Some topics in analysis related to Banach algebras, 3

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Abstract

Some spaces of linear mappings and matrices are discussed, as well as some properties of functions defined by power series.

Contents

Part I
Basic notions

1 Metrics and ultrametrics

Let X be a set, and let $d(x,y)$ be a nonnegative real-valued function $d(x,y)$ defined for $x, y \in X$. As usual, $d(\cdot,\cdot)$ is said to be a *metric* on X if it satisfies the following three conditions: first,

$$d(x,y) = 0 \quad \text{if and only if} \quad x = y; \tag{1.1}$$

second,

$$d(x,y) = d(y,x) \tag{1.2}$$

for every $x, y \in X$; and third,

$$d(x,z) \le d(x,y) + d(y,z) \tag{1.3}$$

for every $x, y, z \in X$. If $d(\cdot,\cdot)$ satisfies (1.1), (1.2), and

$$d(x,z) \le \max(d(x,y), d(y,z)) \tag{1.4}$$

for every $x, y, z \in X$, then $d(\cdot,\cdot)$ is said to be an *ultrametric* on X. Note that (1.4) implies (1.3), so that ultrametrics are metrics.

The *discrete metric* can be defined on any set X by putting $d(x,y)$ equal to 1 when $x \ne y$, and to 0 when $x = y$. It is easy to see that this defines an ultrametric on X.

If a is a positive real number with $a \le 1$, then it is well known that

$$(r+t)^a \le r^a + t^a \tag{1.5}$$

for all nonnegative real numbers r, t. To see this, observe first that

$$\max(r,t)^a = \max(r^a, t^a) \le r^a + t^a \tag{1.6}$$

for every $a > 0$, so that

$$\max(r,t) \le (r^a + t^a)^{1/a}. \tag{1.7}$$

If $a \le 1$, then it follows that

$$\begin{aligned} r + t &\le \max(r,t)^{1-a}\,(r^a + t^a) \\ &\le (r^a + t^a)^{(1-a)/a}\,(r^a + t^a)^{1/a} = (r^a + t^a)^{1/a}, \end{aligned} \tag{1.8}$$

which implies (1.5).

Let $d(x, y)$ be a metric on a set X, and let a be a positive real number. If $a \le 1$, then it is easy to see that

$$d(x,y)^a \tag{1.9}$$

also defines a metric on X, using (1.5). If $d(x, y)$ is an ultrametric on X, then (1.9) is an ultrametric on X for every $a > 0$.

Let $d(x, y)$ be a metric on a set X again. As usual, the *open ball* in X centered at a point $x \in X$ with radius $r > 0$ with respect to $d(\cdot,\cdot)$ is defined by

$$B(x,r) = B_d(x,r) = \{y \in X : d(x,y) < r\}. \tag{1.10}$$

Similarly,

$$\overline{B}(x,r) = \overline{B}_d(x,r) = \{y \in X : d(x,y) \le r\} \tag{1.11}$$

is the *closed ball* in X centered at $x \in X$ with radius $r \ge 0$ with respect to $d(\cdot,\cdot)$. Remember that one can define a topology on X using $d(\cdot,\cdot)$ in a standard way. It is well known that open balls in X with respect to $d(\cdot,\cdot)$ are open sets with respect to this topology. More precisely, the collection of open balls in X with respect to $d(\cdot,\cdot)$ forms a base for this topology on X. One can also verify that closed balls in X with respect to $d(\cdot,\cdot)$ are closed sets with respect to this topology.

If (1.9) is a metric on X as well for some $a > 0$, then the corresponding open and closed balls in X are given by

$$B_{d^a}(x, r^a) = B_d(x,r) \tag{1.12}$$

for every $x \in X$ and $r > 0$, and

$$\overline{B}_{d^a}(x, r^a) = \overline{B}_d(x,r) \tag{1.13}$$

for every $x \in X$ and $r \geq 0$, respectively. It follows from (1.12) that the topologies determined on X by $d(\cdot, \cdot)$ and (1.9) are the same.

Suppose for the moment that $d(x, y)$ is an ultrametric on X. If $x, y \in X$ satisfy $d(x, y) < r$ for some $r > 0$, then one can check that

$$B(x, r) \subseteq B(y, r), \tag{1.14}$$

using the ultrametric version of the triangle inequality. This is symmetric in x and y, so that

$$B(x, r) = B(y, r) \tag{1.15}$$

when $d(x, y) < r$. Similarly, if $d(x, y) \leq r$ for some $r \geq 0$, then

$$\overline{B}(x, r) \subseteq \overline{B}(y, r), \tag{1.16}$$

and hence

$$\overline{B}(x, r) = \overline{B}(y, r). \tag{1.17}$$

In particular, this implies that closed balls in X with positive radius are open sets with respect to the topology determined by $d(\cdot, \cdot)$. One can verify that open balls in X are closed sets too. Of course, the topology determined on X by the discrete metric is the discrete topology.

2 Absolute value functions

Let k be a field. A nonnegative real-valued function $|x|$ defined on k is said to be an *absolute value function* on k if it satisfies the following three conditions: first, for each $x \in k$,

$$|x| = 0 \quad \text{if and only if} \quad x = 0; \tag{2.1}$$

second,

$$|x\,y| = |x|\,|y| \tag{2.2}$$

for every $x, y \in k$; and third,

$$|x + y| \leq |x| + |y| \tag{2.3}$$

for every $x, y \in k$. Of course, the standard absolute value functions on the fields $\mathbf{R}$ of real numbers and $\mathbf{C}$ of complex numbers are absolute value functions in this sense.

Let k be a field, and let $|\cdot|$ be a nonnegative real-valued function on k that satisfies (2.1) and (2.2). It is easy to see that $|1| = 1$, where the first 1 is the multiplicative identity element in k, and the second 1 is the multiplicative identity element in $\mathbf{R}$. This uses the facts that $|1| > 0$ by (2.1), and $|1| = |1^2| = |1|^2$, by (2.2). Similarly, if $x \in k$ satisfies $x^n = 1$ for some positive integer n, then $|x| = 1$. It follows that $|-1| = 1$, where -1 is the additive inverse of 1 in k, because $(-1)^2 = 1$. If $|\cdot|$ is an absolute value function on k, then

$$d(x, y) = |x - y| \tag{2.4}$$

defines a metric on k. More precisely, (2.4) is symmetric in x and y, because $|-1| = 1$.

A nonnegative real-valued function $|\cdot|$ on a field k is said to be an *ultrametric absolute value function* on k if it satisfies (2.1), (2.2), and

$$|x + y| \le \max(|x|, |y|) \tag{2.5}$$

for every $x, y \in k$. Clearly (2.5) implies (2.3), so that an ultrametric absolute value function is an ordinary absolute value function. If $|\cdot|$ is an ultrametric absolute value function on k, then (2.4) is an ultrametric on k.

If k is any field, then the *trivial absolute value function* is defined on k by putting $|x| = 1$ for every $x \in k$ with $x \ne 0$, and $|0| = 0$. It is easy to see that this defines an ultrametric absolute value function on k. The ultrametric on k associated to the trivial absolute value function as in (2.4) is the discrete metric on k.

The *p-adic absolute value function* $|x|_p$ is defined on the field $\mathbf{Q}$ of rational numbers for each prime number p as follows. Let $x \in \mathbf{Q}$ be given, and suppose that $x \ne 0$, since we should put $|0|_p = 0$. We can express x as $p^j\,(a/b)$, where a, b, and j are integers, $a, b \ne 0$, and neither a nor b is a multiple of p. Note that j is uniquely determined by x, and put

$$|x|_p = p^{-j}. \tag{2.6}$$

It is not difficult to verify that this defines an ultrametric absolute value function on $\mathbf{Q}$.

Let k be any field again, and let $|\cdot|$ be an absolute value function on k. If a is a positive real number less than or equal to 1, then

$$|x|^a \tag{2.7}$$

also defines an absolute value function on k. This uses (1.5) to get that (2.7) satisfies the triangle inequality on k. If $|\cdot|$ is an ultrametric absolute value function on k, then (2.7) is an ultrametric absolute value function on k for every $a > 0$. If $|\cdot|$ is the standard Euclidean absolute value function on $\mathbf{Q}$, then (2.7) does not satisfy the triangle inequality on $\mathbf{Q}$ when $a > 1$.

Let $|\cdot|_1$, $|\cdot|_2$ be absolute value functions on a field k. If there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{2.8}$$

for every $x \in k$, then $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* on k. This implies that the topologies determined on k by the metrics associated to $|\cdot|_1$ and $|\cdot|_2$ are the same. Conversely, if the topologies determined on k by the metrics associated to $|\cdot|_1$ and $|\cdot|_2$ are the same, then it is well known that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k in the sense of (2.8). A famous theorem of Ostrowski says that any absolute value function on $\mathbf{Q}$ is either trivial, or equivalent to the standard Euclidean absolute value function, or equivalent to the p-adic absolute value function for some prime number p.

3 Uniform continuity and completeness

Let $(X, d(x,y))$ be a metric space, let a be a positive real number, and suppose that $d(x,y)^a$ also defines a metric on X. As in Section 1, the topologies determined on X by $d(x,y)$ and $d(x,y)^a$ are the same. More precisely, the identity mapping on X is uniformly continuous as a mapping from X equipped with $d(x,y)$ into X equipped with $d(x,y)^a$, and from X equipped with $d(x,y)^a$ into X equipped with $d(x,y)$. Note that a sequence of elements of X is a Cauchy sequence with respect to $d(x,y)$ if and only if it is a Cauchy sequence with respect to $d(x,y)^a$. It follows that X is complete as a metric space with respect to $d(x,y)$ if and only if X is complete with respect to $d(x,y)^a$.

Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose that d_X^a, d_Y^b are also metrics on X, Y, respectively, for some positive real numbers a, b. If a mapping f from X into Y is uniformly continuous with respect to d_X, d_Y, respectively, then f is uniformly continuous with respect to d_X^a, d_Y^b, respectively. This can be verified directly, or using the remark about uniform continuity in the previous paragraph, and the fact that compositions of uniformly continuous mappings are uniformly continuous. Remember that uniformly continuous mappings send Cauchy sequences in the domain to Cauchy sequences in the range, by a simple argument.

Let (X, d_X) and (Y, d_Y) be metric spaces again, let E be a dense subset of X, and suppose that f is a uniformly continuous mapping from E into Y. If Y is complete, then it is well known that f has a unique extension of a uniformly continuous mapping from X into Y. Of course, uniqueness of the extension only requires ordinary continuity.

If a metric space (X, d) is not already complete, then it is well known that X has a completion, which can be given as an isometric embedding of X onto a dense subset of a complete metric space. The completion is unique up to isometric equivalence, because of the extension theorem mentioned in the previous paragraph. Note that a closed subset of a complete metric space is also complete with respect to the restriction of the metric to the subset, by a standard argument.

Let (X, d) be a metric space, and let E be a subset of X. Of course, the restriction of $d(x,y)$ to $x, y \in E$ defines a metric on E. Similarly, if $d(x,y)$ is an ultrametric on X, then its restriction to E is an ultrametric on E. If $d(x,y)$ is any metric on X, $E \subseteq X$ is dense in X with respect to $d(x,y)$, and the restriction of $d(x,y)$ to $x, y \in E$ is an ultrametric on E, then one can check that $d(x,y)$ is an ultrametric on X. In particular, if $d(x,y)$ is an ultrametric on X, then the corresponding metric on a completion of X is an ultrametric as well.

Let k be a field with an absolute value function $|\cdot|$, and let k_0 be a subfield of k. As before, the restriction of $|x|$ to $x \in k_0$ defines an absolute value function on k_0. If $|x|$ is an ultrametric absolute value function on k, then its restriction to $x \in k_0$ is an ultrametric absolute value function on k_0. Let $|\cdot|$ be any absolute value function on k again, let k_0 be a subfield of k, and suppose that k_0 is dense in k with respect to the metric associated to $|\cdot|$. If the restriction of $|x|$ to $x \in k_0$ is an ultrametric absolute value function on k_0, then one can verify that

$|\cdot|$ is an ultrametric absolute value function on k.

Let k be a field with an absolute value function $|\cdot|$ again, which leads to a metric on k as in (2.4). If k is not already complete with respect to this metric, then one can pass to a completion of k as a metric space, as before. One can show that the field operations on k can be extended continuously to the completion, so that the completion is also a field. Similarly, $|\cdot|$ can be extended continuously to the completion, which corresponds to the distance to 0 in the completion. This extension of $|\cdot|$ defines an absolute value on the completion, and the metric on the completion corresponds to this extension of $|\cdot|$ in the usual way. This completion of k with respect to $|\cdot|$ is unique up to isomorphic isometric equivalence, as before. If $|\cdot|$ is an ultrametric absolute value function on k, then the corresponding absolute value function on the completion is an ultrametric absolute value function as well.

In particular, one can apply this to the p-adic absolute value function $|\cdot|_p$ on $\mathbf{Q}$, for any prime number p. This leads to the field $\mathbf{Q}_p$ of *p-adic numbers*. The extension of $|x|_p$ to $x \in \mathbf{Q}_p$ is denoted the same way, and defines an ultrametric absolute value function on $\mathbf{Q}_p$.

4 The archimedean property and discreteness

Let k be a field, and let $|\cdot|$ be an absolute value function on k. Also let $\mathbf{Z}_+$ be the set of positive integers, as usual. If $x \in k$ and $n \in \mathbf{Z}_+$, then we let $n \cdot x$ be the sum of n x's in k. Suppose that there is an $n \in \mathbf{Z}_+$ such that $|n \cdot 1| > 1$, where the first 1 is the multiplicative identity element in k, and the second 1 is the multiplicative identity element in $\mathbf{R}$. If $j \in \mathbf{Z}_+$, then one can check that $n^j \cdot 1 = (n \cdot 1)^j$, so that

$$|n^j \cdot 1| = |(n \cdot 1)^j| = |n \cdot 1|^j \to \infty \quad \text{as } j \to \infty. \tag{4.1}$$

In this case, $|\cdot|$ is said to be *archimedean* on k. Otherwise, if $|n \cdot 1| \le 1$ for every $n \in \mathbf{Z}_+$, then $|\cdot|$ is said to be *non-archimedean* on k. Equivalently, if there is a finite upper bound for $|n \cdot 1|$, $n \in \mathbf{Z}_+$, then $|\cdot|$ is non-archimedian on k, by the previous remark. If $|\cdot|$ is an ultrametric absolute value function on k, then it is easy to see that $|\cdot|$ is non-archimedean on k. Conversely, if $|\cdot|$ is a non-archimedean absolute value function on k, then it is well known that $|\cdot|$ is an ultrametric absolute value function on k.

Let $|\cdot|$ be an absolute value function on a field k again, and observe that

$$\{|x| : x \in k,\ x \neq 0\} \tag{4.2}$$

is a subgroup of the multiplicative group $\mathbf{R}_+$ of positive real numbers. If the real number 1 is a limit point of (4.2) with respect to the standard topology on $\mathbf{R}$, then one can check that (4.2) is dense in $\mathbf{R}_+$ with respect to the topology induced by the standard topology on $\mathbf{R}$. Otherwise, $|\cdot|$ is said to be *discrete* on k if 1 is not a limit point of (4.2) with respect to the standard topology on $\mathbf{R}$.

Let ρ_1 be the nonnegative real number defined by

$$\rho_1 = \sup\{|x| : x \in k, |x| < 1\}, \tag{4.3}$$

and note that $\rho_1 \le 1$. One can check that $\rho_1 = 0$ if and only if $|\cdot|$ is the trivial absolute value function on k, and that $\rho_1 < 1$ if and only if $|\cdot|$ is discrete on k. If $|\cdot|$ is nontrivial and discrete on k, so that $0 < \rho_1 < 1$, then one can verify that the supremum in (4.3) is attained. More precisely, (4.2) consists of integer powers of ρ_1 in this case.

If k has positive characteristic, then there are only finitely many elements of k of the form $n \cdot 1$, with $n \in \mathbf{Z}_+$. This implies that any absolute value function on k is non-archimedean. Suppose that k has characteristic 0, so that there is a natural embedding of $\mathbf{Q}$ into k. Let $|\cdot|$ be an absolute value function on k, which leads to an absolute value function on $\mathbf{Q}$, using the natural embedding of $\mathbf{Q}$ into k. If $|\cdot|$ is a archimedean on k, then it is easy to see that the corresponding absolute value function on $\mathbf{Q}$ is archimedean. This implies that the corresponding absolute value function on $\mathbf{Q}$ is equivalent to the standard Euclidean absolute value function on $\mathbf{Q}$, by Ostrowski's theorem, as in Section 2. In particular, this means that the corresponding absolute value function on $\mathbf{Q}$ is not discrete on $\mathbf{Q}$. It follows that $|\cdot|$ is not discrete on k when $|\cdot|$ is archimedean on k. If $|\cdot|$ is a discrete absolute value function on a field k, then we get that $|\cdot|$ is non-archimedean on k.

Suppose that $|\cdot|$ is an archimedean absolute value function on a field k, and that k is complete with respect to the metric associated to $|\cdot|$. Under these conditions, another famous theorem of Ostrowski states that k is isomorphic to $\mathbf{R}$ of $\mathbf{C}$, in such a way that $|\cdot|$ corresponds to an absolute value function on $\mathbf{R}$ or $\mathbf{C}$ that is equivalent to the standard one.

5 Norms and ultranorms

Let k be a field, and let $|\cdot|$ be an absolute value function on k. Also let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *seminorm* on V with respect to $|\cdot|$ on k if it satisfies the following two conditions: first,

$$N(t\,v) = |t|\,N(v) \tag{5.1}$$

for every $t \in k$ and $v \in V$; and second,

$$N(v + w) \le N(v) + N(w) \tag{5.2}$$

for every $v, w \in V$. Note that (5.1) implies that $N(0) = 0$, by taking $t = 0$. If N also satisfies $N(v) > 0$ for every $v \in V$ with $v \ne 0$, then N is said to be a *norm* on V with respect to $|\cdot|$ on k.

Similarly, a nonnegative real-valued function N on V is said to be a *semi-ultranorm* with respect to $|\cdot|$ if it satisfies (5.1) and

$$N(v + w) \le \max(N(v), N(w)) \tag{5.3}$$

for every $v, w \in V$. Clearly (5.3) implies (5.2), so that semi-ultranorms are seminorms. A semi-ultranorm that is also a norm is called an *ultranorm.* If N is a semi-ultranorm on V with respect to $|\cdot|$ on k, and if $N(v) > 0$ for some $v \in V$, then it is easy to see that $|\cdot|$ is an ultrametric absolute value function on k. More precisely, one can get the ultrametric version of the triangle inequality (2.5) for $|\cdot|$ on k using (5.1) and (5.3) in this case.

If N is a norm on V with respect to $|\cdot|$ on k, then

$$d(v, w) = N(v - w) \tag{5.4}$$

defines a metric on k. This uses the fact that $|-1| = 1$, as in Section 2, to get that (5.4) is symmetric in v and w. If N is an ultranorm on V, then (5.4) is an ultrametric on V.

Of course, k may be considered as a one-dimensional vector space over itself. Similarly, $|\cdot|$ may be considered as a norm on k, with respect to itself. If $|\cdot|$ is an ultrametric absolute value function on k, then $|\cdot|$ may be considered as an ultranorm on k.

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on a field k. If V is any vector space over k, then the *trivial ultranorm* is defined on V by putting $N(0) = 0$, and $N(v) = 1$ for every $v \in V$ with $v \neq 0$. It is easy to see that this defines an ultranorm on V. The ultrametric on V corresponding to N as in (5.4) is the discrete metric.

Let $|\cdot|$ be any absolute value function on a field k again, and let n be a positive integer. Consider the space k^n of n-tuples of elements of k, which is the Cartesian product of n copies of k. This is a vector space over k, with respect to coordinatewise addition and scalar multiplication, as usual. If $v = (v_1, \ldots, v_n) \in k^n$, then put

$$\|v\|_1 = \sum_{j=1}^{n} |v_j| \tag{5.5}$$

and

$$\|v\|_\infty = \max_{1 \le j \le n} |v_j|. \tag{5.6}$$

One can check that (5.5) and (5.6) define norms on k^n, with respect to $|\cdot|$ on k. If $|\cdot|$ is an ultrametric absolute value function on k, then (5.6) is an ultranorm on k^n. Note that

$$\|v\|_\infty \le \|v\|_1 \le n\,\|v\|_\infty \tag{5.7}$$

for every $v \in k^n$.

Let $v \in k^n$ and a positive real number be given, and consider the open ball in k^n centered at v with radius r with respect to the metric associated to (5.6). This is the same as the Cartesian product of the balls in k centered at v_j with radius r with respect to the metric associated to $|\cdot|$ for $j = 1, \ldots, n$. Using this, it is easy to see that the topology determined on k^n by the metric associated to (5.6) is the same as the product topology corresponding to the topology determined on k by the metric associated to $|\cdot|$ on each factor. The topology determined on k^n by the metric associated to (5.5) is the same as the topology determined by the metric associated to (5.6), because of (5.7).

6 Comparing topologies

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k. If N is a norm on V, then put

$$B_N(v, r) = \{w \in V : N(v - w) < r\} \tag{6.1}$$

for each $v \in V$ and positive real number r. This is the same as the open ball in V centered at v with radius r with respect to the metric (5.4) associated to N, as in (1.10).

Let N_1, N_2 be norms on V with respect to $|\cdot|$ on k. Suppose for the moment that there is a positive real number C such that

$$N_1(v) \le C\, N_2(v) \tag{6.2}$$

for every $v \in V$. This implies that

$$B_{N_2}(v, r) \subseteq B_{N_1}(v, C\, r) \tag{6.3}$$

for every $v \in V$ and $r > 0$, using the notation (6.1) for open balls in V with respect to N_1 and N_2. If $U \subseteq V$ is an open set with respect to the topology determined by the metric associated to N_2, then it follows from (6.3) that U is also an open set with respect to the topology determined by the metric associated to N_1.

Conversely, suppose that the topology determined on V by the metric associated to N_2 is at least as strong as the topology determined on V by the metric associated to N_1. Let $r_1 > 0$ be given, and remember that $B_{N_1}(0, r_1)$ is an open set in V with respect to the topology determined by the metric associated to N_1. Thus $B_{N_1}(0, r_1)$ is also an open set in V with respect to the topology determined by the metric associated to N_2, by hypothesis. This implies that there is an $r_2 > 0$ such that

$$B_{N_2}(0, r_2) \subseteq B_{N_1}(0, r_1), \tag{6.4}$$

because 0 is an element of $B_{N_1}(0, r_1)$. Of course, we could simply take $r_1 = 1$ here.

Let $v \in V$ be given, and suppose that $t \in k$ satisfies

$$N_2(v) < |t|\, r_2. \tag{6.5}$$

Equivalently, this means that $t \neq 0$, and that $N_2(t^{-1}\, v) = |t|^{-1}\, N_2(v) < r_2$. Thus (6.4) implies that

$$|t|^{-1}\, N_1(v) = N_1(t^{-1}\, v) < r_1, \tag{6.6}$$

so that $N_1(v) < |t|\, r_1$.

Suppose that $|\cdot|$ is not trivial on k, which implies that there is a $t_0 \in k$ such that $|t_0| > 1$. If $v \in V$ and $v \neq 0$, then there is an integer j such that

$$|t_0|^{j-1}\, r_2 \le N_2(v) < |t_0|^j\, r_2. \tag{6.7}$$

Thus we can apply the remarks in the previous paragraph to $t = t_0^j$, to get that

$$N_1(v) < |t_0|^j \, r_1 \le |t_0| \, (r_1/r_2) \, N_2(v), \tag{6.8}$$

using the first inequality in (6.7) in the second step. This shows that (6.2) holds with $C = |t_0| \, (r_1/r_2)$ when $v \ne 0$, and of course (6.2) is trivial when $v = 0$.

7 Finite support

Let k be a field, and let V be a vector space over k. Also let X be a nonempty set, and let $c(X, V)$ be the space of V-valued functions on X. This is a vector space over k with respect to pointwise addition and scalar multiplication. If $f \in c(X, V)$, then the *support* of f is defined to be the set of $x \in X$ such that $f(x) \ne 0$. Let $c_{00}(X, V)$ be the subset of $c(X, V)$ consisting of V-valued functions f on X whose supports have only finitely many elements. It is easy to see that $c_{00}(X, V)$ is a linear subspace of $c(X, V)$. Of course, if X has only finitely many elements, then $c_{00}(X, V)$ is equal to $c(X, V)$.

Let $|\cdot|$ be an absolute value function on k, and let N be a norm on V with respect to $|\cdot|$ on k. If $f \in c_{00}(X, V)$, then put

$$\|f\|_1 = \sum_{x \in X} N(f(x)), \tag{7.1}$$

where the sum on the right reduces to a finite sum of nonnegative real numbers. Similarly,

$$\|f\|_\infty = \max_{x \in X} N(f(x)) \tag{7.2}$$

reduces to the maximum of finitely many nonnegative real numbers. One can check that (7.1) and (7.2) define norms on $c_{00}(X, V)$ with respect to $|\cdot|$ on k. If N is an ultranorm on V, then (7.2) is an ultranorm on $c_{00}(X, V)$.

Clearly

$$\|f\|_\infty \le \|f\|_1 \tag{7.3}$$

for every $f \in c_{00}(X, V)$. Thus the topology determined on $c_{00}(X, V)$ by the metric associated to $\|f\|_1$ is at least as strong as the topology determined on $c_{00}(X, V)$ by the metric associated to $\|f\|_\infty$, as in the previous section. If X has only finitely many elements, then

$$\|f\|_1 \le (\#X) \, \|f\|_\infty \tag{7.4}$$

for every $f \in c_{00}(X, V)$, where $\#X$ denotes the number of elements in X. In this case, the topologies determined on $c_{00}(X, V)$ by the metrics associated to $\|f\|_1$ and $\|f\|_\infty$ are the same.

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k, and that N is the trivial ultranorm on V. Observe that $\|f\|_\infty$ is the same as the trivial ultranorm on $c_{00}(X, V)$ in this situation. In particular, the ultrametric on $c_{00}(X, V)$ corresponding to $\|f\|_\infty$ is the discrete metric, which determines the discrete topology on $c_{00}(X, V)$. It follows that the topology determined on $c_{00}(X, V)$ by the metric associated to $\|f\|_1$ is at least as strong as the discrete topology, as in the previous paragraph. Of course, this means that the topology determined on $c_{00}(X, V)$ by the metric associated to $\|f\|_1$ is the discrete topology too.

Let $|\cdot|$ be any absolute value function on k again, and let N be any norm with respect to $|\cdot|$ on k. If $f \in c_{00}(X, V)$, then

$$\sum_{x \in X} f(x) \tag{7.5}$$

can be defined as an element of V, by reducing to a finite sum. It is easy to see that this defines a linear mapping from $c_{00}(X, V)$ into V. Observe that

$$N\Big(\sum_{x \in X} f(x)\Big) \le \|f\|_1 \tag{7.6}$$

for every $f \in c_{00}(X, V)$, because of the triangle inequality. Similarly, if N is an ultranorm on V, then

$$N\Big(\sum_{x \in X} f(x)\Big) \le \|f\|_\infty \tag{7.7}$$

for every $f \in c_{00}(X, V)$.

8 Bounded functions

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let X be a nonempty set, and let f be a V-valued function on X. As usual, f is said to be *bounded* on X with respect to N on V if there is a finite upper bound for $N(f(x))$, $x \in X$. Let $\ell^\infty(X, V) = \ell^\infty_N(X, V)$ be the space of bounded V-valued functions f on X, and put

$$\|f\|_\infty = \|f\|_{\ell^\infty(X,V)} = \sup_{x \in X} N(f(x)) \tag{8.1}$$

for each such function f. One can check that $\ell^\infty(X, V)$ is a linear subspace of the space $c(X, V)$ of all V-valued functions on X, and that (8.1) defines a norm on $\ell^\infty(X, V)$ with respect to $|\cdot|$ on k. If N is an ultranorm on V, then (8.1) defines an ultranorm on $\ell^\infty(X, V)$. If $|\cdot|$ is the trivial absolute value function on k, and N is the trivial ultranorm on V, then (8.1) is the trivial ultranorm on $\ell^\infty(X, V)$.

A V-valued function f on X is said to *vanish at infinity* with respect to N if for each $\epsilon > 0$,

$$N(f(x)) < \epsilon \tag{8.2}$$

for all but finitely many $x \in X$. Let $c_0(X, V) = c_{0,N}(X, V)$ be the space of these functions on X. It is easy to see that

$$c_0(X, V) \subseteq \ell^\infty(X, V), \tag{8.3}$$

and that $c_0(X, V)$ is a linear subspace of $\ell^\infty(X, V)$. One can also check that $c_0(X, V)$ is a closed set in $\ell^\infty(X, V)$, with respect to the metric associated to the supremum norm. Of course, if X has only finitely many elements, then every V-valued function on X vanishes at infinity.

Clearly

$$c_{00}(X, V) \subseteq c_0(X, V). \tag{8.4}$$

If $f \in c_0(X, V)$, then f can be approximated by V-valued functions on X with finite support uniformly on X. Thus $c_0(X, V)$ is the same as the closure of $c_{00}(X, V)$ in $\ell^\infty(X, V)$ with respect to the supremum metric. If $|\cdot|$ is the trivial absolute value function on k, N is the trivial ultranorm on V, and $f \in c_0(X, V)$, then f has finite support in X.

If N is any norm on V, and $f \in c_0(X, V)$, then the support of f has at most finitely or countably many elements. More precisely, for each $n \in \mathbf{Z}_+$, there are at most finitely many $x \in X$ such that $N(f(x)) \geq 1/n$, and the support of f is the union of this sequence of finite sets. Note that (8.1) is the same as (7.2) when $f \in c_{00}(X, V)$. If $f \in c_0(X, V)$, then the supremum on the right side of (8.1) is attained. This is trivial when $f(x) = 0$ for every $x \in X$, and otherwise the supremum can be reduced to the maximum over a finite subset of X.

If V is complete with respect to the metric associated to N, then $\ell^\infty(X, V)$ is complete with respect to the metric associated to (8.1), by standard arguments. Indeed, if $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $\ell^\infty(X, V)$ with respect to the supremum metric, then $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in V for each $x \in X$, with respect to the metric associated to N. If V is complete, then it follows that $\{f_j(x)\}_{j=1}^\infty$ converges to an element $f(x)$ of V for each $x \in X$. The Cauchy condition for $\{f_j\}_{j=1}^\infty$ implies in particular that $\{f_j\}_{j=1}^\infty$ is bounded in $\ell^\infty(X, V)$, which can be used to show that f is bounded on X. One can use the Cauchy condition for $\{f_j\}_{j=1}^\infty$ in $\ell^\infty(X, V)$ again to get that $\{f_j\}_{j=1}^\infty$ converges to f with respect to the supremum metric, as desired.

9 Summable functions

Let X be a nonempty set, and let f be a nonnegative real-valued function on X. The sum

$$\sum_{x \in X} f(x) \tag{9.1}$$

is defined as a nonnegative extended real number as the supremum of the sums

$$\sum_{x \in A} f(x) \tag{9.2}$$

over all nonempty finite subsets A of X. If (7.5) is finite, then f is said to be *summable* on X. Of course, (9.1) reduces to a finite sum when X has only finitely many elements, or when f has finite support in X. If f is summable on X, then it is easy to see f vanishes at infinity on X, with respect to the standard absolute value function on $\mathbf{R}$.

If f is summable on X, and t is a nonnegative real number, then one can check that $t\,f(x)$ is summable on X as well, with

$$\sum_{x\in X} t\,f(x) = t \sum_{x\in X} f(x). \tag{9.3}$$

Similarly, if g is another nonnegative real-valued summable function on X, then $f+g$ is summable on X, with

$$\sum_{x\in X} (f(x)+g(x)) = \sum_{x\in X} f(x) + \sum_{x\in X} g(x). \tag{9.4}$$

Both statements can be obtained directly from the definitions, using the analogous properties of finite sums. There are also versions of (9.3) and (9.4) for arbitrary nonnegative real-valued functions on X, with suitable interpretations for extended real numbers. In particular, if f is not summable on X, then one may apply (9.3) to positive real numbers t, or interpret the right side of (9.3) as being equal to 0 when $t=0$.

If $\{f_j\}_{j=1}^\infty$ is a sequence of nonnegative real-valued functions on X that converges pointwise to a nonnegative real-valued function f on X, then

$$\sum_{x\in X} f(x) \le \sup_{j\ge 1} \Big(\sum_{x\in X} f_j(x)\Big). \tag{9.5}$$

This is a simplified version of Fatou's lemma for sums. More precisely, the supremum on the right side of (9.5) is defined as a nonnegative extended real number, and of course (9.5) is trivial when the supremum is $+\infty$. To get (9.5), let A be a nonempty finite subset of X, and observe that

$$\sum_{x\in A} f(x) = \lim_{j\to\infty} \Big(\sum_{x\in A} f_j(x)\Big) \le \sup_{j\ge 1} \Big(\sum_{x\in A} f_j(x)\Big) \le \sup_{j\ge 1} \Big(\sum_{x\in X} f_j(x)\Big). \tag{9.6}$$

This implies (9.5), by the definition of the sum (9.1).

Let k be a field with an absolute value function $|\cdot|$, let V be a vector space over k, and let N be a norm on V with respect to $|\cdot|$ on k. A V-valued function f on X is said to be *summable* on X with respect to N if $N(f(x))$ is summable as a nonnegative real-valued function on X. Let $\ell^1(X,V) = \ell^1_N(X,V)$ be the space of V-valued functions f on X that are summable with respect to N, and put

$$\|f\|_1 = \|f\|_{\ell^1(X,V)} = \sum_{x\in X} N(f(x)) \tag{9.7}$$

for all such functions f. One can verify that $\ell^1(X,V)$ is a linear subspace of the space $c(X,V)$ of all V-valued functions on X, and that (9.7) defines a norm on $\ell^1(X,V)$ with respect to $|\cdot|$ on k. Clearly

$$c_{00}(X,V) \subseteq \ell^1(X,V), \tag{9.8}$$

and (9.7) is the same as (7.1) when $f \in c_{00}(X, V)$. One can check that $c_{00}(X, V)$ is dense in $\ell^1(X, V)$ with respect to the metric associated to the ℓ^1 norm. This uses nonempty finite subsets A of X such that

$$\sum_{x \in A} N(f(x)) \tag{9.9}$$

approximates $\|f\|_1$. If $f \in \ell^1(X, V)$, then $N(f(x))$ is summable as a nonnegative real-valued function on X, and hence $N(f(x))$ vanishes at infinity on X, as before. This is the same as saying that f vanishes at infinity as a V-valued function on X, so that

$$\ell^1(X, V) \subseteq c_0(X, V). \tag{9.10}$$

Note that

$$\|f\|_\infty \le \|f\|_1 \tag{9.11}$$

for every $f \in \ell^1(X, V)$.

If V is complete with respect to the metric associated to N, then $\ell^1(X, V)$ is complete with respect to the metric associated to (9.7). To see this, let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence with respect to the ℓ^1 metric. This implies that $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in V with respect to the metric associated to N for each $x \in X$, so that $\{f_j(x)\}_{j=1}^\infty$ converges to an element $f(x)$ of V for each $x \in X$, because V is complete. It follows that $\{N(f_j(x))\}_{j=1}^\infty$ converges to $N(f(x))$ in $\mathbf{R}$ for every $x \in X$, by standard arguments. Thus

$$\sum_{x \in X} N(f(x)) \le \sup_{j \ge 1} \Big(\sum_{x \in X} N(f_j(x)) \Big), \tag{9.12}$$

as in (9.5). The Cauchy condition for $\{f_j\}_{j=1}^\infty$ in $\ell^1(X, V)$ implies that the right side of (9.12) is finite, so that $f \in \ell^1(X, V)$. Similarly, if $l \in \mathbf{Z}_+$, then

$$\|f - f_l\|_1 \le \sup_{j \ge l} \|f_j - f_l\|_1, \tag{9.13}$$

because $\{f_j - f_l\}_{j=l}^\infty$ converges to $f - f_l$ pointwise on X. This implies that $\{f_l\}_{l=1}^\infty$ converges to f with respect to the ℓ^1 metric, using the Cauchy condition for $\{f_j\}_{j=1}^\infty$ in $\ell^1(X, V)$ again, as desired.

10 Infinite series

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. As usual, an infinite series

$$\sum_{j=1}^\infty v_j \tag{10.1}$$

with terms in V is said to *converge* in V with respect to N if the corresponding sequence of partial sums $\sum_{j=1}^n v_j$ converges in V with respect to the metric associated to N. Of course, the value of the sum (10.1) is defined to be the limit of the sequence of partial sums in this case. If (10.1) converges in V, then the corresponding sequence of partial sums is bounded, and

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \sup_{n \ge 1} N\Big(\sum_{j=1}^{n} v_j\Big). \tag{10.2}$$

This follows from the analogous statements for convergent sequences.

If $\sum_{j=1}^{\infty} a_j$ is an infinite series of nonnegative real numbers, then the corresponding sequence of partial sums increases monotonically. It follows that $\sum_{j=1}^{\infty} a_j$ converges in $\mathbf{R}$ with respect to the standard absolute value function if and only if the sequence of partial sums is bounded, in which case

$$\sum_{j=1}^{\infty} a_j = \sup_{n \ge 1} \Big(\sum_{j=1}^{n} a_j\Big), \tag{10.3}$$

by the analogous statements for monotonically increasing sequences in $\mathbf{R}$. Otherwise, the right side of (10.3) may be interpreted as being $+\infty$, and one may take the left side of (10.3) to be $+\infty$ as well. One can check that this is equivalent to the sum $\sum_{j \in \mathbf{Z}_+} a_j$, as defined in the previous section.

Let k, V, and N be as before, and let (10.1) be an infinite series with terms in V again. Note that the corresponding sequence of partial sums is a Cauchy sequence in V with respect to the metric associated to N if and only if for every $\epsilon > 0$ there is a positive integer L such that

$$N\Big(\sum_{j=l}^{n} v_j\Big) < \epsilon \tag{10.4}$$

for every $n \ge l \ge 1$. This implies that

$$\lim_{j \to \infty} N(v_j) = 0, \tag{10.5}$$

by taking $l = n$, as usual.

If $\sum_{j=1}^{\infty} N(v_j)$ converges as an infinite series of nonnegative real numbers, then (10.1) is said to converge *absolutely* with respect to N. Observe that

$$N\Big(\sum_{j=l}^{n} v_j\Big) \le \sum_{j=l}^{n} N(v_j) \tag{10.6}$$

for every $n \ge l \ge 1$, by the triangle inequality. If (10.1) converges absolutely with respect to N, then it follows that the corresponding sequence of partial sums is a Cauchy sequence in V with respect to the metric associated to N, because of (10.4). If V is complete with respect to the metric associated to N, then we get that (10.1) converges in V. We also have that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \sum_{j=1}^{\infty} N(v_j) \tag{10.7}$$

under these conditions, by (10.2).

Suppose now that N is an ultranorm on V, so that

$$N\Big(\sum_{j=l}^{n} v_j\Big) \le \max_{l \le j \le n} N(v_j) \tag{10.8}$$

for every $n \ge l \ge 1$. If (10.5) holds, then the sequence of partial sums corresponding to (10.1) is a Cauchy sequence in V with respect to N, by (10.4). This implies that (10.1) converges in V with respect to N when V is complete with respect to the ultrametric associated to N. In this situation, we also have that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \max_{j \ge 1} N(v_j), \tag{10.9}$$

by (10.2). The maximum on the right side of (10.9) is attained when (10.5) holds, as before.

11 Bounded linear mappings

Let k be a field with an absolute value function $|\cdot|$, and let V, W be vector spaces over k. Also let N_V, N_W be norms on V, W, respectively, with respect to $|\cdot|$ on k. A linear mapping T from V into W is said to be *bounded* with respect to N_V, N_W if there is a nonnegative real number C such that

$$N_W(T(v)) \le C\, N_V(v) \tag{11.1}$$

for every $v \in V$. In this case, we have that

$$N_W(T(u) - T(v)) = N_W(T(u - v)) \le C\, N_V(u - v) \tag{11.2}$$

for every $u, v \in V$. This implies that T is uniformly continuous with respect to the metrics on V, W associated to N_V, N_W, respectively.

Let T be a linear mapping from V into W again, and suppose that there are positive real numbers r_V, r_W such that

$$N_W(T(v)) < r_W \tag{11.3}$$

for every $v \in V$ with $N_V(v) < r_V$. Of course, if T is continuous at 0 with respect to the metrics on V, W associated to N_V, N_W, respectively, then for each $r_W > 0$ there is an $r_V > 0$ with this property. Suppose that $|\cdot|$ is not trivial on k, and let t_0 be an element of k with $|t_0| > 1$. If $v \in V$ and $v \ne 0$, then there is an integer j such that

$$|t_0|^{j-1}\, r_V \le N_V(v) < |t_0|^j\, r_V. \tag{11.4}$$

Thus $N_V(t_0^{-j}\, v) = |t_0|^{-j}\, N_V(v) < r_V$, so that

$$|t_0|^{-j}\, N_W(T(v)) = N_W(T(t_0^{-j}\, v)) < r_W, \tag{11.5}$$

by (11.3). This implies that

$$N_W(T(v)) < |t_0|^j \, r_W \le |t_0| \, (r_W/r_V) \, N_V(v), \tag{11.6}$$

using the first inequality in (11.4) in the second step. Hence (11.1) holds with $C = |t_0| \, (r_W/r_V)$ when $v \ne 0$, and (11.1) is trivial when $v = 0$.

If T is a bounded linear mapping from V into W with respect to N_V, N_W, respectively, then we put

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \ge 0 : \text{ (11.1) holds}\}. \tag{11.7}$$

More precisely, the infimum is taken over all nonnegative real numbers C for which (11.1) holds. One can check that the infimum is attained, so that (11.1) holds with $C = \|T\|_{op}$. Let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from V into W with respect to N_V, N_W, respectively. It is easy to see that $\mathcal{BL}(V, W)$ is a vector space over k with respect to pointwise addition and scalar multiplication. Similarly, (11.7) defines a norm on $\mathcal{BL}(V, W)$, which is the *operator norm* associated to N_V, N_W. If N_W is an ultranorm on W, then (11.7) is an ultranorm on $\mathcal{BL}(V, W)$.

Let Z be another vector space over k, and let N_Z be a norm on Z with respect to $|\cdot|$ on k. Suppose that T_1 is a bounded linear mapping from V into W with respect to N_V, N_W, and that T_2 is a bounded linear mapping from W into Z with respect to N_W, N_Z. Observe that the composition $T_2 \circ T_1$ of T_1 and T_2 is a bounded linear mapping from V into Z with respect to N_V, N_Z, with

$$\|T_2 \circ T_1\|_{op,VZ} \le \|T_1\|_{op,VW} \, \|T_2\|_{op,WZ}. \tag{11.8}$$

Let V_0 be a linear subspace of V, and suppose that V_0 is dense in V with respect to the metric associated to N_V. Also let T_0 be a bounded linear mapping from V_0 into W, using the restriction of N_V to V_0. Thus T_0 is uniformly continuous with respect to the metrics associated to N_V and N_W, as before. If W is complete with respect to the metric associated to N_W, then there is a unique extension of T_0 to a uniformly continuous mapping from V into W, as in Section 3. One can check that this extension is a bounded linear mapping from V into W under these conditions, with the same operator norm as on V_0.

12 Banach spaces

Let k be a field with an absolute value function $|\cdot|$, let V be a vector space over k, and let N be a norm on V with respect to $|\cdot|$ on k. If V is complete with respect to the metric associated to N, then V is said to be a *Banach space* with respect to N. Otherwise, one can pass to a completion of V, as usual. The vector space operations on V can be extended continuously to the completion, so that the completion is a vector space over k. Similarly, N can be extended continuously to the completion of V, which corresponds to the distance to 0 in the completion. This extension of N defines a norm on the completion, which corresponds to the metric on the completion in the usual way. The completion of V with respect to N is unique up to isometric linear equivalence, because of the existence of extensions of bounded linear mappings from dense linear subspaces into Banach spaces mentioned in the previous section.

If V_0 is a linear subspace of V, then the restriction of N to V_0 defines a norm on V_0 with respect to $|\cdot|$ on k. If N is an ultranorm on V, then the restriction of N to V_0 is an ultranorm on V_0. If N is any norm on V with respect to $|\cdot|$ on k, V_0 is a dense linear subspace of V, and the restriction of N to V_0 is an ultranorm on V_0, then one can check that N is an ultranorm on V. In particular, if N is an ultranorm on V, then the extension of N to the completion of V mentioned in the preceding paragraph is an ultranorm as well. Note that closed linear subspaces of Banach spaces are Banach spaces too, with respect to the restriction of the norm, because closed subsets of complete metric spaces are complete with respect to the restriction of the metric to the subset.

Remember that k may be considered as a one-dimensional vector space over itself, and that $|\cdot|$ may be considered as a norm on k with respect to itself, as in Section 5. If $v \in V$, then

$$t \mapsto t\,v \tag{12.1}$$

defines a bounded linear mapping from k into V, with operator norm equal to $N(v)$. If V is complete with respect to the metric associated to N, but k is not complete with respect to the metric associated to $|\cdot|$, then (12.1) extends to a bounded linear mapping from the completion of k into V, as before. One can verify that V is a vector space over the completion of k with respect to this extension of scalar multiplication on V to the completion of k, and that N is a norm on V as a vector space over the completion of k. One could also include completeness of k in the definition of a Banach space.

Suppose for the moment that k is complete with respect to the metric associated to $|\cdot|$, so that k may be considered as a one-dimensional Banach space over itself. If n is a positive integer, then it is easy to see that k^n is a Banach space with respect to the norms $\|v\|_1$ and $\|v\|_\infty$ defined in (5.5) and (5.6), respectively. In both cases, a sequence of elements of k^n is a Cauchy sequence with respect to the metric associated to the norm if and only if the corresponding n sequences of coordinates in k are Cauchy sequences with respect to the metric associated to $|\cdot|$. If X is a nonempty set, then the spaces $\ell^\infty(X,k)$ and $\ell^1(X,k)$ discussed in Sections 8 and 9, respectively, are Banach spaces with respect to the corresponding norms defined earlier. It follows that $c_0(X,k)$ is a Banach space with respect to the supremum norm as well, because it is a closed subspace of $\ell^\infty(X,k)$.

Let V, W be vector spaces over k, with norms N_V, N_W with respect to $|\cdot|$ on k, respectively. If W is complete with respect to the metric associated to N_W, then $\mathcal{BL}(V,W)$ is complete with respect to the metric associated to the corresponding operator norm. More precisely, let $\{T_j\}_{j=1}^\infty$ be a Cauchy sequence in $\mathcal{BL}(V,W)$ with respect to the operator norm. If $v \in V$, then it is easy to see that $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in W with respect to the metric associated to N_W. It follows that $\{T_j(v)\}_{j=1}^\infty$ converges to an element $T(v)$ of W for every $v \in V$, because W is complete. One can check that T is a linear mapping from

V into W, because the T_j's are linear. The Cauchy condition for $\{T_j\}_{j=1}^\infty$ with respect to the operator norm implies that the operator norms of the T_j's are bounded, which can be used to show that T is bounded as a linear mapping from V into W. One can use the Cauchy condition for $\{T_j\}_{j=1}^\infty$ with respect to the operator norm again to get that $\{T_j\}_{j=1}^\infty$ converges to T with respect to the operator norm, as desired.

13 Sums of vectors

Let k be a field, let V be a vector space over k, and let X be a nonempty set. As in Section 7,

$$f \mapsto \sum_{x \in X} f(x) \tag{13.1}$$

defines a linear mapping from $c_{00}(X, V)$ into V. Let $|\cdot|$ be an absolute value function on k, and let N be a norm on V. Using (7.6), we get that (13.1) is a bounded linear mapping from $c_{00}(X, V)$ into V, with respect to the ℓ^1 norm on $c_{00}(X, V)$ as in (9.7), and N on V. Similarly, if N is an ultranorm on V, then (7.7) implies that (13.1) is a bounded linear mapping from $c_{00}(X, V)$ into V, with respect to the supremum ultranorm on $c_{00}(X, V)$ as in (8.1), and N on V. More precisely, (7.6) and (7.7) say that the corresponding operator norms of (13.1) are less than or equal to 1. In both cases, it is easy to see that the operator norm is equal to 1, unless $V = \{0\}$.

Suppose now that V is complete with respect to the metric associated to N, so that V is a Banach space. As in the Section 11, there is a unique extension of (13.1) to a bounded linear mapping from $\ell^1(X, V)$ into V. This uses the fact that $c_{00}(X, V)$ is dense in $\ell^1(X, V)$, as in Section 9. If N is an ultranorm on V, then there is a unique extension of (13.1) to a bounded linear mapping from $c_0(X, V)$ into V, using the supremum ultranorm (8.1) on $c_0(X, V)$. This uses the fact that $c_{00}(X, V)$ is dense in $c_0(X, V)$ with respect to the supremum metric, as in Section 8. These extensions can be used to define (13.1) in these situations. In both cases, the operator norm of the extension of (13.1) is equal to 1 when $V \neq \{0\}$.

Of course, if X has only finitely many elements, then $c_{00}(X, V)$ is the same as $\ell^1(X, V)$ and $c_0(X, V)$. Suppose that X has infinitely many elements, and let $\{x_j\}_{j=1}^\infty$ be a sequence of distinct elements in X. If f is a V-valued function on X whose support is contained in the set of x_j's, then $\sum_{x \in X} f(x)$ corresponds formally to the infinite series

$$\sum_{j=1}^\infty f(x_j). \tag{13.2}$$

If $f \in \ell^1(X, V)$, then one can check that (13.2) converges absolutely with respect to N. This implies that (13.2) converges in V when V is complete with respect to the metric associated to N, as in Section 10. One can verify that the value of (13.2) is the same as what one gets by extending (13.1) to a bounded linear mapping from $\ell^1(X, V)$ into V. If f is any element of $\ell^1(X, V)$, then f vanishes at infinity on X with respect to N on V, and hence the support of f has only finitely or countably many elements.

Let f be a V-valued function on X whose support is contained in the set of x_j's again. If $f \in c_0(X, V)$, then it is easy to see that $N(f(x_j)) \to 0$ as $j \to \infty$. This implies that (13.2) converges in V with respect to N when N is an ultranorm on V and V is complete with respect to the corresponding ultrametric, as in Section 10. In this case, one can check that the value of (13.2) is the same as what one gets by extending (13.1) to a bounded linear mapping from $c_0(X, V)$ into V, as before. If f is any element of $c_0(X, V)$, then the support of f has only finitely or countable many elements, and hence can be enumerated by a finite or infinite sequence.

If f is a summable real-valued function on X, then f can be expressed as a difference of nonnegative real-valued summable functions on X. This permits one to define $\sum_{x \in X} f(x)$ using the definition of the sum for nonnegative real-valued functions in Section 9. Similarly, if f is a summable complex-valued function on X, then one can apply the previous remark to the real and imaginary parts of f. This approach to the sum is also compatible with the ones just mentioned in this situation.

14 Banach algebras

Let k be a field, and let $\mathcal{A}$ be an (associative) *algebra* over k. This means that $\mathcal{A}$ is a vector space over k equipped with a binary operation of multiplication. Multiplication on $\mathcal{A}$ should be bilinear as a mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, and satisfy the associative law. Let $|\cdot|$ be an absolute value function on k, and let N be a seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k. We say that N is *submultiplicative* on $\mathcal{A}$ if

$$N(x\,y) \le N(x)\,N(y) \tag{14.1}$$

for every $x, y \in \mathcal{A}$, and that N is *multiplicative* on $\mathcal{A}$ if

$$N(x\,y) = N(x)\,N(y) \tag{14.2}$$

for every $x, y \in \mathcal{A}$.

Let $\|\cdot\|$ be a submultiplicative norm on $\mathcal{A}$ with respect to $|\cdot|$ on k. One can check that multiplication on $\mathcal{A}$ is continuous as a mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$, using the topology determined on $\mathcal{A}$ by the metric associated to $\|\cdot\|$, and that corresponding product topology on $\mathcal{A} \times \mathcal{A}$. If $\mathcal{A}$ is complete with respect to the metric associated to $\|\cdot\|$, then $\mathcal{A}$ is said to be a *Banach algebra* with respect to $\|\cdot\|$ over k. Otherwise, one can pass to a completion of $\mathcal{A}$, as in Section 12. One can verify that multiplication extends continuously to the completion of $\mathcal{A}$, so that the completion of $\mathcal{A}$ is an algebra over k, and so that the extension of the norm to the completion is submultiplicative on the completion.

Similarly, if $\mathcal{A}$ is complete with respect to the metric associated to the norm $\|\cdot\|$, but k is not complete with respect to the metric associated to $|\cdot|$, then scalar multiplication on $\mathcal{A}$ can be extended to the completion of k, as in Section

12. It is easy to see that multiplication on $\mathcal{A}$ is also bilinear with respect to this extension of scalar multiplication on $\mathcal{A}$ to the completion of k. One can include completeness of k in the definition of a Banach algebra, as in the case of Banach spaces. One may also ask that $\mathcal{A}$ have a multiplicative identity element e with $\|e\| = 1$, and we shall normally do that here.

Let X be a nonempty set, and observe that the space $c(X, k)$ is a commutative algebra over k with respect to pointwise multiplication of functions. Also let $\mathbf{1}_X$ be the function on X whose value at every point is the multiplicative identity element 1 in k, which is the multiplicative identity element in $c(X, k)$. Remember that $\ell^\infty(X, k)$ denotes the space of bounded k-valued functions on X, as in Section 8, and let $\|f\|_\infty$ be the supremum norm on $\ell^\infty(X, k)$ corresponding to $|\cdot|$ on k. If $f, g \in \ell^\infty(X, k)$, then their product $f\,g$ is bounded on X too, with

$$\|f\,g\|_\infty \le \|f\|_\infty\,\|g\|_\infty. \tag{14.3}$$

Thus $\ell^\infty(X, k)$ is a subalgebra of $c(X, k)$, and $\|\cdot\|_\infty$ is submultiplicative on $\ell^\infty(X, k)$. Of course, constant functions on X are bounded, and

$$\|\mathbf{1}_X\|_\infty = |1| = 1. \tag{14.4}$$

If k is complete with respect to the metric associated to $|\cdot|$, then $\ell^\infty(X, k)$ is complete with respect to the corresponding supremum metric, as in Section 8.

Let V be a vector space over k, and let N_V be a norm on V with respect to $|\cdot|$ on V. Also let $\mathcal{BL}(V) = \mathcal{BL}(V, V)$ be the space of bounded linear mappings from V into itself, with respect to N_V on the domain and range. This is an algebra over k, with composition of linear mappings as multiplication. The corresponding operator norm $\|T\|_{op} = \|T\|_{op,VV}$ is submultiplicative on $\mathcal{BL}(V)$, as in (11.8). The identity mapping $I = I_V$ on V is the multiplicative identity element in $\mathcal{BL}(V)$. If $V \ne \{0\}$, then it is easy to see that $\|I\|_{op} = 1$. If V is complete with respect to the metric associated to N_V, then $\mathcal{BL}(V)$ is complete with respect to the metric associated to the operator norm, as in Section 12.

15 Cauchy products

Let k be a field, and let $\mathcal{A}$ be an algebra over k. Also let $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ be infinite series with terms in $\mathcal{A}$, considered formally for the moment. Thus

$$c_n = \sum_{j=0}^n a_j\, b_{n-j} \tag{15.1}$$

is defined as an element of $\mathcal{A}$ for every nonnegative integer n, and the corresponding series $\sum_{n=0}^\infty c_n$ is called the *Cauchy product* of $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$. It is easy to see that

$$\sum_{n=0}^\infty c_n = \Big(\sum_{j=0}^\infty a_j\Big)\Big(\sum_{l=0}^\infty b_l\Big) \tag{15.2}$$

formally. More precisely, both sides of (15.2) correspond formally to the sum of $a_j\, b_l$ over all nonnegative integers j, l.

Let us look at the partial sums corresponding to these infinite series. If N_1, N_2 are nonnegative integers, then

$$\Big(\sum_{j=0}^{N_1} a_j\Big)\Big(\sum_{l=0}^{N_2} b_l\Big) = \sum\{a_j\, b_l : j = 0, \ldots, N_1,\ l = 0, \ldots, N_2\}. \tag{15.3}$$

Similarly, if N is a nonnegative integer, then

$$\sum_{n=0}^{N} c_n = \sum_{n=0}^{N}\Big(\sum_{j=0}^{n} a_j\, b_{n-j}\Big) = \sum\{a_j\, b_l : j, l \in \mathbf{Z},\ j, l \ge 0,\ j + l \le N\}. \tag{15.4}$$

If $N_1 + N_2 \le N$, then each of the terms in the right side of (15.3) occur in the right side of (15.4). If $N \le N_1, N_2$, then each of the terms on right side of (15.4) also occurs in the right side of (15.3).

Suppose for the moment that there are nonnegative integers J, L such that $a_j = 0$ when $j \ge J$ and $b_l = 0$ when $l \ge L$. In this case, one can check that $c_n = 0$ when $n \ge J + L - 1$. Thus the infinite series $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$, and $\sum_{n=0}^{\infty} c_n$ reduce to finite sums in $\mathcal{A}$, and one can check that (15.2) holds in this situation. More precisely, (15.3) is equal to (15.4) when $J-1 \le N_1$, $L-1 \le N_2$, and $J + L - 2 \le N$. This is because the terms $a_j\, b_l$ that occur in one of (15.3) or (15.4) and not the other are equal to 0.

Suppose now that a_j, b_l are nonnegative real numbers for every $j, l \ge 0$, so that c_n is a nonnegative real number for every $n \ge 0$. Using the earlier remarks about (15.3) and (15.4), we get that

$$\Big(\sum_{j=0}^{N_1} a_j\Big)\Big(\sum_{l=0}^{N_2} b_l\Big) \le \sum_{n=0}^{N} c_n \tag{15.5}$$

when $N_1 + N_2 \le N$. Similarly,

$$\sum_{n=0}^{N} c_n \le \Big(\sum_{j=0}^{N_1} a_j\Big)\Big(\sum_{l=0}^{N_2} b_l\Big) \tag{15.6}$$

when $N \le N_1, N_2$. In this case, $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$, and $\sum_{n=0}^{\infty} c_n$ can be defined as nonnegative extended real numbers. The product of the first two series can also be defined as a nonnegative extended real number when both sums are finite or both sums are positive. One can use (15.5) and (15.6) to get that (15.2) holds as an equality between nonnegative extended real numbers when the right side of (15.2) is defined. Otherwise, if $a_j = 0$ for every $j \ge 0$, or if $b_l = 0$ for every $l \ge 0$, then $c_n = 0$ for every $n \ge 0$.

Let k be a field with an absolute value function $|\cdot|$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative norm $\|\cdot\|$ with respect to $|\cdot|$ on k. Thus

$$\|a_j\, b_l\| \le \|a_j\|\, \|b_l\| \tag{15.7}$$

for every $j, l \geq 0$, so that

$$\|c_n\| \leq \sum_{j=0}^{n} \|a_j\, b_{n-j}\| \leq \sum_{j=0}^{n} \|a_j\|\, \|b_{n-j}\| \tag{15.8}$$

for every $n \geq 0$. Suppose that $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ converge absolutely with respect to $\|\cdot\|$ on $\mathcal{A}$, so that $\sum_{j=0}^{\infty} \|a_j\|$ and $\sum_{l=0}^{\infty} \|b_l\|$ converge as infinite series of nonnegative real numbers. Observe that

$$\sum_{n=0}^{\infty} \|c_n\| \leq \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} \|a_j\|\, \|b_{n-j}\| \Big) = \Big(\sum_{j=0}^{\infty} \|a_j\| \Big) \Big(\sum_{l=0}^{\infty} \|b_l\| \Big), \tag{15.9}$$

using (15.8) in the first step. The second step in (15.9) uses the fact that the right side of (15.8) is the same as the nth term of the Cauchy product of $\sum_{j=0}^{\infty} \|a_j\|$ and $\sum_{l=0}^{\infty} \|b_l\|$, as infinite series of nonnegative real numbers. Hence the sum over n is equal to the product of the two sums, as in the preceding paragraph. In particular, $\sum_{n=0}^{\infty} c_n$ converges absolutely with respect to $\|\cdot\|$ under these conditions.

If $\mathcal{A}$ is complete with respect to the metric associated to $\|\cdot\|$, then absolute convergence of these series implies convergence in $\mathcal{A}$, as in Section 10. One can check that (15.2) also holds in this situation. Basically, one can compare (15.3) and (15.4) as before, and verify that the norms of the errors are small when N, N_1, N_2 are sufficiently large.

Suppose now that $\|\cdot\|$ is an ultranorm on $\mathcal{A}$, so that

$$\|c_n\| \leq \max_{0 \leq j \leq n} \|a_j\, b_{n-j}\| \leq \max_{0 \leq j \leq n} (\|a_j\|\, \|b_{n-j}\|) \tag{15.10}$$

for every $n \geq 0$. If

$$\lim_{j \to \infty} \|a_j\| = \lim_{l \to \infty} \|b_l\| = 0, \tag{15.11}$$

then one can check that

$$\lim_{n \to \infty} \|c_n\| = 0, \tag{15.12}$$

using (15.10). If $\mathcal{A}$ is also complete with respect to the ultrametric associated to $\|\cdot\|$, then it follows that $\sum_{j=0}^{\infty} a_j$, $\sum_{l=0}^{\infty} b_l$, and $\sum_{n=0}^{\infty} c_n$ converge in $\mathcal{A}$ with respect to $\|\cdot\|$, as in Section 10. Under these conditions, one can verify that (15.2) holds again. This is analogous to the previous case, but using the ultranorm version of the triangle inequality to estimate the errors.

Part II
Linear mappings, matrices, and involutions

16 Some linear mappings

Let k be a field, and let X be a nonempty set. If $x, y \in X$, then put

$$\begin{aligned}
\delta_x(y) &= 1 \quad \text{when } x = y \\
&= 0 \quad \text{when } x \neq y.
\end{aligned} \tag{16.1}$$

This defines $\delta_x(y)$ as a k-valued function of y on X for each $x \in X$. Of course, δ_x is in the space $c_{00}(X, k)$ of k-valued functions on X with finite support for every $x \in X$. It is easy to see that the collection of δ_x, $x \in X$, is a basis for $c_{00}(X, k)$ as a vector space over k.

Let V be a vector space over k, and let a be a V-valued function on X, so that $a \in c(X, V)$. If $f \in c_{00}(X, k)$, then

$$T_a(f) = \sum_{x \in X} a(x)\, f(x) \tag{16.2}$$

is defined as an element of V, because $a(x)\, f(x)$ is a V-valued function on X with finite support. Clearly T_a defines a linear mapping from $c_{00}(X, k)$ into V. Note that

$$T_a(\delta_x) = a(x) \tag{16.3}$$

for every $x \in X$. One can check that every linear mapping from $c_{00}(X, k)$ into V is of this form, because the δ_x's form a basis for $c_{00}(X, k)$.

Let $|\cdot|$ be an absolute value function on k, and remember that k may be considered as a one-dimensional vector space over itself, with $|\cdot|$ as a norm on k. Thus $\ell^\infty(X, k)$ and $c_0(X, k)$ may be defined as in Section 8, and $\ell^1(X, k)$ may be defined as in Section 9. Of course,

$$\|\delta_x\|_{\ell^1(X,k)} = \|\delta_x\|_{\ell^\infty(X,k)} = 1 \tag{16.4}$$

for each $x \in X$.

Let N be a norm on V with respect to $|\cdot|$ on k, and suppose that a is bounded on X with respect to N. If $f \in \ell^1(X, k)$, then it is easy to see that $a\, f \in \ell^1(X, V)$, with

$$\|a\, f\|_{\ell^1(X,V)} \leq \|a\|_{\ell^\infty(X,V)}\, \|f\|_{\ell^1(X,k)}. \tag{16.5}$$

Similarly, if $f \in \ell^\infty(X, k)$, then $a\, f \in \ell^\infty(X, V)$, with

$$\|a\, f\|_{\ell^\infty(X,V)} \leq \|a\|_{\ell^\infty(X,V)}\, \|f\|_{\ell^\infty(X,k)}. \tag{16.6}$$

If $f \in c_0(X,k)$, then $a\,f \in c_0(X,V)$.

If $f \in c_{00}(X,k)$, then $a\,f \in c_{00}(X,V)$, and $T_a(f) \in V$ is defined as in (16.2). We also have that

$$N(T_a(f)) \le \|a\,f\|_{\ell^1(X,V)} \le \|a\|_{\ell^\infty(X,V)}\,\|f\|_{\ell 1(X,k)}, \tag{16.7}$$

using (7.6) in the first step, and (16.5) in the second step. If N is an ultranorm on V, then

$$N(T_a(f)) \le \|a\,f\|_{\ell^\infty(X,V)} \le \|a\|_{\ell^\infty(X,V)}\,\|f\|_{\ell^\infty(X,k)}, \tag{16.8}$$

using (7.7) in the first step, and (16.6) in the second step.

Of course, (16.7) says that T_a is bounded as a linear mapping from $c_{00}(X,k)$ into V, with respect to the ℓ^1 norm on $c_{00}(X,k)$. If N is an ultranorm on V, then (16.8) says that T_a is bounded as a linear mapping from $c_{00}(X,k)$ into V, with respect to the supremum norm on $c_{00}(X,k)$. These inequalities also imply that the corresponding operator norms of T_a are less than or equal to $\|a\|_{\ell^\infty(X,V)}$. The operator norm of T_a is equal to $\|a\|_{\ell^\infty(X,V)}$ in both cases, because of (16.3) and (16.4).

Suppose now that V is complete with respect to the metric associated to N. If $f \in \ell^1(X,k)$, then $a\,f \in \ell^1(X,V)$, and the sum on the right side of (16.2) can be defined as an element of V as in Section 13. This defines T_a as a bounded linear mapping from $\ell^1(X,k)$ into V, with operator norm equal to $\|a\|_{\ell^\infty(X,V)}$, as before. If N is an ultranorm on V, and $f \in c_0(X,k)$, then the sum on the right side of (16.2) can be defined as an element of V again, as in Section 13. This defines T_a as a bounded linear mapping from $c_0(X,k)$ into V in this situation, with operator norm equal to $\|a\|_{\ell^\infty(X,V)}$.

Every linear mapping from $c_{00}(X,k)$ into V is of the form T_a for some $a \in c(X,V)$, as mentioned earlier. If T_a is bounded with respect to the ℓ^1 or supremum norm on $c_{00}(X,V)$, then it is easy to see that $a \in \ell^\infty(X,V)$, using (16.3) and (16.4). If T is a bounded linear mapping from $\ell^1(X,k)$, then T is determined by its restriction to $c_{00}(X,k)$, because $c_{00}(X,k)$ is dense in $\ell^1(X,k)$, as in Section 9. It follows that every bounded linear mapping from $\ell^1(X,k)$ into V is of the form T_a for some $a \in \ell^\infty(X,V)$ when V is complete. Similarly, if N is an ultranorm on V, and T is a bounded linear mapping from $c_0(X,k)$ into V with respect to the supremum norm on $c_0(X,k)$, then T is of the form T_a for some $a \in \ell^\infty(X,V)$.

17 Bounded linear functionals

Let k be a field, and let V be a vector space over k. As usual, a *linear functional* on V is a linear mapping from V into k, where k is considered as a one-dimensional vector space over itself. Let $|\cdot|$ be an absolute value function on k, and let N be a norm on V with respect to $|\cdot|$ on k. A *bounded linear functional* is a linear functional on V that is bounded as a linear mapping from V into k, using $|\cdot|$ as a norm on k. The space of bounded linear functionals on V is denoted V', which is the same as $\mathcal{BL}(V, k)$ in the notation of Section 11. This is the *dual space* associated to V, and it is a vector space over k with respect to pointwise addition and scalar multiplication. The *dual norm* N' on V' associated to N on V is the same as the operator norm in this situation. Thus if $\lambda \in V'$, then

$$|\lambda(v)| \le N'(\lambda)\, N(v) \tag{17.1}$$

for every $v \in V$, and N' is the smallest nonnegative real number with this property. If $|\cdot|$ is an ultrametric absolute value function on k, then N' is an ultranorm on V', as in Section 11. If k is complete with respect to the metric associated to $|\cdot|$, then V' is complete with respect to the metric associated to the dual norm, as in Section 12.

Let us continue to ask that k be complete with respect to the metric associated to $|\cdot|$. Let X be a nonempty set, and let a be a bounded k-valued function on X. If $f \in \ell^1(X, k)$, then put

$$\lambda_a(f) = \sum_{x \in X} a(x)\, f(x), \tag{17.2}$$

where the sum on the right side of (17.2) can be defined as in Section 13. This is the same as in the previous section, with $V = k$. This defines a bounded linear functional on $\ell^1(X, k)$, with dual norm equal to $\|a\|_{\ell^\infty(X,k)}$. Every bounded linear functional on $\ell^1(X, k)$ is of this form, so that the dual of $\ell^1(X, k)$ can be identified with $\ell^\infty(X, k)$. The cases where $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value functions are of particular concern here.

Suppose that $|\cdot|$ is an ultrametric absolute value function on a field k, and that k is complete with respect to the corresponding ultrametric, as before. If $a \in \ell^\infty(X, k)$, then (17.2) defines a bounded linear functional on $c_0(X, k)$, with respect to the supremum norm on $c_0(X, k)$. This follows from the remarks in the previous section again, with $V = k$. As before, the dual norm of (17.2) on $c_0(X, k)$ is equal to $\|a\|_{\ell^\infty(X,k)}$, and every bounded linear functional on $c_0(X, k)$ is of this form. Thus the dual of $c_0(X, k)$ can be identified with $\ell^\infty(X, k)$ in this situation.

Let us now take $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. If $a \in \ell^1(X, k)$ and $f \in \ell^\infty(X, k)$, then $a\, f \in \ell^1(X, k)$, with

$$\|a\, f\|_{\ell^1(X,k)} \le \|a\|_{\ell^1(X,k)}\, \|f\|_{\ell^\infty(X,k)}, \tag{17.3}$$

as in (16.5). This implies that $\lambda_a(f)$ can be defined as an element of k as in (17.2), as discussed in Section 13, with

$$|\lambda_a(f)| \le \|a\, f\|_{\ell^1(X,k)} \le \|a\|_{\ell^1(X,k)}\, \|f\|_{\ell^\infty(X,k)}. \tag{17.4}$$

Thus λ_a defines a bounded linear functional on $\ell^\infty(X, k)$, with dual norm less than or equal to $\|a\|_{\ell^1(X,k)}$. The dual norm of λ_a on $\ell^\infty(X, k)$ is equal to $\|a\|_{\ell^1(X,k)}$ in this case, because there is an $f \in \ell^\infty(X, k)$ such that $\|f\|_{\ell^\infty(X,k)} = 1$,

$$a(x)\, f(x) = |a(x)| \tag{17.5}$$

for every $x \in X$, and hence

$$\lambda_a(f) = \sum_{x \in X} |a(x)| = \|a\|_{\ell^1(X,k)}. \tag{17.6}$$

The restriction of λ_a to $c_0(X,k)$ defines a bounded linear functional on $c_0(X,k)$, with respect to the restriction of the supremum norm to $c_0(X,k)$. The dual norm of the restriction of λ_a to $c_0(X,k)$ is clearly less than or equal to $\|a\|_{\ell^1(X,k)}$, and we would like to verify that they are equal. To do this, let E be a nonempty finite subset of X, and let f be a k-valued function on X with support contained in E such that $\|f\|_{\ell^\infty(X,k)} = 1$ and (17.5) holds for every $x \in E$. Observe that

$$\lambda_a(f) = \sum_{x \in E} |a(x)|, \tag{17.7}$$

and that this is less than or equal to the dual norm of the restriction of λ_a to $c_0(X,k)$. This implies that the dual norm of the restriction of λ_a to $c_0(X,k)$ is equal to $\|a\|_{\ell^1(X,k)}$, by taking the supremum over all nonempty finite subsets E of X.

Let λ be any bounded linear functional on $c_0(X,k)$, with respect to the restriction of the supremum norm to $c_0(X,k)$. Put

$$a(x) = \lambda(\delta_x) \tag{17.8}$$

for each $x \in X$, where δ_x is as in (16.1). This defines a k-valued function on X, which determines a linear functional λ_a on $c_{00}(X,k)$ as in (17.2). If $f \in c_{00}(X,k)$, then it is easy to see that

$$\lambda_a(f) = \lambda(f), \tag{17.9}$$

by expressing f as a linear combination of the δ_x's. Using this, one can check that $a \in \ell^1(X,k)$, with $\|a\|_{\ell^1(X,k)}$ less than or equal to the dual norm of λ on $c_0(X,k)$, by the same type of argument as in the preceding paragraph. In particular, λ_a also defines a bounded linear functional on $c_0(X,k)$, as before. It follows that (17.9) holds for every $f \in c_0(X,k)$, because $c_{00}(X,k)$ is dense in $c_0(X,k)$ with respect to the supremum metric. This shows that the dual of $c_0(X,k)$ can be identified with $\ell^1(X,k)$ when $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function.

Let k be a field with an ultrametric absolute value function $|\cdot|$ again, and suppose that k is complete with respect to the ultrametric associated to $|\cdot|$. If $a \in c_0(X,k)$ and $f \in \ell^\infty(X,k)$, then $a\,f \in c_0(X,k)$, as in the previous section. This implies that the sum on the right side of (17.2) can be defined as an element of k, as in Section 13. Thus λ_a defines a linear functional on $\ell^\infty(X,k)$, with

$$|\lambda_a(f)| \le \|a\,f\|_{\ell^\infty(X,k)} \le \|a\|_{\ell^\infty(X,k)}\,\|f\|_{\ell^\infty(X,k)} \tag{17.10}$$

for every $f \in \ell^\infty(X,k)$. This means that λ_a is a bounded linear functional on $\ell^\infty(X,k)$, with dual ultranorm less than or equal to $\|a\|_{\ell^\infty(X,k)}$. The dual ultranorm of λ_a on $\ell^\infty(X,k)$ is equal to $\|a\|_{\ell^\infty(X,k)}$, because

$$\lambda_a(\delta_x) = a(x) \tag{17.11}$$

for every $x \in X$. More precisely, the dual ultranorm of the restriction of λ_a to $c_0(X, k)$ is equal to $\|a\|_{\ell^\infty(X,k)}$ too, using the restriction of the supremum norm to $c_0(X, k)$.

18 Inner product spaces

Suppose for the moment that V and W are vector spaces over the complex numbers, and remember that V, W can also be considered as vector spaces over the real numbers. A mapping T from V into W is said to be *real* or *complex linear* if T is linear when V, W are considered to be vector spaces over the real or complex numbers, respectively. Note that a real-linear mapping T from V into W is complex linear exactly when

$$T(i\,v) = i\,T(v) \tag{18.1}$$

for every $v \in V$. A real-linear mapping T from V into V is said to be *conjugate linear* if

$$T(i\,v) = -i\,T(v) \tag{18.2}$$

for every $v \in V$. This implies that

$$T(a\,v) = \overline{a}\,T(v) \tag{18.3}$$

for every $a \in \mathbf{C}$ and $v \in V$, where $\overline{a}$ is the complex-conjugate of a.

Now let V be a vector space over the real or complex numbers, and let $\langle v, w\rangle$ be a real or complex-valued function, as appropriate, defined for $v, w \in V$. As usual, $\langle v, w\rangle$ is said to be an *inner product* on V if it satisfies the following three conditions. The first condition asks that

$$\lambda_w(v) = \langle v, w\rangle \tag{18.4}$$

be a linear functional on V, as a function of v, for every $w \in V$. The second condition asks that

$$\langle v, w\rangle = \langle w, v\rangle \tag{18.5}$$

for every $v, w \in V$ in the real case, and that

$$\langle v, w\rangle = \overline{\langle v, w\rangle} \tag{18.6}$$

for every $v, w \in V$ in the complex case. It follows that $\langle v, w\rangle$ is linear as a function of w for every $v \in V$ in the real case, and conjugate-linear in the complex case. Note that

$$\langle v, v\rangle \in \mathbf{R} \tag{18.7}$$

for every $v \in V$ in the complex case, by (18.6). The third condition asks that

$$\langle v, v\rangle > 0 \tag{18.8}$$

for every $v \in V$ with $v \neq 0$.

Suppose that $\langle v, w \rangle$ is an inner product on V, and put

$$\|v\| = \langle v, v \rangle^{1/2} \tag{18.9}$$

for every $v \in V$. The *Cauchy–Schwarz inequality* states that

$$|\langle v, w \rangle| \le \|v\| \, \|w\| \tag{18.10}$$

for every $v, w \in V$. One can use this to show that (18.9) defines a norm on V with respect to the standard Euclidean absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate. The standard inner product on $\mathbf{R}^n$ is defined for each positive integer n by

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j \, w_j \tag{18.11}$$

for every $v, w \in \mathbf{R}^n$. Similarly, the standard inner product is defined on $\mathbf{C}^n$ by

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j \, \overline{w_j} \tag{18.12}$$

for every $v, w \in \mathbf{C}^n$.

Let $\langle v, w \rangle$ be an inner product on a vector space V over the real or complex numbers again. Also let $w \in V$ be given, so that (18.4) defines a linear functional λ_w on V. The Cauchy–Schwarz inequality implies that λ_w is a bounded linear functional on V with respect to the norm (18.9), with dual norm less than or equal to $\|w\|$. It is easy to see that the dual norm of λ_w is equal to $\|w\|$, because

$$\lambda_w(w) = \langle w, w \rangle = \|w\|^2. \tag{18.13}$$

Similarly, one can check that

$$\|v\| = \sup\{|\langle v, w \rangle| : w \in V, \ \|w\| \le 1\} \tag{18.14}$$

for every $v \in V$.

19 Hilbert spaces

Let $(V, \langle \cdot, \cdot \rangle_V)$ be a real or complex inner product space, and let $\| \cdot \|_V$ be the corresponding norm, as in (18.9). If V is complete with respect to the metric associated to this norm, then V is said to be a *Hilbert space*. Otherwise, one can pass to a completion, as in Section 12. More precisely, $\langle \cdot, \cdot \rangle_V$ can be extended continuously to the completion of V, and this extension defines an inner product on the completion of V. The norm associated to the extension of $\langle \cdot, \cdot \rangle_V$ to the completion of V is the same as the continuous extension of $\| \cdot \|_V$ to the completion of V, so that the completion of V is a Hilbert space too.

Remember that (18.4) defines a bounded linear functional on V for every $w \in V$. If V is a Hilbert space, then it is well known that every bounded linear functional on V is of this form for some $w \in V$. It is easy to see that this representation is unique.

Suppose that $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ are Hilbert spaces, both real or both complex, and with the corresponding norms $\| \cdot \|_V$ and $\| \cdot \|_W$, respectively. If T is a bounded linear mapping from V into W, and if $w \in W$, then

$$v \mapsto \langle T(v), w \rangle_W \tag{19.1}$$

defines a bounded linear functional on V. Indeed,

$$|\langle T(v), w \rangle_W| \le \|T(v)\|_V \, \|w\|_W \le \|T\|_{op,VW} \, \|v\|_V \, \|w\|_W \tag{19.2}$$

for every $v \in V$, where $\|T\|_{op,VW}$ is the operator norm of T with respect to $\| \cdot \|_V$ and $\| \cdot \|_W$. This implies that (19.1) is a bounded linear functional on V with respect to $\| \cdot \|_V$, with dual norm less than or equal to $\|T\|_{op,VW} \, \|w\|_W$. It follows that there is a unique element $T^*(w)$ of V such that

$$\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V \tag{19.3}$$

for every $v \in V$, because V is a Hilbert space. We also have that

$$\|T^*(w)\|_V \le \|T\|_{op,VW} \, \|w\|_W, \tag{19.4}$$

because the dual norm of (19.1) is equal to $\|T^*(w)\|_V$, and less than or equal to the right side of (19.4). One can check that T^* defines a linear mapping from W into V, which is known as the *adjoint* of T. More precisely, T^* is a bounded linear mapping from W into V, with

$$\|T^*\|_{op,WV} \le \|T\|_{op,VW}, \tag{19.5}$$

by (19.4). Similarly,

$$|\langle T(v), w \rangle_W| = |\langle v, T^*(w) \rangle_V| \le \|v\| \, \|T^*(w)\|_V \le \|v\| \, \|T^*\|_{op,WV} \, \|w\| \tag{19.6}$$

for every $v \in V$ and $w \in W$, which implies that

$$\|T(v)\|_W \le \|v\| \, \|T^*\|_{op,WV} \tag{19.7}$$

for every $v \in V$. This shows that $\|T\|_{op,VW} \le \|T^*\|_{op,WV}$, so that

$$\|T^*\|_{op,WV} = \|T\|_{op,VW}. \tag{19.8}$$

The adjoint $(T^*)^*$ of T^* is defined as a bounded linear mapping from V into W in the same way, and one can verify that

$$(T^*)^* = T. \tag{19.9}$$

The mapping $T \mapsto T^*$ is linear as a mapping from $\mathcal{BL}(V, W)$ into $\mathcal{BL}(W, V)$ in the real case, and conjugate-linear in the complex case. The identity mapping

I_V on V defines a bounded linear mapping from V into itself, and it is easy to see that

$$I_V^* = I_V \tag{19.10}$$

as bounded linear mappings from V into itself.

Let $(Z, \langle \cdot, \cdot \rangle_Z)$ be another Hilbert space, which is real or complex depending on whether V and W are real or complex. If T_1 is a bounded linear mapping from V into W, and T_2 is a bounded linear mapping from W into Z, then $T_2 \circ T_1$ is a bounded linear mapping from V into Z, as in Section 11. One can check that

$$(T_2 \circ T_1)^* = T_1^* \circ T_2^* \tag{19.11}$$

as bounded linear mappings from Z into V.

20 Isometric linear mappings

Let k be a field with an absolute value function $|\cdot|$, and let V, W be vector spaces over k with norms N_V, N_W, respectively. A linear mapping T from V into W is said to be an *isometry* with respect to N_V, N_W if

$$N_W(T(v)) = N_V(v) \tag{20.1}$$

for every $v \in V$. In particular, this implies that the kernel of T is trivial, so that T is injective. Of course, (20.1) implies that T is a bounded linear mapping from V into W, with operator norm equal to 1 when $V \neq \{0\}$.

Suppose that T is a one-to-one linear mapping from V onto W, so that the inverse mapping T^{-1} is defined as a linear mapping from W onto V. In this case, T is an isometry from V onto W if and only if T^{-1} is an isometry from W onto V. If T is a bounded linear mapping from V onto W, T^{-1} is a bounded linear mapping from W onto V, and

$$\|T\|_{op,VW}, \ \|T^{-1}\|_{op,WV} \le 1, \tag{20.2}$$

then one can check that T is an isometry.

Let us now take $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function for the rest of the section. Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be Hilbert spaces again, both real or both complex, and with corresponding norms $\| \cdot \|_V$ and $\| \cdot \|_W$, respectively. Suppose that a linear mapping T from V into W satisfies

$$\langle T(u), T(v) \rangle_W = \langle u, v \rangle_V \tag{20.3}$$

for every $u, v \in V$. This implies that T is an isometry with respect to $\| \cdot \|_V$ and $\| \cdot \|_W$, by taking $u = v$. It is well known that the converse holds, because of polarization identities. If T also maps V onto W, then T is said to be *unitary*. In the real case, T may be called an *orthogonal* linear transformation.

Now let T be any bounded linear mapping from V into itself, so that the adjoint T^* of T is defined as a bounded linear mapping from W into V, as in the previous section. If $u, v \in V$, then

$$\langle T^*(T(u)), v \rangle_V = \langle T(u), T(v) \rangle_W. \tag{20.4}$$

Using this, one can check that (20.3) holds if and only if

$$T^* \circ T = I_V, \tag{20.5}$$

where I_V denotes the identity mapping on V, as before. If T maps V onto W, then (20.5) implies that T is invertible as a linear mapping from V onto W, with $T^{-1} = T^*$.

If T is any bounded linear mapping from V into W and $v \in V$, then

$$\langle T^*(T(v)), v\rangle_V = \|T(v)\|_W^2, \tag{20.6}$$

by taking $u = v$ in (20.4). This implies that

$$\|T(v)\|_W^2 \le \|T^*(T(v))\|_V \, \|v\|_V \le \|T^* \circ T\|_{op,VV} \, \|v\|_V^2 \tag{20.7}$$

for every $v \in V$, using the Cauchy–Schwarz inequality in the first step, and the definition of the operator norm in the second step. It follows that

$$\|T\|_{op,VW}^2 \le \|T^* \circ T\|_{op,VV}. \tag{20.8}$$

We also have that

$$\|T^* \circ T\|_{op,VV} \le \|T\|_{op,VW} \, \|T^*\|_{op,WV} = \|T\|_{op,VW}^2, \tag{20.9}$$

using (11.8) in the first step, and (19.8) in the second step. Thus

$$\|T^* \circ T\|_{op,VV} = \|T\|_{op,VW}^2, \tag{20.10}$$

by combining (20.8) and (20.9).

21 $n \times n$ Matrices

Let R be a ring, and let n be a positive integer. The space of $n \times n$ matrices with entries in R is denoted $M_n(R)$. Of course, $M_n(R)$ is a commutative group with respect to addition of matrices, which is defined using addition on R in each entry. If $A = [a_{j,l}]$ and $B = [b_{j,l}]$ are elements of $M_n(R)$, then their *product* is defined to be the element $C = [c_{j,l}]$ of $M_n(R)$ given by

$$c_{j,l} = \sum_{r=1}^{n} a_{j,r} \, b_{r,l} \tag{21.1}$$

for every $j, l = 1, \ldots, n$, as usual. It is well known and easy to see that $M_n(R)$ is a ring with respect to matrix multiplication. If R has a multiplicative identity element e, then the *identity matrix* I in $M_n(R)$ is defined to be the matrix with diagonal entries equal to e, and off-diagonal entries equal to 0. This is the multiplicative identity element in $M_n(R)$ in this case.

Let k be a field, and let $\mathcal{A}$ be an algebra over k. Note that $M_n(\mathcal{A})$ is a vector space over k, where addition and scalar multiplication of matrices is defined entry-wise. More precisely, $M_n(\mathcal{A})$ is an algebra over k with respect to matrix multiplication. Let $|\cdot|$ be an absolute value function on k, and let N be a seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k. If $A = [a_{j,l}] \in M_n(\mathcal{A})$, then put

$$N_\infty(A) = \max_{1 \le j,l \le n} N(a_{j,l}). \tag{21.2}$$

It is easy to see that this defines a seminorm on $M_n(\mathcal{A})$ with respect to $|\cdot|$ on k, which is a norm when N is a norm on $\mathcal{A}$. If N is a semi-ultranorm on $\mathcal{A}$, then N_∞ is a semi-ultranorm on $M_n(\mathcal{A})$ with respect to $|\cdot|$ on k. If N is a submultiplicative semi-ultranorm on $\mathcal{A}$, then one can check that N_∞ is submultiplicative on $M_n(\mathcal{A})$.

Similarly, if $A = [a_{j,l}] \in M_n(\mathcal{A})$, then put

$$N_{1,\infty}(A) = \max_{1 \le l \le n} \Big(\sum_{j=1}^n N(a_{j,l}) \Big). \tag{21.3}$$

One can verify that this also defines a seminorm on $M_n(\mathcal{A})$ with respect to $|\cdot|$ on k, which is a norm when N is a norm on $\mathcal{A}$. Observe that

$$N_\infty(A) \le N_{1,\infty}(A) \le n\, N_\infty(A) \tag{21.4}$$

for every $A \in M_n(\mathcal{A})$. Suppose that N is submultiplicative on $\mathcal{A}$, and let us check that $N_{1,\infty}$ is submultiplicative on $M_n(\mathcal{A})$. Let $A = [a_{j,l}]$ and $B = [b_{j,l}]$ be elements of $M_n(\mathcal{A})$, and let $C = [c_{j,l}]$ be as in (21.1). Thus

$$N(c_{j,l}) \le \sum_{r=1}^n N(a_{j,r}\, b_{r,l}) \le \sum_{r=1}^n N(a_{j,r})\, N(b_{r,l}) \tag{21.5}$$

for each $j, l = 1, \ldots, n$. Summing over j, we get that

$$\begin{aligned} \sum_{j=1}^n N(c_{j,l}) &\le \sum_{j=1}^n \Big(\sum_{r=1}^n N(a_{j,r})\, N(b_{r,l}) \Big) \\ &= \sum_{r=1}^n \Big(\sum_{j=1}^n N(a_{j,r})\, N(b_{r,l}) \Big) \\ &\le N_{1,\infty}(A) \Big(\sum_{r=1}^n N(b_{r,l}) \Big) \le N_{1,\infty}(A)\, N_{1,\infty}(B) \end{aligned} \tag{21.6}$$

for each $l = 1, \ldots, n$. This implies that

$$N_{1,\infty}(C) \le N_{1,\infty}(A)\, N_{1,\infty}(B), \tag{21.7}$$

as desired, by taking the maximum over $l = 1, \ldots, n$.

If $A = [a_{j,l}] \in M_n(\mathcal{A})$ again, then put

$$N_{\infty,1}(A) = \max_{1 \le j \le n} \Big(\sum_{l=1}^n N(a_{j,l}) \Big). \tag{21.8}$$

As before, this is a seminorm on $M_n(\mathcal{A})$ with respect to $|\cdot|$ on k, and a norm when N is a norm on $\mathcal{A}$. We also have that

$$N_\infty(A) \le N_{\infty,1}(A) \le n\, N_\infty(A) \tag{21.9}$$

for every $A \in M_n(\mathcal{A})$, as in (21.4). Let us verify that $N_{\infty,1}$ is submultiplicative on $M_n(\mathcal{A})$ when N is submultiplicative on $\mathcal{A}$. Let $A = [a_{j,l}], B = [b_{j,l}] \in M_n(\mathcal{A})$ be given, let $C = [c_{j,l}]$ be as in (21.1), and remember that (21.5) holds for every $j, l = 1, \ldots, n$ under these conditions. In this case, we can sum over l to get that

$$\begin{aligned} \sum_{l=1}^n N(c_{j,l}) &\le \sum_{l=1}^n \Big(\sum_{r=1}^n N(a_{j,r})\, N(b_{r,l})\Big) \\ &= \sum_{r=1}^n \Big(\sum_{l=1}^n N(a_{j,r})\, N(b_{r,l})\Big) \\ &\le \Big(\sum_{r=1}^n N(a_{j,r})\Big)\, N_{\infty,1}(B) \le N_{\infty,1}(A)\, N_{\infty,1}(B) \end{aligned} \tag{21.10}$$

for every $j = 1, \ldots, n$. Hence

$$N_{\infty,1}(C) \le N_{\infty,1}(A)\, N_{\infty,1}(B), \tag{21.11}$$

as desired, by taking the maximum over $j = 1, \ldots, n$.

22 Involutions

Let R be a ring, so that R is a commutative group with respect to addition in particular. Also let

$$x \mapsto x^* \tag{22.1}$$

be a mapping from R into itself that is a group homomorphism with respect to addition. If

$$(x\, y)^* = y^*\, x^* \tag{22.2}$$

for every $x, y \in R$, and

$$(x^*)^* = x \tag{22.3}$$

for every $x \in R$, then (22.1) is said to be an (ring) *involution* on R. Note that the identity mapping on R is a ring involution when R is commutative.

Let n be a positive integer, and let $M_n(R)$ be the ring of $n \times n$ matrices with entries in R, as in the previous section. If $A = [a_{j,l}] \in M_n(R)$, then the *transpose* of A is the matrix $A^t = [a^t_{j,l}] \in M_n(R)$ defined by

$$a^t_{j,l} = a_{l,j} \tag{22.4}$$

for $j, l = 1, \ldots, n$, as usual. Observe that

$$A \mapsto A^t \tag{22.5}$$

is a group isomorphism from $M_n(R)$ into itself with respect to addition, and that

$$(A^t)^t = A \tag{22.6}$$

for every $A \in M_n(R)$.

Let (22.1) be an involution on R. If $A = [a_{j,l}] \in M_n(R)$, then let $A^{t*} = [a^{t*}_{j,l}] \in M_n(R)$ be defined by

$$a^{t*}_{j,l} = (a^t_{j,l})^* = (a_{l,j})^* \tag{22.7}$$

for $j, l = 1, \ldots, n$. It is well known and easy to see that

$$A \mapsto A^{t*} \tag{22.8}$$

defines an involution on $M_n(R)$. If R is commutative, then we can take (22.1) to be the identity mapping on R, so that (22.5) is an involution on $M_n(R)$.

Let k be a field, and let $\mathcal{A}$ be an algebra over k. A linear mapping from $\mathcal{A}$ into itself is said to be an (algebra) *involution* on $\mathcal{A}$ if it is an involution on $\mathcal{A}$ as a ring. If $k = \mathbf{C}$, then a conjugate-linear mapping on $\mathcal{A}$ is said to be a conjugate-linear involution on $\mathcal{A}$ if it is an involution on $\mathcal{A}$ as a ring. In particular, complex-conjugation may be considered as a conjugate-linear involution on $\mathbf{C}$.

Let $|\cdot|$ be an absolute value function on k, and let N be a seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k. Also let (22.1) be an algebra involution on $\mathcal{A}$, which may be conjugate-linear when $k = \mathbf{C}$. A basic compatibility condition between (22.1) and N is that (22.1) be bounded with respect to N, so that

$$N(x^*) \le C\, N(x) \tag{22.9}$$

for some $C \ge 0$ and every $x \in \mathcal{A}$. This implies that

$$N(x) = N((x^*)^*) \le C\, N(x^*) \tag{22.10}$$

for every $x \in \mathcal{A}$, because of (22.3). In particular, if (22.9) holds with $C = 1$, then

$$N(x^*) = N(x) \tag{22.11}$$

for every $x \in \mathcal{A}$.

Let $(V, \langle v, w\rangle_V)$ be a real or complex Hilbert space, and consider the algebra $\mathcal{BL}(V)$ of bounded linear mappings from V into itself. The mapping from $T \in \mathcal{BL}(V)$ to its adjoint T^* defines an involution on $\mathcal{BL}(V)$, which is conjugate-linear in the complex case, as in Section 19. We have also seen that this involution preserves the operator norm, as in (19.8).

Let $\mathcal{A}$ be an algebra over a field k again, let n be a positive integer, and let $M_n(\mathcal{A})$ be the algebra of $n \times n$ matrices with entries in $\mathcal{A}$, as in the previous section. Also let $|\cdot|$ be an absolute value function on k, and let N be a seminorm on $\mathcal{A}$ with respect to $|\cdot|$ on k. Note that (22.5) defines a linear mapping from $M_n(\mathcal{A})$ into itself. If $A \in M_n(\mathcal{A})$, then

$$N_\infty(A^t) = N_\infty(A), \tag{22.12}$$

where N_∞ is as defined in (21.2). Similarly,

$$N_{1,\infty}(A^t) = N_{\infty,1}(A), \tag{22.13}$$

where $N_{1,\infty}$, $N_{\infty,1}$ are as defined in (21.3), (21.8), respectively.

Let (22.1) be an algebra involution on $\mathcal{A}$, which may be conjugate-linear when $k = \mathbf{C}$, and suppose that (22.9) holds for some $C \geq 0$. If $A \in M_n(\mathcal{A})$, and A^{t*} is as in (22.7), then

$$N_\infty(A^{t*}) \leq C\, N_\infty(A), \tag{22.14}$$

$$N_{1,\infty}(A^{t*}) \leq C\, N_{\infty,1}(A), \tag{22.15}$$

and

$$N_{\infty,1}(A^{t*}) \leq C\, N_{1,\infty}(A). \tag{22.16}$$

If $C = 1$, so that (22.11) holds, then equality holds in (22.14), (22.15), and (22.16) as well.

23 Infinite matrices

Let R be a ring, let X be a nonempty set, and let $c(X, R)$ be the space of R-valued functions on X. This is a commutative group with respect to pointwise addition of functions, and a ring with respect to pointwise multiplication. As before, the *support* of $f \in c(X, R)$ is the set of $x \in X$ such that $f(x) \neq 0$, and we let $c_{00}(X, R)$ be the set of $f \in c(X, R)$ whose support has only finitely many elements. In particular, $c_{00}(X, R)$ is a subgroup of $c(X, R)$ with respect to addition.

An element a of $c(X \times X, R)$ may be considered as a matrix with entries in R, using the elements of X as indices. Let $a, b \in c_{00}(X \times X, R)$ be given, and put

$$c(x, y) = \sum_{w \in X} a(x, w)\, b(w, y) \tag{23.1}$$

for every $x, y \in X$. More precisely, for each $x, y \in X$, all but finitely many terms in the sum on the right side of (23.1) are equal to 0, so that the sum defines an element of R. Thus (23.1) defines an R-valued function on $X \times X$, and one can check that this function has finite support too. The *product* of a and b in $c_{00}(X \times X, R)$ is defined to be c in this discussion, in analogy with matrix multiplication. In particular, if n is a positive integer and $X = \{1, \ldots, n\}$, then $c_{00}(X \times X, R) = c(X \times X, R)$ corresponds to the space $M_n(R)$ of $n \times n$ matrices with entries in R, and (23.1) corresponds to (21.1). One can check that $c_{00}(X \times X, R)$ is a ring with respect to this definition of multiplication, and pointwise addition of functions.

Put

$$\begin{aligned} c_{00,1}(X \times X, R) = \{a \in c(X \times X, R) :\ & \text{for each } y \in X,\ a(x, y) = 0 \\ & \text{for all but finitely many } x \in X\}. \end{aligned} \tag{23.2}$$

Equivalently, $c_{00,1}(X \times X, R)$ consists of $a \in c(X \times X, R)$ such that for each $y \in X$, $a(x, y)$ has finite support as an R-valued function of $x \in X$. Note that $c_{00,1}(X \times X, R)$ is a subgroup of $c(X \times X, R)$ with respect to pointwise addition of functions, and that

$$c_{00}(X \times X, R) \subseteq c_{00,1}(X \times X, R). \tag{23.3}$$

Let $a, b \in c_{00,1}(X \times X, R)$ be given, and let us check that (23.1) defines an element of $c_{00,1}(X \times X, R)$ as well. To do this, let $y \in X$ be given, and put

$$B_y = \{w \in X : b(w, y) \neq 0\}, \tag{23.4}$$

so that B_y has only finitely many elements, because $b \in c_{00,1}(X \times X, R)$. Thus (23.1) reduces to

$$c(x, y) = \sum_{w \in B_y} a(x, w)\, b(w, y), \tag{23.5}$$

which defines an element of R for every $x \in X$, and is interpreted as being equal to 0 when $B_y = \emptyset$. We also have that (23.5) is equal to 0 for all but finitely many $x \in X$, because $a \in c_{00,1}(X \times X, R)$. This shows that (23.1) defines an element of $c_{00,1}(X \times X, R)$, and we take c to be the product of a and b, as before. One can verify that $c_{00,1}(X \times X, R)$ is a ring with respect to pointwise addition of functions and this definition of multiplication.

Similarly, put

$$\begin{aligned} c_{00,2}(X \times X, R) \quad = \quad & \{a \in c(X \times X, R) : \text{ for each } x \in X,\ a(x, y) = 0 \\ & \qquad \text{for all but finitely many } y \in X\}. \end{aligned} \tag{23.6}$$

This is a subgroup of $c(X \times X, R)$ with respect to pointwise addition of functions, and

$$c_{00}(X \times X, R) \subseteq c_{00,2}(X \times X, R), \tag{23.7}$$

as before. If $a, b \in c_{00,2}(X \times X, R)$, then one can check that (23.1) defines an element of $c_{00,2}(X \times X, R)$, using the same type of argument as in the preceding paragraph. This makes $c_{00,2}(X \times X, R)$ into a ring, with respect to pointwise addition of functions, and this definition of multiplication.

If $a \in c(X \times X, R)$, then let $a^t \in c(X \times X, R)$ be defined by

$$a^t(x, y) = a(y, x) \tag{23.8}$$

for every $x, y \in X$. Clearly

$$a \mapsto a^t \tag{23.9}$$

is a group isomorphism from $c(X \times X, R)$ onto itself with respect to addition, with

$$(a^t)^t = a \tag{23.10}$$

for every $a \in c(X \times X, R)$. Of course, (23.9) corresponds to the transpose of a matrix in this situation. Note that (23.9) maps $c_{00}(X \times X, R)$ onto itself, and maps $c_{00,1}(X \times X, R)$ onto $c_{00,2}(X \times X, R)$.

Put

$$c_{00,12}(X \times X, R) = c_{00,1}(X \times X, R) \cap c_{00,2}(X \times X, R), \tag{23.11}$$

which is a subring of $c_{00,1}(X \times X, R)$ and $c_{00,2}(X \times X, R)$. Using (23.3) and (23.7), we get that

$$c_{00}(X \times X, R) \subseteq c_{00,12}(X \times X, R). \tag{23.12}$$

We also have that (23.9) maps $c_{00,12}(X \times X, R)$ onto itself, because it maps $c_{00,1}(X \times X, R)$ onto $c_{00,2}(X \times X, R)$. If R has a multiplicative identity element e, then let $\delta \in c(X \times X, R)$ be defined by

$$\begin{aligned} \delta(x, y) &= e \quad \text{when } x = y \\ &= 0 \quad \text{when } x \neq y. \end{aligned} \tag{23.13}$$

This is an element of $c_{00,12}(X \times X, R)$, and the multiplicative identity element in $c_{00,1}(X \times X, R)$ and $c_{00,2}(X \times X, R)$.

Let $r \mapsto r^*$ be an involution on R. If $a \in c(X \times X, R)$, then let a^{t*} in $c(X \times X, R)$ be defined by

$$a^{t*}(x, y) = (a^t(x, y))^* = a(y, x)^* \tag{23.14}$$

for every $x, y \in X$. Observe that

$$a \mapsto a^{t*} \tag{23.15}$$

is a group isomorphism from $c(X \times X, R)$ onto itself with respect to addition, with

$$(a^{t*})^{t*} = a \tag{23.16}$$

for every $a \in c(X \times X, R)$. As before, (23.15) maps $c_{00}(X \times X, R)$ onto itself, $c_{00,1}(X \times X, R)$ onto $c_{00,2}(X \times X, R)$, and $c_{00,12}(X \times X, R)$ onto itself. If a, b are elements of $c_{00,1}(X \times X, R)$, then it is easy to see that

$$(a\, b)^{t*} = b^{t*}\, a^{t*}, \tag{23.17}$$

where multiplication is defined as in (23.1). Of course, this also works when $a, b \in c_{00,2}(X \times X, R)$. In particular, (23.15) is an involution on $c_{00,12}(X \times X, R)$, and hence on $c_{00}(X \times X, R)$.

24 Double sums

Let X be a nonempty set, and let A be a commutative group, with the group operations expressed additively. The space $c(X, A)$ of A-valued functions on X is also a commutative group with respect to pointwise addition of functions. As usual, the *support* of $f \in c(X, A)$ is defined to be the set of $x \in X$ such that $f(x) \neq 0$. Let $c_{00}(X, A)$ be the set of $f \in c(X, A)$ with finite support in X, which is a subgroup of $c(X, A)$. If $f \in c_{00}(X, A)$, then $\sum_{x \in X} f(x)$ can be defined as an element of A, and this defines a group homomorphism from $c_{00}(X, A)$ into A.

Let X, Y be nonempty sets, so that their Cartesian product $X \times Y$ is a nonempty set as well. If $f \in c_{00}(X \times Y, A)$, then

$$f_X(x) = \sum_{y \in Y} f(x, y) \tag{24.1}$$

is defined as an element of A for each $x \in X$, and

$$f_Y(y) = \sum_{x \in X} f(x, y) \tag{24.2}$$

is defined as an element of A for each $y \in Y$. It is easy to see that f_X and f_Y have finite support as A-valued functions on X and Y, respectively, so that

$$\sum_{x \in X} f_X(x) \tag{24.3}$$

and

$$\sum_{y \in Y} f_Y(y) \tag{24.4}$$

are defined as elements of A too. The double sum

$$\sum_{(x,y) \in X \times Y} f(x, y) \tag{24.5}$$

can also be defined as an element of A. Of course, the iterated sums (24.3) and (24.4) are equal to (24.5).

Now let f be a nonnegative real-valued function on $X \times Y$. The sums (24.1), (24.2), and (24.5) can be defined as nonnegative extended real numbers, as in Section 9. The iterated sums (24.3) and (24.4) can also be defined as nonnegative extended real numbers, with the convention that the sum is automatically $+\infty$ when any of the terms is $+\infty$. One can check that (24.3), (24.4), and (24.5) are the same in this situation too, by approximating all of these sums by finite subsums.

Let k be a field with an absolute value function $|\cdot|$, let V be a vector space over k, and let N be a norm on V with respect to $|\cdot|$ on k. Suppose that V is complete with respect to the metric associated to N. Let f be a V-valued function on $X \times Y$ that is summable with respect to N. Thus the double sum (24.5) can be defined as an element of V, as in Section 13. Observe that $f(x, y)$ is summable as a V-valued function on $x \in X$ for each $y \in Y$, and similarly that $f(x, y)$ is summable as a V-valued function of $y \in Y$ for each $x \in X$. This permits us to define $f_X(x)$ and $f_Y(y)$ as V-valued functions on X and Y, respectively, as in (24.1) and (24.2), and using the remarks in Section 13 again. We also have that

$$N(f_X(x)) \le \sum_{y \in Y} N(f(x, y)) \tag{24.6}$$

for every $x \in X$, and

$$N(f_Y(y)) \le \sum_{x \in X} N(f(x, y)) \tag{24.7}$$

for every $y \in Y$. It follows that

$$\sum_{x \in X} N(f_X(x)) \le \sum_{x \in X} \Big(\sum_{y \in Y} N(f(x, y)) \Big) = \sum_{(x,y) \in X \times Y} N(f(x, y)) \tag{24.8}$$

and

$$\sum_{y \in Y} N(f_Y(y)) \le \sum_{y \in Y} \Big(\sum_{x \in X} N(f(x, y)) \Big) = \sum_{(x,y) \in X \times Y} N(f(x, y)), \tag{24.9}$$

using the previous remarks about double sums of nonnegative real numbers. In particular, f_X is summable as a V-valued function on X, and f_Y is summable as a V-valued function on Y. This means that the sums (24.3) and (24.4) can also be defined as elements of V, as in Section 13. Under these conditions, one can check that the iterated sums (24.3) and (24.4) are equal to the double sum (24.5). More precisely, this was mentioned earlier when f has finite support in $X \times Y$, and otherwise one can approximate f by V-valued functions with finite support in $X \times Y$ with respect to the ℓ^1 norm.

Suppose now that N is an ultranorm on V, and that V is still complete with respect to the ultrametric associated to N. Let f be a V-valued function on $X \times Y$ that vanishes at infinity with respect to N. In this case, the double sum (24.5) can be defined as an element of V, as in Section 13 again. It is easy to see that $f(x, y)$ vanishes at infinity as a V-valued function of $x \in X$ for every $y \in Y$, and as a function of $y \in Y$ for every $x \in X$. Hence $f_X(x)$ and $f_Y(y)$ may be defined as V-valued functions on X and Y, respectively, as in (24.1) and (24.2), using the remarks in Section 13. In this situation,

$$N(f_X(x)) \le \max_{y \in Y} N(f(x, y)) \tag{24.10}$$

for every $x \in X$, and

$$N(f_Y(y)) \le \max_{x \in X} N(f(x, y)) \tag{24.11}$$

for every $y \in Y$. One can verify that the right side of (24.10) vanishes at infinity as a nonnegative real-valued function of $x \in X$, because $f(x, y)$ vanishes at infinity on $X \times Y$, by hypothesis. Similarly, the right side of (24.11) vanishes at infinity as a nonnegative real-valued function of $y \in Y$. This implies that the iterated sums (24.3) and (24.4) can be defined as elements of V, as in Section 13. One can check that the iterated sums (24.3) and (24.4) are equal to the double sum (24.5), by approximating f by V-valued functions with finite support in $X \times Y$ with respect to the supremum norm.

25 A summability condition

Let k be a field with an absolute value function $|\cdot|$, let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k, and suppose that

$\mathcal{A}$ is complete with respect to the metric associated to N. Also let X be a nonempty set, and remember that the space $c(X \times X, \mathcal{A})$ of $\mathcal{A}$-valued functions on $X \times X$ is a vector space over k with respect topointwise addition and scalar multiplication. If a is an element of $c(X \times X, \mathcal{A})$, then

$$N_\infty(a) = \sup_{x,y \in X} N(a(x,y)) \tag{25.1}$$

and

$$N_{1,\infty}(a) = \sup_{y \in X} \Big(\sum_{x \in X} N(a(x,y)) \Big) \tag{25.2}$$

are defined as nonnegative extended real numbers. Note that (25.1) and (25.2) correspond to (21.2) and (21.3) when $X = \{1, \ldots, n\}$ for some positive integer n. Of course, (25.1) is finite exactly when a is bounded on $X \times X$, in which case (25.1) is the same as the supremum norm of a. Put

$$M_X^{1,\infty}(\mathcal{A}) = \{a \in c(X \times X, \mathcal{A}) : N_{1,\infty}(a) < \infty\}. \tag{25.3}$$

One can check that (25.3) is a linear subspace of $c(X \times X, \mathcal{A})$, and that (25.2) defines a norm on (25.3) with respect to $|\cdot|$ on k. Clearly

$$N_\infty(a) \le N_{1,\infty}(a) \tag{25.4}$$

for every $a \in c(X \times X, \mathcal{A})$, so that (25.3) is contained in $\ell^\infty(X \times X, \mathcal{A})$. One can also verify that (25.3) is complete with respect to the metric associated to (25.2), because $\mathcal{A}$ is complete by hypothesis.

Let $a, b \in M_X^{1,\infty}(\mathcal{A})$ be given. We would like to put

$$c(x,y) = \sum_{w \in X} a(x,w)\, b(w,y) \tag{25.5}$$

for every $x, y \in X$, as in Section 23. Of course,

$$\sum_{w \in X} N(a(x,w)\, b(w,y)) \le \sum_{w \in X} N(a(x,w))\, N(b(w,y)) \tag{25.6}$$

for every $x, y \in X$, because N is submultiplicative on $\mathcal{A}$. The right side of (25.6) is finite for every $x, y \in X$, because $b \in M_X^{1,\infty}(\mathcal{A})$, and $N(a(x,w))$ is bounded. Thus the right side of (25.5) may be defined as an element of $\mathcal{A}$ for every $x, y \in X$, as in Section 13. Note that the norm of (25.5) with respect to N is less than or equal to the left side of (25.6) for every $x, y \in X$. This implies that

$$\sum_{x \in X} N(c(x,y)) \le \sum_{x \in X} \Big(\sum_{w \in X} N(a(x,w))\, N(b(w,y)) \Big) \tag{25.7}$$

for every $y \in X$. It follows that

$$\sum_{x \in X} N(c(x,y)) \le \sum_{w \in X} \Big(\sum_{x \in X} N(a(x,w))\, N(b(w,y)) \Big) \tag{25.8}$$

for every $y \in X$, by interchanging the order of summation on the right side of (25.7), as in the previous section. Hence

$$\sum_{x \in X} N(c(x,y)) \le N_{1,\infty}(a) \sum_{w \in X} N(b(w,y)) \le N_{1,\infty}(a)\, N_{1,\infty}(b) \tag{25.9}$$

for every $y \in X$, by the definition (25.2) of $N_{1,\infty}$. This shows that $c \in M_X^{1,\infty}(\mathcal{A})$, with

$$N_{1,\infty}(c) \le N_{1,\infty}(a)\, N_{1,\infty}(b). \tag{25.10}$$

As before, we take c to be the *product* of a and b, which may be expressed as

$$c = a\, b. \tag{25.11}$$

It is easy to see that this operation of multiplication is bilinear on $M_X^{1,\infty}(\mathcal{A})$.

Now let a, b, c be any elements of $M_X^{1,\infty}(\mathcal{A})$. In order to show that multiplication is associative on $M_X^{1,\infty}(\mathcal{A})$, one would like to verify that

$$\sum_{z \in X} \Big(\sum_{w \in X} a(x,w)\, b(w,z) \Big)\, c(z,y) = \sum_{w \in X} a(x,w) \Big(\sum_{z \in X} b(w,z)\, c(z,y) \Big) \tag{25.12}$$

for every $x, y \in X$. More precisely, the left side of (25.12) is $(a\, b)\, c$ evaluated at x, y, and the right side is $a\, (b\, c)$ evaluated at x, y. Equivalently, this means that

$$\sum_{z \in X} \Big(\sum_{w \in X} a(x,w)\, b(w,z)\, c(z,y) \Big) = \sum_{w \in X} \Big(\sum_{z \in X} a(x,w)\, b(w,z)\, c(z,y) \Big) \tag{25.13}$$

for every $x, y \in X$, so that one would like to interchange the order of summation. If $x, y \in X$, then

$$\begin{aligned} \sum_{z \in X} \Big(\sum_{w \in X} & N(a(x,w)\, b(w,z)\, c(z,y)) \Big) \\ \le \quad & \sum_{z \in X} \Big(\sum_{w \in X} N(a(x,w))\, N(b(w,z))\, N(c(z,y)) \Big) \\ \le \quad & N_\infty(a) \sum_{z \in X} \Big(\sum_{w \in X} N(b(w,z))\, N(c(z,y)) \Big) \\ \le \quad & N_\infty(a)\, N_{1,\infty}(b) \sum_{z \in X} N(c(z,y)) \le N_\infty(a)\, N_{1,\infty}(b)\, N_{1,\infty}(c). \end{aligned} \tag{25.14}$$

This implies that for each $x, y \in X$,

$$a(x,w)\, b(w,z)\, c(z,y) \tag{25.15}$$

is summable with respect to N as an $\mathcal{A}$-valued function of w and z on $X \times X$. This permits one to interchange the order of summation in (25.13), by the remarks in the previous section.

Suppose that $\mathcal{A}$ has a multiplicative identity element e, and let δ be the $\mathcal{A}$-valued function on $X \times X$ defined by

$$\begin{aligned} \delta(x, y) &= e \quad \text{when } x = y \\ &= 0 \quad \text{when } x \neq y. \end{aligned} \tag{25.16}$$

Observe that

$$N_{1,\infty}(\delta) = N(e). \tag{25.17}$$

In particular, $\delta \in M_X^{1,\infty}(\mathcal{A})$, and it is easy to see that δ is the multiplicative identity element in $M_X^{1,\infty}(\mathcal{A})$, as in Section 23.

26 Another summability condition

Let us continue with the same notation and hypotheses as in the previous section. If $a \in c(X \times X, \mathcal{A})$, then

$$N_{\infty,1}(a) = \sup_{x \in X} \Big(\sum_{y \in X} N(a(x, y)) \Big) \tag{26.1}$$

is defined as a nonnegative extended real number, which corresponds to (21.8) when $X = \{1, \ldots, n\}$ for some positive integer n. As before, one can verify that

$$M_X^{\infty,1}(\mathcal{A}) = \{a \in c(X \times X, \mathcal{A}) : N_{\infty,1}(a) < \infty\} \tag{26.2}$$

is a linear subspace of $c(X \times X, \mathcal{A})$, and that (26.1) defines a norm on (26.2) with respect to $|\cdot|$ on k. Note that

$$N_\infty(a) \le N_{\infty,1}(a) \tag{26.3}$$

for every $a \in c(X \times X, \mathcal{A})$, where $N_\infty(a)$ is as in (25.1), so that (26.2) is contained in $\ell^\infty(X \times X, \mathcal{A})$. One can check that (26.2) is complete with respect to the metric associated to (26.1), because $\mathcal{A}$ is supposed to be complete, as in the preceding section.

If $a, b \in M_X^{\infty,1}(\mathcal{A})$, then we would like to define $c(x, y)$ as an element of $\mathcal{A}$ for every $x, y \in X$ as in (25.5). In this situation, the right side of (25.6) is finite for every $x, y \in X$, because $a \in M_X^{\infty,1}(\mathcal{A})$ and $N(b(w, y))$ is bounded. This permits us to define $c(x, y)$ as an element of $\mathcal{A}$ as in (25.5) for every $x, y \in X$, using the remarks in Section 13 again. As before, the norm of $c(x, y)$ with respect to N is less than or equal to the left side of (25.6) for every $x, y \in X$. Thus

$$\sum_{y \in X} N(c(x, y)) \le \sum_{y \in X} \Big(\sum_{w \in X} N(a(x, w))\, N(b(w, y)) \Big) \tag{26.4}$$

for every $x \in X$. We can interchange the order of summation on the right side, as in Section 24, to get that

$$\sum_{y \in Y} N(c(x, y)) \le \sum_{w \in Y} \Big(\sum_{y \in X} N(a(x, w))\, N(b(w, y)) \Big) \tag{26.5}$$

for every $x \in X$. It follows that

$$\sum_{y \in Y} N(c(x,y)) \le N_{\infty,1}(b) \sum_{w \in X} N(a(x,w)) \le N_{\infty,1}(a)\, N_{\infty,1}(b) \tag{26.6}$$

for every $x \in X$, by the definition (26.1) of $N_{\infty,1}$. Hence $c \in M_X^{\infty,1}(\mathcal{A})$, with

$$N_{\infty,1}(c) \le N_{\infty,1}(a)\, N_{\infty,1}(b). \tag{26.7}$$

We take c to be the *product* of a and b again, which may be expressed as $a\,b$. This operation of multiplication is clearly bilinear on $M_X^{\infty,1}(\mathcal{A})$.

In order to show that multiplication is associative on $M_X^{\infty,1}(\mathcal{A})$, one should check that (25.12) holds for every $a, b, c \in M_X^{\infty,1}(\mathcal{A})$ and $x, y \in X$. This is the same as (25.13), as before. In this situation, we observe that

$$\begin{aligned} \sum_{w \in X} \Big(\sum_{z \in X} N(a(x,w)\, b(w,z)\, c(z,y))\Big) \hspace{-6em} & \\ & \le \sum_{w \in X} \Big(\sum_{z \in X} N(a(x,w))\, N(b(w,z))\, N(c(z,y))\Big) \\ & \le N_\infty(c) \sum_{w \in X} \Big(\sum_{z \in X} N(a(x,w))\, N(b(w,z))\Big) \\ & \le N_{\infty,1}(b)\, N_\infty(c) \sum_{w \in X} N(a(x,w)) \le N_{\infty,1}(a)\, N_{\infty,1}(b)\, N_\infty(c) \end{aligned} \tag{26.8}$$

for every $x, y \in X$. It follows that (25.15) is summable with respect to N as an $\mathcal{A}$-valued function of $w, z \in X$ for every $x, y \in X$. Thus one can interchange the order of summation in (25.13), as in Section 24.

If $\mathcal{A}$ has a multiplicative identity element e, then we define $\delta \in c(X \times X, \mathcal{A})$ as in (25.16). Clearly

$$N_{\infty,1}(\delta) = N(e), \tag{26.9}$$

so that $\delta \in M_X^{\infty,1}(\mathcal{A})$. It is easy to see that δ is the multiplicative identity element in $M_X^{\infty,1}(\mathcal{A})$, as before.

If $a \in c(X \times X, \mathcal{A})$, then $a^t \in c(X \times X, \mathcal{A})$ is defined by

$$a^t(x,y) = a(y,x) \tag{26.10}$$

for every $x, y \in X$, as in Section 23. Of course,

$$a \mapsto a^t \tag{26.11}$$

is a linear mapping from $c(X \times X, \mathcal{A})$ onto itself, and

$$(a^t)^t = a \tag{26.12}$$

for every $a \in c(X \times X, \mathcal{A})$, as before. Observe that (26.11) maps $\ell^\infty(X \times X, \mathcal{A})$ onto itself, with

$$N_\infty(a^t) = N_\infty(a) \tag{26.13}$$

for every $a \in c(X \times X, \mathcal{A})$. Similarly, (26.11) maps $M_X^{1,\infty}(\mathcal{A})$ onto $M_X^{\infty,1}(\mathcal{A})$, with

$$N_{\infty,1}(a^t) = N_{1,\infty}(a) \tag{26.14}$$

for every $a \in c(X \times X, \mathcal{A})$.

Let $x \mapsto x^*$ be an involution on $\mathcal{A}$, which may be conjugate-linear when $k = \mathbf{C}$. If $a \in c(X \times X, \mathcal{A})$, then $a^{t*} \in c(X \times X, \mathcal{A})$ is defined by

$$a^{t*}(x, y) = (a^t(x, y))^* = a(y, x)^* \tag{26.15}$$

for every $x, y \in X$, as in Section 23 again. Clearly

$$a \mapsto a^{t*} \tag{26.16}$$

is a linear mapping from $c(X \times X, \mathcal{A})$ onto itself when $x \mapsto x^*$ is linear, and otherwise (26.16) is conjugate-linear when $k = \mathbf{C}$ and $x \mapsto x^*$ is conjugate-linear. We also have that

$$(a^{t*})^{t*} = a \tag{26.17}$$

for every $a \in c(X \times X, \mathcal{A})$, as before.

Suppose that there is a positive real number C such that

$$N(x^*) \le C\, N(x) \tag{26.18}$$

for every $x \in \mathcal{A}$. In this case, we have that

$$N_\infty(a^{t*}) \le C\, N_\infty(a), \tag{26.19}$$

$$N_{\infty,1}(a^{t*}) \le C\, N_{1,\infty}(a), \tag{26.20}$$

and

$$N_{1,\infty}(a^{t*}) \le C\, N_{\infty,1}(a) \tag{26.21}$$

for every $a \in c(X \times X, \mathcal{A})$. In particular, (26.16) maps $\ell^\infty(X \times X, \mathcal{A})$ onto itself, and $M_X^{1,\infty}(\mathcal{A})$ onto $M_X^{\infty,1}(\mathcal{A})$. If $C = 1$, then equality holds in (26.18), as in Section 22. This implies that equality holds in (26.19), (26.20), and (26.21) as well.

If $a, b \in M_X^{1,\infty}(\mathcal{A})$, then

$$(a\, b)^{t*} = b^{t*}\, a^{t*}, \tag{26.22}$$

where mutliplication is defined as in (25.5), as before. This also works when $a, b \in M_X^{\infty,1}(\mathcal{A})$. Note that

$$M_X^{1,\infty}(\mathcal{A}) \cap M_X^{\infty,1}(\mathcal{A}) \tag{26.23}$$

is a subalgebra of $M_X^{1,\infty}(\mathcal{A})$ and $M_X^{\infty,1}(\mathcal{A})$, and that (26.16) maps (26.23) onto itself, by the remarks in the preceding paragraph. It follows that (26.16) defines an algebra involution on (26.23), which is conjugate-linear when $k = \mathbf{C}$ and $x \mapsto x^*$ is conjugate-linear on $\mathcal{A}$.

27 A c_0 condition

Let k be a field with an absolute value function $|\cdot|$ again, let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k, and let X be a nonempty set. Put

$$\begin{aligned} c_{0,1}(X \times X, \mathcal{A}) \quad &= \quad \{a \in \ell^\infty(X \times X, \mathcal{A}) : \text{ for each } y \in X,\ a(x,y) \text{ vanishes} \\ &\qquad \text{at infinity as a function of } x \in X\}. \end{aligned} \tag{27.1}$$

More precisely, this means that $a(x,y)$ vanishes at infinity with respect to N as an $\mathcal{A}$-valued function of $x \in X$ for each $y \in X$, so that $a(\cdot, y) \in c_0(X, \mathcal{A})$ for every $y \in X$. It is easy to see that (27.1) is a closed linear subspace of $\ell^\infty(X \times X, \mathcal{A})$ with respect to the supremum metric, because $c_0(X, \mathcal{A})$ is a closed linear subspace of $\ell^\infty(X, \mathcal{A})$. If a is an element of the space $M_X^{1,\infty}(\mathcal{A})$ defined in (25.3), then $a \in \ell^\infty(X \times X, \mathcal{A})$, and $a(\cdot, y) \in \ell^1(X, \mathcal{A})$ for every $y \in X$. This implies that $a(\cdot, y) \in c_0(X, \mathcal{A})$ for every $y \in X$, so that $a \in c_{0,1}(X \times X, \mathcal{A})$. Of course, if X has only finitely many elements, then (27.1) is the same as the space $c(X \times X, \mathcal{A})$ of all $\mathcal{A}$-valued functions on $X \times X$.

Let us suppose from now on in this section that N is an ultranorm on $\mathcal{A}$, and that $\mathcal{A}$ is complete with respect to the corresponding ultrametric. If $a, b \in c_{0,1}(X \times X, \mathcal{A})$, then we would like to put

$$c(x,y) = \sum_{w \in X} a(x,w)\, b(w,y) \tag{27.2}$$

for every $x, y \in X$, as before. Observe that $a(x,w)\, b(w,y)$ vanishes at infinity as a function of $w \in X$ for every $x, y \in X$, because $a(x,w)$ is bounded, and $b(w,y)$ vanishes at infinity as a function of $w \in X$, by hypothesis. This implies that the sum on the right side of (27.2) may be defined as an element of $\mathcal{A}$ for every $x, y \in X$, as in Section 13. We also have that

$$N(c(x,y)) \le \max_{w \in X} N(a(x,w)\, b(w,y)) \le \max_{w \in X} (N(a(x,w))\, N(b(w,y))) \tag{27.3}$$

for every $x, y \in X$ in this situation. In particular, it follows that $c(x,y)$ is bounded on $X \times X$, with

$$N_\infty(c) \le N_\infty(a)\, N_\infty(b), \tag{27.4}$$

where N_∞ is the supremum ultranorm, as in (25.1). We would like to check that $c \in c_{0,1}(X \times X, \mathcal{A})$. To do this, let $y \in X$ be given, and let us verify that $c(x,y)$ vanishes at infinity as a function of $x \in X$.

If E is a nonempty finite subset of X, then

$$c_E(x,y) = \sum_{w \in E} a(x,w)\, b(w,y) \tag{27.5}$$

vanishes at infinity as a function of $x \in X$, because $a(x,w)$ vanishes at infinity as a function of $x \in X$ for every $w \in X$, by hypothesis. Of course,

$$c(x,y) - c_E(x,y) = \sum_{w \in X \setminus E} a(x,w)\, b(w,y), \tag{27.6}$$

where the sum on the right is defined as an element of $\mathcal{A}$ for the same reasons as before. It follows that

$$(27.7)\qquad \begin{aligned} N(c(x,y) - c_E(x,y)) &\le \max_{w \in X \setminus E} N(a(x,w)\, b(w,y)) \\ &\le \max_{w \in X \setminus E} (N(a(x,w))\, N(b(w,y))) \\ &\le N_\infty(a) \Big(\max_{w \in X \setminus E} N(b(w,y)) \Big) \end{aligned}$$

for every $x \in X$. We can choose E so that the right side of (27.7) is as small as we want, because $b(w,y)$ vanishes at infinity as a function of $w \in X$. This implies that $c(x,y)$ vanishes at infinity as a function of $x \in X$, because it can be approximated uniformly by functions that vanish at infinity on X.

As in the earlier situations, we take (27.2) to be the product of a and b in $c_{0,1}(X \times X, \mathcal{A})$, which may be expressed as $a\, b$. This defines a bilinear operation of multiplication on $c_{0,1}(X \times X, \mathcal{A})$, which we would like to show is associative. Let $a, b, c \in c_{0,1}(X \times X, \mathcal{A})$ be given, and let us verify that

$$(27.8)\quad \sum_{z \in X} \Big(\sum_{w \in X} a(x,w)\, b(w,z) \Big) c(z,y) = \sum_{w \in X} \Big(\sum_{z \in X} b(w,z)\, c(z,y) \Big) c(z,y)$$

for every $x, y \in X$. As before, this is the same as saying that

$$(27.9)\quad \sum_{z \in X} \Big(\sum_{w \in X} a(x,w)\, b(w,z)\, c(z,y) \Big) = \sum_{w \in X} \Big(\sum_{z \in X} a(x,w)\, b(w,z)\, c(z,y) \Big)$$

for every $x, y \in X$, so that we want to interchange the order of summation. Let $x, y \in X$ be given, and let us check that

$$(27.10)\qquad a(x,w)\, b(w,z)\, c(z,y)$$

vanishes at infinity with respect to N as an $\mathcal{A}$-valued function of w and z on $X \times X$. Observe that

$$(27.11)\quad \begin{aligned} N(a(w,x)\, b(w,z)\, c(z,y)) &\le N(a(x,w))\, N(b(w,z))\, N(c(z,y)) \\ &\le N_\infty(a)\, N(b(w,z))\, N(c(z,y)) \end{aligned}$$

for every $w, z \in X$. Because $c(z,y)$ vanishes at infinity as a function of $z \in X$, there are finite subsets E of X such that $N(c(z,y))$ is as small as we want when $z \in X \setminus E$. This implies that the right side of (27.11) is as small as we want for every $w \in X$ and $z \in X \setminus E$, because $b(w,z)$ is bounded on $X \times X$. If $z \in E$, then the right side of (27.11) is as small as we want for all but finitely many $w \in X$, because $b(w,z)$ vanishes at infinity as a function of $w \in X$. It follows that (27.10) vanishes at infinity as a function of w and z on $X \times X$, as desired. This permits us to interchange the order of summation in (27.9), as in Section 24.

If $\mathcal{A}$ has a multiplicative identity element e, then we let $\delta(x, y)$ be the $\mathcal{A}$-valued function on $X \times X$ equal to e when $x = y$, and to 0 when $x \neq y$, as before. Note that $\delta \in c_{0,1}(X \times X, \mathcal{A})$, with

$$N_\infty(\delta) = N(e). \tag{27.12}$$

In fact, δ is the multiplicative identity element in $c_{0,1}(X \times X, \mathcal{A})$.

28 Another c_0 condition

Let k be a field with an absolute value function $|\cdot|$, let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k, and let X be a nonempty set, as in the preceding section. Put

$$\begin{aligned} c_{0,2}(X \times X, \mathcal{A}) \quad = \quad \{a \in \ell^\infty(X \times X, \mathcal{A}) : & \text{ for each } x \in X,\, a(x, y) \text{ vanishes} \\ & \text{at infinity as a function of } y \in X\}. \end{aligned} \tag{28.1}$$

As before, this means that $a(x, y)$ vanishes at infinity with respect to N as an $\mathcal{A}$-valued function of y for each $x \in X$, so that $a(x, \cdot) \in c_0(X, \mathcal{A})$ for every $x \in X$. This also defines a closed linear subapce of $\ell^\infty(X \times X, \mathcal{A})$ with respect to the supremum metric, because $c_0(X, \mathcal{A})$ is a closed linear subspace of $\ell^\infty(X, \mathcal{A})$. The space $M_X^{\infty,1}(\mathcal{A})$ defined in (26.2) is contained in (28.1), because $\ell^1(X, \mathcal{A})$ is contained in $c_0(X, \mathcal{A})$.

As in the previous section, we suppose from now on in this section that N is an ultranorm on $\mathcal{A}$, and that $\mathcal{A}$ is complete with respect to the corresponding ultrametric. Let $a, b \in c_{0,2}(X \times X, \mathcal{A})$ be given, and let us define $c(x, y)$ as an element of $\mathcal{A}$ for every $x, y \in X$ as in (27.2) again. More precisely, for each $x, y \in X$, $a(x, w)\, b(w, y)$ vanishes at infinity as a function of $w \in X$, because $a(x, w)$ vanishes at infinity as a function of $w \in X$, and $b(w, y)$ is bounded, by hypothesis. Hence the sum on the right side of (27.2) may be defined as an element of $\mathcal{A}$, as in Section 13. The simple estimates (27.3) and (27.4) also hold in this situation, for the same reasons as before.

In order to check that $c \in c_{0,2}(X \times X, \mathcal{A})$, let $x \in X$ be given, and let us verify that $c(x, y)$ vanishes at infinity as a function of $y \in X$. If E is a nonempty finite subset of X and $c_E(x, y)$ is defined as in (27.5), then $c_E(x, y)$ vanishes at infinity as a function of $y \in X$, because $b(w, y)$ vanishes at infinity as a function of $y \in X$ for every $w \in X$, by hypothesis. Observe that

$$N(c(x, y) - c_E(x, y)) \le \Big(\max_{w \in X \setminus E} N(a(x, w)) \Big)\, N_\infty(b) \tag{28.2}$$

for every $y \in X$, for essentially the same reasons as for (27.7). As before, we can choose E so that the right side of (28.2) is as small as we want, because $a(x, w)$ vanishes at infinity as a function of $w \in X$, by hypothesis. It follows that $c(x, y)$ vanishes at infinity as a function of $y \in X$, because it can be approximated uniformly by functions that vanish at infinity on X.

As usual, we take (27.2) to be the product of a and b in $c_{0,2}(X \times X, \mathcal{A})$, which may be expressed as $a\,b$. To show that this bilinear operation of multiplication is associative, we should check that (27.8) holds for every $a, b, c \in c_{0,2}(X \times X, \mathcal{A})$ and $x, y \in X$. It suffices to verify that

$$a(x,w)\,b(w,z)\,c(z,y) \tag{28.3}$$

vanishes at infinity with respect to N as an $\mathcal{A}$-valued function of $w, z \in X$, as before. In this case, we use the fact that

$$\begin{aligned} N(a(x,w)\,b(w,z)\,c(z,y)) &\le N(a(x,w))\,N(b(w,z))\,N(c(z,y)) \\ &\le N(a(x,w))\,N(b(w,z))\,N_\infty(c) \end{aligned} \tag{28.4}$$

for every $w, z \in X$. Because $a(x,w)$ vanishes at infinity as a function of $w \in X$, there are finite sets $E \subseteq X$ such that $N(a(x,w))$ is as small as we want when $w \in X \setminus E$, which implies that the right side of (28.4) is as small as we want when $w \in X \setminus E$ and $z \in X$, because $b(w,z)$ is bounded. If $w \in E$, then the right side of (28.4) is as small as we want for all but finitely many $z \in X$, because $b(w,z)$ vanishes at infinity as a function of $z \in X$. Hence (28.3) vanishes at infinity as a function of $w, z \in X$, which permits us to interchange the order of summation in (27.9), as in Section 24.

If $\mathcal{A}$ has a multiplicative identity element e, then we define $\delta \in c(X \times X, \mathcal{A})$ as in the previous section. It is easy to see that $\delta \in c_{0,2}(X \times X, \mathcal{A})$, and that δ is the multiplicative identity element in $c_{0,2}(X \times X, \mathcal{A})$.

As before, if $a \in c(X \times X, \mathcal{A})$, then $a^t \in c(X \times X, \mathcal{A})$ is defined by

$$a^t(x,y) = a(y,x) \tag{28.5}$$

for every $x, y \in X$. Remember that

$$a \mapsto a^t \tag{28.6}$$

is a linear mapping from $c(X \times X, \mathcal{A})$ onto itself, and that

$$(a^t)^t = a \tag{28.7}$$

for every $a \in c(X \times X, \mathcal{A})$. We have also seen that (28.6) maps $\ell^\infty(X \times X, \mathcal{A})$ isometrically onto itself. Observe that (28.6) maps $c_{0,1}(X \times X, \mathcal{A})$ as defined in (27.1) onto $c_{0,2}(X \times X, \mathcal{A})$.

Let $x \mapsto x^*$ be an involution on $\mathcal{A}$. If $a \in c(X \times X, \mathcal{A})$, then $a^{t*} \in c(X \times X, \mathcal{A})$ is defined by

$$a^{t*}(x,y) = (a^t(x,y))^* = a(y,x)^* \tag{28.8}$$

for every $x, y \in X$, as before. Remember that

$$a \mapsto a^{t*} \tag{28.9}$$

is a linear mapping from $c(X \times X, \mathcal{A})$ onto itself, and that

$$(a^{t*})^{t*} = a \tag{28.10}$$

for every $a \in c(X \times X, \mathcal{A})$.

Suppose that there is a positive real number C such that

$$N(x^*) \le C\, N(x) \tag{28.11}$$

for every $x \in \mathcal{A}$, so that

$$N_\infty(a^{t*}) \le C\, N_\infty(a) \tag{28.12}$$

for every $a \in c(X \times X, \mathcal{A})$, as in (26.19). Thus (28.9) maps $\ell^\infty(X \times X, \mathcal{A})$ onto itself, and it is easy to see that (28.9) maps $c_{0,1}(X \times X, \mathcal{A})$ onto $c_{0,2}(X \times X, \mathcal{A})$. If $a, b \in c_{0,1}(X \times X, \mathcal{A})$, then

$$(a\, b)^{t*} = b^{t*}\, a^{t*}, \tag{28.13}$$

where multiplication is defined as in (27.2), and this also works when a and b are elements of $c_{0,2}(X \times X, \mathcal{A})$. Put

$$c_{0,12}(X \times X, \mathcal{A}) = c_{0,1}(X \times X, \mathcal{A}) \cap c_{0,2}(X \times X, \mathcal{A}), \tag{28.14}$$

which is a subalgebra of $c_{0,1}(X \times X, \mathcal{A})$ and $c_{0,2}(X \times X, \mathcal{A})$. Under these conditions, (28.10) maps (28.14) onto itself, and defines an algebra involution on (28.14).

Part III

Lipschitz mappings and power series

29 Lipschitz mappings

Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping f from X into Y is said to be *Lipschitz* if there is a nonnegative real number C such that

$$d_Y(f(x), f(x')) \le C\, d_X(x, x') \tag{29.1}$$

for every $x, x' \in X$. In this case, we may also say that f is Lipschitz with constant C. Note that Lipschitz mappings are uniformly continuous. A mapping f from X to Y is Lipschitz with constant $C = 0$ if and only if f is constant as a mapping on X.

Let $x_0 \in X$ be given, and put

$$f_0(x) = d_X(x, x_0) \tag{29.2}$$

for every $x \in X$. Observe that

$$f_0(x) \le f_0(x') + d_X(x, x') \tag{29.3}$$

for every $x, x' \in X$, by the triangle inequality. Equivalently,

$$f_0(x) - f_0(x') \le d_X(x, x') \tag{29.4}$$

for every $x, x' \in X$, and the same inequality holds with the roles of x and x' interchanged. This implies that

$$|f_0(x) - f_0(x')| \le d_X(x, x') \tag{29.5}$$

for every $x, x' \in X$, using the standard absolute value function on $\mathbf{R}$ on the left side of the inequality. Thus f_0 is Lipschitz with constant $C = 1$ as a mapping from X into $\mathbf{R}$, using the standard Euclidean metric on $\mathbf{R}$.

Let (Z, d_Z) be another metric space, and suppose that f_1 is a Lipschitz mapping from X into Y with constant $C_1 \ge 0$, and that f_2 is a Lipschitz mapping from Y into Z with constant $C_2 \ge 0$. If $x, x' \in X$, then

$$d_Z(f_2(f_1(x)), f_2(f_1(x'))) \le C_2\, d_Y(f_1(x), f_1(x')) \le C_1\, C_2\, d_X(x, x'). \tag{29.6}$$

Thus the composition $f_2 \circ f_1$ of f_1 and f_2 is Lipschitz as a mapping from X into Z, with constant $C_1\, C_2$.

Let f be a Lipschitz mapping from X into Y, and consider

$$\inf\{C \ge 0 : (29.1) \text{ holds}\}, \tag{29.7}$$

where more precisely the infimum is taken over all nonnegative real numbers C for which (29.1) holds. This is the infimum of a nonempty set of nonnegative real numbers, by hypothesis, so that the infimum exists and is nonnegative too. One can check that f is Lipschitz with constant equal to (29.7), so that the infimum is automatically attained. Equivalently, if X has at least two elements, then (29.7) is equal to

$$\sup\left\{\frac{d_Y(f(x), f(x'))}{d_X(x, x')} : x, x' \in X,\ x \ne x'\right\}. \tag{29.8}$$

If X has only one element, then (29.7) is always equal to 0.

Let k be a field with an absolute value function $|\cdot|$. Note that $|x|$ is Lipschitz with constant $C = 1$ as a real-valued function on k, with respect to the metric on k associated to $|\cdot|$, and the standard Euclidean metric on $\mathbf{R}$. This follows from the earlier remarks about (29.2), with $X = k$ and $x_0 = 0$. Similarly, if V is a vector space over k with norm N with respect to $|\cdot|$ on k, then N is Lipschitz with constant $C = 1$ as a real-valued function on V with respect to the metric associated to N.

30 Lipschitz seminorms

Let (X, d_X) be a nonempty metric space, let k be a field with an absolute value function $|\cdot|$, and let W be a vector space over k with a seminorm N_W with respect to $|\cdot|$ on k. As before, a mapping f from X into W is said to be *Lipschitz* if there is a nonnegative real number C such that

$$N_W(f(x) - f(x')) \le C\, d_X(x, x') \tag{30.1}$$

for every $x, x' \in X$. If N_W is a norm on W, then this corresponds exactly to (29.1), with $Y = W$, and d_Y taken to be the metric associated to N_W. We may also say that f is Lipschitz with constant C when (30.1) holds. In particular, if f is a constant mapping from X into W, then f is Lipschitz with constant $C = 0$.

Let $\mathrm{Lip}(X, W)$ be the space of Lipschitz mappings from X into W. If f is an element of $\mathrm{Lip}(X, W)$, then put

$$\|f\|_{\mathrm{Lip}} = \|f\|_{\mathrm{Lip}(X,W)} = \inf\{C \ge 0 : (30.1) \text{ holds}\}, \tag{30.2}$$

where more precisely the infimum is taken over all nonnegative real numbers C for which (30.1) holds. One can check that f is Lipschitz with constant equal to (30.2), as in the previous section. Equivalently, if X has at least two elements, then

$$\|f\|_{\mathrm{Lip}} = \sup\left\{\frac{N_W(f(x) - f(y))}{d_X(x, x')} : x, x' \in X, x \ne x'\right\}, \tag{30.3}$$

as before. If X has only one element, then (30.2) is automatically equal to 0.

One can verify that $\mathrm{Lip}(X, W)$ is a vector space over k, with respect to pointwise addition and scalar multiplication. Moreover, (30.2) defines a seminorm on $\mathrm{Lip}(X, W)$ with respect to $|\cdot|$ on k. If N_W is a semi-ultranorm on W, then (30.2) is a semi-ultranorm on $\mathrm{Lip}(X, W)$. If N_W is a norm on W, then $\|f\|_{\mathrm{Lip}} = 0$ if and only if f is a constant mapping from X into W.

Let V be another vector space over k, and let N_V be a norm on V with respect to $|\cdot|$ on k. If N_W is a norm on W and T is a bounded linear mapping from V into W, then T is Lipschitz with respect to the metrics on V and W associated to N_V and N_W, respectively. More precisely, if (11.1) holds for some $C \ge 0$, then (11.2) says exactly that T is Lipschitz with constant C. In this situation, the operator norm $\|T\|_{op,VW}$ of T corresponds exactly to the Lipschitz seminorm of T, as in (30.2).

If X and Y are topological spaces, then the space of continuous mappings from X into Y may be denoted $C(X, Y)$, as usual. Suppose that N_W is a norm on W, so that W may be considered as a topological space with respect to the topology determined by the metric associated to N_W. If X is a nonempty topological space, then the space $C(X, W)$ of continuous mappings from X into W is a vector space over k with respect to pointwise addition and scalar multiplication. Similarly, if (X, d_X) and (Y, d_Y) are metric spaces, then the space of uniformly continuous mappings from X into Y may be denoted $UC(X, Y)$, and the space of Lipschitz mappings from X into Y may be denoted $\mathrm{Lip}(X, Y)$. Thus

$$\mathrm{Lip}(X, Y) \subseteq UC(X, Y) \subseteq C(X, Y). \tag{30.4}$$

If X is a nonempty metric space, then it is easy to see that $UC(X, W)$ is a linear subspace of $C(X, W)$. In this case, $\mathrm{Lip}(X, W)$ may be considered as a linear subspace of $UC(X, W)$.

31 Bounded continuous functions

Let (Y, d_Y) be a metric space, and let E be a subset of Y. As usual, E is said to be *bounded* in Y if there is a finite upper bound for $d(y, z)$ with $y, z \in E$. If y_0 is any element of Y, then E is bounded in Y if and only if E is contained in a ball in Y centered at y_0 with finite radius. If E is compact with respect to the topology determined on Y by d_Y, then E is bounded in Y.

A mapping f from a set X into Y is said to be *bounded* if $f(X)$ is a bounded set in Y. Let $B(X, Y)$ be the set of bounded mappings from X into Y. If $f, g \in B(X, Y)$, then $d_Y(f(x), g(x))$ is bounded as a nonnegative real-valued function on X. If $X \neq \emptyset$, then it follows that

$$\theta(f, g) = \sup_{x \in X} d_Y(f(x), g(x)) \tag{31.1}$$

is defined as a nonnegative real number. One can check that (31.1) defines a metric on $B(X, Y)$, which is the *supremum metric*.

If $\{f_j\}_{j=1}^\infty$ is a sequence of elements of $B(X, Y)$, and $f \in B(X, Y)$, then $\{f_j\}_{j=1}^\infty$ converges to f with respect to the supremum metric if and only if $\{f_j\}_{j=1}^\infty$ converges to f uniformly on X with respect to d_Y on Y. Let $\{f_j\}_{j=1}^\infty$ be any sequence of mappings from X into Y that converges to a mapping f from X into Y uniformly on X. If f_j is bounded for each j, then it is easy to see that f is bounded as well.

If Y is complete with respect to d_Y, then it is well known that $B(X, Y)$ is complete with respect to the supremum metric. More precisely, if $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $B(X, Y)$ with respect to the supremum metric, then $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in Y for each $x \in X$. This implies that $\{f_j(x)\}_{j=1}^\infty$ converges to an element $f(x)$ in Y for each $x \in X$, because Y is complete. One can verify that $\{f_j\}_{j=1}^\infty$ converges to f uniformly on X, using the fact that $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the supremum metric. It follows that f is bounded too, as in the preceding paragraph.

Let X be a nonempty topological space, and let

$$C_b(X, Y) = C(X, Y) \cap B(X, Y) \tag{31.2}$$

be the space of bounded continuous mappings from X into Y. It is well known that $C_b(X, Y)$ is a closed set in $B(X, Y)$ with respect to the supremum metric. If X is compact, then $C_b(X, Y)$ is the same as $C(X, Y)$, because continuous mappings send compact sets to compact sets.

Similarly, if (X, d_X) is a nonempty metric space, then let

$$UC_b(X, Y) = UC(X, Y) \cap B(X, Y) \tag{31.3}$$

be the space of bounded uniformly continuous mappings from X into Y. It is also well known that $UC_b(X, Y)$ is a closed set in $B(X, Y)$ with respect to the supremum metric. If X is compact, then another well-known theorem states that continuous mappings from X into Y are uniformly continuous.

Let k be a field with an absolute value function $|\cdot|$, and let W be a vector space over k with a norm N_W with respect to $|\cdot|$ on k. If X is a nonempty set, then $B(X, W)$ is the same as the space $\ell^\infty(X, W)$ discussed in Section 8. This is a vector space over k with respect to pointwise addition and scalar multiplication, and the supremum metric on this space is the same as the metric associated to the supremum norm discussed earlier. If X is a nonempty topological space, then $C_b(X, W)$ is a linear subspace of $\ell^\infty(X, W)$. Similarly, if (X, d_X) is a nonempty metric space, then $UC_b(X, W)$ is a linear subspace of $C_b(X, W)$.

Let $\mathcal{A}$ be an algebra over k. If X is a nonempty set, then the space $c(X, \mathcal{A})$ of $\mathcal{A}$-valued functions on X is an algebra over k with respect to pointwise multiplication of functions. If $\mathcal{A}$ has a multiplicative identity element e, then the constant function on X equal to e at every point in X is the multiplicative identity element in $c(X, \mathcal{A})$.

Let N be a a submultiplicative norm on $\mathcal{A}$ with respect to $|\cdot|$ on k. It is easy to see that $\ell^\infty(X, \mathcal{A})$ is a subalgebra of $c(X, \mathcal{A})$. Let $\|\cdot\|_\infty$ be the supremum norm on $\ell^\infty(X, \mathcal{A})$ corresponding to N on $\mathcal{A}$, as in Section 8. If $f, g \in \ell^\infty(X, \mathcal{A})$, then

$$\|f\,g\|_\infty = \sup_{x \in X} N(f(x)\,g(x)) \le \sup_{x \in X} (N(f(x))\,N(g(x))) \le \|f\|_\infty\,\|g\|_\infty, \tag{31.4}$$

so that $\|\cdot\|_\infty$ is submultiplicative on $\ell^\infty(X, \mathcal{A})$.

If X is a nonempty topological space, then the space $C(X, \mathcal{A})$ of continuous mappings from X into $\mathcal{A}$ is subalgebra of $c(X, \mathcal{A})$. Similarly,

$$C_b(X, \mathcal{A}) = C(X, \mathcal{A}) \cap \ell^\infty(X, \mathcal{A}) \tag{31.5}$$

is a subalgebra of $\ell^\infty(X, \mathcal{A})$. The supremum norm of $f \in C_b(X, \mathcal{A})$ may also be denoted $\|f\|_{sup}$.

Let (X, d_X) be a nonempty metric space, and let $f, g \in UC_b(X, \mathcal{A})$ be given. Observe that

$$f(x)\,g(x) - f(y)\,g(y) = (f(x) - f(y))\,g(x) + f(y)\,(g(x) - g(y)) \tag{31.6}$$

for every $x, y \in X$, so that

$$\begin{aligned} &N(f(x)\,g(x) - f(y)\,g(y)) \\ &\quad\le\ N(f(x) - f(y))\,N(g(x)) + N(f(y))\,N(g(x) - g(y)) \\ &\quad\le\ N(f(x) - f(y))\,\|g\|_{sup} + \|f\|_{sup}\,N(g(x) - g(y)) \end{aligned} \tag{31.7}$$

for every $x, y \in X$. Using this, it is easy to see that $f\,g$ is uniformly continuous on X, so that $UC_b(X, \mathcal{A})$ is a subalgebra of $C_b(X, \mathcal{A})$. Similarly, if N is an ultranorm on $\mathcal{A}$, then

$$\begin{aligned} &N(f(x)\,g(x) - f(y)\,g(y)) \\ &\quad\le\ \max(N(f(x) - f(y))\,N(g(x)),\ N(f(y))\,N(g(x) - g(y))) \\ &\quad\le\ \max(N(f(x) - f(y))\,\|g\|_{sup},\ \|f\|_{sup}\,N(g(x) - g(y))) \end{aligned} \tag{31.8}$$

for every $x, y \in X$.

32 Bounded Lipschitz functions

Let k be a field with an absolute value function $|\cdot|$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k. Also let (X, d_X) be a nonempty metric space, and let

$$\mathrm{Lip}_b(X, \mathcal{A}) = \mathrm{Lip}(X, \mathcal{A}) \cap \ell^\infty(X, \mathcal{A}) \tag{32.1}$$

be the space of $\mathcal{A}$-valued functions on X that are bounded and Lipschitz. If $f, g \in \mathrm{Lip}_b(X, \mathcal{A})$, then one can check that $f\,g$ is Lipschitz on X as well, using (31.7). To make this more precise, let $\|\cdot\|_{\mathrm{Lip}}$ be the seminorm on $\mathrm{Lip}(X, \mathcal{A})$ defined in Section 30, and let $\|\cdot\|_{sup}$ be the supremum norm on $C_b(X, \mathcal{A})$, as in the previous section. If $f, g \in \mathrm{Lip}_b(X, \mathcal{A})$, then it is easy to see that

$$\|f\,g\|_{\mathrm{Lip}} \le \|f\|_{\mathrm{Lip}}\,\|g\|_{sup} + \|f\|_{sup}\,\|g\|_{\mathrm{Lip}}, \tag{32.2}$$

using (31.7). If N is an ultranorm on $\mathcal{A}$, then one can verify that

$$\|f\,g\|_{\mathrm{Lip}} \le \max(\|f\|_{\mathrm{Lip}}\,\|g\|_{sup},\ \|f\|_{sup}\,\|g\|_{\mathrm{Lip}}), \tag{32.3}$$

using (31.8). Thus $\mathrm{Lip}_b(X, \mathcal{A})$ is a subalgebra of $UC_b(X, \mathcal{A})$.

If r is a nonnegative real number, then

$$\|f\|_{sup} + r\,\|f\|_{\mathrm{Lip}} \tag{32.4}$$

and

$$\max(\|f\|_{sup}, r\,\|f\|_{\mathrm{Lip}}) \tag{32.5}$$

define norms on $\mathrm{Lip}_b(X, \mathcal{A})$ with respect to $|\cdot|$ on k. One can check that (32.4) is submultiplicative on $\mathrm{Lip}_b(X, \mathcal{A})$ for every $r \ge 0$, using (32.2). If N is an ultranorm on $\mathcal{A}$, then (32.5) is submultiplicative on $\mathrm{Lip}_b(X, \mathcal{A})$ too, because of (32.3). In this case, (32.5) is an ultranorm on $\mathrm{Lip}_b(X, \mathcal{A})$, because $\|f\|_{sup}$ is an ultranorm on $\mathrm{Lip}_b(X, \mathcal{A})$, and $\|f\|_{\mathrm{Lip}}$ is a semi-ultranorm on $\mathrm{Lip}(X, \mathcal{A})$, as in Section 30.

Note that (32.5) is less than or equal to (32.4) for every $r \ge 0$, and that (32.4) is less than or equal to two times (32.5). Of course, (32.4) and (32.5) both increase monotonically in r. If $0 < r \le r' < \infty$, then the analogue of (32.4) with r replaced by r' is bounded by r'/r times (32.4), and similarly for (32.5).

If $\mathcal{A}$ is complete with respect to the metric associated to N, and $r > 0$, then $\mathrm{Lip}_b(X, \mathcal{A})$ is complete with respect to the metric associated to (32.4) or (32.5). To see this, let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence in $\mathrm{Lip}_b(X, \mathcal{A})$ with respect to the metric associated to (32.4) or (32.5). In particular, $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the supremum metric, and hence converges uniformly to a bounded $\mathcal{A}$-valued function f on X, as in the previous section. One can check that f is Lipschitz on X, because Cauchy sequences are bounded, so that $\|f_j\|_{\mathrm{Lip}}$ is bounded. One can use the Cauchy condition in $\mathrm{Lip}_b(X, \mathcal{A})$ again to get that $\{f_j\}_{j=1}^\infty$ converges to f with respect to the Lipschitz seminorm under these conditions.

Let $f \in \mathrm{Lip}_b(X, \mathcal{A})$ be given, so that $f(x)^j$ is an element of $\mathrm{Lip}_b(X, \mathcal{A})$ for each $j \in \mathbf{Z}_+$. One can check that

$$\|f^j\|_{\mathrm{Lip}} \le j \, \|f\|_{sup}^{j-1} \, \|f\|_{\mathrm{Lip}} \tag{32.6}$$

for every $j \ge 1$, using (32.2) and induction. If N is an ultranorm on $\mathcal{A}$, then

$$\|f^j\|_{\mathrm{Lip}} \le \|f\|_{sup}^{j-1} \, \|f\|_{\mathrm{Lip}} \tag{32.7}$$

for every $j \ge 1$, because of (32.3).

33 Polynomials

Let k be a field, and let $a_0, \ldots, a_n$ be $n+1$ elements of k for some nonnegative integer n. Also let T be an indeterminate, so that

$$f(T) = \sum_{j=0}^{n} a_j \, T^j \tag{33.1}$$

is a *formal polynomial* in T with coefficients in k. As in [1, 5], we use upper-case letters like T for indeterminates, and lower-case letters for elements of k, or elements of algebras over k. Let $\mathcal{A}$ be an algebra over k, and suppose that $\mathcal{A}$ has a multiplicative identity element e. If $x \in \mathcal{A}$, then

$$f(x) = \sum_{j=0}^{n} a_j \, x^j \tag{33.2}$$

defines an element of $\mathcal{A}$, where x^j is interpreted as being equal to e when $j = 0$.

Let $|\cdot|$ be an absolute value function on k, and let N be a submultiplicative norm on $\mathcal{A}$ such that $N(e) = 1$. Observe that

$$N(f(x)) \le \sum_{j=0}^{n} |a_j| \, N(x^j) \le \sum_{j=0}^{n} |a_j| \, N(x)^j \tag{33.3}$$

for every $x \in \mathcal{A}$. If N is an ultranorm on $\mathcal{A}$, then

$$N(f(x)) \le \max_{0 \le j \le n} (|a_j| \, N(x^j)) \le \max_{0 \le j \le n} (|a_j| \, N(x)^j) \tag{33.4}$$

for every $x \in \mathcal{A}$. Of course, (33.2) defines a continuous mapping from $\mathcal{A}$ into itself, with respect to the topology determined on $\mathcal{A}$ by the metric associated to N.

Let r be a positive real number, and let

$$\overline{B}_r = \overline{B}(0, r) = \{x \in \mathcal{A} : N(x) \le r\} \tag{33.5}$$

be the closed ball in $\mathcal{A}$ centered at 0 with radius r with respect to the metric associated to N. We may consider $\overline{B}_r$ as a metric space, using the restriction of the metric on $\mathcal{A}$ associated to N to $\overline{B}_r$. Let $\|\cdot\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})}$ be the seminorm defined on $\mathrm{Lip}(\overline{B_r},\mathcal{A})$ as in Section 30. If j is a nonnegative integer, then the restriction of x^j to $\overline{B}_r$ defines a Lipschitz function on $\overline{B}_r$ with values in $\mathcal{A}$, with

$$\|x^j\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})} \le j\, r^{j-1}. \tag{33.6}$$

More precisely, if $j = 0$, then x^j is the constant function equal to e, which is Lipschitz with constant 0. If $j = 1$, then x^j is the identity mapping, which is Lipschitz with constant 1. If $j \ge 2$, then (33.6) follows from (32.6), because the supremum norm of x on $\overline{B}_r$ is less than or equal to r. If N is an ultranorm on $\mathcal{A}$, then we get that

$$\|x^j\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})} \le r^{j-1} \tag{33.7}$$

for every $j \ge 1$, using (32.7).

Similarly, the restriction of (33.2) to $x \in \overline{B}_r$ defines a Lipschitz function on $\overline{B}_r$ with values in $\mathcal{A}$. Using (33.6), we get that

$$\|f\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})} \le \sum_{j=0}^{n} |a_j|\, \|x^j\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})} \le \sum_{j=1}^{n} j\, |a_j|\, r^{j-1}. \tag{33.8}$$

If N is an ultranorm on $\mathcal{A}$, then $\|\cdot\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})}$ is a semi-ultranorm on $\mathrm{Lip}(\overline{B}_r,\mathcal{A})$, as in Section 30. In this case, we have that

$$\|f\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})} \le \max_{0\le j\le n} (|a_j|\, \|x^j\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})}) \le \max_{1\le j\le n} (|a_j|\, r^{j-1}), \tag{33.9}$$

using (33.7) in the second step. Of course, the right sides of (33.8) and (33.9) should be interpreted as being equal to 0 when $n = 0$.

34 Power series

Let k be a field, and let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of elements of k. If T is an indeterminate, then

$$f(T) = \sum_{j=0}^{\infty} a_j\, T^j \tag{34.1}$$

is a *formal power series* in T with coefficients in k. We would like to consider associated functions on algebras over k, under suitable convegence conditions. Let $|\cdot|$ be an absolute value function on k, and let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k. Suppose that $\mathcal{A}$ has a multiplicative identity element e, with $N(e) = 1$, and that $\mathcal{A}$ is complete with respect to the metric associated to N.

Let r be a positive real number, and suppose for the moment that

$$\sum_{j=0}^{\infty} |a_j|\, r^j \tag{34.2}$$

converges. If $x \in \mathcal{A}$ and $N(x) \le r$, then

$$\sum_{j=0}^{\infty} N(a_j\, x^j) \le \sum_{j=0}^{\infty} |a_j|\, N(x)^j \le \sum_{j=0}^{\infty} |a_j|\, r^j. \tag{34.3}$$

Under these conditions, we put

$$f(x) = \sum_{j=0}^{\infty} a_j\, x^j, \tag{34.4}$$

where the convergence of the series on the right follows from the remarks in Section 10. We also have that

$$N(f(x)) \le \sum_{j=0}^{\infty} N(a_j\, x^j) \le \sum_{j=0}^{\infty} |a_j|\, r^j \tag{34.5}$$

when $N(x) \le r$. Let $\overline{B}_r$ be the closed ball in $\mathcal{A}$ centered at 0 with radius r, as in (33.5). In this situation, the partial sums of the series on the right side of (34.4) converge uniformly on $\overline{B}_r$, by the well-known criterion of Weierstrass. It follows that (34.4) is uniformly continuous on $\overline{B}_r$, because the partial sums are uniformly continuous on $\overline{B}_r$.

Let us suppose for the rest of the section that N is an ultranorm on $\mathcal{A}$. Let r be a positive real number again, and suppose for the moment that

$$\lim_{j \to \infty} |a_j|\, r^j = 0. \tag{34.6}$$

If $x \in \mathcal{A}$ and $N(x) \le r$, then

$$N(a_j\, x^j) = |a_j|\, N(x^j) \le |a_j|\, N(x)^j \le |a_j|\, r^j \tag{34.7}$$

for each j, so that

$$\lim_{j \to \infty} N(a_j\, x^j) = 0. \tag{34.8}$$

This implies that the series on the right side of (34.4) converges in $\mathcal{A}$, as in Section 10. If $f(x)$ is the value of the sum, as before, then

$$N(f(x)) \le \max_{j \ge 0} N(a_j\, x^j) \le \max_{j \ge 0}(|a_j|\, r^j). \tag{34.9}$$

Suppose now that $|a_j|\, r^j$ has a finite upper bound, instead of (34.6). If $x \in \mathcal{A}$, then

$$\begin{aligned} N(a_j\, x^j) \le |a_j|\, N(x)^j &= (|a_j|\, r^j)\, (N(x)/r)^j \\ &\le \Big(\sup_{l \ge 0}(|a_l|\, r^l)\Big)\, (N(x)/r)^j \end{aligned} \tag{34.10}$$

for each j. This implies that (34.8) holds when $N(x) < r$, so that the series on the right side of (34.4) converges in $\mathcal{A}$, as in Section 10. If $f(x)$ is the value of the sum again, then

$$N(f(x)) \le \max_{j \ge 0} N(a_j\, x^j) \le \sup_{j \ge 0}(|a_j|\, r_j). \tag{34.11}$$

35 Lipschitz conditions

Let k be a field with an absolute value function $|\cdot|$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k. Suppose that $\mathcal{A}$ has a multiplicative identity element e, and that $\mathcal{A}$ is complete with respect to the metric associated to N. To be consistent with the previous sections, one might ask that $N(e) = 1$, but this will not really be needed in this section. This was also not needed for the Lipschitz conditions mentioned in Section 33. Let $f(T) = \sum_{j=0}^{\infty} a_j\, T^j$ be a formal power series in an indeterminate T with coefficients in k, and let a positive real number r be given. As in (33.5), we let $\overline{B}_r$ be the closed ball in $\mathcal{A}$ centered at 0 with radius r with respect to N. Similarly, let

$$B_r = B(0,1) = \{x \in \mathcal{A} : N(x) < r\} \tag{35.1}$$

be the open ball in $\mathcal{A}$ centered at 0 with radius r with respect to N.

Suppose for the moment that

$$\sum_{j=1}^{\infty} j\, |a_j|\, r^j \tag{35.2}$$

converges, which implies that (34.2) converges. Thus the series on the right side of (34.4) converges absolutely for every $x \in \mathcal{A}$ with $N(x) \le r$, as before, so that (34.4) defines an $\mathcal{A}$-valued function f on $\overline{B}_r$. In this case, one can check that f is Lipschitz on $\overline{B}_r$, with respect to the restriction of the metric on $\mathcal{A}$ associated to N to $\overline{B}_r$. More precisely, we have that

$$\|f\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})} \le \sum_{j=1}^{\infty} j\, |a_j|\, r^{j-1}, \tag{35.3}$$

as in (33.8).

Let us now consider the case where N is an ultranorm on $\mathcal{A}$. If (34.6) holds, then (34.8) holds for every $x \in \mathcal{A}$ such that $N(x) \le r$, which implies that the series on the right side of (34.4) converges in $\mathcal{A}$, as before. It follows that (34.4) defines an $\mathcal{A}$-valued function f on $\overline{B}_r$ again, and one can verify that f is Lipschitz on $\overline{B}_r$. More precisely,

$$\|f\|_{\mathrm{Lip}(\overline{B}_r,\mathcal{A})} \le \max_{j\ge 1}(|a_j|\, r^{j-1}), \tag{35.4}$$

as in (33.9).

If $|a_j|\, r^j$ has a finite upper bound, then (34.8) holds for every $x \in \mathcal{A}$ with $N(x) < r$, as in the previous section. Hence the series on the right side of (34.4) converges in $\mathcal{A}$ for every $x \in \mathcal{A}$ with $N(x) < r$, so that (34.4) defines an $\mathcal{A}$-valued function f on B_r. As usual, one can check that f is Lipschitz on B_r, with respect to the restriction of the metric on $\mathcal{A}$ associated to N to B_r. We also have that

$$\|f\|_{\mathrm{Lip}(B_r,\mathcal{A})} \le \sup_{j\ge 1}(|a_j|\, r^{j-1}), \tag{35.5}$$

as in (33.9) again.

Let r_0 be a positive real number strictly less than r, so that

$$\overline{B}_{r_0} \subseteq B_r. \tag{35.6}$$

If $|a_j|\, r^j$ has a finite upper bound, then

$$\lim_{j \to \infty} |a_j|\, r_0^j = 0, \tag{35.7}$$

because $|a_j|\, r_0^j = |a_j|\, r^j\, (r_0/r)^j$ is bounded by a constant times $(r_0/r)^j$. If f is defined on B_r as in the preceding paragraph, then the restriction of f to $\overline{B}_{r_0}$ can be treated as in the paragraph before that.

36 Some isometric mappings

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative ultranorm N with respect to $|\cdot|$ on k. To be consistent with the previous sections, one might ask $\mathcal{A}$ to have a multiplicative identity element, but this is not really needed here, because we shall consider power series with constant term equal to 0. Let $g(T)$ be a formal power series in an indeterminate T with coefficients in k of the form

$$g(T) = \sum_{j=2}^{\infty} a_j\, T^j, \tag{36.1}$$

and put

$$f(T) = T + g(T) = T + \sum_{j=2}^{\infty} a_j\, T^j. \tag{36.2}$$

Suppose that $\mathcal{A}$ is complete with respect to the ultrametric associated to N, and let a positive real number r be given. Also let B_r and $\overline{B}_r$ be the open and closed balls in $\mathcal{A}$ centered at 0 with radius r with respect to N, as in (35.1) and (33.5), respectively.

Suppose that

$$\lim_{j \to \infty} |a_j|\, r^j = 0, \tag{36.3}$$

and put

$$g(x) = \sum_{j=2}^{\infty} a_j\, x^j \tag{36.4}$$

for every $x \in \mathcal{A}$ with $N(x) \leq r$, as in Section 34. This defines a Lipschitz mapping from $\overline{B}_r$ into $\mathcal{A}$, with

$$\|g\|_{\mathrm{Lip}(\overline{B}_r, \mathcal{A})} \leq \max_{j \geq 2} (|a_j|\, r^{j-1}), \tag{36.5}$$

as in (35.4). In particular, if

$$|a_j|\, r^{j-1} \le 1 \tag{36.6}$$

for each $j \ge 2$, then g is Lipschitz with constant 1 on $\overline{B}_r$ with respect to the ultrametric associated to N on $\mathcal{A}$. Similarly, put

$$f(x) = x + g(x) = x + \sum_{j=2}^{\infty} a_j\, x^j \tag{36.7}$$

for each $x \in \mathcal{A}$ with $N(x) \le r$. This defines a Lipschitz mapping from $\overline{B}_r$ into $\mathcal{A}$ with constant 1 when (36.6) holds. This can be derived from (35.4), or from the analogous statements for g and the identity mapping on $\overline{B}_r$, using the fact that the Lipschitz seminorm is a semi-ultranorm in this case, as in Section 30. Note that

$$N(f(x)) \le N(x) \tag{36.8}$$

for every $x \in \mathcal{A}$ with $N(x) \le r$ when (36.6) holds. This can be obtained from the fact that f is Lipschitz with constant 1 and $f(0) = 0$, or by estimating $N(f(x))$ as in the first step in (34.9). It follows that

$$f(\overline{B}_r) \subseteq \overline{B}_r \tag{36.9}$$

when (36.6) holds.

Suppose that

$$\max_{j \ge 2}(|a_j|\, r^{j-1}) < 1, \tag{36.10}$$

in addition to (36.3), which also ensures that the maximum on the right side of (36.10) is attained. Let $x, y \in \mathcal{A}$ with $N(x), N(y) \le r$ be given, and let us check that

$$N(f(x) - f(y)) = N(x - y). \tag{36.11}$$

The left side of (36.11) is less than or equal to the right side, because f is Lipschitz with constant 1 on $\overline{B}_r$, as in the preceding paragraph. Thus it suffices to verify that the right side of (36.11) is less than or equal to the left side. Of course, this is trivial when $x = y$, and so we may suppose that $x \ne y$. Observe that

$$x - y = f(x) - f(y) - (g(x) - g(y)), \tag{36.12}$$

so that

$$N(x - y) \le \max(N(f(x) - f(y)),\, N(g(x) - g(y))), \tag{36.13}$$

by the ultranorm version of the triangle inequality. We also have that

$$N(g(x) - g(y)) \le \Big(\max_{j \ge 2}(|a_j|\, r^{j-1})\Big)\, N(x - y), \tag{36.14}$$

by (36.5). If $x \ne y$, then (36.10) implies that the right side of (36.14) is strictly less that $N(x - y)$. This means that

$$N(g(x) - g(y)) < N(x - y), \tag{36.15}$$

by (36.14). It follows that

$$N(x - y) \le N(f(x) - f(y)), \tag{36.16}$$

by (36.13).

Suppose now that (36.6) holds for each $j \ge 2$, but not necessarily (36.3) or (36.10). If $x \in \mathcal{A}$ and $N(x) < r$, then we can define $g(x)$ and $f(x)$ as elements of $\mathcal{A}$ as in (36.4) and (36.7), as mentioned in Section 34. In this situation, g is Lipschitz as a mapping from B_r into $\mathcal{A}$, with

$$\|g\|_{\mathrm{Lip}(B_r, \mathcal{A})} \le \sup_{j \ge 2} (|a_j| \, r^{j-1}) \le 1, \tag{36.17}$$

as in (35.5). Similarly, f is Lipschitz as a mapping from B_r into $\mathcal{A}$, with constant 1. This can be obtained from (35.5), or using (36.17) and the fact the identity mapping on B_r is Lipschitz with constant 1, as before. We also have that (36.8) holds for every $x \in \mathcal{A}$ with $N(x) < r$, for essentially the same reasons as before. This implies that

$$f(B_r) \subseteq B_r. \tag{36.18}$$

Let r_0 be a positive real number with $r_0 < r$, so that (35.7) holds. Observe that

$$\max_{j \ge 2} (|a_j| \, r_0^{j-1}) \le (r_0/r) \Big(\sup_{j \ge 2} (|a_j| \, r^{j-1} |) \Big) \le r_0/r < 1, \tag{36.19}$$

using (36.6) in the second step. Thus the earlier remarks can be applied to the restrictions of f and g to $\overline{B}_{r_0}$. In particular, (36.11) holds for every $x, y \in \mathcal{A}$ with $N(x), N(y) \le r_0$, as before. It follows that (36.11) holds for every $x, y \in \mathcal{A}$ with $N(x), N(y) < 1$, by choosing r_0 so that $N(x), N(y) \le r_0$.

37 Some contractions

Let k be a field with an ultrametric absolute value function $| \cdot |$ again, and let $\mathcal{A}$ be an algebra over k with a submultiplicative ultranorm N with respect to $| \cdot |$ on k. Also let $g(T)$ be a formal power series in an indeterminate T with coefficients in k of the form

$$g(T) = \sum_{j=2}^{\infty} a_j \, T^j, \tag{37.1}$$

and suppose that $\mathcal{A}$ is complete with respect to the ultrametric associated to N. Let r be a positive real number, and suppose that

$$\lim_{j \to \infty} |a_j| \, r^j = 0. \tag{37.2}$$

If $x \in \mathcal{A}$ and $N(x) \le r$, then put

$$g(x) = \sum_{j=2}^{\infty} a_j \, x^j, \tag{37.3}$$

as in Section 34. Note that $\mathcal{A}$ does not need to have a multiplicative identity element here, as in the previous section. Let $\overline{B}_r$ be the closed ball in $\mathcal{A}$ centered at 0 with radius r, as in (33.5), so that g defines an $\mathcal{A}$-valued function on $\overline{B}_r$. More precisely, g is Lipschitz with respect to the ultrametric associated to N on $\mathcal{A}$ and its restriction to $\overline{B}_r$, with

$$\|g\|_{\mathrm{Lip}(\overline{B}_r, \mathcal{A})} \le \max_{j \ge 2} (|a_j| \, r^{j-1}), \tag{37.4}$$

as in (35.4).

Let z_0 be an element of $\mathcal{A}$, and put

$$h_0(x) = z_0 - g(x) \tag{37.5}$$

for every $x \in \overline{B}_r$. This defines a Lipschitz mapping from $\overline{B}_r$ into $\mathcal{A}$, with

$$\|h_0\|_{\mathrm{Lip}(\overline{B}_r, \mathcal{A})} = \|g\|_{\mathrm{Lip}(\overline{B}_r, \mathcal{A})}. \tag{37.6}$$

If the right side of (37.4) is less than or equal to 1, and

$$N(z_0) \le r, \tag{37.7}$$

then

$$h_0(\overline{B}_r) \subseteq \overline{B}_r. \tag{37.8}$$

More precisely, $N(h_0(x)) \le r$ for every $x \in \overline{B}_r$, because $h_0(0) = z_0$, h_0 is Lipschitz with constant 1 on $\overline{B}_r$, and N is an ultranorm on $\mathcal{A}$. One can also verify this more directly from the definitions of g and h_0.

Suppose that (37.7) holds and

$$\max_{j \ge 2} (|a_j| \, r^{j-1}) < 1, \tag{37.9}$$

so that h_0 is a Lipschitz mapping from $\overline{B}_r$ into itself with constant strictly less than 1, by (37.4) and (37.6). In this case, the contraction mapping principle implies that there is an $x_0 \in \overline{B}_r$ such that

$$h_0(x_0) = x_0. \tag{37.10}$$

This uses the fact that $\overline{B}_r$ is complete as a metric space with respect to the restriction of the ultrametric on $\mathcal{A}$ associated to N, because $\mathcal{A}$ is complete, by hypothesis, and $\overline{B}_r$ is a closed set in $\mathcal{A}$. If f is the $\mathcal{A}$-valued function defined on $\overline{B}_r$ as in (36.7), then we get that

$$f(x_0) = x_0 + g(x_0) = z_0, \tag{37.11}$$

because of (37.5) and (37.10). It follows that

$$f(\overline{B}_r) = \overline{B}_r \tag{37.12}$$

under these conditions, because we can take z_0 to be any element of $\overline{B}_r$.

Suppose now that

$$|a_j|\, r^{j-1} \le 1 \tag{37.13}$$

for every $j \ge 2$, without asking that (37.2) or (37.9) hold. If $x \in \mathcal{A}$ and $N(x) < 1$, then $g(x)$ can be defined as an element of $\mathcal{A}$ as in (37.3), as mentioned in Section 34. Let B_r be the open ball in $\mathcal{A}$ centered at 0 with radius r, as in (35.1), so that g defines an $\mathcal{A}$-valued function on B_r. In fact, g is Lipschitz with respect to the ultrametric associated to N on $\mathcal{A}$ and its restriction to B_r, with

$$\|g\|_{\mathrm{Lip}(B_r,\mathcal{A})} \le \sup_{j\ge 2}(|a_j|\, r^{j-1}), \tag{37.14}$$

as in (35.5). Let $z_0 \in \mathcal{A}$ be given again, and let $h_0(x)$ be defined for $x \in B_r$ as in (37.5). Note that h_0 is Lipschitz as a mapping from B_r into $\mathcal{A}$, with

$$\|h_0\|_{\mathrm{Lip}(B_r,\mathcal{A})} = \|g\|_{\mathrm{Lip}(B_r,\mathcal{A})}. \tag{37.15}$$

If

$$N(z_0) < 1, \tag{37.16}$$

then

$$h_0(B_r) \subseteq B_r. \tag{37.17}$$

More precisely, one can check that $N(h_0(x)) < r$ for every $x \in B_r$, because $h_0(0) = z_0$, h_0 is Lipschitz with constant 1 on B_r, and N is an ultranorm on $\mathcal{A}$. One can also get this more directly from the definitions of g and h_0, as before.

Remember that B_r is a closed set in $\mathcal{A}$, because N is an ultranorm on $\mathcal{A}$, as in Section 1. This implies that B_r is complete as a metric space with respect to the restriction of the ultrametric on $\mathcal{A}$ associated to N, because $\mathcal{A}$ is complete. If (37.16) holds, and the right side of (37.14) is strictly less than 1, then one can apply the contraction mapping principle to h_0 on B_r, to get that there is an $x_0 \in B_r$ that satisfies (37.10). However, we can also get this using (37.13), without asking that the right side of (37.14) be less than 1, as follows. Let r_0 be a positive real number such that $r_0 < r$ and

$$N(z_0) \le r_0. \tag{37.18}$$

Note that (35.7) and (36.19) hold, because $r_0 < r$, as before. Thus we can apply the earlier remarks to the restrictions of g and h_0 to $\overline{B}_{r_0}$, to get that there is an $x_0 \in \overline{B}_{r_0}$ that satisfies (37.10). If f is the $\mathcal{A}$-valued function defined on B_r as in (36.7), then (37.11) holds, as before, so that

$$f(B_r) = B_r. \tag{37.19}$$

Part IV
The exponential function

38 The real and complex cases

Let k be a field of characteristic 0, so that there is a natural embedding of $\mathbf{Q}$ into k, and we shall simply think of $\mathbf{Q}$ as being a subfield of k. If T is an indeterminate, then

$$\exp(T) = \sum_{j=0}^{\infty} (1/j!)\, T^j \tag{38.1}$$

may be considered as a formal power series in T with coefficients in k, which is the formal power series corresponding to the exponential function. Of course, $j!$ is "j factorial", the product of the positive integers from 1 to j, which is interpreted as being equal to 1 when $j = 0$. We shall consider the convergence of the corresponding power series in algebras over k in various situations.

Suppose first that r is a nonnegative real number, and put

$$\exp(r) = \sum_{j=0}^{\infty} r^j/j!, \tag{38.2}$$

where the sum on the right is considered as an infinite series of nonnegative real numbers. It is well known and not difficult to show that this series converges, so that (38.2) is defined as a nonnegative real number. Note that $\exp(0) = 1$, and that $\exp(r)$ increases monotonically in r, because r^j increases monotonically in $r \geq 0$ for each $j \geq 0$. We also have that $\exp(r) \to +\infty$ as $r \to +\infty$, because of the analogous property of r^j when $j \geq 1$.

Suppose now that $k = \mathbf{R}$ or $\mathbf{C}$, equipped with the standard absolute value function. Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e and a submultiplicative norm N with respect to $|\cdot|$, and suppose that $\mathcal{A}$ is complete with respect to the metric associated to N. It is also convenient to ask that $N(e) = 1$. If $x \in \mathcal{A}$, then we would like to define

$$\exp(x) = \sum_{j=0}^{\infty} (1/j!)\, x^j \tag{38.3}$$

as an element of $\mathcal{A}$. Observe that

$$\sum_{j=0}^{\infty} N((1/j!)\, x^j) \leq \sum_{j=0}^{\infty} (1/j!)\, N(x)^j = \exp(N(x)), \tag{38.4}$$

so that the series on the right side of (38.3) converges absolutely. Hence this series converges in $\mathcal{A}$, because $\mathcal{A}$ is complete with respect to the metric associated to N, as in Section 10. We also get that

$$N(\exp(x)) \le \sum_{j=0}^{\infty} N((1/j!)\, x^j) \le \exp(N(x)). \tag{38.5}$$

Let x, y be commuting elements of $\mathcal{A}$, so that $x\,y = y\,x$. It is well known that

$$(x+y)^l = \sum_{j=0}^{l} \binom{l}{j} x^j\, y^{l-j} \tag{38.6}$$

for each nonnegative integer j, where

$$\binom{l}{j} = \frac{l!}{j!\,(l-j)!} \tag{38.7}$$

is the usual binomial coefficient. Thus

$$\exp(x+y) = \sum_{l=0}^{\infty} (1/l!)\,(x+y)^l = \sum_{l=0}^{\infty} \Big(\sum_{j=0}^{l} \frac{1}{j!\,(l-j)!}\, x^j\, y^{l-j} \Big). \tag{38.8}$$

The sum on the right corresponds exactly to the Cauchy product of the series used to define $\exp(x)$ and $\exp(y)$. It follows that

$$\exp(x+y) = \exp(x)\, \exp(y) \tag{38.9}$$

under these conditions, as in Section 15.

Note that

$$\exp(0) = e \tag{38.10}$$

in $\mathcal{A}$. If x is any element of $\mathcal{A}$, then x commutes with $-x$, so that

$$\exp(x)\, \exp(-x) = \exp(x-x) = e, \tag{38.11}$$

as in (38.9). Similarly,

$$\exp(-x)\, \exp(x) = e. \tag{38.12}$$

Thus $\exp(-x)$ is the multiplicative inverse of $\exp(x)$ in $\mathcal{A}$.

39 Trivial absolute values on Q

Let k be a field of characteristic 0 again, so that $\mathbf{Q}$ may be considered as a subfield of k. Also let $|\cdot|$ be an absolute value function on k, and suppose that the restriction of $|\cdot|$ to $\mathbf{Q}$ is the trivial absolute value function on $\mathbf{Q}$. In particular, this implies that $|\cdot|$ is non-archimedean on k, so that $|\cdot|$ is an ultrametric absolute value function on k, as in Section 4. Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e and a submultiplicative ultranorm N, and suppose that $\mathcal{A}$ is complete with respect to the ultrametric associated to N. As before, it is convenient to ask that $N(e) = 1$ too.

If $x \in \mathcal{A}$ and $N(x) < 1$, then we can define

$$\exp(x) = \sum_{j=0}^{\infty} (1/j!)\, x^j \tag{39.1}$$

as an element of $\mathcal{A}$, as in Section 34. More precisely,

$$N((1/j!)\, x^j) = |1/j!|\, N(x^j) \le N(x)^j \tag{39.2}$$

for every nonnegative integer j, using the hypothesis that $|\cdot|$ is trivial on $\mathbf{Q}$ in the second step. This implies that

$$\lim_{j \to \infty} N((1/j!)\, x^j) = 0 \tag{39.3}$$

when $N(x) < 1$. Using (39.3), we get that the series on the right side of (39.1) converges in $\mathcal{A}$, because N is an ultranorm on $\mathcal{A}$, and $\mathcal{A}$ is complete with respect to the ultrametric associated to N, as in Section 10. We also have that

$$N(\exp(x)) \le \max_{j \ge 0} N((1/j!)\, x^j) = 1, \tag{39.4}$$

using (39.2) and $N(e) = 1$ in the second step. Similarly,

$$N(\exp(x) - e) = N\Big(\sum_{j=1}^{\infty} (1/j!)\, x^j\Big) \le \max_{j \ge 1} N((1/j!)\, x^j) \le N(x), \tag{39.5}$$

using (39.2) in the second step again. The ultranorm version of the triangle inequality implies that

$$1 = N(e) \le \max(N(e - \exp(x)), N(\exp(x))). \tag{39.6}$$

It follows that $N(\exp(x)) \ge 1$, because the right side of (39.5) is strictly less than 1, by hypothesis. Combining this with (39.4), we obtain that

$$N(\exp(x)) = 1 \tag{39.7}$$

for every $x \in \mathcal{A}$ with $N(x) < 1$.

Let x, y be commuting elements of $\mathcal{A}$ with $N(x), N(y) < 1$. Thus

$$N(x + y) \le \max(N(x), N(y)) < 1, \tag{39.8}$$

by the ultranorm version of the triangle inequality. Under these conditions, we have that

$$\exp(x + y) = \exp(x)\, \exp(y), \tag{39.9}$$

where each of the exponentials is defined as an element of $\mathcal{A}$ as in the preceding paragraph. More precisely, the series defining $\exp(x + y)$ corresponds to the Cauchy product of the series defining $\exp(x)$ and $\exp(y)$, as in (38.8). This implies (39.9), as in Section 15 again. In particular, x automatically commutes with $-x$, so that

$$\exp(x)\,\exp(-x) = \exp(-x)\,\exp(x) = e, \tag{39.10}$$

as in the previous section. Thus $\exp(-x)$ is the multiplicative inverse of $\exp(x)$ in $\mathcal{A}$ for every $x \in \mathcal{A}$ with $N(x) < 1$.

If $x, y \in \mathcal{A}$ and $N(x), N(y) < 1$, then

$$N(\exp(x) - \exp(y)) = N(x - y). \tag{39.11}$$

This basically corresponds to (36.11), in the situation described in the paragraphs after the one containing (36.11) in Section 36, with $r = 1$. More precisely, the exponential function is Lipschitz with constant 1 as a mapping from the open unit ball in $\mathcal{A}$ into $\mathcal{A}$, with respect to the ultrametric associated to N and its restriction to the open unit ball, as in (35.5). This implies that the left side of (39.11) is less than or equal to the right side of (39.11), and the opposite inequality is discussed in Section 36. Of course, the constant terms in the definitions of $\exp(x)$ and $\exp(y)$ cancel each other in the left side of (39.11), and they were not included in Section 36.

Observe that the exponential function maps

$$B_1 = B(0, 1) = \{x \in \mathcal{A} : N(x) < 1\} \tag{39.12}$$

into

$$B(e, 1) = \{z \in \mathcal{A} : N(z - e) < 1\}, \tag{39.13}$$

by (39.5). In fact,

$$\exp(B(0, 1)) = B(e, 1), \tag{39.14}$$

essentially by (37.19), with $r = 1$. This takes the constant term into account for the exponential function, which was taken to be 0 in Section 37.

40 Some preliminary facts

Let p be a prime number, and let $|\cdot|_p$ be the p-adic absolute value function on $\mathbf{Q}$, as in Section 2. In order to deal with the exponential function in p-adic situations, we need to estimate

$$|1/l!|_p = 1/|l!|_p \tag{40.1}$$

for nonnegative integers l, as in [1, 5]. Equivalently, we want to estimate the total number of factors of p in $l!$. This is the same as the sum of the factors of p in the positive integers less than or equal to l. In particular, there are no factors of p in $l!$ when $l < p$, so that (40.1) is equal to 1.

Let $[r]$ denote the integer part of a nonnegative real number r, which is the largest integer less than or equal to r. If l is a nonnegative integer, then the total number of factors of p in $l!$ can be given by

$$\sum_{j=1}^{\infty} \left[\frac{l}{p^j}\right]. \tag{40.2}$$

Of course, $[l/p^j] = 0$ when $l < p^j$, so that all but finitely many terms in the sum are equal to 0. The first term $[l/p]$ is the same as the number of positive integers less than or equal to l that are multiples of p. These are the positive integers less than or equal to l that lead to factors of p in $l!$. However, these positive integers less than or equal to l may have more than one factor of p, which lead to additional factors of p in $l!$. The second term $[l/p^2]$ in (40.2) is the number of positive integers less than or equal to l that are multiples of p^2. Each of these positive integers less than or equal to l lead to at least one additional factor of p in $l!$. If j is any positive integer, then $[l/p^j]$ is the number of positive integers less than or equal to l that are multiples of p^j, each of which leads to at least j factors of p in $l!$. In this way, the sum over j is the total number number of factors of p in $l!$.

Observe that

$$\sum_{j=1}^{\infty} \left[\frac{l}{p^j}\right] < \sum_{j=1}^{\infty} \frac{l}{p^j} \tag{40.3}$$

for each positive integer l, because $l/p^j > 0$ for every j, but $[l/p^j] = 0$ when $l < p^j$. Of course, we can sum the geometric series, to get that

$$\sum_{j=1}^{\infty} \frac{l}{p^j} = (l/p) \sum_{j=0}^{\infty} p^{-j} = (l/p)\,(1 - (1/p))^{-1} = \frac{l}{p-1}. \tag{40.4}$$

It follows that

$$|l!|_p > p^{-l/(p-1)} \tag{40.5}$$

for every $l \geq 1$, so that

$$1/|l!|_p < p^{l/(p-1)}. \tag{40.6}$$

If n is a positive integer, then we have that

$$\sum_{j=1}^{\infty} \left[\frac{p^n}{p^j}\right] = \sum_{j=1}^{n} p^{n-j} = \sum_{j=0}^{n-1} p^j = \frac{p^n - 1}{p - 1}. \tag{40.7}$$

This also works when $n = 0$, with the two sums in the middle interpreted as being equal to 0.

Using (40.3) and (40.4), we get that

$$(p-1) \sum_{j=1}^{\infty} \left[\frac{l}{p^j}\right] < l \tag{40.8}$$

for every $l \geq 1$. This implies that

$$(p-1) \sum_{j=1}^{\infty} \left[\frac{l}{p^j}\right] \leq l - 1, \tag{40.9}$$

because the left side is an integer. Thus

$$\sum_{j=1}^{\infty} \left[\frac{l}{p^j}\right] \leq \frac{l-1}{p-1} \tag{40.10}$$

for every $l \geq 1$, so that

$$|l!|_p \geq p^{-(l-1)/(p-1)}, \tag{40.11}$$

and hence

$$1/|l!|_p \leq p^{(l-1)/(p-1)}. \tag{40.12}$$

41 The p-adic case

Let k be a field of characteristic 0 again, and let $|\cdot|$ be an absolute value function on k. As before, we can think of $\mathbf{Q}$ as being a subfield of k, so that the restriction of $|\cdot|$ to $\mathbf{Q}$ defines an absolute value function on $\mathbf{Q}$. Let p be a prime number, and suppose that the restriction of $|\cdot|$ to $\mathbf{Q}$ is the same as the p-adic absolute value function $|\cdot|_p$ on $\mathbf{Q}$. This implies that $|\cdot|$ is non-archimedean on k, so that $|\cdot|$ is an ultrametric absolute value function on k, as in Section 4. Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e and a submultiplicative ultranorm N, and suppose that $\mathcal{A}$ is complete with respect to the ultrametric associated to N. As usual, it is convenient to also ask that $N(e) = 1$. If k is not already complete with respect to the metric associated to $|\cdot|$, then one can pass to a completion, which contains $\mathbf{Q}_p$ as a subfield in this situation. One can consider $\mathcal{A}$ as an algebra over the completion of k, because $\mathcal{A}$ is complete, and in particular one can consider $\mathcal{A}$ as an algebra over $\mathbf{Q}_p$.

If $x \in \mathcal{A}$, then we would like to put

$$\exp(x) = \sum_{j=0}^{\infty} (1/j!)\, x^j \tag{41.1}$$

under suitable conditions on $N(x)$. Observe that

$$\begin{aligned} N((1/j!)\, x^j) = |1/j!|_p\, N(x^j) &\leq (1/|j!|_p)\, N(x)^j \\ &\leq p^{j/(p-1)}\, N(x)^j = (p^{1/(p-1)}\, N(x))^j \end{aligned} \tag{41.2}$$

for every nonnegative integer j, using (40.6) in the first step on the second line. If

$$N(x) < p^{-1/(p-1)}, \tag{41.3}$$

then the right side of (41.2) tends to 0 as $j \to \infty$. This implies that the series on the right side of (41.1) converges in $\mathcal{A}$, as in Section 10. Thus $\exp(x)$ may be defined as an element of $\mathcal{A}$ when (41.3) holds, in which case we have that

$$N(\exp(x)) \leq \max_{j \geq 0} N((1/j!)\, x^j) = 1, \tag{41.4}$$

using (41.2) and $N(e) = 1$ in the second step.

Similarly, if $x \in \mathcal{A}$ satisfies (41.3), then

$$N(\exp(x) - e) \leq \max_{j \geq 1} N((1/j!)\, x^j) \leq \max_{j \geq 1}((1/|j!|_p)\, N(x)^j), \tag{41.5}$$

using the first line in (41.2) in the second step. If j is a positive integer, then

$$(41.6) \quad (1/|j!|_p)\, N(x)^{j-1} \le p^{(j-1)/(p-1)}\, N(x)^{j-1} = (p^{1/(p-1)}\, N(x))^{j-1},$$

using (40.12) in the first step. Combining this with (41.5), we get that

$$(41.7) \qquad N(\exp(x) - e) \le N(x),$$

because of (41.3). It follows that

$$(41.8) \qquad \begin{aligned} 1 = N(e) &\le \max(N(\exp(x) - e), N(\exp(x))) \\ &\le \max(N(x), N(\exp(x))), \end{aligned}$$

using the ultranorm version of the triangle inequality in the first step. Note that $N(x) < 1$, by (41.3). Thus (41.8) implies that $N(\exp(x)) \ge 1$. Hence

$$(41.9) \qquad N(\exp(x)) = 1$$

when $x \in \mathcal{A}$ satisfies (41.3), by (41.4).

Let x, y be commuting elements of $\mathcal{A}$ satisfying (41.3) and

$$(41.10) \qquad N(y) < p^{-1/(p-1)}.$$

Note that

$$(41.11) \qquad N(x + y) \le \max(N(x), N(y)) < p^{-1/(p-1)}$$

too, by the ultranorm version of the triangle inequality. Under these conditions, we have that

$$(41.12) \qquad \exp(x + y) = \exp(x)\, \exp(y),$$

where each of the exponentials is defined as an element of $\mathcal{A}$, as before. Indeed, the series defining $\exp(x + y)$ corresponds to the Cauchy products of the series defining $\exp(x)$ and $\exp(y)$, as in (38.8). This implies (41.12), as in Section 10. In particular, x automatically commutes with $-x$, so that

$$(41.13) \qquad \exp(x)\, \exp(-x) = \exp(-x)\, \exp(x) = e,$$

as before. This implies that $\exp(-x)$ is the multiplicative inverse of $\exp(x)$ in $\mathcal{A}$ when $x \in \mathcal{A}$ satisfies (41.3).

As usual, $B(x, r)$ denotes the open ball in $\mathcal{A}$ centered at $x \in \mathcal{A}$ with radius $r > 0$ with respect to the ultrametric associated to N. Thus the exponential function defines an $\mathcal{A}$-valued function on $B(0, p^{-1/(p-1)})$. If j is a positive integer, then

$$(41.14) \qquad |1/j!|_p\, (p^{-1/(p-1)})^{j-1} \le 1,$$

as in (40.12). This implies that the exponential function is Lipschitz with constant 1 on $B(0, p^{-1/(p-1)})$, as in (35.5). More precisely, the exponential function corresponds to taking $a_j = 1/j!$ in (35.5), and we take $r = p^{-1/(p-1)}$.

If $x, y \in B(0, p^{-1/(p-1)})$, then

$$(41.15) \qquad N(\exp(x) - \exp(y)) = N(x - y).$$

Of course, the left side of (41.15) is less than or equal to the right side, because the exponential function is Lipschitz with constant 1 on $B(0, p^{-1/(p-1)})$, as in the previous paragraph. The fact that equality holds in (41.15) basically corresponds to (36.11), in the situation described in the paragraphs after the one containing (36.11) in Section 36, with $r = p^{-1/(p-1)}$. Note that (36.6) corresponds to (41.14) here. As before, the constant terms in the definitions of $\exp(x)$ and $\exp(y)$ cancel out in the left side of (41.15), and they were not included in Section 36.

The exponential function maps $B(0, p^{-1/(p-1)})$ into $B(e, p^{-1/(p-1)})$, because of (41.7). We actually have that

$$\exp(B(0, p^{-1/(p-1)})) = B(e, p^{-1/(p-1)}), \tag{41.16}$$

as in (37.19), with $r = p^{-1/(p-1)}$. This takes the constant term into account for the exponential function, as before, while the constant term was taken to be 0 in Section 37. The condition (37.13) corresponds to (41.14) here, as in the preceding paragraph.

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