to get that X is regular, a bit more directly than for arbitrary locally compact Hausdorff spaces. Similarly, complete regularity of X can be obtained more directly from the complete regularity of $\mathbf{R}^n$.

5.4.3 Topological manifolds

An *n-dimensional topological manifold* is often defined as

(5.4.11) a Hausdorff topological space X that is locally Euclidean of dimension n and satisfies the second countability condition.

There is also a notion of an *n-dimensional topological manifold with boundary*, although we shall not get into the details of this here. A basic example is a closed ball of positive radius in $\mathbf{R}^n$, with respect to the standard Euclidean metric, equipped with the induced topology. Another example is the set of $x \in \mathbf{R}^n$ with first coordinate $x_1 \geq 0$, equipped with the topology induced by the standard topology on $\mathbf{R}^n$.

5.4.4 Baire's theorem

Let X be any topological space again. If $U_1, \ldots, U_n$ are finitely many dense open subsets of X, then one can check that their intersection $\bigcap_{j=1}^n U_j$ is a dense open set in X too. If $U_1, U_2, U_3, \ldots$ is an infinite sequence of dense open sets in X, then a famous theorem of Baire gives conditions under which

$$\bigcap_{j=1}^{\infty} U_j \text{ is dense in } X. \tag{5.4.12}$$

One condition for this to hold is that there be a metric $d(\cdot, \cdot)$ on X such that X is equipped with the topology determined by $d(\cdot, \cdot)$, and that

$$X \text{ is complete with respect to } d(\cdot, \cdot), \tag{5.4.13}$$

which is to say that every Cauchy sequence in X with respect to $d(\cdot, \cdot)$ converges to an element of X. Another condition for this to hold is that X be a locally compact Hausdorff space.

5.5 σ-Compactness

Let X be a topological space. It is easy to see that

(5.5.1) the union of finitely many compact subsets of X is compact as well.

Similarly,

(5.5.2) the union of finitely or countably many subsets of X with the Lindelöf property has the Lindelöf property too.

A subset E of X is said to be *σ-compact* if there is a sequence $K_1, K_2, K_3, \ldots$ of compact subsets of X such that

$$E = \bigcup_{j=1}^{\infty} K_j. \tag{5.5.3}$$

This implies that E has the Lindelöf property, as before.

5.5.1 Some criteria for σ-compactness

Suppose that X is locally compact, so that X can be covered by open sets that are contained in compact sets. If X also has the Lindelöf property, then it follows that X can be covered by finitely or countably many open sets, each of which is contained in a compact set. In particular, this means that

(5.5.4) X is σ-compact.

Remember that X has the Lindelöf property when X satisfies the second countability condition, by Lindelöf's theorem. It follows that

(5.5.5) topological manifolds are σ-compact,

for instance.

5.5.2 Some bases from open coverings

Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of X, and suppose that for each $\alpha \in A$, $\mathcal{B}_\alpha$ is a base for the topology induced on U_α by the topology on X. Note that the elements of $\mathcal{B}_\alpha$ are open subsets of X for each $\alpha \in A$, because the U_α's are open subsets of X. Thus

$$\mathcal{B} = \bigcup_{\alpha \in A} \mathcal{B}_\alpha \tag{5.5.6}$$

is a collection of open subsets of X. Under these conditions, one can check that

(5.5.7) $\mathcal{B}$ is a base for the topology of X.

Of course, if A has only finitely or countably many elements, and if $\mathcal{B}_\alpha$ has only finitely or countably many elements for each $\alpha \in A$, then

(5.5.8) $\mathcal{B}$ has only finitely or countably many elements.

If X has the Lindelöf property, then one can automatically reduce to the case where A has only finitely or countably many elements.

Suppose that X is locally Euclidean of dimension n for some positive integer n. Thus X can be covered by open sets that are homeomorphic to open subsets of $\mathbf{R}^n$. If X has the Lindelöf property, then it follows that X can be covered by finitely or countably many open sets that are homeomorphic to open subsets of $\mathbf{R}^n$. Each of these open sets has a countable base for its topology, because of the analogous property for open subsets of $\mathbf{R}^n$. This leads to

(5.5.9) a countable base for the topology of X,

as in the preceding paragraph.

5.5.3 An example in R

One can show that the set $\mathbf{R} \setminus \mathbf{Q}$ of irrational numbers is not σ-compact with respect to the standard topology on $\mathbf{R}$, using the Baire category theorem. Note that every subset of the real line has the Lindelöf property, by Lindelöf's theorem.

5.6 Quotient spaces

Let X be a set. A binary relation $\sim$ on X is said to be an *equivalence relation* on X if it is reflexive, symmetric, and transitive on X, as usual. In this case, if $x \in X$, then

$$[x] = \{y \in X : x \sim y\} \tag{5.6.1}$$

is called an *equivalence class* in X. This is a subset of X that contains x, by reflexivity. If $x, w \in X$, then $[x] = [w]$ if and only if $x \sim w$, and otherwise $[x]$ and $[w]$ are disjoint.

Conversely, let $\mathcal{P}$ be a *partition* of X, which is to say a collection of pairwise-disjoint nonempty subsets of X whose union is equal to X. If $x, y \in X$, then put

(5.6.2) $x \sim_{\mathcal{P}} y$ when x and y lie in the same element of $\mathcal{P}$.

This defines an equivalence relation on X, for which the corresponding equivalence classes are the elements of $\mathcal{P}$.

If $\sim$ is any equivalence relation on X, then let

$$X/\sim \tag{5.6.3}$$

be the corresponding *quotient set*, which is the set of equivalence classes in X. In this situation, there is a natural *quotient mapping* from X onto $X/\sim$, which sends $x \in X$ to $[x] \in X/\sim$.

Let Y be another set, and let f be a mapping from X onto Y. If $x, w \in X$, then put

(5.6.4) $x \sim_f w$ when $f(x) = f(w)$.

This defines an equivalence relation on X, for which the corresponding equivalence classes are the sets of the form $f^{-1}(\{y\})$, $y \in Y$.

5.6.1 The quotient or identification topology

Suppose now that X is a topological space. Let us say that

$$V \subseteq Y \text{ is an open set} \tag{5.6.5}$$

exactly when

$$f^{-1}(V) \text{ is an open set in } X. \tag{5.6.6}$$

One can check that this defines a topology on Y, which is called the *quotient topology* or *identification topology* associated to f. Equivalently, this means that

$$E \subseteq Y \text{ is a closed set if and only if } f^{-1}(E) \text{ is a closed set in } X. \tag{5.6.7}$$

Note that f is automatically continuous with respect to the quotient topology on Y.

Remember that Y satisfies the first separation condition if and only if for each $y \in Y$, $\{y\}$ is a closed set in Y. Thus Y satisfies the first separation condition with respect to the quotient topology associated to f if and only if

$$\text{for each } y \in Y,\ f^{-1}(\{y\}) \text{ is a closed set in } X. \tag{5.6.8}$$

Let Z be another topological space, and let g be a mapping from Y into Z. One can check that g is continuous with respect to the quotient topology on Y associated to f if and only if $g \circ f$ is continuous as a mapping from X into Z.

5.7 Local connectedness

Let X be a topological space, and let x be an element of X. Let us say that X is *locally connected at* x if for every open subset W of X that contains x, there is an open subset U_0 of X such that $x \in U_0$, $U_0 \subseteq W$, and

$$\text{every element of } U_0 \text{ can be connected to } x \text{ in } W. \tag{5.7.1}$$

More precisely, this means that

$$\begin{aligned} &\text{for every } y \in U_0 \text{ there is a connected set } E(x,y) \subseteq X \\ &\text{such that } x, y \in E(x,y) \text{ and } E(x,y) \subseteq W. \end{aligned} \tag{5.7.2}$$

If there is a connected open subset U of X such that $x \in U$ and $U \subseteq W$, then we can take $U_0 = U$.

Alternatively, let W be an open subset of X that contains x, and let U_1 be

$$\text{the union of all the connected subsets of } W \text{ that contain } x. \tag{5.7.3}$$

Thus $x \in U_1$, because $\{x\}$ is a connected set that contains x, $U_1 \subseteq W$ by construction, and U_1 is connected, by Proposition 1.15.3. In this situation, local connectedness of X at x says exactly that

$$x \text{ is an element of the interior of } U_1. \tag{5.7.4}$$

If X is locally connected at every point in X, then X is said to be *locally connected* as a topological space. Let W be an open subset of X that contains x again, and let U_1 be as in the preceding paragraph. If X is locally connected, then one can check that

$$U_1 \text{ is an open set in } X. \tag{5.7.5}$$

5.7.1 Local path connectedness

Similarly, X is said to be *locally path connected* at x if for every open subset W of X that contains x, there is an open subset V_0 of X such that $x \in V_0$, $V_0 \subseteq W$, and

$$\begin{aligned} &\text{every element of } V_0 \text{ can be connected to } x \\ &\text{by a continuous path in } W. \end{aligned} \tag{5.7.6}$$

If there is a path-connected open subset V of X such that $x \in V$ and $V \subseteq W$, then we can take $V_0 = V$. Note that local path connectedness at x implies local connectedness at x.

Let W be an open subset of X that contains x, and let V_1 be the set of points in W that can be connected to x by a continuous path in W. Note that V_1 contains the images of these paths, so that V_1 is a path-connected subset of W. Local path connectedness of X at x says exactly that

$$x \text{ is in the interior of } V_1, \tag{5.7.7}$$

as before.

If X is locally path connected at every point, then X is said to be *locally path connected* as a topological space. Thus local path connectedness implies local connectedness. Let W be an open subset of X that contains x, and let V_1 be as in the previous paragraph. If X is locally path connected, then one can verify that

$$V_1 \text{ is an open set in } X. \tag{5.7.8}$$

5.7.2 Some examples and basic properties

It is easy to see that

$$\mathbf{R}^n \text{ is locally path connected} \tag{5.7.9}$$

for every positive integer n, with respect to the standard topology. Similarly, locally Euclidean spaces of dimension n are locally path connected.

If X_0 is an open subset of X, and if X is locally connected or path connected, then X_0 has the same property, with respect to the induced topology.

If X is connected and locally path connected, then one can show that X is path connected. In particular, connected open subsets of $\mathbf{R}^n$ are path connected.

5.8 Connected and path-connected components

Let X be a topological space, and consider the binary relation $\sim_c$ defined on X by putting $x \sim_c y$ when $x, y \in X$ and there is a connected subset $E(x, y)$ of X such that

$$x, y \in E(x, y). \tag{5.8.1}$$

One can check that this defines an equivalence relation on X, using Proposition 1.15.3 to get that $\sim_c$ is transitive.

One can also verify that

(5.8.2) the equivalence classes in X corresponding to $\sim_c$ are connected subsets of X,

using Proposition 1.15.3 or 1.15.6. These equivalence classes are called the *connected components* of X. It is easy to see that the connected components of X are the maximal connected subsets of X, with respect to inclusion.

Observe that

(5.8.3) the connected components of X are closed sets in X,

by Proposition 1.15.1. If X is locally connected, then

(5.8.4) the connected components of X are open sets in X.

5.8.1 Another equivalence relation on X

Similarly, consider the binary relation $\sim_p$ defined on X by putting $x \sim_p y$ when $x, y \in X$ are connected by a continuous path in X. This means that there are real numbers a, b with $a < b$ and a continuous mapping p from $[a, b]$ into X such that

$$p(a) = x, \ p(b) = y. \tag{5.8.5}$$

This uses the topology induced on $[a, b]$ by the standard topology on $\mathbf{R}$. Note that one can always reduce to the case where $a = 0$ and $b = 1$, by reparameterizing the path.

It is not too difficult to show that $\sim_p$ defines an equivalence relation on X. The equivalence classes in X corresponding to $\sim_p$ are path-connected subsets of X, by construction. These equivalence classes are called the *path-connected components* of X. As before, these are the maximal path-connected subsets of X, with respect to inclusion.

If $x, y \in X$, then

$$x \sim_p y \text{ implies } x \sim_c y. \tag{5.8.6}$$

Indeed, if $a, b \in \mathbf{R}$ and $a < b$, then $[a, b]$ is connected as a subset of $\mathbf{R}$ with respect to the standard topology. This implies that $[a, b]$ is connected as a subset of itself, with respect to the induced topology. If p is a continuous mapping from $[a, b]$ into X, then it follows that $p([a, b])$ is a connected set in X.

If X is locally path connected, then

(5.8.7) the path-connected components of X are open subsets of X.

If X is also connected, then one can check that X is path connected.

5.9 One-point compactifications

Let X be a Hausdorff topological space that is not compact. The *one-point compactification* of X can be defined initially as a set by

(5.9.1) $$X^* = X \cup \{p_*\},$$

where $p_* \notin X$. One often refers to p_* as the *point at infinity* in X^*. Let us say that $U \subseteq X^*$ is an open set if either

(5.9.2) $$U \text{ is an open subset of } X,$$

or if

(5.9.3) $$p_* \in U \text{ and } X^* \setminus U \text{ is a compact subset of } X.$$

Remember that compact subsets of X are closed sets in X, because X is Hausdorff, by hypothesis. If $U \subseteq X^*$ is an open set and $p_* \in U$, then it follows that $X^* \setminus U$ is a closed set in X. Note that

(5.9.4) $$X \setminus (U \cap X) = X \setminus U = X^* \setminus U,$$

because $p_* \in U$. Thus

(5.9.5) $$U \cap X \text{ is an open set in } X$$

in this case.

5.9.1 Checking that this defines a topology on X^*

Of course, the empty set is an open subset of X^*, because it is an open subset of X. We also have that X^* is an open set in itself, because it contains p_* and its complement in itself is the empty set, which is a compact subset of X.

If $U_1, \ldots, U_n$ are finitely many open sets in X^*, then we would like to check that their intersection is an open set in X^* as well. If $U_l \subseteq X$ for some l, then

(5.9.6) $$\bigcap_{j=1}^{n} U_j \subseteq X,$$

and it suffices to verify that $\bigcap_{j=1}^{n} U_j$ is an open set in X. This follows from the fact that $U_j \cap X$ is an open set in X for each $j = 1, \ldots, n$, so that

(5.9.7) $$\bigcap_{j=1}^{n} U_j = \bigcap_{j=1}^{n} (U_j \cap X)$$

is an open set in X. Otherwise, if $p_* \in U_j$ for each $j = 1, \ldots, n$, then we should verify that

$$X^* \setminus \Big(\bigcap_{j=1}^{n} U_j\Big) = \bigcup_{j=1}^{n} (X^* \setminus U_j) \tag{5.9.8}$$

is a compact subset of X. In fact, it is easy to see that the union of finitely many compact subsets of any topological space is compact, as mentioned in Section 5.5.

Let $\{U_\alpha\}_{\alpha \in A}$ be a family of open subsets of X^*, and let us check that their union is an open set in X^*. If $U_\alpha \subseteq X$ for every $\alpha \in A$, then U_α is an open set in X for each $\alpha \in A$. This implies that $\bigcup_{\alpha \in A} U_\alpha$ is an open set in X, and hence in X^*. Otherwise, suppose that $p_* \in U_{\alpha_0}$ for some $\alpha_0 \in A$, so that $p_* \in \bigcup_{\alpha \in A} U_\alpha$. In this case, we would like to verify that

$$X^* \setminus \Big(\bigcup_{\alpha \in A} U_\alpha\Big) \text{ is a compact subset of } X. \tag{5.9.9}$$

Observe that

$$X^* \setminus \Big(\bigcup_{\alpha \in A} U_\alpha\Big) = X \setminus \Big(\bigcup_{\alpha \in A} U_\alpha\Big) = \bigcap_{\alpha \in A} (X \setminus U_\alpha). \tag{5.9.10}$$

This is a closed set in X, because $X \setminus U_\alpha$ is a closed set in X for every $\alpha \in A$. We also have that $X \setminus U_{\alpha_0} = X^* \setminus U_{\alpha_0}$ is compact in X, because $p_* \in U_{\alpha_0}$. It follows that (5.9.9) holds, because it is a closed set that is contained in the compact set $X \setminus U_{\alpha_0}$.

This shows that the collection of open subsets of X^* defined earlier is a topology on X^*. By construction, X is an open subset of X^*, and the topology induced on X by this topology on X^* is the same as the given topology on X. Note that

$$p_* \text{ is a limit point of } X \text{ in } X^*, \tag{5.9.11}$$

because X is not compact, by hypothesis.

5.9.2 Compactness of X^*

Let us check that

$$X^* \text{ is compact} \tag{5.9.12}$$

with respect to this topology. Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of X^* in itself. In particular, there is an $\alpha_0 \in A$ such that $p_* \in U_{\alpha_0}$. Thus

$$X \setminus U_{\alpha_0} = X^* \setminus U_{\alpha_0} \text{ is a compact subset of } X. \tag{5.9.13}$$

We may consider $\{U_\alpha \cap X\}_{\alpha \in A}$ as an open covering of $X \setminus U_{\alpha_0}$ in X, because $U_\alpha \cap X$ is an open set in X for each $\alpha \in A$, and the union of these open sets is X. Because $X \setminus U_{\alpha_0}$ is compact in X, there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that

$$X^* \setminus U_{\alpha_0} = X \setminus U_{\alpha_0} \subseteq \bigcup_{j=1}^{n} (U_{\alpha_j} \cap X). \tag{5.9.14}$$

This implies that

$$X^* \subseteq \bigcup_{j=0}^{n} U_{\alpha_j}, \tag{5.9.15}$$

as desired.

5.9.3 When X^* is Hausdorff

Observe that X^* is Hausdorff if and only if for every $x \in X$ there are disjoint open subsets U and V of X^* such that $x \in U$ and $p_* \in V$, because X is Hausdorff by hypothesis. One can verify that this happens exactly when

$$X \text{ is locally compact.} \tag{5.9.16}$$

The one-point compactification of a topological space X can also be defined a bit more carefully when X is not necessarily Hausdorff.

5.10 Homotopic mappings

Let X and Y be topological spaces, and let f, g be continuous mappings from X into Y. A *homotopy* between f and g is a mapping H from $X \times [0,1]$ into Y that satisfies the following two conditions. First, H should be continuous with respect to the product topology on $X \times [0,1]$. Of course, this uses the topology induced on $[0,1]$ by the standard topology on $\mathbf{R}$. Second,

$$H(x,0) = f(x), \tag{5.10.1}$$

$$H(x,1) = g(x) \tag{5.10.2}$$

for every $x \in X$. In this case, one may say that f is *homotopic* to g. It is well known and not too difficult to show that this defines an equivalence relation on the space $C(X,Y)$ of continuous mappings from X into Y.

5.10.1 Additional conditions on f, g

In many cases, one may wish to include additional conditions on f and g, and the same conditions on $H(x,t)$ for each $t \in [0,1]$. Suppose for instance that $X = [0,1]$, and let $p, q \in Y$ be given. Let f, g be continuous mappings from $[0,1]$ into Y such that

$$f(0) = g(0) = p, \ f(1) = g(1) = q. \tag{5.10.3}$$

If a homotopy $H(x,t)$ between f and g also satisfies

$$H(0,t) = p, \ H(1,t) = q \tag{5.10.4}$$

for every $t \in [0,1]$, then one says that f and g are *homotopic with endpoints fixed*. One can show that this defines an equivalence relation on the space of continuous paths in Y from p to q.

5.10.2 Convex sets Y in $\mathbf{R}^n$

Suppose now that Y is a convex subset of $\mathbf{R}^n$ for some positive integer n. This means that for every $y, z \in Y$, the line segment in $\mathbf{R}^n$ connecting y and z is contained in Y too. We consider Y as a topological space, using the topology induced by the standard topology on $\mathbf{R}^n$. Let X be any topological space again, and let f, g be continuous mappings from X into Y. One can check that

$$H(x,t) = (1-t)\, f(x) + t\, g(x) \tag{5.10.5}$$

defines a homotopy between f and g, as continuous mappings from X into Y.

Let $p, q \in Y$ be given again, and suppose that f, g are continuous mappings from $[0,1]$ into Y that satisfy (5.10.3). Note that (5.10.5) satisfies (5.10.4) in this case.

Let Z be another topological space, and let ϕ be a continuous mapping from Y into Z. If f, g are homotopic continuous mappings from X into Y, with homotopy H, then $\phi \circ H$ is a homotopy between $\phi \circ f$ and $\phi \circ g$, as continuous mappings from X into Z. There is an analogous statement for homotopic paths with endpoints fixed, as usual.

5.11 Countable unions and intersections

A subset E of a topological space X is said to be an F_σ *set* if

(5.11.1) $\qquad E$ can be expressed as the union of a sequence of closed subsets of X.

Similarly, a subset A of X is said to be a G_δ *set* if

(5.11.2) $\qquad A$ can be expressed as the intersection of a sequence of open subsets of X.

It is easy to see that

$$E \subseteq X \text{ is an } F_\sigma \text{ set if and only if } X \setminus E \text{ is a } G_\delta \text{ set.} \tag{5.11.3}$$

Of course, closed sets are F_σ sets, and open sets are G_δ sets.

Suppose for the moment that X satisfies the first separation condition. Let x be an element of X, and let $\mathcal{B}(x)$ be a local base for the topology of X at x. Under these conditions, the intersection of the elements of $\mathcal{B}(x)$ is equal to $\{x\}$. If $\mathcal{B}(x)$ has only finitely many elements, then $\{x\}$ is an open set in X. If $\mathcal{B}(x)$ is countable, then $\{x\}$ is a G_δ set in X.

5.11.1 Some subsets of semimetric spaces

Suppose now that $d(x,y)$ is a semimetric on X, and that X is equipped with the topology determined by $d(\cdot,\cdot)$. If A is any subset of X and r is a positive real number, then put

$$A_r = \bigcup_{x \in A} B(x, r). \tag{5.11.4}$$

This is an open subset of X that contains A. One can check that

$$\overline{A} = \bigcap_{j=1}^{\infty} A_{1/j}. \tag{5.11.5}$$

This implies that every closed set in X is a G_δ set, and hence that every open set in X is an F_σ set.

5.11.2 Some remarks about inverse images

Let X and Y be arbitrary topological spaces, and let f be a continuous mapping from X into Y. If $E \subseteq Y$ is an F_σ set, then

$$f^{-1}(E) \text{ is an } F_\sigma \text{ set in } X. \tag{5.11.6}$$

Similarly, if $A \subseteq Y$ is a G_δ set, then

$$f^{-1}(A) \text{ is a } G_\delta \text{ set in } X. \tag{5.11.7}$$

In particular, suppose that f is a continuous real-valued function on X, with respect to the standard topology on $\mathbf{R}$. If $t \in \mathbf{R}$, then $\{t\}$ is a G_δ set in $\mathbf{R}$, so that

$$f^{-1}(\{t\}) \text{ is a } G_\delta \text{ set in } X. \tag{5.11.8}$$

5.11.3 Perfectly normal spaces

Let us say that X is *perfectly normal in the strict sense* if X is normal in the strict sense, and every closed set in X is a G_δ set. If X also satisfies the first separation condition, then we say that X is *perfectly normal in the strong sense.* One may say that X is *perfectly normal* when X is perfectly normal in the strict sense, and *perfectly* T_4 when X is perfectly normal in the strong sense, but the opposite convention is used sometimes too.

5.11.4 Nowhere dense sets and category

A subset E of X is said to be *nowhere dense* if the interior of the closure $\overline{E}$ of E in X is empty. Equivalently, this means that $X \setminus \overline{E}$ is dense in X. This is the same as saying that the interior of $X \setminus E$ is dense in X. Remember that the intersection of finitely many dense open subsets of X is a dense open set in X, as in Subsection 5.4.4. One can use this to check that the union of finitely many nowhere dense subsets of X is nowhere dense in X.

A subset A of X is said to be *of the first category* or *meager* in X if A can be expressed as the union of a sequence of nowhere dense sets. Otherwise, A is said to be *of the second category* or *non-meager* in X. The theorem of Baire mentioned in Subsection 5.4.4 is equivalent to saying that if A is of the first category in X, then

$$X \setminus A \text{ is dense in } X, \tag{5.11.9}$$

under the same conditions as before. This is the same as saying that the interior of A in X is empty. In particular, it follows that X is of second category in this case.

5.12 Distance functions

Let X be a set with a semimetric $d(x, y)$, and let A be a nonempty subset of X. If $x \in X$, then the *distance from x to A with respect to* $d(\cdot, \cdot)$ is defined by

$$\operatorname{dist}(x, A) = \inf\{d(x, a) : a \in A\}, \tag{5.12.1}$$

which is a nonnegative real number. One can check that $\operatorname{dist}(x, A) = 0$ if and only if x is an element of the closure $\overline{A}$ of A with respect to the topology determined on X by $d(\cdot, \cdot)$.

5.12.1 Continuity of distance functions

Observe that

$$\operatorname{dist}(x, A) \le d(x, a) \le d(x, y) + d(y, a) \tag{5.12.2}$$

for every $x, y \in X$ and $a \in A$. Using this, one can check that

$$\operatorname{dist}(x, A) \le d(x, y) + \operatorname{dist}(y, A) \tag{5.12.3}$$

for every $x, y \in X$. One can use this to verify that $\operatorname{dist}(x, A)$ is continuous as a real-valued function on X, with respect to the standard topology on $\mathbf{R}$.

5.12.2 Separated sets and open sets

Suppose that A and B are nonempty separated subsets of X. Put

$$U = \{x \in X : \operatorname{dist}(x, A) < \operatorname{dist}(x, B)\} \tag{5.12.4}$$

and

$$V = \{x \in X : \operatorname{dist}(x, B) < \operatorname{dist}(x, A)\}. \tag{5.12.5}$$

It is easy to see that these are open subsets of X. We also have that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. This is another way to look at the complete normality of X in the strict sense, with respect to the topology determined by $d(\cdot, \cdot)$.

5.12.3 Urysohn's lemma for semimetric spaces

Suppose now that A and B are disjoint nonempty closed subsets of X. Note that

$$\operatorname{dist}(x, A) + \operatorname{dist}(x, B) > 0 \tag{5.12.6}$$

for every $x \in X$. Thus

$$\frac{\operatorname{dist}(x, A)}{\operatorname{dist}(x, A) + \operatorname{dist}(x, B)} \tag{5.12.7}$$

defines a nonnegative real-valued function on X, which is less than or equal to 1 for every $x \in X$. This function is continuous on X, equal to 0 on A, and equal to 1 on B. Of course, this corresponds to Urysohn's lemma in this case.

5.13 Local finiteness

Let X be a topological space, let A be a nonempty set, and let E_α be a subset of X for every $\alpha \in A$. Let us say that $\{E_\alpha\}_{\alpha \in A}$ is *locally finite* at a point $x \in X$ if there is an open subset U of X such that $x \in U$ and

$$E_\alpha \cap U = \emptyset \tag{5.13.1}$$

for all but finitely many $\alpha \in A$. If $\{E_\alpha\}_{\alpha \in A}$ is locally finite at every $x \in X$, then $\{E_\alpha\}_{\alpha \in A}$ is said to be *locally finite in* X.

5.13.1 Local finiteness and adherent points

Of course, if $x \in X$ is adherent to E_β for some $\beta \in A$, then x is adherent to $\bigcup_{\alpha \in A} E_\alpha$.

Let $x \in X$ be given, and suppose that for each $\alpha \in A$, x is not adherent to E_α. Suppose also that $\{E_\alpha\}_{\alpha \in A}$ is locally finite at x, so that there is an open set $U \subseteq X$ such that $x \in U$ and (5.13.1) holds for all but finitely many $\alpha \in A$. Let $\alpha_1, \ldots, \alpha_n$ be a list of the finitely many $\alpha \in A$ such that E_α intersects U, if there are any. By hypothesis, for each $j = 1, \ldots, n$, x is not adherent to E_{α_j}, and so there is an open subset V_j of X such that $x \in V_j$ and $E_{\alpha_j} \cap V_j = \emptyset$.

It follows that

$$W = U \cap \Big(\bigcap_{j=1}^{n} V_j\Big) \tag{5.13.2}$$

is an open subset of X such that $x \in W$ and

$$W \cap \Big(\bigcup_{\alpha \in A} E_\alpha\Big) = \emptyset. \tag{5.13.3}$$

This means that x is not adherent to $\bigcup_{\alpha \in A} E_\alpha$. Equivalently, if $x \in X$ is adherent to $\bigcup_{\alpha \in A} E_\alpha$, and if $\{E_\alpha\}_{\alpha \in A}$ is locally finite at x, then x is adherent to E_α for some $\alpha \in A$.

5.13.2 Local finiteness and closures

If $\{E_\alpha\}_{\alpha \in A}$ is locally finite in X, then it follows that

$$\overline{\Big(\bigcup_{\alpha \in A} E_\alpha\Big)} = \bigcup_{\alpha \in A} \overline{E_\alpha}. \tag{5.13.4}$$

Of course, the right side is automatically contained in the left side, as before.

If $\alpha \in A$ satisfies (5.13.1) for some open set $U \subseteq X$, then

$$\overline{E_\alpha} \cap U = \emptyset. \tag{5.13.5}$$

If $\{E_\alpha\}_{\alpha \in A}$ is locally finite at x, then we get that $\{\overline{E_\alpha}\}_{\alpha \in A}$ is locally finite at x too. If $\{E_\alpha\}_{\alpha \in A}$ is locally finite in X, then $\{\overline{E_\alpha}\}_{\alpha \in A}$ is locally finite in X as well.

5.13.3 Local finiteness and induced topologies

Let X_0 be a subset of X, equipped with the induced topology. If $\{E_\alpha\}_{\alpha \in A}$ is locally finite at $x \in X_0$, then $\{E_\alpha \cap X_0\}_{\alpha \in A}$ is locally finite at x, as a family of subsets of X_0. Similarly, if $\{E_\alpha\}_{\alpha \in A}$ is locally finite in X, then $\{E_\alpha \cap X_0\}_{\alpha \in A}$ is locally finite in X_0.

Suppose for the moment that $\{E_\alpha\}_{\alpha \in A}$ is a family of subsets of X_0. If $\{E_\alpha\}_{\alpha \in A}$ is locally finite at $x \in X_0$, as a family of subsets of X_0, then it is easy to see that $\{E_\alpha\}_{\alpha \in A}$ is locally finite at x, as a family of subsets of X.

5.13.4 Local finiteness and compactness

Let $\{E_\alpha\}_{\alpha \in A}$ be a family of subsets of X again, and let K be a subset of X. Suppose that $\{E_\alpha\}_{\alpha \in A}$ is locally finite at every $x \in K$. Thus, for each $x \in K$, there is an open subset $U(x)$ of X such that $x \in U(x)$ and $E_\alpha \cap U(x) = \emptyset$ for all but finitely many $\alpha \in A$. If K is compact, then there are finitely many elements $x_1, \ldots, x_n$ of K such that

$$K \subseteq \bigcup_{j=1}^{n} U(x_j). \tag{5.13.6}$$

We also have that

$$E_\alpha \cap \Big(\bigcup_{j=1}^{n} U(x_j)\Big) = \emptyset \tag{5.13.7}$$

for all but finitely many $\alpha \in A$ in this situation.

In particular, suppose that $\{E_\alpha\}_{\alpha \in A}$ is locally finite in X, and that X is compact. Under these conditions, we get that $E_\alpha = \emptyset$ for all but finitely many $\alpha \in A$.

5.14 Sums of real-valued functions

Let X be a set, let I be a nonempty set, and let ϕ_j be a real-valued function on X for each $j \in I$. If $x \in X$, then put

$$I(x) = \{j \in I : \phi_j(x) \neq 0\}. \tag{5.14.1}$$

If $I(x)$ has only finitely many elements, then the sum

$$\Phi(x) = \sum_{j \in I} \phi_j(x) \tag{5.14.2}$$

can be defined as a real number, by reducing to a finite sum. More precisely, (5.14.2) is the same as the sum over any nonempty finite subset of I that contains $I(x)$.

Put

$$E_j = \{x \in X : \phi_j(x) \neq 0\} \tag{5.14.3}$$

for each $j \in I$. If $x \in X$, then $I(x)$ is the same as the set of $j \in I$ such that $x \in E_j$. If U_0 is a subset of X, then put

$$I(U_0) = \bigcup_{x \in U_0} I(x) = \{j \in I : \phi_j(x) \neq 0 \text{ for some } x \in U_0\}. \tag{5.14.4}$$

This is the same as the set of $j \in I$ such that $E_j \cap U_0 \neq \emptyset$.

5.14.1 Local finiteness and continuity

Suppose now that X is a topological space. Observe that $\{E_j\}_{j \in I}$ is locally finite at a point $x_0 \in X$ if and only if there is an open set $U_0 \subseteq X$ such that $x_0 \in U_0$ and $I(U_0)$ has only finitely many elements. In particular, this implies that for each $x \in U_0$, $I(x)$ has only finitely many elements.

Suppose that ϕ_j is continuous on X for each $j \in I$, with respect to the standard topology on $\mathbf{R}$. If $\{E_j\}_{j \in I}$ is locally finite in X, then (5.14.2) defines a real-valued function on X. It is easy to see that this function is continuous at every point in X, under these conditions.

5.14.2 Some partitions of unity

Of course, if $\phi_j(x) \geq 0$ for every $j \in I$ and $x \in X$, then $\Phi(x) \geq 0$ for every $x \in X$. If we also have that for each $x \in X$ there is a $j \in I$ such that $\phi_j(x) > 0$, then we get that $\Phi(x) > 0$ for every $x \in X$. Note that $1/\Phi(x)$ is continuous on X in this case. Put

$$\psi_l(x) = \phi_l(x)/\Phi(x) \tag{5.14.5}$$

for every $l \in I$ and $x \in X$, which defines a continuous real-valued function on X. By construction, $\psi_l(x) \geq 0$ for every $l \in I$ and $x \in X$, and

$$\{x \in X : \psi_l(x) > 0\} = E_l. \tag{5.14.6}$$

Thus, for each $x \in X$, $\psi_l(x) > 0$ for only finitely many $l \in I$, and

$$\sum_{l \in I} \psi_l(x) = \Big(\sum_{l \in I} \phi_l(x)/\Phi(x) \Big) = \Phi(x)/\Phi(x) = 1. \tag{5.14.7}$$

The family of functions ψ_l, $l \in I$, is said to be a *partition of unity* on X.

5.15 Local compactness and σ-compactness

Let X be a locally compact topological space. If K is a compact subset of X, then there is an open set $U \subseteq X$ such that $K \subseteq U$ and U is contained in a compact subset of X. Indeed, every element of K is contained in an open set that is contained in a compact subset of X, because X is locally compact. It follows that K is contained in the union of finitely many of these open sets, because K is compact. By construction, the union of these finitely many open sets is contained in the union of finitely many compact subsets of X, which is compact as well.

If X is Hausdorff, then every element of X is contained in an open set whose closure in X is compact, as in Section 5.4. Similarly, if $K \subseteq X$ is compact, then there is an open set $U \subseteq X$ such that $K \subseteq U$ and $\overline{U}$ is compact.

5.15.1 Some sequences of compact and open sets

Suppose now that X is σ-compact, so that there is a sequence $K_1, K_2, K_3, \ldots$ of compact subsets of X with

$$X = \bigcup_{j=1}^{\infty} K_j. \tag{5.15.1}$$

If X is locally compact too, then there is an open set $U_1 \subseteq X$ and a compact set $E_1 \subseteq X$ such that $K_1 \subseteq U_1$ and $U_1 \subseteq E_1$, as before.

Similarly, for each $j \geq 1$, we can choose an open set $U_j \subseteq X$ and a compact set $E_j \subseteq X$ such that

$$\bigcup_{l=1}^{j} K_l \subseteq U_j \subseteq E_j, \tag{5.15.2}$$

and so that

$$E_{j-1} \subseteq U_j \tag{5.15.3}$$

when $j \geq 2$. More precisely, suppose that U_j and E_j have been chosen in this way for some j, and let us see how we can choose U_{j+1} and E_{j+1}.

Note that $E_j \cup K_{j+1}$ is compact in X, because E_j and K_{j+1} are compact. Thus there is an open set $U_{j+1} \subseteq X$ such that $E_j \cup K_{j+1} \subseteq U_{j+1}$ and U_{j+1} is contained in a compact set $E_{j+1} \subseteq X$, as before. It follows that $\bigcup_{l=1}^{j+1} K_l \subseteq U_{j+1}$, because of (5.15.2). If X is Hausdorff, then we can take $E_j = \overline{U_j}$ for each j.

5.15.2 A locally finite sequence of compact sets

Of course,

$$X = \bigcup_{j=1}^{\infty} U_j = \bigcup_{j=1}^{\infty} E_j, \tag{5.15.4}$$

by (5.15.1) and (5.15.2). Put

$$A_1 = E_1 \quad \text{and} \quad A_j = E_j \setminus U_{j-1} \text{ when } j \geq 2. \tag{5.15.5}$$

It is easy to see that A_j is compact in X for every j, because $A_j = E_j \cap (X \setminus U_{j-1})$ is the intersection of a compact set and a closed set when $j \geq 2$. One can check that

$$X = \bigcup_{j=1}^{\infty} A_j, \tag{5.15.6}$$

using (5.15.4).

More precisely, if $x \in X$ and j_0 is the smallest positive integer such that $x \in E_{j_0}$, then $x \in A_{j_0}$. In this case, $x \notin A_j$ when $j < j_0$, and when $j > j_0 + 1$. By construction,

$$A_j \cap U_l = \emptyset \tag{5.15.7}$$

when $j > l$, which implies that the family of A_j's is locally finite in X.

5.15.3 A locally finite sequence of open sets

Suppose now that X is Hausdorff, so that we can take $E_j = \overline{U}_j$ for each j, as before. Clearly $A_j \subseteq U_{j+1}$ for every j, by (5.15.3), and in fact

$$A_j \subseteq U_{j+1} \setminus \overline{U_{j-2}} \tag{5.15.8}$$

when $j \geq 3$. Put

$$V_2 = U_3 \quad \text{and} \quad V_j = U_{j+1} \setminus \overline{U_{j-2}} \text{ when } j \geq 3, \tag{5.15.9}$$

so that V_j is an open subset of X for every $j \geq 2$. It is easy to see that

$$X = \bigcup_{j=2}^{\infty} V_j, \tag{5.15.10}$$

using (5.15.3) and (5.15.4).

Observe that

$$V_j \cap V_l = \emptyset \tag{5.15.11}$$

when $j \geq l + 3$, and in particular that the family of V_j's is locally finite in X.

Chapter 6

Some spaces of mappings

6.1 Bounded sets and mappings

Let Y be a set with a semimetric $d_Y(\cdot,\cdot)$. A subset A of Y is said to be *bounded* with respect to $d_Y(\cdot,\cdot)$ if A is contained in a ball in Y with respect to $d_Y(\cdot,\cdot)$. It is convenient to consider $A = \emptyset$ as a bounded subset of Y even when $Y = \emptyset$.

6.1.1 Countable compactness implies boundedness

If $A \subseteq Y$ is countably compact with respect to the topology determined on Y by $d_Y(\cdot,\cdot)$, then it is easy to see that A is bounded in Y. More precisely, this can be obtained by covering A by open balls in Y centered at a fixed point in Y and with positive integer radii.

6.1.2 Bounded mappings

A mapping f from a set X into Y is said to be *bounded* if $f(X)$ is a bounded set in Y.

Suppose that X is a topological space, and that f is a continuous mapping from X into Y, with respect to the topology determined on Y by $d_Y(\cdot,\cdot)$. If $K \subseteq X$ is countably compact, then $f(K)$ is countably compact in Y, and hence $f(K)$ is bounded in Y. In particular, if X is countably compact, then f is bounded.

6.1.3 Some related real-valued functions

Let p be an element of Y, let f be a mapping from X into Y, and let E be a subset of X. One can check that $f(E)$ is bounded in Y if and only if

$$d_Y(p, f(x)) \tag{6.1.1}$$

is bounded as a real-valued function on E.

If X is a topological space and f is continuous with respect to the topology determined on Y by $d_Y(\cdot,\cdot)$, then (6.1.1) is continuous as a real-valued function on X, with respect to the standard topology on $\mathbf{R}$. Indeed, (6.1.1) is the same as the composition of f with the real-valued function on Y defined by $d_Y(p,\cdot)$, which is continuous with respect to the topology determined on Y by $d_Y(\cdot,\cdot)$.

6.1.4 Supremum semimetrics

Let X be a set, and let E be a nonempty subset of X. Consider the space $\mathcal{B}_E(X,Y)$ of mappings f from X into Y such that $f(E)$ is a bounded set in Y.

If $f,g \in \mathcal{B}_E(X,Y)$, then one can check that $d_Y(f(x),g(x))$ is bounded as a real-valued function on E. In this case, we put

$$\theta_E(f,g) = \sup\{d_Y(f(x),g(x)) : x \in E\}. \tag{6.1.2}$$

One can verify that this defines a semimetric on $\mathcal{B}_E(X,Y)$, which is called the *supremum semimetric* on $\mathcal{B}_E(X,Y)$ associated to $d_Y(\cdot,\cdot)$.

If $E = X$, then we may use $\mathcal{B}(X,Y)$ instead of $\mathcal{B}_X(X,Y)$, which is the space of all bounded mappings from X into Y. Similarly, we may use $\theta(f,g)$ instead of $\theta_X(f,g)$. Note that this is a metric on $\mathcal{B}(X,Y)$ when $d_Y(\cdot,\cdot)$ is a metric on Y.

6.1.5 Some supremum semimetrics on $C(X,Y)$

Let X be a topological space, and let $C(X,Y)$ be the space of continuous mappings from X into Y, as before. Also let K be a nonempty countably compact subset of X, so that

$$C(X,Y) \subseteq \mathcal{B}_K(X,Y), \tag{6.1.3}$$

as before. Thus the restriction of $\theta_K(f,g)$ to $f,g \in C(X,Y)$ defines a semimetric on $C(X,Y)$.

6.1.6 Existence of the maximum

If $f,g \in C(X,Y)$, then one can check that

$$d_Y(f(x),g(x)) \tag{6.1.4}$$

is continuous as a real-valued function on X, with respect to the standard topology on $\mathbf{R}$. If K is a nonempty compact subset of X, then the maximum of (6.1.4) on K is attained, by the extreme value theorem.

This also works when K is countably compact, by analogous arguments. More precisely, one can use the fact that countably compact subsets of the real line are compact, because there is a countable base for the standard topology on $\mathbf{R}$. One can also show more directly that countably compact subsets of $\mathbf{R}$ are closed and bounded.

6.1.7 Bounded continuous mappings

Let

(6.1.5) $$C_b(X,Y) = C(X,Y) \cap \mathcal{B}(X,Y)$$

be the space of mappings from X into Y that are both bounded and continuous. If X is countably compact, then this is the same as $C(X,Y)$, as before.

If X is any topological space, then $C_b(X,Y)$ is a closed set in $\mathcal{B}(X,Y)$ with respect to the supremum semimetric $\theta(\cdot,\cdot)$, as in Subsection 6.2.8.

6.2 Uniform convergence

Let X be a nonempty set, and let Y be a topological space. Also let $(A, \preceq)$ be a nonempty directed system, let $\{f_a\}_{a \in A}$ be a net of mappings from X into Y indexed by A, and let f be another mapping from X into Y. We say that $\{f_a\}_{a \in A}$ converges to f *pointwise* on X if for every $x \in X$, $\{f_a(x)\}_{a \in A}$ converges to $f(x)$ as a net of elements of Y.

6.2.1 Relation with the product topology

The space Y^X of mappings from X into Y may be considered as the Cartesian product of copies of Y indexed by X. Thus the product and strong product topologies can be defined on Y^X in the usual way, using the topology determined on Y by $d_Y(\cdot,\cdot)$.

Note that pointwise convergence of a net of mappings from X into Y is the same as convergence with respect to the product topology. The product topology on Y^X is sometimes also called the *topology of pointwise convergence.*

6.2.2 Using a semimetric on Y

Now let $d_Y(\cdot,\cdot)$ be a semimetric on Y, and suppose that Y is equipped with the topology determined by $d_Y(\cdot,\cdot)$. A net $\{f_a\}_{a \in A}$ of mappings from X into Y is said to converge *uniformly* on X to a mapping f from X into Y with respect to $d_Y(\cdot,\cdot)$ on Y if for every $\epsilon > 0$ there is a $b \in A$ such that

(6.2.1) $$d_Y(f_a(x), f(x)) < \epsilon$$

for every $x \in X$ and $a \in A$ with $b \preceq a$. Clearly uniform convergence implies pointwise convergence, and the converse holds when X has only finitely many elements.

6.2.3 Some sets $N_\epsilon(f)$ and $\widetilde{N}_\epsilon(f)$

Let f be a mapping from X into Y, and let ϵ be a positive real number. Consider the set $N_\epsilon(f)$ of all mappings g from X into Y such that

(6.2.2) $$d_Y(f(x), g(x)) < \epsilon \quad \text{for every } x \in X.$$

Similarly, let $\widetilde{N}_\epsilon(f)$ be the set of all mappings g from X into Y such that $d_Y(f(x), g(x))$ is bounded as a real-valued function on X, with

$$\sup\{d_Y(f(x), g(x)) : x \in X\} < \epsilon. \tag{6.2.3}$$

Clearly

$$\widetilde{N}_\epsilon(f) \subseteq N_\epsilon(f), \tag{6.2.4}$$

because (6.2.3) implies (6.2.2). If η is any positive real number greater than ϵ, then

$$N_\epsilon(f) \subseteq \widetilde{N}_\eta(f). \tag{6.2.5}$$

This is because (6.2.2) implies that

$$\sup\{d_Y(f(x), g(x)) : x \in X\} \le \epsilon. \tag{6.2.6}$$

Of course, if the supremum on the left is attained, then (6.2.2) implies (6.2.3).

6.2.4 The topology of uniform convergence

Let us say that $W \subseteq Y^X$ is an open set with respect to the *topology of uniform convergence* if for every $f \in W$ there is an $\epsilon > 0$ such that

$$N_\epsilon(f) \subseteq W. \tag{6.2.7}$$

This is equivalent to saying that for every $f \in W$ there is an $\eta > 0$ such that

$$\widetilde{N}_\eta(f) \subseteq W, \tag{6.2.8}$$

by the remarks in the previous paragraph. One can check that this defines a topology on Y^X.

It is easy to see that this topology is at least as strong as the product topology on Y^X. The strong product topology on Y^X is at least as strong as the topology of uniform convergence on Y^X.

6.2.5 More on $\widetilde{N}_\eta(f)$

Let f, g, and h be mappings from X into Y, and suppose that $d_Y(f(x), g(x))$ and $d_Y(g(x), h(x))$ are bounded as real-valued functions on X. Observe that

$$\begin{aligned} d_Y(f(x), h(x)) &\le d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \\ &\le \sup_{w \in X} d_Y(f(w), g(w)) + \sup_{w \in X} d_Y(g(w), h(w)) \end{aligned} \tag{6.2.9}$$

for every $x \in X$, using the triangle inequality in the first step. This implies that $d_Y(f(x), h(x))$ is bounded as a real-valued function on X, with

$$\sup_{x \in X} d_Y(f(x), h(x)) \le \sup_{x \in X} d_Y(f(x), g(x)) + \sup_{x \in X} d_Y(g(x), h(x)). \tag{6.2.10}$$

Suppose that $g \in \widetilde{N}_\eta(f)$ for some $\eta > 0$, so that

$$\delta = \eta - \sup_{x \in X} d_Y(f(x), g(x)) > 0. \tag{6.2.11}$$

Under these conditions, one can check that

$$\widetilde{N}_\delta(g) \subseteq \widetilde{N}_\eta(f), \tag{6.2.12}$$

using (6.2.10). This implies that $\widetilde{N}_\eta(f)$ is an open set in Y^X, with respect to the topology of uniform convergence.

6.2.6 Uniformly convergent nets

Let $\{f_a\}_{a \in A}$ be a net of mappings from X into Y indexed by A again, and let f be another mapping from X into Y. One can check that $\{f_a\}_{a \in A}$ converges to f uniformly on X with respect to $d_Y(\cdot, \cdot)$ if and only if $\{f_a\}_{a \in A}$ converges to f with respect to the topology of uniform convergence on Y^X.

6.2.7 Relation with the supremum semimetric on $\mathcal{B}(X, Y)$

Let $\theta(f, g)$ be the supremum semimetric on the space $\mathcal{B}(X, Y)$ of bounded mappings from X into Y with respect to $d_Y(\cdot, \cdot)$, as in the previous section. One can verify that the topology determined on $\mathcal{B}(X, Y)$ by $\theta(\cdot, \cdot)$ is the same as the topology induced on $\mathcal{B}(X, Y)$ by the topology of uniform convergence on Y^X.

One can also check that $\mathcal{B}(X, Y)$ is a closed set in Y^X with respect to the topology of uniform convergence. Equivalently, if a net of elements of $\mathcal{B}(X, Y)$ converges uniformly to a mapping f from X into Y, then f is bounded on X too.

6.2.8 Uniform convergence and continuity

Suppose that X is a topological space, and let $x_0 \in X$ be given. Also let $\{f_a\}_{a \in A}$ be a net of mappings from X into Y indexed by A that converges uniformly to a mapping f from X into Y. If, for each $a \in A$, f_a is continuous at x_0, then it is well known and not too difficult to show that f is continuous at x_0. Similarly, $C(X, Y)$ is a closed set in Y^X with respect to the topology of uniform convergence.

6.3 Infinite series of functions

Let X be a set, and let $a_1(x), a_2(x), a_3(x), \ldots$ be an infinite sequence of real-valued functions on X. Also let $A_1, A_2, A_3, \ldots$ be an infinite sequence of nonnegative real numbers such that

$$|a_j(x)| \le A_j \tag{6.3.1}$$

for every $j \geq 1$ and $x \in X$. Suppose that $\sum_{j=1}^{\infty} A_j$ converges as an infinite series of nonnegative real numbers, so that $\sum_{j=1}^{\infty} a_j(x)$ converges absolutely for every $x \in X$, by the comparison test.

Under these conditions, it is not too difficult to show that the sequence of partial sums $\sum_{j=1}^{n} a_j(x)$ converges to $\sum_{j=1}^{\infty} a_j(x)$ uniformly on X. This is a well-known criterion of Weierstrass.

Suppose now that X is a topological space. If $a_j(x)$ is continuous on X for each $j \geq 1$, with respect to the standard topology on $\mathbf{R}$, in addition to the conditions mentioned in the preceding paragraph, then it follows that $\sum_{j=1}^{\infty} a_j(x)$ is continuous on X as well.

6.3.1 Some continuous functions defined using infinite series

Suppose that X is normal in the strict sense, and let $E \subseteq X$ be a closed G_δ set. Thus there is an infinite sequence $U_1, U_2, U_3, \ldots$ of open subsets of X such that $E = \bigcap_{j=1}^{\infty} U_j$. Urysohn's lemma implies that for each $j \geq 1$ there is a continuous real-valued function f_j on X such that $0 \leq f_j(x) \leq 1$ for every $j \geq 1$ and $x \in X$, $f_j(x) = 0$ for every $j \geq 1$ and $x \in E$, and $f_j(x) = 1$ for every $j \geq 1$ and $x \in X \setminus U_j$.

Put

$$f(x) = \sum_{j=1}^{\infty} 2^{-j} f_j(x) \tag{6.3.2}$$

for every $x \in X$, where the series on the right converges by comparison with the convergent geometric series $\sum_{j=1}^{\infty} 2^{-j} = 1$. Note that $f(x) = 0$ for every $x \in E$, and that $0 \leq f(x) \leq 1$ for every $x \in X$, by construction. If $x \in X \setminus E$, then $x \in X \setminus U_j$ for some $j \geq 1$, which means that $f_j(x) = 1$. This implies that $f(x) > 0$ for every $x \in X \setminus E$. We also have that f is continuous on X, as before.

6.3.2 Perfect normality implies complete normality

Suppose that X is perfectly normal in the strict sense, and let A and B be separated subsets of X. The closures $\overline{A}$, $\overline{B}$ of A, B in X are closed sets in X, and G_δ sets in X too, by hypothesis. It follows that there are nonnegative continuous real-valued functions f_A, f_B on X such that $f_A(x) = 0$ if and only if $x \in A$, and $f_B(x) = 0$ if and only if $x \in B$, as in the preceding paragraph.

Put

$$U = \{x \in X : f_A(x) < f_B(x)\} \tag{6.3.3}$$

and

$$V = \{x \in X : f_B(x) < f_A(x)\}. \tag{6.3.4}$$

These are open subsets of X, because f_A and f_B are continuous on X. It is easy to see that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$, by construction. This shows that X is completely normal in the strict sense.

6.4 Tietze's extension theorem

Let X be a topological space that is normal in the strict sense, and let E be a closed set in X. Also let f be a continuous real-valued function on E, with respect to the induced topology on E and the standard topology on $\mathbf{R}$, that satisfies $|f(x)| \le 1$ for every $x \in E$. Under these conditions, *Tietze's extension theorem* implies that f can be extended to a continuous real-valued function on X that takes values in the interval $[-1, 1]$ as well.

6.4.1 An initial approximation

To see this, observe that

$$A_0 = \{x \in E : f(x) \le -1/3\} \tag{6.4.1}$$

and

$$B_0 = \{x \in E : f(x) \ge 1/3\} \tag{6.4.2}$$

are disjoint closed subsets of X. More precisely, A_0 and B_0 are closed subsets of E, with respect to the induced topology, because f is continuous on E. This implies that A_0 and B_0 are closed subsets of X, because E is a closed set in X.

Using Urysohn's lemma, one can find a continuous real-valued function g_0 on X such that $g_0 = -1/3$ on A_0, $g_0 = 1/3$ on B_0, and $|g_0| \le 1/3$ on X. If $x \in E$, then one can check that

$$|f(x) - g_0(x)| \le 2/3. \tag{6.4.3}$$

6.4.2 Some more precise approximations

Of course, $f - g_0$ defines a continuous real-valued function on E. One can repeat the process, to get a sequence $\{g_j\}_{j=0}^{\infty}$ of continuous real-valued functions on X such that $|g_j| \le (1/3)\,(2/3)^j$ on X for each $j \ge 0$, and

$$\left| f - \sum_{j=0}^{n} g_j \right| \le (2/3)^{n+1} \tag{6.4.4}$$

on E for every $n \ge 0$. In particular, $\sum_{j=0}^{\infty} g_j(x)$ converges as an infinite series of nonnegative real numbers for every $x \in X$, by comparison with the convergent series $\sum_{j=0}^{\infty} (1/3)\,(2/3)^j$.

More precisely, the sequence of partial sums $\sum_{j=0}^{n} g_j$ converges uniformly on X, by Weierstrass' criterion. This implies that $\sum_{j=0}^{\infty} g_j$ is continuous on X, as before. Note that

$$\left| \sum_{j=0}^{\infty} g_j(x) \right| \le \sum_{j=0}^{\infty} |g_j(x)| \le \sum_{j=0}^{\infty} (1/3)\,(2/3)^j = 1 \tag{6.4.5}$$

for every $x \in X$. If $x \in E$, then $\sum_{j=0}^{\infty} g_j(x) = f(x)$, by (6.4.4).

6.4.3 A refinement of this argument

If $|f(x)| < 1$ for every $x \in E$, then one might wish to have an extension of f to a continuous mapping from X into $(-1, 1)$. This can obtained by multiplying the previous extension by a suitable continuous real-valued function on X. More precisely, note that the set where the previous extension is ± 1 is a closed set in X. This set is disjoint from E, because $|f| < 1$ on E. Thus one can use Urysohn's lemma to find a continuous mapping from X into $[0, 1]$ that is equal to 1 on E and to 0 on the set where the previous extension is equal to ± 1.

6.4.4 Another version of Tietze's theorem

Now let f be any continuous real-valued function on E, with respect to the induced topology on E, and the standard topology on $\mathbf{R}$. Also let ϕ be a homeomorphism from the real line onto $(-1, 1)$, with respect to the standard topology on $\mathbf{R}$, and the induced topology on $(-1, 1)$. It follows that $\phi \circ f$ is a continuous mapping from E into $(-1, 1)$, which can be extended to a continuous mapping from X into $(-1, 1)$, as in the preceding paragraph. The composition of this extension with the inverse of ϕ is a continuous extension of f on X. This corresponds to Exercise 6 on p78 of [156].

6.5 Uniform convergence on compact sets

Let X and Y be sets, and let $d_Y(\cdot, \cdot)$ be a semimetric on Y. Also let E be a nonempty subset of X, let f be a mapping from X into Y, and let ϵ be a positive real number. Consider the set $N_{E,\epsilon}(f)$ of all mappings g from X into Y such that

$$d_Y(f(x), g(x)) < \epsilon \quad \text{for every } x \in E. \tag{6.5.1}$$

As before, let $\widetilde{N}_{E,\epsilon}(f)$ be the set of all mappings g from X into Y such that $d_Y(f(x), g(x))$ is bounded on E, with

$$\sup_{x \in E} d_Y(f(x), g(x)) < \epsilon. \tag{6.5.2}$$

Thus

$$\widetilde{N}_{E,\epsilon}(f) \subseteq N_{E,\epsilon}(f), \tag{6.5.3}$$

and

$$N_{E,\epsilon}(f) \subseteq \widetilde{N}_{E,\eta}(f) \tag{6.5.4}$$

for every $\eta > \epsilon$, as in Subsection 6.2.2.

6.5.1 More on $N_{E,\epsilon}(f)$ and $\widetilde{N}_{E,\epsilon}(f)$

If f, g, and h are mappings from X into Y such that $d_Y(f(x), g(x))$ and $d_Y(g(x), h(x))$ are bounded on E, then $d_Y(f(x), h(x))$ is bounded on E, and

$$\sup_{x \in E} d_Y(f(x), h(x)) \le \sup_{x \in E} d_Y(f(x), g(x)) + \sup_{x \in E} d_Y(g(x), h(x)), \tag{6.5.5}$$

as in Subsection 6.2.5. If $g \in \widetilde{N}_{E,\eta}(f)$ for some $\eta > 0$, then

$$\delta = \eta - \sup_{x \in E} d_Y(f(x), g(x)) > 0, \tag{6.5.6}$$

and

$$\widetilde{N}_{E,\delta}(g) \subseteq \widetilde{N}_{E,\eta}(f), \tag{6.5.7}$$

as before.

Let E_0 be a nonempty subset of X such that

$$E_0 \subseteq E. \tag{6.5.8}$$

If f is a mapping from X into Y, then

$$N_{E,\epsilon}(f) \subseteq N_{E_0,\epsilon}(f) \tag{6.5.9}$$

and

$$\widetilde{N}_{E,\epsilon}(f) \subseteq \widetilde{N}_{E_0,\epsilon}(f) \tag{6.5.10}$$

for every $\epsilon > 0$.

6.5.2 The topology of uniform convergence on compact sets

Suppose now that X is a nonempty topological space. A subset W of the space Y^X of all mappings from X into Y is said to be an open set with respect to the *topology of uniform convergence on compact subsets of* X if for every $f \in W$ there are a nonempty compact subset K of X and an $\epsilon > 0$ such that

$$N_{K,\epsilon}(f) \subseteq W. \tag{6.5.11}$$

This is equivalent to saying that for every $f \in W$ there are a nonempty compact set $K \subseteq X$ and an $\eta > 0$ such that

$$\widetilde{N}_{K,\epsilon}(f) \subseteq W. \tag{6.5.12}$$

One can check that this defines a topology on Y^X. More precisely, this uses the fact that the union of finitely many compact subsets of X is compact, to get that the intersection of finitely many open subsets of Y^X is an open set with respect to this topology.

6.5.3 Some properties of this topology

Of course, finite subsets of X are automatically compact. This implies that the topology of uniform convergence on compact subsets of X is at least as strong as the topology of pointwise convergence. If X is equipped with the discrete topology, then every compact subset of X is a finite set, and these two topologies on Y^X are the same.

The topology of uniform convergence on Y^X is at least as strong as the topology of uniform convergence on compact subsets of X. If X is compact, then these two topologies on Y^X are the same.

6.5.4 Convergent nets with respect to this topology

If K is a nonempty compact subset of X, f is a mapping from X into Y, and η is a positive real number, then $\widetilde{N}_{K,\eta}(f)$ is an open set in Y^X with respect to the topology of uniform convergence on compact subsets of X. This follows from (6.5.7).

Let $(A, \preceq)$ be a nonempty directed system, let $\{f_a\}_{a \in A}$ be a net of mappings from X into Y indexed by A, and let f be another mapping from X into Y. One can check that $\{f_a\}_{a \in A}$ converges to f with respect to the topology of uniform convergence on compact subsets of X if and only if for every compact subset K of X, the restriction of f_a to K converges uniformly to the restriction of f to K, as a net of mappings from K into Y indexed by A.

6.6 The compact-open topology

Let X and Y be nonempty topological spaces. If K is a subset of X and V is a subset of Y, then let $N(K, V)$ be the collection of mappings f from X into Y such that

$$f(K) \subseteq V. \tag{6.6.1}$$

A subset W of the space Y^X of all mappings from X into Y is said to be an open set with respect to the *compact-open topology* if there are finitely many compact subsets $K_1, \ldots, K_n$ of X and open sets $V_1, \ldots, V_n$ of Y such that

$$f(K_j) \subseteq V_j \tag{6.6.2}$$

for each $j = 1, \ldots, n$, and

$$\bigcap_{j=1}^{n} N(K_j, V_j) \subseteq W. \tag{6.6.3}$$

One can verify that this defines a topology on Y^X.

If K is a compact subset of X, and V is an open subset of Y, then $N(K, V)$ is an open subset of Y^X with respect to the compact-open topology. The collection of these open sets forms a subbase for the compact-open topology on Y^X.

6.6.1 Some properties of the compact-open topology

Remember that Y^X is the same as the Cartesian product of copies of Y indexed by X. It is easy to see that the compact-open topology on Y^X is at least as strong as the product topology, because finite subsets of X are compact. If X is equipped with the discrete topology, then one can check that the compact-open topology on Y^X is the same as the product topology, because compact subsets of X are finite. The strong product topology on Y^X is at least as strong as the compact-open topology.

6.6.2 The compact-open topology on $C(X, Y)$

Sometimes one may be particularly concerned with the topology induced on the space $C(X, Y)$ of continuous mappings from X into Y by the compact-open topology on Y^X. This may be called the *compact-open topology on* $C(X, Y)$. If K is a compact subset of X and V is an open subset of Y, then

$$N^C(K, V) = N(K, V) \cap C(X, Y) \tag{6.6.4}$$

is relatively open in $C(X, Y)$, and the collection of these relatively open subsets of $C(X, Y)$ is a subbase for the compact-open topology on $C(X, Y)$.

6.6.3 Using a semimetric on Y again

Let $d_Y(\cdot, \cdot)$ be a semimetric on Y, and suppose from now on in this section that Y is equipped with the topology determined by $d_Y(\cdot, \cdot)$. Under these conditions, it is well known that the compact-open topology on $C(X, Y)$ is the same as the topology induced by the topology of uniform convergence on compact subsets of X. Before showing this, let us mention a property of compact subsets of Y.

6.6.4 A property of compact subsets of Y

Let A be a compact subset of Y, and let V be an open subset of Y that contains K. It is well known that there is a positive real number r such that

$$\bigcup_{a \in A} B_Y(a, r) \subseteq V, \tag{6.6.5}$$

where $B_Y(a, r)$ is the open ball in Y centered at a with radius r with respect to $d_Y(\cdot, \cdot)$.

Indeed, if $a \in A$, then $a \in V$, and there is a positive real number $r(a)$ such that $B_Y(a, r(a)) \subseteq V$. The collection of open balls of the form $B_Y(a, r(a)/2)$, $a \in A$, is an open covering of A in Y, and so there are finitely many elements $a_1, \ldots, a_n$ of A such that

$$A \subseteq \bigcup_{j=1}^{n} B_Y(a_j, r(a_j)/2), \tag{6.6.6}$$

because A is compact. One can check that (6.6.5) holds with

$$r = \min_{1 \le j \le n} r(a_j)/2. \tag{6.6.7}$$

6.6.5 Some remarks about continuous mappings and compact sets

Let f be a continuous mapping from X into Y, let K be a nonempty compact subset of X, and let V be an open subset of Y such that $f(K) \subseteq V$. Because

$f(K)$ is a compact subset of Y, there is a positive real number r such that

$$\bigcup_{y \in f(K)} B_Y(y, r) \subseteq V, \tag{6.6.8}$$

as in the previous subsection. If g is a continuous mapping from X into Y such that

$$d_Y(f(x), g(x)) < r \quad \text{for every } x \in K, \tag{6.6.9}$$

then it follows that $g(K) \subseteq V$ too.

This shows that $N^C(K, V)$ is a open subset of $C(X, Y)$ with respect to the topology induced by the topology of uniform convergence on compact subsets of X. Using this, it is easy to see that the topology induced on $C(X, Y)$ by the topology of uniform convergence on compact subsets of X is at least as strong as the compact-open topology.

6.6.6 More on continuous mappings and compact sets

Let f be a continous mapping from X into Y again, let K be a nonempty compact subset of X, and let $\epsilon > 0$ be given. If $x \in K$, then there is an open subset $U(x)$ of X such that $x \in U(x)$ and

$$f(U(x)) \subseteq B_Y(f(x), \epsilon/3), \tag{6.6.10}$$

because f is continuous at x. The collection of open sets of this type, with $x \in K$, forms an open covering of K in X, and so there are finitely many elements $x_1, \ldots, x_n$ of K such that

$$K \subseteq \bigcup_{j=1}^{n} U(x_j), \tag{6.6.11}$$

because K is compact.

6.6.7 Finitely many smaller compact sets

Put

$$E_j = \{w \in X : d_Y(f(w), f(x_j)) \le \epsilon/3\} \tag{6.6.12}$$

for each $j = 1, \ldots, n$. One can check that this is a closed set in X for every $j = 1, \ldots, n$, because f is continuous. This implies that $K \cap E_j$ is a compact subset of X for each $j = 1, \ldots, n$. Note that $U(x_j) \subseteq E_j$ for every $j = 1, \ldots, n$, by (6.6.10). It follows that

$$K = \bigcup_{j=1}^{n} (K \cap E_j), \tag{6.6.13}$$

because of (6.6.11).

6.6.8 Using the compact-open topology

By construction,

$$f(K \cap E_j) \subseteq f(E_j) \subseteq B_Y(f(x_j), \epsilon/2) \tag{6.6.14}$$

for every $j = 1, \ldots, n$. Let g be any mapping from X into Y such that

$$g(K \cap E_j) \subseteq B_Y(f(x_j), \epsilon/2) \tag{6.6.15}$$

for each $j = 1, \ldots, n$. Let $w \in K$ be given, so that $w \in U(x_{j_0})$ for some $j_0 \in \{1, \ldots, n\}$, by (6.6.11). It follows that

$$d_Y(f(w), g(w)) \le d_Y(f(w), f(x_{j_0})) + d_Y(f(x_{j_0}), g(w)) < \epsilon/3 + \epsilon/2, \tag{6.6.16}$$

using (6.6.10) and (6.6.15) in the second step. This implies that the compact-open topology on $C(X, Y)$ is at least as strong as the topology induced by the topology of uniform convergence on compact subsets of X.

6.7 Continuity on compact sets

Let X and Y be topological spaces, and let f be a mapping from X into Y. If f is continuous on X, then the restriction of f to any subset X_0 of X is continuous as a mapping from X_0 into Y, with respect to the topology induced on X_0 by the topology on X. Similarly, if f is continuous at a point $x_0 \in X$, as a mapping on X, then the restriction of f to any subset X_0 of X that contains x_0 is continuous at x_0 too, with respect to the induced topology on X_0.

In some situations, one may be concerned with mappings on X whose restrictions to some subsets of X have some continuity properties, such as restrictions to compact subsets of X. One may also wish to obtain continuity properties on X from continuity properties of these restrictions to some subsets of X, under suitable conditions.

6.7.1 Local compactness and continuity on compact sets

Let x_0 be an element of X, let U_0 be an open subset of X that contains x_0, and let K_0 be a compact subset of X that contains x_0. If f is a mapping from X into Y, and the restriction of f to K_0 is continuous at x_0 with respect to the induced topology on K_0, then it is easy to see that f is continuous at x_0. In particular, if X is locally compact, and if the restriction of f to any compact subset K of X is continuous with respect to the induced topology on K, then f is continuous on X.

6.7.2 Some remarks about sequential continuity

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements of X that converges to an element x of X. Remember that $K = \{x_j : j \in \mathbf{Z}_+\} \cup \{x\}$ is a compact subset of X in this case. Note that $\{x_j\}_{j=1}^{\infty}$ converges to x with respect to the induced topology on K.

If the restriction of f to K is sequentially continuous at x, with respect to the induced topology on K, then $\{f(x_j)\}_{j=1}^{\infty}$ converges to $f(x)$ in Y. If the restriction of f to any compact subset of X that contains x is sequentially continuous at x, with respect to the induced topology, then it follows that f is sequentially continuous at x.

6.7.3 Using the first countability condition

If there is a local base for the topology of X at x with only finitely or countably many elements, then sequential continuity at x implies continuity at x. In this situation, we get that f is continuous at x when the restriction of f to any compact subset of X that contains x is continuous at x, with respect to the induced topology. If X satisfies the first countability condition, and if the restriction of f to any compact subset of X is continuous, with respect to the induced topology, then f is continuous on X.

6.7.4 Uniform convergence and continuity conditions

Let $d_Y(\cdot, \cdot)$ be a semimetric on Y, and suppose that Y is equipped with the topology determined by $d_Y(\cdot, \cdot)$. Also let $(A, \preceq)$ be a nonempty directed system, and let $\{f_a\}_{a \in A}$ be a net of mappings from X into Y indexed by A that converges to a mapping f from X into Y uniformly on compact subsets of X. Suppose that for each $a \in A$, f_a is continuous at a point x_0 in X. If K_0 is a compact subset of X that contains x_0, then the restriction of f_a to K_0 is continuous at x_0 for every $a \in A$, with respect to the induced topology on K_0. This implies that the restriction of f to K_0 is continuous at x_0 as well, with respect to the induced topology on K_0.

6.8 Supports of real-valued functions

Let X be a topological space, and let f be a real-valued function on X. The *support* of f in X is defined by

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}}, \tag{6.8.1}$$

which is the closure in X of the set where f is not 0.

If g is another real-valued function on X, then it is easy to see that

$$\operatorname{supp}(f+g) \subseteq (\operatorname{supp} f) \cup (\operatorname{supp} g) \tag{6.8.2}$$

and

$$\operatorname{supp}(f\, g) \subseteq (\operatorname{supp} f) \cap (\operatorname{supp} g). \tag{6.8.3}$$

6.8.1 Compactly-supported functions

In some situations, one may be concerned with continuous real-valued functions on X whose support is a compact subset of X. If f is a continuous real-valued function on X, then

$$\{x \in X : f(x) \neq 0\} \tag{6.8.4}$$

is an open set in X. If, for every $x \in X$, there is a continuous real-valued function f on X such that f has compact support in X and $f(x) \neq 0$, then X has to be locally compact.

6.8.2 Another version of Urysohn's lemma

Suppose that X is a locally compact Hausdorff space. If $K \subseteq X$ is compact, $U \subseteq X$ is an open set, and $K \subseteq U$, then a version of Urysohn's lemma implies that there is a continuous real-valued function f on X with compact support contained in U such that

$$f \equiv 1 \text{ on } K \text{ and } 0 \leq f \leq 1 \text{ on } X. \tag{6.8.5}$$

6.9 Functions with finite support

Let X be a nonempty set, and let $c(X)$ be the space of all real-valued functions on X. Let us say that $f \in c(X)$ has *finite support* in X if $f(x) = 0$ for all but finitely many $x \in X$. If we take X to be equipped with the discrete topology, then the support of $f \in c(X)$ in X as defined in the previous section is the same as the set of $x \in X$ such that $f(x) \neq 0$.

Let $c_{00}(X)$ be the subset of $c(X)$ consisting of functions on X with finite support. Note that $c_{00}(X) = c(X)$ exactly when X has finitely many elements.

6.9.1 Relation with Cartesian products

Equivalently, $c(X)$ is the same as the Cartesian product of copies of $\mathbf{R}$ indexed by X. Using the standard topology on the real line, we get the corresponding product topology and strong product topology on $c(X)$.

It is easy to see that $c_{00}(X)$ is dense in $c(X)$ with respect to the product topology. It is not too difficult to show that $c_{00}(X)$ is a closed set in $c(X)$ with respect to the strong product topology.

6.9.2 Using positive weights on X

Let a be a positive real-valued function on X. If $f, g \in c_{00}(X)$, then put

$$d_a(f, g) = \max\{a(x)\,|f(x) - g(x)| : x \in X\}. \tag{6.9.1}$$

Note that the maximum is automatically attained, because it reduces to the maximum of finitely many nonnegative real numbers. One can check that this defines a metric on $c_{00}(X)$.

It is easy to see that the topology determined on $c_{00}(X)$ by (6.9.1) is at least as strong as the topology induced on $c_{00}(X)$ by the product topology on $c(X)$. One can also verify that the topology induced on $c_{00}(X)$ by the strong product topology on $c(X)$ is at least as strong as the topology determined on $c_{00}(X)$ by (6.9.1).

If b is another positive real-valued function on X such that

$$a(x) \le b(x) \quad \text{for every } x \in X, \tag{6.9.2}$$

then

$$d_a(f,g) \le d_b(f,g) \tag{6.9.3}$$

for every $f, g \in c_{00}(X)$.

6.9.3 Another metric associated to a

Similarly, if $f, g \in c_{00}(X)$, then put

$$\widetilde{d}_a(f,g) = \sum_{x \in X} a(x)\,|f(x) - g(x)|, \tag{6.9.4}$$

where the sum on the right reduces to a finite sum of nonnegative real numbers. It is easy to see that this defines a metric on $c_{00}(X)$ as well.

Observe that

$$d_a(f,g) \le \widetilde{d}_a(f,g) \tag{6.9.5}$$

for every $f, g \in c_{00}(X)$. If b is another positive real-valued function on X that satisfies (6.9.2), then

$$\widetilde{d}_a(f,g) \le \widetilde{d}_b(f,g) \tag{6.9.6}$$

for every $f, g \in c_{00}(X)$.

6.9.4 The case where $X = \mathbf{Z}_+$

If $X = \mathbf{Z}_+$, and $\sum_{j=1}^{\infty} a(j)/b(j)$ converges as an infinite series of positive real numbers, then

$$\widetilde{d}_a(f,g) \le \Big(\sum_{j=1}^{\infty} a(j)/b(j)\Big)\, d_b(f,g) \tag{6.9.7}$$

for every $f, g \in c_{00}(\mathbf{Z}_+)$.

6.10 Some subspaces of $c(X)$

Let X be a nonempty set again, and let E be a subset of X. Put

$$c^E(X) = \{f \in c(X) : f(x) = 0 \text{ for every } x \in X \setminus E\} \tag{6.10.1}$$

and

$$c_{00}^E(X) = c_{00}(X) \cap c^E(X). \tag{6.10.2}$$

Of course, if E has only finitely many elements, then $c^E(X) \subseteq c_{00}(X)$, and $c^E(X) = c_{00}^E(X)$.

6.10.1 Relation with $c(E)$ and $c_{00}(E)$

Suppose that $E \neq \emptyset$, so that $c(E)$ and $c_{00}(E)$ can be defined as in the previous section. Of course, if $f \in c(X)$, then the restriction of f to E defines an element of $c(E)$. Similarly, if $f \in c_{00}(X)$, then the restriction of f to E defines an element of $c_{00}(E)$. Any real-valued function on E can be extended to a real-valued function on X, by setting it equal to 0 on $X \backslash E$. This defines a one-to-one mapping from $c(E)$ onto $c^E(X)$, which maps $c_{00}(E)$ onto $c_{00}^E(X)$.

6.10.2 Relation with Cartesian products again

Remember that $c(E)$ and $c(X)$ are the same as Cartesian products of copies of $\mathbf{R}$ indexed by E and X, respectively. This leads to the corresponding product topologies and strong product topologies on $c(E)$ and $c(X)$, using the standard topology on $\mathbf{R}$.

Note that $c^E(X)$ is a closed set in $c(X)$ with respect to the product topology. One can check that the product topology on $c(E)$ corresponds exactly to the topology induced on $c^E(X)$ by the product topology on $c(X)$, with respect to the one-to-one mapping from $c(E)$ onto $c^E(X)$ mentioned in the preceding paragraph.

Similarly, the strong product topology on $c(E)$ corresponds exactly to the topology induced on $c^E(X)$ by the strong product topology on $c(X)$, with respect to this mapping.

In particular, if E has only finitely many elements, then the product and strong product topologies on $c(E)$ are the same. In this case, the topologies induced on $c^E(X)$ by the product and strong product topologies on $c(X)$ are the same.

6.10.3 Using positive weights again

Let a be a positive real-valued function on X, and let $d_a(f, g)$ and $\widetilde{d}_a(f, g)$ be the metrics defined on $c_{00}(X)$ in (6.9.1) and (6.9.4), respectively. Suppose that E has only finitely many elements, and put

$$C(a, E) = \sum_{x \in E} a(x). \tag{6.10.3}$$

If $f, g \in c^E(X)$, then it is easy to see that

$$\widetilde{d}_a(f, g) \leq C(a, E)\, d_a(f, g). \tag{6.10.4}$$

In this situation, the topologies determined on $c^E(X)$ by the restrictions of $d_a(f, g)$ and $\widetilde{d}(f, g)$ to $f, g \in c^E(X)$ are the same as the topology induced on $c^E(X)$ by the product topology on $c(X)$.

6.11 Some mappings into products

Let A be a nonempty set, let Y_α be a set for each $\alpha \in A$, and put $Y = \prod_{\alpha \in A} Y_\alpha$. If $\beta \in A$, then let p_β be the corresponding coordinate mapping from Y into Y_β, so that $p_\beta(f) = f(\beta)$ for every $f \in Y$.

Let X be another set, and let ϕ be a mapping from X into Y. Thus, for each $\beta \in A$, $p_\beta \circ \phi$ is a mapping from X into Y_β. Conversely, given a mapping from X into Y_β for each $\beta \in A$, there is a unique mapping from X into Y whose composition with p_β is the given mapping from X into Y_β for every $\beta \in A$.

6.11.1 Continuous mappings into Y

Suppose now that Y_α is a topological space for every $\alpha \in A$, and that X is a topological space as well. If ϕ is a continuous mapping from X into Y, with respect to the corresponding product topology on Y, then $p_\beta \circ \phi$ is continuous as a mapping from X into Y_β for every $\beta \in A$. Conversely, if $p_\beta \circ \phi$ is continuous as a mapping from X into Y_β for every $\beta \in A$, then one can check that ϕ is continuous as a mapping from X into Y, with respect to the product topology on Y.

6.11.2 Mappings into $c_{00}(A)$

Let us take Y_α to be the real line for every $\alpha \in A$, so that Y is the same as the space $c(A)$ of all real-valued functions on A. Remember that $c_{00}(A)$ is the subset of $c(A)$ consisting of functions on A with finite support, as in Section 6.9. A mapping ϕ from X into $c(A)$ takes values in $c_{00}(A)$ if and only if for every $x \in X$,

$$p_\beta(\phi(x)) = 0 \quad \text{for all but finitely many } \beta \in A. \tag{6.11.1}$$

6.11.3 A local finiteness condition

Suppose that for every $x \in X$ there is an open set $U(x) \subseteq X$ and a finite subset $A(x)$ of A such that $x \in U(x)$ and

$$p_\beta(\phi(y)) = 0 \quad \text{for every } y \in U(x) \text{ and } \beta \in A \setminus A(x). \tag{6.11.2}$$

This is the same as saying that the family of subsets of X where $p_\beta \circ \phi \neq 0$ is locally finite in X, as in Section 5.13. In particular, this condition implies that ϕ maps X into $c_{00}(A)$.

Suppose also that $p_\beta \circ \phi$ is continuous as a real-valued function on X for every $\beta \in A$, with respect to the standard topology on $\mathbf{R}$. Under these conditions, one can verify that ϕ is continuous as a mapping from X into $c_{00}(A)$, with respect to the topology induced on $c_{00}(A)$ by the corresponding strong product topology on $c(A)$, as in the previous sections.

6.11.4 Continuity of ϕ

More precisely, if B is a subset of A, then let $c^B(A)$ be the space of $f \in c(A)$ such that $f(\alpha) = 0$ for every $\alpha \in A \setminus B$, as in the previous section. If B has only finitely many elements, then the topologies induced on $c^B(A)$ by the product and strong product topologies on $c(A)$ are the same, as before. Using this notation, (6.11.2) says that $\phi(y)$ is an element of $c^{A(x)}(A)$ for every $y \in U(x)$.

Let a be a positive real-valued function on A, and remember that

$$\widetilde{d}_a(f, g) = \sum_{\alpha \in A} a(\alpha)\, |f(\alpha) - g(\alpha)| \tag{6.11.3}$$

defines a metric on $c_{00}(A)$, as in Subsection 6.9.3. If $B \subseteq A$ has only finitely many elements, then the topology determined on $c^B(A)$ by the restriction of (6.11.3) to $f, g \in c^B(A)$ is the same as the topology induced on $c^B(A)$ by the product topology on $c(A)$, as in the previous section.

Suppose as before that for every $x \in X$ there is an open set $U(x) \subseteq X$ and a finite set $A(x) \subseteq A$ such that (6.11.2) holds, and that $p_\beta \circ \phi$ is continuous as a real-valued function on X for every $\beta \in A$. Under these conditions, ϕ is continuous as a mapping from X into $c_{00}(A)$, with respect to the topology determined on $c_{00}(A)$ by (6.11.3).

Chapter 7

Filters and ultrafilters

7.1 Filters

Let X be a nonempty set. A nonempty collection $\mathcal{F}$ of nonempty subsets of X is said to be a *filter* on X if it satisfies the following two conditions. First,

$$\text{if } A, B \in \mathcal{F}, \text{ then } A \cap B \in \mathcal{F}. \tag{7.1.1}$$

Second,

$$\text{if } A \in \mathcal{F},\ E \subseteq X, \text{ and } A \subseteq E, \text{ then } E \in \mathcal{F}. \tag{7.1.2}$$

7.1.1 Some basic examples of filters

If X_0 is a nonempty subset of X, then the collection of all subsets of X that contain X_0 is a filter on X.

If X has infinitely many elements, then one can check that the collection of all subsets A of X such that $X \setminus A$ has only finitely many elements is a filter on X.

If X is a topological space and $x \in X$, then

$$\{A \subseteq X : \text{there is an open set } U \subseteq X \text{ such that } x \in U \text{ and } U \subseteq A\} \tag{7.1.3}$$

is a filter on X. In this situation, a filter $\mathcal{F}$ on X is said to *converge* to x if for every open subset U of X with $x \in U$, we have that

$$U \in \mathcal{F}. \tag{7.1.4}$$

7.1.2 Pre-filters or filter bases

Let $\mathcal{B}$ be a nonempty collection of nonempty subsets of a nonempty set X. We say that $\mathcal{B}$ is a *pre-filter* or a *filter base* on X if

$$\text{for every } A, B \in \mathcal{B} \text{ there is a } C \in \mathcal{B} \text{ such that } C \subseteq A \cap B. \tag{7.1.5}$$

In this case, one can check that

$$\mathcal{F} = \{E \subseteq X : \text{there is an } A \in \mathcal{B} \text{ such that } A \subseteq E\} \tag{7.1.6}$$

is a filter on X.

Conversely, if there is a filter $\mathcal{F}$ on X such that (7.1.6) holds, then it is easy to see that $\mathcal{B}$ is a filter base on X, and $\mathcal{B}$ is said to be a *base* for $\mathcal{F}$.

If X is a topological space and $x \in X$, then a local base for the topology of X at x is the same as a base for the filter (7.1.3).

7.1.3 Filter bases from nets

Let $(A, \preceq)$ be a nonempty directed system, and let $\{x_a\}_{a \in A}$ be a net of elements of X indexed by A. If $a \in A$, then put

$$E_a = \{x_b : b \in A,\ a \preceq b\}. \tag{7.1.7}$$

It is easy to see that the collection of these subsets E_a, $a \in A$, of X form a filter base on X, and we let $\mathcal{F}$ be the corresponding filter on X.

Suppose that X is a topological space, and let x be an element of X. One can check that $\{x_a\}_{a \in A}$ converges to x as a net of elements of X if and only if the corresponding filter $\mathcal{F}$ converges to x in X.

7.1.4 Some nets related to filters

Let $\mathcal{F}$ be a filter on a nonempty set X, and let $\mathcal{B}$ be a base for $\mathcal{F}$. If $A, B \in \mathcal{B}$, then put $A \preceq B$ when $B \subseteq A$. This defines a partial ordering on $\mathcal{B}$, which makes $\mathcal{B}$ a directed system, because of (7.1.5).

Suppose that X is a topological space again, and that $x \in X$. If $\mathcal{F}$ converges to x on X, and if $\{x_A\}_{A \in \mathcal{B}}$ is a net of elements of X indexed by $\mathcal{B}$ such that

$$x_A \in A \tag{7.1.8}$$

for every $A \in \mathcal{B}$, then $\{x_A\}_{A \in \mathcal{B}}$ converges to x in X.

Conversely, if $\mathcal{F}$ does not converge to x in X, then there is an open set $U \subseteq X$ such that $x \in U$ and $U \not\in \mathcal{F}$. This implies that for each $A \in \mathcal{B}$, $A \not\subseteq U$. If $x_A \in A \setminus U$ for every $A \in \mathcal{B}$, then $\{x_A\}_{A \in \mathcal{B}}$ does not converge to x, as a net of elements of X.

7.1.5 Uniqueness of the limit in Hausdorff spaces again

If X is a Hausdorff topological space, then a filter $\mathcal{F}$ on X can converge to at most one element of X. Indeed, suppose for the sake of a contradiction that $\mathcal{F}$ converges to distinct elements x and y of X. Because X is Hausdorff, there are disjoint open subsets U and V of X such that $x \in U$ and $y \in V$. It follows that $U, v \in \mathcal{F}$, so that $\emptyset = U \cap V \in \mathcal{F}$, which is a contradiction.

Conversely, if X is not Hausdorff, then there are distinct elements x, y of X such that for all open subsets U and V of X with $x \in U$ and $y \in V$, we have that $U \cap V \neq \emptyset$. Observe that the collection of subsets of X of the form $U \cap V$, where $U, V \subseteq X$ are open sets with $x \in U$ and $y \in V$, is a filter base on X. It is easy to see that the corresponding filter on X converges to both x and y.

7.1.6 Adherent points and filters

Let X be a topological space, let $\mathcal{F}$ be a filter on X that converges to x, and let E be an element of $\mathcal{F}$. If U is an open subset of X that contains x, then $U \in \mathcal{F}$, and hence

$$E \cap U \in \mathcal{F}. \tag{7.1.9}$$

This implies that $E \cap U \neq \emptyset$, so that

$$x \in \overline{E}. \tag{7.1.10}$$

Conversely, suppose that $x \in X$ is adherent to $E \subseteq X$, and let $\mathcal{B}(x)$ be a local base for the topology of X at x. One can check that

$$\{E \cap U : U \in \mathcal{B}(x)\} \tag{7.1.11}$$

is a filter base on X, for which the corresponding filter on X converges to x and contains E as an element.

7.1.7 Filters and induced topologies

Let $\mathcal{F}$ be a filter on a nonempty set X. If $Y \in \mathcal{F}$, then

$$\mathcal{F}_Y = \{A \in \mathcal{F} : A \subseteq Y\} \tag{7.1.12}$$

may be considered as a filter on Y. Note that any filter on a nonempty subset of X may be considered as a filter base on X.

Suppose now that X is a topological space, $Y \in \mathcal{F}$, and $x \in Y$. Under these conditions, one can check that $\mathcal{F}$ converges to x if and only if (7.1.12) converges to x with respect to the induced topology on Y.

7.2 Continuity and refinements

Let X and Y be nonempty sets, and let f be a mapping from X into Y. If $\mathcal{F}$ is a filter on X, then one can check that

$$\{f(A) : A \in \mathcal{F}\} \tag{7.2.1}$$

is a filter base on Y. More precisely, this uses the fact that if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and

$$f(A \cap B) \subseteq f(A) \cap f(B). \tag{7.2.2}$$

7.2.1 Pushing filters forward

Let $f_*(\mathcal{F})$ be the filter on Y obtained from (7.2.1) as in the previous section. Equivalently, one can verify that

$$f_*(\mathcal{F}) = \{E \subseteq Y : f^{-1}(E) \in \mathcal{F}\}. \tag{7.2.3}$$

Suppose now that X and Y are topological spaces. If $\mathcal{F}$ converges to $x \in X$, and if f is continuous at x, then one can check that $f_*(\mathcal{F})$ converges to $f(x)$ in Y.

Conversely, if $\mathcal{F}$ is the filter (7.1.3) obtained from the open sets in X that contain x, and if $f_*(\mathcal{F})$ converges to $f(x)$ in Y, then it is easy to see that f is continuous at x.

7.2.2 Refinements of filters

Let X be a nonempty set again, and let $\mathcal{F}$ be a filter on X. A filter $\mathcal{F}'$ on X is said to be a *refinement* of $\mathcal{F}$ if

$$\mathcal{F} \subseteq \mathcal{F}', \tag{7.2.4}$$

as collections of subsets of X.

If X is a topological space, $x \in X$, and $\mathcal{F}$ is a filter on X that converges to x, then every filter on X that is a refinement of $\mathcal{F}$ converges to x too. Note that a filter $\mathcal{F}$ on X converges to x exactly when $\mathcal{F}$ is a refinement of the filter (7.1.3) obtained from the open subsets of X that contain x.

7.2.3 Convergent refinements

Suppose that a filter $\mathcal{F}$ on X has a refinement $\mathcal{F}'$ that converges to $x \in X$. If $E \in \mathcal{F}'$, then $x \in \overline{E}$, as in the previous section. In particular, this holds for $E \in \mathcal{F}$, so that

$$x \in \bigcap_{E \in \mathcal{F}} \overline{E}. \tag{7.2.5}$$

Conversely, suppose that (7.2.5) holds, and let $\mathcal{B}(x)$ be a local base for the topology of X at x. One can verify that

$$\{E \cap U : E \in \mathcal{F},\, U \in \mathcal{B}(x)\} \tag{7.2.6}$$

is a filter base on X, for which the corresponding filter is a refinement of $\mathcal{F}$ that converges to x.

7.2.4 Filters and compact sets

Let K be a compact subset of X, and let $\mathcal{F}$ be a filter on X such that $K \in \mathcal{F}$. If $E_1, \ldots, E_n$ are finitely many elements of $\mathcal{F}$, then

$$\Big(\bigcap_{j=1}^{n} E_j\Big) \cap K \in \mathcal{F}, \tag{7.2.7}$$

so that $\left(\bigcap_{j=1}^n E_j\right) \cap K \neq \emptyset$. In particular,

$$\left(\bigcap_{j=1}^n \overline{E_j}\right) \cap K \neq \emptyset, \tag{7.2.8}$$

which means that the collection of closed sets $\overline{E}$, $E \in \mathcal{F}$, has the finite intersection property with respect to K. It follows that

$$\left(\bigcap_{E \in \mathcal{F}} \overline{E}\right) \cap K \neq \emptyset, \tag{7.2.9}$$

because K is compact. This implies that $\mathcal{F}$ has a refinement that converges to an element of K in X, as in the preceding paragraph.

7.2.5 Filter bases and the finite intersection property

Let X be a set, let K be a subset of X, and let $\{E_j\}_{j \in I}$ be a nonempty family of subsets of X with the finite intersection property with respect to K. Consider the collection of subsets of X of the form

$$\left(\bigcap_{l=1}^n E_{j_l}\right) \cap K, \tag{7.2.10}$$

where $j_1, \ldots, j_n$ are finitely many elements of I. It is easy to see that this collection is a filter base on X.

Suppose that X is a topological space, and that $K \subseteq X$ has the property that every filter on X that contains K as an element has a refinement that converges to an element of K. In order to show that K is compact, let $\{E_j\}_{j \in I}$ be any nonempty family of closed subsets of X with the finite intersection property with respect to K. Consider the filter $\mathcal{F}$ on X for which the collection of sets of the form (7.2.10) is a base. By construction, $K \in \mathcal{F}$, so that $\mathcal{F}$ has a refinement that converges to an element x of K. This implies that

$$x \in \left(\bigcap_{j \in I} E_j\right) \cap K, \tag{7.2.11}$$

by (7.2.5), so that K is compact, as desired.

7.3 Ultrafilters

A filter $\mathcal{F}$ on a nonempty set X is said to be an *ultrafilter* if $\mathcal{F}$ is maximal with respect to refinement. This means that if $\mathcal{F}'$ is a filter on X that is a refinement of $\mathcal{F}$, then $\mathcal{F} = \mathcal{F}'$.

7.3.1 Some ultrafilters

If p is any element of X, then it is easy to see that the collection of all subsets of X that contain p as an element is an ultrafilter on X.

If $\mathcal{F}$ is any filter on X, then one can use Zorn's lemma or Hausdorff's maximality principle to show that $\mathcal{F}$ has a refinement which is an ultrafilter.

7.3.2 Compactness and ultrafilters

Let X be a topological space. If $K \subseteq X$ is compact and $\mathcal{F}$ is a filter on X that contains K as an element, then there is a refinement of $\mathcal{F}$ that converges to an element of K, as in the previous section. If $\mathcal{F}$ is an ultrafilter, then it follows that $\mathcal{F}$ converges to an element of K.

Conversely, suppose that $K \subseteq X$ has the property that every ultrafilter on X that contains K as an element converges to an element of K. If $\mathcal{F}$ is any filter on X that contains K, then $\mathcal{F}$ has a refinement that is an ultrafilter, as before. This refinement contains K, and thus converges to an element of K, by hypothesis. This implies that K is compact, as in the previous section.

7.3.3 Some refinements of filters

Let $\mathcal{F}$ be a filter on a nonempty set X, and suppose that $E \subseteq X$ has the property that $A \cap E \neq \emptyset$ for every $A \in \mathcal{F}$. It is easy to see that

$$\{A \cap E : A \in \mathcal{F}\} \tag{7.3.1}$$

is a filter base on X.

The filter on X for which (7.3.1) is a base is a refinement of $\mathcal{F}$ that contains E as an element. If $\mathcal{F}$ is an ultrafilter on X, then it follows that $E \in \mathcal{F}$.

7.3.4 A characterization of ultrafilters

If $\mathcal{F}$ is an ultrafilter on X and $E \subseteq X$, then either E or $X \setminus E$ is an element of $\mathcal{F}$. More precisely, if $X \setminus E$ is not an element of $\mathcal{F}$, then no element A of $\mathcal{F}$ is contained in $X \setminus E$. This means that every element A of $\mathcal{F}$ intersects E, so that E is an element of $\mathcal{F}$, as in the preceding paragraph.

Conversely, if $\mathcal{F}$ is a filter on X, and for every $E \subseteq X$, either E or $X \setminus E$ is an element of $\mathcal{F}$, then one can verify that $\mathcal{F}$ is an ultrafilter on X.

7.3.5 Pushing ultrafilters forward

Let f be a mapping from X into a nonempty set Y. If $\mathcal{F}$ is an ultrafilter on X, then the corresponding filter $f_*(\mathcal{F})$ on Y defined in the previous section is an ultrafilter. To see this, let $E \subseteq Y$ be given, and let us check that E or $Y \setminus E$ is an element of $f_*(\mathcal{F})$. This is the same as saying that $f^{-1}(E)$ or $f^{-1}(Y \setminus E)$ is an element of $\mathcal{F}$, by (7.2.3). This holds because $f^{-1}(Y \setminus E) = X \setminus f^{-1}(E)$, as in the previous subsection.

7.3.6 Filters and the product topology

Let I be a nonempty set, and let X_j be a nonempty topological space for every $j \in I$. Also let $\mathcal{F}$ be a filter on the Cartesian product $\prod_{j \in I} X_j$, and let x be an element of $\prod_{j \in I} X_j$. One can check that $\mathcal{F}$ converges to x with respect to the product topology on $\prod_{j \in I} X_j$ if and only if $(p_l)_*(\mathcal{F})$ converges to $p_l(x)$ on X_l for every $l \in I$, where p_l is the usual coordinate mapping from $\prod_{j \in I} X_j$ into X_l. Of course, the "only if" part follows from the continuity of p_l.

7.3.7 Tychonoff's theorem using ultrafilters

If $K_j \subseteq X_j$ is compact for every $j \in I$, then Tychonoff's theorem says that $\prod_{j \in I} K_j$ is compact in $\prod_{j \in I} X_j$ with respect to the product topology. To show this, let $\mathcal{F}$ be an ultrafilter on $\prod_{j \in I} X_j$ that contains $\prod_{j \in I} K_j$ as an element. If $l \in I$, then K_l is an element of $(p_l)_*(\mathcal{F})$. This implies that $(p_l)_*(\mathcal{F})$ converges to an element of K_l, because $(p_l)_*(\mathcal{F})$ is an ultrafilter on X_l. One can use this to get that $\mathcal{F}$ converges to an element of $\prod_{j \in I} K_j$.

7.4 Filters and relations

Let I be a nonempty set, and let X_j be a set for each $j \in I$. Also let $\mathcal{F}$ be a filter on I. If $f, g \in \prod_{j \in I} X_j$, then put

$$f \sim_{\mathcal{F}} g \tag{7.4.1}$$

when

$$\{j \in I : f(j) = g(j)\} \in \mathcal{F}. \tag{7.4.2}$$

7.4.1 An equivalence relation associated to the filter $\mathcal{F}$

Note that $f \sim_{\mathcal{F}} f$ automatically, because $I \in \mathcal{F}$. Clearly (7.4.1) is symmetric in f and g. If $h \in \prod_{j \in I} X_j$ too, then

$$\{j \in I : f(j) = g(j)\} \cap \{j \in I : g(j) = h(j)\} \subseteq \{j \in I : f(j) = h(j)\}. \tag{7.4.3}$$

If (7.4.1) holds and $g \sim_{\mathcal{F}} h$, then it follows that $f \sim_{\mathcal{F}} h$.

7.4.2 The corresponding quotient space

Thus (7.4.1) defines an equivalence relation on $\prod_{j \in I} X_j$. Let

$$\prod_{j \in I} X_j / \sim_{\mathcal{F}} \tag{7.4.4}$$

be the corresponding quotient space of equivalence classes in $\prod_{j \in I} X_j$ with respect to (7.4.1), as in Section 5.6. If $f \in \prod_{j \in I} X_j$, then the equivalence class containing f may be denoted $[f]_{\mathcal{F}}$.

7.4.3 The case where $\mathcal{F}$ is an ultrafilter

If $f, g \in \prod_{j \in I} X_j$, then

$$\{j \in I : f(j) = g(j)\} \cup \{j \in I : f(j) \neq g(j)\} = I. \tag{7.4.5}$$

If $\mathcal{F}$ is an ultrafilter on I, then either (7.4.1) holds, or

$$\{j \in I : f(j) \neq g(j)\} \in \mathcal{F}. \tag{7.4.6}$$

7.4.4 Some related partial orderings

Let $\preceq_j$ be a partial ordering on X_j for every $j \in I$. If $f, g \in \prod_{j \in I} X_j$, then put

$$f \preceq_{\mathcal{F}} g \tag{7.4.7}$$

when

$$\{j \in I : f(j) \preceq_j g(j)\} \in \mathcal{F}. \tag{7.4.8}$$

Of course,

$$\{j \in I : f(j) = g(j)\} = \{j \in I : f(j) \preceq_j g(j)\} \cap \{j \in I : g(j) \preceq_j f(j)\}, \tag{7.4.9}$$

because $\preceq_j$ is a partial ordering on X_j for each $j \in I$. This implies that (7.4.1) holds exactly when (7.4.7) and $g \preceq_{\mathcal{F}} f$ hold. In particular, $f \preceq_{\mathcal{F}} f$ holds automatically.

If $h \in \prod_{j \in I} X_j$ as well, then

$$\begin{aligned} \{j \in I : f(j) \preceq_j g(j)\} \cap \{j \in I : g(j) \preceq_j h(j)\} \\ \subseteq \quad \{j \in I : f(j) \preceq_j h(j)\}, \end{aligned} \tag{7.4.10}$$

because $\preceq_j$ is transitive on X_j for each $j \in I$. If (7.4.7) holds and $g \preceq_{\mathcal{F}} h$, then we get that $f \preceq_{\mathcal{F}} h$. This means that $\preceq_{\mathcal{F}}$ is transitive as a binary relation on $\prod_{j \in I} X_j$, so that it defines a pre-order on $\prod_{j \in I} X_j$.

Suppose that $\widetilde{f}, \widetilde{g} \in \prod_{j \in I} X_j$ satisfy

$$f \sim_{\mathcal{F}} \widetilde{f}, \quad g \sim_{\mathcal{F}} \widetilde{g}. \tag{7.4.11}$$

Under these conditions, it is easy to see that (7.4.7) holds if and only if $\widetilde{f} \preceq_{\mathcal{F}} \widetilde{g}$.

If (7.4.7) holds, then put

$$[f]_{\mathcal{F}} \preceq_{\mathcal{F}} [g]_{\mathcal{F}}. \tag{7.4.12}$$

This does not depend on the particular choices f and g of the equivalence classes, by the previous remark. This defines $\preceq_{\mathcal{F}}$ as a binary relation on the quotient space (7.4.4). We use the same notation as for the binary relation (7.4.7) on $\prod_{j \in I} X_j$, for simplicity.

Observe that (7.4.12) is reflexive and transitive on the quotient space (7.4.4), because of the analogous properties of (7.4.7) on $\prod_{j \in I} X_j$.

If (7.4.12) and $[g]_{\mathcal{F}} \preceq_{\mathcal{F}} [f]_{\mathcal{F}}$ hold, then (7.4.7) and $g \preceq_{\mathcal{F}} f$ hold. This implies that (7.4.1) holds, as before, so that $[f]_{\mathcal{F}} = [g]_{\mathcal{F}}$. Thus (7.4.12) defines a partial ordering on the quotient space (7.4.4).

Suppose that X_j is linearly ordered by $\preceq_j$ for each $j \in I$. If $f, g \in \prod_{j \in I} X_j$, then it follows that

$$\{j \in I : f(j) \preceq_j g(j)\} \cup \{j \in I : g(j) \preceq_j f(j)\} = I. \tag{7.4.13}$$

If $\mathcal{F}$ is an ultrafilter on I, then either (7.4.7) holds, or $g \preceq_{\mathcal{F}} f$. This means that either (7.4.12) holds, or $[g]_{\mathcal{F}} \preceq_{\mathcal{F}} [f]_{\mathcal{F}}$. This implies that the quotient space (7.4.4) is linearly ordered by (7.4.12) in this case.

Chapter 8

Families and coverings

8.1 Coverings and refinements

Let X be a set, and let E be a subset of X. Also let A be a nonempty set, and let U_α be a subset of X for every $\alpha \in A$. As usual, $\{U_\alpha\}_{\alpha \in A}$ is said to be a *covering* of E in X if $E \subseteq \bigcup_{\alpha \in A} U_\alpha$. If A_0 is a subset of A and

$$E \subseteq \bigcup_{\alpha \in A_0} U_\alpha, \tag{8.1.1}$$

then $\{U_\alpha\}_{\alpha \in A_0}$ is said to be a *subcovering* of E from $\{U_\alpha\}_{\alpha \in A}$.

8.1.1 Refinements of families of sets

Let B be another nonempty set, and let $\{V_\beta\}_{\beta \in B}$ be a family of subsets of X indexed by B. We say that $\{V_\beta\}_{\beta \in B}$ is a *refinement* of $\{U_\alpha\}_{\alpha \in A}$ if for every $\beta \in B$ there is an $\alpha \in A$ such that

$$V_\beta \subseteq U_\alpha. \tag{8.1.2}$$

Of course, this implies that

$$\bigcup_{\beta \in B} V_\beta \subseteq \bigcup_{\alpha \in A} U_\alpha. \tag{8.1.3}$$

If $\{U_\alpha\}_{\alpha \in A}$ covers E, then we may be particularly interested in refinements that cover E too.

If $A_0 \subseteq A$, then $\{U_\alpha\}_{\alpha \in A_0}$ may be considered as a refinement of $\{U_\alpha\}_{\alpha \in A}$.

8.1.2 Refinements of refinements

Let C be another nonempty set, and let $\{W_\gamma\}_{\gamma \in C}$ be a family of subsets of X indexed by C. If $\{V_\beta\}_{\beta \in B}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$, and $\{W_\gamma\}_{\gamma \in C}$ is a refinement of $\{V_\beta\}_{\beta \in B}$, then it is easy to see that $\{W_\gamma\}_{\gamma \in C}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$.

8.1.3 Some subfamilies associated to refinements

Suppose that $\{V_\beta\}_{\beta \in B}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$ again. Thus, for each $\beta \in B$, we can choose an $\alpha(\beta) \in A$ such that $V_\beta \subseteq U_{\alpha(\beta)}$. Let

$$A_1 = \{\alpha(\beta) : \beta \in B\} \tag{8.1.4}$$

be the set of indices in A that have been chosen in this way. By construction, $A_1 \subseteq A$, and

$$\bigcup_{\beta \in B} V_\beta \subseteq \bigcup_{\beta \in B} U_{\alpha(\beta)} = \bigcup_{\alpha \in A_1} U_\alpha. \tag{8.1.5}$$

If $\{V_\beta\}_{\beta \in B}$ covers E, then $\{U_\alpha\}_{\alpha \in A}$ covers E, and $\{U_\alpha\}_{\alpha \in A_1}$ is a subcovering of E from $\{U_\alpha\}_{\alpha \in A}$.

We also have that

$$\# A_1 \le \# B. \tag{8.1.6}$$

In particular, if B has only finitely many elements, or only finitely or countably many elements, then A_1 has the same property.

8.1.4 Some common refinements

Let $\{U_\alpha\}_{\alpha \in A}$ and $\{V_\beta\}_{\beta \in B}$ be arbitrary families of subsets of X, not necessarily related by refinement. It is easy to see that

$$\{U_\alpha \cap V_\beta\}_{(\alpha,\beta) \in A \times B} \tag{8.1.7}$$

is a refinement of both $\{U_\alpha\}_{\alpha \in A}$ and $\{V_\beta\}_{\beta \in B}$.

Observe that

$$\Big(\bigcup_{\alpha \in A} U_\alpha\Big) \cap \Big(\bigcup_{\beta \in B} V_\beta\Big) = \bigcup_{\alpha \in A} \bigcup_{\beta \in B} (U_\alpha \cap V_\beta). \tag{8.1.8}$$

If $\{U_\alpha\}_{\alpha \in A}$ and $\{V_\beta\}_{\beta \in B}$ cover E, then it follows that (8.1.7) covers E too.

8.2 Open coverings

Let X be a topological space, and let $\mathcal{B}$ be a base for the topology of X. Also let $\{U_\alpha\}_{\alpha \in A}$ be a family of open subsets of X. If $\alpha \in A$, then put

$$\mathcal{B}_\alpha = \{V \in \mathcal{B} : V \subseteq U_\alpha\}. \tag{8.2.1}$$

Thus

$$U_\alpha = \bigcup_{V \in \mathcal{B}_\alpha} V, \tag{8.2.2}$$

because $\mathcal{B}$ is a base for the topology of X.

Put

$$\widetilde{\mathcal{B}} = \bigcup_{\alpha \in A} \mathcal{B}_\alpha, \tag{8.2.3}$$

so that

$$\bigcup_{V \in \widetilde{\mathcal{B}}} V = \bigcup_{\alpha \in A} \bigcup_{V \in \mathcal{B}_\alpha} V = \bigcup_{\alpha \in A} U_\alpha. \tag{8.2.4}$$

Note that $\widetilde{B}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$, by construction. If $\{U_\alpha\}_{\alpha \in A}$ is an open covering of a subset E of X, then $\widetilde{\mathcal{B}}$ is an open covering of E in X too.

8.2.1 Another look at Lindelöf's theorem

This type of refinement was already used in the proof of Lindelöf's theorem, Theorem 3.6.6. More precisely, if $\{U_\alpha\}_{\alpha \in A}$ is any nonempty family of open subsets of X, then $\{U_\alpha\}_{\alpha \in A}$ may be considered as a covering of its union $E = \bigcup_{\alpha \in A} U_\alpha$.

As in the previous section, there is a subset A_1 of A that satisfies (8.1.5) and (8.1.6). In this situation, (8.1.5) implies that

$$\bigcup_{\alpha \in A_1} U_\alpha = \bigcup_{\alpha \in A} U_\alpha, \tag{8.2.5}$$

because the left side is automatically contained in the right side. If $\mathcal{B}$ has only finitely or countably many elements, then (8.1.6) implies that A_1 has only finitely or countably many elements, as before.

8.2.2 Another look at compactness with respect to a base

This type of refinement was used in the proof of Proposition 3.9.2 as well. In that case, compactness of $E \subseteq X$ with respect to $\mathcal{B}$ implies that there is a finite subcover of E from $\widetilde{\mathcal{B}}$. This finite subcovering from $\widetilde{\mathcal{B}}$ may be considered as a refinement of $\{U_\alpha\}_{\alpha \in A}$, which leads to a finite subcovering of E from $\{U_\alpha\}_{\alpha \in A}$, as before.

8.2.3 A theorem of Lebesgue

Let $d(x, y)$ be a semimetric on X, and suppose now that X is equipped with the topology determined by $d(\cdot, \cdot)$. Suppose that $E \subseteq X$ is compact, and let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of E in X.

A well-known result going back to Lebesgue implies that there is a positive real number r such that for each $x \in E$ there is an $\alpha \in A$ with

$$B(x, r) \subseteq U_\alpha. \tag{8.2.6}$$

This means that the covering of E by the open balls $B(x, r)$ in X centered at elements of E with radius r is a refinement of $\{U_\alpha\}_{\alpha \in A}$.

8.2.4 The proof of Lebesgue's theorem

To see this, let $x \in E$ be given, so that there is an index $\alpha_0(x) \in A$ such that $x \in U_{\alpha_0(x)}$. This implies that there is a positive real number $r_0(x)$ such that

$$B(x, r_0(x)) \subseteq U_{\alpha_0(x)}, \tag{8.2.7}$$

because $U_{\alpha_0(x)}$ is an open set in X.

Consider the covering of E by open balls of the form $B(x, r_0(x)/2)$, where $x \in E$ and $r_0(x) > 0$ is as before. If E is compact, then there are finitely many elements $x_1, \ldots, x_n$ of E such that

$$E \subseteq \bigcup_{j=1}^{n} B(x_j, r_0(x_j)/2). \tag{8.2.8}$$

Put

$$r = \min_{1 \le j \le n} r_0(x_j)/2. \tag{8.2.9}$$

If $x \in E$, then $x \in B(x_j, r_0(x_j)/2)$ for some $j = 1, \ldots, n$, by (8.2.8). This implies that

$$B(x, r) \subseteq B(x_j, r_0(x_j)/2 + r) \subseteq B(x_j, r_0(x_j)) \subseteq U_{\alpha_0(x_j)}, \tag{8.2.10}$$

using the triangle inequality in the first step.

8.3 Paracompactness

Let X be a topological space.

8.3.1 Paracompactness in the strict sense

Let us say that X is *paracompact in the strict sense* if for every open covering $\{U_\alpha\}_{\alpha \in A}$ of X there is a refinement $\{V_\beta\}_{\beta \in B}$ of $\{U_\alpha\}_{\alpha \in A}$ such that $\{V_\beta\}_{\beta \in B}$ is a locally finite open covering of X.

If X is compact, then X is paracompact in the strict sense.

8.3.2 Paracompactness in the strong sense

Let us say that X is *paracompact in the strong sense* if X is paracompact in the strict sense and X is Hausdorff. One may simply say that X is *paracompact* in this case, but this term is sometimes used for paracompactness in the strict sense. If X is equipped with the discrete topology, then it is easy to see that X is paracompact in the strong sense.

8.3.3 Another version of paracompactness

Paracompactness of X is defined on p156 of [240] to mean that X is both paracompact in the strict sense and regular in the strict sense. If X is paracompact in the strong sense, then it is well known that X is regular, as we shall see in a moment.

8.3.4 A paracompactness property of closed sets in X

Let E be a closed set in X, and let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of E in X. Thus $X \setminus E$ is an open set in X, and the collection of open subsets of X consisting of the U_α's, $\alpha \in A$, together with $X \setminus E$, is an open covering of X.

If X is paracompact in the strict sense, then there is an open covering $\{V_\beta\}_{\beta \in B}$ of X that is a refinement of the collection of U_α's, $\alpha \in A$, and $X \setminus E$. Put

$$B_0 = \{\beta \in B : V_\beta \cap E \neq \emptyset\}, \tag{8.3.1}$$

and observe that $\{V_\beta\}_{\beta \in B_0}$ is an open covering of E in X. It is easy to see that $\{V_\beta\}_{\beta \in B_0}$ is locally finite in X, because $\{V_\beta\}_{\beta \in B}$ if locally finite in X.

If $\beta \in B$, then either $V_\beta \subseteq X \setminus E$, or there is an $\alpha \in A$ such that $V_\beta \subseteq U_\alpha$, because $\{V_\beta\}_{\beta \in B}$ is a refinement of the collection of U_α's, $\alpha \in A$, and $X \setminus E$. If $\beta \in B_0$, then it follows that there is an $\alpha \in A$ such that $V_\beta \subseteq U_\alpha$, because $V_\beta \cap E \neq \emptyset$. This means that $\{V_\beta\}_{\beta \in B_0}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$.

8.3.5 Paracompactness and induced topologies

In particular, one can use this to check that E is paracompact, with respect to the induced topology. More precisely, any open covering of E by relatively open subsets of E can be obtained from an open covering of E in X, by taking intersections of open subsets of X with E to get relatively open subsets of E.

The remarks in the preceding subsection imply that an open covering of E in X has a locally finite refinement which is an open covering of E in X too. One can take the intersections of the open subsets of X in this refinement with E, to get a refinement of the initial covering of E by relatively open subsets of E that is locally finite in E and a covering of E by relatively open subsets of itself.

8.3.6 Paracompactness in the strong sense implies regularity

Suppose that X is paracompact in the strong sense, and let us verify that X is regular. Let E be a closed set in X, and let x be an element of X not in E.

If $y \in E$, then there is an open subset U_y of X such that $y \in U_y$ and $x \notin \overline{U_y}$, because X is Hausdorff. Thus the collection of open subsets of X that do not contain x in their closures is an open covering of E in X.

It follows that there is a locally finite open covering $\{W_\gamma\}_{\gamma \in C}$ of E in X that is a refinement of the previous open covering of E, because X is paracompact in the strict sense, as before.

If $\gamma \in C$, then $x \notin \overline{W_\gamma}$, because W_γ is contained in a subset of X whose closure does not contain x. This implies that

$$x \notin \overline{\Big(\bigcup_{\gamma \in C} W_\gamma\Big)}, \tag{8.3.2}$$

as in Subsection 5.13.2. This is equivalent to what we want for regularity, because $\bigcup_{\gamma \in C} W_\gamma$ is an open subset of X that contains E.

8.3.7 Paracompactness and normality

Similarly, if X is paracompact in the strict sense and regular in the strict sense, then it is well known that X is normal in the strict sense. To see this, let E be a closed set in X, and let W be an open subset of X that contains E.

Every element of E is contained in an open subset of X whose closure is contained in W, because X is regular in the strict sense. This means that the collection of open subsets of X whose closures are contained in W is an open covering of E in X.

If X is paracompact in the strict sense, then there is locally finite refinement $\{V_\beta\}_{\beta \in B}$ of the open covering of E just mentioned that is an open covering of E too, as before.

If $\beta \in B$, then $\overline{V_\beta} \subseteq W$, because V_β is contained in a subset of X whose closure is contained in W. This implies that

$$\overline{\Big(\bigcup_{\beta \in B} V_\beta\Big)} = \bigcup_{\beta \in B} \overline{V_\beta} \subseteq W, \tag{8.3.3}$$

where the first step is as in Subsection 5.13.2. Thus $\bigcup_{\beta \in B} V_\beta$ is an open subset of X that contains E and whose closure is contained in W, as desired.

8.3.8 Some criteria for paracompactness

It is well known and not too difficult to show that if a locally compact Hausdorff space X is σ-compact, then X is paracompact. However, more precise results are known, and in particular X is paracompact in the strict sense when X is regular in the strict sense and X has the Lindelöf property. It is also well known that X is paracompact in the strict sense when the topology on X is determined by a semimetric.

8.4 Closed refinements

Let X be a topological space again.

8.4.1 A variant of paracompactness

Consider the following condition:

(8.4.1) if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of X, then there is a locally finite covering $\{E_\beta\}_{\beta \in B}$ of X that is a refinement of $\{U_\alpha\}_{\alpha \in A}$.

Paracompactness in the strict sense is the same as saying that (8.4.1) holds, with E_β an open subset of X for every $\beta \in B$. In particular, paracompactness in the strict sense implies (8.4.1) automatically.

If X is regular in the strict sense, then it is well known that (8.4.1) implies that X is paracompact in the strict sense.

8.4.2 Another variant of paracompactness

A family $\{E_\beta\}_{\beta \in B}$ of subsets of X is said to be *closed* if E_β is a closed set in X for every $\beta \in B$. Consider the following variant of (8.4.1):

(8.4.2) if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of X, then there is a locally finite closed covering $\{E_\beta\}_{\beta \in B}$ of X that is a refinement of $\{U_\alpha\}_{\alpha \in A}$.

This condition obviously implies (8.4.1).

8.4.3 Using the first variant

Suppose that X is regular in the strict sense and satisfies (8.4.1), and let us check that (8.4.2) holds. Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of X. Let $x \in X$ be given, so that $x \in U_\alpha$ for some $\alpha \in A$. Because X is regular in the strict sense, there is an open subset W of X such that $x \in W$ and $\overline{W} \subseteq U_\alpha$. Thus the collection of open subsets W of X such that $\overline{W} \subseteq U_\alpha$ for some $\alpha \in A$ is an open covering of X.

By hypothesis, there is a locally finite covering $\{E_\beta\}_{\beta \in B}$ of X that is a refinement of the covering of X just mentioned. This implies that for each $\beta \in B$ there is an open subset W of X and an $\alpha \in A$ such that $E_\beta \subseteq W$ and $\overline{W} \subseteq U_\alpha$. Of course, this means that $\overline{E_\beta} \subseteq \overline{W} \subseteq U_\alpha$, so that $\{\overline{E_\beta}\}_{\beta \in B}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$. We also have that $\{\overline{E_\beta}\}_{\beta \in B}$ is locally finite in X, because $\{E_\beta\}_{\beta \in B}$ is locally finite in X, as in Subsection 5.13.2.

8.4.4 Using the second variant

Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of X again, and suppose that $\{E_\beta\}_{\beta \in B}$ is a locally finite closed covering of X that is a refinement of $\{U_\alpha\}_{\alpha \in A}$. Thus, for each $\beta \in B$, we can choose $\alpha(\beta) \in A$ such that

$$E_\beta \subseteq U_{\alpha(\beta)}. \tag{8.4.3}$$

Let $x \in X$ be given, so that there is an open subset $W(x)$ of X such that $x \in W(x)$ and $W(x) \cap E_\beta = \emptyset$ for all but finitely many $\beta \in B$, because $\{E_\beta\}_{\beta \in B}$ is locally finite at x, by hyothesis. Put

$$B(x) = \{\beta \in B : x \in E_\beta\} \tag{8.4.4}$$

and

$$B_1(x) = \{\beta \in B : W(x) \cap E_\beta \neq \emptyset\}, \tag{8.4.5}$$

so that $B_1(x)$ is a finite subset of B that contains $B(x)$.

Consider

$$W_1(x) = W(x) \cap \Big(\bigcap_{\beta \in B(x)} U_{\alpha(\beta)} \Big) \cap \Big(\bigcap_{\beta \in B_1(x) \setminus B(x)} (X \setminus E_\beta) \Big), \tag{8.4.6}$$

where the intersection over $\beta \in B_1(x) \setminus B(x)$ can be omitted when $B(x) = B_1(x)$. Of course, $B(x) \neq \emptyset$, because $\{E_\beta\}_{\beta \in B}$ covers X.

Note that $x \in W_1(x)$, by construction. Clearly $W_1(x)$ is an open set in X, because it is the intersection of finitely many open sets. If $\beta \in B \setminus B(x)$, then

$$W_1(x) \cap E_\beta = \emptyset, \tag{8.4.7}$$

because β is an element of $B \setminus B_1(x)$ or $B_1(x) \setminus B(x)$.

8.5 Star refinements

Let X be a set, let C be a nonempty set, and let W_γ be a subset of X for each $\gamma \in C$. Let $x \in X$ be given, and put

$$C(x) = \{\gamma \in C : x \in W_\gamma\}. \tag{8.5.1}$$

8.5.1 The star of x with respect to $\{W_\gamma\}_{\gamma \in C}$

The *star* of x with respect to $\{W_\gamma\}_{\gamma \in C}$ is defined to be

$$\bigcup_{\gamma \in C(x)} W_\gamma. \tag{8.5.2}$$

This union is interpreted as being the empty set when $C(x) = \emptyset$. If $\{W_\gamma\}_{\gamma \in C}$ covers X, then $C(x) \neq \emptyset$ for every $x \in X$.

8.5.2 Star refinements of a family of subsets of X

Let $\{U_\alpha\}_{\alpha \in A}$ be another nonempty family of subsets of X.

We say that $\{W_\gamma\}_{\gamma \in C}$ is a *star refinement* of $\{U_\alpha\}_{\alpha \in A}$ if for each $x \in X$ there is an $\alpha \in A$ such that the star of x with respect to $\{W_\gamma\}_{\gamma \in C}$ is contained in U_α. Equivalently, this means that the collection of subsets of X that are stars of elements of X with respect to $\{W_\gamma\}_{\gamma \in C}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$. It is easy to see that $\{W_\gamma\}_{\gamma \in C}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$ in this case, because W_γ is contained in the star of each of its elements.

8.5.3 Fully normal spaces in the strict sense

A topological space X is said to be *fully normal in the strict sense* if for every open covering $\{U_\alpha\}_{\alpha \in A}$ of X there is an open covering $\{W_\gamma\}_{\gamma \in C}$ that is a star refinement of $\{U_\alpha\}_{\alpha \in A}$.

8.5.4 A criterion for full normality in the strict sense

If X satisfies (8.4.2), then it is well known that X is fully normal in the strict sense. More precisely, let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of X, and let $\{E_\beta\}_{\beta \in B}$ be a locally finite closed covering of X that is a refinement of $\{U_\alpha\}_{\alpha \in A}$. If $x \in X$, then let $B(x)$ and $W_1(x)$ be as in the previous section. The family of sets $W_1(x)$, $x \in X$, is an open covering of X, and we would like to check that it is a star refinement of $\{U_\alpha\}_{\alpha \in A}$.

8.5.5 Checking that $W_1(x)$, $x \in X$, is a star refinement of $\{U_\alpha\}_{\alpha \in A}$

Let $\beta_0 \in B$ be given. If $x \in X$ and $W_1(x) \cap E_{\beta_0} \neq \emptyset$, then $\beta_0 \in B(x)$, as before. This implies that

$$W_1(x) \subseteq U_{\alpha(\beta_0)}, \tag{8.5.3}$$

by the definition of $W_1(x)$.

Let $y \in X$ be given, and let β_0 be an element of B such that $y \in E_{\beta_0}$. If $x \in X$ and $y \in W_1(x)$, then $W_1(x) \cap E_{\beta_0} \neq \emptyset$, so that (8.5.3) holds. It follows that the star of y with respect to the family of sets $W_1(x)$, $x \in X$, is contained in $U_{\alpha(\beta_0)}$, as desired.

8.5.6 Normality from full normality

Suppose that X is fully normal in the strict sense, and let us check that X is normal in the strict sense. Let A and B be disjoint closed subsets of X, so that their complements $X \setminus A$, $X \setminus B$ form an open covering of X.

By hypothesis, there is an open covering $\{W_\gamma\}_{\gamma \in C}$ of X that is a star refinement of $\{X \setminus A, X \setminus B\}$. Put

$$U = \bigcup \{W_\gamma : \gamma \in C,\ W_\gamma \cap A \neq \emptyset\} \tag{8.5.4}$$

and

$$V = \bigcup \{W_\gamma : \gamma \in C,\ W_\gamma \cap B \neq \emptyset\}. \tag{8.5.5}$$

These are open subsets of X, with $A \subseteq U$ and $B \subseteq V$.

Suppose for the sake of a contradiction that there is a point x in $U \cap V$. This means that there are $\gamma_1, \gamma_2 \in C$ such that $x \in W_{\gamma_1} \cap W_{\gamma_2}$, $W_{\gamma_1} \cap A \neq \emptyset$, and $W_{\gamma_2} \cap B \neq \emptyset$. Thus W_{γ_1} and W_{γ_2} are both contained in the star of x with respect to $\{W_\gamma\}_{\gamma \in C}$.

It follows that the star of x with respect to $\{W_\gamma\}_{\gamma \in C}$ is not contained in either $X \setminus A$ or $X \setminus B$. This contradicts the hypothesis that $\{W_\gamma\}_{\gamma \in C}$ be a star refinement of $\{X \setminus A, X \setminus B\}$.

8.5.7 Fully normal spaces in the strong sense

Let us say that X is *fully normal in the strong sense* if X is fully normal in the strict sense and X satisfies the first separation condition. One may say that that X is *fully normal* when X is fully normal in the strict sense, and that X is *fully* T_4 when X is if fully normal in the strong sense, but the opposite convention may be used as well.

8.6 Using another refinement

Let X be a set, and let $\{W_\gamma\}_{\gamma \in C}$ be a nonempty family of subsets of X. If E is a subset of X, then put

$$C(E) = \{\gamma \in C : W_\gamma \cap E \neq \emptyset\}. \tag{8.6.1}$$

8.6.1 The star of E with respect to $\{W_\gamma\}_{\gamma in C}$

The *star* of E with respect to $\{W_\gamma\}_{\gamma \in C}$ is defined to be

$$\bigcup_{\gamma \in C(E)} W_\gamma, \tag{8.6.2}$$

which is interpreted as being the empty set when $C(E) = \emptyset$. Of course, if $\{W_\gamma\}_{\gamma \in C}$ covers X, then $C(E) \neq \emptyset$ when $E \neq \emptyset$. In this case, E is contained in its star with respect to $\{W_\gamma\}_{\gamma \in C}$.

If $x \in X$, then $C(\{x\})$ is the same as $C(x)$ in the previous section, and the star of $\{x\}$ with respect to $\{W_\gamma\}_{\gamma \in C}$ is the same as the star of x. Observe that $C(E)$ is the same as the union of $C(x)$ over $x \in E$, so that the star of E with respect to $\{W_\gamma\}_{\gamma \in C}$ is the same as the union of the stars of the elements of E.

8.6.2 The star of the star of $x \in X$ with respect to $\{W_\gamma\}_{\gamma \in C}$

The star of $x \in X$ with respect to $\{W_\gamma\}_{\gamma \in C}$ is the set of $y \in X$ for which there is a $\gamma \in C$ such that $x, y \in W_\gamma$. This condition is symmetric in x and y, so that y is in the star of x if and only if x is in the star of y.

The star of the star of $x \in X$ with respect to $\{W_\gamma\}_{\gamma \in C}$ is the union of the stars of the elements y of the star of x. This is the same as the union of the stars of $y \in X$ such that x is in the star of y, as in the preceding paragraph.

8.6.3 Star refinements of star refinements

Let $\{U_\alpha\}_{\alpha \in A}$, $\{V_\beta\}_{\beta \in B}$, and $\{W_\gamma\}_{\gamma \in C}$ be nonempty families of subsets of X. Suppose that $\{V_\beta\}_{\beta \in B}$ is a star refinement of $\{U_\alpha\}_{\alpha \in A}$, and that $\{W_\gamma\}_{\gamma \in C}$ is a star refinement of $\{V_\beta\}_{\beta \in B}$.

Let $x \in X$ be given, and suppose that $y \in X$ is an element of the star of x with respect to $\{W_\gamma\}_{\gamma \in C}$. Equivalently, this means that x is an element of the star of y with respect to $\{W_\gamma\}_{\gamma \in C}$, as before.

Because $\{W_\gamma\}_{\gamma \in C}$ is a star refinement of $\{V_\beta\}_{\beta \in B}$, for each such y there is a $\beta(y) \in B$ such that

(8.6.3) $\quad V_{\beta(y)}$ contains the star of y with respect to $\{W_\gamma\}_{\gamma \in C}$.

It follows that $x \in V_{\beta(y)}$ for these points y, so that

(8.6.4) $\quad V_{\beta(y)}$ is contained in the star of x with respect to $\{V_\beta\}_{\beta \in B}$.

Because $\{V_\beta\}_{\beta \in B}$ is a star refinement of $\{U_\alpha\}_{\alpha \in A}$, there is an $\alpha \in A$ such that U_α contains the star of x with respect to $\{V_\beta\}_{\beta \in B}$. This means that

(8.6.5) $\quad U_\alpha$ contains the star of the star of x with respect to $\{W_\gamma\}_{\gamma \in C}$

in this situation.

8.6.4 Using full normality in the strict sense

Now let X be a topological space, and let $\{E_\beta\}_{\beta \in B}$ be a locally finite family of subsets of X. Thus, for each $x \in X$, there is an open subset $V(x)$ of X such that $x \in V(x)$ and $V(x) \cap E_\beta = \emptyset$ for all but finitely many $\beta \in B$. Of course, the family of such open sets $V(x)$ is an open covering of X.

Suppose that X is fully normal in the strict sense, so that this open covering has a star refinement that is an open covering of X. Using full normality again, we can get an open covering $\{W_\gamma\}_{\gamma \in C}$ of X that is a star refinement of the previous open covering of X that is a star refinement of the family of $V(x)$'s, $x \in X$.

8.6.5 The star of E_β with resect to $\{W_\gamma\}_{\gamma \in C}$

If $\beta \in B$, then let Z_β be the star of E_β with respect to $\{W_\gamma\}_{\gamma \in C}$. Note that

$$E_\beta \subseteq Z_\beta, \tag{8.6.6}$$

and that Z_β is an open set in X.

Let $x_0 \in X$ be given, and let γ_0 be an element of C such that W_{γ_0} contains x_0. Suppose that

$$W_{\gamma_0} \cap Z_\beta \neq \emptyset, \tag{8.6.7}$$

and let y be an element of $W_{\gamma_0} \cap Z_\beta$. By definition of Z_β, there is a $\gamma \in C$ such that $W_\gamma \cap E_\beta \neq \emptyset$ and $y \in W_\gamma$.

Thus y is in the star of x_0 with respect to $\{W_\gamma\}_{\gamma \in C}$, and E_β intersects the star of the star of x_0 with respect to $\{W_\gamma\}_{\gamma \in C}$.

8.6.6 Local finiteness of $\{Z_\beta\}_{\beta \in B}$

Because $\{W_\gamma\}_{\gamma \in C}$ is a star refinement of a star refinement of the family of $V(x)$'s, $x \in X$, there is an $x_1 \in X$ such that the star of the star of x_0 with respect to $\{W_\gamma\}_{\gamma \in C}$ is contained in $V(x_1)$, as before.

Remember that $V(x_1) \cap E_\beta \neq \emptyset$ for all but finitely many $\beta \in B$, so that the star of the star of x_0 with respect to $\{W_\gamma\}_{\gamma \in C}$ intersects E_β for only finitely many $\beta \in B$. This implies that (8.6.7) holds for only finitely many $\beta \in B$. This means that $\{Z_\beta\}_{\beta \in B}$ is locally finite in X as well.

8.6.7 Staring with an open covering of X

Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of X, and suppose that $\{E_\beta\}_{\beta \in B}$ is also a refinement of $\{U_\alpha\}_{\alpha \in A}$. Thus, for each $\beta \in B$, we can choose $\alpha(\beta) \in A$ such that $E_\beta \subseteq U_{\alpha(\beta)}$. If Z_β is as before, then

$$Z_\beta \cap U_{\alpha(\beta)} \tag{8.6.8}$$

is an open subset of X that contains E_β and is contained in $U_{\alpha(\beta)}$.

Of course, the family of sets (8.6.8), with $\beta \in B$, is locally finite in X, because $\{Z_\beta\}_{\beta \in B}$ is locally finite in X. If $\{E_\beta\}_{\beta \in B}$ covers X, then the family of sets (8.6.8), with $\beta \in B$, covers X as well.

8.6.8 Using this to get paracompactness in the strict sense

Suppose that every open covering $\{U_\alpha\}_{\alpha \in A}$ of X has a locally finite refinement $\{E_\beta\}_{\beta \in B}$ that covers X, as in (8.4.1). If X is fully normal in the strict sense, then we get a locally finite open covering of X that is a refinement of $\{U_\alpha\}_{\alpha \in A}$, as in the previous subsection. This means that X is paracompact in the strict sense.

If every open covering of X has a locally finite closed refinement that covers X, as in (8.4.2), then X satisfies (8.4.1) and is fully normal in the strict sense, as in the previous sections.

If X is regular in the strict sense and satisfies (8.4.1), then we have seen that X satisfies (8.4.2), so that X is fully normal in the strict sense, and paracompact in the strict sense.

8.7 σ-Local finiteness

A family $\{E_\beta\}_{\beta \in B}$ of subsets of a topological space X is said to be *σ-locally finite* in X if there is a sequence $B_1, B_2, B_3, \ldots$ of subsets of B such that

$$B = \bigcup_{j=1}^{\infty} B_j \tag{8.7.1}$$

and $\{E_\beta\}_{\beta \in B_j}$ is locally finite in X for every $j \geq 1$. Of course, if $\{E_\beta\}_{\beta \in B}$ is locally finite in X, then $\{E_\beta\}_{\beta \in B}$ is σ-locally finite in X.

8.7.1 σ-Locally finite refinements

Consider the following condition:

(8.7.2) if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of X, then there is a σ-locally finite open covering $\{V_\beta\}_{\beta \in B}$ of X that is a refinement of $\{U_\alpha\}_{\alpha \in A}$.

If X is paracompact in the strict sense, then X satisfies (8.7.2), because local finiteness implies σ-local finiteness. If X satisfies (8.7.2), then it is well known that X satisfies (8.4.1), which is to say that every open covering of X has a locally finite refinement that covers X.

8.7.2 Using σ-locally finite refinements

To see this, let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of X, so that there is a σ-locally finite open covering $\{V_\beta\}_{\beta \in B}$ of X that is a refinement of $\{U_\alpha\}_{\alpha \in A}$. Let $B_1, B_2, B_3, \ldots$ be a sequence of subsets of B whose union is B such that $\{V_\beta\}_{\beta \in B_j}$ is locally finite in X for each $j \ge 1$. We may as well suppose that the B_j's are pairwise-disjoint, by replacing B_j with $B_j \setminus \left(\bigcup_{l=1}^{j-1} B_l\right)$ when $j \ge 2$.

8.7.3 A subset E_β of V_β

If $\beta \in B$, then put $E_\beta = V_\beta$ when $\beta \in B_1$, and

$$E_\beta = V_\beta \setminus \Big(\bigcup_{l=1}^{j-1} \bigcup_{\gamma \in B_l} V_\gamma\Big) \tag{8.7.3}$$

when $\beta \in B_j$ for some $j \ge 2$. Thus

$$E_\beta \subseteq V_\beta \tag{8.7.4}$$

for every $\beta \in B$. This implies that $\{E_\beta\}_{\beta \in B}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$, because $\{V_\beta\}_{\beta \in B}$ is a refinement of $\{U_\alpha\}_{\alpha \in A}$, by hypothesis.

8.7.4 The E_β's, $\beta \in B$, cover X

Let $x \in X$ be given, so that $x \in \bigcup_{\beta \in B} V_\beta = \bigcup_{j=1}^\infty \bigcup_{\beta \in B_j} V_\beta$. Let $j_0(x)$ be the smallest positive integer j such that $x \in \bigcup_{\beta \in B_j} V_\beta$, and let $\beta_0(x)$ be an element of $B_{j_0(x)}$ such that $x \in V_{\beta_0(x)}$. Observe that

$$x \in E_{\beta_0(x)}, \tag{8.7.5}$$

because $j_0(x)$ is as small as possible. Thus $\{E_\beta\}_{\beta \in B}$ covers X. If $\beta \in B_l$ for some $l \ge j_0(x) + 1$, then

$$E_\beta \cap V_{\beta_0(x)} = \emptyset, \tag{8.7.6}$$

by the definition of E_β.

8.7.5 Local finiteness of $\{E_\beta\}_{\beta \in B}$

If $1 \le l \le j_0(x)$, then there is an open subset W_l of X such that $x \in W_l$ and $W_l \cap V_\beta = \emptyset$ for all but finitely many $\beta \in B_l$, because $\{V_\beta\}_{\beta \in B_l}$ is locally finite in X. Put

$$W = \Big(\bigcap_{l=1}^{j_0(x)} W_l\Big) \cap V_{\beta_0(x)}, \tag{8.7.7}$$

so that W is an open subset of X that contains x. It is easy to see that $W \cap E_\beta = \emptyset$ for all but finitely many $\beta \in B$. This means that $\{E_\beta\}_{\beta \in B}$ is locally finite in X, as desired.

If X satisfies (8.7.2) and X is regular in the strict sense, then it follows that X is paracompact in the strict sense, as in the previous section.

8.8 Semimetrics and full normality

Let X be a set with a semimetric $d(x, y)$, and let us show that X is fully normal in the strict sense with respect to the topology determined by $d(\cdot, \cdot)$.

8.8.1 Some collections of open balls in X

Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of X. Consider the collection of open balls in X of the form $B(w, r)$, where $w \in X$, $0 < r \le 1$, and

$$B(w, 5\, r) \subseteq U_\alpha \tag{8.8.1}$$

for some $\alpha \in A$. It is easy to see that the collection of these open balls is an open covering of X, and we would like to verify that it is a star refinement of $\{U_\alpha\}_{\alpha \in A}$.

Let $x \in X$ be given, so that we would like to check that the star of x with respect to this collection of open balls is contained in U_α for some $\alpha \in A$. In this situation, the star of x is the union of the open balls of the form $B(w, r)$, where $w \in X$, $0 < r \le 1$, (8.8.1) holds for some $\alpha \in A$, and $x \in B(w, r)$. In particular, there are $w_0 \in X$ and $r_0 \in (0, 1]$ with these properties, and so that r_0 is strictly larger that one-half the supremum of the set of $r \in (0, 1]$ that occur in this way.

Suppose that $w \in X$ and $r \in (0, 1]$ have the properties mentioned in the preceding paragraph. Thus

$$r < 2\, r_0, \tag{8.8.2}$$

by construction. Observe that

$$d(w_0, w) \le d(w_0, x) + d(x, w) < r_0 + r < 3\, r_0, \tag{8.8.3}$$

because x is an element of $B(w_0, r_0)$ and $B(w, r)$. This implies that

$$B(w, r) \subseteq B(w_0, 3\, r_0 + r) \subseteq B(w_0, 5\, r_0). \tag{8.8.4}$$

It follows that the star of x with respect to this collection of open balls is contained in $B(w_0, 5\,r_0)$. By construction, $B(w_0, 5\,r_0)$ is contained in U_{α_0} for some $\alpha_0 \in A$. This means that this collection of open balls is a star refinement of $\{U_\alpha\}_{\alpha\in A}$, as desired.

8.9 Discrete families of sets

Let $\{E_\beta\}_{\beta\in B}$ be a family of subsets of a topological space X. Let us say that $\{E_\beta\}_{\beta\in B}$ is *discrete* at a point $x \in X$ if there is an open subset V of X such that $x \in V$ and

$$E_\beta \cap V = \emptyset \tag{8.9.1}$$

for all but at most one $\beta \in B$. Of course, this implies that $\{E_\beta\}_{\beta\in B}$ is locally finite at x.

If $\{E_\beta\}_{\beta\in B}$ is discrete at every $x \in X$, then $\{E_\beta\}_{\beta\in B}$ is said to be *discrete in* X. This implies that $\{E_\beta\}_{\beta\in B}$ is locally finite in X, and that the E_β's are pairwise disjoint.

Note that (8.9.1) implies that

$$\overline{E_\beta} \cap V = \emptyset, \tag{8.9.2}$$

because V is an open set. If $\{E_\beta\}_{\beta\in B}$ is discrete at $x \in X$, then $\{\overline{E_\beta}\}_{\beta\in B}$ is discrete at x as well. If $\{E_\beta\}_{\beta\in B}$ is discrete in X, then $\{\overline{E_\beta}\}_{\beta\in B}$ is discrete in X too.

8.9.1 Discrete families and induced topologies

Let X_0 be a subset of X. If $\{E_\beta\}_{\beta\in B}$ is discrete at a point $x \in X_0$, then $\{E_\beta \cap X_0\}_{\beta\in B}$ is discrete at x as a family of subsets of X_0, with respect to the induced topology. If $\{E_\beta\}_{\beta\in B}$ is discrete in X, then $\{E_\beta \cap X_0\}_{\beta\in B}$ is discrete in X_0. If a family of subsets of X_0 is discrete at $x \in X_0$, with respect to the induced topology on X_0, then this family is discrete at x as a family of subsets of X.

8.9.2 Discrete families and full normality

Suppose that $\{E_\beta\}_{\beta\in B}$ is discrete in X, and that X is fully normal in the strict sense. Under these conditions, one can find an open subset Z_β of X for each $\beta \in B$ such that $E_\beta \subseteq Z_\beta$, and $\{Z_\beta\}_{\beta\in B}$ is discrete in X too. This uses essentially the same argument as in Section 8.6.

Suppose that $\{E_\beta\}_{\beta\in B}$ is also a refinement of an open covering $\{U_\alpha\}_{\alpha\in A}$ of X. If $\beta \in B$, then we can choose $\alpha(\beta) \in A$ such that $E_\beta \subseteq U_{\alpha(\beta)}$. This implies that $Z_\beta \cap U_{\alpha(\beta)}$ is an open subset of X that contains E_β and is contained in $U_{\alpha(\beta)}$. Clearly $\{Z_\beta \cap U_{\alpha(\beta)}\}_{\beta\in B}$ is discrete in X, because $\{Z_\beta\}_{\beta\in B}$ is discrete in X.

8.9.3 σ-Discrete families of sets

A family $\{E_\beta\}_{\beta \in B}$ of subsets of a topological space X is said to be *σ-discrete* in X if there is a sequence $B_1, B_2, B_3, \ldots$ of subsets of B such that $B = \bigcup_{j=1}^\infty B_j$ and $\{E_\beta\}_{\beta \in B_j}$ is discrete in X for every $j \geq 1$. In this case, $\{E_\beta\}_{\beta \in B}$ is σ-locally finite in X.

Suppose that $\{E_\beta\}_{\beta \in B}$ is a σ-discrete family of subsets of X that is a refinement of an open convering $\{U_\alpha\}_{\alpha \in A}$ of X. If X is fully normal in the strict sense, then for each $\beta \in B$ we can find an open subset of X that contains E_β, so that the resulting family of open sets is a σ-discrete refinement of $\{U_\alpha\}_{\alpha \in A}$. This follows from the previous argument for discrete families.

8.10 Sequences of star refinements

Let X be a topological space, and suppose that X is fully normal in the strict sense. If $\mathcal{B}_0$ is an open covering of X, then there is an open covering $\mathcal{B}_1$ of X that is a star refinement of an open covering of X that is a star refinement of $\mathcal{B}_0$.

Continuing in this way, we get a sequence of open coverings $\mathcal{B}_j$ of X such that $\mathcal{B}_{j+1}$ is a star refinement of an open covering of X that is a star refinement of $\mathcal{B}_j$ for each $j \geq 0$. If $x \in X$ and $j \in \mathbf{Z}_+$, then

(8.10.1) the star of the star of x with respect to $\mathcal{B}_j$ is contained in an element of $\mathcal{B}_{j-1}$,

as in Section 8.6.

8.10.1 The star of $E \in \mathcal{B}_j$ with respect to $\mathcal{B}_j$

If $E \in \mathcal{B}_j$ for some $j \geq 1$, and $x \in E$, then E is contained in the star of x with respect to $\mathcal{B}_j$. This implies that the star of E with respect to $\mathcal{B}_j$ is contained in the star of the star of x with respect to $\mathcal{B}_j$. It follows that

(8.10.2) the star of E with respect to $\mathcal{B}_j$ is contained in an element of $\mathcal{B}_{j-1}$,

by (8.10.1).

8.10.2 The star of the star of E with respect to $\mathcal{B}_{j+1}$

Let E be a subset of X, and let j be a positive integer. Remember that the star of E with respect to $\mathcal{B}_j$ is the same as the union of the stars of the elements of E. The star of the star of E with respect to $\mathcal{B}_j$ is the same as the union of the stars of the stars of the elements of E with respect to $\mathcal{B}_j$. Using (8.10.1), we get that

(8.10.3) the star of the star of E with respect to $\mathcal{B}_{j+1}$ is contained in the star of E with respect to $\mathcal{B}_j$.

8.10.3 Some more collections of subsets of X

Put $\mathcal{C}_1 = \mathcal{B}_1$. If $\mathcal{C}_j$ has been defined for some $j \in \mathbf{Z}_+$, then let $\mathcal{C}_{j+1}$ be the collection of subsets of X obtained by taking the star with respect to $\mathcal{B}_{j+1}$ of an element of $\mathcal{C}_j$. Similarly, let $\widetilde{\mathcal{C}}_{j+1}$ be the collection of subsets of X obtained by taking the star with respect to $\mathcal{B}_{j+1}$ of an element of $\mathcal{C}_{j+1}$.

Equivalently, the elements of $\widetilde{\mathcal{C}}_{j+1}$ can be obtained by taking the star of the star of an element of $\mathcal{C}_j$, with respect to $\mathcal{B}_{j+1}$. Thus

(8.10.4) every element of $\widetilde{\mathcal{C}}_{j+1}$ is contained in the star with respect to $\mathcal{B}_j$ of an element of $\mathcal{C}_j$,

by (8.10.3).

If $j \geq 2$, then the elements of $\mathcal{C}_j$ can be obtained as the star with respect to $\mathcal{B}_j$ of an element of $\mathcal{C}_{j-1}$. This implies that every element of $\widetilde{\mathcal{C}}_{j+1}$ is contained in the star of the star of an element of $\mathcal{C}_{j-1}$, with respect to $\mathcal{B}_j$. This means that every element of $\widetilde{\mathcal{C}}_{j+1}$ is contained in an element of $\widetilde{\mathcal{C}}_j$, so that

$$\widetilde{\mathcal{C}}_{j+1} \text{ is a refinement of } \widetilde{\mathcal{C}}_j. \tag{8.10.5}$$

If $l \geq 2$, then we get that

$$\widetilde{\mathcal{C}}_l \text{ is a refinement of } \widetilde{\mathcal{C}}_2. \tag{8.10.6}$$

Every element of $\widetilde{\mathcal{C}}_2$ is contained in the star with respect to $\mathcal{B}_1$ of an element of $\mathcal{B}_1 = \mathcal{C}_1$, as in (8.10.4). The star with respect to $\mathcal{B}_1$ of an element of $\mathcal{B}_1$ is contained in an element of $\mathcal{B}_0$, as in (8.10.3). Thus every element of $\widetilde{\mathcal{C}}_2$ is contained in an element of $\mathcal{B}_0$, which is to say that

$$\widetilde{\mathcal{C}}_2 \text{ is a refinement of } \mathcal{B}_0. \tag{8.10.7}$$

It follows that

$$\widetilde{\mathcal{C}}_l \text{ is a refinement of } \mathcal{B}_0 \tag{8.10.8}$$

for every $l \geq 2$.

Note that

$$\text{the star of } E \subseteq X \text{ with respect to } \mathcal{B}_j \text{ contains } E \tag{8.10.9}$$

for every $j \geq 0$, because $\mathcal{B}_j$ covers X, by hypothesis. This implies that $\mathcal{C}_l$ is a refinement of $\widetilde{\mathcal{C}}_l$ for every $l \geq 2$. Thus

$$\mathcal{C}_l \text{ is a refinement of } \mathcal{B}_0 \tag{8.10.10}$$

for every $l \geq 2$. Of course, this also holds when $l = 1$, because $\mathcal{B}_1 = \mathcal{C}_1$ is a refinement of $\mathcal{B}_0$, by construction.

Similarly, $\mathcal{C}_j$ is a refinement of $\mathcal{C}_{j+1}$ for every $j \geq 1$, by (8.10.9). This implies that $\mathcal{B}_1 = \mathcal{C}_1$ is a refinement of $\mathcal{C}_j$ for each $j \geq 1$. Thus $\mathcal{C}_j$ covers X for every $j \geq 1$, because $\mathcal{B}_1$ covers X. If $E \subseteq X$ and $j \geq 0$, then it is easy to see that

(8.10.11) the star of E with respect to $\mathcal{B}_j$ is an open set,

because the elements of $\mathcal{B}_j$ are open subsets of X. This means that $\mathcal{C}_l$ is an open covering of X for each $l \geq 1$.

8.11 σ-Discreteness and full normality

Let X be a topological space that is fully normal in the strict sense again. If $\{U_\alpha\}_{\alpha \in A}$ is an open covering of X, then it is well known that there is a σ-discrete open covering of X that is a refinement of $\{U_\alpha\}_{\alpha \in A}$. It suffices to show that there is a σ-discrete covering of X that is a refinement of $\{U_\alpha\}_{\alpha \in A}$, as in Section 8.9.

Let $\mathcal{B}_0$ be an open covering of X that is a star refinement of $\{U_\alpha\}_{\alpha \in A}$. As in the previous section, we can find an open covering $\mathcal{B}_j$ of X for each positive integer j, so that $\mathcal{B}_j$ is a star refinement of an open covering of X that is a star refinement of $\mathcal{B}_{j-1}$. Let $\mathcal{C}_j$ be as in the previous section for each $j \in \mathbf{Z}_+$, which is an open covering of X. Remember that $\mathcal{C}_j$ is a refinement of $\mathcal{B}_0$ for every $j \geq 1$. This implies that $\mathcal{C}_j$ is a star refinement of $\{U_\alpha\}_{\alpha \in A}$.

If $x \in X$ and $l \in \mathbf{Z}_+$, then let $W_l(x)$ be the star of x with respect to $\mathcal{C}_l$. This is the union of the elements of $\mathcal{C}_l$ that contain x. The star of each of these elements of $\mathcal{C}_l$ with respect to $\mathcal{B}_{l+1}$ is an element of $\mathcal{C}_{l+1}$ that contains x. This implies that

(8.11.1) the star of $W_l(x)$ with respect to $\mathcal{B}_{l+1}$ is contained in $W_{l+1}(x)$.

8.11.1 Using a well-ordering on X to get some more sets in X

Let $\preceq$ be a well-ordering on X. Put

$$\widetilde{W}_l(x) = W_l(x) \setminus \bigcup \{W_{l+1}(y) : y \in X,\, y \preceq x,\, y \neq x\}. \tag{8.11.2}$$

We would like to check that the collection of sets $\widetilde{W}_l(x)$, $x \in X$, is discrete in X for each $l \in \mathbf{Z}_+$.

Let $l \geq 1$ be given, and let x, y be distinct elements of X. We would like to verify that

(8.11.3) the star of $\widetilde{W}_l(y)$ with respect to $\mathcal{B}_{l+1}$ is disjoint from $\widetilde{W}_l(x)$.

This is the same as saying that

(8.11.4) no element of $\mathcal{B}_{l+1}$ intersects both $\widetilde{W}_l(x)$ and $\widetilde{W}_l(y)$.

Of course, (8.11.4) is symmetric in x and y, so that (8.11.3) is equivalent to the star of $\widetilde{W}_l(x)$ with respect to $\mathcal{B}_{l+1}$ being disjoint from $\widetilde{W}_l(y)$.

Note that the star of $\widetilde{W}_l(y)$ with respect to $\mathcal{B}_{l+1}$ is contained in the star of $W_l(y)$ with respect to $\mathcal{B}_{l+1}$, because $\widetilde{W}_l(y) \subseteq W_l(y)$. This implies that the star of $\widetilde{W}_l(y)$ with respect to $\mathcal{B}_{l+1}$ is contained in $W_{l+1}(y)$, by (8.11.1). If $y \preceq x$, then $W_{l+1}(y)$ is disjoint from $\widetilde{W}_l(x)$, by construction. It follows that (8.11.3) holds when $y \preceq x$.

Similarly, if $x \preceq y$, then the star of $\widetilde{W}_l(x)$ with respect to $\mathcal{B}_{l+1}$ is disjoint from $\widetilde{W}_l(y)$. This implies (8.11.3), as before. Thus (8.11.3) holds, which means that (8.11.4) holds. This implies that the collection of sets $\widetilde{W}_l(x)$, $x \in X$, is discrete in X, because $\mathcal{B}_{l+1}$ is an open covering of X.

Let $y \in X$ be given, and note that $y \in W_l(y)$ for every $l \geq 1$. Let z be the smallest element of X, with respect to $\preceq$, such that $y \in W_l(z)$ for some $l \geq 1$. It is easy to see that $y \in \widetilde{W}_l(z)$ in this case. This shows that the collection of sets $\widetilde{W}_l(x)$, with $x \in X$ and $l \in \mathbf{Z}_+$, covers X.

If $x \in X$ and $l \in \mathbf{Z}_+$, then there is an $\alpha_0 \in A$ such that $W_l(x) \subseteq U_{\alpha_0}$, because $\mathcal{C}_l$ is a star refinement of $\{U_\alpha\}_{\alpha \in A}$. This implies that $\widetilde{W}_l(x) \subseteq U_{\alpha_0}$. Thus the collection of sets $\widetilde{W}_l(x)$, with $x \in X$ and $l \in \mathbf{Z}_+$, is a refinement of $\{U_\alpha\}_{\alpha \in A}$. This collection is σ-discrete in X, because the collection of sets $\widetilde{W}_l(x)$, $x \in X$, is discrete in X for each $l \geq 1$, as before.

8.12 Point finiteness

A family $\{E_\alpha\}_{\alpha \in A}$ of subsets of a set X is said to be *point finite* in X if for every $x \in X$ there are only finitely many $\alpha \in A$ such that $x \in E_\alpha$. Suppose now that X is a topological space. If $\{E_\alpha\}_{\alpha \in A}$ is locally finite in X, then $\{E_\alpha\}_{\alpha \in A}$ is point finite in X.

8.12.1 Metacompactness

If every open covering of X has a refinement that is a point finite open covering of X, then X is said to be *metacompact*. If X is paracompact in the strict sense, then X is metacompact.

8.12.2 Using point finiteness

Suppose that X is normal in the strict sense, and that $\{V_\alpha\}_{\alpha \in A}$ is a point finite open covering of X. Under these conditions, one can choose an open subset U_α of X for each $\alpha \in A$ such that $\overline{U_\alpha} \subseteq V_\alpha$ and $\{U_\alpha\}_{\alpha \in A}$ is an open covering of X, as in Problem V (a) on p171 of [240]. More precisely, this can be obtained using Zorn's lemma or Hausdorff's maximality principle, as follows.

Let A_0 be a subset of A, and let ϕ_0 be a mapping from A_0 into the set of all open subsets of X. Let us say that (A_0, ϕ_0) is *admissible* if

$$\overline{\phi_0(\alpha)} \subseteq V_\alpha \tag{8.12.1}$$

for every $\alpha \in A_0$, and

$$X = \Big(\bigcup_{\alpha \in A_0} \phi_0(\alpha)\Big) \cup \Big(\bigcup_{\alpha \in A \setminus A_0} V_\alpha\Big). \tag{8.12.2}$$

Let $\mathcal{A}$ be the collection of admissible pairs (A_0, ϕ_0). If $(A_1, \phi_1), (A_2, \phi_2) \in \mathcal{A}$, then put

$$(A_1, \phi_1) \preceq (A_2, \phi_2) \tag{8.12.3}$$

when $A_1 \subseteq A_2$ and $\phi_1 = \phi_2$ on A_1. This defines a partial ordering on $\mathcal{A}$.

Let $\mathcal{C}$ be a chain in $\mathcal{A}$, and put

$$A_{\mathcal{C}} = \bigcup_{(A_0, \phi_0) \in \mathcal{C}} A_0, \tag{8.12.4}$$

which is a subset of A. If $\alpha \in A_{\mathcal{C}}$, then there is an element (A_0, ϕ_0) in $\mathcal{C}$ such that $\alpha \in A_0$, and we would like to put

$$\phi_{\mathcal{C}}(\alpha) = \phi_0(\alpha). \tag{8.12.5}$$

One can check that this does not depend on the particular element (A_0, ϕ_0) of $\mathcal{C}$ with $\alpha \in A_0$. Thus $\phi_{\mathcal{C}}$ is a well-defined mapping from $A_{\mathcal{C}}$ into the set of all open subsets of X. Note that

$$\overline{\phi_{\mathcal{C}}(\alpha)} \subseteq V_\alpha \tag{8.12.6}$$

for every $\alpha \in A_{\mathcal{C}}$, because of the analogous property (8.12.1) of every (A_0, ϕ_0) in $\mathcal{C}$.

We would like to verify that

$$X = \Big(\bigcup_{\alpha \in A_{\mathcal{C}}} \phi_{\mathcal{C}}(\alpha)\Big) \cup \Big(\bigcup_{\alpha \in A \setminus A_{\mathcal{C}}} V_\alpha\Big). \tag{8.12.7}$$

Let $x \in X$ be given, and let us check that x is an element of the right side of (8.12.7). Of course, $x \in V_\alpha$ for some $\alpha \in A$, because $\{V_\alpha\}_{\alpha \in A}$ covers X, by hypothesis. If $x \in V_\alpha$ for some $\alpha \in A \setminus A_{\mathcal{C}}$, then x is an element of the right side of (8.12.7). Suppose now that for each $\alpha \in A \setminus A_{\mathcal{C}}$, we have that $x \notin V_\alpha$.

Remember that there are only finitely many $\alpha \in A$ such that $x \in V_\alpha$, because $\{V_\alpha\}_{\alpha \in A}$ is point finite in X, by hypothesis. In this situation, $\mathcal{A}_{\mathcal{C}}$ contains all of the $\alpha \in A$ with $x \in V_\alpha$, by construction. This means that each of these α's is an element of A_0 for some $(A_0, \phi_0) \in \mathcal{C}$, by definition of $A_{\mathcal{C}}$. More precisely, there is an $(A_0, \phi_0) \in \mathcal{C}$ such that A_0 contains every $\alpha \in A$ with $x \in A_\alpha$. This uses the facts that there are only finitely many of these α's, and that $\mathcal{C}$ is a chain in $\mathcal{A}$.

Of course, (A_0, ϕ_0) satisfies (8.12.2), because $(A_0, \phi_0) \in \mathcal{A}$. If $\alpha \in A \setminus A_0$, then $x \notin V_\alpha$, as in the preceding paragraph. Using (8.12.2), we get that there is an $\alpha \in A_0$ such that $x \in \phi_0(\alpha)$. This α is contained in $A_{\mathcal{C}}$ and satisfies $x \in \phi_{\mathcal{C}}(\alpha)$, by definition of $(A_{\mathcal{C}}, \phi_{\mathcal{C}})$. It follows that x is an element of the right side of (8.12.7).

Thus (8.12.7) holds, so that $(A_{\mathcal{C}}, \phi_{\mathcal{C}})$ is an element of $\mathcal{A}$. Using Zorn's lemma or Hausdorff's maximality principle, one can get a maximal element (A_1, ϕ_1) of $\mathcal{A}$. We would like to verify that $A_1 = A$. Suppose for the sake of a contradiction that $A_1 \neq A$, and let α_2 be an element of $A \setminus A_1$.

Put $A_2 = A_1 \cup \{\alpha_2\}$ and

$$E_{\alpha_2} = X \setminus \Big(\Big(\bigcup_{\alpha \in A_1} \phi_1(\alpha) \Big) \cup \Big(\bigcup_{\alpha \in A \setminus A_2} V_\alpha \Big) \Big). \tag{8.12.8}$$

Note that E_{α_2} is a closed set in X, because $\phi_1(\alpha)$ is an open set for every $\alpha \in A_1$, and V_α is an open set for every $\alpha \in A$. We also have that

$$E_{\alpha_2} \subseteq V_{\alpha_2}, \tag{8.12.9}$$

because of the analogue of (8.12.2) for (A_1, ϕ_1). Thus there is an open subset W_{α_2} of X such that

$$E_{\alpha_2} \subseteq W_{\alpha_2} \quad \text{and} \quad \overline{W_{\alpha_2}} \subseteq V_{\alpha_2}, \tag{8.12.10}$$

because X is normal in the strict sense, by hypothesis.

Put $\phi_2(\alpha) = \phi_1(\alpha)$ for every $\alpha \in A_1$, and $\phi_2(\alpha_2) = W_{\alpha_2}$. This defines ϕ_2 as a mapping from A_2 into the set of all open subsets of X. Observe that $\overline{\phi_2(\alpha)} \subseteq V_\alpha$ for every $\alpha \in A_2$, because of the analogous property of ϕ_1 on A_1, and the second part of (8.12.10). Using the first part of (8.12.10), we get that

$$X = \Big(\bigcup_{\alpha \in A_2} \phi_2(\alpha) \Big) \cup \Big(\bigcup_{\alpha \in A \setminus A_2} V_\alpha \Big). \tag{8.12.11}$$

This means that (A_2, ϕ_2) is admissible, and hence an element of $\mathcal{A}$. By construction, (8.12.3) holds, and $(A_1, \phi_1) \neq (A_2, \phi_2)$. This contradicts the maximality of (A_1, ϕ_1), so that $A_1 = A$, as desired.

8.13 Partitions of unity

Let X be a topological space, and suppose that X is normal in the strict sense. Also let $\{V_\alpha\}_{\alpha \in A}$ be a locally finite open covering of X, so that $\{V_\alpha\}_{\alpha \in A}$ is point finite in X in particular. Thus, for each $\alpha \in A$, we can choose an open subset U_α of X such that $\overline{U_\alpha} \subseteq V_\alpha$ and $\{U_\alpha\}_{\alpha \in A}$ is an open covering of X, as in the previous section.

If $\alpha \in A$, then we can use Urysohn's lemma to get a continuous real-valued function ϕ_α on X such that $\phi_\alpha(x) = 1$ for every $x \in \overline{U_\alpha}$, $\phi_\alpha(x) = 0$ for every $x \in X \setminus V_\alpha$, and $0 \leq \phi_\alpha(x) \leq 1$ for every $x \in X$. In this case,

$$\{x \in X : \phi_\alpha(x) \neq 0\} \tag{8.13.1}$$

is contained in V_α for every $\alpha \in A$, so that the family of these subsets of X is locally finite too.

It follows that

(8.13.2)
$$\Phi(x) = \sum_{\alpha \in A} \phi_\alpha(x)$$

defines a continuous real-valued function on X, as in Subsection 5.14.1. If $x \in \overline{U_{\alpha_0}}$ for some $\alpha_0 \in A$, then

(8.13.3)
$$\Phi(x) \ge \phi_{\alpha_0}(x) \ge 1,$$

because $\phi_\alpha(x) \ge 0$ for every $\alpha \in A$, by construction. This means that $\Phi(x) \ge 1$ for every $x \in X$, because X is covered by $\{U_\alpha\}_{\alpha \in A}$.

If we put

(8.13.4)
$$\psi_\beta(x) = \phi_\beta(x)/\Phi(x)$$

for every $\beta \in A$ and $x \in X$, then we get a partition of unity on X, as in Subsection 5.14.2. This corresponds to Problem W on p171 of [240].

8.14 Minimal coverings

Let X be a set, and let $\{E_\alpha\}_{\alpha \in A}$ be a point finite covering of X. Under these conditions, there is a minimal subcovering of X from $\{E_\alpha\}_{\alpha \in A}$, as in Problem V (b) on p171 of [240]. This is also mentioned on p23 of [408], for topological spaces, although the topology does not play a role for this part.

More precisely, let $\mathcal{A}$ be the collection of subsets B of A such that $\{E_\alpha\}_{\alpha \in B}$ covers X. We would like to show that $\mathcal{A}$ has a minimal element, with respect to inclusion. Let $\preceq$ be the partial ordering on $\mathcal{A}$ defined by putting $A_1 \preceq A_2$ when $A_1, A_2 \in \mathcal{A}$ satisfy $A_2 \subseteq A_1$. We would like to show that $\mathcal{A}$ has a maximal element with respect to $\preceq$, using Zorn's lemma or Hausdorff's maximality principle.

Let $\mathcal{C}$ be a chain in $\mathcal{A}$ with respect to $\preceq$, which is the same as saying that $\mathcal{C}$ is a chain in $\mathcal{A}$ with respect to inclusion. Put

(8.14.1)
$$A_{\mathcal{C}} = \bigcap_{A_0 \in \mathcal{C}} A_0,$$

which is a subset of A. We would like to check that

(8.14.2)
$$\bigcup_{\alpha \in A_{\mathcal{C}}} E_\alpha = X,$$

so that $A_{\mathcal{C}} \in \mathcal{A}$.

Let $x \in X$ be given, and remember that $x \in E_\alpha$ for only finitely many $\alpha \in A$, by hypothesis. If x is not an element of the union on the left side of (8.14.2), then $A_{\mathcal{C}}$ does not contain any $\alpha \in A$ such that $x \in E_\alpha$. This means that for each $\alpha \in A$ with $x \in E_\alpha$, there is an element of $\mathcal{C}$ that does not contain α. Because $\mathcal{C}$ is a chain, and there are only finitely many of these α's, there is an

$A_0 \in \mathcal{C}$ such that A_0 does not contain any $\alpha \in A$ such that $x \in E_\alpha$. This implies that x is not an element of $\bigcup_{\alpha \in A_0} E_\alpha$, contradicting the fact that $A_0 \in \mathcal{A}$.

Thus (8.14.2) holds, so that $A_{\mathcal{C}} \in \mathcal{A}$. One can use this and Zorn's lemma or Hausdorff's maximality principle to get a maximal element of $\mathcal{A}$ with respect to $\preceq$. This is the same as a minimal element of $\mathcal{A}$ with respect to inclusion, as desired.

8.14.1 Using metacompactness

Now let X be a topological space, and suppose that X is metacompact, as in Section 8.12. If X is also countably compact, then X is compact, as on p23f of [408]. This corresponds to Problem V (c) on p171 of [240], where X is asked to satisfy the first separation condition too. However, this additional hypothesis appears to be included because of the way that some other results were stated, and is not really needed. In fact, the argument indicated in [240] is the same as the one in [408].

Let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary open covering of X. Because X is metacompact, there is a point finite refinement $\{V_\beta\}_{\beta \in B}$ of $\{U_\alpha\}_{\alpha \in A}$ that is an open covering of X. As before, there is a minimal subset B_0 of B such that $\{V_\beta\}_{\beta \in B_0}$ covers X. If B_0 has only finitely many elements, then we can get a finite subcovering of X from $\{U_\alpha\}_{\alpha \in A}$.

If $\beta \in B_0$, then there is a point $x_\beta \in V_\beta$ that is not in V_γ for any other $\gamma \in B_0$, because B_0 is minimal. Let us choose such a point x_β for every $\beta \in B_0$, and let L be the set of points x_β, $\beta \in B_0$, that have been chosen in this way. Note that these points are distinct, by construction. Suppose that B_0 has infinitely many elements, so that L has infinitely many elements too. Remember that countable compactness is equivalent to the strong limit point property, which implies that there is a point $x \in X$ that is a strong limit point of L. Because $\{V_\beta\}_{\beta \in B_0}$ covers X, there is a $\beta_0 \in B_0$ such that $x \in V_{\beta_0}$. It follows that V_{β_0} contains infinitely many elements of L, because x is a strong limit point of L, and V_{β_0} is an open set. This contradicts the fact that x_{β_0} is the only element of L in V_{β_0}.

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Index