Some topics in analysis related to Banach algebras

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Abstract

Some basic topics related to Banach algebras over fields with absolute value functions are discussed, in connection with Fourier series in particular.

Contents

Part I
Some basic notions

1 Some inequalities

Let X be a nonempty finite set, and let f be a nonnegative real-valued function on X. Put

$$\|f\|_r = \Big(\sum_{x \in X} f(x)^r\Big)^{1/r} \tag{1.1}$$

for each positive real number r, and

$$\|f\|_\infty = \max_{x \in X} f(x). \tag{1.2}$$

Note that

$$\|t\, f\|_r = t\, \|f\|_r \tag{1.3}$$

for every nonnegative real number t and $0 < r \le \infty$, by construction. We also have that

$$\|f\|_\infty \le \|f\|_r \le (\#X)^{1/r}\, \|f\|_\infty \tag{1.4}$$

for every $0 < r < \infty$, where $\#X$ denotes the number of elements of X. In particular,

$$\lim_{r \to \infty} \|f\|_r = \|f\|_\infty, \tag{1.5}$$

because of the well-known fact that $a^{1/r} \to 1$ as $r \to \infty$ for any positive real number a. If $0 < r_1 \le r_2 < \infty$, then

$$\|f\|_{r_2}^{r_2} = \sum_{x \in X} f(x)^{r_2} \le \sum_{x \in X} \|f\|_\infty^{r_2 - r_1}\, f(x)^{r_1} = \|f\|_\infty^{r_2 - r_1}\, \|f\|_{r_1}^{r_1}. \tag{1.6}$$

This implies that

$$\|f\|_{r_2} \le \|f\|_\infty^{1-(r_1/r_2)} \, \|f\|_{r_1}^{r_1/r_2} \le \|f\|_{r_1}, \tag{1.7}$$

using the first inequality in (1.4) in the second step.

If a, b are nonnegative real numbers, then

$$\max(a, b) \le (a^r + b^r)^{1/r} \le 2^{1/r} \, \max(a, b) \tag{1.8}$$

for every positive real number r. This corresponds to (1.4), where X is a set with two elements. Using (1.5) or (1.8), we get that

$$\lim_{r \to \infty} (a^r + b^r)^{1/r} = \max(a, b). \tag{1.9}$$

Similarly, if $0 < r_1 \le r_2 < \infty$, then

$$(a^{r_2} + b^{r_2})^{1/r_2} \le (a^{r_1} + b^{r_1})^{1/r_1}, \tag{1.10}$$

as in (1.7). If $0 < r \le 1$, then it follows that

$$(a + b)^r \le a^r + b^r, \tag{1.11}$$

by taking $r_1 = r$ and $r_2 = 1$ in (1.10), and taking the rth power of both sides of the inequality.

If f, g are nonnegative real-valued functions on X and $1 \le r \le \infty$, then

$$\|f + g\|_r \le \|f\|_r + \|g\|_r. \tag{1.12}$$

This is *Minkowski's inequality* for finite sums. Of course, equality holds trivially in (1.12) when $r = 1$, and the $r = \infty$ case can be verified directly. If $0 < r \le 1$, then

$$\|f + g\|_r^r = \sum_{x \in X} (f(x) + g(x))^r \le \sum_{x \in X} f(x)^r + \sum_{x \in X} g(x)^r = \|f\|_r^r + \|g\|_r^r, \tag{1.13}$$

using (1.11) in the second step.

If f is a nonnegative real-valued function on X and $1 \le r < \infty$, then

$$\Big(\frac{1}{\# X} \sum_{x \in X} f(x) \Big)^r \le \frac{1}{\# X} \sum_{x \in X} f(x)^r. \tag{1.14}$$

This follows from the convexity of the function t^r on the set $[0, \infty)$ of nonnegative real numbers. If $0 < r_1 \le r_2 < \infty$, then we get that

$$\Big(\frac{1}{\# X} \sum_{x \in X} f(x)^{r_1} \Big)^{r_2/r_1} \le \frac{1}{\# X} \sum_{x \in X} f(x)^{r_2}, \tag{1.15}$$

by applying (1.14) with $r = r_2/r_1$ and $f(x)^{r_1}$ in place of $f(x)$. Equivalently, this means that

$$\|f\|_{r_1} \le (\# X)^{(1/r_1) - (1/r_2)} \, \|f\|_{r_2}. \tag{1.16}$$

2 q-Metrics and q-semimetrics

Let X be a set, and let q be a positive real number. A nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ is said to be a *q-semimetric* on X if it satisfies the following three conditions. First,

$$d(x, x) = 0 \quad \text{for every } x \in X. \tag{2.1}$$

Second,

$$d(x, y) = d(y, x) \quad \text{for every } x, y \in X. \tag{2.2}$$

Third,

$$d(x, z)^q \le d(x, y)^q + d(y, z)^q \quad \text{for every } x, y, z \in X. \tag{2.3}$$

If we also have that

$$d(x, y) > 0 \quad \text{when } x \ne y, \tag{2.4}$$

then $d(\cdot, \cdot)$ is said to be a *q-metric* on X. A q-metric or q-semimetric with $q = 1$ is also simply known as a *metric* or *semimetric*, respectively. Note that $d(x, y)$ is a q-metric or q-semimetric on X if and only if

$$d(x, y)^q \tag{2.5}$$

is a metric or semimetric on X, respectively.

Equivalently, (2.3) says that

$$d(x, z) \le (d(x, y)^q + d(y, z)^q)^{1/q} \tag{2.6}$$

for every $x, y, z \in X$. A nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ is said to be a *semi-ultrametric* on X if it satisfies (2.1), (2.2), and

$$d(x, z) \le \max(d(x, y), d(y, z)) \tag{2.7}$$

for every $x, y, z \in X$. If $d(\cdot, \cdot)$ satisfies (2.4) as well, then $d(\cdot, \cdot)$ is said to be an *ultrametric* on X. An ultrametric or semi-ultrametric on X may be considered as a q-metric or q-semimetric on X, respectively, with $q = \infty$, because of (1.9). If $0 < q_1 \le q_2 \le \infty$ and $d(\cdot, \cdot)$ is a q_2-metric or q_2-semimetric on X, then $d(\cdot, \cdot)$ is a q_1-metric or q_1-semimetric on X, as appropriate, because of the first inequality in (1.8) and (1.10).

The *discrete metric* on X is defined as usual by putting $d(x, y)$ equal to 1 when $x \ne y$, and to 0 when $x = y$. It is easy to see that this defines an ultrametric on X. Now let $q > 0$ be given, and let $d(x, y)$ be any q-metric or q-semimetric on X. If a is a positive real number, then one can check that

$$d(x, y)^a \tag{2.8}$$

defines a (q/a)-metric or (q/a)-semimetric on X, as appropriate. If $q = \infty$, then q/a is interpreted as being ∞ too, as usual.

Let $d(\cdot,\cdot)$ be a q-semimetric on X for some $q > 0$. If $x \in X$ and r is a positive real number, then the *open ball* centered at x with radius r with respect to $d(\cdot,\cdot)$ in X is defined as usual by

$$B(x,r) = B_d(x,r) = \{y \in X : d(x,y) < r\}. \tag{2.9}$$

Similarly, the *closed ball* in X centered at $x \in X$ with radius $r \geq 0$ with respect to $d(\cdot,\cdot)$ is defined by

$$\overline{B}(x,r) = \overline{B}_d(x,r) = \{y \in X : d(x,y) \leq r\}. \tag{2.10}$$

If a is a positive real number, then (2.8) is a (q/a)-semimetric on X, as in the preceding paragraph. It is easy to see that

$$B_{d^a}(x,r^a) = B_d(x,r) \tag{2.11}$$

for every $x \in X$ and $r > 0$, and that

$$\overline{B}_{d^a}(x,r^a) = \overline{B}_d(x,r) \tag{2.12}$$

for every $x \in X$ and $r \geq 0$.

3 q-Absolute value functions

Let k be a field, and let q be a positive real number. A nonnegative real-valued function $|\cdot|$ defined on k is said to be a *q-absolute value function* on k if it satisfies the following three conditions. First, for each $x \in k$,

$$|x| = 0 \quad \text{if and only if } x = 0. \tag{3.1}$$

Second,

$$|x\,y| = |x|\,|y| \quad \text{for every } x,y \in k. \tag{3.2}$$

Third,

$$|x+y|^q \leq |x|^q + |y|^q \quad \text{for every } x,y \in k. \tag{3.3}$$

A q-absolute value function with $q = 1$ is also known as an *absolute value function*. Observe that $|x|$ is a q-absolute value function on k if and only if $|x|^q$ is an absolute value function on k. It is well known that the standard absolute value functions on the real numbers $\mathbf{R}$ and complex numbers $\mathbf{C}$ are absolute value functions in this sense.

Let $|x|$ be a nonnegative real-valued function on k that satisfies (3.1) and (3.2). One can check that

$$|1| = 1, \tag{3.4}$$

where the first 1 is the multiplicative identity element in k, and the second 1 is the usual real number. If $x \in k$ satisfies $x^n = 1$ for some positive integer n, then

$$|x| = 1, \tag{3.5}$$

because $|x|^n = |x^n| = 1$. If $-y$ is the additive inverse of $y \in k$, then $-y = (-1)\,y$, and in particular $(-1)^2 = 1$. Hence $|-1| = 1$, by (3.5).

As before, (3.3) is the same as saying that

$$|x + y| \le (|x|^q + |y|^q)^{1/q} \tag{3.6}$$

for every $x, y \in k$. A nonnegative real-valued function $|\cdot|$ on k is said to be an *ultrametric absolute value function* on k if it satisfies (3.1), (3.2), and

$$|x + y| \le \max(|x|, |y|) \tag{3.7}$$

for every $x, y \in k$. An ultrametric absolute value function may be considered as a q-absolute value function with $q = \infty$, because of (1.9). If $0 < q_1 \le q_2 \le \infty$ and $|\cdot|$ is a q_2-absolute value function on k, then $|\cdot|$ is a q_1-absolute value function on k, because of the first inequality in (1.8) and (1.10). If $|\cdot|$ is a q-absolute value function on k for some $q > 0$ and a is a positive real number, then it is easy to see that

$$|x|^a \tag{3.8}$$

defines a (q/a)-absolute value function on k, where q/a is interpreted as being ∞ when $q = \infty$, as before.

If $|\cdot|$ is a q-absolute value function on k for some $q > 0$, then

$$d(x, y) = |x - y| \tag{3.9}$$

defines a q-metric on k. This uses the fact that $|-1| = 1$ to get that (3.9) satisfies the symmetry condition (2.2). The *trivial absolute value function* is defined on k be putting $|x|$ equal to 1 when $x \ne 0$, and to 0 when $x = 0$. This defines an ultrametric absolute value function on k. The ultrametric on k corresponding to the trivial absolute value function as in (3.9) is the same as the discrete metric on k.

If p is a prime number, then the *p-adic absolute value* $|x|_p$ of a rational number x is defined as follows. Of course, $|0|_p = 0$. If $x \ne 0$, then x can be expressed as $p^j\,(a/b)$ for some integers a, b, and j, where $a, b \ne 0$ and neither a nor b is a multiple of p. In this case, we put

$$|x|_p = p^{-j}. \tag{3.10}$$

Note that j is uniquely determined by x here, so that $|x|_p$ is well defined. One can check that this defines an ultrametric absolute value function on the field $\mathbf{Q}$ of rational numbers. The corresponding ultrametric

$$d_p(x, y) = |x - y|_p \tag{3.11}$$

is known as the *p-adic metric* on $\mathbf{Q}$.

4 Associated topologies

Let X be a set, and let $d(\cdot,\cdot)$ be a q-semimetric on X for some $q > 0$. As usual, a subset U of X is said to be an *open set* with respect to $d(\cdot,\cdot)$ if for every $x \in U$ there is an $r > 0$ such that

$$B(x,r) \subseteq U, \tag{4.1}$$

where $B(x,r)$ is the open ball in X centered at x with radius r with respect to $d(\cdot,\cdot)$, as in (2.9). It is easy to see that this defines a topology on X. If a is a positive real number, then $d(\cdot,\cdot)^a$ defines a (q/a)-semimetric on X, as in Section 2. The topology determined on X by $d(\cdot,\cdot)^a$ is the same as the topology determined by $d(\cdot,\cdot)$, because of (2.11). In particular, this permits one to reduce to the case of ordinary semimetrics, by replacing $d(\cdot,\cdot)$ with $d(\cdot,\cdot)^q$ when $q \le 1$. One can use this to get that open balls in X with respect to $d(\cdot,\cdot)$ are open sets, for instance, by reducing to standard results for ordinary semimetrics, although analogous arguments could be used for any $q > 0$. Similarly, closed balls in X with respect to $d(\cdot,\cdot)$ are closed sets. If $q = \infty$, then one can check that open balls in X with respect to $d(\cdot,\cdot)$ are closed sets, and that closed balls in X with respect to $d(\cdot,\cdot)$ with positive radius are open sets. If $d(\cdot,\cdot)$ is a q-metric on X, then X is Hausdorff with respect to the topology determined by $d(\cdot,\cdot)$.

Let Y be a subset of X, and observe that the restriction of $d(x,y)$ to $x, y \in Y$ defines a q-semimetric on Y. The topology determined on Y by the restriction of $d(x,y)$ to $x, y \in Y$ is the same as the topology induced on Y by the topology determined on X by $d(\cdot,\cdot)$. More precisely, if $U \subseteq X$ is an open set, then it is easy to see that $U \cap Y$ is an open set in Y with respect to the topology determined by the restriction of $d(x,y)$ to $x, y \in Y$. In the other direction, an open ball in Y with respect to the restriction of $d(x,y)$ to $x, y \in Y$ is the same as the intersection of Y with the open ball in X with the same center and radius, which implies that open balls in Y are open sets with respect to the induced topology. Any open set in Y with respect to the topology determined by the restriction of $d(x,y)$ to $x, y \in Y$ is a union of open balls, and hence is an open set with respect to the induced topology.

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. The associated q-metric (3.9) determines a topology on k, as before. One can check that addition and multiplication on k define continuous mappings from $k \times k$ into k, by standard arguments. This uses the product topology on $k \times k$ corresponding to the topology just mentioned on k. Similarly, $x \mapsto 1/x$ is continuous as a mapping from $k \setminus \{0\}$ into itself, with respect to the topology induced on $k \setminus \{0\}$ by the topology on k just mentioned.

Now let $|\cdot|_1$, $|\cdot|_2$ be q_1, q_2-absolute value functions on k for some $q_1, q_2 > 0$. If there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{4.2}$$

for every $x \in k$, then $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* on k. Of course, this implies that

$$|x - y|_2 = |x - y|_1^a \tag{4.3}$$

for every $x, y \in k$. This means that the q_1, q_2-metrics on k associated to $| \cdot |_1$, $| \cdot |_2$ as in (3.9) correspond to each other in the same way. It follows that the q_1, q_2-metrics on k associated to $| \cdot |_1$, $| \cdot |_2$ determine the same topology on k, as before. Conversely, if the topologies determined on k by the q_1, q_2-metrics associated to $| \cdot |_1$, $| \cdot |_2$ are the same, then it is well known that $| \cdot |_1$ and $| \cdot |_2$ are equivalent on k in this sense. Let us sketch some steps in the proof of this statement. Observe that $x \in k$ satisfies $|x|_1 < 1$ if and only if $x^n \to 0$ as $n \to \infty$ with respect to the topology determined on k by the q_1-metric associated to $|\cdot|_1$, and similarly for $| \cdot |_2$. If the topologies determined on k by the q_1, q_2-metrics associated to $| \cdot |_1$, $| \cdot |_2$ are the same, then it follows that the corresponding open unit balls in k are the same. We also have that $x \in k$ satisfies $|x|_1 > 1$ if and only $|x|_2 > 1$ when the topologies are the same, by applying the previous statement to $1/x$. Combining these two statements, we get that $x \in k$ satisfies $|x|_1 = 1$ if and only if $|x|_2 = 1$ when the topologies are the same. If $y, z \in k$, $z \neq 0$, and $m, n \in \mathbf{Z}_+$, then

$$(4.4) \quad |y|_1^m/|z|_1^n = |y^m/z^n|_1 < 1 \quad \text{if and only if} \quad |y|_2^m/|z|_2^n = |y^m/z^n|_2 < 1$$

when the topologies are the same. Using this, one can show that $| \cdot |_1$ and $| \cdot |_2$ are equivalent on k.

Let $|\cdot|$ be a q-absolute value function on $\mathbf{Q}$ for some $q > 0$. A famous theorem of Ostrowski implies that either $| \cdot |$ is the trivial absolute value function on $\mathbf{Q}$, or $| \cdot |$ is equivalent to the standard (Euclidean) absolute value function on $\mathbf{Q}$, or $| \cdot |$ is equivalent to the p-adic absolute value function on $\mathbf{Q}$ for some prime p. Some aspects of the proof of this theorem will be mentioned later.

5 Bounded sets and product q-semimetrics

Let X be a set, and let $d(\cdot, \cdot)$ be a q-semimetric on X for some $q > 0$. A subset E of X is said to be *bounded* with respect to $d(\cdot, \cdot)$ if there is a finite upper bound for $d(x, y)$ with $x, y \in E$. If x_0 is any element of X, then $E \subseteq X$ is bounded with respect to $d(\cdot, \cdot)$ if and only if E is contained in a ball in X centered at x_0 with finite radius with respect to $d(\cdot, \cdot)$. If $E \subseteq X$ is bounded with respect to $d(\cdot, \cdot)$, and if a is any positive real number, then E is also bounded with respect to $d(\cdot, \cdot)^a$ as a (q/a)-semimetric on X. If $K \subseteq X$ is compact with respect to the topology determined by $d(\cdot, \cdot)$, then K is bounded with respect to $d(\cdot, \cdot)$, by standard arguments.

Let n be a positive integer, and suppose that for each $j = 1, \ldots, n$, $d_j(\cdot, \cdot)$ is a q_j-semimetric on X for some $q_j > 0$. Put

$$(5.1) \qquad q = \min(q_1, \ldots, q_n),$$

so that d_j is a q-semimetric on X for each j, as in Section 2. One can check that

$$(5.2) \qquad d(x, y) = \max_{1 \leq j \leq n} d_j(x, y)$$

defines a q-semimetric on X as well. If $x \in X$, then

$$B_d(x, r) = \bigcap_{j=1}^{n} B_{d_j}(x, r) \tag{5.3}$$

for every $r > 0$, and

$$\overline{B}_d(x, r) = \bigcap_{j=1}^{n} \overline{B}_{d_j}(x, r) \tag{5.4}$$

for every $r \geq 0$, where these open and closed balls are defined as in (2.9) and (2.10). Note that $E \subseteq X$ is bounded with respect to $d(\cdot, \cdot)$ if and only if E is bounded with respect to $d_j(\cdot, \cdot)$ for each $j = 1, \ldots, n$.

Let n be a positive integer again, let $X_1, \ldots, X_n$ be sets, and let $X = \prod_{j=1}^{n} X_j$ be their Cartesian product. Thus X is the set of n-tuples $x = (x_1, \ldots, x_n)$ such that $x_j \in X_j$ for each $j = 1, \ldots, n$. Suppose that for each $j = 1, \ldots, n$, $d_j(x_j, y_j)$ is a q_j-semimetric on X_j for some $q_j > 0$. It is easy to see that

$$\widetilde{d}_j(x, y) = d_j(x_j, y_j) \tag{5.5}$$

defines a q_j-semimetric on X for each $j = 1, \ldots, n$. Hence

$$d(x, y) = \max_{1 \leq j \leq n} \widetilde{d}_j(x, y) = \max_{1 \leq j \leq n} d_j(x_j, y_j) \tag{5.6}$$

defines a q-semimetric on X, where q is as in (5.1), as in the preceding paragraph. If $x \in X$, then

$$B_{X,d}(x, r) = \prod_{j=1}^{n} B_{X_j, d_j}(x_j, r) \tag{5.7}$$

for every $r > 0$, where the additional subscripts of X and X_j indicate the spaces in which the corresponding balls are defined. Similarly,

$$\overline{B}_{X,d}(x, r) = \prod_{j=1}^{n} \overline{B}_{X_j, d_j}(x_j, r) \tag{5.8}$$

for every $x \in X$ and $r > 0$. It follows from (5.7) that the topology determined on X by $d(\cdot, \cdot)$ is the same as the product topology corresponding to the topology determined on X_j by $d_j(\cdot, \cdot)$ for each $j = 1, \ldots, n$. If $E_j \subseteq X_j$ is a bounded set with respect to $d_j(\cdot, \cdot)$ for $j = 1, \ldots, n$, then $E = \prod_{j=1}^{n} E_j$ is a bounded subset of X with respect to $d(\cdot, \cdot)$. If $d_j(\cdot, \cdot)$ is a q_j-metric on X_j for each $j = 1, \ldots, n$, then $d(\cdot, \cdot)$ is a q-metric on X.

Of course, one can also combine semimetrics using sums, or sums of powers.

6 Uniform continuity

Let X, Y be sets, and let d_X, d_Y be q_X, q_Y-semimetrics on X, Y, respectively, for some $q_X, q_Y > 0$. As usual, a mapping f from X into Y is said to be *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$d_Y(f(x), f(x')) < \epsilon \tag{6.1}$$

for every $x, x' \in X$ with $d_X(x, x') < \delta$. Of course, this implies that f is continuous with respect to the topologies determined on X, Y by d_X, d_Y, respectively, as in Section 4. Let Z be another set with a q_Z-semimetric d_Z for some $q_Z > 0$. If f is a uniformly continuous mapping from X into Y, and g is a uniformly continuous mapping from Y into Z, then their composition $g \circ f$ is uniformly continuous as a mapping from X into Z.

Let a, b be positive real numbers, so that d_X^a, d_Y^b define (q_X/a), (q_Y/b)-semimetrics on X, Y, respectively, as in Section 2. It is easy to see that a mapping f from X into Y is uniformly continuous with respect to d_X, d_Y if and only if f is uniformly continuous with respect to d_X^a, d_Y^b. As usual, this can be used to reduce to the case of ordinary semimetrics, by taking $a = q_X$ when $q_X \le 1$, and $b = q_Y$ when $q_Y \le 1$.

If f is any continuous mapping from X into Y with respect to the topologies determined by d_X, d_Y, and if X is compact with respect to the topology determined by d_X, then one can show that f is uniformly continuous, using standard arguments. In particular, one can reduce to the case of ordinary semimetrics, as in the preceding paragraph.

Let q be a positive real number such that $q \le q_X$, so that d_X may be considered as a q-semimetric on X, as in Section 2. Also let $w \in X$ be given, and put

$$f_{w,q}(x) = d_X(w, x)^q \tag{6.2}$$

for every $x \in X$. Observe that

$$f_{w,q}(x) \le f_{w,q}(x') + d_X(x, x')^q \tag{6.3}$$

and

$$f_{w,q}(x') \le f_{w,q}(x) + d_X(x, x')^q \tag{6.4}$$

for every $x, x' \in X$, by the q-semimetric version of the triangle inequality. This implies that

$$|f_{w,q}(x) - f_{w,q}(x')| \le d_X(x, x')^q \tag{6.5}$$

for every $x, x' \in X$, where $|\cdot|$ is the standard absolute value function on $\mathbf{R}$. It follows that (6.2) is uniformly continuous as a real-valued function of x on X, with respect to d_X on X, and the standard Euclidean metric on $\mathbf{R}$.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$. Also let q be a positive real number with $q \le q_k$, so that $|\cdot|$ may be considered as a q-absolute value function on k, as in Section 3. Put

$$f_q(x) = |x|^q \tag{6.6}$$

for each $x \in k$, which defines a nonnegative real-valued function on k. This is the same as (6.2), with $X = k$, d_X equal to the q_k-metric (3.9) associated to $|\cdot|$

on k, and $w = 0$. Thus (6.6) is uniformly continuous as a mapping from k into $\mathbf{R}$, with respect to (3.9) on k, and the standard Euclidean metric on $\mathbf{R}$.

As in Section 4, $x \mapsto 1/x$ defines a continuous mapping from $k \setminus \{0\}$ into itself, with respect to the topology induced on $k \setminus \{0\}$ by the topology determined on k by (3.9). If r is a positive real number, then one can check that $x \mapsto 1/x$ is uniformly continuous as a mapping from

$$\{x \in k : |x| \geq r\} \tag{6.7}$$

into k, using (3.9) on k and its restriction to (6.7).

Similarly, addition on k defines a continuous mapping from $k \times k$ into k, using the corresponding product topology on $k \times k$. Using the q_k-metric (3.9) on k associated to $|\cdot|$, we can get a q_k-metric on $k \times k$, as in (5.6). One can verify that addition on k is uniformly continuous as a mapping from $k \times k$ into k, with respect to this q_k-metric on $k \times k$.

Multiplication defines a continuous mapping from $k \times k$ into k as well. The restriction of this mapping to bounded subsets of $k \times k$ is uniformly continuous with respect to the q_k-metric on $k \times k$ mentioned in the preceding paragraph, by standard arguments.

7 Completeness

Let X be a set, and let $d(\cdot, \cdot)$ be a q-metric on X for some $q > 0$. As usual, a sequence $\{x_j\}_{j=1}^\infty$ of elements of X is said to be a *Cauchy sequence* in X with respect to $d(\cdot, \cdot)$ if

$$d(x_j, x_l) \to 0 \quad \text{as } j, l \to \infty. \tag{7.1}$$

If a is a positive real number, then $d(\cdot, \cdot)^a$ is a (q/a)-metric on X, as in Section 2. It is easy to see that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $d(\cdot, \cdot)^a$ if and only if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $d(\cdot, \cdot)$. One can check that convergent sequences in X are Cauchy sequences, as in the case of ordinary metric spaces, and one can reduce to that case using $d(\cdot, \cdot)^q$ when $q \leq 1$. If every Cauchy sequence of elements of X converges to an element of X, then X is said to be *complete* with respect to $d(\cdot, \cdot)$. If $0 < a < \infty$, then X is complete with respect to $d(\cdot, \cdot)$ if and only if X is complete with respect to $d(\cdot, \cdot)^a$.

Let Y be a subset of X, so that the restriction of $d(x, y)$ to $x, y \in Y$ defines a q-metric on Y. Note that a sequence of elements $\{y_j\}_{j=1}^\infty$ of Y is a Cauchy sequence in Y with respect to the restriction of $d(\cdot, \cdot)$ to Y if and only if $\{y_j\}_{j=1}^\infty$ is a Cauchy sequence in X with respect to $d(\cdot, \cdot)$. If X is complete with respect to $d(\cdot, \cdot)$, and Y is a closed set in X with respect to the topology determined by $d(\cdot, \cdot)$, then it is easy to see that Y is complete with respect to the restriction of $d(\cdot, \cdot)$ to Y. In the other direction, if Y is complete with respect to the restriction of $d(\cdot, \cdot)$, then one can check that Y is a closed set in X with respect to the topology determined by $d(\cdot, \cdot)$.

Let X, Y be sets, and let d_X, d_Y be q_X, q_Y-metrics on X, Y, respectively, for some $q_X, q_Y > 0$. Let E be a dense subset of X, and let f be a uniformly continuous mapping from E into Y, with respect to the restriction of d_X to E. If Y is complete with respect to d_Y, then there is a unique extension of f to a uniformly continuous mapping from X into Y. This is well known for ordinary metric spaces, and it can be shown in this situation in essentially the same way, or by reducing to the case of ordinary metric spaces. Of course, the uniqueness of the extension only requires ordinary continuity.

Let X be a set with a q-metric $d(\cdot,\cdot)$ for some $q > 0$ again. If X is not complete with respect to $d(\cdot,\cdot)$, then one can pass to a completion. More precisely, this means that there is an isometric embedding of X onto a dense subset of a complete q-metric space. This is well known when $q = 1$, and one can reduce to that case when $q < \infty$, using a completion of X with respect to the metric $d(\cdot,\cdot)^q$. Alternatively, if $1 \le q \le \infty$, then $d(\cdot,\cdot)$ is a metric on X, and one can use a completion of X as a metric space with respect to $d(\cdot,\cdot)$. In this case, one can check that the metric on the completion is a q-metric too. One can also verify that the completion of X is unique up to isometric equivalence, using the extension theorem mentioned in the previous paragraph.

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. If k is not already complete with respect to the associated q-metric (3.9), then one can pass to a completion of k. One can start with a completion of k as a q-metric space, as before. One can check that the field operations on k can be extended continuously to the completion, so that the completion is a field as well. The extension of $|\cdot|$ to the completion corresponds to the distance to 0 in the completion, and defines a q-absolute value function on the completion. One can also obtain the completion more directly in this situation, as a field with a q-absolute value function. As before, the completion of k with respect to $|\cdot|$ is unique up to isometric isomorphic equivalence. If p is a prime number, then the field $\mathbf{Q}_p$ of *p-adic numbers* is obtained by completing $\mathbf{Q}$ with respect to the p-adic absolute value function $|\cdot|_p$. One also uses $|\cdot|_p$ to denote the corresponding extension of the p-adic absolute value function to $\mathbf{Q}_p$.

8 Separation conditions

A topological space X is said to be *regular in the strict sense* if for every $x \in X$ and closed set $E \subseteq X$ with $x \notin E$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $E \subseteq V$. This is equivalent to asking that for every $x \in X$ and open set $W \subseteq X$ with $x \in W$ there be an open set $U \subseteq X$ such that $x \in U$ and the closure $\overline{U}$ of U in X is contained in W. If X is regular in the strict sense and X satisfies the first or even 0th separation condition, then we may simply say that X is regular, or that X satisfies the third separation condition. In this case, it is easy to see that X is Hausdorff. If the topology on X is determined by a q-semimetric $d(\cdot,\cdot)$ for some $q > 0$, then X is regular in the strict sense.

Similarly, a topological space X is *completely regular in the strict sense* if for every $x \in X$ and closed set $E \subseteq X$ with $x \notin E$ there is a continuous real-valued function f on X such that $f(y) = 0$ for every $y \in E$ and $f(x) \neq 0$. Of course, this uses the standard topology on $\mathbf{R}$ as the range of f. If X is completely regular in the strict sense, then X is regular in the strict sense, because the real line is Hausdorff with respect to the standard topology. If X is completely regular in the strict sense and X satisfies the first or even 0th separation condition, then we may simply say that X is completely regular, or that X satisfies separation condition number three and a half. If the topology on X is determined by a q-semimetric $d(\cdot, \cdot)$ for some $q > 0$, then X is completely regular in the strict sense, because of the continuity of the functions defined in (6.2).

A topological space X is said to be *normal in the strict sense* if for every pair A, B of disjoint closed subsets of X there are disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. If X is normal in the strict sense and X satisfies the first separation condition, then we may simply say that X is normal in this case, or that X satisfies the fourth separation condition. It is easy to see that normal topological spaces are regular. If the topology on X is determined by a q-semimetric $d(\cdot, \cdot)$ for some $q > 0$, then it is well known that X is normal in the strict sense. It is also well known that compact Hausdorff topological spaces are normal.

If a topological space X is normal in the strict sense, and if A, B are disjoint closed subsets of X, then there is a continuous real-valued function f on X such that $f(x) = 0$ for every $x \in A$ and $f(y) = 1$ for every $y \in B$, by Urysohn's lemma. In particular, normal topological spaces are completely regular. It is well known that a locally compact Hausdorff topological space X is completely regular. This can be derived from the normality of the one-point compactification of X, or using a suitable version of the proof of Urysohn's lemma.

A topological space X is said to be *completely Hausdorff* if for every pair x, y of distinct elements of X there are open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and the closures of U and V in X are disjoint. In this case, we also say that X satisfies separation condition number two and a half. Of course, completely Hausdorff spaces are Hausdorff. It is easy to see that regular topological spaces are completely Hausdorff. A topological space X is said to be a *Urysohn space* if continuous real-valued functions on X separate points in X. Thus completely regular topological spaces are Urysohn spaces. Urysohn spaces are completely Hausdorff, because the real line is completely Hausdorff with respect to the standard topology.

Let X be a set, and let τ_1, τ_2 be topologies on X, with

$$\tau_1 \subseteq \tau_2. \tag{8.1}$$

If (X, τ_1) satisfies the 0th, first, or second separation condition, then it is easy to see that (X, τ_2) has the same property. This also works for the completely Hausdorff and Urysohn conditions mentioned in the previous paragraph. However, this type of simple argument does not work for regularity, complete regularity, or normality.

Let X be a topological space again, and let Y be a subset of X, equipped with the induced topology. It is well known and not difficult to show that if X satisfies the 0th, first, or second separation conditions, then Y has the same property. There are analogous statements for regularity, complete regularity, completely Hausdorff spaces, and Urysohn spaces, but not normality.

9 Topological dimension 0

A topological space X has *topological dimension* 0 *in the strict sense* if for every $x \in X$ and open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$, $U \subseteq W$, and U is a closed set in X too. Sometimes one also asks that $X \neq \emptyset$, in connection with inductive notions of topological dimension. If X has topological dimension 0 in the strict sense and X satisfies the first or even 0th separation condition, then we may simply say that X has topological dimension 0. If the topology on X is determined by a semi-ultrametric $d(\cdot, \cdot)$, then X has topological dimension 0 in the strict sense. It is easy to see that the set $\mathbf{Q}$ of rational numbers has topological dimension 0 with respect to the topology induced by the standard topology on $\mathbf{R}$.

A topological space X is said to be *totally separated* if for every pair x, y of distinct elements of X there are disjoint open sets U, V such that $x \in U$, $y \in V$, and $U \cup V = X$. If X has topological dimension 0, then X is totally separated. Of course, totally separated spaces are completely Hausdorff. If τ_1 and τ_2 are topologies on a set X such that $\tau_1 \subseteq \tau_2$ and (X, τ_1) is totally separated, then it is easy to see that (X, τ_2) is totally separated. However, this type of argument does not work for topological dimension 0.

Let X be a totally separated topological space. If $x \in X$, $K \subseteq X$ is compact, and $x \not\in K$, then one can check that there is an open set $U \subseteq X$ such that $x \in U$, $U \cap K = \emptyset$, and U is a closed set in X. If X is also locally compact, then one can use this to show that X has topological dimension 0.

Let X be a topological space, and let Y be a subset of X, equipped with the induced topology. If X is totally separated or has topological dimension 0, then one can verify that Y has the same property. Note that a totally separated topological space with at least two elements is not connected. It follows that a totally separated topological space is totally disconnected, in the sense that every subset of the space with at least two elements is not connected.

Let X, Y be topological spaces, and let $C(X, Y)$ be the space of continuous mappings from X into Y. If $U \subseteq X$ is both open and closed, and if $y_1, y_2 \in Y$, then

$$\begin{aligned} f(x) &= y_1 \quad \text{when } x \in U \\ &= y_2 \quad \text{when } x \in X \setminus U \end{aligned} \tag{9.1}$$

defines a continuous mapping from X into Y. If X is totally separated and Y has at least two elements, then it follows that $C(X, Y)$ separates points in X. In particular, totally separated topological spaces are Urysohn spaces. In the other direction, if Y is totally separated and $C(X, Y)$ separates points in X, then it is easy to see that X is totally separated.

Suppose that X has topological dimension 0 in the strict sense. If $x \in X$, $E \subseteq X$ is a closed set, and $x \not\in E$, then there is an open set $U \subseteq X$ such that $x \in U$, $U \cap E = \emptyset$, and U is a closed set in X. This is equivalent to the earlier definition, with $W = X \setminus E$. If $y_1, y_2 \in Y$, then (9.1) defines a continuous mapping from X into Y that is constant on E. In particular, this implies that X is completely regular in the strict sense.

Let X, Y be topological spaces again. Suppose that for every $x \in X$ and closed set $E \subseteq X$ with $x \notin E$ there is a continuous mapping f from X into Y such that f is constant on E and this constant value is different from $f(x)$. If Y is totally separated, then X has topological dimension 0 in the strict sense.

10 The archimedean property

Let k be a field, and let $\mathbf{Z}_+$ be the set of positive integers. If $x \in k$ and $n \in \mathbf{Z}_+$, then let $n \cdot x$ be the sum of n x's in k. Observe that

$$m \cdot (n \cdot x) = (m\, n) \cdot x \tag{10.1}$$

for every $m, n \in \mathbf{Z}_+$ and $x \in k$, and that

$$n \cdot (x\, y) = (n \cdot x)\, y \tag{10.2}$$

for every $n \cdot \mathbf{Z}_+$ and $x, y \in k$. In particular,

$$n \cdot y = (n \cdot 1)\, y \tag{10.3}$$

for every $n \in \mathbf{Z}_+$ and $y \in k$, where 1 is the multiplicative identity element in k. Using this, one can check that

$$(n \cdot 1)^j = n^j \cdot 1 \tag{10.4}$$

for every $j, n \in \mathbf{Z}_+$.

Let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. If there are positive integers n such that $|n \cdot 1|$ is arbitrarily large, then $|\cdot|$ is said to be *archimedean* on k. Otherwise, $|\cdot|$ is *non-archimedean* on k, which means that there is a nonnegative real number C such that

$$|n \cdot 1| \le C \tag{10.5}$$

for every $n \in \mathbf{Z}_+$. If $n \in \mathbf{Z}_+$ and $|n \cdot 1| > 1$, then

$$|(n^j \cdot 1)| = |(n \cdot 1)^j| = |n \cdot 1|^j \to \infty \quad \text{as } j \to \infty, \tag{10.6}$$

so that $|\cdot|$ is archimedean on k. If $|\cdot|$ is non-archimedean on k, then it follows that (10.5) holds with $C = 1$.

If $|\cdot|$ is an ultrametric absolute value function on k, then it is easy to see that (10.5) holds with $C = 1$. Conversely, if $|\cdot|$ is non-archimedean on k, then it is well known that $|\cdot|$ is an ultrametric absolute value function on k. To see this, suppose that $q < \infty$, and that (10.5) holds for some $C \geq 1$. If $x, y \in k$ and $n \in \mathbf{Z}_+$, then

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} \cdot x^j\, y^{n-j}, \tag{10.7}$$

where $\binom{n}{j} \in \mathbf{Z}_+$ are the usual binomial coefficients. It follows that

$$|x+y|^{n\,q} = |(x+y)^n|^q \le \sum_{j=0}^{n} \left| \binom{n}{j} \cdot x^j \, y^{n-j} \right|^q, \tag{10.8}$$

using (3.3) in the second step. Thus

$$\begin{aligned} |x+y|^{n\,q} &\le \sum_{j=0}^{n} \left| \binom{n}{j} \cdot 1 \right|^q |x|^{j\,q} \, |y|^{(n-j)\,q} \\ &\le C^q \, (n+1) \, \max(|x|, |y|)^{n\,q}, \end{aligned} \tag{10.9}$$

using (10.5) in the second step. This implies that

$$|x+y| \le C^{1/n} \, (n+1)^{1/(n\,q)} \, \max(|x|, |y|). \tag{10.10}$$

To get the ultrametric version (3.7) of the triangle inequality, one can take the limit as $n \to \infty$ of the right side of (10.10).

Let $|\cdot|$ be a q-absolute value function on $\mathbf{Q}$ for some $q > 0$, and suppose that

$$|n| \le 1 \tag{10.11}$$

for every $n \in \mathbf{Z}_+$. This implies that $|\cdot|$ is non-archimedean on $\mathbf{Q}$, so that $|\cdot|$ is an ultrametric absolute value function on $\mathbf{Q}$. If $|n| = 1$ for every $n \in \mathbf{Z}_+$, then it is easy to see that $|\cdot|$ is the trivial absolute value function on $\mathbf{Q}$. Otherwise, suppose that $|n| < 1$ for some $n \in \mathbf{Z}_+$. Let p be the smallest positive integer with $|p| < 1$. Of course, $p > 1$, because $|1| = 1$, as in (3.4). On can check that p has to be a prime number, using (3.2). Under these conditions, one can show that $|\cdot|$ is equivalent to the p-adic absolute function on $\mathbf{Q}$. This is part of the theorem of Ostrowski mentioned in Section 4.

11 Some related conditions

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some positive real number q. One can check that

$$|n \cdot 1|^q \le n \tag{11.1}$$

for every $n \in \mathbf{Z}_+$, using (3.3). Suppose that $q \le 1$, and

$$|n \cdot 1| \le C\, n \tag{11.2}$$

for some real number $C \ge 1$ and every $n \in \mathbf{Z}_+$. If $j, n \in \mathbf{Z}_+$, then we get that

$$|n \cdot 1|^j = |(n \cdot 1)^j| = |n^j \cdot 1| \le C\, n^j, \tag{11.3}$$

using (10.4) in the second step. This implies that

$$|n \cdot 1| \le C^{1/j}\, n \tag{11.4}$$

for every $j, n \in \mathbf{Z}_+$. It follows that

$$|n \cdot 1| \le n \tag{11.5}$$

for every $n \in \mathbf{Z}_+$, because $C^{1/j} \to 1$ as $j \to \infty$. Let us show that (11.2) implies that $|\cdot|$ is an ordinary absolute value function on k.

If $x, y \in k$ and $n \in \mathbf{Z}_+$, then

$$|x+y|^{n\,q} \le \sum_{j=0}^{n} \left| \binom{n}{j} \cdot x^j\, y^{n-j} \right|^q \le (n+1) \max_{0 \le j \le n} \left| \binom{n}{j} \cdot x^j\, y^{n-j} \right|^q, \tag{11.6}$$

using (10.8) in the first step. This implies that

$$|x+y|^n \le (n+1)^{1/q} \max_{0 \le j \le n} \left(\left| \binom{n}{j} \cdot 1 \right| |x|^j\, |y|^{n-j} \right), \tag{11.7}$$

by taking the qth root of both sides of (11.6). It follows that

$$\begin{aligned} |x+y|^n &\le (n+1)^{1/q} \sum_{j=0}^{n} \left| \binom{n}{j} \cdot 1 \right| |x|^j\, |y|^{n-j} \\ &\le C\,(n+1)^{1/q} \sum_{j=0}^{n} \binom{n}{j} |x|^j\, |y|^{n-j} = C\,(n+1)^{1/q}\,(|x|+|y|)^n, \end{aligned} \tag{11.8}$$

using (11.2) in the second step, and the binomial theorem in the third step. Thus

$$|x+y| \le C^{1/n}\,(n+1)^{1/(q\,n)}\,(|x|+|y|) \tag{11.9}$$

by taking the nth roots of both sides of (11.8). This implies that $|\cdot|$ satisfies the ordinary version of the triangle inequality on k, by taking the limit as $n \to \infty$ on the right side of (11.9).

Let k be any field again, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. If k has positive characteristic, then there are only finitely many elements of k of the form $n \cdot 1$ with $n \in \mathbf{Z}_+$, so that $|\cdot|$ is automatically non-archimedean on k. Suppose now that k has characteristic 0, so that there is a natural embedding of $\mathbf{Q}$ into k. Thus $|\cdot|$ leads to a q-absolute value function on $\mathbf{Q}$. It is easy to see that $|\cdot|$ is archimedean on k exactly when the induced q-absolute value function on $\mathbf{Q}$ is archimedean. In this case, the theorem of Ostrowski mentioned in Section 4 implies that the induced absolute value function on $\mathbf{Q}$ is equivalent to the standard Euclidean absolute value function on $\mathbf{Q}$. Let us suppose that the induced absolute value function on $\mathbf{Q}$ is equal to the standard absolute value function on $\mathbf{Q}$, which can be arranged by replacing $|\cdot|$ on k with a suitable positive power of itself. In this situation, the earlier discussion implies that $|\cdot|$ is an ordinary absolute value function on k.

Let $|\cdot|$ be an archimedean q-absolute value function on a field k for some $q > 0$ again, and suppose that k is complete with respect to the associated q-metric. Under these conditions, another famous theorem of Ostrowski implies that k

is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, where $|\cdot|$ corresponds to a q-absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, that is equivalent to the standard absolute value function. As in the preceding paragraph, we can replace $|\cdot|$ on k with a positive power of itself, if necessary, to get that induced absolute value function on $\mathbf{Q}$ is equal to the standard absolute value function on $\mathbf{Q}$. With this normalization, $|\cdot|$ corresponds exactly to the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, under the isomorphism just mentioned.

12 q-Norms and q-seminorms

Let k be a field, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. Also let V be a vector space over k, and let q be a positive real number. A nonnegative real-valued function N on V is said to be a *q-seminorm* on V with respect to $|\cdot|$ on k if it satisfies the following two conditions. First,

$$N(t\,v) = |t|\,N(v) \tag{12.1}$$

for every $v \in V$ and $t \in k$. Note that this implies that $N(0) = 0$, by taking $t = 0$. Second,

$$N(v+w)^q \le N(v)^q + N(w)^q \tag{12.2}$$

for every $v, w \in V$. If we also have that

$$N(v) > 0 \quad \text{when } v \ne 0, \tag{12.3}$$

then N is said to be a *q-norm* on V with respect to $|\cdot|$ on k. A q-norm or q-seminorm with $q = 1$ is also known as a *norm* or *seminorm*, respectively.

As usual, (12.2) is the same as saying that

$$N(v+w) \le (N(v)^q + N(w)^q)^{1/q} \tag{12.4}$$

for every $v, w \in V$. A nonnegative real-valued function N on V is said to be a *semi-ultranorm* on V with respect to $|\cdot|$ on k if it satisfies (12.1) and

$$N(v+w) \le \max(N(v), N(w)) \tag{12.5}$$

for every $v, w \in V$. If N satisfies (12.3) too, then N is said to be an *ultranorm* on V with respect to $|\cdot|$ on k. An ultranorm or semi-ultranorm may be considered as a q-norm or q-seminorm with $q = \infty$, respectively, because of (1.9). If $0 < q_1 \le q_2 \le \infty$ and N is a q_2-norm or q_2-seminorm on V with respect to $|\cdot|$ on k, then N is a q_1-norm or q_1-seminorm on V with respect to $|\cdot|$ on k, as appropriate, because of the first inequality in (1.8) and (1.10).

If N is a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$, then

$$d(v,w) = d_N(v,w) = N(v-w) \tag{12.6}$$

defines a q-semimetric on V. This uses the fact that $|-1| = 1$, as in Section 3, to get that (12.6) satisfies the symmetry condition (2.2). If N is a q-norm on V, then (12.6) defines a q-metric on V.

The *trivial ultranorm* is defined on V by putting $N(v)$ equal to 1 when $v \neq 0$ and to 0 when $v = 0$. This is an ultranorm on V with respect to the trivial absolute value function on k. The ultrametric on V corresponding to the trivial ultranorm as in (12.6) is the same as the discrete metric on V.

Let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$, and suppose that $N(v) > 0$ for some $v \in V$. Under these conditions, one can check that $|\cdot|$ is a q-absolute value function on k.

Let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$ again, and let a be a positive real number. Remember that $|\cdot|^a$ is a (q_k/a)-absolute value function on k, as in Section 3. Similarly,

$$N(v)^a \tag{12.7}$$

is a (q/a)-seminorm on V with respect to $|\cdot|^a$ on k. If N is a q-norm on V with respect to $|\cdot|$ on k, then (12.7) is a (q/a)-norm on V with respect to $|\cdot|^a$ on k. Of course, the (q/a)-metric or semimetric on V associated to (12.7) is the ath power of the q-metric or q-semimetric associated to N on V.

Let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$ again. Thus the corresponding q-semimetric (12.6) determines a topology on V, as in Section 4. It is easy to see that addition of vectors in V defines a continuous mapping from $V \times V$ into V, using the product topology on $V \times V$ associated to the topology on V just mentioned. Similarly, scalar multiplication on V determines a continuous mapping from $k \times V$ into V. This uses the product topology on $k \times V$ associated to the topology determined on k by the q_k-metric corresponding to $|\cdot|$ and the topology on V just mentioned.

Using (12.6), we can get a q-semimetric on $V \times V$, as in (5.6). One can check that addition of vectors in V defines a uniformly continuous mapping from $V \times V$ into V with respect to this q-semimetric. Similarly, we can get a $\min(q_k, q)$-semimetric on $k \times V$ using the q_k-metric on k associated to $|\cdot|$ and (12.6). One can verify that the mapping from $k \times V$ into V corresponding to scalar multiplication is uniformly continuous on bounded subsets of $k \times V$, with respect to this $\min(q_k, q)$-semimetric on $k \times V$.

If q_0 is a positive real number with $q_0 \leq q$, then N may be considered as a q_0-seminorm on V with respect to $|\cdot|$ on k, as before. Put

$$f_{q_0}(v) = N(v)^{q_0} \tag{12.8}$$

for each $v \in V$, which defines a nonnegative real-valued function on V. This is the same as (6.2), with $X = V$, d_X equal to (12.6), $w = 0$, and q replaced with q_0. Hence (12.8) is uniformly continuous as a mapping from V into $\mathbf{R}$, with respect to (12.6) on V, and the standard Euclidean metric on $\mathbf{R}$.

Let N be a q-norm on V with respect to $|\cdot|$ on k for some $q > 0$. If V is not already complete as a q-metric space with respect to (12.6), then one can pass to a completion of V. One can start by completing V as a q-metric space, as in Section 2. The vector space operations on V can be extended continuously to the completion of V, so that the completion of V becomes a vector space over k too. The extension of N to the completion corresponds to the distance to 0 in the completion, and defines a q-norm on the completion with respect to $|\cdot|$ on k. The completion of V as a vector space over k with a q-norm with respect to $|\cdot|$ on k can be obtained more directly as well. If V is complete as a q-metric space with respect to (12.6), and k is not complete as a q_k-metric space with respect to (3.9), then scalar multiplication on V can be extended continuously to the completion of k. This makes V into a vector space over the completion of k, and N becomes a q-norm on V as a vector space over the completion of k.

If V is complete with respect to the q-metric associated to N, then V is said to be a *q-Banach space* with respect to N. In this case, if $q = 1$, then we may simply say that V is a *Banach space* with respect to N. One may prefer to include the completeness of k in the definition of a q-Banach space.

13 Supremum q-semimetrics and q-seminorms

Let X, Y be nonempty sets, and let d_Y be a q_Y-semimetric on Y for some $q_Y > 0$. A mapping f from X into Y is said to be *bounded* if $f(X)$ is bounded as a subset of Y, with respect to d_Y. Let $B(X, Y)$ be the collection of bounded mappings from X into Y. If $f, g \in B(X, Y)$, then it is easy to see that $d_Y(f(x), g(x))$ is bounded as a nonnegative real-valued function on X. Thus means that

$$\theta(f, g) = \sup_{x \in X} d_Y(f(x), g(x)) \tag{13.1}$$

is defined as a nonnegative real number. One can check that (13.1) defines a q_Y-semimetric on $B(X, Y)$, which is the *supremum q-semimetric* associated to d_Y. If d_Y is a q_Y-metric on Y, then (13.1) defines a q_Y-metric on $B(X, Y)$. In this case, if Y is also complete with respect to d_Y, then $B(X, Y)$ is complete with respect to (13.1), by standard arguments.

If X is a topological space, then we let $C(X, Y)$ denote the space of continuous mappings from X into Y, as in Section 9, using the topology determined on Y by d_Y. Let $C_b(X, Y)$ be the space of bounded continuous mappings from X into Y, so that

$$C_b(X, Y) = B(X, Y) \cap C(X, Y). \tag{13.2}$$

One can verify that $C_b(X, Y)$ is a closed set in $B(X, Y)$ with respect to (13.1), using standard arguments. If X is equipped with the discrete topology, then $C_b(X, Y)$ is the same as $B(X, Y)$. If X is a compact topological space, and f is a continuous mapping from X into Y, then $f(X)$ is a compact subset of Y, which implies that $f(X)$ is bounded in Y. Thus $C_b(X, Y)$ is the same as $C(X, Y)$ when X is compact. If d_Y is a q_Y-metric on Y, and Y is complete with respect to d_Y, then $C_b(X, Y)$ is complete with respect to (13.1). This follows from the completeness of $B(X, Y)$ with respect to (13.1), and the fact that $C_b(X, Y)$ is a closed set in $B(X, Y)$ with respect to the topology determined by (13.1).

Let X be a nonempty set again, let k be a field, and let V be a vector space over k. Observe that the space $c(X, V)$ of all V-valued functions on X is a vector space over k with respect to pointwise addition and scalar multiplication of functions.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$. Let $\ell^\infty(X, V) = \ell^\infty_N(X, V)$ be the space of V-valued functions on X that are bounded with respect to N, in the sense that $N(f(x))$ is bounded as a nonnegative real-valued function on X. This is the same as saying that f is bounded with respect to the q-semimetric on V associated to N, so that $\ell^\infty(X, V)$ is the same as $B(X, V)$ in this situation. It is easy to see that $\ell^\infty(X, V)$ is a linear subspace of $c(X, V)$. Put

$$\|f\|_\infty = \|f\|_{\ell^\infty(X,V)} = \|f\|_{\ell^\infty_N(X,V)} = \sup_{x \in X} N(f(x)) \tag{13.3}$$

for every $f \in \ell^\infty(X, V)$. One can check that this defines a q-seminorm on $\ell^\infty(X, V)$ with respect to $|\cdot|$ on k. This is known as the *supremum q-seminorm* on $\ell^\infty(X, V)$ associated to N. Note that the q-semimetric on $\ell^\infty(X, V)$ associated to (13.3) is the same as the supremum q-semimetric corresponding to the q-semimetric on V associated to N as in (13.1). If N is a q-norm on V, then (13.3) is a q-norm on $\ell^\infty(X, V)$.

Now let X be a nonempty topological space, and let $C(X, V)$ be the space of all continuous V-valued functions on X, as in Section 9. This uses the topology determined on V by the q-semimetric associated to N. It is easy to see that $C(X, V)$ is a linear subspace of the space $c(X, V)$ of all V-valued functions on X. Let E be a nonempty compact subset of X. If $f \in C(X, V)$, then f is bounded on E with respect to N, by standard results. Thus

$$\|f\|_{sup,E} = \|f\|_{sup,E,N} = \sup_{x \in E} N(f(x)) \tag{13.4}$$

is defined as a nonnegative real number, and in fact the supremum is attained. One can check that (13.4) defines a q-seminorm on $C(X, V)$ with respect to $|\cdot|$ on k. This is the *supremum q-seminorm* on $C(X, V)$ associated to E.

Let $C_b(X, V)$ be the space of V-valued functions on X that are bounded and continuous with respect to N, as before. This is a linear subspace of both $C(X, V)$ and $\ell^\infty(X, V)$. The supremum q-seminorm (13.3) may also be denoted $\|f\|_{sup}$, which corresponds to (13.4) with $E = X$.

If X is a nonempty topological space and $E \subseteq X$ is nonempty and compact, then the q-semimetric on $C(X, V)$ corresponding to (13.4) is the *supremum q-semimetric* associated to E. This q-semimetric determines a topology τ_E on $C(X, V)$, as in Section 4. Let τ be the topology on $C(X, V)$ which is generated by the collection of topologies τ_E, where E is a nonempty compact subset of X. Of course, finite subsets of X are compact, so that this collection is nonempty. More precisely, the union of τ_E over all nonempty compact subsets E of X is a sub-base for τ. In this situation, one can check that the union of τ_E over all nonempty compact subsets E of X is a base for τ. This uses the fact that if $E_1, \ldots E_n$ are finitely many nonempty compact subsets of X, then their union $\bigcup_{j=1}^n E_j$ is compact as well. The supremum q-seminorm on $C(X, V)$ associated to $\bigcup_{j=1}^n E_j$ is the same as the maximum of the supremum q-seminorms associated to $E_1, \ldots, E_n$. If E is any nonempty compact subset of X, then the collection of open balls in $C(X,V)$ corresponding to the supremum q-semimetric associated to (13.4) is a base for τ_E. Thus the collection of all open balls in $C(X,V)$ corresponding to supremum q-semimetrics associated to nonempty compact subsets E of X is a base for τ. If X is compact, then $C(X,V) = C_b(X,V)$, and τ is the same as the topology τ_X determined by the supremum q-semimetric associated to $E = X$.

14 Bounded linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k. Also let N_V, N_W be q_V, q_W-seminorms on V, W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. A linear mapping T from V into W is said to be *bounded* with respect to N_V, N_W if there is a nonnegative real number C such that

$$N_W(T(v)) \le C\, N_V(v) \tag{14.1}$$

for every $v \in V$. This implies that

$$N_W(T(u) - T(v)) = N_W(T(u-v)) \le C\, N_V(u-v) \tag{14.2}$$

for every $u, v \in V$, and hence that T is uniformly continuous as a mapping from V into W, with respect to the q_V, q_W-semimetrics associated to N_V, N_W on V, W, respectively. In the other direction, let T be a linear mapping from V into W, and suppose that $|\cdot|$ is nontrivial on k. If $N_W(T(v))$ is bounded on a ball in V centered at 0 with positive radius with respect to N_V, then one can check that T is bounded as a linear mapping with respect to N_V, N_W. In particular, this condition holds when T is continuous at 0 with respect to the topologies determined on V, W by the q_V, q_W-semimetrics associated to N_V, N_W, respectively.

If T is a bounded linear mapping from V into W with respect to N_V, N_W, then put

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \ge 0 : (14.1) \text{ holds}\}, \tag{14.3}$$

where more precisely the infimum is taken over all nonnegative real numbers C such that (14.1) holds. It is easy to see that the infimum is automatically attained, so that (14.1) holds with $C = \|T\|_{op}$. Let $\mathcal{BL}(V,W)$ be the space of all bounded linear mappings from V into W with respect to N_V, N_W. One can verify that $\mathcal{BL}(V,W)$ is a vector space over k with respect to pointwise addition and scalar multiplication. Moreover, (14.3) defines a q_W-seminorm on $\mathcal{BL}(V,W)$. If N_W is a q_W-norm on W, then (14.3) defines a q_W-norm on $\mathcal{BL}(V,W)$. In this case, if W is complete with respect to the q_W-metric associated to N_W, then $\mathcal{BL}(V,W)$ is complete with respect to the q_W-metric associated to (14.3), by standard arguments.

Let Z be another vector space over k, and let N_Z be a q_Z-seminorm on Z with respect to $|\cdot|$ on k, for some $q_Z > 0$. If T_1 is a bounded linear mapping from V into W with respect to N_V, N_W, and T_2 is a bounded linear mapping from W into Z with respect to N_W, N_Z, then their composition $T_2 \circ T_1$ is a bounded linear mapping from V into Z with respect to N_V, N_Z, with

$$\|T_2 \circ T_1\|_{op,VZ} \le \|T_1\|_{op,VW} \, \|T_2\|_{op,WZ}. \tag{14.4}$$

Let $\mathcal{BL}(V)$ be the space of bounded linear mappings from V into itself, using N_V on V as both the domain and range of the mapping. Note that the identity mapping $I = I_V$ on V is a bounded linear mapping from V into itself. If $N_V(v) > 0$ for some $v \in V$, then $\|I\|_{op} = 1$.

Suppose now that N_V, N_W are q_V, q_W-norms on V, W, respectively, and let V_0 be a linear subspace of V that is dense in V with respect to the q_V-metric associated to N_V. Let T_0 be a bounded linear mapping from V_0 into W, using the restriction of N_V to V_0 as a q_V-norm on V_0. In particular, T_0 is uniformly continuous with respect to the q_V, q_W-metrics associated to N_V, N_W, respectively, as before. If W is complete with respect to the q_W-metric associated to N_W, then there is a unique extension of T_0 to a uniformly continuous mapping from V into W, as in Section 7. In this situation, one can check that this extension is a bounded linear mapping from V into W, with the same operator q_W-norm as on V_0.

15 Bilinear mappings

Let k be a field, and let V, W, and Z be vector spaces over k. Also let $b(v, w)$ be a Z-valued function of $v \in V$ and $w \in W$. As usual, b is said to be *bilinear* if $b(v, w)$ is linear in each variable. More precisely, this means that $b(v, w)$ is linear in v for each $w \in W$, and that $b(v, w)$ is linear in w for each $v \in V$. Thus

$$b_{1,v}(w) = b(v, w) \tag{15.1}$$

may be considered as a linear mapping from W into Z for each $v \in V$, and

$$b_{2,w}(v) = b(v, w) \tag{15.2}$$

may be considered as a linear mapping from V into Z for each $w \in W$. Let $\mathcal{L}(V, Z)$ be the space of linear mappings from V into Z, and similarly for $\mathcal{L}(W, Z)$. These are vector spaces over k with respect to pointwise addition and scalar multiplication. The bilinearity of b implies that

$$v \mapsto b_{1,v} \tag{15.3}$$

defines a linear mapping from V into $\mathcal{L}(W, Z)$, and that

$$w \mapsto b_{2,w} \tag{15.4}$$

defines a linear mapping from W into $\mathcal{L}(V, Z)$. Conversely, a linear mapping from V into $\mathcal{L}(W, Z)$ or from W into $\mathcal{L}(V, Z)$ corresponds to a bilinear mapping from $V \times W$ into Z in this way.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N_V, N_W, and N_Z be q_V, q_W, and q_Z-seminorms on V, W, and Z, respectively, for some $q_V, q_W, q_Z > 0$, and with respect to $|\cdot|$ on k. A bilinear mapping b from $V \times W$ into Z is said to be *bounded* with respect to N_V, N_W, and N_Z if there is a nonnegative real number C such that

$$N_Z(b(v, w)) \le C\, N_V(v)\, N_W(w) \tag{15.5}$$

for every $v \in V$ and $w \in W$. In this case, one can check that b is continuous as a mapping from $V \times W$ into Z, with respect to the topologies determined on V, W, and Z by the q-semimetrics associated to N_V, N_W, and N_Z, and the corresponding product topology on $V \times W$. More precisely, remember that one can get a $\min(q_V, q_W)$-semimetric on $V \times W$ from the q_V, q_W-semimetrics on V, W associated to N_V, N_W, respectively, as in (5.6). One can verify that b is uniformly continuous on bounded subsets of $V \times W$ with respect to this $\min(q_V, q_W)$-semimetric, using standard arguments.

Suppose for the moment that $|\cdot|$ is nontrivial on k. If $N_Z(b(v, w))$ is bounded for v, w in balls in V, W centered at 0 with positive radius with respect to N_V, N_W, respectively, then it is easy to see that b is bounded as a bilinear mapping from $V \times W$ into Z with respect to N_V, N_W, and N_Z. In particular, this condition holds when b is continuous as a mapping from $V \times W$ into Z at $(0, 0)$.

Suppose that b is a bounded bilinear mapping from $V \times W$ into Z with respect to N_V, N_W, and N_Z, so that (15.5) holds for some $C \ge 0$. If $v \in V$ and $b_{1,v}$ is as in (15.1), then it follows that $b_{1,v}$ is a bounded linear mapping from W into Z, with

$$\|b_{1,v}\|_{op,WZ} \le C\, N_V(v). \tag{15.6}$$

This implies that (15.3) is a bounded linear mapping from V into $\mathcal{BL}(W, Z)$, with respect to the corresponding operator q_Z-seminorm $\|\cdot\|_{op,WZ}$ on $\mathcal{BL}(W, Z)$. Similarly, if $w \in W$ and $b_{2,w}$ is as in (15.2), then $b_{2,w}$ is a bounded linear mapping from V into Z, with

$$\|b_{2,w}\|_{op,VZ} \le C\, N_W(w). \tag{15.7}$$

This means that (15.4) is a bounded linear mapping from W into $\mathcal{BL}(V, Z)$, with respect to the corresponding operator q_Z-seminorm $\|\cdot\|_{op,VZ}$.

Suppose now that N_V, N_W, and N_Z are q_V, q_W, and q_Z-norms on V, W, and Z, respectively. Let V_0, W_0 be dense linear subspaces of V, W, respectively, with respect to the topologies determined by the q_V, q_W-metrics associated to N_V, N_W. Also let b_0 be a bilinear mapping from $V_0 \times W_0$ into Z that is bounded with respect to the restrictions of N_V, N_W to V_0, W_0, respectively. If Z is complete with respect to the q_Z-metric associated to N_Z, then there is a unique extension of b_0 to a bounded bilinear mapping from $V \times W$ into Z. To get the existence of this extension, one can use the uniform continuity of b_0 on bounded subsets of $V_0 \times W_0$, as before. Alternatively, the extension can be obtained one variable at a time, using the analogous statement for bounded linear mappings in the previous section. Note that the constant for the boundedness of the extension of b_0 to $V \times W$ is the same as for b_0 on $V_0 \times W_0$.

16 Associative algebras

Let k be a field, and let $\mathcal{A}$ be a vector space over k. Suppose that for each $x, y \in \mathcal{A}$, the product $x\,y$ is defined as an element of $\mathcal{A}$. More precisely, this corresponds to a mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$. As usual, this operation is said to be *associative* on $\mathcal{A}$ if

$$(x\,y)\,z = z\,(y\,z) \tag{16.1}$$

for every $x, y, z \in \mathcal{A}$. If multiplication on $\mathcal{A}$ is both associative and bilinear, then $\mathcal{A}$ is said to be an (associative) *algebra* over k. If

$$x\,y = y\,x \tag{16.2}$$

for every $x, y \in \mathcal{A}$, then multiplication on $\mathcal{A}$ is said to be *commutative*. If $\mathcal{A}$ is an algebra over k and multiplication on $\mathcal{A}$ is commutative, then $\mathcal{A}$ is said to be a *commutative algebra* over k.

Let $\mathcal{A}$ be an algebra over k. An element e of $\mathcal{A}$ is said to be the *multiplicative identity element* in $\mathcal{A}$ if

$$e\,x = x\,e = x \tag{16.3}$$

for every $x \in \mathcal{A}$. If there is a multiplicative identity element in $\mathcal{A}$, then it is easy to see that it is unique.

If V is a vector space over k, then the space $\mathcal{L}(V)$ of linear mappings from V into itself is a vector space with respect to pointwise addition and scalar multiplication. In fact, $\mathcal{L}(V)$ is an algebra over k with respect to composition of linear mappings. The identity mapping $I = I_V$ on V is the multiplicative identity element in $\mathcal{L}(V)$.

If X is a nonempty set, then the space $c(X, k)$ of all k-valued functions on X is a commutative algebra over k with respect to pointwise multiplication of functions. Let $\mathbf{1}_X$ be the k-valued function on X whose value at every point in X is the multiplicative identity element 1 in k. This is the multiplicative identity element in $c(X, k)$.

Let $\mathcal{A}$, $\mathcal{B}$ be algebras over k. A linear mapping ϕ from $\mathcal{A}$ into $\mathcal{B}$ is said to be an (algebra) *homomorphism* if

$$\phi(x\,y) = \phi(x)\,\phi(y) \tag{16.4}$$

for every $x, y \in \mathcal{A}$. Of course, this uses multiplication on $\mathcal{A}$ on the left side, and multiplication on $\mathcal{B}$ on the right side.

Let $\mathcal{A}$ be an algebra over k again, and let a be an element of $\mathcal{A}$. Put

$$M_a(x) = a\,x \tag{16.5}$$

for each $x \in \mathcal{A}$, so that M_a defines a linear mapping from $\mathcal{A}$ into itself. This is the (left) *multiplication operator* on $\mathcal{A}$ associated to a. If b is another element of $\mathcal{A}$, then

$$M_a(M_b(x)) = M_a(b\,x) = a\,(b\,x) = (a\,b)\,x = M_{a\,b}(x) \tag{16.6}$$

for every $x \in \mathcal{A}$. Thus

$$M_a \circ M_b = M_{a\,b} \tag{16.7}$$

as linear mappings from $\mathcal{A}$ into itself.

It is easy to see that

$$a \mapsto M_a \tag{16.8}$$

defines a linear mapping from $\mathcal{A}$ into the space $\mathcal{L}(\mathcal{A})$ of linear mappings from $\mathcal{A}$ into itself. More precisely, this is an algebra homomorphism from $\mathcal{A}$ into $\mathcal{L}(\mathcal{A})$, because of (16.7). If $\mathcal{A}$ has a multiplicative identity element e, then M_e is the identity mapping on $\mathcal{A}$. In this case, we also have that

$$M_a(e) = a\,e = a \tag{16.9}$$

for every $a \in \mathcal{A}$, which implies that (16.8) is injective.

17 Submultiplicative q-seminorms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k. Also let N be a q-seminorm on $\mathcal{A}$ as a vector space over k for some $q > 0$, with respect to $|\cdot|$ on k. If

$$N(x\,y) \le N(x)\,N(y) \tag{17.1}$$

for every $x, y \in \mathcal{A}$, then N is said to be *submultiplicative* on $\mathcal{A}$. If $\mathcal{A}$ has a multiplicative identity element e, then we get that

$$N(x) = N(e\,x) \le N(e)\,N(x) \tag{17.2}$$

for every $x \in \mathcal{A}$. This implies that $N(e) \ge 1$ when $N(x) > 0$ for some $x \in \mathcal{A}$.

Let X be a nonempty topological space, and let $C(X, k)$ be the space of continuous k-valued functions on X, as in Section 9. This uses the topology determined on k by the q_k-metric associated to $|\cdot|$. As in Section 13, $C(X, k)$ is a vector space over k with respect to pointwise addition and scalar multipliction of functions, and in fact $C(X, k)$ is a commutative algebra with respect to pointwise multiplication of functions. More precisely, $C(X, k)$ is a subalgebra of the algebra $c(X, k)$ of all k-valued functions on X. If E is a nonempty compact subset of X, then

$$\|f\|_{sup,E} = \sup_{x \in E} |f(x)| \tag{17.3}$$

defines a q_k-seminorm on $C(X, k)$ with respect to $|\cdot|$ on k, as in Section 13. It is easy to see that

$$\|f\,g\|_{sup,E} \le \|f\|_{sup,E}\,\|g\|_{sup,E} \tag{17.4}$$

for every $f, g \in C(X, k)$, so that (17.3) is submultiplicative on $C(X, k)$. Of course, constant functions on X are continuous, including the function $\mathbf{1}_X$ whose value at every point in X is the multiplicative identity element 1 in k. Note that

$$\|\mathbf{1}_X\|_{sup,E} = |1| = 1 \tag{17.5}$$

for every nonempty compact set $E \subseteq X$, using (3.4) in the second step.

Let X be a nonempty set, and let $\ell^\infty(X,k)$ be the space of bounded k-valued functions on X, as in Section 13. This is a vector space over k with respect to pointwise addition and scalar multiplication of functions, and in fact $\ell^\infty(X,k)$ is a subalgebra of $c(X,k)$. Remember that

$$\|f\|_\infty = \|f\|_{\ell^\infty(X,k)} = \sup_{x \in X} |f(x)| \tag{17.6}$$

defines a q_k-norm on $\ell^\infty(X,k)$, as in (13.3). If $f,g \in \ell^\infty(X,k)$, then

$$\|f\,g\|_\infty \le \|f\|_\infty\,\|g\|_\infty, \tag{17.7}$$

so that (17.6) is submultiplicative on $\ell^\infty(X,k)$. Constant k-valued functions on X are bounded, and

$$\|\mathbf{1}_X\|_\infty = |1| = 1, \tag{17.8}$$

using (3.4) again.

Let X be a nonempty topological space again, and let $C_b(X,k)$ be the space of bounded continuous k-valued functions on X, as in Section 13. This is the same as the intersection of $C(X,k)$ and $\ell^\infty(X,k)$, and in particular $C_b(X,k)$ is a subalgebra of both $C(X,k)$ and $\ell^\infty(X,k)$. As in Section 13, (17.6) may also be denoted $\|f\|_{sup}$, which corresponds to (17.3) with $E = X$. Remember that $C_b(X,k)$ is the same as $C(X,k)$ when X is compact, and that $C_b(X,k)$ is the same as $\ell^\infty(X,k)$ when X is equipped with the discrete topology.

Let V be a vector space over k, and let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k for some $q_V > 0$. As in Section 14, the space $\mathcal{BL}(V)$ of bounded linear mappings on V with respect to N_V is a vector space with respect to pointwise addition and scalar multiplication. More precisely, $\mathcal{BL}(V)$ is a subalgebra of the algebra $\mathcal{L}(V)$ of all linear mappings from V into itself, with composition of linear mappings as multiplication. Let $\|\cdot\|_{op}$ be the corresponding operator q_V-seminorm on $\mathcal{BL}(V)$, as in (14.3). We have seen that $\|\cdot\|_{op}$ is submultiplicative on $\mathcal{BL}(V)$, as in (14.4).

18 Continuity of multiplication

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let $\mathcal{A}$ be an algebra over k, and let N be a q-seminorm on $\mathcal{A}$ as a vector space over k for some $q > 0$, with respect to $|\cdot|$ on k. Suppose that there is a nonnegative real number C such that

$$N(x\,y) \le C\,N(x)\,N(y) \tag{18.1}$$

for every $x, y \in \mathcal{A}$. This is the same as saying that multiplication is bounded as a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, as in Section 15. In particular, this implies that multiplication on $\mathcal{A}$ is continuous as a mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, with respect to the topology determined on $\mathcal{A}$ by the q-metric associated to N, and the corresponding product topology on $\mathcal{A}$. Using this q-semimetric on $\mathcal{A}$, we can get a q-semimetric on $\mathcal{A} \times \mathcal{A}$, as in (5.6). The restriction of the mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ corresponding to multiplication on $\mathcal{A}$ to bounded subsets of $\mathcal{A} \times \mathcal{A}$ is uniformly continuous with respect to the q-semimetrics just mentioned, as in Section 15. Of course, (18.1) is the same as (17.1) when $C = 1$.

Let $a \in \mathcal{A}$ be given, and let M_a be the corresponding left multiplication operator on $\mathcal{A}$, as in (16.5). Using (18.1), we get that M_a is a bounded linear mapping from $\mathcal{A}$ into itself with respect to N, with

$$\|M_a\|_{op} \le C\, N(a). \tag{18.2}$$

Here $\|\cdot\|_{op}$ is the operator q-seminorm on $\mathcal{BL}(\mathcal{A})$ corresponding to N, as in (14.3). Thus $a \mapsto M_a$ defines a bounded linear mapping from $\mathcal{A}$ into $\mathcal{BL}(\mathcal{A})$, with respect to $\|\cdot\|_{op}$ on $\mathcal{BL}(\mathcal{A})$.

Suppose that $\mathcal{A}$ has a multiplicative identity element e. Using (18.1), we get that

$$N(x) = N(e\, x) \le C\, N(e)\, N(x) \tag{18.3}$$

for every $x \in \mathcal{A}$, which implies that $C\, N(e) \ge 1$ when $N(x) > 0$ for some $x \in \mathcal{A}$. If $a \in \mathcal{A}$, and M_a is the corresponding left multiplication operator on $\mathcal{A}$ again, then

$$N(a) = N(a\, e) = N(M_a(e)) \le \|M_a\|_{op}\, N(e). \tag{18.4}$$

In particular, if $C = 1$ and $N(e) = 1$, then it follows that $\|M_a\|_{op} = N(a)$ for every $a \in \mathcal{A}$.

Let X be a nonempty topological space, and remember that the space $C(X, k)$ of continuous k-valued functions on X is a commutative algebra over k. If E is a nonempty compact subset of X, then the corresponding supremum q_k-seminorm $\|f\|_{sup,E}$ on $C(X, k)$ is defined as in (17.3), and $\|f\|_{sup,E}$ is submultiplicative on $C(X, k)$, as in (17.4). Let τ_E be the topology determined on $C(X, k)$ by the q_k-semimetric on $C(X, k)$ associated to $\|f\|_{sup,E}$, as in Section 13. As before, multiplication on $C(X, k)$ defines a continuous mapping from $C(X, k) \times C(X, k)$ into $C(X, k)$ with respect to τ_E on $C(X, k)$ and the corresponding product topology on $C(X, k) \times C(X, k)$. Let τ be the topology on $C(X, k)$ generated by the collection of topologies τ_E, where E is a nonempty compact subset of X, as in Section 13 again. One can check that multiplication on $C(X, k)$ defines a continuous mapping from $C(X, k) \times C(X, k)$ into $C(X, k)$ with respect to τ on $C(X, k)$ and the corresponding product topology on $C(X, k) \times C(X, k)$. Of course, there are analogous statements for continuity of addition and scalar multiplication on $C(X, k)$ with respect to τ.

19 Multiplicativity conditions

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, let $\mathcal{A}$ be an algebra over k, and let N be a q-seminorm on $\mathcal{A}$ as a vector space over k for some $q > 0$, with respect to $|\cdot|$ on k. If

$$N(x\, y) = N(x)\, N(y) \tag{19.1}$$

for every $x, y \in \mathcal{A}$, then N is said to be *multiplicative* on $\mathcal{A}$.

Now let N be a submultiplicative q-seminorm on $\mathcal{A}$ for some $q > 0$, with respect to $|\cdot|$ on k. If $x \in \mathcal{A}$, then

$$N(x^n) \le N(x)^n \tag{19.2}$$

for every $n \in \mathbf{Z}_+$. If N is multiplicative on $\mathcal{A}$, then

$$N(x^n) = N(x)^n \tag{19.3}$$

for every $x \in \mathcal{A}$ and $n \in \mathbf{Z}_+$.

Suppose for the moment that $\mathcal{A}$ has a multiplicative identity element e. If we take $x = e$ in (19.3), then we get that

$$N(e) = N(e^n) = N(e)^n. \tag{19.4}$$

This implies that $N(e) = 1$ when $N(e) > 0$ and $n \ge 2$.

If X is a nonempty set, then it is easy to see that the supremum q_k-norm $\|f\|_\infty$ satisfies (19.3) on the algebra $\ell^\infty(X, k)$ of bounded k-valued functions on X. Similarly, if X is a nonempty topological space, and E is a nonempty compact subset of X, then the corresponding supremum q_k-seminorm $\|f\|_{sup,E}$ satisfies (19.3) on $C(X, k)$.

Let $\mathcal{A}$ be an algebra over k again, and let N be a submultiplicative q-seminorm on $\mathcal{A}$ for some $q > 0$. Suppose that (19.3) holds for some $x \in \mathcal{A}$ and $n \in \mathbf{Z}_+$. If $j \in \mathbf{Z}_+$ and $j < n$, then

$$N(x)^n = N(x^n) = N(x^j\, x^{n-j}) \le N(x^j)\, N(x^{n-j}) \le N(x^j)\, N(x)^{n-j}. \tag{19.5}$$

It follows that

$$N(x^j) = N(x)^j \tag{19.6}$$

when $N(x) > 0$, and this can be verified more directly when $N(x) = 0$. If (19.3) holds for some arbitrarily large positive integers n, then this implies that (19.3) holds for every $n \in \mathbf{Z}_+$.

Suppose that for each $x \in \mathcal{A}$ there is an $n \in \mathbf{Z}_+$ such that $n \ge 2$ and (19.3) holds. If $x \in \mathcal{A}$, then one can apply the hypothesis repeatedly to powers of x, to get that (19.3) holds for some arbitrarily large positive integers n. This implies that (19.3) holds for every $n \in \mathbf{Z}_+$, as in the preceding paragraph.

If a is a positive real number, then $|\cdot|^a$ defines a (q_k/a)-absolute value function on k, as in Section 3. If N is a q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k, then

$$N(x)^a \tag{19.7}$$

defines a (q/a)-seminorm on $\mathcal{A}$ with respect to $|\cdot|^a$ on k, as in (12.7). If N is submultiplicative on $\mathcal{A}$, then (19.7) is clearly submultiplicative on $\mathcal{A}$ too. Similarly, if N is multiplicative on $\mathcal{A}$, then (19.7) is multiplicative on $\mathcal{A}$ as well. If N satisfies (19.3) for every $x \in \mathcal{A}$ and $n \in \mathbf{Z}_+$, then (19.7) satisfies the analogous condition on $\mathcal{A}$ for each $n \in \mathbf{Z}_+$.

20 Consequences for q

Let k be a field, and let $\mathcal{A}$ be an algebra over k. If $n \in \mathbf{Z}_+$ and $t \in k$, then we let $n \cdot t$ be the sum of n t's in k, as in Section 10. Similarly, if $x \in \mathcal{A}$, then let $n \cdot x$ be the sum of n x's in $\mathcal{A}$. As before, we have that

$$m \cdot (n \cdot x) = (m\, n) \cdot x \tag{20.1}$$

for every $m, n \in \mathbf{Z}_+$ and $x \in \mathcal{A}$, and

$$n \cdot (x\, y) = (n \cdot x)\, y \tag{20.2}$$

for every $n \in \mathbf{Z}_+$ and $x, y \in \mathcal{A}$. If $n \in \mathbf{Z}_+$, $t \in k$, and $x \in \mathcal{A}$, then

$$n \cdot (t\, x) = (n \cdot t)\, x, \tag{20.3}$$

using scalar multiplication on $\mathcal{A}$.

Let us suppose from now on in this section that $\mathcal{A}$ is a commutative algebra over k. If $x, y \in \mathcal{A}$ and $n \in \mathbf{Z}_+$, then

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} \cdot x^j\, y^{n-j}, \tag{20.4}$$

where $\binom{n}{j}$ are the usual binomial coefficients. Here $x^j\, y^{n-j}$ is interpreted as being x^n when $j = n$, and as being y^n when $j = 0$. Equivalently,

$$(x + y)^n = \sum_{j=0}^{n} \Big(\binom{n}{j} \cdot 1\Big)\, x^j\, y^{n-j}, \tag{20.5}$$

where $\binom{n}{j} \cdot 1$ is the sum of $\binom{n}{j}$ 1's in k, as before.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a submultiplicative q-seminorm on $\mathcal{A}$ for some $q > 0$ with respect to $|\cdot|$ on k. If $q < \infty$, then we get that

$$N((x + y)^n)^q = N\Big(\sum_{j=0}^{n} \binom{n}{j} \cdot x^j\, y^{n-j}\Big)^q \le \sum_{j=0}^{n} N\Big(\binom{n}{j} \cdot x^j\, y^{n-j}\Big)^q, \tag{20.6}$$

using the q-seminorm version (12.2) of the triangle inequality in the second step. We also have that

$$\begin{aligned} N\Big(\binom{n}{j} \cdot x^j\, y^{n-j}\Big) &= N\Big(\Big(\binom{n}{j} \cdot 1\Big)\, x^j\, y^{n-j}\Big) \\ &= \Big|\binom{n}{j} \cdot 1\Big|\, N(x^j\, y^{n-j}) \\ &\le \Big|\binom{n}{j} \cdot 1\Big|\, N(x)^j\, N(y)^{n-j} \end{aligned} \tag{20.7}$$

for each $j = 0, 1, \ldots, n$, using the submultiplicativity of N in the third step. Thus

$$N((x+y)^n)^q \le \sum_{j=0}^n \left| \binom{n}{j} \cdot 1 \right|^q N(x)^{j\,q}\, N(y)^{(n-j)\,q}. \tag{20.8}$$

Suppose that for the moment that $q_k = \infty$, so that $|\cdot|$ is an ultrametric absolute value function on k. In this case, (20.8) implies that

$$\begin{aligned} N((x+y)^n)^q &\le \sum_{j=0}^n N(x)^{j\,q}\, N(y)^{(n-j)\,q} \\ &\le (n+1)\, \max(N(x), N(y))^{n\,q}. \end{aligned} \tag{20.9}$$

Suppose that N satisfies (19.3) on $\mathcal{A}$ for all $n \in \mathbf{Z}_+$, so that

$$N((x+y)^n) = N(x+y)^n \tag{20.10}$$

for every $n \in \mathbf{Z}_+$. Combining this with (20.9), we get that

$$N(x+y) = N((x+y)^n)^{1/n} \le (n+1)^{1/(n\,q)}\, \max(N(x), N(y)) \tag{20.11}$$

for every $n \in \mathbf{Z}_+$. It follows that

$$N(x+y) \le \max(N(x), N(y)) \tag{20.12}$$

for every $x, y \in \mathcal{A}$, by taking the limit as $n \to \infty$ on the right side of (20.11). Thus N is a semi-ultranorm on $\mathcal{A}$ under these conditions. Of course, this is very similar to the argument for non-archimedean absolute value functions in Section 10.

Suppose now that $k = \mathbf{R}$, $\mathbf{C}$, or simply $\mathbf{Q}$ with the standard absolute value function. Using (20.8), we get that

$$\begin{aligned} N((x+y)^n)^q &\le \sum_{j=0}^n \binom{n}{j}^q N(x)^{j\,q}\, N(y)^{(n-j)\,q} \\ &\le (n+1) \max_{0 \le j \le n} \Big(\binom{n}{j}^q N(x)^{j\,q}\, N(y)^{(n-j)\,q} \Big). \end{aligned} \tag{20.13}$$

Hence

$$\begin{aligned} N((x+y)^n) &\le (n+1)^{1/q} \max_{0 \le j \le n} \Big(\binom{n}{j} N(x)^j\, N(y)^{n-j} \Big) \\ &\le (n+1)^{1/q} \sum_{j=0}^n \binom{n}{j} N(x)^j\, N(y)^{n-j} \\ &\le (n+1)^{1/q}\, (N(x) + N(y))^n, \end{aligned} \tag{20.14}$$

by taking the qth root of both sides of (20.13) in the first step, and using the binomial theorem in the third step. Suppose again that N satisfies (19.3) on $\mathcal{A}$

for each $n \in \mathbf{Z}_+$, so that (20.10) holds for every $n \in \mathbf{Z}_+$. Using this and (20.14), we get that

$$N(x+y) = N((x+y)^n)^{1/n} \le (n+1)^{1/(n\,q)}\,(N(x)+N(y)) \tag{20.15}$$

for each $n \in \mathbf{Z}_+$. This implies that

$$N(x+y) \le N(x) + N(y) \tag{20.16}$$

for every $x, y \in \mathcal{A}$, by taking the limit as $n \to \infty$ on the right side of (20.15). This shows that N is an ordinary seminorm on $\mathcal{A}$ under these conditions, so that we can take $q = 1$. This is analogous to some of the remarks in Section 11.

Part II

Some more basic notions

21 Nonnegative sums

Let X be a nonempty set, and let f be a nonnegative real-valued function on X. The sum

$$\sum_{x \in X} f(x) \tag{21.1}$$

is defined as a nonnegative extended real number to be the supremum of the sums

$$\sum_{x \in A} f(x) \tag{21.2}$$

over all nonempty finite subsets A of X. Thus (21.1) is finite if and only if there is a finite upper bound for the finite subsums (21.2), in which case f is said to be *summable* on X. Of course, if X has only finitely many elements, then (21.1) can be defined as a finite sum directly. It is sometimes convenient to allow f to take values in the set of nonnegative extended real numbers, where (21.1) is automatically interpreted as being $+\infty$ when $f(x) = +\infty$ for any $x \in X$.

If f is a nonnegative extended real-valued function on X and t is a positive real number, then

$$\sum_{x \in X} t\, f(x) = t \sum_{x \in X} f(x), \tag{21.3}$$

where t times $+\infty$ is interpreted as being $+\infty$. Similarly, if g is another nonnegative extended real-valued function on X, then

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x), \tag{21.4}$$

where the sum of any nonnegative extended real number and $+\infty$ is interpreted as being $+\infty$. Both statements can be verified directly from the definitions, by approximating the various sums by the corresponding finite subsums.

Let f be a nonnegative real-valued function on X, and let r be a positive real number. If $f(x)^r$ is summable on X, then f is said to be *r-summable* on X. In this case,

$$\|f\|_r = \Big(\sum_{x \in X} f(x)^r\Big)^{1/r} \tag{21.5}$$

is defined as a nonnegative real number, and otherwise (21.5) may be interpreted as being $+\infty$. Similarly,

$$\|f\|_\infty = \sup_{x \in X} f(x) \tag{21.6}$$

is defined as a nonnegative extended real number, which is finite exactly when f has a finite upper bound on X. Observe that

$$\|t\, f\|_r = t\, \|f\|_r \tag{21.7}$$

for every positive real number t and $0 < r \le \infty$. If $0 < r_1 \le r_2 \le \infty$, then

$$\|f\|_{r_2} \le \|f\|_{r_1}, \tag{21.8}$$

as in (1.4) and (1.7). More precisely, this can be verified in the same way as in Section 1, or by reducing to that situation.

Let g be another nonnegative real-valued function on X. If $1 \le r \le \infty$, then

$$\|f + g\|_r \le \|f\|_r + \|g\|_r. \tag{21.9}$$

This is *Minkowski's inequality* for arbitrary sums. This can be shown in essentially the same way as for finite sums, or by reducing to that case. If $0 < r \le 1$, then

$$\|f + g\|_r^r \le \|f\|_r^r + \|g\|_r^r, \tag{21.10}$$

as in (1.13).

Let r_0 be a positive real number, and let f be a nonnegative real-valued r_0-summable function on X. If $r_0 \le r < \infty$, then

$$\|f\|_\infty \le \|f\|_r \le \|f\|_\infty^{1-(r_0/r)}\, \|f\|_{r_0}^{r_0/r}, \tag{21.11}$$

where the first step is the same as the first inequality in (1.4), and the second step is the same as the first inequality in (1.7). As usual, these inequalities can be obtained in the same way as for finite sums, or by reducing to that case. In particular, f is r-summable on X when $r_0 \le r < \infty$. Using (21.11), one can check that

$$\lim_{r \to \infty} \|f\|_r = \|f\|_\infty, \tag{21.12}$$

as in (1.5).

22 r-Summable functions

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let V be a vector space over k, and let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$. Also let X be a nonempty set, and let r be a positive real number. A V-valued function f on X is said to be *r-summable* with respect to N if $N(f(x))$ is r-summable as a nonnegative real-valued function on X, which means that $N(f(x))^r$ is summable on X, as in the previous section. Let $\ell^r(X,V) = \ell^r_N(X,V)$ be the space of V-valued functions on X that are r-summable with respect to N. If $f \in \ell^r(X,V)$, then we put

$$\|f\|_r = \|f\|_{\ell^r(X,V)} = \|f\|_{\ell^r_N(X,V)} = \Big(\sum_{x\in X} N(f(x))^r\Big)^{1/r}, \tag{22.1}$$

where the sum is defined as in the previous section. If f is not r-summable on X, then one may consider (22.1) as being equal to $+\infty$. If f is r-summable on X and $t \in k$, then it is easy to see that $t\,f(x)$ is r-summable on X with respect to N, with

$$\|t\,f\|_r = |t|\,\|f\|_r. \tag{22.2}$$

Let f, g be V-valued functions on X. If $r \le q$, then one can check that

$$\|f+g\|_r^r \le \|f\|_r^r + \|g\|_r^r. \tag{22.3}$$

More precisely, if $r \le q$, then N may be considered as an r-seminorm on V, as in Section 12. Using this, (22.3) can be verified directly from the definitions, as in (1.13) and (21.10). If $q \le r$, then

$$\|f+g\|_r^q \le \|f\|_r^q + \|g\|_r^q. \tag{22.4}$$

This can be obtained from Minkowski's inequality for sums, with exponent $r/q \ge 1$. In particular, if f and g are r-summable on X for any positive real number r, then $f+g$ is r-summable on X too. It follows that $\ell^r(X,V)$ is a vector space with respect to pointwise addition and scalar multiplication for every positive real number r. If $r \le q$, then $\|f\|_r$ defines an r-seminorm on $\ell^r(X,V)$, by (22.2) and (22.3). Similarly, if $q \le r$, then $\|f\|_r$ is a q-seminorm on $\ell^r(X,V)$, by (22.2) and (22.4).

As in Section 13, $\ell^\infty(X,V)$ is the space of V-valued functions f on X such that $N(f(x))$ is bounded on X. In this case, $\|f\|_\infty$ is defined as in (13.3), and determines a q-seminorm on $\ell^\infty(X,V)$. If $N(f(x))$ does not have a finite upper bound on X, then one can take $\|f\|_\infty$ to be $+\infty$, as usual. If f is any V-valued function on X and $0 < r_1 \le r_2 \le \infty$, then

$$\|f\|_{r_2} \le \|f\|_{r_1}, \tag{22.5}$$

as in (21.8). In particular, $\ell^{r_1}(X,V) \subseteq \ell^{r_2}(X,V)$ when $r_1 \le r_2$.

If N is a q-norm on V, then $\|f\|_r$ is an r-norm on $\ell^r(X,V)$ when $r \le q$, and $\|f\|_r$ is a q-norm on $\ell^r(X,V)$ when $q \le r$. If V is also complete with respect to the q-metric associated to N, then $\ell^r(X,V)$ is complete with respect to the q or r-metric associated to $\|f\|_r$, as appropriate, by standard arguments.

23 Vanishing at infinity

Let k be a field, let V be a vector space over k, and let X be a nonempty set. The *support* of a V-valued function f on X is defined to be the set

$$\operatorname{supp} f = \{x \in X : f(x) \neq 0\}. \tag{23.1}$$

Let $c_{00}(X, V)$ be the space of V-valued functions f on X such that the support of f has only finitely many elements. This is a linear subspace of the space $c(X, V)$ of all V-valued functions on X. Of course, if X has only finitely many elements, then $c_{00}(X, V)$ is the same as $c(X, V)$.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$. If f is a V-valued function on X, then the support of $N(f(x))$, as a real-valued function on X, is contained in the support of f. Let $c_{00,N}(X, V)$ be the space of V-valued functions f on X such that $N(f(x))$ has finite support in X. This is a linear subspace of $c(X, V)$ that contains $c_{00}(X, V)$. If N is a q-norm on V and $f \in c(X, V)$, then the supports of f and $N(f(x))$ in X are the same, so that $c_{00}(X, V)$ is equal to $c_{00,N}(X, V)$.

A V-valued function f on X is said to *vanish at infinity* on X with respect to N if for every $\epsilon > 0$ we have that

$$N(f(x)) < \epsilon \tag{23.2}$$

for all but finitely many $x \in X$. Let $c_0(X, V) = c_{0,N}(X, V)$ be the space of V-valued functions on X that vanish at infinity with respect to N. If f vanishes at infinity on X with respect to N, then it is easy to see that f is bounded on X with respect to N, so that

$$c_0(X, V) \subseteq \ell^\infty(X, V). \tag{23.3}$$

More precisely, $c_0(X, V)$ is a linear subspace of $\ell^\infty(X, V)$. One can check that $c_0(X, V)$ is also a closed set in $\ell^\infty(X, V)$, with respect to the supremum q-semimetric associated to N.

If $N(f(x))$ has finite support in X as a real-valued function on X, then f vanishes at infinity on X with respect to N. Thus

$$c_{00,N}(X, V) \subseteq c_0(X, V). \tag{23.4}$$

In particular, $c_{00}(X, V)$ is contained in $c_0(X, V)$. One can verify that $c_0(X, V)$ is the same as the closure of $c_{00}(X, V)$ in $\ell^\infty(X, V)$, with respect to the supremum q-semimetric associated to N.

If r is a positive real number, then

$$\ell^r(X, V) \subseteq c_0(X, V). \tag{23.5}$$

Equivalently, if f is a V-valued function on X that does not vanish at infinity with respect to N, then f is not r-summable with respect to N.

Clearly

$$c_{00,N}(X,V) \subseteq \ell^r(X,V) \tag{23.6}$$

for every $r > 0$, so that $c_{00}(X,V)$ is contained in $\ell^r(X,V)$ in particular. If $r < \infty$, then $c_{00}(X,V)$ is dense in $\ell^r(X,V)$ with respect to the q or r-semimetric associated to $\|f\|_r$. This can be seen by approximating $\|f\|_r^r$ by finite subsums.

Let f be a V-valued function on X that vanishes at infinity. Thus

$$\{x \in X : N(f(x)) \geq \epsilon\} \tag{23.7}$$

has only finitely many elements for each $\epsilon > 0$. It follows that the support of $N(f(x))$ has only finitely or countably many elements, by applying the previous statement to $\epsilon = 1/j$, with $j \in \mathbf{Z}_+$.

24 Infinite series

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let V be a vector space over k, and let N be a q-norm on V with respect to $|\cdot|$ on k for some $q > 0$. As usual, an infinite series

$$\sum_{j=1}^{\infty} v_j \tag{24.1}$$

with terms in V is said to *converge* in V if the corresponding sequence

$$\sum_{j=1}^{n} v_j \tag{24.2}$$

of partial sums converges to an element of V with respect to the q-metric associated to N. In this case, the value of the sum (24.1) is defined to be the limit of the sequence of partial sums (24.2). It is easy to see that the sequence of partial sums (24.2) is a Cauchy sequence in V with respect to the q-metric associated to N if and only if for every $\epsilon > 0$ there is a positive integer L such that

$$N\Big(\sum_{j=l}^{n} v_j\Big) < \epsilon \tag{24.3}$$

when $n \geq l \geq L$. In particular, this implies that

$$\lim_{j \to \infty} N(v_j) = 0, \tag{24.4}$$

by taking $l = n$.

Let $\sum_{j=1}^{\infty} a_j$ be an infinite series of nonnegative real numbers, so that the corresponding sequence $\sum_{j=1}^{n} a_j$ of partial sums is monotonically increasing. It is well known that the series converges in $\mathbf{R}$ with respect to the standard absolute value function on $\mathbf{R}$ if and only if the sequence of partial sums has a finite upper bound, in which case the sequence of partial sums converges to its supremum in $\mathbf{R}$. One can also define

$$\sum_{j \in \mathbf{Z}_+} a_j \tag{24.5}$$

as a nonnegative extended real number as in Section 21. If $\sum_{j=1}^{\infty} a_j$ converges in $\mathbf{R}$, then one can check that (24.5) is finite, and that the two sums are the same. Otherwise, if the partial sums $\sum_{j=1}^{n} a_j$ do not have a finite upper bound in $\mathbf{R}$, then (24.5) is equal to $+\infty$.

Suppose for the moment that $q < \infty$, so that

$$N\Big(\sum_{j=l}^{n} v_j\Big)^q \le \sum_{j=l}^{n} N(v_j)^q \tag{24.6}$$

for every $n \ge l \ge 1$. Let us say that (24.1) converges *q-absolutely* if

$$\sum_{j=1}^{\infty} N(v_j)^q \tag{24.7}$$

converges as an infinite series of nonnegative real numbers. This implies that the sequence (24.2) of partial sums is a Cauchy sequence in V with respect to the q-metric on V associated to N, because of (24.6). If V is complete with respect to the q-metric associated to N, then it follows that (24.1) converges in V. Under these conditions, one can also use (24.6) to get that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big)^q \le \sum_{j=1}^{\infty} N(v_j)^q. \tag{24.8}$$

Suppose now that $q = \infty$, so that

$$N\Big(\sum_{j=l}^{n} v_j\Big) \le \max_{l \le j \le n} N(v_j) \tag{24.9}$$

for every $n \ge l \ge 1$. In this situation, (24.4) implies that the sequence (24.2) of partial sums is a Cauchy sequence in V with respect to the ultrametric associated to N. If V is complete with respect to the ultrametric associated to N, then it follows that (24.1) converges in V. As before, one can also use (24.9) to get that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \max_{j \ge 1} N(v_j) \tag{24.10}$$

under these conditions. More precisely, the maximum on the right side of (24.10) is attained, because of (24.4).

If $\{w_j\}_{j=1}^{\infty}$ is any sequence of vectors in V, then

$$\sum_{j=l}^{n} (w_{j+1} - w_j) = w_{n+1} - w_l \tag{24.11}$$

for every $n \geq l \geq 1$. This implies that $\{w_j\}_{j=1}^\infty$ converges as a sequence of vectors in V if and only if $\sum_{j=1}^\infty (w_{j+1} - w_j)$ converges as an infinite series in V, in which case

$$\sum_{j=1}^\infty (w_{j+1} - w_j) = \lim_{j \to \infty} w_j - w_1. \tag{24.12}$$

Of course, if $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence in V, then we have that

$$\lim_{j \to \infty} N(w_{j+1} - w_j) = 0 \tag{24.13}$$

in particular. If V has the property that an infinite series (24.1) converges in V when (24.4) holds, then it follows that V is complete with respect to the q-metric associated to N. Similarly, suppose that $q < \infty$, and that V has the property that every q-absolutely convergent series in V converges in V. If $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence in V, thenit is easy to see that there is a subsequence $\{w_{j_r}\}_{r=1}^\infty$ of $\{w_j\}_{j=1}^\infty$ such that $\sum_{r=1}^\infty (w_{j_{r+1}} - w_{j_r})$ converges q-absolutely. This implies that this series converges in V, by hypothesis, so that $\{w_{j_r}\}_{r=1}^\infty$ converges as a sequence in V, as before. Because $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence in V, we get that $\{w_j\}_{j=1}^\infty$ converges to the same limit in V, by standard arguments. This shows that V is complete with respect to the q-metric associated to N under these conditions.

25 Sums of vectors

Let k be a field, let V be a vector space over k, and let X be a nonempty set. If f is a V-valued function on X with finite support, then

$$\sum_{x \in X} f(x) \tag{25.1}$$

can be defined as an element of V, by reducing to a finite sum. Moreover,

$$f \mapsto \sum_{x \in X} f(x) \tag{25.2}$$

defines a linear mapping from $c_{00}(X, V)$ into V.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$. If $f \in c_{00}(X, V)$, then it is easy to see that

$$N\Big(\sum_{x \in X} f(x)\Big) \leq \|f\|_q, \tag{25.3}$$

using the q-seminorm version of the triangle inequality. Here $\|f\|_q$ is as in (13.3) when $q = \infty$, and as in (22.1) when $q < \infty$. This implies that (25.2) is a bounded linear mapping from $c_{00}(X, V)$ into V, with respect to $\|f\|_q$ on $c_{00}(X, V)$ and

N on V. The corresponding operator q-seminorm of (25.2) is equal to 1 when $N(v) > 0$ for some $v \in V$.

Suppose now that N is a q-norm on V, and that V is complete with respect to the corresponding q-metric. If $q < \infty$, then there is a unique extension of (25.2) to a bounded linear mapping from $\ell^q(X, V)$ into V, as in Section 14. This uses the fact that $c_{00}(X, V)$ is dense in $\ell^q(X, V)$ with respect to the q-metric associated to $\|f\|_q$ when $q < \infty$, as in Section 23. Similarly, if $q = \infty$, then there is a unique extension of (25.2) to a bounded linear mapping from $c_0(X, V)$ into V, using the restriction of the supremum ultranorm $\|f\|_\infty$ associated to N to $c_0(X, V)$. This uses the fact that $c_{00}(X, V)$ is dense in $c_0(X, V)$ with respect to the supremum ultrametric, as before. These extensions can be used to define (25.1) as an element of V when $q < \infty$ and $f \in \ell^q(X, V)$, and when $q = \infty$ and $f \in c_0(X, V)$. More precisely, these extensions also satisfy (25.3), as in Section 14.

Let $\{x_j\}_{j=1}^\infty$ be a sequence of distinct elements of X, and let f be a V-valued function on X whose support is contained in the set of x_j's. In this situation, the sum (25.1) basically corresponds to the infinite series

$$\sum_{j=1}^\infty f(x_j). \tag{25.4}$$

In particular, if f has finite support in X, then all but finitely many terms in (25.4) are equal to 0. If $q < \infty$ and $f \in \ell^q(X, V)$, then it is easy to see that (25.4) converges q-absolutely with respect to N. Similarly, if $q = \infty$ and $f \in c_0(X, V)$, then one can check that $N(f(x_j)) \to 0$ as $j \to \infty$. In both cases, if V is complete with respect to the q-metric associated to N, then it follows that (25.4) converges in V, as in the previous section. One can verify that this approach to the sum is equivalent to the one mentioned in the preceding paragraph.

Suppose for the moment that $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. If f is a nonnegative real-valued function on X, then the sum (25.1) can be defined as in Section 21. One can also use this to define (25.1) as a real or complex number when f is a summable real or complex-valued function on X, by expressing f as a linear combination of summable nonnegative real-valued functions on X. One can check that this approach to the sum is equivalent to the previous ones in this situation.

26 Sums of sums

Let I, X be nonempty sets, and let $\{E_j\}_{j\in I}$ be a family of pairwise-disjoint nonempty subsets of X indexed by I. If f is a nonnegative extended real-valued function on X, then

$$\sum_{x\in E_j} f(x) \tag{26.1}$$

can be defined as a nonnegative extended real number for every $j \in I$, as in Section 21. This defines a nonnegative extended real-valued function of j on I, so that

$$\sum_{j \in I} \Big(\sum_{x \in E_j} f(x) \Big) \tag{26.2}$$

can be defined as a nonnegative extended real number as well. Put

$$E = \bigcup_{j \in I} E_j, \tag{26.3}$$

so that

$$\sum_{x \in E} f(x) \tag{26.4}$$

can be defined as a nonnegative extended real number too. One can check that (26.2) is equal to (26.4), directly from the definitions, by comparing the corresponding finite subsums.

Let k be a field, and let V be a vector space over k. If f is a V-valued function on X with finite support, then the restriction of f to any nonempty subset of X has finite support in that subset. Thus the sum of $f(x)$ over any nonempty subset of X is defined as an element of V. In particular, (26.1) can be defined as an element of V for each $j \in I$, which defines a V-valued function of j on I. It is easy to see that this function has finite support in I, because the E_j's are pairwise disjoint. This implies that (26.2) is defined as an element of V, and (26.4) is defined as an element of V as well. As before, (26.2) is equal to (26.4) in this situation.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, let N be a q-norm on V with respect to $|\cdot|$ on k for some $q > 0$, and suppose that V is complete with respect to the q-metric associated to N. Suppose for the moment that $q < \infty$, and let $f \in \ell^q(X, V)$ be given. Observe that the restriction of f to any nonempty subset of X is q-summable on that set. Hence the sum of $f(x)$ over any nonempty subset of X can be defined as an element of V, as in the previous section. Let us apply this to E_j for each $j \in I$, and observe that

$$N\Big(\sum_{x \in E_j} f(x) \Big)^q \le \sum_{x \in E_j} N(f(x))^q \tag{26.5}$$

for every $j \in I$, as in (25.3). It follows that

$$\sum_{j \in I} N\Big(\sum_{x \in E_j} f(x) \Big)^q \le \sum_{j \in I} \Big(\sum_{x \in E_j} N(f(x))^q \Big) = \sum_{x \in E} N(f(x))^q, \tag{26.6}$$

using the earlier remarks for nonnegative real-valued functions in the second step. Thus (26.1) is q-summable as a V-valued function of j on I, so that (26.2) can be defined as an element of V, as in the previous section. As usual, one can check that (26.2) is equal to (26.4) under these conditions.

Suppose now that $q = \infty$, and let $f \in c_0(X, V)$ be given. It is easy to see that the restriction of f to any nonempty subset of X vanishes at infinity on that set. This implies that the sum of $f(x)$ over any nonempty subset of X can be defined as an element of V, as in the previous section. If we apply this to E_j for each $j \in I$, then we get that

$$N\Big(\sum_{x \in E_j} f(x)\Big) \le \max_{x \in E_j} N(f(x)) \tag{26.7}$$

for every $j \in I$, as in (25.3). Note that the maximum on the right side of (26.7) is attained, because $N(f(x))$ vanishes at infinity as a nonnegative real-valued function on E_j. Using (26.7), one can verify that (26.1) vanishes at infinity as a V-valued function on I, because the E_j's are pairwise disjoint. This implies that (26.2) can be defined as an element of V, as in the previous section. One can check that (26.2) is equal to (26.4) in this situation too, as before.

Let Y, Z be nonempty sets, and let $X = Y \times Z$ be their Cartesian product. Thus X is partitioned by the family of subsets of the form $\{y\} \times Z$ with $y \in Y$, and by the family of subsets of the form $Y \times \{z\}$ with $z \in Z$. In this situation, one can use the previous remarks to show that sums over X are the same as iterated sums over Y and Z under suitable conditions. More precisely, one can sum over Y and then Z, or over Z and then Y. In particular, the equality of these iterated sums with the corresponding sum over X implies that these two iterated sums are the same.

27 Some linear mappings

Let k be a field, and let X be a nonempty set. If $x \in X$, then let δ_x be the k-valued function defined on X by

$$\begin{aligned} \delta_x(y) &= 1 \quad \text{when } x = y \\ &= 0 \quad \text{when } x \ne y. \end{aligned} \tag{27.1}$$

Remember that $c_{00}(X, k)$ denotes the space of k-valued functions on X with finite support, as in Section 23. Observe that $\delta_x \in c_{00}(X, k)$ for each $x \in X$, and that the collection of δ_x, $x \in X$, is a basis for $c_{00}(X, k)$ as a vector space over k.

Let V be a vector space over k, and let a be a V-valued function on X. If $f \in c_{00}(X, k)$, then $a(x)\, f(x)$ defines a V-valued function of x with finite support in X, so that

$$T_a(f) = \sum_{x \in X} a(x)\, f(x) \tag{27.2}$$

is defined as an element of V. Observe that T_a defines a linear mapping from $c_{00}(X, k)$ into V, and that

$$T_a(\delta_x) = a(x) \tag{27.3}$$

for every $x \in X$. If T is any linear mapping from $c_{00}(X, k)$ into V, then

$$a(x) = T(\delta_x) \tag{27.4}$$

defines a V-valued function on X. It is easy to see that $T = T_a$ in this case, because the δ_x's form a basis for $c_{00}(X, k)$.

If a is any V-valued function on X again, then

$$f \mapsto a\,f \tag{27.5}$$

defines a linear mapping from $c_{00}(X, k)$ into $c_{00}(X, V)$. Note that (27.2) is the same as the composition of (27.5) with the mapping from $c_{00}(X, V)$ into V defined by summing over X, as in Section 25. Of course, (27.5) also defines a linear mapping from $c(X, k)$ into $c(X, V)$, where $c(X, V)$ is as in Section 13.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. We may consider k as a one-dimensional vector space over itself, and $|\cdot|$ as a q_k-norm on k. Thus $\ell^r(X, k)$ may be defined for $r > 0$ as in Sections 13 and 22, and $c_0(X, k)$ may be defined as in Section 23. Observe that

$$\|\delta_x\|_{\ell^r(X,k)} = 1 \tag{27.6}$$

for every $x \in X$ and $r > 0$.

Let N be a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$, and let a be a V-valued function on X that is bounded with respect to N. If $f \in \ell^r(X, k)$ for some $r > 0$, then it is easy to see that $a\,f \in \ell^r(X, V)$, with

$$\|a\,f\|_{\ell^r(X,V)} \le \|a\|_{\ell^\infty(X,V)}\,\|f\|_{\ell^r(X,k)}. \tag{27.7}$$

Thus (27.5) defines a bounded linear mapping from $\ell^r(X, k)$ into $\ell^r(X, V)$, with the corresponding operator q or r-seminorm less than or equal to $\|a\|_{\ell^\infty(X,V)}$. If $x \in X$, then

$$\|a\,\delta_x\|_{\ell^r(X,V)} = N(a(x)) \tag{27.8}$$

for every $r > 0$. This implies that for each $r > 0$, the operator q or r-seminorm of (27.5) as a bounded linear mapping from $\ell^r(X, k)$ into $\ell^r(X, V)$ is equal to $\|a\|_{\ell^\infty(X,V)}$. Similarly, if $f \in c_0(X, k)$, then $a\,f \in c_0(X, V)$, so that (27.5) defines a linear mapping from $c_0(X, k)$ into $c_0(X, V)$. This is a bounded linear mapping with respect to the corresponding supremum q_k-norm and q-seminorm, with operator q-seminorm equal to $\|a\|_{\ell^\infty(X,V)}$.

If $f \in c_{00}(X, k)$, then $a\,f \in c_{00}(X, V)$, $T_a(f)$ is defined as an element of V as in (27.2), and

$$N(T_a(f)) \le \|a\,f\|_{\ell^q(X,V)} \le \|a\|_{\ell^\infty(X,V)}\,\|f\|_{\ell^q(X,k)}. \tag{27.9}$$

This uses (25.3) in the first step, and (27.7) in the second step. Thus T_a defines a bounded linear mapping from $c_{00}(X, k)$ equipped with the $\ell^q(X, k)$ q or q_k-norm into V, with operator q-seminorm less than or equal to $\|a\|_{\ell^\infty(X,V)}$. It is easy to see that the operator q-seminorm of T_a is equal to $\|a\|_{\ell^\infty(X,V)}$, using (27.3) and (27.6).

Suppose now that N is a q-norm on V, and that V is complete with respect to the associated q-metric. If $q < \infty$ and $f \in \ell^q(X, k)$, then $a\,f \in \ell^q(X, V)$, and $T_a(f)$ can be defined as an element of V as in (27.2), using the remarks in Section 25. We also have (27.9) in this situation, so that T_a is a bounded linear mapping from $\ell^q(X,k)$ into V. The corresponding operator q-norm of T_a is equal to $\|a\|_{\ell^\infty(X,V)}$, as before. Similarly, if $q = \infty$ and $f \in c_0(X,k)$, then $a\,f \in c_0(X,V)$, and $T_a(f)$ can be defined as an element of V as in (27.2), using the remarks in Section 25 again. We have (27.9) in this situation too, so that T_a defines a bounded linear mapping from $c_0(X,k)$ into V, with respect to the supremum q_k-norm on $c_0(X,k)$. The corresponding operator ultranorm of T_a is equal to $\|a\|_{\ell^\infty(X,V)}$, as in the previous cases.

28 Holomorphic functions

In this section, we take $k = \mathbf{C}$, with the standard absolute value function. Let U be a nonempty open subset of $\mathbf{C}$, and let $H(U)$ be the space of complex-valued functions on U that are holomorphic on U. Remember that the space $C(U) = C(U,\mathbf{C})$ of complex-valued continuous functions on U is a commutative algebra over $\mathbf{C}$ with respect to pointwise addition and multiplication of functions. Of course, $H(U) \subseteq C(U)$, because holomorphic functions are continuous. It is well known that $H(U)$ is a subalgebra of $C(U)$, because sums and products of holomorphic functions are holomorphic too.

The collection of supremum seminorms on $C(U)$ associated to nonempty compact subsets of U determines a natural topology on $C(U)$, as in Section 13. It is well known that $H(U)$ is a closed set in $C(U)$ with respect to this topology. Basically, this means that if $f \in C(U)$ can be approximated uniformly on compact subsets of U by holomorphic functions, then f is holomorphic on U. This uses the fact that a continuous complex-valued function on U is holomorphic if and only if it can be expressed locally as in the Cauchy integral formula. If $f \in C(U)$ can be approximated by holomorphic functions on U uniformly on compact subsets of U, then it is easy to see that f can be expressed locally as in the Cauchy integral formula, so that f is holomorphic on U.

Let $H^\infty(U)$ be the space of bounded holomorphic functions on U. This is a subalgebra of the algebra $C_b(U) = C_b(U,\mathbf{C})$ of bounded continuous complex-valued functions on U. Equivalently,

$$H^\infty(U) = H(U) \cap C_b(U). \tag{28.1}$$

As before, $H^\infty(U)$ is a closed set in $C_b(U)$ with respect to the topology determined on $C_b(U)$ by the supremum metric. Of course, this topology is stronger than the one determined by the collection of supremum semimetrics associated to nonempty compact subsets of U. Remember that $C_b(U)$ is complete with respect to the supremum metric, as in Section 13. This implies that $H^\infty(U)$ is complete with respect to the supremum metric as well, because $H^\infty(U)$ is a closed set in $C_b(U)$.

Suppose now that U is also a bounded set in $\mathbf{C}$, so that the closure $\overline{U}$ of U in $\mathbf{C}$ is compact. Let $A(U)$ be the collection of continuous complex-valued functions on $\overline{U}$ that are holomorphic on U. This is a subalgebra of the algebra $C(\overline{U})$ of continuous complex-valued functions on $\overline{U}$. Of course, continuous complex-valued functions on $\overline{U}$ are automatically bounded on $\overline{U}$, because $\overline{U}$ is compact. As in the previous situations, $A(U)$ is a closed set in $C(\overline{U})$ with respect to the topology determined on $C(\overline{U})$ by the supremum metric. We also have that $C(\overline{U})$ is complete with respect to the supremum metric, as in Section 13. This implies that $A(U)$ is complete with respect to the supremum metric too, because $A(U)$ is a closed set in $C(\overline{U})$.

Let R be the mapping that sends $f \in C(\overline{U})$ to its restriction to ∂U, so that R is an algebra homomorphism from $C(\overline{U})$ into the algebra $C(\partial U)$ of continuous complex-valued functions on ∂U. Note that R maps $C(\overline{U})$ onto $C(\partial U)$, by well-known results. Of course,

$$\sup_{z \in \partial U} |f(z)| \le \sup_{z \in \overline{U}} |f(z)| \tag{28.2}$$

for every $f \in C(\overline{U})$, because $\partial U \subseteq \overline{U}$. This implies that R is bounded as a linear mapping from $C(\overline{U})$ into $C(\partial U)$ with respect to the corresponding supremum norms. It is easy to see that the operator norm of R is equal to 1, by considering constant functions on $\overline{U}$. If $f \in A(U)$, then it is well known that

$$\sup_{z \in \partial U} |f(z)| = \sup_{z \in \overline{U}} |f(z)|, \tag{28.3}$$

by the maximum principle. Thus R embeds $A(U)$ isometrically into $C(\partial U)$. In particular, R is injective on $A(U)$. Observe that $R(A(U))$ is complete with respect to the restriction of the supremum metric on $C(\partial U)$ to $R(A(U))$, because $A(U)$ is complete with respect to the restriction of the supremum metric on $C(\overline{U})$ to $A(U)$. It follows that $R(A(U))$ is a closed set in $C(\partial U)$ with respect to the topology determined on $C(\partial U)$ by the supremum metric.

29 Cauchy products

Let k be a field, and let $\mathcal{A}$ be an (associative) algebra over k. Also let $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ be infinite series with terms in $\mathcal{A}$, considered formally for the moment. Put

$$c_n = \sum_{j=0}^n a_j \, b_{n-j} \tag{29.1}$$

for each nonnegative integer n, which is an element of $\mathcal{A}$ too. The series $\sum_{n=0}^\infty c_n$ is called the *Cauchy product* of $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$, as formal series with terms in $\mathcal{A}$. It is well known that

$$\sum_{n=0}^\infty c_n = \Big(\sum_{j=0}^\infty a_j\Big) \Big(\sum_{l=0}^\infty b_l\Big) \tag{29.2}$$

formally, and that this can be made precise in some situations. To see this, put

$$X = (\mathbf{Z}_+ \cup \{0\}) \times (\mathbf{Z}_+ \cup \{0\}), \tag{29.3}$$

and

$$f(j,l) = a_j\, b_l \tag{29.4}$$

for each $(j,l) \in X$, which defines an $\mathcal{A}$-valued function on X. If we put

$$E_n = \{(j,l) \in X : j + l = n\} \tag{29.5}$$

for each nonnegative integer n, then the E_n's are pairwise-disjoint nonempty subsets of X such that

$$\bigcup_{n=0}^{\infty} E_n = X. \tag{29.6}$$

By construction,

$$c_n = \sum_{(j,l) \in E_n} f(j,l) \tag{29.7}$$

for every $n \ge 0$, so that

$$\sum_{n=0}^{\infty} c_n = \sum_{(j,l) \in X} f(j,l) \tag{29.8}$$

formally again. We also have that

$$\sum_{(j,l) \in X} f(j,l) = \Big(\sum_{j=0}^{\infty} a_j\Big) \Big(\sum_{l=0}^{\infty} b_l\Big) \tag{29.9}$$

formally, by identifying the sum over X with iterated sums over j and l.

Suppose for the moment that $a_j = 0$ for all but finitely many $j \ge 0$, and that $b_l = 0$ for all but finitely many $l \ge 0$. This implies that (29.4) is equal to 0 for all but finitely many $(j,l) \in X$. Similarly, (29.1) is equal to 0 for all but finitely many $n \ge 0$, which can be verified directly, or obtained from the previous statement using (29.7). Thus

$$\sum_{j=0}^{\infty} a_j, \ \sum_{l=0}^{\infty} b_l, \ \sum_{n=0}^{\infty} c_n, \text{ and } \sum_{(j,l) \in X} f(j,l) \tag{29.10}$$

reduce to finite sums in this situation. In particular, (29.8) and (29.9) hold, as in Section 26, which implies that (29.2) holds as well.

Suppose now that $\mathcal{A} = k = \mathbf{R}$, and that a_j and b_l are nonnegative real numbers for every $j, l \ge 0$. In this case, each of the sums in (29.10) is defined as a nonnegative extended real number, as in Section 21. We also have (29.8) in this situation, as in Section 26. Similarly, the left side of (29.9) can be expressed in terms of iterated sums over j and l. These iterated sums reduce to the right side of (29.9) when both sums on the right side of (29.9) are finite, and when both sums are positive. Otherwise, if either $a_j = 0$ for every $j \ge 0$, or $b_l = 0$ for every $l \ge 0$, then (29.4) is equal to 0 for every $(j,l) \in X$. This implies that the left side of (29.9) is equal to 0 too.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a q-norm N with respect to $|\cdot|$ on k for some

$q > 0$. Suppose that $\mathcal{A}$ is complete with respect to the q-metric associated to N, and that N satisfies the weaker submultiplicativity condition (18.1) for some $C \geq 0$. This implies that

$$N(f(j,l)) = N(a_j\, b_l) \leq C\, N(a_j)\, N(b_l) \tag{29.11}$$

for every $j, l \geq 0$. Suppose for the moment that $q < \infty$, and that

$$\sum_{j=0}^{\infty} N(a_j)^q, \ \sum_{l=0}^{\infty} N(b_l)^q \tag{29.12}$$

converge as infinite series of nonnegative real numbers. Observe that

$$\begin{aligned} \sum_{(j,l)\in X} N(f(j,l))^q &\leq C^q \sum_{(j,l)\in X} N(a_j)^q\, N(b_l)^q \\ &= C^q \Big(\sum_{j=0}^{\infty} N(a_j)^q\Big) \Big(\sum_{l=0}^{\infty} N(b_l)^q\Big), \end{aligned} \tag{29.13}$$

using (29.11) in the first step, and the remarks in the previous paragraph in the second step. Thus $f(j,l)$ is q-summable as an $\mathcal{A}$-valued function on X, with respect to N. We also have that

$$N(c_n)^q \leq \sum_{j=0}^{n} N(a_j\, b_{n-j})^q = \sum_{(j,l)\in E_n} N(f(j,l))^q \tag{29.14}$$

for every $n \geq 0$, using the definition (29.1) of c_n and the q-norm version of the triangle inequality (12.2) in the first step, and the definition (29.5) of E_n in the second step. It follows that

$$\sum_{n=0}^{\infty} N(c_n)^q \leq \sum_{n=0}^{\infty} \Big(\sum_{(j,l)\in E_n} N(f(j,l))^q \Big) = \sum_{(j,l)\in X} N(f(j,l))^q, \tag{29.15}$$

using (29.6) in the second step. Hence each of the sums in (29.10) can be defined as an element of V in this situation, as in Sections 24 and 25. These sums also satisfy (29.8) and (29.9), as in Section 26.

Suppose instead that $q = \infty$, and that

$$\lim_{j\to\infty} N(a_j) = \lim_{l\to\infty} N(b_l) = 0. \tag{29.16}$$

Using this and (29.11), one can check that

$$f(j,l) \in c_0(X, \mathcal{A}). \tag{29.17}$$

Observe that

$$N(c_n) \leq \max_{0\leq j\leq n} N(a_j\, b_{n-j}) = \max_{(j,l)\in E_n} N(f(j,l)) \tag{29.18}$$

for every $n \geq 0$, because of the ultranorm version of the triangle inequality. One can use this to verify that

$$\lim_{n \to \infty} N(c_n) = 0, \tag{29.19}$$

using also (29.11) and (29.16), or (29.17). Thus each of the sums in (29.10) is defined as an element of V, as in Sections 24 and 25, and they satisfy (29.8) and (29.9), as in Section 26.

30 q-Banach algebras

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative q-norm $\|\cdot\|$ with respect to $|\cdot|$ on k for some $q > 0$. If $\mathcal{A}$ is complete with respect to the q-metric associated to $\|\cdot\|$, then $\mathcal{A}$ is said to be *q-Banach algebra* with respect to $\|\cdot\|$. If $q = 1$, then we may simply say that $\mathcal{A}$ is a *Banach algebra* with respect to $\|\cdot\|$. If $\mathcal{A}$ is not complete with respect to the q-metric associated to $\|\cdot\|$, then one can pass to a completion, as in Sections 7 and 12. Multiplication on $\mathcal{A}$ can be extended to a bounded bilinear mapping on the completion as in Section 15. It is easy to see that this extension of multiplication to the completion of $\mathcal{A}$ is associative. Thus the completion of $\mathcal{A}$ is a q-Banach algebra.

If k is not already complete with respect to the q_k-metric associated to $|\cdot|$, then we can pass to a completion, as in Section 7. If $\mathcal{A}$ is complete, then scalar multiplication on $\mathcal{A}$ can be extended continuously to the completion of k, as mentioned in Section 12. In this way, $\mathcal{A}$ becomes a vector space over the completion of k, and N is a q-norm on $\mathcal{A}$ as a vector space over the completion of k. It is easy to see that multiplication on $\mathcal{A}$ is also bilinear with respect to the completion of k, so that $\mathcal{A}$ is also a Banach algebra with respect to the completion of k. One may prefer to include the completeness of k in the definition of a q-Banach algebra.

Sometimes one includes the condition that $\mathcal{A}$ have a multiplicative identity element e with $\|e\| = 1$ in the definition of a q-Banach algebra, and we shall do that here. As usual, an element a of $\mathcal{A}$ is said to be *invertible* in $\mathcal{A}$ if there is an element b of $\mathcal{A}$ such that

$$a\,b = b\,a = e. \tag{30.1}$$

If such an element b of $\mathcal{A}$ exists, then it is unique, and it is denoted a^{-1}. In this case, a^{-1} is invertible in $\mathcal{A}$, with $(a^{-1})^{-1} = a$. If x, y are invertible elements of $\mathcal{A}$, then $x\,y$ is invertible in $\mathcal{A}$, and

$$(x\,y)^{-1} = y^{-1}\,x^{-1}. \tag{30.2}$$

If $x \in \mathcal{A}$ and n is a nonnegative integer, then

$$(e - x) \sum_{j=0}^{n} x^j = \Big(\sum_{j=0}^{n} x^j\Big)(e - x) = e - x^{n+1}, \tag{30.3}$$

by a standard argument. Here x^j is interpreted as being equal to e when $j = 0$, as usual. Suppose for the moment that

$$\sum_{j=0}^{\infty} x^j \tag{30.4}$$

converges as an infinite series in $\mathcal{A}$, which implies in particular that

$$\lim_{j \to \infty} \|x^j\| = 0. \tag{30.5}$$

In this case, we can take the limit as $n \to \infty$ in (30.3), to get that

$$(e - x) \sum_{j=0}^{\infty} x^j = \Big(\sum_{j=0}^{\infty} x^j\Big)(e - x) = e. \tag{30.6}$$

This implies that $e - x$ is invertible in $\mathcal{A}$, with

$$(e - x)^{-1} = \sum_{j=0}^{\infty} x^j. \tag{30.7}$$

Of course,

$$\|x^j\| \le \|x\|^j \tag{30.8}$$

for each j, by the submultiplicativity of $\|\cdot\|$ on $\mathcal{A}$. Suppose that $\|x\| < 1$, so that (30.5) follows from (30.8). If $q < \infty$, then

$$\sum_{j=0}^{\infty} \|x^j\|^q \le \sum_{j=0}^{\infty} \|x\|^{q\,j} = (1 - \|x\|^q)^{-1}, \tag{30.9}$$

by summing the geometric series in the second step. Thus (30.4) converges q-absolutely with respect to $\|\cdot\|$. If $\mathcal{A}$ is complete with respect to the q-metric associated to $\|\cdot\|$, then it follows that (30.4) converges in $\mathcal{A}$, as in Section 24. Similarly, if $q = \infty$, and $\mathcal{A}$ is complete with respect to the ultrametric associated to $\|\cdot\|$, then the convergence of (30.4) in $\mathcal{A}$ follows from (30.5), as in Section 24 again. In both cases, the remarks in the previous paragraph imply that $e - x$ is invertible in $\mathcal{A}$.

31 Submultiplicative sequences

Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of nonnegative real numbers that is *submultiplicative*, in the sense that

$$a_{j+l} \le a_j\, a_l \tag{31.1}$$

for every $j, l \in \mathbf{Z}_+$. Put

$$\alpha = \inf_{j \ge 1} a_j^{1/j}, \tag{31.2}$$

so that α is a nonnegative real number too. Under these conditions, it is well known that

$$\lim_{j \to \infty} a_j^{1/j} = \alpha. \tag{31.3}$$

Of course,

$$\alpha \le \liminf_{j \to \infty} a_j^{1/j}, \tag{31.4}$$

by construction. Thus it suffices to show that

$$\limsup_{j \to \infty} a_j^{1/j} \le \alpha. \tag{31.5}$$

Let $j_0, j \in \mathbf{Z}_+$ be given, and observe that j can be expressed as

$$j = j_0\, l_0 + r_0, \tag{31.6}$$

where l_0, r_0 are nonnegative integers, at least one of l_0 and r_0 is positive, and $r_0 < j_0$. Using (31.1), we get that

$$a_j = a_{j_0\, l_0 + r_0} \le a_{j_0}^{l_0}\, a_1^{r_0}, \tag{31.7}$$

and hence

$$a_j^{1/j} \le (a_{j_0}^{1/j_0})^{j_0\, l_0/j}\, a_1^{r_0/j} = (a_{j_0}^{1/j_0})^{1-(r_0/j)}\, a_1^{r_0/j}. \tag{31.8}$$

It follows that

$$\limsup_{j \to \infty} a_j^{1/j} \le a_{j_0}^{1/j_0} \tag{31.9}$$

for every $j_0 \in \mathbf{Z}_+$, because $b^{1/j} \to 1$ as $j \to \infty$ for every positive real number b. This implies (31.5), by taking the infimum over $j_0 \in \mathbf{Z}_+$.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative q-seminorm N with respect to $|\cdot|$ on k for some $q > 0$. If $x \in \mathcal{A}$, then

$$N(x^{j+l}) = N(x^j\, x^l) \le N(x^j)\, N(x^l) \tag{31.10}$$

for every $j, l \in \mathbf{Z}_+$, so that $a_j = N(x^j)$ defines a submultiplicative sequence of nonnegative real numbers. Put

$$N_\rho(x) = \inf_{j \ge 1} N(x^j)^{1/j}, \tag{31.11}$$

which corresponds to (31.2) in this situation. Thus

$$\lim_{j \to \infty} N(x^j)^{1/j} = N_\rho(x) \tag{31.12}$$

for every $x \in \mathcal{A}$, as in (31.3). Note that

$$N_\rho(x) \le N(x) \tag{31.13}$$

for every $x \in \mathcal{A}$, by taking $j = 1$ in the right side of (31.11).

If $x \in \mathcal{A}$ and $t \in k$, then it is easy to see that

$$N_\rho(t\,x) = |t|\,N_\rho(x), \tag{31.14}$$

using the analogous property of N. If $x \in \mathcal{A}$ and $n \in \mathbf{Z}_+$, then

$$\lim_{l \to \infty} N(x^{l\,n})^{1/(l\,n)} = \lim_{j \to \infty} N(x^j)^{1/j} = N_\rho(x), \tag{31.15}$$

because $\{N(x^{l\,n})^{1/(l\,n)}\}_{l=1}^\infty$ is a subsequence of $\{N(x^j)^{1/j}\}_{j=1}^\infty$. Hence

$$N((x^n)^l)^{1/l} = N(x^{l\,n})^{1/l} \to N_\rho(x)^n \tag{31.16}$$

as $l \to \infty$, so that

$$N_\rho(x^n) = N_\rho(x)^n. \tag{31.17}$$

Of course, if $N(x^j) = N(x)^j$ for each $j \geq 1$, then

$$N_\rho(x) = N(x). \tag{31.18}$$

Let $\widetilde{N}$ be a submultiplicative $\widetilde{q}$-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $\widetilde{q} > 0$, and let $\widetilde{N}_\rho$ be as in (31.11). Suppose that there is a positive real number C such that

$$N(x) \leq C\,\widetilde{N}(x) \tag{31.19}$$

for every $x \in \mathcal{A}$. This implies that

$$N(x^j) \leq C\,\widetilde{N}(x^j) \tag{31.20}$$

for every $x \in \mathcal{A}$ and $j \in \mathbf{Z}_+$, and hence

$$N(x^j)^{1/j} \leq C^{1/j}\,\widetilde{N}(x^j)^{1/j} \tag{31.21}$$

for every $x \in \mathcal{A}$ and $j \in \mathbf{Z}_+$. It follows that

$$N_\rho(x) \leq \widetilde{N}_\rho(x) \tag{31.22}$$

for every $x \in \mathcal{A}$, by taking the limit as $j \to \infty$ on both sides of (31.21), and using (31.12).

32 Multiplicative inverses

Let k be a field, and let $\mathcal{A}$ be an algebra over k with a nonzero multiplicative identity element e. Also let w, z be commuting elements of $\mathcal{A}$, so that $w\,z = z\,w$. If w is invertible in $\mathcal{A}$, then it is easy to see that w^{-1} commutes with z as well.

Let b be an element of $\mathcal{A}$. An element a of $\mathcal{A}$ is said to be a *left inverse* of b if

$$a\,b = e. \tag{32.1}$$

Similarly, $c \in \mathcal{A}$ is a *right inverse* of b if

$$b\,c = e. \tag{32.2}$$

If a is a left inverse of b, and c is a right inverse of b, then one can check that $a = c$, so that b is invertible in $\mathcal{A}$.

Let x, y be elements of $\mathcal{A}$. If $x\,y$ is invertible in $\mathcal{A}$, then

$$x\,(y\,(x\,y)^{-1}) = (x\,y)\,(x\,y)^{-1} = e,\ \ ((x\,y)^{-1}\,x)\,y = (x\,y)^{-1}\,(x\,y) = e, \tag{32.3}$$

so that x has a right inverse in $\mathcal{A}$, and y has a left inverse in $\mathcal{A}$. Similarly, if $y\,x$ is invertible in $\mathcal{A}$, then

$$y\,(x\,(y\,x)^{-1}) = (y\,x)\,(y\,x)^{-1} = e,\ \ ((y\,x)^{-1}\,y)\,x = (y\,x)^{-1}\,(y\,x) = e, \tag{32.4}$$

so that y has a right inverse in $\mathcal{A}$, and x has a left inverse in $\mathcal{A}$. If $x\,y$ and $y\,x$ are both invertible in $\mathcal{A}$, then it follows that x and y are invertible in $\mathcal{A}$.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a submultiplicative q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. Also let N_ρ be as in (31.11), and let $x \in \mathcal{A}$ be given. Suppose that

$$N_\rho(x) < 1, \tag{32.5}$$

and let r be a real number such that

$$N_\rho(x) < r < 1. \tag{32.6}$$

Using (31.12), we get that there is a positive integer L such that

$$N(x^j)^{1/j} < r \tag{32.7}$$

for every $j \ge L$. Equivalently, this means that

$$N(x^j) < r^j \tag{32.8}$$

for every $j \ge L$. In particular, this implies that

$$\lim_{j \to \infty} N(x^j) = 0. \tag{32.9}$$

Note that (32.5) holds when $N(x^j) < 1$ for some $j \in \mathbf{Z}_+$, by (31.11).

Suppose now that N is a q-norm on $\mathcal{A}$, and that $\mathcal{A}$ is a q-Banach algebra with respect to N. If $x \in \mathcal{A}$ satisfies (32.5), then

$$\sum_{j=0}^{\infty} x^j \tag{32.10}$$

converges in $\mathcal{A}$. More precisely, if $q < \infty$, then (32.10) converges q-absolutely, by comparison with a convergent geometric series, because of (32.8). If $q = \infty$, then the convergence of (32.10) follows from (32.9), as in Section 24. In both cases, the convergence of (32.10) implies that $e - x$ is invertible in $\mathcal{A}$, as in Section 30. Alternatively, one can use (32.5) to get that

$$N(x^{n+1}) < 1 \tag{32.11}$$

for some nonnegative integer n, because of (31.11). This implies that $e - x^{n+1}$ is invertible in $\mathcal{A}$, as in Section 30. Using this and (30.3), it follows that $x - e$ is invertible in $\mathcal{A}$, by the remarks at the beginning of the section.

33 Some additional properties

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let $\mathcal{A}$ be an algebra over k with a submultiplicative q-seminorm N with respect to $|\cdot|$ on k for some $q > 0$. Also let N_ρ be as in (31.11), so that N_ρ is a nonnegative real-valued function on $\mathcal{A}$. Let $x, y \in \mathcal{A}$ be given, and suppose that x and y commute in $\mathcal{A}$, so that

$$x\,y = y\,x. \tag{33.1}$$

If $j \in \mathbf{Z}_+$, then $(x\,y)^j = x^j\,y^j$, and hence

$$N((x\,y)^j)^{1/j} = N(x^j\,y^j)^{1/j} \le N(x^j)^{1/j}\,N(y^j)^{1/j}, \tag{33.2}$$

using the submultiplicativity of N in the second step. This implies that

$$N_\rho(x\,y) \le N_\rho(x)\,N_\rho(y), \tag{33.3}$$

by taking the limit as $j \to \infty$, as in (31.12).

We would like to estimate $N_\rho(x+y)$ in terms of $N_\rho(x)$ and $N_\rho(y)$. Of course, if $N \equiv 0$ on $\mathcal{A}$, then $N_\rho \equiv 0$ on $\mathcal{A}$, and there is nothing to do. Otherwise, if $N \not\equiv 0$ on $\mathcal{A}$, then we have seen that $|\cdot|$ is a q-absolute value function on k, as in Section 12. Thus we may as well suppose that

$$q \le q_k. \tag{33.4}$$

We may also restrict our attention to the case where

$$x, y \neq 0, \tag{33.5}$$

since otherwise $x+y$ is equal to x or y. Let r_x, r_y be positive real numbers such that

$$N_\rho(x) < r_x,\ N_\rho(y) < r_y. \tag{33.6}$$

Using (31.12) for both x and y, we get that there is an $L \in \mathbf{Z}_+$ such that

$$N(x^l)^{1/l} < r_x,\ N(y^l)^{1/l} < r_y \tag{33.7}$$

for every $l \ge L$. Equivalently, this means that

$$N(x^l) < r_x^l,\ N(y^l) < r_y^l \tag{33.8}$$

for every $l \ge L$.

Suppose for the moment that $q = \infty$, and let $n \in \mathbf{Z}_+$ be given. Observe that

$$N((x+y)^n) \le \max_{0 \le j \le n} N(x^j\, y^{n-j}), \tag{33.9}$$

where $j = 0, 1, \ldots, n$ is an integer, because $(x+y)^n$ can be expressed as a sum of the monomials $x^j\, y^{n-j}$. Thus

$$N((x+y)^n)^{1/n} \le \max_{0 \le j \le n} N(x^j\, y^{n-j})^{1/n} \le \max_{0 \le j \le n} (N(x^j)\, N(y^{n-j}))^{1/n}, \tag{33.10}$$

using the submultiplicativity of N in the second step. If $j \ge L$ and $n - j \ge L$, then we have that

$$(N(x^j)\, N(y^{n-j}))^{1/n} < (r_x^j\, r_y^{n-j})^{1/n} \le \max(r_x, r_y), \tag{33.11}$$

by (33.8). If $j \ge L$ and $n - j < L$, then we shall use the estimate

$$\begin{aligned} (N(x^j)\ N(y^{n-j}))^{1/n} \le (r_x^j\, N(y^{n-j}))^{1/n} &= (r_x^n\, r_x^{j-n}\, N(y^{n-j}))^{1/n} \\ &= r_x\, (r_x^{j-n}\, N(y^{n-j}))^{1/n}, \end{aligned} \tag{33.12}$$

where the first step follows from the first part of (33.8). Similarly, if $j < L$ and $n - j \ge L$, then we shall use the estimate

$$(N(x^j)\, N(y^{n-j}))^{1/n} \le (N(x^j)\, r_y^{n-j})^{1/n} = (N(x^j)\, r_y^{-j})^{1/n}\, r_y, \tag{33.13}$$

where the first step follows from the second part of (33.8). Of course, if $n \ge 2\,L$, then either $j \ge L$ or $n - j \ge L$ for any j.

Put

$$C = \max\Big(1, \max_{0 \le l < L} (r_x^{-l}\, N(y^l)), \max_{0 \le j < L} (N(x^j)\, r_y^{-j})\Big), \tag{33.14}$$

and note that this does not depend on n. Combining the estimates in the previous paragraph, we get that

$$N((x+y)^n)^{1/n} \le C^{1/n}\, \max(r_x, r_y) \tag{33.15}$$

for every $n \ge 2\,L$. This implies that

$$N_\rho(x+y) \le \max(r_x, r_y), \tag{33.16}$$

because $C^{1/n} \to 1$ as $n \to \infty$. It follows that

$$N_\rho(x+y) \le \max(N_\rho(x), N_\rho(y)) \tag{33.17}$$

when $q = \infty$, by taking r_x, r_y arbitrarily close to $N_\rho(x)$, $N_\rho(y)$, respectively.

Suppose from now in this section that $q < \infty$, and let $n \in \mathbf{Z}_+$ be given again. Remember that

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} \cdot x^j\, y^{n-j}, \tag{33.18}$$

as in the binomial theorem. Thus

$$N((x+y)^n)^q \le \sum_{j=0}^{n} N\Big(\binom{n}{j} \cdot x^j\, y^{n-j}\Big)^q, \tag{33.19}$$

by the q-seminorm version (12.2) of the triangle inequality. Observe that

$$\begin{aligned} N\Big(\binom{n}{j} \cdot x^j\, y^{n-j}\Big) &= N\Big(\Big(\binom{n}{j} \cdot 1\Big)\, x^j\, y^{n-j}\Big) \\ &= \left|\binom{n}{j} \cdot 1\right| N(x^j\, y^{n-j}) \\ &\le \left|\binom{n}{j} \cdot 1\right| N(x^j)\, N(y^{n-j}) \end{aligned} \tag{33.20}$$

for each $j = 0, 1, \ldots, n$, using (20.3) in the first step, and the submultiplicativity of N in the third step. Hence

$$N((x+y)^n)^q \le \sum_{j=0}^{n} \left|\binom{n}{j} \cdot 1\right|^q N(x^j)^q\, N(y^{n-j})^q. \tag{33.21}$$

Suppose for the moment that $q_k = \infty$, so that

$$\begin{aligned} N((x+y)^n)^q &\le \sum_{j=0}^{n} N(x^j)^q\, N(y^{n-j})^q \\ &\le (n+1) \max_{0 \le j \le n} (N(x^j)^q\, N(y^{n-j})^q), \end{aligned} \tag{33.22}$$

using (33.21) in the first step. This implies that

$$N((x+y)^n)^{1/n} \le (n+1)^{1/(n\,q)} \max_{0 \le j \le n} (N(x^j)\, N(y^{n-j}))^{1/n}, \tag{33.23}$$

which is analogous to (33.10). Remember that (33.8) holds when $j \ge L$, and let C be as in (33.14). In this situation, we have that

$$N((x+y)^n)^{1/n} \le (n+1)^{1/(n\,q)}\, C^{1/n} \max(r_x, r_y) \tag{33.24}$$

for every $n \ge 2\,L$, for essentially the same reasons as in (33.15). More precisely, this uses (33.11), (33.12), (33.13), (33.23), and the definition of C. As before, one can use (33.24) to get (33.16), by taking the limit as $n \to \infty$. Using (33.16), one gets (33.17) by taking r_x, r_y arbitrarily close to $N_\rho(x)$, $N_\rho(y)$ again.

Suppose now that $k = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{Q}$ with the standard absolute value function, so that $q_k = 1$. As in (33.4), we may as well suppose that $q \le 1$. Let $n \in \mathbf{Z}_+$ be given again, and observe that

$$N((x+y)^n)^q \le (n+1) \max_{0 \le j \le n} \Big(\binom{n}{j}^q N(x^j)^q\, N(y^{n-j})^q\Big), \tag{33.25}$$

by (33.21). This implies that

$$N((x+y)^n) \le (n+1)^{1/q} \max_{0 \le j \le n} \Big(\binom{n}{j} N(x^j)\, N(y^{n-j}) \Big), \tag{33.26}$$

by taking the qth root of both sides of (33.25).

Remember that (33.8) holds when $l \ge L$, and that

$$\binom{n}{j} = \binom{n}{n-j} \tag{33.27}$$

for each $j = 0, 1, \ldots, n$. If $L \le j \le n$ and $n - j \ge L$, then

$$\binom{n}{j} N(x^j)\, N(y^{n-j}) < \binom{n}{j} r_x^j\, r_y^{n-j} \le \sum_{l=0}^{n} \binom{n}{l} r_x^l\, r_y^{n-l} = (r_x + r_y)^n, \tag{33.28}$$

by (33.8). If $L \le j \le n$ and $n - j < L$, then we shall use the estimate

$$\binom{n}{j} N(x^j)\, N(y^{n-j}) \le \binom{n}{j} r_x^j\, N(y^{n-j}) = \binom{n}{n-j} r_x^n\, r_x^{j-n}\, N(y^{n-j}), \tag{33.29}$$

where the first step follows from the first part of (33.8), and the second step follows from (33.27). Similarly, if $0 \le j \le n$, $j < L$, and $n - j \ge L$, then we shall use the estimate

$$\binom{n}{j} N(x^j)\, N(y^{n-j}) \le \binom{n}{j} N(x^j)\, r_y^{n-j} = \binom{n}{j} N(x^j)\, r_y^{-j}\, r_y^n, \tag{33.30}$$

where the first step follows from the second part of (33.8). As before, if $n \ge L$, then either $j \ge L$ or $n - j \ge L$ for any j.

Put

$$C(n) = \max\Big(1, \max_{0 \le l < L} \Big(\binom{n}{l} r_x^l\, N(y^l) \Big), \max_{0 \le j < L} \Big(\binom{n}{j} N(x^j)\, r_y^{-j} \Big)\Big) \tag{33.31}$$

for each $n \ge L$. Using (33.26) and the estimates in the previous paragraph, one can check that

$$N((x+y)^n) \le (n+1)^{1/q}\, C(n)\, (r_x + r_y)^n \tag{33.32}$$

for every $n \ge 2L$. Equivalently, this means that

$$N(x+y)^{1/n} \le (n+1)^{1/(n\,q)}\, C(n)^{1/n}\, (r_x + r_y) \tag{33.33}$$

when $n \ge 2L$. Because $C(n) = O(n^{L-1})$, one can use this to get that

$$N_\rho(x+y) \le r_x + r_y. \tag{33.34}$$

It follows that

$$N_\rho(x+y) \le N_\rho(x) + N_\rho(y) \tag{33.35}$$

in this situation, by taking r_x, r_y arbitrarily close to $N_\rho(x)$, $N_\rho(y)$, respectively.

34 Hölder's inequality

Let X be a nonempty set, and let f, g be nonnegative real-valued functions on X. Remember that $\|f\|_r$ is defined as a nonnegative extended real number when $0 < r \le \infty$, as in Section 21. Suppose that $0 < r_1, r_2, r_3 \le \infty$ satisfy

$$1/r_3 = 1/r_1 + 1/r_2, \tag{34.1}$$

where $1/\infty$ is interpreted as being 0, as usual. Under these conditions, *Hölder's inequality* for sums implies that

$$\|f\, g\|_{r_3} \le \|f\|_{r_1} \, \|g\|_{r_2}. \tag{34.2}$$

More precisely, if either of the factors on the right side of (34.2) is equal to 0, then the corresponding function is equal to 0 on X, so that $f(x)\, g(x) = 0$ for every $x \in X$. This implies that the left side of (34.2) is equal to 0. Thus one may interpret the left side of (34.2) as being equal to 0 when either of the factors is equal to 0, even if the other factor is $+\infty$. Otherwise, the right side of (34.2) is defined as a nonnegative extended real number in the usual way. If $r_1 = \infty$ or $r_2 = \infty$, then (34.2) can be verified directly. Hölder's inequality is often stated for $r_3 = 1$, and it is not difficult to reduce to that case when $r_3 < \infty$. If $r_1 = r_2 = 2$, so that $r_3 = 1$, then (34.2) is a version of the Cauchy–Schwarz inequality.

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. Of course, k may be considered as a one-dimensional vector space over itself, and $|\cdot|$ may be considered as a q-norm on k. Thus $\ell^r(X, k)$ may be defined as in Section 13 when $r = \infty$, and as in Section 22 when $0 < r < \infty$. If $f \in \ell^r(X, k)$ for some $r > 0$, then $\|f\|_r$ is defined as a nonnegative real number, as in the sections just mentioned. If $r_1, r_2, r_3 > 0$ satisfy (34.1), $f \in \ell^{r_1}(X, k)$, and $g \in \ell^{r_2}(X, k)$, then

$$f\, g \in \ell^{r_3}(X, k), \tag{34.3}$$

and (34.2) holds. This follows from the remarks in the preceding paragraph, applied to $|f(x)|$, $|g(x)|$ as nonnegative real-valued functions on X. Of course, the case where $r_1 = r_2 = r_3 = \infty$ has been discussed previously.

Let $r > 0$ be given, and suppose that $r_1, r_2, r_3 > 0$ satisfy (34.1) and

$$r_3 \le r \le r_1, r_2. \tag{34.4}$$

One can take $r_1 = r_2 = r$, for instance, in which case $r_3 = r/2 \le r$. Alternatively, if $r_3 = r$, then (34.1) implies that $r \le r_1, r_2$. If $f, g \in \ell^r(X, k)$, then it follows that $f \in \ell^{r_1}(X, k)$ and $g \in \ell^{r_2}(X, k)$, as in Section 22. Similarly, (34.3) implies that

$$f\, g \in \ell^r(X, k), \tag{34.5}$$

and (34.2) implies that

$$\|f\, g\|_r \le \|f\, g\|_{r_3} \le \|f\|_{r_1} \, \|g\|_{r_2} \le \|f\|_r \, \|g\|_r \tag{34.6}$$

in this situation. It follows that $\ell^r(X,k)$ is a commutative algebra over k with respect to pointwise multiplication of functions, and that $\|\cdot\|_r$ is submultiplicative on $\ell^r(X,k)$. As before, this has been discussed previously when $r=\infty$.

If $f \in \ell^r(X,\mathbf{Z})$ and $n \in \mathbf{Z}_+$, then it is easy to see that $f(x)^n \in \ell^{r/n}(X,k)$, with

$$\|f^n\|_{r/n} = \|f\|_r^n, \tag{34.7}$$

directly from the definitions. In particular, this implies that $f(x)^n \in \ell^r(X,k)$, as in Section 22 again, and in fact

$$\|f^n\|_r^{1/n} = \|f\|_{n\,r}. \tag{34.8}$$

Note that

$$\lim_{n\to\infty} \|f\|_{n\,r} = \|f\|_\infty, \tag{34.9}$$

using (21.12) when $r<\infty$.

Part III
Some additional basic notions

35 Inner product spaces

Let V, W be vector spaces over the complex numbers. Of course, V and W can also be considered as vector spaces over the real numbers. If T is a linear mapping from V into W as complex vector spaces, then T may be considered as a linear mapping from V into W as real vector spaces as well. We may refer to real and complex-linear mappings to be precise. A mapping T from V into W is said to be *conjugate-linear* if

$$T(v+v') = T(v) + T(v') \tag{35.1}$$

for every $v, v' \in V$, and

$$T(a\,v) = \overline{a}\,T(v) \tag{35.2}$$

for every $a \in \mathbf{C}$ and $v \in \mathbf{C}$, where $\overline{a}$ is the complex-conjugate of a. Conjugate-linear mappings are real-linear too. A real-linear mapping T from V into W is complex-linear when

$$T(i\,v) = i\,T(v) \tag{35.3}$$

for every $v \in V$, and T is conjugate-linear when

$$T(i\,v) = -i\,T(v) \tag{35.4}$$

for every $v \in V$.

Now let V be a vector space over the real or complex numbers. An *inner product* on V is a real or complex-valued function $\langle v, w\rangle$, as appropriate, defined for $v, w \in V$, and satisfying the following three properties. First, for each $w \in W$,

$$v \mapsto \langle v, w \rangle \tag{35.5}$$

should be a linear mapping from V into $\mathbf{R}$ or $\mathbf{C}$, as appropriate. More precisely, this means that (35.5) should be real-linear in the real case, and complex-linear in the complex case. Second,

$$\langle v, w \rangle = \langle w, v \rangle \tag{35.6}$$

for every $v, w \in V$ in the real case, and

$$\langle v, w \rangle = \overline{\langle w, v \rangle} \tag{35.7}$$

for every $v, w \in V$ in the complex case. This implies that for each $v \in V$,

$$w \mapsto \langle v, w \rangle \tag{35.8}$$

is linear in the real case, and conjugate-linear in the complex case. We also get that

$$\langle v, v \rangle \in \mathbf{R} \tag{35.9}$$

for every $v \in V$ in the complex case, which is trivial in the real case. The third condition is that

$$\langle v, v \rangle > 0 \tag{35.10}$$

for every $v \in V$ with $v \neq 0$. Of course, $\langle v, w \rangle = 0$ when either $v = 0$ or $w = 0$, by the previous conditions.

Let $\langle v, w \rangle$ be an inner product on V, and put

$$\|v\| = \langle v, v \rangle^{1/2} \tag{35.11}$$

for each $v \in V$. Observe that

$$\|t\, v\| = |t|\, \|v\| \tag{35.12}$$

for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, where $|\cdot|$ is the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$. It is well known that

$$|\langle v, w \rangle| \leq \|v\|\, \|w\| \tag{35.13}$$

for every $v, w \in V$, which is the *Cauchy–Schwarz inequality*. Using this, one can check that

$$\|v + w\| \leq \|v\| + \|w\| \tag{35.14}$$

for every $v, w \in V$. Thus $\|\cdot\|$ defines a norm on V, with respect to the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate.

Let $v \in V$ be given, and let us check that

$$\|v\| = \sup\{|\langle v, w \rangle| : w \in V,\ \|w\| \leq 1\}. \tag{35.15}$$

The Cauchy–Schwarz inequality implies that the right side of (35.15) is less than or equal to the left side. If $v = 0$, then both sides of (35.15) are equal to 0. If $v \neq 0$ and $w_0 = v/\|v\|$, then $\|w_0\| = 1$ and

$$\langle v, w_0 \rangle = \langle v, v \rangle / \|v\| = \|v\|. \tag{35.16}$$

This implies that the left side of (35.15) is less than or equal to the right side.

36 Hilbert spaces

Let $(V, \langle v, w \rangle)$ be a real or complex inner product space, and let $\|v\|$ be the corresponding norm, as in (35.11). Thus

$$d(v, w) = \|v - w\| \tag{36.1}$$

defines a metric on V, as in Section 12. If V is also complete with respect to this metric, then V is said to be a *Hilbert space*. As usual, if V is not already complete with respect to (36.1), then one can pass to a completion of V. The inner product on V has a unique continuous extension to the completion, which makes the completion of V a Hilbert space.

Let X be a nonempty set, and let f, g be real or complex-valued functions on X. Suppose that f and g are 2-summable, as in Section 22, wth respect to the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate. This implies that $f\,g$ is summable on X, as in Section 34. Alternatively, if a and b are nonnegative real numbers, then

$$2\,a\,b \le a^2 + b^2, \tag{36.2}$$

because $(a - b)^2 \ge 0$. It follows that

$$2\,|f(x)|\,|g(x)| \le |f(x)|^2 + |g(x)|^2 \tag{36.3}$$

for every $x \in X$, so that

$$2 \sum_{x \in X} |f(x)|\,|g(x)| \le \sum_{x \in X} |f(x)|^2 + \sum_{x \in X} |g(x)|^2. \tag{36.4}$$

Put

$$\langle f, g \rangle = \langle f, g \rangle_{\ell^2(X, \mathbf{R})} = \sum_{x \in X} f(x)\,g(x) \tag{36.5}$$

in the real case, and

$$\langle f, g \rangle = \langle f, g \rangle_{\ell^2(X, \mathbf{C})} = \sum_{x \in X} f(x)\,\overline{g(x)} \tag{36.6}$$

in the complex case. It is easy to see that (36.5) defines an inner product on $\ell^2(X, \mathbf{R})$ as a real vector space, and that (36.6) defines an inner product on $\ell^2(X, \mathbf{C})$ as a complex vector space.

In both cases, we have that

$$\langle f, f \rangle = \sum_{x \in X} |f(x)|^2 = \|f\|_2^2, \tag{36.7}$$

where $\|f\|_2$ is as in Section 22. This means that the norms on $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$ associated to the inner products (36.5) and (36.6), respectively, are the same as the corresponding ℓ^2 norms defined previously. Of course, $\mathbf{R}$ and $\mathbf{C}$ are complete with respect to the metrics associated to their standard absolute value functions. It follows that $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$ are complete with respect to the metrics associated to their ℓ^2 norms, as in Section 22. Thus $\ell^2(X, \mathbf{R})$ and $\ell^2(X, \mathbf{C})$ are Hilbert spaces with respect to the inner products (36.5) and (36.6), respectively.

Let $(V, \langle v, w \rangle)$ be a real or complex inner product space again, with corresponding norm $\|v\|$. If $w \in V$, then

$$\lambda_w(v) = \langle v, w \rangle \tag{36.8}$$

defines a linear functional on V, which is to say a linear mapping from V into $\mathbf{R}$ or $\mathbf{C}$, as appropriate. Of course, the Cauchy–Schwarz inequality implies that

$$|\lambda_w(v)| \le \|v\| \, \|w\| \tag{36.9}$$

for every $v \in V$. This means that λ_w is bounded as a linear functional on V, using the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, as the norm on the range of λ_w. More precisely, (36.9) implies that the corresponding operator norm of λ_w, which is also known as the dual norm of λ_w, is less than or equal to $\|w\|$. It is easy to see that the dual norm of λ_w is equal to $\|w\|$, because

$$\lambda_w(w) = \langle w, w \rangle = \|w\|^2. \tag{36.10}$$

If V is a Hilbert space, then it is well known that every bounded linear functional on V is of the form λ_w for some $w \in V$. This representation is unique, basically because $\lambda_w \ne 0$ when $w \ne 0$, as in (36.10).

37 Adjoint mappings

Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be inner product spaces, both real or both complex, and let $\| \cdot \|_V$ and $\| \cdot \|_W$ be the corresponding norms on V and W, respectively. Also let T be a bounded linear mapping from V into W, and let $w \in W$ be given. Observe that

$$\mu_{T,w}(v) = \langle T(v), w \rangle_W \tag{37.1}$$

defines a linear functional on V. We also have that

$$|\mu_{T,w}(v)| \le \|T(v)\|_W \, \|w\|_W \le \|T\|_{op,VW} \, \|v\|_V \, \|w\|_W \tag{37.2}$$

for every $v \in V$, using the Cauchy–Schwarz inequality in the first step, and the definition of the operator norm $\|T\|_{op,VW}$ of T with respect to $\| \cdot \|_V$, $\| \cdot \|_W$ in the second step. Thus $\mu_{T,w}$ is a bounded linear functional on V. Suppose that V is a Hilbert space, so that there is a unique element $T^*(w)$ of V such that

$$\mu_{T,w}(v) = \langle v, T^*(w) \rangle_V \tag{37.3}$$

for every $v \in V$, as in the previous section. This defines a mapping T^* from W into V, which is characterized by the property that

$$\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V \tag{37.4}$$

for every $v \in V$ and $w \in W$. It is easy to see that T^* is linear as a mapping from W into V, using uniqueness of $T^*(w)$. This is known as the *adjoint* of T.

As in (37.2), we have that

$$|\langle v, T^*(w)\rangle_V| = |\langle T(v), w\rangle_W| \le \|T\|_{op,VW}\, \|v\|_V\, \|w\|_W \tag{37.5}$$

for every $v \in V$ and $w \in W$. Using (35.15), we get that

$$\|T^*(w)\|_V = \sup\{|\langle v, T^*(w)\rangle_V| : v \in V,\ \|v\| \le 1\} \le \|T\|_{op}\, \|w\|_W \tag{37.6}$$

for every $w \in W$. This implies that T^* is a bounded linear mapping from W into V, with

$$\|T^*\|_{op,WV} \le \|T\|_{op,VW}. \tag{37.7}$$

Similarly,

$$|\langle T(v), w\rangle_W| = |\langle v, T^*(w)\rangle_V| \le \|T^*\|_{op,WV}\, \|v\|_V\, \|w\|_W \tag{37.8}$$

for every $v \in V$ and $w \in W$. One can use this in the same way as before to get that $\|T\|_{op,VW} \le \|T^*\|_{op,WV}$, so that

$$\|T^*\|_{op,WV} = \|T\|_{op,VW}. \tag{37.9}$$

We may consider

$$T \mapsto T^* \tag{37.10}$$

as a mapping from the space $\mathcal{BL}(V, W)$ of bounded linear mappings from V into W into the analogous space $\mathcal{BL}(W, V)$. One can check that this mapping is linear in the real case, and conjugate-linear in the complex case. Suppose that W is a Hilbert space too, so that there is an analogous mapping from $\mathcal{BL}(W, V)$ into $\mathcal{BL}(V, W)$. If T is a bounded linear mapping from V into W, as before, then T^* is defined as a bounded linear mapping from W into V, and the adjoint $(T^*)^*$ of T^* is defined as a bounded linear mapping from V into W. One can verify that T satisfies the requirements of $(T^*)^*$, so that

$$(T^*)^* = T. \tag{37.11}$$

Let $(Z, \langle\cdot,\cdot\rangle_Z)$ be another inner product space, which is real if V, W are real, and complex if V, W are complex. Also let T_1 be a bounded linear mapping from V into W, and let T_2 be a bounded linear mapping from W into Z, so that their composition $T_2 \circ T_1$ is a bounded linear mapping from V into Z. Thus T_1^* is defined as a bounded linear mapping from W into V, T_2^* is defined as a bounded linear mapping from Z into W, and $(T_2 \circ T_1)^*$ is defined as a bounded linear mapping from Z into V. If $v \in V$ and $z \in Z$, then

$$\langle T_2(T_1(v)), z\rangle_Z = \langle T_1(v), T_2^*(z)\rangle_W = \langle v, T_1^*(T_2^*(z))\rangle_V, \tag{37.12}$$

by definition of T_1^* and T_2^*. This implies that

$$(T_2 \circ T_1)^* = T_1^* \circ T_2^* \tag{37.13}$$

as bounded linear mappings from Z into V. Of course, the identity mapping I_V on V is a bounded linear mapping from V into itself. Hence the adjoint I_V^* of I_V is defined as a bounded linear mapping from V into itself, and it is easy to see that

$$I_V^* = I_V. \tag{37.14}$$

Let T be a bounded linear mapping from V into W again, so that T^* is a bounded linear mapping from W into V, and their composition $T^* \circ T$ is a bounded linear mapping from V into itself. Observe that

$$\|T^* \circ T\|_{op,VV} \le \|T\|_{op,VW}\, \|T^*\|_{op,WV} = \|T\|_{op,VW}^2, \tag{37.15}$$

using (37.9) in the second step. If $v \in V$, then

$$\langle T^*(T(v)), v\rangle_V = \langle T(v), T(v)\rangle_W = \|T(v)\|_W^2 \tag{37.16}$$

and

$$|\langle T^*(T(v)), v\rangle_V| \le \|T^*(T(v))\|_V\, \|v\|_V \le \|T^* \circ T\|_{op,VV}\, \|v\|_V^2. \tag{37.17}$$

Using (37.16) and (37.17), we get that

$$\|T\|_{op,VW}^2 \le \|T^* \circ T\|_{op,VV}. \tag{37.18}$$

It follows that

$$\|T^* \circ T\|_{op,VV} = \|T\|_{op,VW}^2. \tag{37.19}$$

38 Involutions

Let k be a field, and let $\mathcal{A}$ be an algebra over k. A linear mapping

$$x \mapsto x^* \tag{38.1}$$

from $\mathcal{A}$ into itself is said to be an (algebra) *involution* on $\mathcal{A}$ if

$$(x\,y)^* = y^*\,x^* \tag{38.2}$$

for every $x, y \in \mathcal{A}$, and

$$(x^*)^* = x \tag{38.3}$$

for every $x \in \mathcal{A}$. If multiplication on $\mathcal{A}$ is commutative, then the identity mapping on $\mathcal{A}$ satisfies these conditions. If $k = \mathbf{C}$, then we may also be interested in conjugate-linear mappings (38.1) that satisfy (38.2) and (38.3).

Let X be a nonempty set, and consider the commutative algebra $c(X, \mathbf{C})$ of complex-valued functions on X. If $f \in c(X, \mathbf{C})$, then the complex-conjugate $\overline{f(x)}$ is an element of $c(X, \mathbf{C})$ too, and

$$f \mapsto \overline{f} \tag{38.4}$$

defines a conjugate-linear involution on $c(X, \mathbf{C})$.

Let $(V, \langle v, w \rangle)$ be a real or complex Hilbert space. If T is a bounded linear mapping from V into itself, then its adjoint T^* can be defined as a bounded linear mapping from V into itself as well, as in the previous section. Remember that

$$T \mapsto T^* \tag{38.5}$$

is a linear mapping from the algebra $\mathcal{BL}(V)$ of bounded linear mappings from V into itself in the real case, and that (38.5) is conjugate-linear in the complex case. In both cases, we have seen that (38.5) satisfies (38.2) and (38.3).

Let $\mathcal{A}$ be any algebra over a field k again, let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let N be a submultiplicative q-seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k for some $q > 0$. An involution (38.1) on $\mathcal{A}$ is said to be *compatible* with N if

$$N(x^*) = N(x) \tag{38.6}$$

for every $x \in \mathcal{A}$. Of course, if $\mathcal{A}$ is a commutative algebra and (38.1) is the identity mapping on $\mathcal{A}$, then (38.6) holds trivially.

Let X be a nonempty set, and remember that $\ell^r(X, \mathbf{C})$ may be considered as a commutative algebra over $\mathbf{C}$ with respect to pointwise multiplication of functions, as in Section 34. As before, (38.4) defines a conjugate-linear involution on $\ell^r(X, \mathbf{C})$ for each $r > 0$. It is easy to see that (38.4) is compatible with $\|f\|_r$ for every $r > 0$, as in the preceding paragraph. Similarly, if X is a nonempty topological space, then (38.4) defines a conjugate-linear involution on the algebra $C(X, \mathbf{C})$ of continuous complex-valued functions on X. If E is a nonempty compact subset of X, then (38.4) is compatible with the supremum seminorm on $C(X, \mathbf{C})$ associated to E.

Let $(V, \langle v, w \rangle)$ be a real or complex Hilbert space again. We have seen that (38.5) is compatible with the operator norm on $\mathcal{BL}(V)$, as in (37.9).

Let k, $\mathcal{A}$, and N be as before, and let (38.1) be an involution on $\mathcal{A}$ again. Another interesting condition on (38.1) and N asks that

$$N(x^* \, x) = N(x)^2 \tag{38.7}$$

for every $x \in \mathcal{A}$. If $\mathcal{A}$ is a commutative algebra and (38.1) is the identity mapping on $\mathcal{A}$, then (38.7) is the same as saying that

$$N(x^2) = N(x)^2 \tag{38.8}$$

for every $x \in \mathcal{A}$. If X is a nonempty set, then the supremum norm on $\ell^\infty(X, \mathbf{C})$ satisfies (38.7) with respect to (38.4). Similarly, if X is a nonempty topological space, and E is a nonempty compact subset of X, then the supremum seminorm on $C(X, \mathbf{C})$ associated to E satisfies (38.7) with respect to (38.4). If $(V, \langle v, w \rangle)$ is a real or complex inner product space, then the operator norm on $\mathcal{BL}(V)$ satisfies (38.7) with respect to (38.5), as in (37.19). Note that (38.8) does not always hold in this situation.

Let k, $\mathcal{A}$, and N be as before, let (38.1) be an involution on $\mathcal{A}$, and suppose that (38.7) holds for every $x \in \mathcal{A}$. Using submultiplicativity, we get that

$$N(x)^2 = N(x^* \, x) \le N(x^*) \, N(x) \tag{38.9}$$

for every $x \in \mathcal{A}$. This implies that

$$N(x) \le N(x^*) \tag{38.10}$$

when $N(x) > 0$, and (38.10) holds trivially when $N(x) = 0$. Applying (38.10) to x^*, we get that

$$N(x^*) \le N((x^*)^*) = N(x) \tag{38.11}$$

for every $x \in \mathcal{A}$. Thus (38.6) holds for every $x \in \mathcal{A}$ under these conditions.

Suppose for the moment that $x \in \mathcal{A}$ is *self-adjoint* with respect to (38.1), in the sense that

$$x^* = x. \tag{38.12}$$

Of course, (38.7) is the same as (38.8) in this case. It is easy to see that x^j is self-adjoint for every positive integer j, so that

$$N(x^{2\,j}) = N(x^j)^2 \tag{38.13}$$

for every $j \in \mathbf{Z}_+$. This implies that

$$N(x^n) = N(x)^n \tag{38.14}$$

when $n = 2^l$ for some $l \in \mathbf{Z}_+$. Using this, we get that (19.3, 2) holds for every $n \in \mathbf{Z}_+$, as in Section 19.

If x is any element of $\mathcal{A}$, then it is easy to see that $x + x^*$ and $x^*\,x$ are self-adjoint with respect to (38.1). It follows that

$$N((x^*\,x)^n) = N(x^*\,x)^n = N(x)^{2\,n} \tag{38.15}$$

for every $x \in \mathcal{A}$ and $n \in \mathbf{Z}_+$, by applying (38.14) to $x^*\,x$.

Suppose now that $x \in \mathcal{A}$ is *normal* with respect to (38.1), in the sense that

$$x\,x^* = x^*\,x. \tag{38.16}$$

This implies that

$$(x^*\,x)^n = (x^*)^n\,x^n \tag{38.17}$$

for every $n \in \mathbf{Z}_+$. Combining this with (38.15), we get that

$$N(x)^{2\,n} = N((x^*\,x)^n) = N((x^*)^n\,x^n) \le N((x^*)^n)\,N(x^n) \tag{38.18}$$

for every $n \in \mathbf{Z}_+$. It is easy to see that $(x^*)^n = (x^n)^*$ for each $n \in \mathbf{Z}_+$, so that

$$N(x)^{2\,n} \le N((x^n)^*)\,N(x^n) = N(x^n)^2 \tag{38.19}$$

for every $n \ge 1$. It follows that (38.14) holds for every $n \in \mathbf{Z}_+$ in this case too, because $N(x^n) \le N(x)^n$ for each $n \ge 1$, by submultiplicativity.

39 Orthogonal vectors

Let $(V, \langle v, w\rangle)$ be a real or complex inner product space, and let $\|v\|$ be the corresponding norm on V. As usual, two elements v, w of V are said to be *orthogonal* in V if $\langle v, w\rangle = 0$. In this case,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2. \tag{39.1}$$

Let $\sum_{j=1}^{\infty} v_j$ be an infinite series of pairwise-orthogonal elements of V, so that v_j is orthogonal to v_l when $j \neq l$. If $n \geq l \geq 1$, then

$$\Big\|\sum_{j=l}^{n} v_j\Big\|^2 = \sum_{j=l}^{n} \|v_j\|^2, \tag{39.2}$$

as in (39.1). In particular, the partial sums

$$\sum_{j=1}^{n} v_j \tag{39.3}$$

are bounded in V if and only if the sums $\sum_{j=1}^{n} \|v_j\|^2$ have a finite upper bound. Of course, this happens if and only if

$$\sum_{j=1}^{\infty} \|v_j\|^2 \tag{39.4}$$

converges as an infinite series of nonnegative real numbers. If (39.4) converges, then it is easy to see that the sequence of partial sums (39.3) is a Cauchy sequence in V with respect to the metric associated to the norm, using (39.2). If V is a Hilbert space, then it follows that (39.2) converges in V. In this case, one can check that

$$\Big\|\sum_{j=1}^{\infty} v_j\Big\|^2 = \sum_{j=1}^{\infty} \|v_j\|^2, \tag{39.5}$$

using (39.2).

Now let $v_1, \ldots, v_n$ be finitely many orthonormal vectors in V, so that v_j is orthogonal to v_l when $j \neq l$, and $\|v_j\| = 1$ for each $j = 1, \ldots, n$. Also let $v \in V$ be given, and put

$$w = \sum_{j=1}^{n} \langle v, v_j\rangle\, v_j. \tag{39.6}$$

By construction,

$$\langle w, v_l\rangle = \langle v, v_l\rangle \tag{39.7}$$

for each $l = 1, \ldots, n$, which means that $v - w$ is orthogonal to v_l for each $l = 1, \ldots, n$. It follows that $v - w$ is orthogonal to w, so that

$$\|v\|^2 = \|v - w\|^2 + \|w\|^2, \tag{39.8}$$

as in (39.1). We also have that

$$\|w\|^2 = \sum_{j=1}^{n} |\langle v, v_j \rangle|^2, \tag{39.9}$$

using (39.1) again, so that

$$\|v\|^2 = \|v - w\|^2 + \sum_{j=1}^{n} |\langle v, v_j \rangle|^2. \tag{39.10}$$

Suppose that $u \in V$ is in the linear span of $v_1, \ldots, v_n$. If v, w are as in the previous paragraph, then $w - u$ is in the linear span of $v_1, \ldots, v_n$ as well, and hence $v - w$ is orthogonal to $w - u$. This implies that

$$\|v - u\|^2 = \|v - w\|^2 + \|w - u\|^2, \tag{39.11}$$

using (39.1) again. In particular,

$$\|v - u\| \geq \|v - w\| \tag{39.12}$$

under these conditions.

40 Lipschitz mappings

Let X, Y be sets, and let d_X, d_Y be q_X, q_Y-semimetrics on X, Y, respectively, for some $q_X, q_Y > 0$. Also let f be a mapping from X into Y, and let α be a positive real number. If there is a nonnegative real number C such that

$$d_Y(f(x), f(x')) \leq C \, d_X(x, x')^\alpha \tag{40.1}$$

for every $x, x' \in X$, then f is said to be *Lipschitz* of order α with constant C. Of course, constant mappings are Lipschitz of any order α, with constant $C = 0$. If f satisfies (40.1) with $C = 0$ for some α, and if d_Y is a q_Y-metric on Y, then f is a constant mapping.

Let a, b be positive real numbers, and remember that d_X^a, d_Y^b define (q_X/a), (q_Y/b)-semimetrics on X, Y, respectively, as in Section 2. Observe that (40.1) holds if and only if

$$d_Y(f(x), f(x'))^b \leq C^b \, (d_X(x, x')^a)^{\alpha \, b/a} \tag{40.2}$$

for every $x, x' \in X$. Thus f is Lipschitz of order α with constant C with respect to d_X, d_Y if and only if f is Lipschitz of order $\alpha \, b/a$ with constant C^b with respect to d_X^a, d_Y^b.

Note that Lipschitz mappings of any order are uniformly continuous. Let $UC(X, Y)$ be the space of uniformly continuous mappings from X into Y, which is a subset of the space $C(X, Y)$ of continuous mappings from X into Y. Also let $\mathrm{Lip}_\alpha(X, Y)$ be the space of Lipschitz mappings of order α from X into Y, which is contained in $UC(X, Y)$.

Suppose again that f is Lipschitz of order α with constant $C = C_f$. Let Z be another set, and let d_Z be a q_Z-semimetric on Z for some $q_Z > 0$. Suppose that g is a mapping from Y into Z that is Lipschitz of order $\beta > 0$ with constant $C_g \geq 0$ with respect to d_Y, d_Z. If $x, x' \in X$, then

$$(40.3) \quad d_Z(g(f(x)), g(f(x'))) \leq C_g \, d_Y(f(x), f(x'))^\beta \leq C_g \, C_f^\beta \, d_X(x, x')^{\alpha\,\beta}.$$

This means that the composition $g \circ f$ is Lipschitz of order $\alpha\,\beta$ with constant $C_g \, C_f^\beta$ as a mapping from X into Z.

Let q be a positive real number with $q \leq q_X$, and let $w \in X$ be given. Put

$$(40.4) \qquad f_{w,q}(x) = d_X(w, x)^q$$

for each $x \in X$, which defines a nonnegative real-valued function on X. The inequality (6.5) says exactly that $f_{w,q}$ is Lipschitz of order q with constant 1 with respect to d_X on X and the standard Euclidean metric on $\mathbf{R}$.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let q be a positive real number such that $q \leq q_k$. As before, $|x|^q$ defines a nonnegative real-valued function on k, which is the same as (40.4) with $X = k$, d_X equal to the q_k-metric associated to $|\cdot|$ on k, and $w = 0$. It follows that $|x|^q$ is Lipschitz of order q with constant 1 with respect to the q_k-metric associated to $|\cdot|$ on k and the standard Euclidean metric on $\mathbf{R}$.

Similarly, let V be a vector space over k, and let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k for some $q_V > 0$. Also let q be a positive real number with $q \leq q_V$, so that $N_V(v)^q$ defines a nonnegative real-valued function on V. This is the same as (40.4), with $X = V$, d_X equal to the q_V-semimetric associated to N_V on V, and $w = 0$. Hence $N_V(v)^q$ is Lipschitz of order q with respect to the q_V-semimetric associated to N_V on V and the standard Euclidean on $\mathbf{R}$.

41 Lipschitz q-seminorms

Let X be a nonempty set with a q_X-semimetric d_X for some $q_X > 0$, let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let W be a vector space over k with a q_W-seminorm N_W with respect to $|\cdot|$ on k for some $q_W > 0$. As in the previous section, $UC(X, W)$ denotes the space of uniformly continuous mappings from X into W, with respect to the q_W-semimetric associated to N_W on W. It is easy to see that $UC(X, W)$ is a linear subspace of the vector space $C(X, W)$ of continuous mappings from X into W. Similarly, for each positive real number α, $\mathrm{Lip}_\alpha(X, W)$ denotes the space of Lipschitz mappings from X into W of order α, with respect to the q_W-semimetric associated to N_W on W. One can check that $\mathrm{Lip}_\alpha(X, W)$ is a linear subspace of $UC(X, W)$ for each $\alpha > 0$.

Let a positive real number α and $f \in \mathrm{Lip}_\alpha(X, W)$ be given. If $x, y \in X$ satisfy $d_X(x, y) = 0$, then we have that

$$N_W(f(x) - f(y)) = 0, \tag{41.1}$$

which is trivial when d_X is a q_X-metric on X. If $d_X(x, y) > 0$ for some $x, y \in X$, then put

$$\begin{aligned} \|f\|_{\mathrm{Lip}_\alpha} &= \|f\|_{\mathrm{Lip}_\alpha(X,W)} \\ &= \sup\left\{\frac{N_W(f(x) - f(y))}{d(x, y)^\alpha} : x, y \in X,\ d_X(x, y) > 0\right\}. \end{aligned} \tag{41.2}$$

More precisely, the hypothesis on X implies that the supremum is taken over a nonempty set, and the Lipschitz condition on f implies that the nonnegative real numbers whose supremum is being taken have a finite upper bound, so that the supremum is defined as a nonnegative real number. If $d_X(x, y) = 0$ for every $x, y \in X$, then we take $\|f\|_{\mathrm{Lip}_\alpha}$ to be 0.

By construction, f is Lipschitz of order α with constant $C \geq 0$ if and only if

$$\|f\|_{\mathrm{Lip}_\alpha} \leq C. \tag{41.3}$$

In particular, f is Lipschitz of order α with constant $C = \|f\|_{\mathrm{Lip}_\alpha}$. Equivalently, $\|f\|_{\mathrm{Lip}_\alpha}$ is the same as the infimum of the set of nonnegative real numbers C such that f is Lipschitz of order α with constant C. One can check that $\|f\|_{\mathrm{Lip}_\alpha}$ defines a q_W-seminorm on $\mathrm{Lip}_\alpha(X, W)$ with respect to $|\cdot|$ on k. Of course, this uses the analogous properties of N_W on W.

Let V be another vector space over k, and let N_V be a q_V-seminorm on V with respect to $|\cdot|$ on k for some $q_V > 0$. If T is a bounded linear mapping from V into W with respect to N_V and N_W, then T is Lipschitz of order 1 with respect to the q_V, q_W-semimetrics on V, W associated to N_V, N_W, respectively, as in (14.2). Thus $\mathcal{BL}(V, W)$ may be considered as a linear subspace of $\mathrm{Lip}_1(V, W)$. If $T \in \mathcal{BL}(V, W)$, then the corresponding operator q_W-seminorm $\|T\|_{op,VW}$ of T is the same as the Lipschitz q_W-seminorm $\|T\|_{\mathrm{Lip}_1(V,W)}$.

42 Bounded Lipschitz functions

Let X be a nonempty set with a q_X-semimetric d_X for some $q_X > 0$, and let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$. It is easy to see that the space $UC_b(X, k)$ of bounded uniformly continuous k-valued functions on X is a linear subspace of $UC(X, k)$. If f, g are bounded uniformly continuous k-valued functions on X, then one can check that their product $f\,g$ is uniformly continuous as well. This uses the fact that

$$f(x)\,g(x) - f(y)\,g(y) = (f(x) - f(y))\,g(x) + f(y)\,(g(x) - g(y)) \tag{42.1}$$

for every $x, y \in X$. Of course, $f\,g$ is bounded on X too, so that $UC_b(X, k)$ is a subalgebra of the algebra $C_b(X, k)$ of bounded continuous k-valued functions on

X. One can also verify that $UC_b(X, k)$ is a closed set in $C_b(X, k)$ with respect to the supremum metric. This is because uniform limits of uniformly continuous functions are uniformly continuous, by standard arguments.

Let α be a positive real number, and let $\mathrm{Lip}_{\alpha,b}(X, k)$ be the space of bounded k-valued functions on X that are Lipschitz of order α. This is a linear subspace of $\mathrm{Lip}_\alpha(X, k)$, and one can check that it is a subalgebra of $UC_b(X, k)$, using (42.1). To make this more precise, it is helpful to consider the following two cases.

Suppose first that $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. Remember that $\|f\|_{\mathrm{Lip}_\alpha}$ defines a seminorm on $\mathrm{Lip}_\alpha(X, k)$, as in the previous section. If $f, g \in \mathrm{Lip}_{\alpha,b}(X, k)$, then it is easy to see that

$$\|f\,g\|_{\mathrm{Lip}_\alpha} \le \|f\|_{\mathrm{Lip}_\alpha}\,\|g\|_{sup} + \|f\|_{sup}\,\|g\|_{\mathrm{Lip}_\alpha}, \tag{42.2}$$

using (42.1). Here $\|f\|_{sup}$ is the supremum norm on $C_b(X, k)$, as in Section 13. Using (42.2), one can check that

$$\|f\|_{sup} + \|f\|_{\mathrm{Lip}_\alpha} \tag{42.3}$$

is a submultiplicative norm on $\mathrm{Lip}_{\alpha,b}(X, k)$. If $f \in \mathrm{Lip}_{\alpha,b}(X, k)$, then $f(x)^n$ is an element of $\mathrm{Lip}_{\alpha,b}(X, k)$ for each positive integer n, and one can verify that

$$\|f^n\|_{\mathrm{Lip}_\alpha} \le n\,\|f\|_{sup}^{n-1}\,\|f\|_{\mathrm{Lip}_\alpha}, \tag{42.4}$$

using (42.2). One can use this to check that

$$\lim_{n\to\infty} (\|f^n\|_{sup} + \|f^n\|_{\mathrm{Lip}_\alpha})^{1/n} = \|f\|_{sup}. \tag{42.5}$$

This also uses the fact that

$$\|f^n\|_{sup} = \|f\|_{sup}^n \tag{42.6}$$

for every $n \ge 1$.

Suppose now that k is any field with an ultrametric absolute value function $|\cdot|$, so that $\|f\|_{sup}$ is an ultranorm on $C_b(X, k)$, and $\|f\|_{\mathrm{Lip}_\alpha}$ defines a semi-ultranorm on $\mathrm{Lip}_\alpha(X, k)$. In this case, we get that

$$\|f\,g\|_{\mathrm{Lip}_\alpha} \le \max(\|f\|_{\mathrm{Lip}_\alpha}\,\|g\|_{sup}, \|f\|_{sup}\,\|g\|_{\mathrm{Lip}_\alpha}), \tag{42.7}$$

using (42.1) again. This implies that

$$\max(\|f\|_{sup}, \|f\|_{\mathrm{Lip}_\alpha}) \tag{42.8}$$

is a submultiplicative ultranorm on $\mathrm{Lip}_{\alpha,b}(X, k)$. If $f \in \mathrm{Lip}_{\alpha,b}$, then f^n is an element of $\mathrm{Lip}_{\alpha,b}(X, k)$ for each $n \in \mathbf{Z}_+$, and

$$\|f^n\|_{\mathrm{Lip}_\alpha} \le \|f\|_{sup}^{n-1}\,\|f\|_{\mathrm{Lip}_\alpha}. \tag{42.9}$$

As before, it follows that

$$\lim_{n\to\infty} (\max(\|f^n\|_{sup}, \|f^n\|_{\mathrm{Lip}_\alpha}))^{1/n} = \|f\|_{sup}. \tag{42.10}$$

43 C^1 Functions on R

Let I be an interval of positive length in the real line, which may be open, closed, or half-open and half-closed. We also allow I to be unbounded, so that I may be the real line, or an open or closed half-line. Let f be a real or complex-valued function on I. If x is an element of the interior of I, then the derivative $f'(x)$ of f at x can be defined in the usual way, when it exists. If x is an endpoint of I, if there is one, then one can take $f'(x)$ to be the appropriate one-sided derivative of f at x, when it exists. Let us say that f is differentiable on I when $f'(x)$ exists for every $x \in I$. Of course, this implies that f is continuous on I, by standard arguments. If f is differentiable on I, and if $f'(x)$ is continuous on I, then f is said to be continuously differentiable on I. Let $C^1(I, \mathbf{R})$ and $C^1(I, \mathbf{C})$ be the spaces of continuously differentiable real and complex-valued functions on I, respectively. If f and g are differentiable real or complex-valued functions on I, then it is well known that their product $f\,g$ is differentiable on I as well, with

$$(f\,g)' = f'\,g + f\,g'. \tag{43.1}$$

In particular, if f and g are continuously differentiable on I, then $f\,g$ is continuously differentiable as well. Thus $C^1(I, \mathbf{R})$ and $C^1(I, \mathbf{C})$ are subalgebras of $C(I, \mathbf{R})$ and $C(I, \mathbf{C})$, respectively.

Let f be a differentiable real or complex-valued function on I, such that f' is bounded on I. Under these conditions, f is Lipschitz of order 1 on I with respect to the standard metrics on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, and the restriction of the standard metric on $\mathbf{R}$ to I. More precisely, one can take the Lipschitz constant of f to be the supremum norm $\|f'\|_{sup}$ of f' on I. This follows from the mean value theorem when f is real-valued. If f is complex-valued, then one can reduce to the real case, by considering the real part of $a\,f(x)$ for $a \in \mathbf{C}$ with $|a| = 1$. If f is continuously differentiable, then the same conclusion can be obtained from the fundamental theorem of calculus, in both the real and complex cases. It is easy to see directly that $|f'(x)|$ is less than or equal to the corresponding Lipschitz seminorm $\|f\|_{\mathrm{Lip}_1}$ of f on I for each $x \in I$, so that

$$\|f\|_{\mathrm{Lip}_1} = \|f'\|_{sup}. \tag{43.2}$$

Let $C^1_b(I, \mathbf{R})$ and $C^1_b(I, \mathbf{C})$ be the spaces of continuously differentiable real and complex-valued functions f on I, respectively, such that f and f' are both bounded on I. It is easy to see that these are subalgebras of $C_b(I, \mathbf{R})$ and $C_b(I, \mathbf{C})$, respectively, using (43.1). More precisely, f and g are elements of $C^1_b(I, \mathbf{R})$ or $C^1_b(I, \mathbf{C})$, then

$$\|(f\,g)'\|_{sup} = \|f'\,g + f\,g'\|_{sup} \le \|f'\|_{sup}\,\|g\|_{sup} + \|f\|_{sup}\,\|g'\|_{sup}. \tag{43.3}$$

Using this, one can check that

$$\|f\|_{sup} + \|f'\|_{sup} \tag{43.4}$$

defines a submultiplicative norm on each of $C^1_b(I, \mathbf{R})$ and $C^1_b(I, \mathbf{C})$. Of course, these statements correspond to ones in the previous section, because of (43.2).

If f is a differentiable real or complex-valued function on I, then $f(x)^n$ is differentiable on I for every positive ineteger n, with

$$(f^n)' = n\, f^{n-1}\, f'. \tag{43.5}$$

Thus

$$\|(f^n)'\|_{sup} = n\, \|f^{n-1}\, f'\|_{sup} \le n\, \|f\|_{sup}^{n-1}\, \|f'\|_{sup}. \tag{43.6}$$

As before, one can use this to verify that

$$\lim_{n \to \infty} (\|f^n\|_{sup} + \|(f^n)'\|_{sup})^{1/n} = \|f\|_{sup}. \tag{43.7}$$

These statements correspond to ones in the previous section as well, by (43.2).

44 Formal power series

Let k be a field, and let T be an indeterminate. As in [4, 14], we use upper-case letters like T for indeterminates, and lower-case letters like t for elements of k. A *formal power series* in T with coefficients in k may be expressed as

$$f(T) = \sum_{j=0}^{\infty} f_j\, T^j, \tag{44.1}$$

where $f_j \in k$ for each nonnegative integer j. The space of formal power series in T with cefficients in k is denoted $k[[T]]$. More precisely, $k[[T]]$ can be defined as the space $c(\mathbf{Z}_+ \cup \{0\}, k)$ of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$, where $f(T)$ corresponds to f_j as a k-valued function of j. As usual, $c(\mathbf{Z}_+ \cup \{0\}, k)$ is a vector space over k with respect to pointwise addition and scalar multiplication, which corresponds to termwise addition and scalar multiplication of formal power series expressed as in (44.1). If $f(T) \in k[[T]]$ satisfies $f_j = 0$ for all but finitely many $j \ge 0$, then $f(T)$ is considered to be a *formal polynomial* in T with coefficients in k. The space of these formal polynomials is denoted $k[T]$, which may be defined more precisely as the space $c_{00}(\mathbf{Z}_+ \cup \{0\}, k)$ of k-valued functions on $\mathbf{Z}_+ \cup \{0\}$ with finite support.

Let $f(T)$ and $g(T) = \sum_{j=0}^{\infty} g_j(T)$ be elements of $k[[T]]$. Their product is defined to be the formal power series

$$f(T)\, g(T) = h(T) = \sum_{n=0}^{\infty} h_n\, T^n, \tag{44.2}$$

where

$$h_n = \sum_{j=0}^{n} f_j\, g_{n-j} \tag{44.3}$$

for each nonnegative integer n. Thus $h(T)$ corresponds to the Cauchy product of $f(T)$ and $g(T)$, as in Section 29. One can check that $k[[T]]$ is a commutative algebra over k with respect to this operation of multiplication. If $f(T)$, $g(T)$ are formal polynomials in T, then it is easy to see that their product is a formal polynomial too, so that $k[T]$ is a subalgebra of $k[[T]]$. We can identify elements of k with formal polynomials for which all but the first coefficient is equal to 0, so that k corresponds to a subalgebra of $k[T]$. In particular, the multiplicative identity element 1 in k corresponds to the multiplicative identity element in $k[[T]]$.

Let $a(T) \in k[[T]]$ be given. If $l \in \mathbf{Z}_+$, then the lth power $a(T)^l$ of $a(T)$ can be defined as an element of $k[[T]]$ as in the preceding paragraph. As usual, $a(T)^l$ is interpreted as being equal to 1 when $l = 0$. If j is a nonnegative integer, then the coefficient of T^j in $a(T)^l \, T^l$ is equal to 0 when $j < l$. This implies that the coefficient of T^j in

$$\sum_{l=0}^{n} a(T)^l \, T^l \tag{44.4}$$

does not depend on n when $n \ge j$. The infinite series

$$\sum_{l=0}^{\infty} a(T)^l \, T^l \tag{44.5}$$

can be defined as a formal power series in T, by taking the coefficient of T^j in (44.5) to be the same as the coefficient of (44.4) when $n \ge j$. If n is any nonnegative integer, then

$$(1 - a(T)\, T) \sum_{l=0}^{n} a(T)^l \, T^l = 1 - a(T)^{n+1} \, T^{l+1} \tag{44.6}$$

by a standard argument. Using this, it is easy to see that

$$(1 - a(T)\, T) \sum_{l=0}^{\infty} a(T)^l \, T^l = 1. \tag{44.7}$$

Thus (44.5) is the multiplicative inverse of $1 - a(T)\, T$ in $k[[T]]$.

If $f(T), g(T) \in k[[T]]$ satisfy $f(T)\, g(T) = 1$, then $f_0 \, g_0 = 1$, and hence $f_0, g_0 \ne 0$. In the other direction, let $f(T)$ be any element of $k[[T]]$ such that $f_0 \ne 0$. This permits us to express $f(T)$ as $f_0\,(1-a(T)\, T)$ for some $a(T) \in k[[T]]$. It follows that $f(T)$ has a multiplicative inverse in $k[[T]]$, because $1 - a(T)\, T$ has a multiplicative inverse in $k[[T]]$, as in the previous paragraph.

45 Formal Laurent series

Let k be a field, and let T be an indeterminate. A *formal Laurent series* in T with coefficients in k may be expressed as

$$f(T) = \sum_{j=-\infty}^{\infty} f_j \, T^j, \tag{45.1}$$

where $f_j \in k$ for each $j \in \mathbf{Z}$. Thus f_j defines a k-valued function on $\mathbf{Z}$, so that the space of formal Laurent series in T with coefficients in k can be defined precisely as $c(\mathbf{Z}, k)$. Remember that $c(\mathbf{Z}, k)$ is a vector space over k with respect to pointwise addition and scalar multiplication, which corresponds to termwise addition and scalar multiplication of formal Laurent series expressed as in (45.1). A formal power series in T with coefficients in k may be considered as a formal Laurent series $f(T)$ with $f_j = 0$ when $j < 0$, which corresponds to identifying $c(\mathbf{Z}_+ \cup \{0\}, k)$ with the subspace of $c(\mathbf{Z}, k)$ consisting of functions on $\mathbf{Z}$ that vanish on negative integers.

Let $k((T))$ be the space of formal Laurent series $f(T)$ in T with coefficients in k such that $f_j = 0$ for all but finitely many $j < 0$. In this case, $f(T)$ may be expressed as

$$f(T) = \sum_{j>>-\infty} f_j \, T^j \tag{45.2}$$

to indicate that $f_j = 0$ for all but finitely many $j < 0$, as in [4]. More precisely, $k((T))$ can be defined as the linear subspace of $c(\mathbf{Z}, k)$ consisting of functions on $\mathbf{Z}$ that are equal to 0 at all but finitely many negative integers. As before, $k[[T]]$ may be considered as a linear subspace of $k((T))$.

If $f(T)$ and $g(T) = \sum_{j>>-\infty} g_j \, T^j$ are elements of $k((T))$, then their product is defined by

$$f(T) \, g(T) = h(T) = \sum_{n=-\infty}^{\infty} h_n \, T^n, \tag{45.3}$$

where

$$h_n = \sum_{j=-\infty}^{\infty} f_j \, g_{n-j} \tag{45.4}$$

for each $n \in \mathbf{Z}$. More precisely, all but finitely many terms in the sum on the right side of (45.4) are equal to 0, because $f_j = g_j = 0$ for all but finitely many $j < 0$. Thus h_n is defined as an element of k for every $n \in \mathbf{Z}$, and one can check that $h_n = 0$ for all but finitely many $n < 0$, so that $h(T) \in k((T))$ too. One can also verify that $k((T))$ is a commutative algebra over k with respect to this operation of multiplication. This is compatible with the definition of multiplication of formal power series in the previous section, so that $k[[T]]$ is a subalgebra of $k((T))$.

Every nonzero element of $k((T))$ can be expressed as $T^l \, f(T)$ for some $f(T)$ in $k[[T]]$ and $l \in \mathbf{Z}$, where $f_0 \neq 0$. As in the previous section, $f(T)$ has a multiplicative inverse in $k[[T]]$ under these conditions. This implies that $T^l \, f(T)$ has a multiplicative inverse in $k((T))$, which is given by $T^{-l} \, f(T)^{-1}$. It follows that $k((T))$ is a field with respect to this definition of multiplication.

Let r be a positive real number with $r \leq 1$. If $f(T) \in k((T))$ and $f(T) \neq 0$, then there is a unique integer $j_0 = j_0(f)$ such that $f_{j_0} \neq 0$ and $f_j = 0$ when $j < j_0$. In this case, we put

$$|f(T)|_r = r^{j_0}, \tag{45.5}$$

and we put $|f(T)|_r = 0$ when $f(T) = 0$. One can check that

$$|f(T) + g(T)|_r \le \max(|f(T)|_r, |g(T)|_r) \tag{45.6}$$

for every $f(T), g(T) \in k((T))$. This uses the fact that

$$j_0(f(T) + g(T)) \ge \min(j_0(f(T)), j_0(g(T)) \tag{45.7}$$

when $f(T)$, $g(T)$, and $f(T) + g(T)$ are nonzero. Similarly,

$$|f(T)\, g(T)|_r = |f(T)|_r\, |g(T)|_r \tag{45.8}$$

for every $f(T), g(T) \in k((T))$, because

$$j_0(f(T)\, g(T)) = j_0(f(T)) + j_0(g(T)) \tag{45.9}$$

when $f(T), g(T) \ne 0$. Thus $|\cdot|_r$ defines an ultrametric absolute value function on $k((T))$.

If $r = 1$, then $|\cdot|_r$ is the trivial absolute value function on $k((T))$. If $0 < r \le 1$ and a is a positive real number, then

$$|f(T)|_r^a = |f(T)|_{r^a} \tag{45.10}$$

for every $f(T) \in k((T))$. This means that the absolute value functions $|\cdot|_r$ are all equivalent on $k((T))$ when $0 < r < 1$, as in Section 4.

Suppose that $0 < r < 1$, and let $l \in \mathbf{Z}$ be given. The closed ball $\overline{B}(0, r^l)$ in $k((T))$ centered at 0 with radius r^l with respect to the ultrametric associated to $|\cdot|_r$ consists of $f \in k((T))$ such that $f_j = 0$ when $j < l$. Thus $\overline{B}(0, r^l)$ can be identified with the Cartesian product of copies of k indexed by $j \in \mathbf{Z}$ with $j \ge l$. The discrete topology on k leads to a product topology on this Cartesian product. One can check that the topology determined on $\overline{B}(0, r^l)$ by the ultrametric associated to $|\cdot|_r$ corresponds exactly to the product topology just mentioned.

One can also verify that $k((T))$ is complete with respect to the ultrametric associated to $|\cdot|_r$ for each $0 < r \le 1$. Of course, this is trivial when $r = 1$. If $r < 1$, then it is helpful to observe that a Cauchy sequence in $k((T))$ is contained in $\overline{B}(0, r^l)$ for some $l \in \mathbf{Z}$. This uses the fact that a Cauchy sequence in any q-metric space is bounded.

46 Discrete absolute value functions

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. Observe that

$$\{|x| : x \in k,\ x \ne 0\} \tag{46.1}$$

is a subgroup of the multiplicative group $\mathbf{R}_+$ of positive real numbers. If the real number 1 is not a limit point of (46.1) with respect to the standard topology on $\mathbf{R}$, then $|\cdot|$ is said to be *discrete* on k. If 1 is a limit point of (46.1) in $\mathbf{R}$, then one can check that (46.1) is dense in $\mathbf{R}_+$ with respect to the topology induced on $\mathbf{R}_+$ by the standard topology on $\mathbf{R}$. Note that the real number 0 is a limit point of (46.1) unless $|\cdot|$ is the trivial absolute value function on k, in which case (46.1) is the trivial subgroup of $\mathbf{R}_+$.

Put

$$\rho_1 = \sup\{|x| : x \in k,\ |x| < 1\}, \tag{46.2}$$

so that $0 \le \rho_1 \le 1$. It is easy to see that $\rho_1 = 0$ if and only if $|\cdot|$ is the trivial absolute value function on k. Similarly, $\rho_1 < 1$ if and only if $|\cdot|$ is discrete on k. Suppose for the moment that $|\cdot|$ is discrete on k, and not the trivial absolute value function on k, so that $0 < \rho_1 < 1$. Under these conditions, one can verify that the supremum in (46.2) is attained, so that there is an $x \in k$ such that $|x| = \rho_1$. One can also check that (46.1) consists exactly of the integer powers of ρ_1. More precisely, if this were not the case, then there would be a $y \in k$ such that $\rho_1 < |y| < 1$, contradicting the definition of ρ_1.

Suppose now that $|\cdot|$ is an archimedean q-absolute value function on k, as in Section 10. This implies that k has characteristic 0, so that there is a natural embedding of $\mathbf{Q}$ into k. In this case, $|\cdot|$ induces an archimedean q-absolute value function on $\mathbf{Q}$. The theorem of Ostrowski mentioned in Section 4 implies that the induced q-absolute value function on $\mathbf{Q}$ is equivalent to the standard (Euclidean) absolute value function on $\mathbf{Q}$ in this situation. It is easy to see that any q-absolute value function on $\mathbf{Q}$ that is equivalent to the standard absolute value function is not discrete, because the standard absolute value function on $\mathbf{Q}$ is not discrete. It follows that $|\cdot|$ is not discrete on k. This shows that if $|\cdot|$ is a discrete q-absolute value function on a field k, then $|\cdot|$ is non-archimedian on k. Of course, this means that $|\cdot|$ is an ultrametric absolute value function on k, as in Section 10.

Let $|\cdot|$ be any ultrametric absolute value function on a field k. Suppose that there are finitely many elements $x_1, \ldots, x_n$ of k and positive real numbers $r_1, \ldots r_n$ such that $r_j < 1$ for each $j = 1, \ldots, n$ and

$$B(0,1) \subseteq \bigcup_{j=1}^{n} \overline{B}(x_j, r_j). \tag{46.3}$$

The open an closed balls in k are defined as in Section 2, as usual, with respect to the ultrametric associated to $|\cdot|$. We may as well suppose that

$$B(0,1) \cap \overline{B}(x_j, r_j) \ne \emptyset \tag{46.4}$$

for each $j = 1, \ldots, n$, since otherwise $\overline{B}(x_j, r_j)$ is not needed to cover $B(0,1)$. This implies that $|x_j| < 1$ for every $j = 1, \ldots, n$, by the ultrametric version of the triangle inequality. Thus

$$r = \max(|x_1|, \ldots, |x_n|, r_1, \ldots, r_n) < 1, \tag{46.5}$$

and

$$\overline{B}(x_j, r_j) \subseteq \overline{B}(0, r) \tag{46.6}$$

for each $j = 1, \ldots, n$, using the ultrametric version of the triangle inequality again. Combining this with (46.3), we get that

$$B(0,1) \subseteq \overline{B}(0,r), \tag{46.7}$$

which means that $|\cdot|$ is discrete on k.

47 Weighted ℓ^r spaces

Let X be a nonempty set, and let $w(x)$ be a positive real-valued function on X. Also let k be a field, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. If r is a positive real number, then we let $\ell^r_w(X,k)$ be the space of k-valued functions f on X such that $|f(x)|\,w(x)$ is r-summable on X as a nonnegative real-valued function on X. In this case, we put

$$\|f\|_{\ell^r_w(X,k)} = \Big(\sum_{x\in X} |f(x)|^r\, w(x)^r\Big)^{1/r}. \tag{47.1}$$

Similarly, let $\ell^\infty_w(X,k)$ be the space of k-valued functions f on X such that $|f(x)|\,w(x)$ is bounded as a nonnegative real-valued function on X, in which case we put

$$\|f\|_{\ell^\infty_w(X,k)} = \sup_{x\in X}(|f(x)|\,w(x)). \tag{47.2}$$

Equivalently, if $f \in \ell^r_w(X,k)$ for some $0 < r \le \infty$, then

$$\|f\|_{\ell^r_w(X,k)} = \||f|\,w\|_r, \tag{47.3}$$

where $\|\cdot\|_r$ is defined for nonnegative real-valued functions on X as in Section 21. One can check that $\ell^r_w(X,k)$ is a vector space over k with respect to pointwise addition and scalar multiplication for each $0 < r \le \infty$, using the same type of arguments as in the unweighted case. Similarly, $\|f\|_{\ell^r_w(X,k)}$ defines an r-norm on $\ell^r_w(X,k)$ when $r \le q_k$, and $\|f\|_{\ell^r_w(X,k)}$ defines a q_k-norm on $\ell^r_w(X,k)$ when $r \ge q_k$, as before. Of course, if $w(x) = 1$ for every $x \in X$, then $\ell^r_w(X,k)$ is the same as the space $\ell^r(X,k)$ defined previously for every $r > 0$, and $\|f\|_{\ell^r_w(X,k)}$ is the same as $\|f\|_{\ell^r(X,k)}$. If V is a vector space over k with a q-seminorm N with respect to $|\cdot|$ on k for some $q > 0$, then one can deal with V-valued functions on X in the same way, as before.

Let $c_{0,w}(X,k)$ be the space of k-valued functions such that $|f(x)|\,w(x)$ vanishes at infinity on X, as a real-valued function on X. In particular, this implies that $|f(x)|\,w(x)$ is bounded on X, so that

$$c_{0,w}(X,k) \subseteq \ell^\infty_w(X,k). \tag{47.4}$$

As before, $c_{0,w}(X,k)$ is a linear subspace of $\ell^\infty_w(X,k)$, and a closed set in $\ell^\infty_w(X,k)$ with respect to the q_k-metric associated to the $\ell^\infty_w(X,k)$ q_k-norm. We also have that

$$c_{00}(X,k) \subseteq c_{0,w}(X,k), \tag{47.5}$$

and that $c_{0,w}(X,k)$ is the closure of $c_{00}(X,k)$ in $\ell^\infty_w(X,k)$.

If $0 < r_1 \le r_2 \le \infty$, then

$$\ell^{r_1}_w(X,k) \subseteq \ell^{r_2}_w(X,k), \tag{47.6}$$

and

$$\|f\|_{\ell^{r_2}_w(X,k)} \le \|f\|_{\ell^{r_1}_w(X,k)} \tag{47.7}$$

for every $f \in \ell^{r_1}_w(X,k)$, as in (21.8). If $0 < r < \infty$, then it is easy to see that

$$\ell^r_w(X,k) \subseteq c_{0,w}(X,k). \tag{47.8}$$

Of course,

$$c_{00}(X,k) \subseteq \ell^r_w(X,k) \tag{47.9}$$

for every $r > 0$. If $0 < r < \infty$, then one can check that $c_{00}(X,k)$ is dense in $\ell^r_w(X,k)$, with respect to the q_k or r-metric associated to $\|f\|_{\ell^r_w(X,k)}$, as appropriate. If k is complete with respect to the q_k-metric associated to $|\cdot|$, then one can verify that $\ell^r_w(X,k)$ is complete with respect to the q_k or r-metric associated to $\|f\|_{\ell^r_w(X,k)}$, by standard arguments.

Let $a(x)$, $f(x)$ be k-valued functions on X, and put

$$(M_a(f))(x) = a(x)\, f(x) \tag{47.10}$$

for every $x \in X$. This defines $M_a(f)$ as a k-valued function on X, and M_a defines a linear mapping from the space $c(X,k)$ of k-valued functions on X into itself. This is the multiplication operator on $c(X,k)$ associated to a. If $a(x) \ne 0$ for every $x \in X$, then $1/a$ is also a k-valued function on X, so that $M_{1/a}$ defines a linear mapping from $c(X,k)$ into itself too. In this case, M_a is a one-to-one linear mapping from $c(X,k)$ onto itself, with inverse equal to $M_{1/a}$.

Let $w_1(x)$, $w_2(x)$ be positive real-valued functions on X, so that w_1/w_2 and w_2/w_1 are positive real-valued functions on X as well. If $a \in \ell^\infty_{w_1/w_2}(X,k)$, then

$$|a(x)|\,(w_1(x)/w_2(x)) \le \|a\|_{\ell^\infty_{w_1/w_2}(X,k)} \tag{47.11}$$

for each $x \in X$, which means that

$$|a(x)|\, w_1(x) \le \|a\|_{\ell^\infty_{w_1/w_2}(X,k)}\, w_2(x) \tag{47.12}$$

for every $x \in X$. Let $f \in c(X,k)$ be given, and observe that

$$\begin{aligned} |(M_a(f))(x)|\, w_1(x) &= |a(x)|\,|f(x)|\, w_1(x) \\ &\le \|a\|_{\ell^\infty_{w_1/w_2}(X,k)}\, |f(x)|\, w_2(x) \end{aligned} \tag{47.13}$$

for every $x \in X$. This implies that M_a defines a bounded linear mapping from $\ell^r_{w_2}(X,k)$ into $\ell^r_{w_1}(X,k)$ for every $r > 0$, and one can check that the corresponding operator q_k or r-norm of M_a is equal to $\|a\|_{\ell^\infty_{w_1/w_2}(X,k)}$. Similarly, M_a maps $c_{0,w_2}(X,k)$ into $c_{0,w_1}(X,k)$ under these conditions.

Let $w(x)$ be a positive real-valued function on X again, and let a be an element of $\ell^\infty_{1/w}(X, k)$. Thus M_a defines a bounded linear mapping from $\ell^r_w(X, k)$ into $\ell^r(X, k)$ for every $r > 0$, as in the preceding paragraph. Similarly, if $a(x) \neq 0$ for every $x \in X$, and $1/a \in \ell^\infty_w(X, k)$, then $M_{1/a}$ defines a bounded linear mapping from $\ell^r(X, k)$ into $\ell^r_w(X, k)$ for every $r > 0$. Suppose now that $a \in \ell^\infty_{1/w}(X, k)$, $a(x) \neq 0$ for every $x \in X$, and $1/a \in \ell^\infty_w(X, k)$, and let $r > 0$ be given. Under these conditions, M_a is a one-to-one bounded linear mapping from $\ell^r_w(X, k)$ onto $\ell^r(X, k)$, with bounded inverse equal to $M_{1/a}$.

In particular, if $|a(x)| = w(x)$ for every $x \in X$, then $a \in \ell^\infty_{1/w}(X, k)$ with $\|a\|_{\ell^\infty_{1/w}(X,k)} = 1$, and $1/a \in \ell^\infty_w(X, k)$ with $\|1/a\|_{\ell^\infty_w(X,k)} = 1$. In this case, M_a defines an isometric linear mapping from $\ell^r_w(X, k)$ onto $\ell^r(X, k)$ for every $r > 0$. Of course, if $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function, then one can simply take $a(x) = w(x)$ for each $x \in X$. Suppose now that k is any field with a nontrivial q_k-absolute value function $|\cdot|$. In this situation, every positive real number is within a bounded factor of a positive value of $|\cdot|$ on k. This implies that there is k-valued functions $a(x)$ on X such that $|a(x)|$ and $w(x)$ are each bounded by constant multiples of the other. This is the same as saying that $a \in \ell^\infty_{1/w}(x, k)$, $a(x) \neq 0$ for every $x \in X$, and $1/a \in \ell^\infty_w(X, k)$.

Part IV
Fourier series

48 The unit circle

Let $\mathbf{T}$ be the unit circle in the complex plane, so that

$$\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}, \tag{48.1}$$

where $|\cdot|$ is the standard absolute value function on $\mathbf{C}$. It is well known that

$$\int_{\mathbf{T}} z^j \, |dz| = 0 \tag{48.2}$$

for every nonzero integer j, where $|dz|$ indicates the element of arclength on $\mathbf{T}$. We can also define arclength measure on $\mathbf{T}$, using an arclength parameterization of $\mathbf{T}$ to reduce to ordinary Lebesgue measure on the interval $[0, 2\pi)$ in the real line. Of course, the arclength of $\mathbf{T}$ is equal to 2π, and arclength measure on $\mathbf{T}$ is invariant under rotations and reflections. Let $L^r(\mathbf{T})$ be the corresponding Lebesgue space of complex-valued functions on $\mathbf{T}$ for each $r > 0$. If $0 < r < \infty$, then we put

$$\|f\|_r = \|f\|_{L^r(\mathbf{T})} = \Big(\frac{1}{2\pi} \int_{\mathbf{T}} |f(z)|^r \, |dz|\Big)^{1/r} \tag{48.3}$$

for each $f \in L^r(\mathbf{T})$, where now $|dz|$ refers to arclength measure on $\mathbf{T}$. This defines a norm on $L^r(\mathbf{T})$ when $1 \leq r < \infty$, by the integral version of Minkowski's inequality. If $0 < r \le 1$, then $\|f\|_r$ defines an r-norm on $L^r(\mathbf{T})$, by (1.11). The essential supremum norm $\|f\|_\infty = \|f\|_{L^\infty(\mathbf{T})}$ can be defined on $L^\infty(\mathbf{T})$ can also be defined in the usual way.

Suppose that $0 < r_1, r_2, r_3 \le \infty$ satisfy

$$1/r_3 = 1/r_1 + 1/r_2, \tag{48.4}$$

and let $f \in L^{r_1}(\mathbf{T})$, $g \in L^{r_2}(\mathbf{T})$ be given. Hölder's inequality implies that $f\,g \in L^{r_3}(\mathbf{T})$, with

$$\|f\,g\|_{r_3} \le \|f\|_{r_1}\,\|g\|_{r_2}. \tag{48.5}$$

As before, this is often stated in the case where $r_3 = 1$, and one can reduce to this case when $r_3 < \infty$. It is easy to verify (48.5) directly when $r_1 = \infty$ or $r_2 = \infty$, and the case where $r_1 = r_2 = 2$ is another version of the Cauchy–Schwarz inequality.

If $f, g \in L^2(\mathbf{T})$, then $f\,g \in L^1(\mathbf{T})$, as before, and we put

$$\langle f, g\rangle = \langle f, g\rangle_{L^2(\mathbf{T})} = \frac{1}{2\pi}\int_{\mathbf{T}} f(z)\,\overline{g(z)}\,|dz|. \tag{48.6}$$

This defines an inner product on $L^2(\mathbf{T})$, with

$$\langle f, f\rangle = \frac{1}{2\pi}\int_{\mathbf{T}} |f(z)|^2\,|dz| = \|f\|_2^2 \tag{48.7}$$

for each $f \in L^2(\mathbf{T})$. Of course, $L^2(\mathbf{T})$ is complete with respect to the metric associated to the L^2 metric, so that $L^2(\mathbf{T})$ is a Hilbert space. It is easy to see that the functions on $\mathbf{T}$ of the form z^j with $j \in \mathbf{Z}$ are orthogonal with respect to this inner product, because of (48.2). More precisely, these functions are orthonormal with respect to this inner product.

If $0 < r_1 \le r_2 \le \infty$, then it is well known that

$$L^{r_2}(\mathbf{T}) \subseteq L^{r_1}(\mathbf{T}), \tag{48.8}$$

with

$$\|f\|_{r_1} \le \|f\|_{r_2} \tag{48.9}$$

for every $f \in L^{r_2}(\mathbf{T})$. This can be verified directly when $r_2 = \infty$. If $r_2 < \infty$, then (48.9) can be obtained from Jensen's inequality, using the convexity of the function t^r on $[0, \infty)$ when $1 \le r < \infty$. One can also get (48.9) using Hölder's inequality.

Let $C(\mathbf{T}) = C(\mathbf{T}, \mathbf{C})$ be the space of continuous complex-valued functions on $\mathbf{T}$, using the standard metric on $\mathbf{C}$ and its restriction to $\mathbf{T}$. Remember that elements of $C(\mathbf{T})$ are automatically bounded and uniformly continuous on $\mathbf{T}$, because $\mathbf{T}$ is compact. It is well known that $C(\mathbf{T})$ is dense in $L^r(\mathbf{T})$ when $0 < r < \infty$.

49 Fourier coefficients

Let $f \in L^1(\mathbf{T})$ be given. If $j \in \mathbf{Z}$, then the jth *Fourier coefficient* of f is defined by

$$\widehat{f}(j) = \frac{1}{2\pi} \int_{\mathbf{T}} f(z)\, z^{-j}\, |dz|. \tag{49.1}$$

Observe that

$$|\widehat{f}(j)| \le \|f\|_1 \tag{49.2}$$

for every $j \in \mathbf{Z}$. Of course, $\widehat{f}(j)$ is linear in f, so that $f \mapsto \widehat{f}$ defines a bounded linear mapping from $L^1(\mathbf{T})$ into $\ell^\infty(\mathbf{Z}, \mathbf{C})$.

Suppose now that $f \in L^2(\mathbf{T})$, and observe that $\widehat{f}(j)$ can be interpreted as the inner product of f with z^j with respect to (48.6). Put

$$f_n(z) = \sum_{j=-n}^{n} \widehat{f}(j)\, z^j \tag{49.3}$$

for each nonnegative integer n and $z \in \mathbf{T}$. Because of the orthonormality of the z^j's with respect to (48.6), we have that

$$\|f\|_2^2 = \|f - f_n\|_2^2 + \sum_{j=-n}^{n} |\widehat{f}(j)|^2 \tag{49.4}$$

for every $n \ge 0$, as in (39.8). In particular,

$$\sum_{j=-n}^{n} |\widehat{f}(j)|^2 \le \|f\|_2^2 \tag{49.5}$$

for each $n \ge 0$. This implies that

$$\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2 \le \|f\|_2^2 \tag{49.6}$$

where the sum on the left may be considered as a sum of two infinite series, or as a sum over $j \in \mathbf{Z}$, as in Section 21.

Thus $\widehat{f} \in \ell^2(\mathbf{Z}, \mathbf{C})$ when $f \in L^2(\mathbf{T})$. It follows that $\widehat{f} \in c_0(\mathbf{Z}, \mathbf{C})$ when $f \in L^2(\mathbf{T})$, by (23.5). Equivalently, this means that

$$\lim_{|j| \to \infty} \widehat{f}(j) = 0 \tag{49.7}$$

when $f \in L^2(\mathbf{T})$. Remember that $L^2(\mathbf{T})$ is dense in $L^1(\mathbf{T})$ with respect to the metric associated to the L^1 norm. Using this and (49.2), one can show that (49.7) holds for every $f \in L^1(\mathbf{T})$.

Let $f \in L^2(\mathbf{T})$ and a nonnegative integer n be given again, and let $f_n(z)$ be as in (49.3). Also let $a_{-n}, \ldots, a_n \in \mathbf{C}$ be given, and put

$$g_n(z) = \sum_{j=-n}^{n} a_j\, z^j \tag{49.8}$$

for every $z \in \mathbf{T}$. Under these conditions, we have that

$$\|f - f_n\|_2 \le \|f - g_n\|_2, \tag{49.9}$$

as in (39.12).

It is well known that the linear span of the z^j's with $j \in \mathbf{Z}$ is dense in $C(\mathbf{T})$ with respect to the supremum metric. This can be obtained from the theorem of Lebesgue, Stone, and Weierstrass, and another argument will be mentioned later. It follows that the linear span of the z^j's with $j \in \mathbf{Z}$ is dense in $L^2(\mathbf{T})$, because $C(\mathbf{T})$ is dense in $L^2(\mathbf{T})$. If $f \in L^2(\mathbf{T})$ and $f_n(z)$ is as in (49.3), then one can check that

$$\lim_{n \to \infty} \|f - f_n\|_2 = 0, \tag{49.10}$$

using the previous statement and (49.9). Combining this with (49.4), we get that

$$\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2 = \|f\|_2^2 \tag{49.11}$$

for every $f \in L^2(\mathbf{T})$.

50 Convolutions on T

Let f, g be nonnegative measurable functions on $\mathbf{T}$. If $z \in \mathbf{T}$, then the *convolutions* of f and g at z is defined as a nonnegative extended real number by

$$(f * g)(z) = \frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, g(z\, w^{-1})\, |dw|. \tag{50.1}$$

As in the theorems of Fubini and Tonelli, $f * g$ is measurable on $\mathbf{T}$, and

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbf{T}} (f * g)(z)\, |dz| &= \frac{1}{2\pi} \int_{\mathbf{T}} \Big(\frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, g(z\, w^{-1})\, |dw| \Big)\, |dz| \\ &= \frac{1}{2\pi} \int_{\mathbf{T}} \Big(\frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, g(z\, w^{-1})\, |dz| \Big)\, |dw| \\ &= \Big(\frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, |dw| \Big) \Big(\frac{1}{2\pi} \int_{\mathbf{T}} g(z)\, |dz| \Big). \end{aligned} \tag{50.2}$$

Here we use the standard convention in integration theory of interpreting 0 times $+\infty$ as being 0. If f and g are integrable on $\mathbf{T}$ with respect to arclength measure, then it follows that $f * g$ is integrable on $\mathbf{T}$ with respect to arclength measure as well, and in particular $(f * g)(z) < \infty$ for almost every $z \in \mathbf{T}$ with respect to arclength measure.

Now let f, g be complex-valued measurable functions on $\mathbf{T}$. Suppose that $z \in \mathbf{T}$ has the property that

$$(|f| * |g|)(z) = \frac{1}{2\pi} \int_{\mathbf{T}} |f(w)|\, |g(z\, w^{-1})|\, |dw| < \infty. \tag{50.3}$$

This implies that the integral on the right side of (50.1) is defined as the integrable function on $\mathbf{T}$, so that $(f * g)(z)$ is defined as a complex number. We also have that

$$|(f * g)(z)| \le \frac{1}{2\pi} \int_{\mathbf{T}} |f(w)|\,|g(z\,w^{-1})|\,|dw| = (|f| * |g|)(z). \tag{50.4}$$

If $f, g \in L^1(\mathbf{T})$, then (50.3) holds for almost every $z \in \mathbf{T}$ with respect to arclength measure, as in the previous paragraph. Thus $(f * g)(z)$ is defined for almost every $z \in \mathbf{T}$, and in fact $f * g \in L^1(\mathbf{T})$, with

$$\|f * g\|_1 \le \|f\|_1\,\|g\|_1. \tag{50.5}$$

This follows from (50.4), using (50.2) applied to $|f|$ and $|g|$. One can check that $L^1(\mathbf{T})$ is a commutative algebra with respect to convolution as multiplication.

Let $f, g \in L^1(\mathbf{T})$ and $j \in \mathbf{Z}$ be given. The jth Fourier coefficient of $f * g$ is given by

$$\begin{aligned} \widehat{(f * g)}(j) &= \frac{1}{2\pi} \int_{\mathbf{T}} (f * g)(z)\, z^{-j}\,|dz| \\ &= \frac{1}{2\pi} \int_{\mathbf{T}} \Big(\frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, g(z\,w^{-1})\,|dw|\Big)\, z^{-j}\,|dz| \end{aligned} \tag{50.6}$$

Using Fubini's theorem, we get that

$$\begin{aligned} \widehat{(f * g)}(j) &= \frac{1}{2\pi} \int_{\mathbf{T}} \Big(\frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, g(z\,w^{-1})\,(z\,w^{-1})^{-j}\,|dz|\Big)\, w^{-j}\,|dw| \\ &= \widehat{f}(j)\,\widehat{g}(j). \end{aligned} \tag{50.7}$$

This also uses the integrability of $|f(w)|\,|g(z\,w^{-1})|$ on $\mathbf{T} \times \mathbf{T}$ with respect to the product measure corresponding to arclength measure on each of the factors, as in (50.2) applied to $|f|$ and $|g|$.

If $f \in L^1(\mathbf{T})$ and $g \in C(\mathbf{T})$, then $(f * g)(z)$ is defined for every $z \in \mathbf{T}$. One can check that $f * g$ is uniformly continuous on $\mathbf{T}$, using the uniform continuity of g on $\mathbf{T}$. Similarly, if $f \in C(\mathbf{T})$ and $g \in L^1(\mathbf{T})$, then $f * g$ is continuous on $\mathbf{T}$, because convolution is commutative.

Let $1 \le r, r' \le \infty$ be conjugate exponents, so that

$$1/r + 1/r' = 1. \tag{50.8}$$

If $f \in L^r(\mathbf{T})$ and $g \in L^{r'}(\mathbf{T})$, then $(f * g)(z)$ is defined for every $z \in \mathbf{T}$, and

$$|(f * g)(z)| \le \|f\|_r\,\|g\|_{r'}, \tag{50.9}$$

by Hölder's inequality. One can also verify that $f * g$ is continuous on $\mathbf{T}$ in this case, by approximating f by continuous functions with respect to the L^r norm when $r < \infty$, or approximating g by continuous functions when $r' < \infty$.

Let $f \in L^1(\mathbf{T})$ and $g \in L^r(\mathbf{T})$ be given, where $1 \le r < \infty$. Suppose for the moment that $\|f\|_1 = 1$. If $z \in \mathbf{T}$, then

$$\begin{aligned} ((|f| * |g|)(z))^r &= \Big(\frac{1}{2\pi} \int_{\mathbf{T}} |f(w)|\, |g(z\, w^{-1})|\, |dw|\Big)^r \\ &\le \frac{1}{2\pi} \int_{\mathbf{T}} |f(w)|\, |g(z\, w^{-1})|^r\, |dw| = (|f| * |g|^r)(z), \end{aligned} \tag{50.10}$$

using Jensen's inequality or Hölder's inequality in the second step. Integrating over z, we get that

$$\|f * g\|_r^r \le \||f| * (|g|^r)\|_1 \le \|f\|_1\, \||g|^r\|_1 = \|g\|_r^r. \tag{50.11}$$

If we drop the condition that $\|f\|_1 = 1$, then we have that

$$\|f * g\|_r \le \|f\|_1\, \|g\|_r. \tag{50.12}$$

More precisely, this reduces to (50.11) when $\|f\|_1 = 1$. Otherwise, one can reduce to the case where $\|f\|_1 = 1$, by considering $f/\|f\|_1$ when f is not equal to 0 almost everywhere on $\mathbf{T}$. Note that (50.12) also holds when $r = \infty$, as in (50.9). In particular, we get that $f * g \in L^r(\mathbf{T})$ when $f \in L^1(\mathbf{T})$ and $g \in L^r(\mathbf{T})$.

51 The Poisson kernel

Let $f \in L^1(\mathbf{T})$ be given, and let

$$U = \{z \in \mathbf{C} : |z| < 1\} \tag{51.1}$$

be the open unit disk in the complex plane. If $z \in U$, then put

$$h_+(z) = \sum_{j=0}^{\infty} \widehat{f}(j)\, z^j \tag{51.2}$$

and

$$h_-(z) = \sum_{j=1}^{\infty} \widehat{f}(-j)\, \overline{z}^j. \tag{51.3}$$

These series converge absolutely when $|z| < 1$, because of the boundedness of the Fourier coefficients $\widehat{f}(j)$, as in (49.2). Of course, $h_+(z)$ is a holomorphic function on U, and that $h_-(z)$ is conjugate-holomorphic function on U. This implies that

$$h(z) = h_+(z) + h_-(z) \tag{51.4}$$

is a harmonic function on U.

If $z \in U$ and $w \in \mathbf{T}$, then put

$$p_+(z, w) = \sum_{j=0}^{\infty} z^j\, w^{-j} \tag{51.5}$$

and

$$p_-(z,w) = \sum_{j=1}^{\infty} \overline{z}^j\, w^j. \tag{51.6}$$

As before, these series converge absolutely under these conditions, and indeed their sums can be evaluated explicitly. If $0 \le r < 1$, then the partial sums of these series converge uniformly for $|z| \le r$ and $w \in \mathbf{T}$. This follows from a well-known criterion of Weierstrass, and it can also be seen by direct computation in this case. In particular, $p_+(z,w)$ and $p_-(z,w)$ are continuous in z and w, which can be seen by summing the series too. Observe that

$$h_+(z) = \frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, p_+(z,w)\, |dw| \tag{51.7}$$

and

$$h_-(z) = \frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, p_-(z,w)\, |dw| \tag{51.8}$$

for every $z \in U$. This follows from the definitions (51.2) and (51.3) of $h_+(z)$ and $h_-(z)$ and the definition (49.1) of the Fourier coefficients $\widehat{f}(j)$ of f, by interchanging the order of summation and integration. The latter step uses the uniform convergence of the partial sums of the series on the right sides of (51.5) and (51.6) in $w \in \mathbf{T}$. The *Poisson kernel* is defined for $z \in U$ and $w \in \mathbf{T}$ by

$$p(z,w) = p_+(z,w) + p_-(z,w). \tag{51.9}$$

Combining (51.7) and (51.8), we get that

$$h(z) = \frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, p(z,w)\, |dw| \tag{51.10}$$

for every $z \in U$.

Summing the geometric series on the right side of (51.5), we get that

$$p_+(z,w) = (1 - z\, w^{-1})^{-1} \tag{51.11}$$

for every $z \in U$ and $w \in \mathbf{T}$. Observe that

$$p_-(z,w) = \overline{p_+(z,w)} - 1 \tag{51.12}$$

for every $z \in U$ and $w \in \mathbf{T}$. Thus

$$p(z,w) = 2\ \mathrm{Re}\, p_+(z,w) - 1 \tag{51.13}$$

for every $z \in U$ and $w \in \mathbf{T}$, where $\mathrm{Re}\, a$ is the real part of a complex number a. We also have that

$$p_+(z,w) = \frac{1}{1 - z\, w^{-1}}\, \frac{1 - \overline{z}\, w}{1 - \overline{z}\, \overline{w}^{-1}} = \frac{1 - \overline{z}\, w}{|1 - z\, w^{-1}|^2} \tag{51.14}$$

for every $z \in U$ and $w \in \mathbf{T}$. If $z \in \mathbf{C}$ and $w \in \mathbf{T}$, then

$$(51.15) \qquad |1 - z\,w^{-1}|^2 = |1 - z\,\overline{w}|^2 = (1 - z\,\overline{w})\,(1 - \overline{z}\,w) = 1 - z\,\overline{w} - \overline{z}\,w + |z|^2\,|w|^2 = 1 - 2\,\mathrm{Re}(\overline{z}\,w) + |z|^2.$$

Combining this with (51.13) and (51.14), we get that

$$(51.16) \qquad p(z,w) = \frac{1 - |z|^2}{|1 - z\,w^{-1}|^2}$$

for every $z \in U$ and $w \in \mathbf{T}$. In particular,

$$(51.17) \qquad p(z,w) \ge 0$$

for every $z \in U$ and $w \in \mathbf{T}$.

Using the definition (51.5) of $p_+(z,w)$, we get that

$$(51.18) \qquad \frac{1}{2\pi}\int_{\mathbf{T}} p_+(z,w)\,|dw| = \sum_{j=0}^{\infty} \frac{1}{2\pi}\int_{\mathbf{T}} z^j\,w^{-j}\,|dw| = 1$$

for every $z \in U$. More precisely, we can interchange the order of summation and integration in the first step because the partial sums of (51.11) converge uniformly in $w \in \mathbf{T}$, as before. In the second step, the $j = 0$ term is equal to 1, and all of the other terms are equal to 0, by (48.2). It follows that

$$(51.19) \qquad \frac{1}{2\pi}\int_{\mathbf{T}} p(z,w)\,|dw| = 1$$

for every $z \in U$, because of (51.13). Alternatively, one can check that the integral of $p_-(z,w)$ over $w \in \mathbf{T}$ is equal to 0 for every $z \in U$, by interchanging the order of summation and integration again.

52 Abel sums

Let $f \in L^1(\mathbf{T})$ be given, and consider the corresponding *Fourier series*

$$(52.1) \qquad \sum_{j=-\infty}^{\infty} \widehat{f}(j)\,z^j,$$

at least formally, for the moment. Let r be a real number with $0 \le r < 1$, and put

$$(52.2) \qquad A_r(f)(z) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)\,r^{|j|}\,z^j$$

for each $z \in \mathbf{T}$. More precisely, this doubly-infinite series may be treated as a sum of two ordinary infinite series. In this situation, each of these two series converges absolutely, because the Fourier coefficients $\widehat{f}(j)$ are bounded, as in (49.1). The partial sums of these series converge uniformly in $z \in \mathbf{T}$, by the criterion of Weierstrass. The sum (52.2) is the *Abel sum* of the Fourier series (52.1) associated to r. Convergence of $A_r(f)(z)$ as $r \to 1$ is known as *Abel summability* of the Fourier series (52.1).

Observe that

$$A_r(f)(z) = h(r\,z) \tag{52.3}$$

for every $z \in \mathbf{T}$ and $r \in [0, 1)$, where h is as in (51.4). Combining this with (51.10), we get that

$$A_r(f)(z) = \frac{1}{2\pi} \int_{\mathbf{T}} f(w)\, p(r\,z, w)\, |dw| \tag{52.4}$$

for every $z \in \mathbf{T}$ and $0 \le r < 1$, where $p(\cdot, \cdot)$ is as in (51.9). Put

$$p_r(z) = \frac{1 - r^2}{|1 - r\,z|^2} \tag{52.5}$$

for every $z \in \mathbf{T}$ and $0 \le r < 1$, which is another version of the Poisson kernel. Using (51.16), we have that

$$p(r\,z, w) = p_r(z\,w^{-1}) \tag{52.6}$$

for every $z, w \in \mathbf{T}$ and $0 \le r < 1$. Thus (52.4) implies that

$$A_r(f)(z) = (f * p_r)(z) \tag{52.7}$$

for every $z \in \mathbf{T}$ and $0 \le r < 1$, where $f * p_r$ is the convolution of f and p_r, as in (50.1).

Of course,

$$p_r(z) = p_r(\overline{z}) \ge 0 \tag{52.8}$$

for every $z \in \mathbf{T}$ and $0 \le r < 1$. We also have that

$$\frac{1}{2\pi} \int_{\mathbf{T}} p_r(w)\, |dw| = 1 \tag{52.9}$$

for every $0 \le r < 1$, by (51.19). If $f \in L^{r_0}(\mathbf{T})$ for some $r_0 \ge 1$, then

$$\|A_r(f)\|_{r_0} = \|f * p_r\|_{r_0} \le \|f\|_{r_0} \tag{52.10}$$

for every $0 \le r < 1$. This follows from (50.12) and commutativity of convolution.

If $f \in C(T)$, then it is well known that

$$\lim_{r \to 1-} A_r(f)(z) = f(z) \tag{52.11}$$

for every $z \in \mathbf{T}$. This uses the fact that $p_r(z) \to 0$ as $r \to 1-$ when z is not too close to 1, in addition to the other properties of $p_r(z)$ mentioned before. More precisely, the convergence in (52.11) is uniform in $z \in \mathbf{T}$, because f is uniformly continuous on $\mathbf{T}$. Remember that for each $0 \le r < 1$, the partial sums in the definition (52.2) of $A_r(f)(z)$ converge uniformly to $A_r(f)(z)$. It follows that the linear span of the z^j's, $j \in \mathbf{Z}$, is dense in $C(\mathbf{T})$ with respect to the supremum metric.

If $1 \le r_0 < \infty$ and $f \in L^{r_0}(\mathbf{T})$, then it is well known that $A_r(f) \to f$ as $r \to 1-$ with respect to the L^{r_0} norm. This follows from the uniform convergence mentioned in the previous paragraph when $f \in C(\mathbf{T})$, and otherwise one can approximate $f \in L^{r_0}(\mathbf{T})$ by continuous functions with respect to the L^{r_0} norm. This approximation argument uses (52.10) as well. It is also well known that for every $f \in L^1(\mathbf{T})$, (52.11) holds for almost every $z \in \mathbf{T}$ with respect to arclength measure. This is closely related to Lebesgue's differentiation theorem.

53 Square-integrable functions

Let $f \in L^2(\mathbf{T})$ be given, so that $\widehat{f} \in \ell^2(\mathbf{Z}, \mathbf{C})$, as in Section 49. Using the orthonormality of the z^j's in $L^2(\mathbf{T})$, we get that the infinite series

$$\sum_{j=0}^{\infty} \widehat{f}(j)\, z^j, \quad \sum_{j=1}^{\infty} \widehat{f}(-j)\, z^{-j} \tag{53.1}$$

converge in $L^2(\mathbf{T})$, as in Section 39. The Fourier series (52.1) may be considered as the sum of these two series, and thus defines an element of $L^2(\mathbf{T})$. Similarly, if $0 \le r < 1$, then

$$\sum_{j=0}^{\infty} \widehat{f}(j)\, r^j\, z^j, \quad \sum_{j=1}^{\infty} \widehat{f}(-j)\, r^j\, z^{-j} \tag{53.2}$$

converge in $L^2(\mathbf{T})$. More precisely, these two series converge absolutely, and their partial sums converge uniformly, as in the previous section. Of course, uniform convergence of the partial sums implies convergence of the partial sums with respect to the L^2 norm. Note that the sum of the two series in (53.2) is the same as the Abel sum $A_r(f)(z)$ in (52.2).

The series

$$\sum_{j=0}^{\infty} \widehat{f}(j)\, (1 - r^j)\, z^j, \quad \sum_{j=1}^{\infty} \widehat{f}(-j)\, (1 - r^j)\, z^{-j} \tag{53.3}$$

also converge in $L^2(\mathbf{T})$ for each $0 \le r < 1$, and their sums are equal to the differences of the sums of the series in (53.1) and (53.2), respectively. Observe that

$$\Big\| \sum_{j=0}^{\infty} \widehat{f}(j)\, (1 - r^j)\, z^j \Big\|_{L^2(\mathbf{T})}^2 = \sum_{j=0}^{\infty} |\widehat{f}(j)|^2\, (1 - r^j)^2 \tag{53.4}$$

for each $0 \le r < 1$, because of the orthogonality of the z^j's in $L^2(\mathbf{T})$. We also have that

$$\lim_{r \to 1-} \sum_{j=0}^{\infty} |\widehat{f}(j)|^2\, (1 - r^j)^2 = 0. \tag{53.5}$$

This can be verified directly, or using the version of Lebesgue's dominated convergence theorem for sums. The L^2 norm of the second series in (53.3) converges to 0 as $r \to 1-$ as well, by the same argument.

The sum of the two series in (53.3) is equal to

$$\sum_{j=-\infty}^{\infty} \widehat{f}(j)\, z^j - A_r(f)(z) \tag{53.6}$$

for each $0 \le r < 1$. Hence

$$\Big\| \sum_{j=-\infty}^{\infty} \widehat{f}(j)\, z^j - A_r(f)(z) \Big\|_{L^2(\mathbf{T})}^2 = \sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2\, (1 - r^{|j|})^2 \to 0 \tag{53.7}$$

as $r \to 1-$, as in the preceding paragraph. We also saw in the previous section that $A_r(f) \to f$ in $L^2(\mathbf{T})$ as $r \to 1-$. It follows that the Fourier series (52.1) is equal to f as an element of $L^2(\mathbf{T})$. This could also be obtained as in Section 49, using the fact that f can be approximated by linear combinations of the z^j's with respect to the L^2 norm.

Suppose now that $f \in L^1(\mathbf{T})$, and that $\widehat{f} \in \ell^2(\mathbf{Z}, \mathbf{C})$. This condition on $\widehat{f}$ implies that the Fourier series (52.1) defines an element of $L^2(\mathbf{T})$, as before. We also have (53.7) in this situation, for the same reasons as before. The discussion in the previous section implies that $A_r(f) \to f$ in $L^1(\mathbf{T})$ as $r \to 1-$. Of course, the convergence of $A_r(f)(z)$ to the Fourier series (52.1) as $r \to 1-$ with respect to the L^2 norm implies convergence with respect to the L^1 norm as well. This implies that f is the same as the Fourier series (52.1) as an element of $L^1(\mathbf{T})$. It follows that $f \in L^2(\mathbf{T})$ under these conditions, because the Fourier series (52.1) is an element of $L^2(\mathbf{T})$.

54 Convolution powers

If $f \in L^1(\mathbf{T})$ and $n \in \mathbf{Z}_+$, then we let f^{*n} be the nth convolution power of f. This is equal to f when $n = 1$, to $f * f$ when $n = 2$, and to $f^{*(n-1)} * f = f * f^{*(n-1)}$ for every $n \ge 2$. If $1 \le r \le \infty$, then $L^r(\mathbf{T})$ is a commutative algebra with respect to convolution, as in Section 50. In this case, the L^r norm is also submultiplicative with respect to convolution, by (50.12). It follows that

$$\lim_{n \to \infty} \|f^{*n}\|_r^{1/n} \tag{54.1}$$

exists in $\mathbf{R}$ for every $f \in L^r(\mathbf{T})$, as in Section 31. Note that (54.1) increases monotonically in r, because $\|\cdot\|_r$ increases monotonically in r. In particular,

$$\lim_{n \to \infty} \|f^{*n}\|_1^{1/n} \le \lim_{n \to \infty} \|f^{*n}\|_r^{1/n} \tag{54.2}$$

for every $f \in L^r(\mathbf{T})$.

Let $r \ge 1$ and $f \in L^r(\mathbf{T})$ be given, and observe that

$$\|f^{*n}\|_r = \|f^{*(n-1)} * f\|_r \le \|f^{*(n-1)}\|_1\, \|f\|_r \le \|f\|_1^{n-1}\, \|f\|_r \tag{54.3}$$

for every $n \geq 2$. This uses (50.12) in the second step, and (50.5) in the third step. It follows that

$$\|f^{*n}\|_r^{1/n} \leq \|f\|_1^{1-(1/n)}\,\|f\|_r^{1/n} \tag{54.4}$$

for every $n \geq 2$. Using this, one can check that

$$\lim_{n\to\infty} \|f^{*n}\|_r^{1/n} \leq \lim_{n\to\infty} \|f^{*n}\|_1^{1/n}. \tag{54.5}$$

Combining this with (54.2), we get that

$$\lim_{n\to\infty} \|f^{*n}\|_r^{1/n} = \lim_{n\to\infty} \|f^{*n}\|_1^{1/n}. \tag{54.6}$$

Let $f \in L^1(\mathbf{T})$ be given, and observe that

$$(\widehat{f^{*n}})(j) = \widehat{f}(j)^n \tag{54.7}$$

for every $j \in \mathbf{Z}$ and $n \in \mathbf{Z}_+$, by (50.7). This implies that

$$\|\widehat{f}\|_{\ell^\infty(\mathbf{Z},\mathbf{C})}^n = \|(\widehat{f})^n\|_{\ell^\infty(\mathbf{Z},\mathbf{C})} \leq \|f^{*n}\|_{L^1(\mathbf{T})} \tag{54.8}$$

for every $n \in \mathbf{Z}_+$, using (49.2) in the second step. Thus

$$\|\widehat{f}\|_{\ell^\infty(\mathbf{Z},\mathbf{C})} \leq \|f^{*n}\|_{L^1(\mathbf{T})}^{1/n} \tag{54.9}$$

for every $n \in \mathbf{Z}_+$, and hence

$$\|\widehat{f}\|_{\ell^\infty(\mathbf{Z},\mathbf{C})} \leq \lim_{n\to\infty} \|f^{*n}\|_{L^1(\mathbf{T})}^{1/n}. \tag{54.10}$$

Suppose for the moment that $f \in L^2(\mathbf{T})$, so that $f^{*n} \in L^2(\mathbf{T})$ for each $n \in \mathbf{Z}_+$. As in (49.11),

$$\|f^{*n}\|_{L^2(\mathbf{T})} = \|(\widehat{f^{*n}})\|_{\ell^2(\mathbf{Z},\mathbf{C})} \tag{54.11}$$

for every $n \in \mathbf{Z}_+$, which can also be obtained from the discussion in the previous section. Using (54.7), we get that

$$\|(\widehat{f^{*n}})\|_{\ell^2(\mathbf{Z},\mathbf{C})} \leq \|\widehat{f}\|_{\ell^\infty(\mathbf{Z},\mathbf{C})}^{n-1}\,\|\widehat{f}\|_{\ell^2(\mathbf{Z},\mathbf{C})} \tag{54.12}$$

when $n \geq 2$. Thus

$$\|f^{*n}\|_{L^2(\mathbf{T})} \leq \|\widehat{f}\|_{\ell^\infty(\mathbf{Z},\mathbf{C})}^{n-1}\,\|f\|_{L^2(\mathbf{T})} \tag{54.13}$$

for each $n \geq 2$, so that

$$\|f^{*n}\|_{L^2(\mathbf{T})}^{1/n} \leq \|\widehat{f}\|_{\ell^\infty(\mathbf{Z},\mathbf{C})}^{1-(1/n)}\,\|f\|_{L^2(\mathbf{T})}^{1/n}. \tag{54.14}$$

This implies that

$$\lim_{n\to\infty} \|f^{*n}\|_{L^2(\mathbf{T})}^{1/n} \leq \|\widehat{f}\|_{\ell^\infty(\mathbf{Z},\mathbf{C})}. \tag{54.15}$$

It follows that

$$\lim_{n \to \infty} \|f^{*n}\|_{L^1(\mathbf{T})}^{1/n} \le \|\widehat{f}\|_{\ell^\infty(\mathbf{Z},\mathbf{C})} \tag{54.16}$$

when $f \in L^2(\mathbf{T})$, by (54.2). Remember that

$$\lim_{n \to \infty} \|f^{*n}\|_{L^1(\mathbf{T})}^{1/n} \tag{54.17}$$

defines a seminorm on $L^1(\mathbf{T})$ that is less than or equal to the L^1 norm, as in Sections 31 and 33. Using this, one can check that (54.16) holds for every $f \in L^1(\mathbf{T})$, because $L^2(\mathbf{T})$ is dense in $L^1(\mathbf{T})$. Combining this with (54.10), we get that

$$\lim_{n \to \infty} \|f^{*n}\|_{L^1(\mathbf{T})}^{1/n} = \|f\|_{\ell^\infty(\mathbf{Z},\mathbf{C})} \tag{54.18}$$

for every $f \in L^1(\mathbf{T})$. Similarly,

$$\lim_{n \to \infty} \|f^{*n}\|_{L^r(\mathbf{T})}^{1/n} = \|f\|_{\ell^\infty(\mathbf{Z},\mathbf{C})} \tag{54.19}$$

for every $r \ge 1$ and $f \in L^r(\mathbf{T})$, by (54.6).

55 Analytic type

Let $f \in L^1(\mathbf{T})$ be given. If

$$\widehat{f}(j) = 0 \quad \text{for every } j \in \mathbf{Z} \text{ with } j < 0, \tag{55.1}$$

then f is said to be of *analytic type*. This implies that the corresponding function h_- defined on the open unit disk U as in (51.3) is equal to 0 everywhere on U. This means that the function h defined on U as in (51.4) is equal to the function h_+ defined in (51.2), so that h is holomorphic on U.

Let $f \in C(\mathbf{T})$ be given, and let h be defined on U as in (51.4) again. Consider the function on the closed unit disk $\overline{U}$ that is equal to h on U and to f on $\partial U = \mathbf{T}$. It is well known that this function is continuous on $\overline{U}$, for the same type of reasons as in Section 52. If f is of analytic type, then this function on $\overline{U}$ is an element of the algebra $A(U)$ defined in Section 28.

Let ϕ be a holomorphic function on U, and put

$$\phi_r(z) = \phi(r\,z) \tag{55.2}$$

for each $0 \le r < 1$ and $z \in \mathbf{T}$. Thus $\phi_r \in C(\mathbf{T})$ for each $0 \le r < 1$, and one can check that ϕ_r is of analytic type for every $0 \le r < 1$, using Cauchy's theorem. If $\phi \in A(U)$, then ϕ_r can be defined as in (55.2) when $r = 1$. In this case, $\phi_r \to \phi_1$ as $r \to 1-$ uniformly on $\mathbf{T}$, because continuous functions on $\overline{U}$ are uniformly continuous. Using this, one can verify that ϕ_1 is of analtyic type too.

Let $f, g \in L^2(\mathbf{T})$ be given, so that $f\,g \in L^1(\mathbf{T})$. We would like to check that

$$\widehat{(f\,g)}(n) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)\,\widehat{g}(n-j) \tag{55.3}$$

for each $n \in \mathbf{Z}$. Remember that $\widehat{f}.\widehat{g} \in \ell^2(\mathbf{Z}, \mathbf{C})$, as in Section 49. It follows that $\widehat{g}(n-j)$ is square-summable as a function of $j \in \mathbf{Z}$ for each $n \in \mathbf{Z}$, with ℓ^2 norm equal to the ℓ^2 norm of $\widehat{g}$. This implies that $\widehat{f}(j)\,\widehat{g}(n-j)$ is summable as a function of $j \in \mathbf{Z}$ for each $n \in \mathbf{Z}$. The ℓ^1 norm of this function of j is less than or equal to the product of the ℓ^2 norms of $\widehat{f}$ and $\widehat{g}$, by the Cauchy–Schwarz inequality. Thus the right side of (55.3) is defined as a complex number, and its absolute value is less than or equal to the product of the L^2 norms of f and g. One can first verify that (55.3) holds when $g(z)$ is of the form z^l for some $l \in \mathbf{Z}$, directly from the definitions. This implies that (55.3) holds when $g(z)$ is a linear combination of z^l's, by linearity. To deal with any $g \in L^2(\mathbf{T})$, one can approximate $g(z)$ by linear combinations of the z^l's with respect to the L^2 norm.

Suppose now that f, g are of analytic type. In this case, (55.3) reduces to

$$(55.4) \qquad \widehat{(f\,g)}(n) = \sum_{j=0}^{n} \widehat{f}(j)\,\widehat{g}(n-j)$$

when $n \geq 0$. If $n < 0$, then the right side of (55.3) is equal to 0, so that $f\,g$ is of analytic type.

Suppose that $f \in L^1(\mathbf{T})$ is of analytic type, and let $0 \leq r < 1$ be given. In this situation, the Abel sum $A_r(f)(z)$ defined in (52.2) reduces to

$$(55.5) \qquad A_r(f)(z) = \sum_{j=0}^{\infty} \widehat{f}(j)\, r^j\, z^j$$

for each $z \in \mathbf{T}$. As before, the partial sums of (55.5) converge uniformly on $\mathbf{T}$, by the well-known criterion of Weierstrass. Thus $A_r(f)(z)$ can be uniformly approximated on $\mathbf{T}$ by linear combinations of the z^j's with $j \geq 0$. If $f \in C(\mathbf{T})$, then $A_r(f) \to f$ uniformly on $\mathbf{T}$ as $r \to 1-$, as in Section 52. If $f \in C(\mathbf{T})$ is of analytic type, then it follows that $f(z)$ can be uniformly approximated by linear combinations of z^j'z with $j \geq 0$. Similarly, if $1 \leq r_0 < \infty$ and $f \in L^{r_0}(\mathbf{T})$, then $A_r(f) \to f$ as $r \to 1-$ with respect to the L^{r_0} norm, as in Section 52. If f is of analytic type, then $f(z)$ can be approximated by linear combinations of z^j's with $j \geq 0$ with respect to the L^{r_0} norm.

Of course, z^j is of analytic type for each $j \geq 0$, so that linear combinations of the z^j's with $j \geq 0$ are of analytic type. If $1 \leq r_0 \leq \infty$, then the collection of $f \in L^{r_0}(\mathbf{T})$ of analytic type is a linear subspace of $L^{r_0}(\mathbf{T})$ and a closed set with respect to the metric associated to the L^{r_0} norm. The collection of $f \in C(\mathbf{T})$ of analytic type is a closed subalgebra of $C(\mathbf{T})$ with respect to the supremum metric. It is well known that $L^\infty(\mathbf{T})$ can be identified with the dual space of bounded linear functionals on $L^1(\mathbf{T})$, which leads to a corresponding weak* topology on $L^\infty(\mathbf{T})$. It is easy to see that the collection of $f \in L^\infty(\mathbf{T})$ of analytic type is a closed set with respect to this weak* topology.

Let $1 \leq r_0, r_0' \leq \infty$ be conjugate exponents, so that

$$(55.6) \qquad 1/r_0 + 1/r_0' = 1.$$

Also let $f \in L^{r_0}(\mathbf{T})$ and $g \in L^{r_0'}(\mathbf{T})$ be given, so that $f\,g \in L^1(\mathbf{T})$. If f, g are of analytic type, then $f\,g$ is of analytic type as well, and the nth Fourier coefficient of $f\,g$ is given as in (55.4) when $n \geq 0$. To see this, we may as well suppose that $r_0' < \infty$, since otherwise we can interchange the roles of r_0 and r_0'. If $g(z) = z^l$ for some $l \in \mathbf{Z}$ with $l \geq 0$, then one can verify the previous statements directly from the definitions. This implies that these statements also hold when g is a linear combination of z^l's with $l \geq 0$. If g is any element of $L^{r_0'}$ of analytic type, then one can approximate g by linear combinations of z^l's with $l \geq 0$ with respect to the $L^{r_0'}$ norm, as before, because $r_0' < \infty$.

Let f, g be as in the preceding paragraph, and let h_f, h_g, and $h_{f\,g}$ be the holomorphic functions on the open unit disk U corresponding to f, g, and $f\,g$, respectively, as in (51.4). These are the same as the corresponding functions of the form (51.2) in this situation, as before. It is easy to see that

$$h_{f\,g}(z) = h_f(z)\,h_g(z) \tag{55.7}$$

for every $z \in U$. Equivalently,

$$A_r(f\,g)(z) = A_r(f)(z)\,A_r(g)(z) \tag{55.8}$$

for every $0 \leq r < 1$ and $z \in \mathbf{T}$. This uses (55.4) to identify the left sides of (55.7) and (55.8) with the Cauchy products of the series on the right sides of these equations.

Note that $L^\infty(\mathbf{T})$ is a commutative Banach algebra with respect to pointwise multiplication of functions and the L^∞ norm. The collection of $f \in L^\infty(\mathbf{T})$ of analytic type is a subalgebra of $L^\infty(\mathbf{T})$, by the earlier remarks about products of two elements of $L^2(\mathbf{T})$ of analytic type.

If $f \in L^\infty(\mathbf{T})$ and h is as in (51.4), then h is a bounded harmonic complex-valued function on U. More precisely, the supremum norm of h on U is bounded by $\|f\|_{L^\infty(\mathbf{T})}$, as in (52.10). In fact, one can check that the supremum norm of h on U is equal to $\|f\|_{L^\infty(\mathbf{T})}$. Conversely, it is well known that every bounded harmonic complex-valued function on U corresponds to some $f \in L^\infty(\mathbf{T})$ in this way. Similarly, bounded holomorphic functions on U correspond to $f \in L^\infty(\mathbf{T})$ of analytic type.

Part V

Some analysis on Z

56 Another Fourier transform

Let $f(j)$ be a complex-valued summable function on $\mathbf{Z}$. The *Fourier transform* of f is the complex-valued function defined on the unit circle by

$$\widehat{f}(z) = \sum_{j=-\infty}^{\infty} f(j)\,z^j \tag{56.1}$$

for each $z \in \mathbf{T}$. More precisely, the sum on the right side of (56.1) is defined as a complex number, because $f(j)\, z^j$ is summable as a function of $j \in \mathbf{Z}$ when $z \in \mathbf{T}$. Alternatively, if $z \in \mathbf{T}$, then the series

$$\sum_{j=0}^{\infty} f(j)\, z^j, \quad \sum_{j=1}^{\infty} f(-j)\, z^{-j} \tag{56.2}$$

converge absolutely, and the sum on the right side of (56.1) is the same as the sum of the two series in (56.2). We also have that

$$|\widehat{f}(z)| \le \sum_{j=-\infty}^{\infty} |f(j)| \tag{56.3}$$

for each $z \in \mathbf{T}$.

The partial sums of the series in (56.2) converge uniformly on $\mathbf{T}$, by the well-known criterion of Weierstrass. In particular, this implies that $\widehat{f}$ is continuous on $\mathbf{T}$. Of course, (56.3) is the same as saying that

$$\|\widehat{f}\|_{sup,\mathbf{T}} \le \|f\|_{\ell^1(\mathbf{Z},\mathbf{C})}. \tag{56.4}$$

where $\|\widehat{f}\|_{sup,\mathbf{T}}$ is the supremum norm of $\widehat{f}$ on $\mathbf{T}$. One can also consider the sum on the right side of (56.1) as a sum over $j \in \mathbf{Z}$ of elements of the space $C(\mathbf{T}) = C(\mathbf{T},\mathbf{C})$ of continuous complex-valued functions on $\mathbf{T}$ equipped with the supremum norm, as in Section 25. Note that $f \mapsto \widehat{f}$ defines a bounded linear mapping from $\ell^1(\mathbf{Z},\mathbf{C})$ into $C(\mathbf{T})$, by (56.4).

Remember that the functions on $\mathbf{T}$ of the form z^j with $j \in \mathbf{Z}$ are orthonormal with respect to the usual integral inner product on $L^2(\mathbf{T})$, as in Section 48. If $f \in \ell^2(\mathbf{Z},\mathbf{C})$, then the series in (56.2) converge in $L^2(\mathbf{T})$, as in Section 39. Thus $\widehat{f}$ may be defined as an element of $L^2(\mathbf{T})$ as in (56.1). We also get that

$$\|\widehat{f}\|_{L^2(\mathbf{T})} = \|f\|_{\ell^2(\mathbf{Z},\mathbf{C})} \tag{56.5}$$

for every $f \in \ell^2(\mathbf{Z},\mathbf{C})$, because of the orthonormality of the z^j's in $L^2(\mathbf{T})$. If f is an element of $\ell^1(\mathbf{Z},\mathbf{C})$, and hence of $\ell^2(\mathbf{Z},\mathbf{C})$, then it is easy to see that this definition of $\widehat{f}$ as an element of $L^2(\mathbf{T})$ is compatible with the previous definition.

Let $f \in \ell^2(\mathbf{Z},\mathbf{C})$ be given, so that $\widehat{f}$ is defined as an element of $L^2(\mathbf{T})$, as in the previous paragraph. If $l \in \mathbf{Z}$, then the lth Fourier coefficient $\widehat{(\widehat{f})}(l)$ of $\widehat{f}$ can be defined as in Section 49, and is the same as the inner product of $\widehat{f}$ and z^l with respect to the usual inner product on $L^2(\mathbf{T})$. Observe that

$$\widehat{(\widehat{f})}(l) = f(l) \tag{56.6}$$

for every $l \in \mathbf{Z}$, because of the orthonormality of the z^j's in $L^2(\mathbf{T})$. Similarly, if $f \in L^2(\mathbf{T})$, then $\widehat{f} \in \ell^2(\mathbf{Z},\mathbf{C})$, as in Section 49, so that $\widehat{(\widehat{f})}$ can be defined as an element of $L^2(\mathbf{T})$, as in the preceding paragraph. The discussion in Section 53 implies that

$$(56.7) \qquad \widehat{(\widehat{f})} = f$$

as elements of $L^2(\mathbf{T})$.

Suppose that $f \in \ell^2(\mathbf{Z}, \mathbf{C})$ has the property that

$$(56.8) \qquad f(j) = 0 \quad \text{for every } j \in \mathbf{Z} \text{ with } j < 0.$$

This implies that

$$(56.9) \qquad \widehat{f}(z) = \sum_{j=0}^{\infty} f(j)\, z^j$$

is of analytic type, as in the previous section. In this case, the series on the right side of (56.9) converges absolutely when $z \in \mathbf{C}$ satisfies $|z| < 1$, and defines a holomorphic function on the open unit disk U. If $f \in \ell^1(\mathbf{Z}, \mathbf{C})$ satisfies (56.8), then the series on the right side of (56.9) converges absolutely for every $z \in \mathbf{C}$ with $|z| \le 1$, and the partial sums of this series converge uniformly on the closed unit disk $\overline{U}$, by the usual criterion of Weierstrass. Hence this series defines an element of the space $A(U)$ defined in Section 28 in this situation.

57 Convolution of summable functions

Let f, g be complex-valued summable functions on $\mathbf{Z}$. The *convolution* of f and g is the complex-valued function defined on $\mathbf{Z}$ by

$$(57.1) \qquad (f * g)(j) = \sum_{l=-\infty}^{\infty} f(l)\, g(j-l)$$

for every $j \in \mathbf{Z}$. More precisely, for each $j \in \mathbf{Z}$, the right side of (57.1) is the sum of a summable function of l, because summable functions are bounded. Thus the sum is defined as a complex number for each $j \in \mathbf{Z}$, and satisfies

$$(57.2) \qquad |(f * g)(j)| \le \sum_{l=-\infty}^{\infty} |f(l)|\, |g(j-l)| = (|f| * |g|)(j)$$

for every $j \in \mathbf{Z}$. Let us check that $f * g$ is summable on $\mathbf{Z}$ under these conditions.

Using (57.2), we get that

$$(57.3) \qquad \sum_{j=-\infty}^{\infty} |(f * g)(j)| \le \sum_{j=-\infty}^{\infty} \Big(\sum_{l=-\infty}^{\infty} |f(l)|\, |g(j-l)| \Big).$$

As in Section 26, we can interchange the order of summation, to obtain that

$$(57.4) \qquad \sum_{j=-\infty}^{\infty} |(f * g)(j)| \quad \le \quad \sum_{l=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{\infty} |f(l)|\, |g(j-l)| \Big)$$

$$= \sum_{l=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{\infty} |f(l)|\,|g(j)| \Big)$$
$$= \Big(\sum_{l=-\infty}^{\infty} |f(l)| \Big) \Big(\sum_{j=-\infty}^{\infty} |g(j)| \Big).$$

This implies that $f * g$ is summable on $\mathbf{Z}$, with

$$\|f * g\|_{\ell^1(\mathbf{Z},\mathbf{C})} \le \|f\|_{\ell^1(\mathbf{Z},\mathbf{C})}\, \|g\|_{\ell^1(\mathbf{Z},\mathbf{C})}. \tag{57.5}$$

Note that $|f(l)|\,|g(j-l)|$ is summable as a nonnegative real-valued function of (j,l) on $\mathbf{Z} \times \mathbf{Z}$, as in (57.4).

One can check that $\ell^1(\mathbf{Z},\mathbf{C})$ is a commutative algebra over $\mathbf{C}$ with respect to convolution. Let $\delta_0(j)$ be the complex-valued function on $\mathbf{Z}$ equal to 1 when $j = 0$ and to 0 when $j \ne 0$. It is easy to see that

$$f * \delta_0 = \delta_0 * f = f \tag{57.6}$$

for every $f \in \ell^1(\mathbf{Z},\mathbf{C})$. Thus δ_0 is the multiplicative identity element in $\ell^1(\mathbf{Z},\mathbf{C})$ with respect to convolution.

Let $f, g \in \ell^1(\mathbf{Z},\mathbf{C})$ and $z \in \mathbf{T}$ be given, so that

$$(\widehat{f * g})(z) = \sum_{j=-\infty}^{\infty} (f * g)(j)\, z^j = \sum_{j=-\infty}^{\infty} \Big(\sum_{l=-\infty}^{\infty} f(l)\, g(j-l) \Big) z^j. \tag{57.7}$$

Interchanging the order of summation, as in Section 26, we get that

$$\begin{aligned}(\widehat{f * g})(z) &= \sum_{l=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{\infty} f(l)\, g(j-l)\, z^j \Big) \\ &= \sum_{l=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{\infty} f(l)\, g(j)\, z^{j+l} \Big) \\ &= \Big(\sum_{l=-\infty}^{\infty} f(l)\, z^l \Big) \Big(\sum_{j=-\infty}^{\infty} g(j)\, z^j \Big) = \widehat{f}(z)\, \widehat{g}(z).\end{aligned} \tag{57.8}$$

Suppose for the moment that f, g also satisfy

$$f(j) = g(j) = 0 \quad \text{for every } j \in \mathbf{Z} \text{ with } j < 0. \tag{57.9}$$

In this case, (57.1) reduces to

$$(f * g)(j) = \sum_{l=0}^{j} f(l)\, g(j-l) \tag{57.10}$$

when $j \ge 0$, and $(f * g)(j) = 0$ when $j < 0$. Remember that $\widehat{f}(z)$ and $\widehat{g}(z)$ can be defined for every z in the closed unit disk $\overline{U}$ as in (56.9) in this situation. If $z \in \overline{U}$, then $\widehat{f}(z)\, \widehat{g}(z)$ can be treated as a Cauchy product, as in Section 29. This Cauchy product corresponds exactly to $(\widehat{f * g})(z)$, because of (57.10).

58 Some related estimates

Let $0 < r \le 1$ and $f, g \in \ell^r(\mathbf{Z}, \mathbf{C})$ be given. In particular, f and g are summable on $\mathbf{Z}$, as in Section 22. If $j \in \mathbf{Z}$, then

$$(58.1)\quad (|f| * |g|)(j) = \sum_{l=-\infty}^{\infty} |f(l)|\,|g(j-l)| \le \Big(\sum_{l=-\infty}^{\infty} |f(l)|^r\,|g(j-l)|^r\Big)^{1/r},$$

using (21.8) in the second step. This implies that

$$(58.2)\qquad |(f * g)(j)|^r \le (|f| * |g|)(j)^r \le (|f|^r * |g|^r)(j)$$

for every $j \in \mathbf{Z}$, using (57.2) in the first step. Note that $|f|^r$ and $|g|^r$ are summable on $\mathbf{Z}$, so that $|f|^r * |g|^r$ is summable on $\mathbf{Z}$, as in the previous section. It follows that $|f * g|^r$ is summable on $\mathbf{Z}$, by (58.2). More precisely,

$$(58.3)\quad \||f * g|^r\|_{\ell^1(\mathbf{Z},\mathbf{C})} \le \||f|^r * |g|^r\|_{\ell^1(\mathbf{Z},\mathbf{C})} \le \||f|^r\|_{\ell^1(\mathbf{Z},\mathbf{C})}\,\||g|^r\|_{\ell^1(\mathbf{Z},\mathbf{C})},$$

using (57.5) in the second step. Equivalently, $f * g$ is r-summable on $\mathbf{Z}$, with

$$(58.4)\qquad \|f * g\|_{\ell^r(\mathbf{Z},\mathbf{C})} \le \|f\|_{\ell^r(\mathbf{Z},\mathbf{C})}\,\|g\|_{\ell^r(\mathbf{Z},\mathbf{C})}.$$

Thus $\ell^r(\mathbf{Z}, \mathbf{C})$ is a subalgebra of $\ell^1(\mathbf{Z}, \mathbf{C})$ with respect to convolution.

Let f^{*n} be the nth power of $f \in \ell^r(\mathbf{Z}, \mathbf{C})$ with respect to convolution for each $n \in \mathbf{Z}_+$. As before, f^{*n} is equal to f when $n = 1$, and to $f^{*(n-1)} * f = f * f^{*(n-1)}$ when $n \ge 2$. As in Section 31,

$$(58.5)\qquad \lim_{n\to\infty} \|f^{*n}\|_{\ell^r(\mathbf{Z},\mathbf{C})}^{1/n}$$

exists in $\mathbf{R}$ for every $f \in \ell^r(\mathbf{Z}, \mathbf{C})$. We also have that (58.5) decreases monotonically as r increases, because of the corresponding property of $\|\cdot\|_{\ell^r(\mathbf{Z},\mathbf{C})}$, as in (22.5). In particular,

$$(58.6)\qquad \lim_{n\to\infty} \|f^{*n}\|_{\ell^1(\mathbf{Z},\mathbf{C})}^{1/n} \le \lim_{n\to\infty} \|f^{*n}\|_{\ell^r(\mathbf{Z},\mathbf{C})}^{1/n}$$

for every $f \in \ell^r(\mathbf{Z}, \mathbf{C})$.

If $f \in \ell^1(\mathbf{Z}, \mathbf{C})$, $n \in \mathbf{Z}_+$, and $z \in \mathbf{T}$, then

$$(58.7)\qquad (\widehat{f^{*n}})(z) = \widehat{f}(z)^n,$$

by (57.8). Hence

$$(58.8)\qquad \|\widehat{f}\|_{sup,\mathbf{T}}^n = \|(\widehat{f})^n\|_{sup,\mathbf{T}} \le \|f^{*n}\|_{\ell^1(\mathbf{Z},\mathbf{C})}$$

for every $f \in \ell^1(\mathbf{Z}, \mathbf{C})$ and $n \in \mathbf{Z}_+$, using (56.4) in the second step. This implies that

$$(58.9)\qquad \|\widehat{f}\|_{sup,\mathbf{T}} \le \|f^{*n}\|_{\ell^1(\mathbf{Z},\mathbf{C})}^{1/n}$$

for every $f \in \ell^1(\mathbf{Z}, \mathbf{C})$ and $n \in \mathbf{Z}_+$, so that

$$\|\widehat{f}\|_{sup,\mathbf{T}} \le \lim_{n \to \infty} \|f^{*n}\|_{\ell^1(\mathbf{Z},\mathbf{C})}^{1/n}. \tag{58.10}$$

Let $f \in \ell^1(\mathbf{Z}, \mathbf{C})$, $n \in \mathbf{Z}_+$, and $j \in \mathbf{Z}$ be given. As in (56.6),

$$f^{*n}(j) = ((\widehat{\widehat{f^{*n}}}))(j) \tag{58.11}$$

for every $n \in \mathbf{Z}_+$ and $j \in \mathbf{Z}$. It follows that

$$|f^{*n}(j)| = |((\widehat{\widehat{f^{*n}}}))(j)| \le \|(\widehat{f^{*n}})\|_{L^1(\mathbf{T})}, \tag{58.12}$$

using (49.2) in the second step. Combining this with (58.7), we get that

$$|f^{*n}(j)| \le \|(\widehat{f})^n\|_{L^1(\mathbf{Z},\mathbf{C})} \le \|\widehat{f}\|_{sup,\mathbf{T}}^n. \tag{58.13}$$

Let L be a nonnegative integer, and let f be a complex-valued function on $\mathbf{Z}$ such that

$$f(j) = 0 \quad \text{for every } j \in \mathbf{Z} \text{ with } |j| > L. \tag{58.14}$$

If $n \in \mathbf{Z}_+$, then one can check that

$$f^{*n}(j) = 0 \quad \text{for every } j \in \mathbf{Z} \text{ with } |j| > n\,L. \tag{58.15}$$

Let r be a positive real number, and observe that

$$\|f^{*n}\|_{\ell^r(\mathbf{Z},\mathbf{C})} \le (2\,n\,L + 1)^{1/r}\, \|f^{*n}\|_{\ell^\infty(\mathbf{Z},\mathbf{C})} \tag{58.16}$$

for each $n \in \mathbf{Z}_+$. This implies that

$$\|f^{*n}\|_{\ell^r(\mathbf{Z},\mathbf{C})} \le (2\,n\,L + 1)^{1/r}\, \|\widehat{f}\|_{sup,\mathbf{T}}^n \tag{58.17}$$

for every $n \in \mathbf{Z}_+$, because of (58.13). Equivalently,

$$\|f^{*n}\|_{\ell^r(\mathbf{Z},\mathbf{C})}^{1/n} \le (2\,n\,L + 1)^{1/(n\,r)}\, \|\widehat{f}\|_{sup,\mathbf{T}} \tag{58.18}$$

for every $n \in \mathbf{Z}_+$. Using this, we get that

$$\lim_{n \to \infty} \|f^{*n}\|_{\ell^r(\mathbf{Z},\mathbf{C})}^{1/n} \le \|\widehat{f}\|_{sup,\mathbf{T}} \tag{58.19}$$

for every $0 < r \le 1$ when f satisfies (58.14). As in Sections 31 and 33, (58.5) defines a seminorm on $\ell^r(\mathbf{Z}, \mathbf{C})$ when $0 < r \le 1$, and this seminorm is less than or equal to the ℓ^r r-norm. Of course, every $f \in \ell^r(\mathbf{Z}, \mathbf{C})$ can be approximated by functions on $\mathbf{Z}$ with finite support. One can use this to check that (58.19) holds for every $f \in \ell^r(\mathbf{Z}, \mathbf{C})$ when $0 < r \le 1$. It follows that

$$\lim_{n \to \infty} \|f^{*n}\|_{\ell^r(\mathbf{Z},\mathbf{C})}^{1/n} = \|\widehat{f}\|_{sup,\mathbf{T}} \tag{58.20}$$

for every $f \in \ell^r(\mathbf{Z}, \mathbf{C})$ when $0 < r \le 1$, by combining (58.6), (58.10), and (58.19).

59 The ultrametric case

Let k be a field with an ultrametric absolute value function $|\cdot|$, and suppose that k is complete with respect to the associated ultrametric. If $f, g \in c_0(\mathbf{Z}, k)$, then their *convolution* is defined as a k-valued function on $\mathbf{Z}$ by

$$(59.1) \qquad (f * g)(j) = \sum_{l=-\infty}^{\infty} f(l)\, g(j-l)$$

for each $j \in \mathbf{Z}$. More precisely, if $j \in \mathbf{Z}$, then $f(l)\, g(j-l)$ vanishes at infinity as a k-valued function of l on $\mathbf{Z}$. Thus the sum (59.1) may be treated as a sum over $\mathbf{Z}$, as in Section 25. Alternatively, (59.1) may be considered as a sum of two infinite series, for which the convergence in k can be obtained as in Section 24.

In particular, we have that

$$(59.2) \qquad |(f * g)(j)| \le \max_{l \in \mathbf{Z}} (|f(l)|\, |g(j-l)|)$$

for every $j \in \mathbf{Z}$, as in Sections 24 and 25. As usual, the maximum on the right side of (59.2) is attained, because $f(l)\, g(j-l)$ vanishes at infinity as a function of l for each j. It follows that $f * g$ is bounded on $\mathbf{Z}$, with

$$(59.3) \qquad \|f * g\|_{\ell^\infty(\mathbf{Z},k)} \le \|f\|_{\ell^\infty(\mathbf{Z},k)}\, \|g\|_{\ell^\infty(\mathbf{Z},k)}$$

for every $f, g \in c_0(\mathbf{Z}, k)$.

If f, g have finite support in $\mathbf{Z}$, then it is easy to see that $f * g$ has finite support in $\mathbf{Z}$. Similarly, if $f, g \in c_0(\mathbf{Z}, \mathbf{C})$, then one can check that $f * g$ vanishes at infinity on $\mathbf{Z}$. Remember that f and g can be approximated uniformly by k-valued functions on $\mathbf{Z}$ with finite support in this case, as in Section 23. One can use this and (59.3) to show that $f * g$ can be approximated uniformly by k-valued functions on $\mathbf{Z}$ with finite support as well, which implies that $f * g$ vanishes at infinity on $\mathbf{Z}$. The same conclusion can also be obtained more directly from (59.2).

As usual, one can verify that $c_0(\mathbf{Z}, k)$ is a commutative algebra over k with respect to convolution. Let $\delta_0(j)$ be the k-valued function on $\mathbf{Z}$ that is equal to 1 when $j = 0$ and to 0 when $j \ne 0$. It is easy to see that δ_0 is the multiplicative identity element in $c_0(\mathbf{Z}, k)$ with respect to convolution, as before.

Let $f, g \in c_0(\mathbf{Z}, k)$ be given, and let us check that

$$(59.4) \qquad \|f * g\|_{\ell^\infty(\mathbf{Z},k)} = \|f\|_{\ell^\infty(\mathbf{Z},k)}\, \|g\|_{\ell^\infty(\mathbf{Z},k)},$$

using a well-known type of argument. Of course, this is trivial when $f = 0$ or $g = 0$, and so we may suppose that $f, g \ne 0$. Remember that

$$(59.5) \qquad \|f\|_{\ell^\infty(\mathbf{Z},k)} = \max_{j \in \mathbf{Z}} |f(j)|, \quad \|g\|_{\ell^\infty(\mathbf{Z},k)} = \max_{j \in \mathbf{Z}} |g(j)|,$$

where the maxima are attained, because f, g vanish at infinity on $\mathbf{Z}$. Hence there are integers $j_1(f)$, $j_1(g)$ such that

$$(59.6) \qquad |f(j_1(f))| = \|f\|_{\ell^\infty(\mathbf{Z},k)}, \quad |g(j_1(g))| = \|g\|_{\ell^\infty(\mathbf{Z},k)}.$$

We can also choose $j_1(f)$, $j_1(g)$ so that

$$|f(j)| < \|f\|_{\ell^\infty(\mathbf{Z},k)} \quad \text{when } j < j_1(f), \tag{59.7}$$

and

$$|g(j)| < \|g\|_{\ell^\infty(\mathbf{Z},k)} \quad \text{when } j < j_1(g). \tag{59.8}$$

We would like to verify that

$$\|f\|_{\ell^\infty(\mathbf{Z},k)} \, \|g\|_{\ell^\infty(\mathbf{Z},k)} \le |(f * g)(j_1(f) + j_1(g))|. \tag{59.9}$$

By the definition (59.1) of $f * g$, we have that

$$(f * g)(j_1(f) + j_1(g)) = \sum_{l=-\infty}^{\infty} f(l) \, g(j_1(f) + j_1(g) - l). \tag{59.10}$$

The $l = j_1(f)$ term in the sum on the right side of (59.10) is equal to

$$f(j_1(f)) \, g(j_1(g)). \tag{59.11}$$

The sum of the terms with $l > j_1(f)$ is equal to

$$\sum_{l=j_1(f)+1}^{\infty} f(l) \, g(j_1(f) + j_1(g) - l) = \sum_{l=1}^{\infty} f(j_1(f) + l) \, g(j_1(g) - l). \tag{59.12}$$

The sum of the terms with $l < j_1(f)$ is equal to

$$\sum_{l=-\infty}^{j_1(f)-1} f(l) \, g(j_1(f) + j_2(g) - l) = \sum_{l=1}^{\infty} f(j_1(f) - l) \, g(j_1(g) + l). \tag{59.13}$$

Using (59.10), we get that

$$\begin{aligned} f(j_1(f)) \, g(j_1(g)) &= (f * g)(j_1(f) + j_1(g)) - \sum_{l=1}^{\infty} f(j_1(f) + l) \, g(j_1(g) - l) \\ &\quad - \sum_{l=1}^{\infty} f(j_1(f) - l) \, g(j_1(g) + l). \end{aligned} \tag{59.14}$$

Of course,

$$|f(j_1(f))| \, |g(j_1(g))| = \|f\|_{\ell^\infty(\mathbf{Z},k)} \, \|g\|_{\ell^\infty(\mathbf{Z},k)}, \tag{59.15}$$

by (59.6). Observe that

$$\begin{aligned} \biggl|\sum_{l=1}^{\infty} f(j_1(f) + l) \, g(j_1(g) - l)\biggr| &\le \max_{l \ge 1}(|f(j_1(f) + l)| \, |g(j_1(g) - l)|) \\ &< \|f\|_{\ell^\infty(\mathbf{Z},k)} \, \|g\|_{\ell^\infty(\mathbf{Z},k)}. \end{aligned} \tag{59.16}$$

This uses (24.10) in the first step, and (59.8) and the definition of $\|f\|_{\ell^\infty(\mathbf{Z},k)}$ in the second step. Similarly,

$$\begin{aligned}\Big|\sum_{l=1}^{\infty} f(j_1(f)-l)\,g(j_1(g)+l)\Big| &\le \max_{l\ge 1}(|f(j_1(f)-l)|\,|g(j_1(g)+l)|)\\ &< \|f\|_{\ell^\infty(\mathbf{Z},k)}\,\|g\|_{\ell^\infty(\mathbf{Z},k)},\end{aligned} \tag{59.17}$$

using (59.7) and the definition of $\|g\|_{\ell^\infty(\mathbf{Z},k)}$ in the second step.

It follows from (59.14) and the ultrametric version of the triangle inequality that

$$\begin{aligned}|f(j_1(f))|\,|g(j_1(g))| \le \max\Big(&|(f*g)(j_1(f)+j_1(g))|,\\ &\Big|\sum_{l=1}^{\infty} f(j_1(f)+l)\,g(j_1(g)-l)\Big|,\\ &\Big|\sum_{l=1}^{\infty} f(j_1(f)-l)\,g(j_1(g)+l)\Big|\Big).\end{aligned} \tag{59.18}$$

One can get (59.9) from this, using (59.15), (59.16), and (59.17). Clearly (59.9) implies that

$$\|f\|_{\ell^\infty(\mathbf{Z},k)}\,\|g\|_{\ell^\infty(\mathbf{Z},k)} \le \|f*g\|_{\ell^\infty(\mathbf{Z},k)}. \tag{59.19}$$

Combining this with (59.3), we get that (59.4) holds, as desired.

60 r-Summability

Let k be a field with an ultrametric absolute value function $|\cdot|$ such that k is complete with respect to the corresponding ultrametric again. Also let r be a positive real number, and let $f, g \in \ell^r(\mathbf{Z}, k)$ be given. As in Section 23, f and g vanish at infinity on $\mathbf{Z}$, so that $f*g$ can be defined on $\mathbf{Z}$ as in the previous section. If $j \in \mathbf{Z}$, then

$$|(f*g)(j)|^r \le \max_{l\in\mathbf{Z}}(|f(l)|^r\,|g(j-l)|^r) \le \sum_{l=-\infty}^{\infty} |f(l)|^r\,|g(j-l)|^r, \tag{60.1}$$

using (59.2) in the first step. The right side of (60.1) is the same as the convolution $(|f|^r * |g|^r)(j)$ of $|f|^r$ and $|g|^r$, as nonnegative real-valued summable functions on $\mathbf{Z}$.

As in Section 57,

$$\sum_{j=-\infty}^{\infty}\Big(\sum_{l=-\infty}^{\infty}|f(l)|^r\,|g(j-l)|^r\Big) \le \Big(\sum_{l=-\infty}^{\infty}|f(l)|^r\Big)\Big(\sum_{j=-\infty}^{\infty}|g(j)|^r\Big). \tag{60.2}$$

Combining this with (60.1), we get that

$$\sum_{j=-\infty}^{\infty}|(f*g)(j)|^r \le \Big(\sum_{l=-\infty}^{\infty}|f(l)|^r\Big)\Big(\sum_{j=-\infty}^{\infty}|g(j)|^r\Big). \tag{60.3}$$

This implies that $f * g$ is r-summable on $\mathbf{Z}$, with

$$\|f * g\|_{\ell^r(\mathbf{Z},k)} \le \|f\|_{\ell^r(\mathbf{Z},k)} \, \|g\|_{\ell^r(\mathbf{Z},k)}. \tag{60.4}$$

It follows that $\ell^r(\mathbf{Z}, k)$ is a subalgebra of $c_0(\mathbf{Z}, k)$ with respect to convolution.

If $f \in c_0(\mathbf{Z}, k)$ and $n \in \mathbf{Z}_+$, then we let f^{*n} be the nth power of f with respect to convolution, as before. Note that

$$\|f^{*n}\|_{\ell^\infty(\mathbf{Z},k)} = \|f\|^n_{\ell^\infty(\mathbf{Z},k)} \tag{60.5}$$

for every $f \in c_0(\mathbf{Z}, k)$ and $n \in \mathbf{Z}_+$, because of (59.4). Let a positive real number r and $f \in \ell^r(\mathbf{Z}, k)$ be given again, and remember that

$$\lim_{n\to\infty} \|f^{*n}\|^{1/n}_{\ell^r(\mathbf{Z},k)} \tag{60.6}$$

exists in $\mathbf{R}$, as in Section 31. Of course,

$$\|f\|_{\ell^\infty(\mathbf{Z},k)} = \|f^{*n}\|^{1/n}_{\ell^\infty(\mathbf{Z},k)} \le \|f^{*n}\|^{1/n}_{\ell^r(\mathbf{Z},k)} \tag{60.7}$$

for every $n \in \mathbf{Z}_+$, using (60.5) in the first step, and (22.5) in the second step. Hence

$$\|f\|_{\ell^\infty(\mathbf{Z},k)} \le \lim_{n\to\infty} \|f^{*n}\|^{1/n}_{\ell^r(\mathbf{Z},k)}. \tag{60.8}$$

As in Sections 31 and 33, (60.6) defines a semi-ultranorm on $\ell^r(\mathbf{Z}, k)$, and

$$\lim_{n\to\infty} \|f^{*n}\|^{1/n}_{\ell^r(\mathbf{Z},k)} \le \|f\|_{\ell^r(\mathbf{Z},k)} \tag{60.9}$$

for every $f \in \ell^r(\mathbf{Z}, k)$. Using this, one can check that

$$\lim_{n\to\infty} \|f^{*n}\|^{1/n}_{\ell^r(\mathbf{Z},k)} \le \|f\|_{\ell^\infty(\mathbf{Z},k)} \tag{60.10}$$

for every $f \in \ell^r(\mathbf{Z}, k)$. More precisely, if the support of f has at most one element, then (60.10) follows from (60.9), and (60.10) can be verified directly in this case anyway. If f has finite support in $\mathbf{Z}$, then (60.10) can be derived from the previous case, using the fact that (60.6) is a semi-ultranorm on $\ell^r(\mathbf{Z}, k)$. Alternatively, one can use an argument like the one in Section 58 when f has finite support in $\mathbf{Z}$. If f is any element of $\ell^r(\mathbf{Z}, k)$, then one can get (60.10) by approximating f by functions on $\mathbf{Z}$ with finite support. Combining (60.8) and (60.10), we get that

$$\lim_{n\to\infty} \|f^{*n}\|^{1/n}_{\ell^r(\mathbf{Z},k)} = \|f\|_{\ell^\infty(\mathbf{Z},k)} \tag{60.11}$$

for every $f \in \ell^r(\mathbf{Z}, k)$.

61 Other radii

Let r be a positive real number, and put

$$w_r(j) = r^j \tag{61.1}$$

for every $j \in \mathbf{Z}$. This is a positive real-valued function on $\mathbf{Z}$, which can be used to define weighted ℓ^{r_0} and c_0 spaces on $\mathbf{Z}$, as in Section 47. Of course, if $r = 1$, then $w_r(j) = 1$ for every $j \in \mathbf{Z}$.

In particular, let $f(j)$ be a complex-valued function on $\mathbf{Z}$ such that $f(j)\, r^j$ is summable on $\mathbf{Z}$, which is to say that $f \in \ell^1_{w_r}(\mathbf{Z}, \mathbf{C})$. If $z \in \mathbf{C}$ and $|z| = r$, then it follows that $f(j)\, z^j$ is summable as a complex-valued function on $\mathbf{Z}$. Put

$$\widehat{f}(z) = \sum_{j=-\infty}^{\infty} f(j)\, z^j, \tag{61.2}$$

where the sum on the right side may be considered as a sum of a summable function on $\mathbf{Z}$, or as a sum of two absolutely convergent series of complex numbers. Thus $\widehat{f}$ is defined as a complex-valued function on the circle in $\mathbf{C}$ centered at 0 with radius r. This may be considered as the *Fourier transform* of f in this situation.

Let $f(j)$ be any complex-valued function on $\mathbf{Z}$, and put

$$(M_{w_r}(f))(j) = f(j)\, w_r(j) = f(j)\, r^j \tag{61.3}$$

for each $j \in \mathbf{Z}$. This defines $M_{w_r}(f)$ as a complex-valued function on $\mathbf{Z}$. As in Section 47, M_{w_r} defines a one-to-one linear mapping from the space $c(\mathbf{Z}, \mathbf{C})$ of complex-valued functions on $\mathbf{Z}$ onto itself, which is the multiplication operator associated to w_r. It is easy to see that M_{w_r} defines an isometric linear mapping from $\ell^1_{w_r}(\mathbf{Z}, \mathbf{C})$ onto $\ell^1(\mathbf{Z}, \mathbf{C})$, as before. Let $f \in \ell^1_{w_r}(\mathbf{Z}, \mathbf{C})$ be given, so that $M_{w_r}(f) \in \ell^1(\mathbf{Z}, \mathbf{C})$. Thus the Fourier transform $(\widehat{M_{w_r}(f)})$ is defined as a complex-valued function on $\mathbf{T}$, as in Section 56. If $z \in \mathbf{T}$, then

$$(\widehat{M_{w_r}(f)})(z) = \sum_{j=-\infty}^{\infty} (M_{w_r}(f))(j)\, z^j = \sum_{j=-\infty}^{\infty} f(j)\, r^j\, z^j = \widehat{f}(r\, z), \tag{61.4}$$

where $\widehat{f}(r\, z)$ is defined as in (61.2).

Let $f, g \in \ell^1_{w_r}(\mathbf{Z}, \mathbf{C})$ be given, so that $M_{w_r}(f), M_{w_r}(g) \in \ell^1(\mathbf{Z}, \mathbf{C})$. If $j, l \in \mathbf{Z}$, then

$$\begin{aligned} f(l)\, g(j-l) &= r^{-j}\, (f(l)\, r^l)\, (g(j-l)\, r^{j-l}) \\ &= r^{-j}\, (M_{w_r}(f))(l)\, (M_{w_r}(g))(j-l). \end{aligned} \tag{61.5}$$

As in Section 57, (61.5) is summable as a function of l on $\mathbf{Z}$ for every $j \in \mathbf{Z}$. Thus

$$(f * g)(j) = \sum_{l=-\infty}^{\infty} f(l)\, g(j-l) \tag{61.6}$$

can be defined as a complex number for each $j \in \mathbf{Z}$. This defines the convolution $f * g$ of f and g as a complex-valued function on $\mathbf{Z}$. Using (61.5), we get that

$$(61.7) \qquad (f * g)(j)\, r^j = \sum_{l=-\infty}^{\infty} (M_{w_r}(f))(l)\, (M_{w_r}(g))(j-l) = ((M_{w_r}(f)) * (M_{w_r}(g)))(j)$$

for every $j \in \mathbf{Z}$, where the convolution on the right side is defined as in Section 57. This implies that

$$(61.8) \qquad M_{w_r}(f * g) = (M_{w_r}(f)) * (M_{w_r}(g))$$

as functions on $\mathbf{Z}$. Note that the right side of (61.8) is summable on $\mathbf{Z}$, as in Section 57. It follows that $f * g \in \ell^1_{w_r}(\mathbf{Z}, \mathbf{C})$, with

$$(61.9) \qquad \begin{aligned} \|f * g\|_{\ell^1_{w_r}(\mathbf{Z},\mathbf{C})} &= \|M_{w_r}(f * g)\|_{\ell^1(\mathbf{Z},\mathbf{C})} \\ &= \|(M_{w_r}(f)) * (M_{w_r}(g))\|_{\ell^1(\mathbf{Z},\mathbf{C})} \\ &\le \|M_{w_r}(f)\|_{\ell^1(\mathbf{Z},\mathbf{C})}\, \|M_{w_r}(g)\|_{\ell^1(\mathbf{Z},\mathbf{C})} \\ &= \|f\|_{\ell^1_{w_r}(\mathbf{Z},\mathbf{C})}\, \|g\|_{\ell^1_{w_r}(\mathbf{Z},\mathbf{C})}, \end{aligned}$$

using (61.8) in the second step, and (57.5) in the third step.

As in Section 47, M_{w_r} is an isometric linear mapping from $\ell^{r_0}_{w_r}(\mathbf{Z}, \mathbf{C})$ onto $\ell^{r_0}(\mathbf{Z}, \mathbf{C})$ for every $r_0 > 0$. One can use this and the previous remarks to extend other properties of the Fourier transform and convolutions to this setting.

62 Ultrametric absolute values

Let r be a positive real number again, and let w_r be defined on $\mathbf{Z}$ as in (61.1). Also let k be a field with an ultrametric absolute value function $|\cdot|$, and suppose that k is complete with respect to the ultrametric associated to $|\cdot|$. As in Section 47, $c_{0,w_r}(\mathbf{Z}, k)$ denotes the space of k-valued functions f on $\mathbf{Z}$ such that

$$(62.1) \qquad |f(j)|\, w_r(j) = |f(j)|\, r^j$$

vanishes at infinity on $\mathbf{Z}$, as a real-valued function on $\mathbf{Z}$. Similarly, $\ell^\infty_{w_r}(\mathbf{Z}, k)$ denotes the space of k-valued functions f on $\mathbf{Z}$ such that (62.1) is bounded on $\mathbf{Z}$. In this case, $\|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}$ is defined to be the supremum of (62.1) over $j \in \mathbf{Z}$, which defines an ultranorm on $\ell^\infty_{w_r}(\mathbf{Z}, k)$ as a vector space over k.

Let $f, g \in c_{0,w_r}(\mathbf{Z}, k)$ be given, and observe that

$$(62.2) \qquad |f(l)|\, |g(j-l)| = r^{-j}\, (|f(l)|\, r^l)\, (|g(j-l)|\, r^{j-l})$$

for every $j, l \in \mathbf{Z}$. Using this, it is easy to see that

$$(62.3) \qquad f(l)\, g(j-l)$$

vanishes at infinity as a k-valued function of l on $\mathbf{Z}$ for each $j \in \mathbf{Z}$. Put

$$(f * g)(j) = \sum_{l=-\infty}^{\infty} f(l)\, g(j-l) \tag{62.4}$$

for every $j \in \mathbf{Z}$, as usual. The sum on the right may be treated as a sum over $\mathbf{Z}$, as in Section 25, or as a sum of two convergent infinite series, as in Section 24. This defines the convolution $f * g$ of f and g as a k-valued function on $\mathbf{Z}$.

As in Sections 24 and 25,

$$|(f * g)(j)| \le \max_{l \in \mathbf{Z}}(|f(l)|\, |g(j-l)|) \tag{62.5}$$

for every $j \in \mathbf{Z}$. This implies that

$$|(f * g)(j)|\, r^j \le \max_{l \in \mathbf{Z}}((|f(l)|\, r^l)\, (|g(j-l)|\, r^{j-l})) \tag{62.6}$$

for each $j \in \mathbf{Z}$, because of (62.2). In particular, it follows that $f * g \in \ell^\infty_{w_r}(\mathbf{Z}, k)$, with

$$\|f * g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)} \le \|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}\, \|g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}. \tag{62.7}$$

If $f, g \in c_{00}(\mathbf{Z}, k)$, then $f * g \in c_{00}(\mathbf{Z}, k)$, as before. If $f, g \in c_{0,w_r}(\mathbf{Z}, k)$, then one can verify that $f * g \in c_{0,w_r}(\mathbf{Z}, k)$, by approximating f and g by elements of $c_{00}(\mathbf{Z}, k)$. This also uses (62.7) to get that $f*g$ can be approximated by functions with finite support in $\mathbf{Z}$ with respect to the $\ell^\infty_{w_r}$ ultranorm. Alternatively, the same conclusion can be obtained from (62.6).

One can check that $c_{0,w_r}(\mathbf{Z}, k)$ is a commutative algebra over k with respect to convolution. The k-valued function $\delta_0(j)$ on $\mathbf{Z}$ equal to 1 when $j = 0$ and to 0 when $j \ne 0$ is the multiplicative identity element in $c_{0,w_r}(\mathbf{Z}, k)$ with respect to convolution, as before.

Let $f, g \in c_{0,w_r}(\mathbf{Z}, k)$ be given, and let us verify that

$$\|f * g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)} = \|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}\, \|g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}, \tag{62.8}$$

as in Section 59. We may suppose that $f, g \ne 0$, since (62.8) is trivial otherwise. Note that

$$\|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)} = \max_{j \in \mathbf{Z}}(|f(j)|\, r^j), \quad \|g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)} = \max_{j \in \mathbf{Z}}(|g(j)|\, r^j), \tag{62.9}$$

where the maxima are attained in this situation. Thus there are integers $j_r(f)$, $j_r(g)$ such that

$$|f(j_r(f))|\, r^{j_r(f)} = \|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}, \quad |g(j_r(g))|\, r^{j_r(g)} = \|g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}. \tag{62.10}$$

As before, we can also choose $j_r(f)$, $j_r(g)$ so that

$$|f(j)|\, r^j < \|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)} \quad \text{when } j < j_r(f) \tag{62.11}$$

and

$$|g(j)|\, r^j < \|g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)} \quad \text{when } j < j_r(g). \tag{62.12}$$

We would like to check that

$$\|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}\, \|g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)} \le |(f*g)(j_r(f)+j_r(f))|\, r^{j_r(f)+j_r(g)}. \tag{62.13}$$

As in (59.14), we have that

$$\begin{aligned} f(j_r(f))\, g(j_r(g)) &= (f*g)(j_r(f)+j_r(g)) - \sum_{l=1}^\infty f(j_r(f)+l)\, g(j_r(g)-l) \\ &\quad - \sum_{l=1}^\infty f(j_r(f)-l)\, g(j_r(g)+l). \end{aligned} \tag{62.14}$$

Observe that

$$|f(j_r(f))|\, |g(j_r(g))|\, r^{j_r(f)+j_r(g)} = \|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}\, \|g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}, \tag{62.15}$$

by (62.10).

Using (24.10), we get that

$$\begin{aligned} &\left|\sum_{l=1}^\infty f(j_r(f)+l)\, g(j_r(g)-l)\right| r^{j_r(f)+j_r(g)} \\ &\quad \le \Big(\max_{l\ge 1}(|f(j_r(f)+l)|\, |g(j_r(g)-l)|)\Big)\, r^{j_r(f)+j_r(g)} \\ &\quad = \max_{l\ge 1}((|f(j_r(f)+l)|\, r^{j_r(f)+l})\, (|g(j_r(g)-l)|\, r^{j_r(g)-l})). \end{aligned} \tag{62.16}$$

Similarly,

$$\begin{aligned} &\left|\sum_{l=1}^\infty f(j_r(f)-l)\, g(j_r(g)+l)\right| r^{j_r(f)+j_r(g)} \\ &\quad \le \Big(\max_{l\ge 1}(|f(j_r(f)-l)|\, |g(j_r(g)+l)|)\Big)\, r^{j_r(f)+j_r(g)} \\ &\quad = \max_{l\ge 1}((|f(j_r(f)-l)|\, r^{j_r(f)-l})\, (|g(j_r(g)+l)|\, r^{j_r(g)+l})). \end{aligned} \tag{62.17}$$

One can verify that the right sides of (62.16) and (62.17) are both strictly less than

$$\|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}\, \|g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}, \tag{62.18}$$

using (62.11), (62.12), and the definition of the $\ell^\infty_{w_r}$ ultranorm.

As before,

$$\begin{aligned} |f(j_r(j))|\, |g(j_r(g))| \le \max\Big(&|(f*g)(j_r(f)+j_r(g))|, \\ &\left|\sum_{l=1}^\infty f(j_r(f)+l)\, g(j_r(g)-l)\right|, \\ &\left|\sum_{l=1}^\infty f(j_r(f)-l)\, g(j_r(g)+l)\right|\Big), \end{aligned} \tag{62.19}$$

by (62.14) and the ultrametric version of the triangle inequality. Multiplying both sides by $r^{j_r(f)+j_r(g)}$, we get that

$$
\begin{aligned}
(62.20) \qquad & |f(j_r(f))|\,|g(j_r(g))|\,r^{j_r(f)+j_r(g)} \\
&\le \max\Big(|(f*g)(j_r(f)+j_r(g))|\,r^{j_r(f)+j_r(g)}, \\
&\qquad \Big|\sum_{l=1}^{\infty} f(j_r(f)+l)\,g(j_r(g)-l)\Big|\,r^{j_r(f)+j_r(g)}, \\
&\qquad \Big|\sum_{l=1}^{\infty} f(j_r(f)-l)\,g(j_r(g)+l)\Big|\,r^{j_r(f)+j_r(g)}\Big).
\end{aligned}
$$

The second and third expressions in the maximum on the right side of (62.20) are the same as the left sides of (62.16) and (62.17), respectively. Hence the second and third expressions in the maximum on the right side of (62.20) are strictly less than (62.18), as in the preceding paragraph. One can use this to get (62.13), because of (62.15).

Of course, (62.13) implies that

$$
(62.21) \qquad \|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}\,\|g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)} \le \|f*g\|_{\ell^\infty_{w_r}(\mathbf{Z},k)}.
$$

This and (62.7) yield (62.8), as desired.

63 r_0-Summability

As before, we let r be a positive real number, and w_r be defined on $\mathbf{Z}$ as in (61.1). We also let k be a field with an ultrametric absolute value function $|\cdot|$ such that k is complete with respect to the associated ultrametric. Remember that $\ell^{r_0}_{w_r}(\mathbf{Z},k)$ is defined as in Section 47 for every $r_0>0$, as well as $c_{0,w_r}(\mathbf{Z},k)$. Let r_0 be a positive real number, and let $f,g\in\ell^{r_0}_{w_r}(\mathbf{Z},k)$ be given. In particular, $f,g\in c_{0,w_r}(\mathbf{Z},k)$, so that $f*g$ can be defined on $\mathbf{Z}$ as in the preceding section. Using (62.6), we get that

$$
(63.1) \qquad (|(f*g)(j)|\,r^j)^{r_0} \le \max_{l\in\mathbf{Z}}((|f(l)|\,r^l)^{r_0}\,(|g(j-l)|\,r^{j-l})^{r_0})
$$

for every $j\in\mathbf{Z}$. It follows that

$$
(63.2) \qquad (|(f*g)(j)|\,r^j)^{r_0} \le \sum_{l=-\infty}^{\infty}(|f(l)|\,r^l)^{r_0}\,(|g(j-l)|\,r^{j-l})^{r_0}
$$

for every $j\in\mathbf{Z}$.

Observe that $(|f(j)|\,r^j)^{r_0}$ and $(|g(j)|\,r^j)^{r_0}$ are summable as nonnegative real-valued functions of j on $\mathbf{Z}$, because $f,g\in\ell^{r_0}_{w_r}(\mathbf{Z},k)$. The right side of (63.2) is the same as the convolution of these two functions, as summable real-valued functions on $\mathbf{Z}$. Hence

$$\sum_{j=-\infty}^{\infty} \Big(\sum_{l=-\infty}^{\infty} (|f(l)|\, r^l)^{r_0}\, (|g(j-l)|\, r^{j-l})^{r_0} \Big) \tag{63.3}$$
$$\leq \Big(\sum_{l=-\infty}^{\infty} (|f(l)|\, r^l)^{r_0} \Big) \Big(\sum_{j=-\infty}^{\infty} (|g(j)|\, r^j)^{r_0} \Big),$$

as in Section 57. This implies that

$$\sum_{j=-\infty}^{\infty} (|(f*g)(j)|\, r^j)^{r_0} \leq \Big(\sum_{l=-\infty}^{\infty} (|f(l)|\, r^l)^{r_0} \Big) \Big(\sum_{j=-\infty}^{\infty} (|g(j)|\, r^j)^{r_0} \Big), \tag{63.4}$$

by (63.2). Thus $f * g \in \ell^{r_0}_{w_r}(\mathbf{Z}, k)$, with

$$\|f * g\|_{\ell^{r_0}_{w_r}(\mathbf{Z},k)} \leq \|f\|_{\ell^{r_0}_{w_r}(\mathbf{Z},k)}\, \|g\|_{\ell^{r_0}_{w_r}(\mathbf{Z},k)}. \tag{63.5}$$

In particular, $\ell^{r_0}_{w_r}(\mathbf{Z}, k)$ is a subalgebra of $c_{0,w_r}(\mathbf{Z}, k)$ with respect to convolution. As usual, if $f \in c_{0,w_r}(\mathbf{Z}, k)$ and $n \in \mathbf{Z}_+$, then f^{*n} denotes the nth power of f with respect to convolution. Using (62.8), we have that

$$\|f^{*n}\|_{\ell^{\infty}_{w_r}(\mathbf{Z},k)} = \|f\|^n_{\ell^{\infty}_{w_r}(\mathbf{Z},k)} \tag{63.6}$$

for every $f \in c_{0,w_r}(\mathbf{Z}, k)$ and $n \in \mathbf{Z}_+$. Let $0 < r_0 < \infty$ and $f \in \ell^{r_0}_{w_r}(\mathbf{Z}, k)$ be given again, and remember that

$$\lim_{n \to \infty} \|f^{*n}\|^{1/n}_{\ell^{r_0}_{w_r}(\mathbf{Z},k)} \tag{63.7}$$

exists, as in Section 31. We also have that

$$\|f\|_{\ell^{\infty}_{w_r}(\mathbf{Z},k)} = \|f^{*n}\|^{1/n}_{\ell^{\infty}_{w_r}(\mathbf{Z},k)} \leq \|f^{*n}\|^{1/n}_{\ell^{r_0}_{w_r}(\mathbf{Z},k)} \tag{63.8}$$

for every $n \in \mathbf{Z}_+$, by (63.6) and (47.7), so that

$$\|f\|_{\ell^{\infty}_{w_r}(\mathbf{Z},k)} \leq \lim_{n \to \infty} \|f^{*n}\|^{1/n}_{\ell^{r_0}_{w_r}(\mathbf{Z},k)}. \tag{63.9}$$

Remember that (63.7) defines a semi-ultranorm on $\ell^{r_0}_{w_r}(\mathbf{Z}, k)$, and

$$\lim_{n \to \infty} \|f^{*n}\|^{1/n}_{\ell^{r_0}_{w_r}(\mathbf{Z},k)} \leq \|f\|_{\ell^{r_0}_{w_r}(\mathbf{Z},k)} \tag{63.10}$$

for every $f \in \ell^{r_0}_{w_r}(\mathbf{Z}, k)$, as in Sections 31 and 33. As before, one can use this to check that

$$\lim_{n \to \infty} \|f^{*n}\|^{1/n}_{\ell^{r_0}_{w_r}(\mathbf{Z},k)} \leq \|f\|_{\ell^{\infty}_{w_r}(\mathbf{Z},k)} \tag{63.11}$$

for every $f \in \ell^{r_0}_{w_r}(\mathbf{Z}, k)$. This is easy to see when the support of f has at most one element. If f has finite support in $\mathbf{Z}$, then one can get (63.11) from the previous case and the semi-ultranorm property of (63.7). One can also use an argument like the one in Section 58 in this situation. If f is an arbitrary element of $\ell^{r_0}_{w_r}(\mathbf{Z}, k)$, then one can approximate f by functions on $\mathbf{Z}$ with finite support to get (63.11). It follows that

$$\lim_{n \to \infty} \|f^{*n}\|^{1/n}_{\ell^{r_0}_{w_r}(\mathbf{Z},k)} = \|f\|_{\ell^\infty_{w_r}(\mathbf{Z},k)} \tag{63.12}$$

for every $f \in \ell^{r_0}_{w_r}(\mathbf{Z}, k)$, by (63.9) and (63.11).

Let $x \in k$ with $x \neq 0$ be given, and put

$$a(x) = x^j \tag{63.13}$$

for each $j \in \mathbf{Z}$. This defines a k-valued function on $\mathbf{Z}$, with

$$|a(j)| = |x|^j \tag{63.14}$$

for each $j \in \mathbf{Z}$. If $f(j)$ is any k-valued function on $\mathbf{Z}$, then we put

$$(M_a(f))(j) = a(j)\, f(j) = f(j)\, x^j \tag{63.15}$$

for every $j \in \mathbf{Z}$. This defines a one-to-one linear mapping M_a from the space $c(\mathbf{Z}, k)$ of k-valued functions on $\mathbf{Z}$ onto itself, as in Section 47. Let $w_{r/|x|}$ be defined on $\mathbf{Z}$ as in (61.1), so that

$$w_{r/|x|}(j) = (r/|x|)^j = r^j\, |x|^{-j} \tag{63.16}$$

for every $j \in \mathbf{Z}$. As in Section 47, M_a defines an isometric linear mapping from $\ell^{r_0}_{w_r}(\mathbf{Z}, k)$ onto $\ell^{r_0}_{w_{r/|x|}}(\mathbf{Z}, k)$ for every $r_0 > 0$. Similarly, M_a maps $c_{0,w_r}(\mathbf{Z}, k)$ onto $c_{0,w_{r/|x|}}(\mathbf{Z}, k)$. Let $f, g \in c_{0,w_r}(\mathbf{Z}, k)$ be given, so that $M_a(f)$, $M_a(g)$ are elements of $c_{0,w_{r/|x|}}(\mathbf{Z}, k)$. Observe that

$$\begin{aligned} f(l)\, g(j-l)\, x^j &= (f(l)\, x^l)\, (g(j-l)\, x^{j-l}) \\ &= (M_a(f))(l)\, (M_a(g))(j-l) \end{aligned} \tag{63.17}$$

for every $j, l \in \mathbf{Z}$. This implies that

$$\begin{aligned} (f * g)(j)\, x^j &= \sum_{l=-\infty}^{\infty} (M_a(f))(l)\, (M_a(g))(j-l) \\ &= ((M_a(f)) * (M_a(g)))(j) \end{aligned} \tag{63.18}$$

for every $j \in \mathbf{Z}$, so that

$$M_a(f * g) = (M_a(f)) * (M_a(g)) \tag{63.19}$$

as functions on $\mathbf{Z}$.

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