Some topics in analysis related to Banach algebras, 4

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Abstract

These informal notes are concerned with power series in several variables over fields with absolute value functions.

Contents

Part I
Polynomials and power series

1 Vector spaces and algebras

Let k be a field, and let V be a vector space over k. Also let X be a nonempty set, and let $c(X, V)$ be the space of V-valued functions on X. This is a vector space over k too, with respect to pointwise addition and scalar multiplication. If $f \in c(X, V)$, then the *support* of f is defined to be the set of $x \in X$ such that $f(x) \neq 0$. Let $c_{00}(X, V)$ be the collection of $f \in c(X, V)$ such that the support of f has only finitely many elements. It is easy to see that $c_{00}(X, V)$ is a linear subspace of $c(X, V)$. Of course, if X has only finitely many elements, then $c_{00}(X, V)$ is the same as $c(X, V)$.

In particular, we can apply this to $V = k$, considered as a one-dimensional vector space over itself. If $x, y \in X$, then put

$$\begin{aligned} \delta_x(y) &= 1 \quad \text{when } x = y, \\ &= 0 \quad \text{when } x \neq y, \end{aligned} \tag{1.1}$$

where 0 and 1 are the additive and multiplicative identity elements in k, respectively. Thus $\delta_x \in c_{00}(X, k)$ for each $x \in X$, and the collection of δ_x, $x \in X$, is a basis for $c_{00}(X, k)$ as a vector space over k.

If V, W are vector spaces over k, then we let $\mathcal{L}(V, W)$ be the space of all linear mappings from V into W. This is a vector space over k with respect to pointwise addition and scalar multiplication.

Let V, W, and Z be vector spaces over k, and let b be a mapping from the Cartesian product $V \times W$ of V and W into Z. As usual, b is said to be *bilinear* if b is linear in each coordinate. More precisely, this means that for each $w \in W$, $b(v, w)$ is linear as a function of $v \in V$, and that for each $v \in V$, $b(v, w)$ is linear as a function of $w \in W$.

Let $\mathcal{A}$ be a vector space over k, and suppose that $\mathcal{A}$ is equipped with a bilinear mapping

$$(a, b) \mapsto a\, b \tag{1.2}$$

from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$. If the associative law

$$(1.3) \qquad (a\,b)\,c = a\,(b\,c)$$

holds for every $a, b, c \in \mathcal{A}$, then $\mathcal{A}$ is said to be an (associative) *algebra* over k. If the commutative law

$$(1.4) \qquad a\,b = b\,a$$

holds for every $a, b \in \mathcal{A}$ as well, then $\mathcal{A}$ is said to be a *commutative algebra* over k. Of course, k may be considered as a commutative algebra over itself.

An element e of an algebra $\mathcal{A}$ over k is said to be a *multiplicative identity element* in $\mathcal{A}$ if

$$(1.5) \qquad a\,e = e\,a = a$$

for every $a \in \mathcal{A}$. If $\mathcal{A}$ has a multiplicative identity element, then it is easy to see that it is unique.

If V is a vector space over k, then the space $\mathcal{L}(V) = \mathcal{L}(V, V)$ of linear mappings from V into itself is an algebra over k, with composition of linear mappings as multiplication. In this case, the identity mapping $I = I_V$ on V is the multiplicative identity element in $\mathcal{L}(V)$.

Let $\mathcal{A}$ be an algebra over k, and let X be a nonempty set again. The space $c(X, \mathcal{A})$ of $\mathcal{A}$-valued functions on X is an algebra over k too, with respect to pointwise multiplication of functions. If $\mathcal{A}$ is commutative, then $c(X, \mathcal{A})$ is commutative as well. If $\mathcal{A}$ has a multiplicative identity element e, then the constant function on X equal to e at every point is the multiplicative identity element in $c(X, \mathcal{A})$. In particular, the space $c(X, k)$ of k-valued functions on X is a commutative algebra over k, with a multiplicative identity element.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e, and let a be an element of $\mathcal{A}$. If there is an element a^{-1} of $\mathcal{A}$ such that

$$(1.6) \qquad a\,a^{-1} = a^{-1}\,a = e,$$

then a is said to be *invertible* in $\mathcal{A}$. It is easy to see that a^{-1} is uniquely determined by (1.6). If a, b are invertible elements of $\mathcal{A}$, then $a\,b$ is invertible too, with

$$(1.7) \qquad (a\,b)^{-1} = b^{-1}\,a^{-1}.$$

Let $\mathcal{A}_1$, $\mathcal{A}_2$ be algebras over k. A linear mapping ϕ from $\mathcal{A}_1$ into $\mathcal{A}_2$ is said to be an *algebra homomorphism* if

$$(1.8) \qquad \phi(a\,b) = \phi(a)\,\phi(b)$$

for every $a, b \in \mathcal{A}_1$.

2 Polynomials with vector coefficients

If A is a set and n is a positive integer, then we let A^n be the set of n-tuples $a = (a_1, \ldots, a_n)$ with $a_j \in A$ for each $j = 1, \ldots, n$, as usual. Let $\mathbf{Z}$ be the set of integers, and let $\mathbf{Z}_+$ be the set of positive integers. If $n \in \mathbf{Z}_+$ and $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, then α is called a *multi-index*, and we put

$$|\alpha| = \sum_{j=1}^{n} \alpha_j. \tag{2.1}$$

If $\alpha, \beta \in (\mathbf{Z}_+ \cup \{0\})^n$, then their sum $\alpha + \beta$ is defined as an element of $(\mathbf{Z}_+ \cup \{0\})^n$ by coordinatewise addition, and

$$|\alpha + \beta| = |\alpha| + |\beta|. \tag{2.2}$$

Let k be a field, and let V be a vector space over k again. Also let $T_1, \ldots, T_n$ be n commuting indeterminates, for some positive integer n. As in [1, 3], we normally try to use upper-case letters like T for indeterminates, and lower-case letters like t for elements of k, or algebras over k. Put

$$T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n} \tag{2.3}$$

for each $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, which is a formal monomial in $T_1, \ldots, T_n$. A *formal polynomial* in $T_1, \ldots, T_n$ with coefficients in V can be expressed as

$$f(T) = \sum_{|\alpha| \le N} f_\alpha \, T^\alpha, \tag{2.4}$$

where N is a nonnegative integer, the sum is taken over $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$ with $|\alpha| \le N$, and $f_\alpha \in V$ for each such α.

The space of formal polynomials in $T_1, \ldots, T_n$ with coefficients in V may be denoted $V[T_1, \ldots, V_n]$. This is a vector space over k, with respect to termwise addition and scalar multiplication. More precisely, $V[T_1, \ldots, T_n]$ can be defined as the space $c_{00}((\mathbf{Z}_+ \cup \{0\})^n, V)$ of V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ with finite support, where such a function gives the coefficients of the polynomial. In particular, the coefficient f_α of a formal polynomial $f(T)$ may be considered as being defined for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, with $f_\alpha = 0$ for all but finitely many α. Addition and scalar multiplication of formal polynomials correspond exactly to pointwise addition and scalar multiplication of these V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$.

Let $f(T) \in V[T_1, \ldots, T_n]$ be given as in (2.4), and let t be an element of k^n. If $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, then put

$$t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}, \tag{2.5}$$

where $t_j^{\alpha_j}$ is interpreted as being the multiplicative identity element 1 in k when $\alpha_j = 0$. This defines an element of k, so that

$$f(t) = \sum_{|\alpha| \le N} f_\alpha \, t^\alpha \tag{2.6}$$

defines an element of V. Note that

$$f(T) \mapsto f(t) \tag{2.7}$$

defines a linear mapping from $V[T_1, \ldots, T_n]$ into V. If $t = 0$, then $t^\alpha = 0$ when $\alpha \neq 0$, and

$$f(0) = f_0. \tag{2.8}$$

As before, we can apply the previous remarks to $V = k$, considered as a one-dimensional vector space over itself. Note that the monomials T^α, with $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, may be considered as elements of $k[T_1, \ldots, T_n]$, and that they form a basis for $k[T_1, \ldots, T_n]$ as a vector space over k. If $f(T) \in k[T_1, \ldots, T_n]$ and $v \in V$, then $f(T)\, v$ defines an element of $V[T_1, \ldots, T_n]$.

3 Coefficients in an algebra

Let k be a field, and let $\mathcal{A}$ be an algebra over k. Also let $T_1, \ldots, T_n$ be commuting indeterminates, so that $\mathcal{A}[T_1, \ldots, T_n]$ may be defined as a vector space over k as in the previous section. Let $f(T) \in \mathcal{A}[T_1, \ldots, T_n]$ be given as in (2.4), and let

$$g(T) = \sum_{|\beta| \le N'} g_\beta\, T^\beta \tag{3.1}$$

be another element of $\mathcal{A}[T_1, \ldots, T_n]$, where N' is a nonnegative integer, and $g_\beta \in \mathcal{A}$ for every $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$ with $|\beta| \le N'$. The product

$$h(T) = f(T)\, g(T) \tag{3.2}$$

is defined by

$$h(T) = \sum_{|\gamma| \le N+N'} h_\gamma\, T^\gamma, \tag{3.3}$$

where

$$h_\gamma = \sum_{\alpha+\beta=\gamma} f_\alpha\, g_\beta \tag{3.4}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$ with $|\gamma| \le N + N'$. More precisely, the coefficients f_α, g_β of $f(T)$, $g(T)$ may be considered as being defined for all multi-indices α, β, as in the previous section, with $f_\alpha = 0$ when $|\alpha| > N$, and $g_\beta = 0$ when $|\beta| > N'$. If γ is any multi-index, then there are only finitely many pairs of multi-indices α, β such that $\alpha + \beta = \gamma$. The sum on the right side of (3.4) is taken over these pairs of multi-indices, and defines an element of $\mathcal{A}$. It is easy to see that $h_\gamma = 0$ when $|\gamma| > N + N'$. This defines (3.3) as an element of $\mathcal{A}[T_1, \ldots, T_n]$, which defines multiplication on $\mathcal{A}[T_1, \ldots, T_n]$. Of course, multiplication on $\mathcal{A}[T_1, \ldots, T_n]$ is bilinear with respect to k. One can check that multiplication on $\mathcal{A}[T_1, \ldots, T_n]$ is associative, so that $\mathcal{A}[T_1, \ldots, T_n]$ is an algebra over k. If multiplication on $\mathcal{A}$ is commutative, then multiplication on $\mathcal{A}[T_1, \ldots, T_n]$ is commutative as well.

There is a natural embedding of $\mathcal{A}$ into $\mathcal{A}[T_1, \ldots, T_n]$, which sends $a \in \mathcal{A}$ to the polynomial for which the coefficient of T^α is equal to a when $\alpha = 0$, and to 0 when $\alpha \neq 0$. This embedding is an algebra homomorphism, so that we may identify $\mathcal{A}$ with a subalgebra of $\mathcal{A}[T_1, \ldots, T_n]$. If $\mathcal{A}$ has a multiplicative identity element e, then the corresponding polynomial is the multiplicative identity element in $\mathcal{A}[T_1, \ldots, T_n]$.

Let $f(T) \in \mathcal{A}[T_1, \ldots, T_n]$ be given as in (2.4) again, and let $t \in k^n$ be given. Thus t^α is defined as an element of k for each multi-index α, as in (2.5), so that $f(t)$ can be defined as an element of $\mathcal{A}$ as in (2.6). One can verify that $f(T) \mapsto f(t)$ defines an algebra homomorphism from $\mathcal{A}[T_1, \ldots, T_n]$ into $\mathcal{A}$.

The previous remarks can be applied in particular to $\mathcal{A} = k$, considered as a commutative algebra over itself. Thus $k[T_1, \ldots, T_n]$ is a commutative algebra over k, with a multiplicative identity element corresponding to the multiplicative identity element in k. Let $\mathcal{A}_0$ be an algebra over k with a multiplicative identity element e_0, and let $x \in \mathcal{A}_0^n$ be given. Suppose that the coordinates of x commute in $\mathcal{A}_0$, which is to say that

$$x_j \, x_l = x_l \, x_j \tag{3.5}$$

for every $j, l = 1, \ldots, n$. Note that this condition holds automatically when $n = 1$. If $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, then put

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{3.6}$$

where $x_j^{\alpha_j}$ is interpreted as being equal to e_0 when $\alpha_j = 0$. This defines (3.6) as an element of $\mathcal{A}_0$. If $f(T) \in k[T_1, \ldots, T_n]$ is given as in (2.4), then

$$f(x) = \sum_{|\alpha| \le N} f_\alpha \, x^\alpha \tag{3.7}$$

defines an element of $\mathcal{A}_0$. One can check that $f(T) \mapsto f(x)$ defines an algebra homomorphism from $k[T_1, \ldots, T_n]$ into $\mathcal{A}_0$ under these conditions.

4 Formal power series

Let k be a field, let V be a vector space over k, and let $T_1, \ldots, T_n$ be commuting indeterminates. A *formal power series* in $T_1, \ldots, T_n$ with coefficients in V can be expressed as

$$f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, T^\alpha, \tag{4.1}$$

where $f_\alpha \in V$ for each multi-index α. The space $V[[T_1, \ldots, T_n]]$ of these formal power series is a vector space over k with respect to termwise addition and scalar multiplication. As before, this space can be defined more precisely as the space $c((\mathbf{Z}_+ \cup \{0\})^n, V)$ of all V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$. The space $V[T_1, \ldots, T_n]$ of formal polynomials in $T_1, \ldots, T_n$ with coefficients in V may be considered as a linear subspace of $V[[T_1, \ldots, T_n]]$, corresponding to $c_{00}((\mathbf{Z}_+ \cup \{0\})^n, V)$ as a linear subspace of $c((\mathbf{Z}_+ \cup \{0\})^n, V)$.

Let $\mathcal{A}$ be an algebra over k, so that $\mathcal{A}[[T_1, \ldots, T_n]]$ can be defined initially as a vector space over k, as in the preceding paragraph. Let $f(T) \in \mathcal{A}[[T_1, \ldots, T_n]]$

be given as in (4.1), and let

$$g(T) = \sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} g_\beta \, T^\beta \tag{4.2}$$

be another element of $\mathcal{A}[[T_1, \ldots, T_n]]$. The product $h(T) = f(T)\, g(T)$ is defined by

$$h(T) = \sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} h_\gamma \, T^\gamma, \tag{4.3}$$

where

$$h_\gamma = \sum_{\alpha + \beta = \gamma} f_\alpha \, g_\beta \tag{4.4}$$

for each multi-index γ. More precisely, the sum on the right side of (4.4) is taken over all pairs of multi-indices α, β such that $\alpha + \beta = \gamma$, and there are only finitely many such pairs for each multi-index γ, as before. Thus (4.4) defines an element of $\mathcal{A}$ for every multi-index γ, so that (4.3) defines an element of $\mathcal{A}[[T_1, \ldots, T_n]]$. This defines multiplication on $\mathcal{A}[[T_1, \ldots, T_n]]$, which is bilinear with respect to k. One can verify that multiplication on $\mathcal{A}[[T_1, \ldots, T_n]]$ is associative, as before, so that $\mathcal{A}[[T_1, \ldots, T_n]]$ is an algebra over k. If $\mathcal{A}$ is a commutative algebra, then $\mathcal{A}[[T_1, \ldots, T_n]]$ is a commutative algebra too.

The algebra $\mathcal{A}[T_1, \ldots, T_n]$ of formal polynomials in $T_1, \ldots, T_n$ with coefficients in $\mathcal{A}$ may be considered as a subalgebra of $\mathcal{A}[[T_1, \ldots, T_n]]$. In particular, $\mathcal{A}$ can be identified with a subalgebra of $\mathcal{A}[[T_1, \ldots, T_n]]$, as before. If $\mathcal{A}$ has a multiplicative identity element e, then the corresponding formal power series is the multiplicative identity element in $\mathcal{A}[[T_1, \ldots, T_n]]$.

There is a natural mapping from $\mathcal{A}[[T_1, \ldots, T_n]]$ onto $\mathcal{A}$, which sends a formal power series $f(T)$ as in (4.1) to the $\alpha = 0$ coefficient f_0. Note that this mapping is an algebra homomorphism.

Let us suppose from now on in this section that $\mathcal{A}$ has a multiplicative identity element e, for which the corresponding formal power series is the multiplicative identity element in $\mathcal{A}[[T_1, \ldots, T_n]]$, as before. Let $f(T) \in [[T_1, \ldots, T_n]]$ be as in (4.1) again, and suppose that $f(T)$ has a multiplicative inverse in $\mathcal{A}[[T_1, \ldots, T_n]]$. Under these conditions, one can check that f_0 has a multiplicative inverse in $\mathcal{A}[[T_1, \ldots, T_n]]$, using the homomorphism mentioned in the preceding paragraph. It is well known that the converse holds, using geometric series in $\mathcal{A}[[T_1, \ldots, T_n]]$, as follows.

Let

$$b(T) = \sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} b_\beta \, T^\beta \tag{4.5}$$

be an element of $\mathcal{A}[[T_1, \ldots, T_n]]$. If j is a positive integer, then $b(T)^j$ can be defined using multiplication in $\mathcal{A}[[T_1, \ldots, T_n]]$. Suppose that $b_0 = 0$. If $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$ and $j \in \mathbf{Z}_+$ satisfy $|\alpha| < j$, then one can verify that the coefficient of T^α in $b(T)^j$ is equal to 0. Let us interpret $b(T)^j$ as being the power series corresponding to e when $j = 0$, so that

$$\sum_{j=0}^{n} b(T)^j \tag{4.6}$$

can be defined as an element of $\mathcal{A}[[T_1, \ldots, T_n]]$ for each nonnegative integer n. The coefficient of T^α in (4.6) does not depend on n when $n \geq |\alpha|$, because the coefficient of T^α in $b(T)^j$ is 0 when $j > |\alpha|$. This permits us to define

$$\sum_{j=0}^{\infty} b(T)^j \tag{4.7}$$

as an element of $\mathcal{A}[[T_1, \ldots, T_n]]$, where the coefficient of T^α in (4.7) is the same as the coefficient of T^α in (4.6) when $n \geq |\alpha|$.

Of course,

$$(e - b(T)) \Big(\sum_{j=0}^{n} b(T)^j \Big) = \Big(\sum_{j=0}^{n} b(T)^j \Big) (e - b(T)) = e - b(T)^{n+1} \tag{4.8}$$

for each nonnegative integer n, where e is also used to denote the corresponding formal power series. Using this, one can check that

$$(e - b(T)) \Big(\sum_{j=0}^{\infty} b(T)^j \Big) = \Big(\sum_{j=0}^{\infty} b(T)^j \Big) (e - b(T)) = e. \tag{4.9}$$

More precisely, for each multi-index γ, the coefficient of T^γ in either of the products in (4.9) is the same as in the corresponding product in (4.9) when $n \geq |\gamma|$. Hence this coefficient is equal to e when $\gamma = 0$, and to 0 otherwise. This shows that (4.7) is the multiplicative inverse of $e - b(T)$ in $\mathcal{A}[[T_1, \ldots, T_n]]$.

Let $f(T) \in \mathcal{A}[[T_1, \ldots, T_n]]$ be as in (4.1) again, and suppose that f_0 has a multiplicative inverse in $\mathcal{A}$. This permits us to express $f(T)$ as

$$f(T) = f_0 \, (e - b(T)), \tag{4.10}$$

where $b(T) \in \mathcal{A}[[T_1, \ldots, T_n]]$ satisfies $b_0 = 0$. Thus $e - b(T)$ has a multiplicative inverse in $\mathcal{A}[[T_1, \ldots, T_n]]$, as in the preceding paragraph. It follows that $f(T)$ has a multiplicative inverse in $\mathcal{A}[[T_1, \ldots, T_n]]$ as well in this case.

5 Combining indeterminates

Let k be a field, and let V be a vector space over k. Also let m, n be positive integers, and let $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$ be $m + n$ commuting indeterminates. As in the previous section, the space $V[[X_1, \ldots, X_m]]$ of formal power series in $X_1, \ldots, X_m$ with coefficients in V is a vector space over k. Thus the space

$$(V[[X_1, \ldots, X_m]])[[Y_1, \ldots, Y_n]] \tag{5.1}$$

of formal power series in $Y_1, \ldots, Y_n$ with coefficients in $V[[X_1, \ldots, X_m]]$ may be defined as before, and is a vector space over k. An element of (5.1) can be expressed as

$$f(Y) = \sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} f_\beta \, Y^\beta, \tag{5.2}$$

where $f_\beta = f_\beta(X) \in V[[X_1, \ldots, X_m]]$ for each $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$.

Of course, $f_\beta(X)$ can be expressed as

$$f_\beta(X) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^m} f_{\alpha,\beta} \, X^\alpha \tag{5.3}$$

for each $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$, where $f_{\alpha,\beta} \in V$ for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^m$. Let us consider

$$f(X, Y) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^m} \sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} f_{\alpha,\beta} \, X^\alpha \, Y^\beta \tag{5.4}$$

as a formal power series in $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$, with coefficients in V. This uses the obvious identification of $(\mathbf{Z}_+ \cup \{0\})^{m+n}$ with the Cartesian product of $(\mathbf{Z}_+ \cup \{0\})^m$ and $(\mathbf{Z}_+ \cup \{0\})^n$. This leads to a one-to-one linear mapping from (5.1) onto

$$V[[X_1, \ldots, X_m, Y_1, \ldots, Y_n]]. \tag{5.5}$$

More precisely, if (5.4) is any element of (5.5), then (5.3) defines an element of $V[[X_1, \ldots, X_m]]$ for every $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$, so that (5.2) defines an element of (5.1).

If A is a nonempty set, then the space $c(A, V)$ of V-valued functions on A is a vector space over k, as in Section 1. Similarly, if B is another nonempty set, then the space $c(B, c(A, V))$ of functions on B with values in $c(A, V)$ is a vector space over k too. The space $c(A \times B, V)$ of V-valued functions on the Cartesian product $A \times B$ of A and B is a vector space over k as well. If $f(a, b)$ is a V-valued function on $A \times B$, then

$$f_b(a) = f(a, b) \tag{5.6}$$

defines a V-valued function of $a \in A$ for each $b \in B$. Thus

$$b \mapsto f_b \tag{5.7}$$

defines a function on B with values in $c(A, V)$. This defines a one-to-one linear mapping from $c(A \times B, V)$ onto $c(B, c(A, V))$. If we take $A = (\mathbf{Z}_+ \cup \{0\})^m$ and $B = (\mathbf{Z}_+ \cup \{0\})^n$, then $A \times B$ can be identified with $(\mathbf{Z}_+ \cup \{0\})^{m+n}$, as before. In this case, the identification between (5.1) and (5.5) described in the preceding paragraph corresponds to identifying $c(B, c(A, V))$ with $c(A \times B, V)$.

The space $V[X_1, \ldots, X_n]$ of formal polynomials in $X_1, \ldots, X_m$ with coefficients in V may be considered as a linear subspace of $V[[X_1, \ldots, X_m]]$, as before. Similarly, the space

$$(V[X_1, \ldots, X_m])[Y_1, \ldots, Y_n] \tag{5.8}$$

of formal polynomials in $Y_1, \ldots, Y_n$ may be considered as a linear subspace of (5.1). An element $f(Y)$ of (5.8) can be expressed as in (5.2), where $f_\beta = f_\beta(X) \in V[X_1, \ldots, X_m]$ for every $\beta \in (\mathbf{Z}_+ \cup \{0\})^m$, and $f_\beta(X) = 0$ for all but finitely many β. If $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$, then $f_\beta(Y)$ can be expressed as in (5.3), where $f_{\alpha,\beta} = 0$ for all but finitely many $\alpha \in (\mathbf{Z}_+ \cup \{0\})^m$. Under these conditions, (5.4) defines an element of

$$V[X_1, \ldots, X_m, Y_1, \ldots, Y_n], \tag{5.9}$$

considered as a linear subspace of (5.5). Conversely, if (5.4) corresponds to an element of (5.9), then $f_{\alpha,\beta} = 0$ for all but finitely many $\alpha \in (\mathbf{Z}_+ \cup \{0\})^m$ and $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$. This implies that (5.3) defines an element of $V[X_1, \ldots, X_m]$ for every $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$, which is equal to 0 for all but finitely many β. Hence (5.2) defines an element of (5.1), so that (5.8) corresponds exactly to (5.9). One can also look at this in terms of V-valued functions with finite support, as in the previous paragraph.

Let $\mathcal{A}$ be an algebra over k, so that $\mathcal{A}[[X_1, \ldots, X_m]]$ is an algebra over k too. Thus

$$(\mathcal{A}[[X_1, \ldots, X_m]])[[Y_1, \ldots, Y_n]] \tag{5.10}$$

is an algebra over k as well, as is

$$\mathcal{A}[[X_1, \ldots, X_m, Y_1, \ldots, Y_m]]. \tag{5.11}$$

One can verify that the mapping from (5.2) to (5.4) described earlier defines an algebra isomorphism from (5.10) onto (5.11).

6 Adding indeterminates

Let k be a field, and let V be a vector space over k. If n is a positive integer and $v \in V$, then $n \cdot v$ can be defined as the sum of n v's in V. Note that

$$n \cdot (v + w) = n \cdot v + n \cdot w \tag{6.1}$$

for every $n \geq 1$ and $v, w \in V$, and that

$$(m + n) \cdot v = m \cdot v + n \cdot v \tag{6.2}$$

for every $m, n \geq 1$ and $v \in V$. We also have that

$$(m\, n) \cdot v = m \cdot (n \cdot v) \tag{6.3}$$

for every $m, n \geq 1$ and $v \in V$. Similarly, if n is a positive integer and $t \in k$, then $n \cdot t$ can be defined as the sum of n x's in k. If $t \in k$ and $v \in V$, then

$$n \cdot (t\, v) = (n \cdot t)\, v = t\, (n \cdot v) \tag{6.4}$$

In particular, if 1 is the multiplicative identity element in k, then $n \cdot v = (n \cdot 1)\, v$ for every $n \geq 1$ and $v \in V$.

Let X and Y be commuting indeterminates. If j is a nonnegative integer, then $(X+Y)^j$ can be defined as a polynomial in X, Y with integer coefficients. More precisely,

$$(X+Y)^j = \sum_{l=0}^{j} \binom{j}{l} X^l \, Y^{j-l}, \tag{6.5}$$

where $\binom{j}{l}$ are the usual binomial coefficients. One can also consider X and Y as polynomials with coefficients in k, using the multiplicative identity element in k. In this case, $(X+Y)^j$ is defined as a polynomial in X and Y with coefficients in k too, and can be expressed as in (6.5), where positive integer multiples are defined as in the preceding paragraph.

Let T be an indeterminate, and let

$$f(T) = \sum_{j=0}^{\infty} f_j \, T^j \tag{6.6}$$

be a formal power series in T with coefficients in V. Let us consider

$$f(X+Y) = \sum_{j=0}^{\infty} f_j \, (X+Y)^j = \sum_{j=0}^{\infty} \sum_{l=0}^{j} \binom{j}{l} \cdot f_j \, X^l \, Y^{j-l} \tag{6.7}$$

as a formal power series in X and Y, with coefficients in V. This defines a linear mapping from $V[[T]]$ into $V[[X,Y]]$. If $f(T)$ is a formal polynomial in T, then $f(X+Y)$ is a formal polynomial in X and Y. There are analogous statements for formal power series in several commuting indeterminates, which can also be reduced to the case of formal power series in one indeterminate, as in the preceding section.

Using (6.7), we can get a formal power series $a(X,Y)$ in X and Y with coefficients in V such that

$$f(X+Y) = f(X) + a(X,Y)\, Y. \tag{6.8}$$

Here $f(X)$ is defined as a formal power series in X using the same coefficients as in (6.6), and $a(X,Y)\,Y$ is defined as a formal power series in X and Y with coefficients in V by multiplying $a(X,Y)$ by Y termwise in the obvious way. More precisely, the sum of the terms on the right side of (6.7) with $l = j$ gives $f(X)$. The sum of the other terms can be expressed as $a(X,Y)\,Y$, because each of these terms has at least one factor of Y. If $f(T)$ is a formal polynomial in T, then $a(X,Y)$ is a formal polynomial in X and Y.

7 Differentiation

Let k be a field, and let V be a vector space over k. Also let T be an indeterminate, and let

$$f(T) = \sum_{j=0}^{\infty} f_j \, T^j \tag{7.1}$$

be a formal power series in T with coefficients in V. The (formal) *derivative* of $f(T)$ is defined as usual by

$$f'(T) = \sum_{j=1}^{\infty} j \cdot f_j \, T^{j-1} = \sum_{j=0}^{\infty} (j+1) \cdot f_{j+1} \, T^j, \tag{7.2}$$

which is a formal power series in T with coefficients in V as well. Here $j \cdot f_j$ is defined as an element of V as in the previous section. Note that

$$f(T) \mapsto f'(T) \tag{7.3}$$

defines a linear mapping from $V[[T]]$ into itself, which sends $V[T]$ into itself.

Suppose for the moment that k has characteristic 0, so that the field $\mathbf{Q}$ of rational numbers may be considered as a subfield of k. If $f(T) \in V[[T]]$ satisfies

$$f'(T) = 0, \tag{7.4}$$

then it follows that $f_j = 0$ for each $j \geq 1$, so that $f(T)$ is a constant power series. If $g(T) \in V[[T]]$, then there is an $f(T) \in V[[T]]$ such that

$$f'(T) = g(T). \tag{7.5}$$

In this case, $f(T)$ is uniquely determined by $g(T)$, except for the constant term in $f(T)$. Similarly, if $g(T) \in V[T]$, then there is an $f(T) \in V[T]$ that satisfies (7.5).

Let $f(T) \in V[[T]]$ be given again, and let X, Y be commuting indeterminates. Thus $f(X+Y)$ can be defined as formal power series in X and Y with coefficients in V, as in the previous section. One can check that there is a formal power series $b(X,Y)$ in X and Y with coefficients in V such that

$$f(X+Y) = f(X) + f'(X)\, Y + b(X,Y)\, Y^2, \tag{7.6}$$

where $f(X)$ and $f'(X)$ are defined as formal power series in X using the same coefficients as in (7.1) and (7.2). Here $f'(X)\, Y$ and $b(X,Y)\, Y^2$ are defined as formal power series in X and Y with coefficients in V by multiplying $f'(X)$ and $b(X,Y)$ termwise by Y and Y^2, respectively, in the obvious way. As before, the sum of the terms on the right side of (6.7) with $l = j$ gives $f(X)$. Similarly, the sum of the terms on the right side of (6.7) with $j \geq 1$ and $l = j-1$ gives $f'(X)\, Y$. The sum of the other terms can be expressed as $b(X,Y)\, Y^2$, because each of these terms has at least two factors of Y. If $f(T)$ is a formal polynomial in T, then $b(X,Y)$ is a formal polynomial in X and Y.

Let $T_1, \ldots, T_n$ be commuting indeterminates, and let $f(T)$ be a formal power series in $T_1, \ldots, T_n$ with coefficients in V, as in (4.1). The partial derivative

$$(\partial_j f)(T) = \Big(\frac{\partial}{\partial T_j} f\Big)(T) \tag{7.7}$$

of $f(T)$ with respect to T_j can be defined as a formal power series in $T_1, \ldots, T_n$ with coefficients in V for each $j = 1, \ldots, n$, in essentially the same way as before.

One can also reduce to the one-variable case, by identifying $f(T)$ with a formal power series in T_j with coefficients in the vector space of formal power series in the other indeterminates, as in Section 5. In particular, if $(\partial_j f)(T) = 0$ for some j, and if k has characteristic 0, then $f(T)$ reduces to a formal power series in the other $n - 1$ indeterminates.

8 The product rule

Let k be a field again, and let $\mathcal{A}$ be an algebra over k. If n is a positive integer and $x \in \mathcal{A}$, then $n \cdot x$ is defined as a sum of n x's in $\mathcal{A}$, as before. It is easy to see that

$$n \cdot (x\, y) = (n \cdot x)\, y = x\, (n \cdot y) \tag{8.1}$$

for every $n \geq 1$ and $x, y \in \mathcal{A}$.

Let $f(T)$ as in (7.1) and

$$g(T) = \sum_{l=0}^{\infty} g_l\, T^l \tag{8.2}$$

be formal power series in T with coefficients in $\mathcal{A}$. As in Section 4, the product $h(T) = f(T)\, g(T)$ is defined by

$$h(T) = \sum_{n=0}^{\infty} h_n\, T^n, \tag{8.3}$$

where

$$h_n = \sum_{\substack{j,l \geq 0 \\ j+l=n}} f_j\, g_l = \sum_{j=0}^{n} f_j\, g_{n-j} \tag{8.4}$$

for each nonnegative integer n. More precisely, the first sum is taken over all nonnegative integers j and l such that $j + l = n$, which is the same as the second sum. Thus

$$h'(T) = \sum_{n=0}^{\infty} (n+1) \cdot h_{n+1}\, T^n = \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n+1} (n+1) \cdot f_j\, g_{n+1-j} \Big)\, T^n, \tag{8.5}$$

as in the previous section. Under these conditions, we have that

$$h'(T) = f'(T)\, g(T) + f(T)\, g'(T), \tag{8.6}$$

as in the usual product rule.

To see this, remember that $f'(T)$ is given in (7.2), and similarly

$$g'(T) = \sum_{l=0}^{\infty} (l+1) \cdot\, g_{l+1}\, T^l. \tag{8.7}$$

Thus

$$f'(T)\,g(T) = \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} (j+1)\cdot f_{j+1}\,g_{n-j}\Big) T^n \tag{8.8}$$

and

$$f(T)\,g'(T) = \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} f_j\,(n-j+1)\cdot g_{n-j+1}\Big)\,T^n. \tag{8.9}$$

Equivalently,

$$f'(T)\,g(T) = \sum_{n=0}^{\infty} \Big(\sum_{j=1}^{n+1} j\cdot f_j\,g_{n-j+1}\Big)\,T^n = \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n+1} j\cdot f_j\,g_{n+1-j}\Big)\,T^n \tag{8.10}$$

and

$$f(T)\,g'(T) = \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n+1} (n+1-j)\cdot f_j\,g_{n+1-j}\Big)\,T^n. \tag{8.11}$$

The sum of the right sides of (8.10) and (8.11) is equal to the right side of (8.5), as desired.

Now let $T_1, \ldots, T_n$ be commuting indeterminates, and let $f(T)$ and $g(T)$ be formal power series in $T_1, \ldots, T_n$ with coefficients in $\mathcal{A}$. Their product $h(T) = f(T)\,g(T)$ is defined as a formal power series in $T_1, \ldots, T_n$ with coefficients in $\mathcal{A}$ too, as in Section 4. The partial derivatives of $f(T)$, $g(T)$, and $h(T)$ with respect to T_j can be defined as formal power series in $T_1, \ldots, T_n$ with coefficients in $\mathcal{A}$ for each $j = 1, \ldots, n$, as in the previous section. One can check that the product rule

$$(\partial_j h)(T) = (\partial_j f)(T)\,g(T) + f(T)\,(\partial_j g)(T) \tag{8.12}$$

holds for each $j = 1, \ldots, n$. This can be verified directly, in essentially the same way as before. Alternatively, for each $j = 1, \ldots, n$, one can identify $f(T)$, $g(T)$, and $h(T)$ with formal power series in T_j, with coefficients in the algebra of formal power series in the other $n-1$ indeterminates, as in Section 5. This permits one to reduce (8.12) to the previous case.

9 Exponentiation

Let k be a field of characteristic 0, so that $\mathbf{Q}$ may be considered as a subfield of k. The *exponential function* can be defined as a formal power series in an indeterminate T with coefficients in k by

$$\exp(T) = \sum_{j=0}^{\infty} (1/j!)\,T^j, \tag{9.1}$$

as usual. Of course, the derivative of this formal power series is equal to itself. If X and Y are commuting indeterminates, then

$$\exp(X+Y) = \sum_{j=0}^{\infty} (1/j!)\,(X+Y)^j = \sum_{j=0}^{\infty}\sum_{l=0}^{j} (1/j!) \binom{j}{l}\,X^l\,Y^{j-l} \tag{9.2}$$

defines a formal power series in X and Y with coefficients in k, as before. In this case, this reduces to

$$(9.3) \quad \exp(X+Y) = \sum_{j=0}^{\infty} \sum_{l=0}^{j} (1/l!)\,(1/(j-l)!)\,X^l\,Y^{j-l} = \exp(X)\,\exp(Y),$$

using the fact that $\binom{j}{l} = j!/l!\,(j-l)!$ in the first step.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e, and let $a \in \mathcal{A}$ be given. Consider

$$(9.4) \qquad E_a(T) = \sum_{j=0}^{\infty} (a^j/j!)\,T^j$$

as a formal power series in an indeterminate T with coefficients in $\mathcal{A}$, which corresponds to exponentiating $a\,T$. Here a^j is interpreted as being equal to e when $j = 0$, as usual. Observe that

$$(9.5) \qquad E_a'(T) = \sum_{j=0}^{\infty} (j+1)\,(a^{j+1}/(j+1)!)\,T^j = a\,E_a(T).$$

Of course, if $a = 0$, then $E_a(T)$ is the constant power series corresponding to e, which is the multiplicative identity element in $\mathcal{A}[[T]]$.

Let b be another element of $\mathcal{A}$, and suppose that a and b commute. This implies that

$$(9.6) \qquad (a+b)^j = \sum_{l=0}^{j} \binom{j}{l} a^l\,b^{j-l}$$

for each nonnegative integer j, as in the binomial theorem. Thus

$$(9.7) \quad E_{a+b}(T) = \sum_{j=0}^{\infty} ((a+b)^j/j!)\,T^j = \sum_{j=0}^{\infty} \Big(\sum_{l=0}^{j} (a^l/l!)\,(b^{j-l}/(j-l)!) \Big)\,T^j.$$

It follows that

$$(9.8) \qquad E_{a+b}(T) = E_a(T)\,E_b(T)$$

as formal power series in T under these conditions.

In particular, we can apply this to $b = -a$, to get that

$$(9.9) \qquad E_a(T)\,E_{-a}(T) = E_0(T) = e.$$

This means that $E_a(T)$ has a multiplicative inverse in $\mathcal{A}[[T]]$, namely

$$(9.10) \qquad E_a(T)^{-1} = E_{-a}(T).$$

Similarly, we can use (9.8) to get that

$$(9.11) \qquad E_a(T)^j = E_{j\,a}(T)$$

for every positive integer j. This also works for negative integers j, because of (9.10). If $j = 0$, then $E_a(T)^j$ is interpreted as being the multiplicative identity element in $\mathcal{A}[[T]]$, which is the same as $E_0(T)$, so that (9.11) holds in this case too.

10 Binomial series

Let k be a field of characteristic 0 again, and let A be an indeterminate. If j is a positive integer, then

$$\binom{A}{j} = \frac{A\,(A-1)\cdots(A-j+1)}{j!} \tag{10.1}$$

defines a formal polynomial of degree j in A with coefficients in $\mathbf{Q}$, and hence k. If $j = 0$, then (10.1) is interpreted as being the constant polynomial corresponding to 1, as usual. If $a \in k$, then

$$\binom{a}{j} = \frac{a\,(a-1)\cdots(a-j+1)}{j!} \tag{10.2}$$

defines an element of k for each positive integer j, which corresponds to evaluating (10.1) at a. Similarly, (10.2) is interpreted as being equal to 1 for every $a \in k$ when $j = 0$. Of course, if a and j are nonnegative integers with $j \leq a$, then (10.2) is the usual binomial coefficient. If a and j are nonnegative integers and $a < j$, then (10.2) is equal to 0.

Let X be another indeterminate, which commutes with A. We can consider

$$B(A, X) = \sum_{j=0}^{\infty} \binom{A}{j} X^j \tag{10.3}$$

as a formal power series in X with coefficients in $k[A]$, the algebra of formal polynomials in A with coefficients in k. In particular, this corresponds to a formal power series in A and X with coefficients in k, as in Section 5. It is well known that (10.3) corresponds, at least formally, to

$$(1 + X)^A. \tag{10.4}$$

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e. If $a \in \mathcal{A}$, then

$$\binom{a}{j} = \frac{a\,(a-e)\cdots(a-(j-1)\,e)}{j!} \tag{10.5}$$

defines an element of $\mathcal{A}$ for each positive integer j. This corresponds to evaluating (10.1) at a, as in Section 3. Of course, (10.5) reduces to (10.2) when $\mathcal{A} = k$. If $j = 0$, then (10.5) is interpreted as being equal to e for every $a \in \mathcal{A}$, as usual. Thus

$$B(a, X) = \sum_{j=0}^{\infty} \binom{\alpha}{j} X^j \tag{10.6}$$

may be considered as a formal power series in X with coefficients in $\mathcal{A}$ when $a \in \mathcal{A}$. As before, this corresponds formally to

$$(1 + X)^a. \tag{10.7}$$

In particular, we can take $\mathcal{A} = k$, considered as an algebra over itself. If a is a nonnegative integer, then (10.6) reduces to

$$B(a, X) = \sum_{j=0}^{n} \binom{a}{j} X^j = (1 + X)^a, \tag{10.8}$$

using the binomial theorem in the second step. Observe that

$$\binom{-1}{j} = (-1)^j \tag{10.9}$$

for every nonnegative integer j. Thus

$$B(-1, X) = \sum_{j=0}^{\infty} (-1)^j X^j = (1 + X)^{-1}, \tag{10.10}$$

which is the multiplicative inverse of $1 + X$ in $k[[T]]$.

11 Some identities

Let X be an indeterminate. If a and b are nonnegative integers, then

$$(1 + X)^a = \sum_{j=0}^{a} \binom{a}{j} X^j, \tag{11.1}$$

$$(1 + X)^b = \sum_{l=0}^{b} \binom{b}{l} X^l, \quad \text{and} \tag{11.2}$$

$$(1 + X)^{a+b} = \sum_{n=0}^{a+b} \binom{a+b}{n} X^n \tag{11.3}$$

as formal polynomials in X with integer coefficients, as in the binomial theorem. Of course, we also have that

$$(1 + X)^a (1 + X)^b = (1 + X)^{a+b} \tag{11.4}$$

as formal polynomials in X. for all nonnegative integers a, b. It follows that

$$\binom{a+b}{n} = \sum_{j=0}^{n} \binom{a}{j} \binom{b}{n-j} \tag{11.5}$$

for all nonnegative integers a, b, n. Remember that $\binom{u}{v} = 0$ when u and v are nonnegative integers with $u < v$, as in the previous section.

Let A be another indeterminate. If j is a nonnegative integer, then $\binom{A}{j}$ is defined as a formal polynomial in A with rational coefficients, as in (10.1). Similarly, if b and n are nonnegative integers, then

$$\binom{A+b}{n} = \frac{(A + b)\,(A + b - 1) \cdots (A + b - n + 1)}{n!} \tag{11.6}$$

is defined as a formal polynomial in A with rational coefficients, which is interpreted as being the constant polynomial corresponding to 1 when $n = 0$. Using (11.5), we get that

$$\binom{A+b}{n} = \sum_{j=0}^{n} \binom{A}{j} \binom{b}{n-j} \tag{11.7}$$

as formal polynomials in A with rational coefficients for all nonnegative integers b, n. More precisely, (11.5) says that the corresponding polynomial functions take the same values on nonnegative integers, so that these polynomials are the same.

Let B be an indeterminate that commutes with A. If l is a nonnegative integer, then $\binom{B}{l}$ can be defined as a formal polynomial with rational coefficients, as in (10.1) again. If n is a nonnegative integer, then

$$\binom{A+B}{n} = \frac{(A+B)\,(A+B-1)\cdots(A+B-n+1)}{n!} \tag{11.8}$$

defines a formal polynomial in A and B with rational coefficients, which is interpreted as being the constant polynomial corresponding to 1 when $n = 0$, as usual. As in Part (i) of Exercise 8 on p74 of [1],

$$\binom{A+B}{n} = \sum_{j=0}^{n} \binom{A}{j} \binom{B}{n-j} \tag{11.9}$$

as formal polynomials in A and B with rational coefficients for every nonnegative integer n. To see this, one can consider both sides of (11.9) as formal polynomials in A with coefficients in $\mathbf{Q}[B]$, the algebra of formal polynomials in B with rational coefficients. If we evaluate both sides of (11.9) at a nonnegative integer b, then we get an equality between formal polynomials in A, as in the preceding paragraph. This implies that the polynomials in B that occur as coefficients of powers of A in both sides of (11.9) are the same, because they are the same when evaluated at nonnegative integers.

Let k be a field of characteristic 0, and let A_1, A_2, and X be commuting indeterminates. One can define $B(A_1, X)$ and $B(A_2, X)$ as formal power series in A_1, X and A_2, X, respectively, with coefficients in k, as in the previous section. More precisely, $B(A_1, X)$ and $B(A_2, X)$ can be defined as formal power series in X whose coefficients are formal polynomials in A_1 and A_2, respectively, with coefficients in k. Similarly,

$$B(A_1 + A_2, X) = \sum_{j=0}^{\infty} \binom{A_1 + A_2}{j} X^j \tag{11.10}$$

can be defined as a formal power series in X with coefficients in $k[A_1, A_2]$, the algebra of formal polynomials in A_1 and A_2 with coefficients in k. Using (11.9), we get that

$$B(A_1 + A_2, X) = B(A_1, X)\, B(A_2, X), \tag{11.11}$$

as formal power series in X with coefficients in $k[A_1, A_2]$.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e. If $a \in \mathcal{A}$ and j is a nonnegative integer, then $\binom{a}{j}$ can be defined as an element of $\mathcal{A}$, as in the previous section. If a and b are commuting elements of $\mathcal{A}$, then (11.5) holds for all nonnegative integers n, because of (11.8). This implies that

$$B(a+b, X) = B(a, X)\, B(b, X) \tag{11.12}$$

as formal power series in X with coefficients in $\mathcal{A}$. Of course, this also corresponds to evaluating (11.11) at a and b.

12 A differential equation

Let k be a field of characteristic 0, and let A be an indeterminate. If $b \in k$, then $\binom{A+b}{n}$ can be defined as a formal polynomial in A with coefficients in k for each nonnegative integer n, as in (11.6). This also corresponds to evaluating $\binom{A+B}{n}$ at b, where B is another indeterminate that commutes with A, and $\binom{A+B}{n}$ is defined as a formal polynomial in A and B with coefficients in k as in (11.8). Note that (11.7) holds for every nonnegative integer n, as an equality of formal polynomials in A with coefficients in k. This follows from (11.9), by evaluating at b.

Similarly, if X is another indeterminate that commutes with A, then

$$B(A+b, X) = \sum_{j=0}^{\infty} \binom{A+b}{j} X^j \tag{12.1}$$

can be defined as a formal power series in X with coefficients in $k[A]$, the algebra of formal polynomials in A with coefficients in k. This corresponds to (11.10) with $A_1 = A$ and A_2 evaluated at b. As before,

$$B(A+b, X) = B(A, X)\, B(b, X), \tag{12.2}$$

as formal power series in X with coefficients in $k[A]$. This follows from the version of (11.7) mentioned in the preceding paragraph. This can also be obtained from (11.11), with $A_1 = A$ and A_2 evaluated at b again.

Consider

$$\frac{\partial}{\partial X} B(A, X) = \sum_{j=0}^{\infty} (j+1) \binom{A}{j+1} X^j, \tag{12.3}$$

which is another formal power series in X with coefficients in $k[X]$. Observe that

$$\begin{aligned} (j+1)\binom{A}{j+1} &= (j+1)\,\frac{A\,(A-1)\cdots(A-(j+1)+1)}{(j+1)!} \\ &= \frac{A\,(A-1)\cdots(A-j)}{j!} \end{aligned} \tag{12.4}$$

for each nonnegative integer j, using (10.1) in the first step. Hence

$$(12.5)\quad (j+1)\binom{A}{j+1} = A\,\frac{(A-1)\cdots((A-1)-j+1)}{j!} = A\binom{A-1}{j}$$

for every $j \ge 0$, as formal polynomials in A. This implies that

$$(12.6)\qquad \frac{\partial}{\partial X} B(A,X) = A\,B(A-1,X)$$

as formal power series in X with coefficients in $k[A]$. Equivalently, this means that

$$(12.7)\qquad \frac{\partial}{\partial X} B(A,X) = A\,(1+X)^{-1}\,B(A,X),$$

because of (10.10) and (12.2).

Let $\mathcal{A}$ be an algebra over k with multiplicative identity element e. If $a \in \mathcal{A}$, then $B(a,X)$ is defined as a formal power series in X with coefficients in $\mathcal{A}$, as in Section 10, and

$$(12.8)\qquad \frac{d}{dX} B(a,X) = \sum_{j=0}^{\infty} (j+1)\binom{a}{j+1} X^j.$$

As in (12.7), we have that

$$(12.9)\qquad \frac{d}{dX} B(a,X) = a\,(1+X)^{-1}\,B(a,X)$$

as formal power series in X with coefficients in $\mathcal{A}$. More precisely, this can be obtained in the same way as before, or by evaluating (12.7) at a. Of course, we are implicitly identifying elements of k with the corresponding multiples of e in $\mathcal{A}$.

Sometimes (12.7) is expressed as

$$(12.10)\qquad (1+X)\,\frac{\partial}{\partial X} B(A,X) = A\,B(A,X),$$

and similarly (12.9) may be expressed as

$$(12.11)\qquad (1+X)\frac{d}{dX} B(a,X) = a\,B(a,X),$$

as in Exercise 22 on p201 in [8]. To get this more directly, one should verify that

$$(12.12)\qquad B(A-1,X)\,(1+X) = B(A,X),$$

as formal power series in X with coefficients in $k[A]$. This reduces to the fact that

$$(12.13)\qquad \binom{A-1}{j} + \binom{A-1}{j-1} = \binom{A}{j}$$

as formal polynomials in A with coefficients in k for each positive integer j. This corresponds to replacing A in (11.7) with $A - 1$, and taking $b = 1$. One can check (12.13) directly from the definitions as well.

Part II
Absolute values and norms

13 Metrics and ultrametrics

Let X be a set, and let $d(x, y)$ be a nonnegative real-valued function defined for $x, y \in X$. As usual, $d(x, y)$ is said to be a *metric* on X if it satisfies the following three conditions. First,

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{13.1}$$

Second,

$$d(x, y) = d(y, x) \quad \text{for every } x, y \in X. \tag{13.2}$$

Third,

$$d(x, z) \le d(x, y) + d(y, z) \quad \text{for every } x, y, z \in X. \tag{13.3}$$

If $d(\cdot, \cdot)$ satisfies (13.1), (13.2), and

$$d(x, z) \le \max(d(x, y), d(y, z)) \quad \text{for every } x, y, z \in X, \tag{13.4}$$

then $d(\cdot, \cdot)$ is said to be an *ultrametric* on X. Note that (13.4) implies (13.3), so that ultrametrics are metrics. The *discrete metric* is defined on X by putting $d(x, y) = 1$ when $x \neq y$, and $d(x, x) = 0$ for every $x \in X$. It is easy to see that this defines an ultrametric on X.

Let $d(x, y)$ be a metric on X. If $x \in X$ and r is a positive real number, then the *open ball* in X centered at x with radius r is defined by

$$B(x, r) = \{y \in X : d(x, y) < r\}. \tag{13.5}$$

Similarly, if r is a nonnegative real number, then the *closed ball* in X centered at x with radius r is defined by

$$\overline{B}(x, r) = \{y \in X : d(x, y) \le r\}. \tag{13.6}$$

It is well known that open balls in X are open sets with respect to the topology determined by $d(\cdot, \cdot)$, and that closed balls are closed sets. If $d(\cdot, \cdot)$ is an ultrametric on X, then one can check that clsoed balls in X of positive radius are open sets. More precisely, if $x, y \in X$ and $d(x, y) \le r$, then

$$\overline{B}(y, r) \subseteq \overline{B}(x, r). \tag{13.7}$$

In fact,

$$\overline{B}(x, r) = \overline{B}(y, r) \tag{13.8}$$

in this case, as one can see by exchanging the roles of x and y. One can also verify that open balls in X are closed sets when $d(\cdot, \cdot)$ is an ultrametric on X.

Let $d(x, y)$ be a metric on X again, and let Y be a subset of X. Of course, the restriction of $d(x, y)$ to $x, y \in Y$ defines a metric on Y. If $d(x, y)$ is an ultrametric on X, then the restriction of $d(x, y)$ to Y is an ultrametric on Y. If Y is dense in X, and if the restriction of $d(x, y)$ to $x, y \in Y$ is an ultrametric on Y, then one can check that $d(x, y)$ is an ultrametric on X.

Let (X, d_X) and (Y, d_Y) be metric spaces, and let E be a dense subset of X. Also let f be a uniformly continuous mapping from E into Y, with respect to the restriction of d_X to E. If Y is complete, then it is well known that there is a unique extension of f to a uniformly continuous mapping from X into Y. More precisely, the uniqueness of the extension only requires ordinary continuity.

If a metric space (X, d) is not complete, then one can pass to a completion, by standard arguments. This can be defined as an isometric embedding of X onto a dense subset of a complete metric space. The completion is unique up to isometric equivalence, because of the extension theorem mentioned in the preceding paragraph. If $d(\cdot, \cdot)$ is an ultrametric on X, then the metric on the completion is an ultrametric as well, because its restriction to a dense set is an ultrametric.

14 Absolute value functions

Let k be a field. A nonnegative real-valued function $|x|$ defined on k is said to be an *absolute value function* on k if it satisfies the following three conditions. First,

$$|x| = 0 \quad \text{if and only if} \quad x = 0. \tag{14.1}$$

Second,

$$|x\,y| = |x|\,|y| \quad \text{for every } x, y \in k. \tag{14.2}$$

Third,

$$|x + y| \le |x| + |y| \quad \text{for every } x, y \in k. \tag{14.3}$$

It is well known that the standard absolute value functions on the fields $\mathbf{R}$ of real numbers and $\mathbf{C}$ of complex numbers are absolute value functions in this sense. If $|\cdot|$ is an absolute value function on a field k and k_0 is a subfield of k, then the restriction of $|x|$ to $x \in k_0$ defines an absolute value function on k_0.

A nonnegative real-valued function $|\cdot|$ on a field k is said to be an *ultrametric absolute value function* on k if it satisfies (14.1), (14.2), and

$$|x + y| \le \max(|x|, |y|) \quad \text{for every } x, y \in k. \tag{14.4}$$

Clearly (14.4) implies (14.3), so that an ultrametric absolute value function on k is an absolute value function on k. The *trivial absolute value function* is defined on a field k by putting $|x| = 1$ when $x \ne 0$, and $|0| = 0$. This defines an ultrametric absolute value function on k. If $|\cdot|$ is an ultrametric absolute value function on a field k and k_0 is a subfield of k, then the restriction of $|x|$ to $x \in k_0$ defines an ultrametric absolute value function on k_0.

Let $|\cdot|$ be a nonnegative real-valued function on a field k that satisfies (14.1) and (14.2). Observe that

$$|1| = 1, \tag{14.5}$$

where the first 1 is the multiplicative identity element in k, and the second is in $\mathbf{R}$. This uses the facts that $|1| > 0$ and $|1| = |1^2| = |1|^2$, by (14.1) and (14.2). Similarly, if $x \in k$ satisfies $x^n = 1$ for some positive integer n, then

$$|x| = 1, \tag{14.6}$$

because $|x|^n = |x^n| = |1| = 1$. In particular, this holds when $x = -1$, the additive inverse of 1 in k, because $(-1)^2 = -(-1) = 1$ in k.

If $|\cdot|$ is an absolute value function on k, then

$$d(x, y) = |x - y| \tag{14.7}$$

defines a metric on k. This uses (14.6) with $x = -1$ to get that (14.7) is symmetric in x and y. Similarly, if $|\cdot|$ is an ultrametric absolute value function on k, then (14.7) defines an ultrametric on k. If $|\cdot|$ is the trivial absolute value function on k, then (14.7) is the discrete metric on k.

Let $|\cdot|$ be an absolute value function on a field k again, and let k_0 be a subfield of k. Suppose that k_0 is dense in k with respect to the metric (14.7) associated to $|\cdot|$. If the restriction of $|\cdot|$ to k_0 is an ultrametric absolute value function on k_0, then one can check that $|\cdot|$ is an ultrametric absolute value function on k.

If p is a prime number, then the *p-adic absolute value* $|x|_p$ of a rational number x is defined as follows. Of course, $|0|_p = 0$. If $x \neq 0$, then x can be expressed as $p^j\,(a/b)$, where a, b, and j are integers, $a, b \neq 0$, and neither a nor b is a multiple of p. In this case,

$$|x|_p = p^{-j}. \tag{14.8}$$

One can check that this defines an ultrametric absolute value function on the field $\mathbf{Q}$ of rational numbers.

If a field k is not complete with respect to the metric (14.7) associated to an absolute value function $|\cdot|$, then one can pass to a completion of k, by standard arguments. More precisely, this can be defined as an isomorphism between k and a dense subfield of a field with an absolute value function, which is complete with respect to the associated metric. Of course, the restriction of the absolute value function on the completion to the image of k should correspond to $|\cdot|$, so that the embedding is an isometry with respect to the associated metrics. To say that the image of k is dense in the completion means that it is a dense subset of the completion with respect to the metric associated to the absolute value function, as before. The completion of k with respect to $|\cdot|$ is unique up to a suitable isomorphic equivalence. One often identifies k with its image in the completion, so that k may be considered as a subfield of its completion. If $|\cdot|$ is an ultrametric absolute value function on k, then absolute value function on the completion of k is an ultrametric absolute value function as well.

If p is a prime number, then the field $\mathbf{Q}_p$ of *p-adic numbers* is the completion of $\mathbf{Q}$ with respect to the p-adic absolute value function $|\cdot|_p$, as in the preceding paragraph. The extension of $|\cdot|_p$ to $\mathbf{Q}_p$ is known as the p-adic absolute value on $\mathbf{Q}_p$, and is also denoted $|\cdot|_p$.

15 Norms and ultranorms

Let k be a field, and let $|\cdot|$ be an absolute value function on k. Also let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *norm* on V with respect to $|\cdot|$ on k if it satisfies the following three conditions. First,

$$N(v) = 0 \quad \text{if and only if} \quad v = 0. \tag{15.1}$$

Second,

$$N(t\,v) = |t|\,N(v) \quad \text{for every } t \in k \text{ and } v \in V. \tag{15.2}$$

Third,

$$N(v + w) \le N(v) + N(w) \quad \text{for every } v, w \in V. \tag{15.3}$$

If N satisfies (15.1), (15.2), and

$$N(v + w) \le \max(N(v), N(w)) \quad \text{for every } v, w \in V, \tag{15.4}$$

then N is said to be an *ultranorm* on V with respect to $|\cdot|$ on k. Ultranorms are automatically norms, because (15.4) implies (15.3).

Of course, k may be considered as a one-dimensional vector space over itself, and $|\cdot|$ may be considered as a norm on k with respect to itself. If $|\cdot|$ is an ultrametric absolute value function on k, then $|\cdot|$ may be considered as an ultranorm on k, as a one-dimensional vector space over itself.

If N is an ultranorm on a vector space V over k with respect to an absolute value function $|\cdot|$ on k, and if $V \ne \{0\}$, then $|\cdot|$ has to be an ultrametric absolute value function on k. More precisely, (14.4) can be obtained from (15.2) and (15.4) in this case.

Let $|\cdot|$ be an absolute value function on k again, and let N be a norm on a vector space V over k, with respect to $|\cdot|$ on k. It is easy to see that

$$d(v, w) = d_N(v, w) = N(v - w) \tag{15.5}$$

defines a metric on V. If N is an ultranorm on V, then (15.5) is an ultrametric on V.

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k. The *trivial ultranorm* on a vector space V over k is defined by putting $N(v) = 1$ when $v \ne 0$, and $N(0) = 0$. One can verify that this defines an ultranorm on V, for which the corresponding ultrametric is the discrete metric.

Let $|\cdot|$ be an absolute value function on k, and let N be a norm on a vector space V over k. If V_0 is a linear subspace of V, then the restriction of N to V_0 is a norm on V_0. If N is an ultranorm on V, then the restriction of N to V_0 is an ultranorm on V_0. Suppose that V_0 is dense in V, with respect to the metric

(15.5) associated to N. If the restriction of N to V_0 is an ultranorm on V_0, then one can check that N is an ultranorm on V.

Let N be a norm on a vector space V over k again. If V is complete with respect to the metric (15.5) associated to N, then V is said to be a *Banach space* with respect to N. In this case, one may also ask that k be complete with respect to the metric associated to $|\cdot|$. Otherwise, one can pass to the completion of k, as in the previous section. Using the completeness of V, one can check that scalar multiplication on V can be extended to the completion of k, so that V becomes a vector space over the completion of k, and N is a norm on V as a vector space over the completion of k, with respect to the extension of $|\cdot|$ to the completion of k.

If V is not complete with respect to (15.5), then one can pass to a completion, as usual. This can be defined as a linear mapping from V onto a dense linear subspace of a Banach space over k, where the linear mapping also preserves norms. The condition that the image of V be dense in the completion uses the metric associated to the norm on the completion. The completion of V with respect to N is unique up to a suitable linear mapping that preserves norms. If N is an ultranorm on V, then the norm on the completion of V is an ultranorm too.

16 Infinite series

Let k be a field with an absolute value function $|\cdot|$, let V be a vector space over k, and let N be a norm on V with respect to $|\cdot|$ on k. An infinite series $\sum_{j=1}^{\infty} v_j$ with terms in V is said to *converge* if the corresponding sequence of partial sums $\sum_{j=1}^{n} v_j$ converges to an element of V, with respect to the metric associated to the norm N. In this case, the value of the sum $\sum_{j=1}^{\infty} v_j$ is taken to be the limit of the sequence of partial sums. Note that a linear combination of convergent series in V converges too, with sum equal to the corresponding linear combination of the sums of the individual series. This follows from the analogous statement for convergent sequences in V, which can be verified using standard arguments.

Let $\sum_{j=1}^{\infty} v_j$ be an infinite series with terms in V again. One can check that the corresponding sequence of partial sums $\sum_{j=1}^{n} v_j$ is a Cauchy sequence in V with respect to the metric associated to N if and only if for every $\epsilon > 0$ there is a positive integer L such that

$$N\Big(\sum_{j=l}^{n} v_j\Big) < \epsilon \tag{16.1}$$

for every $n \geq l \geq L$. In particular, this implies that

$$\lim_{l \to \infty} N(v_l) = 0, \tag{16.2}$$

by taking $l = n$ in (16.1). Of course, (16.2) is the same as saying that $\{v_j\}_{j=1}^{\infty}$ converges to 0 in V with respect to the metric associated to N. If V is a Banach space with respect to N, then the Cauchy criterion (16.1) implies that $\sum_{j=1}^{\infty} v_j$ converges in V.

Observe that

$$N\Big(\sum_{j=l}^{n} v_j\Big) \le \sum_{j=l}^{n} N(v_j) \tag{16.3}$$

for every $n \ge l \ge 1$, by the triangle inequality for N. If

$$\sum_{j=1}^{\infty} N(v_j) \tag{16.4}$$

converges as an infinite series of nonnegative real numbers, then $\sum_{j=1}^{\infty} v_j$ is said to converge *absolutely* with respect to N. In this case, one can use (16.3) to verify the criterion (16.1), to get that the sequence of partial sums $\sum_{j=1}^{n} v_j$ is a Cauchy sequence in V with respect to the metric associated to N. If V is a Banach space with respect to N, then it follows that $\sum_{j=1}^{\infty} v_j$ converges in V. One can also check that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \sum_{j=1}^{\infty} N(v_j) \tag{16.5}$$

in this situation, using (16.3).

If N is an ultranorm on V, then

$$N\Big(\sum_{j=l}^{n} v_j\Big) \le \max_{l \le j \le n} N(v_j) \tag{16.6}$$

for every $n \ge l \ge 1$. In this case, one can use (16.6) to verify the criterion (16.1) when (16.2) holds, to get that the sequence of partial sums $\sum_{j=1}^{n} v_j$ is a Cauchy sequence in V with respect to the metric associated to N. It follows that $\sum_{j=1}^{\infty} v_j$ converges in V when V is a Banach space with respect to N, as before. Under these conditions, one can check that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \max_{j \ge 1} N(v_j), \tag{16.7}$$

using (16.6). Note that the maximum on the right side of (16.7) is attained when (16.2) holds.

Of course, there are analogous statements for infinite series that start at $j = 0$, or some other integer.

17 Banach algebras

Let k be a field with an absolute value function $|\cdot|$, and let $\mathcal{A}$ be an algebra over k. A norm N on $\mathcal{A}$ is said to be *submultiplicative* on $\mathcal{A}$ if

$$N(a\,b) \le N(a)\,N(b) \tag{17.1}$$

for every $a, b \in \mathcal{A}$. If N is a submultiplicative norm on $\mathcal{A}$, and if $\mathcal{A}$ is complete with respect to the metric associated to N, then $\mathcal{A}$ is said to be a *Banach algebra* with respect to N. If $\mathcal{A}$ has a nonzero multiplicative identity element e, then it is easy to see that $N(e) \geq 1$, using submultiplicativity. Sometimes the condition that there be a multiplicative identity element e with $N(e) = 1$ is included in the definition of a Banach algebra.

If $\mathcal{A}$ is not complete with respect to the metric associated to N, then one can pass to a completion of $\mathcal{A}$, as before. This can be defined as a norm-preserving algebra isomorphism from $\mathcal{A}$ onto a dense subalgebra of a Banach algebra over k. The completion is unique up to a suitable norm-preserving algebra isomorphism, as usual. If $\mathcal{A}$ is complete, but k is not complete with respect to the metric associated to $|\cdot|$, then scalar multiplication on $\mathcal{A}$ can be extended to the completion of k, so that $\mathcal{A}$ becomes a Banach algebra over the completion of k. One may wish to include the completeness of k in the definition of a Banach algebra.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e. If $a \in \mathcal{A}$ and n is a nonnegative integer, then

$$(e - a) \sum_{j=0}^{n} a^j = \Big(\sum_{j=0}^{n} a^j\Big) (e - a) = e - a^{n+1}, \tag{17.2}$$

by a standard argument. Here a^j is interpreted as being equal to e when $j = 0$, as usual. Let N be a submultiplicative norm on $\mathcal{A}$, and observe that

$$N(a^j) \leq N(a)^j \tag{17.3}$$

for every positive integer j. If $N(e) = 1$, then (17.3) also holds when $j = 0$, with $N(a)^j$ interpreted as being equal to 1.

Suppose that $N(a) < 1$, so that

$$\lim_{j \to \infty} N(a^j) = 0 \tag{17.4}$$

by (17.3). In this case,

$$\sum_{j=0}^{\infty} N(a^j) \tag{17.5}$$

converges as an infinite series of nonnegative real numbers, by comparison with a convergent geometric series. More precisely, if $N(e) = 1$, then

$$\sum_{j=0}^{\infty} N(a^j) \leq \sum_{j=0}^{\infty} N(a)^j = (1 - N(a))^{-1}, \tag{17.6}$$

using (17.3) in the first step. Note that

$$\max_{j \geq 0} N(a^j) = N(e), \tag{17.7}$$

by (17.5) and the fact that $N(e) \geq 1$ when $e \neq 0$, as mentioned earlier. Of course, if $e = 0$, then $\mathcal{A} = \{0\}$, and (17.7) is trivial.

If $\mathcal{A}$ is a Banach algebra with respect to N, then it follows that

$$\sum_{j=0}^{\infty} a^j \tag{17.8}$$

converges in $\mathcal{A}$ with respect to the metric associated to N, as in the preceding section. We also have that

$$(e-a)\sum_{j=0}^{\infty} a^j = \Big(\sum_{j=0}^{\infty} a^j\Big)(e-a) = e, \tag{17.9}$$

by taking the limit as $n \to \infty$ in (17.2), using (17.4). This means that (17.8) is the multiplicative inverse of $e - a$ in $\mathcal{A}$. If $N(e) = 1$, then

$$N\Big(\sum_{j=1}^{\infty} a^j\Big) \leq \sum_{j=0}^{\infty} N(a^j) \leq (1 - N(a))^{-1}, \tag{17.10}$$

by (16.5) and (17.6). Similarly, if N is an ultranorm on N, then

$$N\Big(\sum_{j=0}^{\infty} a^j\Big) \leq \max_{j \geq 0} N(a^j) = N(e), \tag{17.11}$$

by (16.7) and (17.7).

18 Bounded functions

Let k be a field with an absolute value function $|\cdot|$, and let X be a nonempty set. Also let V be a vector space over k, with a norm N with respect to $|\cdot|$ on k. A V-valued function f on X is said to be *bounded* with respect to N if $N(f(x))$ is bounded as a nonnegative real-valued function on X. Let $\ell^\infty(X, V)$ be the space of bounded V-valued functions on X, and put

$$\|f\|_\infty = \|f\|_{\ell^\infty(X,V)} = \sup_{x \in X} N(f(x)) \tag{18.1}$$

for every $f \in \ell^\infty(X, V)$. Remember that the space $c(X, V)$ of all V-valued functions on X is a vector space over k with respect to pointwise addition and scalar multiplication, as in Section 1. One can check that $\ell^\infty(X, V)$ is a linear subspace of the space $c(X, V)$, and that (18.1) defines a norm on $\ell^\infty(X, V)$ with respect to $|\cdot|$ on k. If N is an ultranorm on V, then (18.1) is an ultranorm on $\ell^\infty(X, V)$. If V is a Banach space with respect to N, then $\ell^\infty(X, V)$ is a Banach space with respect to (18.1), by standard arguments.

Suppose that X is a nonempty topological space, and let $C(X, V)$ be the space of continuous mappings from X into V, with respect to the topology determined on V by the metric associated to N. This is a vector space over k with respect to pointwise addition and scalar multiplication, by standard arguments. Of course, constant functions on X are automatically continuous. It follows that the space

$$C_b(X, V) = C(X, V) \cap \ell^\infty(X, V) \tag{18.2}$$

of V-valued functions on X that are bounded and continuous is a linear subspace of both $C(X, V)$ and $\ell^\infty(X, V)$. If $f \in C_b(X, V)$, then the supremum norm (18.1) of f may also be denoted

$$\|f\|_{sup} = \|f\|_{\mathrm{sup}, X} = \|f\|_{C_b(X,V)}. \tag{18.3}$$

It is well known that $C_b(X, V)$ is a closed set in $\ell^\infty(X, V)$ with respect to the supremum metric. If X is compact, then every continuous V-valued function f on X is bounded, and in fact the supremum on the right side of (18.1) is attained.

Now let $\mathcal{A}$ be an algebra over k, and let N be a submultiplicative norm on $\mathcal{A}$ with respect to $|\cdot|$ on k. If X is any nonempty set, then the space $c(X, \mathcal{A})$ of all $\mathcal{A}$-valued functions on X is also an algebra over k with respect to pointwise multiplication of functions, as in Section 1. It is easy to see that $\ell^\infty(X, \mathcal{A})$ is a subalgebra of $c(X, \mathcal{A})$, and that (18.1) is submultiplicative on $\ell^\infty(X, \mathcal{A})$. Suppose that $\mathcal{A}$ has a multiplicative identity element e, so that the constant function on x equal to e at every point in X is the multiplicative identity element in $c(X, \mathcal{A})$, as before. Of course, this constant function is bounded on X, with ℓ^∞ norm equal to $N(e)$, and hence is the multiplicative identity element in $\ell^\infty(X, \mathcal{A})$ too.

If X is a nonempty topological space, then the space $C(X, \mathcal{A})$ of continuous $\mathcal{A}$-valued functions on X is also an algebra over k, with respect to pointwise multiplication of functions. Thus

$$C_b(X, \mathcal{A}) = C(X, \mathcal{A}) \cap \ell^\infty(X, \mathcal{A}) \tag{18.4}$$

is a subalgebra of both $C(X, \mathcal{A})$ and $\ell^\infty(X, \mathcal{A})$. If $\mathcal{A}$ has a multiplicative identity element e, then the constant function on X equal to e at every point in X is bounded and continuous, and hence defines the multiplicative identity element in $C(X, \mathcal{A})$ and $C_b(X, \mathcal{A})$.

In particular, we can apply these remarks to $\mathcal{A} = k$, considered as an algebra over itself, and with $|\cdot|$ as the norm. If X is a nonempty set, then $c(X, k)$ is a commutative algebra over k, and $\ell^\infty(X, k)$ is a subalgebra of $c(X, k)$. The constant function on X equal to the multiplicative identity element 1 in k at each point is the multiplicative identity element in $c(X, k)$ and $\ell^\infty(X, k)$. Similarly, if X is a nonempty topological space, then $C(X, k)$ is a commutative algebra over k, and $C_b(X, k)$ is a subalgebra of both $C(X, k)$ and $\ell^\infty(X, k)$. The constant function on X equal to 1 at every point is continuous, and hence is contained in $C(X, k)$ and $C_b(X, k)$.

19 Bounded linear mappings

Let k be a field with an absolute value function $|\cdot|$, and let V, W be vector spaces over k, with norms N_V, N_W with respect to $|\cdot|$ on k, respectively. A linear mapping T from V into W is said to be *bounded* with respect to N_V, N_W if there is a nonnegative real number C such that

$$N_W(T(v)) \le C\, N_V(v) \tag{19.1}$$

for every $v \in V$. Let $\mathcal{BL}(V,W)$ be the space of bounded linear mappings from V into W. If $T \in \mathcal{BL}(V,W)$, then put

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \ge 0 : (19.1) \text{ holds}\}, \tag{19.2}$$

where more precisely the infimum is taken over all nonnegative real numbers C such that (19.1) holds. One can check that the infimum is automatically attained, so that (19.1) holds with $C = \|T\|_{op}$.

It is well known and not difficult to verify that $\mathcal{BL}(V,W)$ is a vector space over k with respect to pointwise addition and scalar multiplication. Moreover, (19.2) defines a norm on $\mathcal{BL}(V,W)$ with respect to $|\cdot|$ on k, which is the *operator norm* associated to N_V and N_W. If N_W is an ultranorm on W, then (19.2) is an ultranorm on $\mathcal{BL}(V,W)$. If W is complete with respect to the metric associated to N_W, then $\mathcal{BL}(V,W)$ is complete with respect to the metric associated to (19.2), by standard arguments.

Let Z be another vector space over k, and let N_Z be a norm on Z with respect to $|\cdot|$ on k. Suppose that T_1 is a bounded linear mapping from V into W, and that T_2 is a bounded linear mapping from W into Z. If $v \in V$, then

$$\begin{aligned} N_Z(T_2(T_1(v))) &\le \|T_2\|_{op,WZ}\, N_W(T_1(v)) \\ &\le \|T_2\|_{op,WZ}\, \|T_1\|_{op,VW}\, N_V(v), \end{aligned} \tag{19.3}$$

where the subscripts on the operator norms indicate the spaces and norms being used. It follows that the composition of T_1 and T_2 is a bounded linear mapping from V into Z, with

$$\|T_2 \circ T_1\|_{op,VZ} \le \|T_1\|_{op,VW}\, \|T_2\|_{op,WZ}. \tag{19.4}$$

Let $\mathcal{BL}(V)$ be the space of bounded linear mappings from V into itself, using N_V on V as both the domain and range of the mapping. This is an algebra over k, using composition of mappings as multiplication. The operator norm $\|T\|_{op} = \|T\|_{op,VV}$ is submultiplicative on $\mathcal{BL}(V)$, by (19.4). The identity mapping $I = I_V$ on V is a bounded linear mapping from V into itself, with $\|I\|_{op} = 1$ when $V \ne \{0\}$. Of course, I is the multiplicative identity element in $\mathcal{BL}(V)$.

Let V_0 be a dense linear subspace of V, with respect to the metric associated to N_V, and let T_0 be a bounded linear mapping from V_0 into W, using the restriction of N_V to V_0. It is easy to see that T_0 is uniformly continuous with respect to the metrics associated to N_V and N_W. If W is complete with respect to the metric associated to N_W, then there is a unique extension of T_0 to a uniformly continuous mapping from V into W, as in Section 13. One can check that this extension is a bounded linear mapping from V into W, with the same operator norm as on V_0.

20 Sums of nonnegative real numbers

Let X be a nonempty set, and let f be a nonnegative real-valued function on X. If A is a nonempty finite subset of X, then

$$\sum_{x \in A} f(x) \tag{20.1}$$

can be defined as a nonnegative real number in the usual way. The sum

$$\sum_{x \in X} f(x) \tag{20.2}$$

is defined as a nonnegative extended real number as the supremum of the finite subsums (20.1), over all nonempty finite subsets A of X. If f is a nonnegative extended real-valued function on X, then (20.1) can still be defined as a nonnegative extended real number, which is equal to $+\infty$ when $f(x) = +\infty$ for any $x \in A$. Similarly, (20.2) can be defined as a nonnegative extended real-number in this situation, which is equal to $+\infty$ when $f(x) = +\infty$ for any $x \in X$.

If $\sum_{j=1}^\infty a_j$ is an infinite series of nonnegative real numbers, then the corresponding sequence of partial sums $\sum_{j=1}^n a_j$ increases monotonically in n. The series converges with respect to the standard absolute value function on $\mathbf{R}$ exactly when the partial sums have a finite upper bound, in which case

$$\sum_{j=1}^\infty a_j = \sup_{n \ge 1} \sum_{j=1}^n a_j. \tag{20.3}$$

Otherwise, $\sum_{j=1}^\infty a_j$ may be considered to be $+\infty$ when the partial sums are not bounded, so that (20.3) holds, with the supremum on the right defined as a nonnegative real number. This is equivalent to the definition of $\sum_{j \in \mathbf{Z}_+} a_j$ as in the preceding paragraph, where $\mathbf{Z}_+$ is the set of positive integers, as usual. This uses the fact that every finite subset of $\mathbf{Z}_+$ is contained in $\{1, \ldots, n\}$ for some $n \in \mathbf{Z}_+$.

Let X be a nonempty set again, and let f, g be nonnegative extended real-valued functions on X. One can check that

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x), \tag{20.4}$$

with the usual conventions for adding nonnegative extended real numbers. Similarly, if t is a positive real number, then

$$\sum_{x \in X} t\, f(x) = t \sum_{x \in X} f(x), \tag{20.5}$$

where $t(+\infty) = +\infty$. More precisely, (20.4) and (20.5) can be obtained from the analogous statements for finite sums, by approximating the sums over X by finite subsums.

Let I be a nonempty set, and suppose that E_j is a nonempty subset of X for each $j \in I$. If f is a nonnegative extended real-valued function on X again, then

$$\sum_{x \in E_j} f(x) \tag{20.6}$$

is defined as a nonnegative extended real number for each $j \in I$, as before. This permits us to define

$$\sum_{j \in I} \Big(\sum_{x \in E_j} f(x) \Big) \tag{20.7}$$

as a nonnegative extended real number in the same way. Put

$$E = \bigcup_{j \in I} E_j, \tag{20.8}$$

so that

$$\sum_{x \in E} f(x) \tag{20.9}$$

is defined as a nonnegative extended real number as well. One can check that (20.9) is less than or equal to (20.7), by approximating (20.9) by finite subsums, as usual. If the E_j's are pairwise disjoint, so that $E_j \cap E_l = \emptyset$ when $j \neq l$, then (20.7) is equal to (20.9). To see this, one can start with the case where I has only finitely many elements, and then use that to deal with the case where I has infinitely many elements.

21 ℓ^1 And c_0 spaces

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. A V-valued function f on a nonempty set X is said to be *summable* on X with respect to N if $N(f(x))$ is summable as a nonnegative real-valued function on X, in the sense that

$$\sum_{x \in X} N(f(x)) < \infty. \tag{21.1}$$

Let $\ell^1(X, V)$ be the space of summable V-valued functions on X, and put

$$\|f\|_1 = \|f\|_{\ell^1(X,V)} = \sum_{x \in X} N(f(x)) \tag{21.2}$$

for each $f \in \ell^1(X, V)$. One can check that $\ell^1(X, V)$ is a vector space over k, with respect to pointwise addition and scalar multiplication. One can also verify that (21.2) is a norm on $\ell^1(X, V)$, with respect to $|\cdot|$ on k.

A V-valued function f on X is said to *vanish at infinity* with respect to N if for each $\epsilon > 0$ we have that

$$N(f(x)) < \epsilon \tag{21.3}$$

for all but finitely many $x \in X$. Let $c_0(X, V)$ be the space of V-valued functions on X that vanish at infinity with respect to N. If $f \in c_0(X, V)$, then it is easy to see that f is bounded on X with respect to N. More precisely, $c_0(X, V)$ is a linear subspace of the space $\ell^\infty(X, V)$ of V-valued functions on X that are bounded with respect to N. In fact, $c_0(X, V)$ is a closed set in $\ell^\infty(X, V)$ with respect to the supremum metric, by standard arguments.

If $f \in \ell^1(X, V)$, then it is easy to see that f is bounded on X with respect to N, with

$$\|f\|_\infty \le \|f\|_1. \tag{21.4}$$

Here $\|f\|_\infty$ is the supremum norm of f on X, with respect to N on V. We also have that f vanishes at infinity on X in this case. Otherwise, there is an $\epsilon > 0$ such that $N(f(x)) \ge \epsilon$ for infinitely many $x \in X$, which implies that f is not summable on X.

Remember that $c_{00}(X, V)$ is the space of V-valued function on X with finite support, as in Section 1. If $f \in c_{00}(X, V)$, then f is clearly summable on X with respect to N, and f vanishes at infinity on X with respect to N.

If $f \in \ell^1(X, V)$, then f can be approximated by elements of $c_{00}(X, V)$ with respect to the ℓ^1 norm. This uses the fact that the right side of (21.2) can be approximated by sums over finite subsets of X.

Similarly, if $f \in c_0(X, V)$, then f can be approximated by elements of $c_{00}(X, V)$ with respect to the supremum metric. Thus $c_0(X, V)$ is the same as the closure of $c_{00}(X, V)$ in $\ell^\infty(X, V)$.

If $f \in c_0(X, V)$, then the support of f has only finitely or countably many elements. To see this, one can use the the condition that f vanishes at infinity on X with $\epsilon = 1/n$ for each positive integer n. Note that this holds in particular when $f \in \ell^1(X, V)$.

If V is complete with respect to the metric associated to N, then $\ell^1(X, V)$ is complete with the metric associated to (21.2), by standard arguments.

22 Sums of vectors

Let k be a field, let V be a vector space over k, and let X be a nonempty set. If f is a V-valued function on X with finite support, then

$$\sum_{x \in X} f(x) \tag{22.1}$$

can be defined as an element of V, by reducing to a finite sum. This defines a linear mapping from the space $c_{00}(X, V)$ of V-valued functions on X with finite support into V.

Let $|\cdot|$ be an absolute value function on k, and let N be a norm on V with respect to $|\cdot|$ on k. If $f \in c_{00}(X, V)$, then

$$N\Big(\sum_{x \in X} f(x)\Big) \le \|f\|_1, \tag{22.2}$$

where $\|f\|_1$ is as in (21.2). Similarly, if N is an ultranorm on V, then

$$N\Big(\sum_{x \in X} f(x)\Big) \le \|f\|_\infty \tag{22.3}$$

for every $f \in c_{00}(X, V)$. Here $\|f\|_\infty$ denotes the supremum norm of f on X with respect to N, as usual.

Let us suppose for the rest of the section that V is complete with respect to the metric associated to N. In this case, the sum (22.1) can be extended to a linear mapping from $\ell^1(X, V)$ into V that satisfies (22.2). This uses the fact that $c_{00}(X, V)$ is dense in $\ell^1(X, V)$, as in the previous section, and the results about extending bounded linear mappings mentioned in Section 19. If N is an ultranorm on V, then the sum (22.1) can be extended to a linear mapping from $c_0(X, V)$ into V that satisfies (22.3). This uses the fact that $c_{00}(X, V)$ is dense in $c_0(X, V)$ with respect to the supremum metric, as before.

Suppose for the moment that $k = \mathbf{R}$ with the standard absolute value function, and that $V = \mathbf{R}$ too. If f is a summable real-valued function on X, then f can be expressed as the difference of nonnegative real-valued summable functions on X. The sums over X of these nonnegative real-valued summable functions can be defined as in Section 20, so that (22.1) can be defined as the difference of these two sums. Similarly, if $k = \mathbf{C}$ with the standard absolute value function, and f is a summable complex-valued function on X, then the real and imaginary parts of f are summable on X, so that (22.1) can be defined using the previous remarks. One can check that these definitions of the sum are compatible with the one mentioned in the preceding paragraph in these situations.

Let k and V be as before, and let $\{x_j\}_{j=1}^\infty$ be a sequence of disctinct elements of X. Also let f be a V-valued function on X, and suppose that the support of f is contained in the set of x_j's, $j \ge 1$. In this case, one can try to deal with the sum (22.1) by considering the infinite series

$$\sum_{j=1}^\infty f(x_j). \tag{22.4}$$

If $f \in \ell^1(X, V)$, then (22.4) converges absolutely with respect to N. Similarly, if $f \in c_0(X, V)$, then $\{f(x_j)\}_{j=1}^\infty$ converges to 0 with respect to N. This permits us to use the criteria for convergence of infinite series mentioned in Section 16. This approach to the sum (22.1) is also compatible with the one described earlier, because of the way that it approximates the sum by finite sums.

23 Sums of sums

Let k be a field, let V be a vector space over k, and let X be a nonempty set again. Also let I be a nonempty set, let E_j be a nonempty subset of X for each $j \in I$, and suppose that the E_j's are pairwise disjoint. If f is a V-valued function on X with finite support, then the restriction of f to E_j has finite support for each $j \in I$, so that

$$\sum_{x \in E_j} f(x) \tag{23.1}$$

can be defined as an element of V. Note that the restriction of f to E_j is identically equal to 0 for all but finitely many $j \in I$, because the E_j's are pairwise disjoint. This implies that (23.1) is equal to 0 for all but finitely many $j \in I$, so that

$$\sum_{j \in I} \Big(\sum_{x \in E_j} f(x) \Big) \tag{23.2}$$

can be defined as an element of V as well. Put

$$E = \bigcup_{j \in I} E_j, \tag{23.3}$$

so that

$$\sum_{x \in E} f(x) \tag{23.4}$$

can be defined as an element of V too. It is easy to see that (23.2) is equal to (23.4) under these conditions.

Let $|\cdot|$ be an absolute value function on k, and let N be a norm on V with respect to $|\cdot|$ on k. If $j \in I$, then

$$N\Big(\sum_{x \in E_j} f(x)\Big) \le \sum_{x \in E_j} N(f(x)), \tag{23.5}$$

as in (22.2). Similarly,

$$N\Big(\sum_{j \in I} \Big(\sum_{x \in E_j} f(x)\Big)\Big) \le \sum_{j \in I} N\Big(\sum_{x \in E_j} f(x)\Big) \le \sum_{j \in I} \Big(\sum_{x \in E_j} N(f(x))\Big) \tag{23.6}$$

and

$$N\Big(\sum_{x \in E} f(x)\Big) \le \sum_{x \in E} N(f(x)). \tag{23.7}$$

Note that

$$\sum_{j \in I} \Big(\sum_{x \in E_j} N(f(x))\Big) = \sum_{x \in E} N(f(x)) \le \sum_{x \in X} N(f(x)), \tag{23.8}$$

as sums of nonnegative real numbers. More precisely, this works for any V-valued function f on X, as in Section 20.

Suppose now that $f \in \ell^1(X, V)$, and that V is complete with respect to the metric associated to N. If $j \in I$, then the restriction of f to E_j is summable on E_j, so that the sum (23.1) can be defined as an element of V as in the previous section. We also have (23.5) for each $j \in I$, which implies that (23.1) is summable as a V-valued function on I, because of (23.8). Thus the sum (23.2) can be defined as an element of V, and satisfies (23.6). Similarly, the sum (23.4) can be defined as an element of V, and satisfies (23.7). One can check that (23.2) is equal to (23.4) in this situation. This uses the fact that $c_{00}(X, V)$ is dense in $\ell^1(X, V)$, as in Section 21.

Suppose from now on in this section that N is an ultranorm on V. If f is a V-valued function on X with finite support, then

$$N\Big(\sum_{x \in E_j} f(x)\Big) \le \max_{x \in E_j} N(f(x)) \tag{23.9}$$

for every $j \in I$, as in (22.3). Similarly,

$$N\Big(\sum_{j \in I}\Big(\sum_{x \in E_j} f(x)\Big)\Big) \le \max_{j \in I} N\Big(\sum_{x \in E_j} f(x)\Big) \le \max_{j \in I}\Big(\max_{x \in E_j} N(f(x))\Big) \tag{23.10}$$

and

$$N\Big(\sum_{x \in E} f(x)\Big) \le \max_{x \in E} N(f(x)) \tag{23.11}$$

in this case. Of course,

$$\max_{j \in I}\Big(\max_{x \in E_j} N(f(x))\Big) = \max_{x \in E} N(f(x)) \le \max_{x \in X} N(f(x)) \tag{23.12}$$

for every $f \in c_{00}(X, V)$. This works for any V-valued function f on X, with the maxima replaced with suprema.

Suppose that $f \in c_0(X, V)$, and that V is complete with respect to the metric associated to N. Observe that the restriction of f to any nonempty subset of X vanishes at infinity on that subset. In particular, the restriction of f to E_j vanishes at infinity on E_j for each $j \in I$, so that the sum (23.1) can be defined as an element of V that satisfies (23.9), as in the preceding section. One can check that (23.1) vanishes at infinity as a V-valued function on I, using (23.9) and the hypothesis that the E_j's are pairwise disjoint. Hence the sum (23.2) can be defined as an element of V too, and satisfies (23.10). Similarly, the restriction of f to E vanishes at infinity on E, so that the sum (23.4) can be defined as an element of V, and satisfies (23.11). As before, one can verify that (23.1) is equal to (23.4), by approximating f by elements of $c_{00}(X, V)$ with respect to the supremum norm.

24 Cauchy products

Let n be a positive integer, and put

$$E_\gamma = \{(\alpha, \beta) \in (\mathbf{Z}_+ \cup \{0\})^n \times (\mathbf{Z}_+ \cup \{0\})^n : \alpha + \beta = \gamma\} \tag{24.1}$$

for each $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$. Thus the E_γ's are pairwise-disjoint nonempty finite sets such that

$$\bigcup_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} E_\gamma = (\mathbf{Z}_+ \cup \{0\})^n \times (\mathbf{Z}_+ \cup \{0\})^n. \tag{24.2}$$

Let k be a field, and let $\mathcal{A}$ be an algebra over k. Also let f, g be $\mathcal{A}$-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$, and put

$$h(\gamma) = \sum_{(\alpha,\beta) \in E_\gamma} f(\alpha)\, g(\beta) \tag{24.3}$$

for each $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$. This defines h as an $\mathcal{A}$-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$. Observe that

$$\sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} h(\gamma) \tag{24.4}$$

is equal to

$$\sum_{(\alpha,\beta) \in (\mathbf{Z}_+ \cup \{0\})^n \times (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha)\, g(\beta), \tag{24.5}$$

at least formally, because of (24.2). This is the same as the iterated sum

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} \Big(\sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha)\, g(\beta) \Big), \tag{24.6}$$

at least formally, which reduces to the product

$$\Big(\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f(\alpha) \Big) \Big(\sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} g(\beta) \Big). \tag{24.7}$$

Suppose for the moment that f and g have finite support in $(\mathbf{Z}_+ \cup \{0\})^n$, so that each of the sums in (24.7) are defined as elements of $\mathcal{A}$. In this case, $f(\alpha)\, g(\beta)$ has finite support as a function of (α, β), so that the sum in (24.5) is also defined as an element of $\mathcal{A}$. It is easy to see that h has finite support in $(\mathbf{Z}_+ \cup \{0\})^n$ in this situation too, so that (24.4) is defined as an element of $\mathcal{A}$. As in the previous section, (24.4) is equal to (24.5) under these conditions. Similarly, (24.5) is equal to (24.7), so that (24.4) is equal to (24.7).

Suppose now that f, g are nonnegative real-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$, so that h is a nonnegative real-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$ as well. This implies that (24.4), (24.5), and each of the sums in (24.7) are defined as nonnegative extended real numbers. If both factors in (24.7) are positive, then (24.7) can be defined as a nonnegative extended real number in the usual way. If one of the factors in (24.7) is equal to 0, then let us take the product to be 0, even if the other factor is $+\infty$. In this case, (24.4), (24.5), and (24.7) are the same, as in Section 20.

Let $|\cdot|$ be an absolute value function on k, and let N be a submultiplicative norm on $\mathcal{A}$, with respect to $|\cdot|$ on k. Let f, g be $\mathcal{A}$-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ again, and note that

$$N(f(\alpha)\, g(\beta)) \le N(f(\alpha))\, N(g(\beta)) \tag{24.8}$$

for every $\alpha, \beta \in (\mathbf{Z}_+ \cup \{0\})^n$. Thus

$$N(h(\gamma)) \le \sum_{(\alpha,\beta)\in E_\gamma} N(f(\alpha))\, N(g(\beta)) \tag{24.9}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$. Suppose that f and g are summable as $\mathcal{A}$-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ with respect to N, so that $N(f(\alpha))$ and $N(g(\beta))$ are summable as nonnegative real-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$. This implies that $N(f(\alpha))\, N(g(\beta))$ is summable as a nonnegative real-valued function of (α, β), as in the preceding paragraph. It follows that $f(\alpha)\, g(\beta)$ is summable as an $\mathcal{A}$-valued function of (α, β) with respect to N, because of (24.8). We also get that the right side of (24.9) is summable as a nonnegative real-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$, so that h is summable as an $\mathcal{A}$-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$ with respect to N.

Suppose from now on in this section that $\mathcal{A}$ is complete with respect to the metric associated to N. Using the summability conditions mentioned in the previous paragraph, the sums (24.4), (24.5), and in both factors in (24.7) can be defined as elements of $\mathcal{A}$, as in Section 22. The remarks in the preceding section imply that (24.4) is equal to (24.5). Similarly, the remarks in the preceding section can be used to treat (24.5) as an iterated sum. It follows that (24.5) is equal to (24.7), so that (24.4) is equal to (24.7).

Suppose now that N is an ultranorm on $\mathcal{A}$. Using (24.8), we get that

$$N(h(\gamma)) \le \max_{(\alpha,\beta)\in E_\gamma} (N(f(\alpha))\, N(g(\beta))) \tag{24.10}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$. Suppose that f and g vanish at infinity as $\mathcal{A}$-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ with respect to N. In particular, this implies that f and g are bounded on $(\mathbf{Z}_+ \cup \{0\})^n$, as before. One can check that $f(\alpha)\, g(\beta)$ vanishes at infinity as an $\mathcal{A}$-valued function of (α, β) with respect to N, using (24.8) again. One can also verify that h vanishes at infinity as an $\mathcal{A}$-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$ with respect to N. More precisely, the right side of (24.10) vanishes at infinity as a nonnegative real-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$.

Because $\mathcal{A}$ is complete, the sums (24.4), (24.5), and in both factors in (24.7) are defined as elements of $\mathcal{A}$ in this situation, as in Section 22. As before, the remarks in the previous section imply that (24.4) is equal to (24.5), and (24.5) can be treated as an iterated sum. Thus (24.5) is equal to (24.7), and hence (24.4) is equal to (24.7).

25 Some weighted conditions

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let X be a nonempty set, and let w be a positive real-valued function defined on X. A V-valued function f on X is said to be *bounded* with respect to N on V and w on X if

$$N(f(x))\, w(x) \tag{25.1}$$

is bounded in the usual sense as a nonnegative real-valued function on X. Let $\ell^\infty_w(X,V)$ be the space of V-valued functions on X that are bounded with respect to N on V and w on X. If $f \in \ell^\infty_w(X,V)$, then put

$$\|f\|_{\infty,w} = \|f\|_{\ell^\infty_w(X,V)} = \sup_{x\in X}(N(f(x))\,w(x)). \tag{25.2}$$

which is the same as the ordinary supremum norm of (25.1). One can check that $\ell^\infty_w(X,V)$ is a vector space over k with respect to pointwise addition and scalar multiplication, and that (25.2) defines a norm on $\ell^\infty_w(X,V)$ with respect to $|\cdot|$ on k. If N is an ultranorm on V, then (25.2) is an ultranorm on $\ell^\infty_w(X,V)$. If V is complete with respect to the metric associated to N, then one can verify that $\ell^\infty_w(X,V)$ is complete with respect to the metric associated to (25.2).

A V-valued function f on X is said to be *summable* with respect to N on V and w on X if (25.1) is summable in the usual sense as a nonnegative real-valued function on X. Let $\ell^1_w(X,V)$ be the space of V-valued functions on X that are summable with respect to N on V and w on X. If $f \in \ell^1_w(X,V)$, then put

$$\|f\|_{1,w} = \|f\|_{\ell^1_w(X,V)} = \sum_{x\in X} N(f(x))\,w(x), \tag{25.3}$$

which is the same as the ordinary ℓ^1 norm of (25.1). One can check that $\ell^1_w(X,V)$ is a vector space over k with respect to pointwise addition and scalar multiplication, and that (25.3) defines a norm on $\ell^1_w(X,V)$ with respect to $|\cdot|$ on k, as usual. If V is complete with respect to the metric associated to N, then $\ell^1_w(X,V)$ is complete with respect to the metric associated to (25.3), by standard arguments.

A V-valued function f on X is said to *vanish at infinity* with respect to N on V and w on X if (25.1) vanishes at infinity as a nonnegative real-valued function on X. Let $c_{0,w}(X,V)$ be the space of V-valued functions on X that vanish at infinity with respect to N on V and w on X. If $f \in c_{0,w}(X,V)$, then f is bounded with respect to N on V and w on X, as before. It is easy to see that $c_{0,w}(X,V)$ is a linear subspace of $\ell^\infty_w(X,V)$. One can also verify that $c_{0,w}(X,V)$ is a closed set in $\ell^\infty_w(X,V)$, with respect to the metric associated to (25.2).

If $f \in \ell^1_w(X,V)$, then f is bounded with respect to N on V and w on X, with

$$\|f\|_{\infty,w} \le \|f\|_{1,w}. \tag{25.4}$$

More precisely, f vanishes at infinity with respect to N on V and w on X, as before. If f is a V-valued function on X with finite support, then f is summable with respect to N on V and w on X, and f vanishes at infinity with respect to N on V and w on X. One can check that $c_{00}(X,V)$ is dense in $\ell^1_w(X,V)$ with respect to the metric associated to (25.3). Similarly, $c_{00}(X,V)$ is dense in $c_{0,w}(X,V)$ with respect to (25.2). If $f \in c_{0,w}(X,V)$, then the support of f has only finitely or countably many elements. This follows from the analogous statement for (25.1), which was mentioned previously.

26 Cartesian products

Let k be a field, let V be a vector space over k, and let X, Y be nonempty sets. If $f(x, y)$ is a V-valued function on the Cartesian product $X \times Y$, then

$$f_y(x) = f(x, y) \tag{26.1}$$

defines a V-valued function of $x \in X$ for each $y \in Y$, as in Section 5. Thus

$$F(y) = f_y \tag{26.2}$$

may be considered as a mapping from Y into the space $c(X, V)$ of V-valued functions on X, as before. Conversely, any mapping from Y into $c(X, V)$ corresponds to a V-valued function on $X \times Y$ in this way. This defines a natural one-to-one linear mapping from $c(X \times Y, V)$ onto $c(Y, c(X, V))$.

Similarly, if $f(x, y)$ is a V-valued function on $X \times Y$ with finite support, then (26.1) has finite support as a function of $x \in X$ for each $y \in Y$. Hence (26.2) may be considered as a mapping from Y into the space $c_{00}(X, V)$ of V-valued functions on X with finite support, and this mapping has finite support in Y in this case. Conversely, one can check that any mapping from Y into $c_{00}(X, V)$ with finite support in Y corresponds to a V-valued function on $X \times Y$ with finite support in this way. This means that $c_{00}(X \times Y, V)$ corresponds exactly to $c_{00}(Y, c_{00}(X, V))$ as in the previous paragraph.

Let $|\cdot|$ be an absolute value function on k, and let N be a norm on V with respect to $|\cdot|$ on k. Also let $w_1(x)$ be a positive real-valued function on X, and let $w_2(y)$ be a positive real-valued function on Y. Put

$$w(x, y) = w_1(x)\, w_2(y), \tag{26.3}$$

which defines a positive real-valued function on $X \times Y$.

Suppose that $f(x, y)$ is a V-valued function on $X \times Y$ that is bounded with respect to N on V and w on $X \times Y$, so that

$$N(f(x, y))w(x, y) = N(f(x, y))\, w_1(x)\, w_2(y) \tag{26.4}$$

is bounded as a nonnegative real-valued function on $X \times Y$. Equivalently, $f(x, y)$ is an element of the space $\ell^\infty_w(X \times Y, V)$ defined in the previous section, and we put

$$\begin{aligned} \|f\|_{\ell^\infty_w(X\times Y,V)} &= \sup_{(x,y)\in X\times Y} (N(f(x, y))\, w(x, y)) \\ &= \sup_{y\in Y} \sup_{x\in X} (N(f(x, y))\, w_1(x)\, w_2(y)), \end{aligned} \tag{26.5}$$

as before. If $y \in Y$ and f_y is the V-valued function defined on X as in (26.1), then f_y is bounded with respect to N on V and w_1 on X. More precisely,

$$N(f_y(x))\, w_1(x) = N(f(x, y))\, w_1(x) \le \|f\|_{\ell^\infty_w(X\times Y,V)}\, w_2(y)^{-1} \tag{26.6}$$

for every $x \in X$, so that f_y is an element of the space $\ell^\infty_{w_1}(X, V)$ defined in the previous section, with

$$(26.7)\quad \|f_y\|_{\ell^\infty_{w_1}(X,V)} = \sup_{x \in X} (N(f_y(x))\, w_1(x)) \le \|f\|_{\ell^\infty_w(X \times Y, V)}\, w_2(y)^{-1}.$$

Of course, this is the same as saying that

$$(26.8)\quad \|f_y\|_{\ell^\infty_{w_1}(X,V)}\, w_2(y) = \sup_{x \in X} (N(f(x,y))\, w_1(x))\, w_2(y) \le \|f\|_{\ell^\infty(X \times Y, V)}$$

for every $y \in Y$. In this situation, we may consider F in (26.2) as a mapping from Y into $\ell^\infty_{w_1}(X, V)$, with

$$(26.9)\quad \|F(y)\|_{\ell^\infty_{w_1}(X,V)}\, w_2(y) = \|f_y\|_{\ell^\infty_{w_1}(X,V)}\, w_2(y) \le \|f\|_{\ell^\infty_w(X \times Y, V)}$$

for every $y \in Y$. This implies that F is bounded with respect to the $\ell^\infty_{w_1}(X, V)$ norm and w_2 on Y, so that F is an element of $\ell^\infty_{w_2}(Y, \ell^\infty_{w_1}(X, V))$. We also have that

$$(26.10)\quad \|F\|_{\ell^\infty_{w_2}(Y, \ell^\infty_{w_1}(X,V))} = \sup_{y \in Y} (\|F(y)\|_{\ell^\infty_{w_1}(X,V)}\, w_2(y)) = \|f\|_{\ell^\infty_w(X \times Y, V)}.$$

Conversely, every element of $\ell^\infty_{w_2}(Y, \ell^\infty_{w_1}(X, V))$ corresponds to an element of $\ell^\infty_w(X \times Y, V)$ in this way.

Suppose now that $f(x, y)$ is a V-valued function on $X \times Y$ that vanishes at infinity with respect to N on V and w on $X \times Y$, so that (26.4) vanishes at infinity as a nonnegative real-valued function on $X \times Y$. Thus, for each $\epsilon > 0$, there is a finite subset E_ϵ of $X \times Y$ such that

$$(26.11)\quad N(f(x, y))\, w_1(x)\, w_2(y) < \epsilon$$

for every $(x, y) \in X \times Y \setminus E_\epsilon$. Note that

$$(26.12)\quad E_\epsilon(y) = \{x \in X : (x, y) \in E_\epsilon\}$$

is a finite subset of X for every $\epsilon > 0$ and $y \in Y$. If $y \in Y$, then

$$(26.13)\quad N(f_y(x))\, w_1(x)\, w_2(y) = N(f(x, y))\, w_1(x)\, w_2(y)$$

vanishes at infinity as a nonnegative real-valued function of $x \in X$, and hence

$$(26.14)\quad N(f_y(x))\, w_1(x)$$

vanishes at infinity as a nonnegative real-valued function of $x \in X$. This means that

$$(26.15)\quad f_y \in c_{0,w_1}(X, V)$$

for every $y \in Y$, where $c_{0,w_1}(X, V)$ is the space of V-valued functions on X that vanish at infinity with respect to N on V and w_1 on X, as in the previous section.

If $y \in Y$ and $E_\epsilon(y) = \emptyset$, then

$$N(f_y(x))\, w_1(x) = N(f(x,y))\, w_1(x) < \epsilon\, w_2(y)^{-1} \tag{26.16}$$

for every $x \in X$, because of (26.11). In this case, we get that

$$\|f_y\|_{\ell^\infty_{w_1}(X,V)} = \sup_{x \in X} (N(f_y(x))\, w_1(x)) \le \epsilon\, w_2(y)^{-1}, \tag{26.17}$$

so that

$$\|f_y\|_{\ell^\infty_{w_1}(X,V)}\, w_2(y) \le \epsilon. \tag{26.18}$$

We may consider F in (26.2) as a mapping from Y into $c_{0,w_1}(X,V)$, by (26.15). Under these conditions, F vanishes at infinity with respect to the $\ell^\infty_{w_1}(X,V)$ norm on $c_{0,w_1}(X,V)$ and w_2 on Y. This uses the fact that for each $\epsilon > 0$, there are only finitely many $y \in Y$ such that $E_\epsilon(y) \ne \emptyset$. Thus

$$F \in c_{0,w_2}(Y, c_{0,w_1}(X,V)), \tag{26.19}$$

using the $\ell^\infty_{w_1}(X,V)$ norm on $c_{0,w_1}(X,V)$ to define $c_{0,w_2}(Y, c_{0,w_1}(X,V))$. Conversely, every element of $c_{0,w_2}(Y, c_{0,w_1}(X,V))$ corresponds to an element of $c_{0,w}(X \times Y, V)$ in this way.

Now let $f(x,y)$ be a V-valued function on $X \times Y$ that is summable with respect to N on V and w on $X \times Y$, so that (26.4) is summable as a nonnegative real-valued function on $X \times Y$. If $y \in Y$, then it follows that (26.13) is summable as a nonnegative real-valued function of $x \in X$, so that (26.14) is summable as a nonnegative real-valued function of $x \in X$. Thus

$$f_y \in \ell^1_{w_1}(X,V) \tag{26.20}$$

for every $y \in Y$, where $\ell^1_{w_1}(X,V)$ is the space of V-valued functions on X that are summable with respect to N on V and w_1 on X, as in the previous section. Of course,

$$\begin{aligned} \|f_y\|_{\ell^1_{w_1}(X,V)}\, w_2(y) &= \sum_{x \in X} N(f_y(x))\, w_1(x)\, w_2(y) \\ &= \sum_{x \in X} N(f(x,y))\, w_1(x)\, w_2(y) \end{aligned} \tag{26.21}$$

for every $y \in Y$. Summing over $y \in Y$, we get that

$$\begin{aligned} \sum_{y \in Y} \|f_y\|_{\ell^1_{w_1}(X,V)}\, w_2(y) &= \sum_{y \in Y} \Big(\sum_{x \in X} N(f(x,y))\, w_1(x)\, w_2(y) \Big) \\ &= \sum_{(x,y) \in X \times Y} N(f(x,y))\, w_1(x)\, w_2(y) \\ &= \|f\|_{\ell^1_w(X \times Y, V)}, \end{aligned} \tag{26.22}$$

using the remarks about sums of sums in Section 20 in the second step. Because of (26.20), we may consider F in (26.2) as a mapping from Y into $\ell^1_{w_1}(X,V)$.

This mapping is summable with respect to the $\ell^1_{w_1}(X, V)$ norm and w_2 on Y, by (26.22). This means that F is an element of $\ell^1_{w_2}(Y, \ell^1_{w_1}(X, V))$, with

$$(26.23)\quad \|F\|_{\ell^1_{w_2}(Y,\ell^1_{w_1}(X,V))} = \sum_{y\in Y} \|F(y)\|_{\ell^1_{w_1}(X,V)}\, w_2(y) = \|f\|_{\ell^1_w(X\times Y,V)}.$$

Conversely, every element of $\ell^1_{w_2}(Y, \ell^1_{w_1}(X, V))$ corresponds to an element of $\ell^1_w(X \times Y, V)$ in this way.

27 Hölder's inequality

Let X be a nonempty set, and let q be a positive real number. A nonnegative real-valued function f on X is said to be *q-summable* on X if $f(x)^q$ is summable on X. In this case, we put

$$(27.1)\qquad \|f\|_q = \Big(\sum_{x\in X} f(x)^q\Big)^{1/q}.$$

Similarly, if f is bounded on X, then we put

$$(27.2)\qquad \|f\|_\infty = \sup_{x\in X} f(x),$$

as before.

Let q_1, q_2, q_3 be positive real numbers such that

$$(27.3)\qquad 1/q_3 = 1/q_1 + 1/q_2.$$

This implies that $1/q_1, 1/q_2 < 1/q_3$, so that $q_3 < q_1, q_2$. Let f, g be nonnegative real-valued functions on X that are q_1, q_2-summable, respectively. Under these conditions, *Hölder's inequality* implies that $f\,g$ is q_3-summable on X, with

$$(27.4)\qquad \|f\,g\|_{q_3} \le \|f\|_{q_1}\,\|g\|_{q_2}.$$

This is often stated for $q_3 = 1$, and it is easy to reduce to that case.

If we take $q_1 = \infty$, then $1/q_1$ is interpreted as being equal to 0, as usual, and (27.3) reduces to saying that $q_2 = q_3$. Let f, g be nonnegative real-valued functions on X again, and suppose that f is bounded on X, and that g is q_2-summable on X for some positive real number q_2. It is easy to see that $f\,g$ is also q_2-summable on X, and that (27.4) holds with $q_3 = q_2$. This is often included in the statement of Hölder's inequality. Of course, one can deal with $q_2 = \infty$ in the same way.

If we take $q_1 = q_2 = \infty$, then (27.4) says that $q_3 = \infty$ as well. If f, g are bounded nonnegative real-valued functions on X, then $f\,g$ is bounded on X too, and (27.4) holds with $q_1 = q_2 = q_3 = \infty$. In this case, if either f or g vanishes at infinity on X, then $f\,g$ vanishes at infinity on X.

Let f be a nonnegative real-valued function on X again, and let w be a positive real-valued function on X. If $f\,w$ is summable on X, then we say that f is *summable* on X with respect to w, as before, and we put

$$\|f\|_{1,w} = \sum_{x \in X} f(x)\, w(x) = \|f\, w\|_1. \tag{27.5}$$

If $f\,w$ is bounded on X, then we say that f is *bounded* with respect to w on X, and we put

$$\|f\|_{\infty,w} = \sup_{x \in X} (f(x)\, w(w)) = \|f\, w\|_\infty. \tag{27.6}$$

If $f\,w$ vanishes at infinity on X, then we may say that f *vanishes at infinity* with respect to w on X. This implies that f is bounded on X with respect to w, as usual.

Let w_0, w_1 be positive real-valued functions on X, and put

$$w_t(x) = w_0(x)^{1-t}\, w_1(x)^t \tag{27.7}$$

for every $x \in X$ and $t \in \mathbf{R}$ with $0 < t < 1$. Note that this reduces to w_0 when $t = 0$, and to w_1 when $t = 1$. If f is a nonnegative real-valued function on X, then

$$\begin{aligned} f(x)\, w_t(x) &= f(x)^{1-t}\, w_0(x)^{1-t}\, f(x)^t\, w_1(x)^t \\ &= (f(x)\, w_0(x))^{1-t}\, (f(x)\, w_1(x))^t \end{aligned} \tag{27.8}$$

for every $x \in X$ and $0 < t < 1$.

Let $0 < t < 1$ be given, and put

$$q_1 = 1/(1-t), \quad q_2 = 1/t. \tag{27.9}$$

Thus $1 < q_1, q_2 < \infty$, and

$$1/q_1 + 1/q_2 = (1-t) + t = 1. \tag{27.10}$$

Let f be a nonnegative real-valued function on X that is summable with respect to w_0 and w_1 on X, so that $f\,w_0$ and $f\,w_1$ are summable on X. This implies that $(f(x)\, w_0(x))^{1-t}$ is q_1-summable on X, with

$$\|(f\, w_0)^{1-t}\|_{q_1} = \Big(\sum_{x \in X} f(x)\, w_0(x)\Big)^{1-t} = \|f\|_{1,w_0}^{1-t}. \tag{27.11}$$

Similarly, $(f(x)\, w_1(x))^t$ is q_2-summable on X, with

$$\|(f\, w_1)^t\|_{q_2} = \Big(\sum_{x \in X} f(x)\, w_1(x)\Big)^t = \|f\|_{1,w_1}^t. \tag{27.12}$$

Let w_t be as in (27.7), so that $f\,w_t$ can be expressed as in (27.8). Using Hölder's inequality, we get that $f\,w_t$ is summable on X, with

$$\|f\|_{1,w_t} = \|f\, w_t\|_1 \le \|(f\, w_0)^{1-t}\|_{q_1}\, \|(f\, w_1)^t\|_{q_2} = \|f\|_{1,w_0}^{1-t}\, \|f\|_{1,w_1}^t. \tag{27.13}$$

Now let f be a nonnegative real-valued function on X that is bounded with respect to w_0 and w_1. If $0 < t < 1$ and w_t is as in (27.7), then it is easy to see that f is bounded with respect to w_t, using (27.8). More precisely,

$$\begin{aligned} \|f\|_{\infty,w_t} = \|f\,w_t\|_\infty &= \|(f\,w_0)^{1-t}\,(f\,w_1)^t\|_\infty \\ &\le \|(f\,w_0)^{1-t}\|_\infty\,\|(f\,w_1)^t\|_\infty \\ &= \|f\,w_0\|_\infty^{1-t}\,\|f\,w_1\|_\infty^t = \|f\|_{\infty,w_0}^{1-t}\,\|f\|_{\infty,w_1}^t. \end{aligned} \tag{27.14}$$

If f also vanishes at infinity on X with respect to w_0 or w_1, then f vanishes at infinity on X with respect to w_t when $0 < t < 1$.

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let f be a V-valued function on X, so that $N(f(x))$ is a nonnegative real-valued function on X, to which the previous remarks can be applied. Let w_0, w_1 be positive real-valued functions on X again, and let w_t be as in (27.7) for $0 < t < 1$. If f is summable with respect to N on V and both w_0 and w_1 on X, then it follows that f is summable with respect to N on V and w_t on X when $0 < t < 1$, with

$$\|f\|_{1,w_t} \le \|f\|_{1,w_0}^{1-t}\,\|f\|_{1,w_1}^t, \tag{27.15}$$

as in (27.13). This uses the terminology and notation in Section 25. Thus

$$\ell^1_{w_0}(X,V) \cap \ell^1_{w_1}(X,V) \subseteq \ell^1_{w_t}(X,V) \tag{27.16}$$

when $0 < t < 1$. Similarly, if f is bounded with respect to N on V and both w_0 and w_1 on X, then f is bounded with respect to N on V and w_t on X when $0 < t < 1$, with

$$\|f\|_{\infty,w_t} \le \|f\|_{\infty,w_0}^{1-t}\,\|f\|_{\infty,w_1}^t, \tag{27.17}$$

as in (27.14). Hence

$$\ell^\infty_{w_0}(X,V) \cap \ell^\infty_{w_1}(X,V) \subseteq \ell^\infty_{w_t}(X,V) \tag{27.18}$$

when $0 < t < 1$. In this case, if f also vanishes at infinity with respect to N on V and either w_0 or w_1 on X, then f vanishes at infinity with respect to N on V and w_t on X when $0 < t < 1$. This means that

$$c_{0,w_0}(X,V) \cap \ell^\infty_{w_1}(X,V),\ \ell^\infty_{w_0}(X,V) \cap c_{0,w_1}(X,V) \subseteq c_{0,w_t}(X,V) \tag{27.19}$$

when $0 < t < 1$.

28 Generalized convergence of sums

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let X be a nonempty set, and let f be a V-valued function on X. If A is a nonempty finite subset of X, then the sum

$$\sum_{x \in A} f(x) \tag{28.1}$$

can be defined as an element of V in the usual way. One way to try to define the sum

$$\sum_{x \in X} f(x) \tag{28.2}$$

is as a limit of the finite subsums (28.1), when the limit exists. More precisely, the collection of all nonempty finite subsets of X is a partially-ordered set with respect to inclusion. If A_1, A_2 are nonempty finite subsets of X, then $A_1 \cup A_2$ is a nonempty finite subset of X that contains A_1 and A_2. This means that the collection of nonempty finite subsets of X is a directed system with respect to inclusion. The family of finite subsums (28.1) may be considered as a net in V, indexed by this directed system. Convergence of this net in V can be defined in the usual way, using the topology determined on V by the metric associated to N.

In this situation, the convergence of the net of finite subsums (28.1) to a vector $v \in V$ means that for every $\epsilon > 0$ there is a nonempty finite subset $A(\epsilon)$ of X such that

$$N\Big(\sum_{x \in A} f(x) - v\Big) < \epsilon \tag{28.3}$$

for every nonempty finite subset A of X with

$$A(\epsilon) \subseteq A. \tag{28.4}$$

One can check that the limit v is unique when it exists, in which case the sum (28.2) is defined to be v. The collection of V-valued functions f on X for which the limit exists is a vector space over k with respect to pointwise addition and scalar multiplication, and the sum (28.2) is linear in f. If f is a nonnegative real-valued summable function on X, then the net of finite subsums (28.1) converges to the supremum of these subsums, so that this definition of the sum (28.2) is compatible with the one in Section 20. If f is a V-valued function on X with finite support, then the finite subsums (28.1) are the same when A contains the support of f, so that the sum (28.2) reduces to this finite sum.

Let f be a V-valued function on X again. The net of finite subsums (28.1) is a Cauchy net with respect to the metric on V associated to N if for every $\epsilon > 0$ there is a nonempty finite subset $A_1(\epsilon)$ of X such that

$$N\Big(\sum_{x \in A_1} f(x) - \sum_{x \in A_2} f(x)\Big) < \epsilon \tag{28.5}$$

for all finite subsets A_1, A_2 of X that contain $A_1(\epsilon)$. If the net of finite sums (28.1) converges in V, as in the preceding paragraph, then this condition holds with $A_1(\epsilon) = A(\epsilon/2)$. If N is an ultranorm on V, then one can take $A_1(\epsilon) = A(\epsilon)$. Alternatively, this Cauchy condition can be formulated as saying that for every $\epsilon > 0$ there is a nonempty finite subset $A_2(\epsilon)$ of X such that

$$N\Big(\sum_{x \in B} f(x)\Big) < \epsilon \tag{28.6}$$

for every nonempty finite subset B of X such that $A_2(\epsilon) \cap B = \emptyset$. More precisely, the previous condition implies this one with $A_2(\epsilon) = A_1(\epsilon)$, by taking $A_1 = A_1(\epsilon) \cup B$ and $A_2 = A_1(\epsilon)$. Conversely, the second formulation implies the first one with $A_1(\epsilon) = A_2(\epsilon/2)$, and with $A_1(\epsilon) = A_2(\epsilon)$ when N is an ultranorm on V. Note that the second formulation implies that f vanishes at infinity on X with respect to N on V, by taking $B = \{x\}$ when $x \in X \setminus A_2(\epsilon)$.

Of course,

$$N\Big(\sum_{x \in B} f(x)\Big) \le \sum_{x \in B} N(f(x)) \tag{28.7}$$

for every nonempty finite subset B of X, by the triangle inequality. If f is summable on X with respect to N on V, then one can use (28.7) to check that the second version of the Cauchy condition in the preceding paragraph holds. If N is an ultranorm on V, then

$$N\Big(\sum_{x \in B} f(x)\Big) \le \max_{x \in B} N(f(x)) \tag{28.8}$$

for every nonempty finite subset B of X. In this case, if f vanishes at infinity on X with respect to N on V, then it follows that the second version of the Cauchy condition in the preceding paragraph holds again.

Suppose that $A_1, A_2, A_3, \ldots$ is an infinite sequence of nonempty finite subsets of X such that

$$A_j \subseteq A_{j+1} \tag{28.9}$$

for each $j \ge 1$, and that the support of f is contained in $\bigcup_{j=1}^\infty A_j$. If the net of all finite subsums (28.1) converges in V, then the sequence of subsums

$$\sum_{x \in A_j} f(x) \tag{28.10}$$

converges to the same element of V. Similarly, if the net of finite subsums (28.1) satisfies the Cauchy condition mentioned earlier, then the sequence of finite subsums (28.10) is a Cauchy sequence in V with respect to the metric associated to N. In particular, if V is complete with respect to this metric, then it follows that the sequence of finite subsums (28.10) converges in V. If the net of finite subsums (28.1) satisfies the Cauchy condition mentioned earlier, and if the sequence of finite subsums (28.10) converges in V, then the net of all finite subsums (28.1) converges to the same limit.

If the net of finite subsums (28.1) satisfies the Cauchy condition mentioned earlier, then f vanishes at infinity on X with respect to N on V, and hence the support of f has only finitely or countably many elements. This implies that the support of f is contained in the union of a sequence of monotonically increasing nonempty finite subsets of X, as in the previous paragraph. If V is complete with respect to the metric associated to N, then it follows that the net of all finite subsums (28.1) converges in V, as before. This can be used as another way to define the sum (28.2) in Section 22.

Suppose that f is a real or complex-valued function on X, using the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate. If f is summable on X, then f can be expressed as a linear combination of nonnegative real-valued summable functions on X. This can be used to define the sum (28.2) as a real or complex number, as appropriate, by reducing to the case of nonnegative real-valued summable functions. In particular, the convergence of the net of finite subsums (28.1) follows from the analogous statement for nonnegative real-valued summable functions on X in this case.

If f is any V-valued function on X, then the Cauchy condition mentioned earlier implies that the finite subsums (28.1) have bounded norm. If f is a real-valued function on X, then the boundedness of the finite subsums (28.1) implies that f is summable on X, by considering nonempty finite subsets of X on which f has constant sign. Similarly, if f is a complex-valued function on X, then the boundedness of the finite subsums (28.1) implies that f is summable on X, by applying the previous statement to the real and imaginary parts of f.

Part III
Norms and power series

29 Some norms on polynomials

Let n be a positive integer, and let $r = (r_1, \ldots, r_n)$ be an n-tuple of positive real numbers. If $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$ is a multi-index, then put

$$w_r(\alpha) = r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n}. \tag{29.1}$$

This defines w_r as a positive real-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$.

Let k be a field, let V be a vector space over k, and let $T_1, \ldots, T_n$ be n commuting indeterminates. Remember that the space $V[T_1, \ldots, T_n]$ of formal polynomials in $T_1, \ldots, T_n$ with coefficients in V is a vector space over k with respect to termwise addition and scalar multiplication, as in Section 2. Let $|\cdot|$ be an absolute value function on k, and let N be a norm on V with respect to $|\cdot|$ on k. Also let

$$f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, T^\alpha \tag{29.2}$$

be a formal polynomial in $T_1, \ldots, T_n$ with coefficients in V, so that $f_\alpha \in V$ for every multi-index α, and $f_\alpha = 0$ for all but finitely many α. Put

$$\|f(T)\|_{1,r} = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha) \, r^\alpha, \tag{29.3}$$

where the sum on the right is a sum of nonnegative real numbers, all but finitely many of which are equal to 0. It is easy to see that this defines a norm on $V[T_1, \ldots, T_n]$, with respect to $|\cdot|$ on k.

Remember that $V[T_1, \ldots, T_n]$ can be defined as the space $c_{00}((\mathbf{Z}_+ \cup \{0\})^n, V)$ of V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ with finite support, where $f(T)$ in (29.2) corresponds to f_α as a V-valued function of α. Using this identification and w_r in (29.1), (29.3) corresponds to the $\ell^1_{w_r}((\mathbf{Z}_+ \cup \{0\})^n, V)$ norm defined as in Section 25, restricted to $c_{00}((\mathbf{Z}_+ \cup \{0\})^n, V)$.

Similarly, put

$$\|f(T)\|_{\infty,r} = \max_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} (N(f_\alpha)\, r^\alpha) \tag{29.4}$$

for each $f(T) \in V[T_1, \ldots, T_n]$, where the maximum on the right is automatically attained, because all but finitely many of the terms are equal to 0. This defines a norm on $V[T_1, \ldots, T_n]$ too, with respect to $|\cdot|$ on k. As in the previous paragraph, (29.4) corresponds exactly to the $\ell^\infty_{w_r}((\mathbf{Z}_+ \cup \{0\})^n, V)$ norm defined as in Section 25, restricted to $c_{00}((\mathbf{Z}_+ \cup \{0\})^n, V)$. If N is an ultranorm on V, then (29.4) is an ultranorm on $V[T_1, \ldots, T_n]$. Note that

$$\|f(T)\|_{\infty,r} \le \|f(T)\|_{1,r} \tag{29.5}$$

for every $f(T) \in V[T_1, \ldots, T_n]$.

If $f(T) \in V[T_1, \ldots, T_n]$ and $t \in k^n$, then

$$f(t) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha\, t^\alpha \tag{29.6}$$

defines an element of V, as in Section 2, because all but finitely many terms on the right side are equal to 0. Of course,

$$N(f(t)) \le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha)\, |t^\alpha|, \tag{29.7}$$

and

$$|t^\alpha| = |t_1^{\alpha_1} \cdots t_n^{\alpha_n}| = |t_1|^{\alpha_1} \cdots |t_n|^{\alpha_n} \tag{29.8}$$

for every multi-index α. If $|t_j| \le r_j$ for each $j = 1, \ldots, n$, then

$$|t^\alpha| \le r^\alpha \tag{29.9}$$

for every multi-index α, so that

$$N(f(t)) \le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha)\, r^\alpha = \|f(T)\|_{1,r}. \tag{29.10}$$

If N is an ultranorm on V, then

$$N(f(t)) \le \max_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} (N(f_\alpha)\, |t^\alpha|) \tag{29.11}$$

for every $t \in k^n$. In this case, if we also have that $|t_j| \le r_j$ for each $j = 1, \ldots, n$, then we get that

$$N(f(t)) \le \max_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} (N(f_\alpha)\, r^\alpha) = \|f(T)\|_{\infty,r}. \tag{29.12}$$

30 Summable power series

Let k be a field, let V be a vector space over k, and let $T_1, \ldots, T_n$ be n commuting indeterminates for some positive integer n. Remember that the space $V[[T_1, \ldots, T_n]]$ of formal power series in $T_1, \ldots, T_n$ with coefficients in V is a vector space over k with respect to pointwise addition and scalar multiplication, as in Section 4. Let $|\cdot|$ be an absolute value function on k, let N be a norm on V with respect to $|\cdot|$ on k, and let $r = (r_1, \ldots, r_n)$ be an n-tuple of positive real numbers. If

$$f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, T^\alpha \tag{30.1}$$

is an element of $V[[T_1, \ldots, T_n]]$, then

$$\|f(T)\|_{1,r} = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha) \, r^\alpha \tag{30.2}$$

can be defined as a nonnegative extended real number, as in Section 20. Put

$$V_r^1[[T_1, \ldots, T_n]] = \{f(T) \in V[[T_1, \ldots, T_n]] : \|f(T)\|_{1,r} < \infty\}. \tag{30.3}$$

More precisely, remember that $V[[T_1, \ldots, T_n]]$ can be defined as the space $c((\mathbf{Z}_+ \cup \{0\})^n, V)$ of V-valued functions on the set $(\mathbf{Z}_+ \cup \{0\})^n$ of multi-indices, as in Section 4. Let w_r be the positive real-valued function defined on $(\mathbf{Z}_+ \cup \{0\})^n$ in (29.1). Thus (30.3) corresponds exactly to the space $\ell^1_{w_r}((\mathbf{Z}_+ \cup \{0\})^n, V)$ of V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ that are summable with respect to N on V and w_r on $(\mathbf{Z}_+ \cup \{0\})^n$, as in Section 25. Similarly, (30.2) corresponds to the $\ell^1_{w_r}((\mathbf{Z}_+ \cup \{0\})^n, V)$ norm defined earlier. In particular, (30.3) is a linear subspace of $V[[T_1, \ldots, T_n]]$, and (30.2) defines a norm on (30.3) with respect to $|\cdot|$ on k.

The space $V[T_1, \ldots, T_n]$ of formal polynomials in $T_1, \ldots, T_n$ with coefficients in V corresponds to the space $c_{00}(\mathbf{Z}_+ \cup \{0\})^n, V)$ of V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$, as before. This is a linear subspace of (30.3), which is dense in (30.3) with respect to the metric associated to the norm (30.2). If V is complete with respect to the metric associated to N, then (30.3) is complete with respect to the metric associated to (30.2), because of the analogous statement for $\ell^1_{w_r}((\mathbf{Z}_+ \cup \{0\})^n, V)$.

Let $t \in k^n$ be given, with

$$|t_j| \le r_j \tag{30.4}$$

for each $j = 1, \ldots, n$, so that (29.9) holds for every multi-index α. Also let $f(T) \in V_r^1[[T_1, \ldots, T_n]]$ be given, and observe that

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha) \, |t^\alpha| \le \|f(T)\|_{1,r}. \tag{30.5}$$

Thus $f_\alpha \, t^\alpha$ defines a summable V-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^n$ with respect to N. Suppose that V is complete with respect to the metric associated to N, and put

$$f(t) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, t^\alpha, \tag{30.6}$$

where the sum on the right can be defined as an element of V as in Section 22. Note that

$$N(f(t)) \le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha) \, |t^\alpha| \le \|f(T)\|_{1,r}. \tag{30.7}$$

Under these conditions,

$$f(T) \mapsto f(t) \tag{30.8}$$

defines a linear mapping from $V_r^1[[T_1, \ldots, T_n]]$ into V. More precisely,

$$f(T) \mapsto f_\alpha \, t^\alpha \tag{30.9}$$

defines a linear mapping from $V_r^1[[T_1, \ldots, T_n]]$ into the space $\ell^1((\mathbf{Z}_+ \cup \{0\})^n, V)$ of summable V-valued functions of α on $(\mathbf{Z}_+ \cup \{0\})^n$ with respect to N. The mapping (30.8) is the same as the composition of (30.9) with the mapping from $\ell^1((\mathbf{Z}_+ \cup \{0\})^n, V)$ into V defined by summing over α, as in Section 22.

31 More spaces of power series

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let $T_1, \ldots, T_n$ be n commuting indeterminates for some $n \in \mathbf{Z}_+$, and let $r = (r_1, \ldots, r_n)$ be an n-tuple of positive real numbers. If $f(T)$ is a formal power series in $T_1, \ldots, T_n$ with coefficients in V as in (30.1), then

$$\|f(T)\|_{\infty,r} = \sup_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} (N(f_\alpha) \, r^\alpha) \tag{31.1}$$

is defined as a nonnegative extended real number. Let $V_r^\infty[[T_1, \ldots, T_n]]$ be the space of $f(T) \in V[[T_1, \ldots, T_n]]$ such that

$$N(f_\alpha) \, r^\alpha \tag{31.2}$$

is bounded as a nonnegative real-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^n$, so that (31.1) is finite. Similarly, let $V_{0,r}[[T_1, \ldots, T_n]]$ be the space of formal power series $f(T)$ in $T_1, \ldots, T_n$ with coefficients in V such that (31.2) vanishes at infinity as a nonnegative real-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^n$.

As before, the space $V[[T_1, \ldots, T_n]]$ of formal power series in $T_1, \ldots, T_n$ with coefficients in V can be defined as the space $c((\mathbf{Z}_+ \cup \{0\})^n, V)$ of all V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$. Let w_r be the positive real-valued function defined on $(\mathbf{Z}_+ \cup \{0\})^n$ as in (29.1) again. The space $V_r^\infty[[T_1, \ldots, T_n]]$ defined in the preceding paragraph corresponds exactly to the space $\ell^\infty_{w_r}((\mathbf{Z}_+ \cup \{0\})^n, V)$ of V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ that are bounded with respect to N on V and w_r on $(\mathbf{Z}_+ \cup \{0\})^n$, as in Section 25. Similarly, $V_{0,r}[[T_1, \ldots, T_n]]$ corresponds exactly to the space $c_{0,w_r}((\mathbf{Z}_+ \cup \{0\})^n, V)$ of V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ that vanish at infinity with respect to N on V and w_r on $(\mathbf{Z}_+ \cup \{0\})^n$, as in Section 25. Note that (31.1) corresponds to the $\ell^\infty_{w_r}((\mathbf{Z}_+ \cup \{0\})^n, V)$ norm defined previously.

In particular, $V_r^\infty[[T_1, \ldots, T_n]]$ and $V_{0,r}[[T_1, \ldots, T_n]]$ are linear subspaces of $V[[T_1, \ldots, T_n]]$. We also have that

$$\begin{aligned} V[T_1, \ldots, T_n] \subseteq V_r^1[[T_1, \ldots, T_n]] &\subseteq V_{0,r}[[T_1, \ldots, T_n]] \\ &\subseteq V_r^\infty[[T_1, \ldots, T_n]], \end{aligned} \tag{31.3}$$

because of the analogous statements for functions that are summable, vanish at infinity, or are bounded. Of course, (31.1) defines a norm on $V_r^\infty[[T_1, \ldots, T_n]]$ with respect to $|\cdot|$ on k. If $f(T) \in V_r^1[[T_1, \ldots, T_n]]$, then

$$\|f(T)\|_{\infty,r} \le \|f(T)\|_{1,r}, \tag{31.4}$$

where $\|f(T)\|_{1,r}$ is as in (30.2). As in Section 25, $V[T_1, \ldots, T_n]$ is dense in $V_{0,r}[[T_1, \ldots, T_n]]$ with respect to the metric associated to (31.1). Similarly, $V_{0,r}[[T_1, \ldots, T_n]]$ is a closed set in $V_r^\infty[[T_1, \ldots, T_n]]$ with respect to this metric. If V is complete with respect to the metric associated to N, then $V_r^\infty[[T_1, \ldots, T_n]]$ is complete with respect to the metric associated to (31.1).

Suppose from now on in this section that N is an ultranorm on V. This implies that (31.1) is an ultranorm on $V_r^\infty[[T_1, \ldots, T_n]]$, as before. Let $t \in k$ be given, with $|t_j| \le r_j$ for each $j = 1, \ldots, n$, so that $|t^\alpha| \le r^\alpha$ for every multi-index α, as in (29.9). Also let $f(T) \in V_{0,r}[[T_1, \ldots, T_n]]$ be given, and observe that

$$N(f_\alpha\, t^\alpha) = N(f_\alpha)\, |t^\alpha| \le N(f_\alpha)\, r^\alpha \tag{31.5}$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$. This implies that $f_\alpha\, t^\alpha$ vanishes at infinity as a V-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^n$ with respect to N.

If V is complete with respect to the ultrametric associated to N, then we can put

$$f(t) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha\, t^\alpha, \tag{31.6}$$

where the sum on the right is defined as an element of V as in Section 22. In this situation, we have that

$$N(f(t)) \le \max_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha\, t^\alpha) \le \|f(T)\|_{\infty,r}, \tag{31.7}$$

because N is an ultranorm on V. As before, $f(T) \mapsto f(t)$ is a linear mapping from $V_{0,r}[[T_1, \ldots, T_n]]$ into V. Indeed, $f(T) \mapsto f_\alpha\, t^\alpha$ is a linear mapping from $V_{0,r}[[T_1, \ldots, T_n]]$ into the space $c_0((\mathbf{Z}_+ \cup \{0\})^n, V)$ of V-valued functions on $(\mathbf{Z}_+ \cup \{0\})^n$ that vanish at infinity with respect to N. By construction, the first mapping $f(T) \mapsto f(t)$ is the composition of the second mapping $f(T) \mapsto f_\alpha\, T^\alpha$ with the mapping from $c_0((\mathbf{Z}_+ \cup \{0\})^n, V)$ into V defined by summing over α, as in Section 22.

32 Comparing r's

Let n be a positive integer, and let $\rho = (\rho_1, \ldots, \rho_n)$ be an n-tuple of positive real numbers with $\rho_j < 1$ for each $j = 1, \ldots, n$. As usual, we put $\rho^\alpha = \rho_1^{\alpha_1} \cdots \rho_n^{\alpha_n}$ for each $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$. Observe that

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} \rho^\alpha = \prod_{j=1}^n \Big(\sum_{\alpha_j = 0}^\infty \rho_j^{\alpha_j} \Big) = \prod_{j=1}^n (1 - \rho_j)^{-1}. \tag{32.1}$$

More precisely, the sum over $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$ on the left can be identified with the iterated sum

$$\sum_{\alpha_1=0}^\infty \Big(\sum_{\alpha_2=0}^\infty \cdots \Big(\sum_{\alpha_n=0}^\infty \rho^\alpha \Big) \cdots \Big), \tag{32.2}$$

as in Section 23. This iterated sum reduces to the product of the geometric series in the middle of (32.1).

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let $T_1, \ldots, T_n$ be commuting indeterminates, and let $r = (r_1, \ldots, r_n)$ and $R = (R_1, \ldots, R_n)$ be n-tuples of positive real numbers. Suppose that

$$r_j \le R_j \tag{32.3}$$

for each $j = 1, \ldots, n$, so that

$$r^\alpha \le R^\alpha \tag{32.4}$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$. It is easy to see that

$$V_R^1[[T_1, \ldots, T_n]] \subseteq V_r^1[[T_1, \ldots, T_n]], \tag{32.5}$$

with

$$\|f(T)\|_{1,r} \le \|f(T)\|_{1,R} \tag{32.6}$$

for every $f(T) \in V_R^1[[T_1, \ldots, T_n]]$. Similarly,

$$V_R^\infty[[T_1, \ldots, T_n]] \subseteq V_r^\infty[[T_1, \ldots, T_n]], \tag{32.7}$$

with

$$\|f(T)\|_{\infty,r} \le \|f(T)\|_{\infty,R} \tag{32.8}$$

for every $f(T) \in V_R^\infty[[T_1, \ldots, T_n]]$. Moreover,

$$V_{0,R}[[T_1, \ldots, T_n]] \subseteq V_{0,r}[[T_1, \ldots, T_n]]. \tag{32.9}$$

Of course,

$$\begin{aligned} &\{t \in k^n : |t_j| \le r_j \text{ for each } j = 1, \ldots, n\} \\ &\quad \subseteq \quad \{t \in k^n : |t_j| \le R_j \text{ for each } j = 1, \ldots, n\}. \end{aligned} \tag{32.10}$$

Suppose now that

$$r_j < R_j \tag{32.11}$$

for each $j = 1, \ldots, n$, and put

$$\rho_j = r_j / R_j, \tag{32.12}$$

$1 \le j \le n$. Thus $\rho_j < 1$ for each $j = 1, \ldots, n$, and we put $\rho = (\rho_1, \ldots, \rho_n)$. Observe that

$$r^\alpha = \rho^\alpha \, R^\alpha \tag{32.13}$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$. If $f(T) \in V_R^\infty[[T_1, \ldots, T_n]]$, then

$$N(f_\alpha) \, r^\alpha = N(f_\alpha) \, \rho^\alpha \, R^\alpha \le \|f(T)\|_{\infty, R} \, \rho^\alpha \tag{32.14}$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, by the definition (31.1) of $\|f(T)\|_{\infty, R}$. This implies that

$$\begin{aligned} \|f(T)\|_{1,r} = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha) \, r^\alpha \quad & \le \quad \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} \|f(T)\|_{\infty, R} \, \rho^\alpha \\ & = \quad \|f(T)\|_{\infty, R} \prod_{j=1}^n (1 - \rho_j)^{-1}, \end{aligned} \tag{32.15}$$

using (32.1) in the third step. It follows that $f(T) \in V_r^1[[T_1, \ldots, T_n]]$, so that

$$V_R^\infty[[T_1, \ldots, T_n]] \subseteq V_r^1[[T_1, \ldots, T_n]]. \tag{32.16}$$

In particular,

$$V_R^\infty[[T_1, \ldots, T_n]] \subseteq V_{0,r}[[T_1, \ldots, T_n]]. \tag{32.17}$$

Note that

$$\begin{aligned} & \{t \in k^n : |t_j| \le r_j \text{ for each } j = 1, \ldots, n\} \\ & \quad \subseteq \quad \{t \in k^n : |t_j| < R_j \text{ for each } j = 1, \ldots, n\}. \end{aligned} \tag{32.18}$$

Let $t \in k^n$ be given, and suppose that

$$|t_j| < R_j \tag{32.19}$$

for each $j = 1, \ldots, n$. Also let $r_1, \ldots, r_n$ be positive real numbers such that

$$|t_j| \le r_j < R_j \tag{32.20}$$

for each j. Suppose that V is complete with respect to the metric associated to N. If $f(T) \in V_R^\infty[[T_1, \ldots, T_n]]$, then $f(T) \in V_r^1[[T_1, \ldots, T_n]]$, as in the preceding paragraph. Under these conditions, $f(t)$ can be defined as an element of V, as in Section 30.

33 Uniform convergence

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let $T_1, \ldots, T_n$ be n commuting indeterminates for some $n \in \mathbf{Z}_+$, and let $r = (r_1, \ldots, r_n)$ be an n-tuple of positive real numbers. Suppose that $f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, T^\alpha$ is an element of the space $V_r^1[[T_1, \ldots, T_n]]$ defined in Section 30, and that V is complete with respect to the metric associated to N. Put

$$\overline{D}(r) = \{t \in k^n : |t_j| \le r_j \text{ for each } j = 1, \ldots, n\}, \tag{33.1}$$

which is the closed polydisk in k^n centered at 0 associated to r. If $t \in \overline{D}(r)$, then $f(t)$ can be defined as an element of V as in Sections 22 and 30.

Let A be a finite subset of $(\mathbf{Z}_+ \cup \{0\})^n$, so that

$$f^A(T) = \sum_{\alpha \in A} f_\alpha \, T^\alpha \tag{33.2}$$

is a formal polynomial in $T_1, \ldots, T_n$ with coefficients in V. More precisely, the right side of (33.2) is interpreted as being equal to 0 when $A = \emptyset$. Of course,

$$f(T) - f^A(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} f_\alpha \, T^\alpha. \tag{33.3}$$

If $t \in \overline{D}(r)$, then

$$f(t) - f^A(t) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} f_\alpha \, t^\alpha, \tag{33.4}$$

where the sum on the right is defined as an element of V as in Section 22. Thus

$$\begin{aligned} N(f(t) - f^A(t)) &\le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} N(f_\alpha) \, |t^\alpha| \\ &\le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} N(f_\alpha) \, r^\alpha = \|f(T) - f^A(T)\|_{1,r}, \end{aligned} \tag{33.5}$$

as in (30.7).

Let $\epsilon > 0$ be given, and remember that $N(f_\alpha) \, r^\alpha$ is summable as a nonnegative real-valued function of $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, by hypothesis. This implies that there is a finite subset $A(\epsilon)$ of $(\mathbf{Z}_+ \cup \{0\})^n$ such that

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha) \, r^\alpha < \sum_{\alpha \in A(\epsilon)} N(f_\alpha) \, r^\alpha + \epsilon. \tag{33.6}$$

It follows that

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A(\epsilon)} N(f_\alpha) \, r^\alpha = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha) \, r^\alpha - \sum_{\alpha \in A(\epsilon)} N(f_\alpha) \, r^\alpha < \epsilon. \tag{33.7}$$

If $A \subseteq (\mathbf{Z}_+ \cup \{0\})^n$ is a finite set, $A(\epsilon) \subseteq A$, and $t \in \overline{D}(r)$, then

$$(33.8) \qquad \begin{aligned} N(f(t) - f^A(t)) &\le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} N(f_\alpha)\, r^\alpha \\ &\le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A(\epsilon)} N(f_\alpha)\, r^\alpha < \epsilon, \end{aligned}$$

by (33.5) and (33.7). Thus $f(t)$ can be approximated uniformly by the $f^A(t)$'s on $\overline{D}(r)$ under these conditions.

Suppose now that N is an ultranorm on V, and that $f(T)$ is an element of the space $V_{0,r}[[T_1, \ldots, T_n]]$ defined in Section 31. We continue to ask that V be complete with respect to the metric associated to N, so that $f(t)$ can be defined as an element of V when $t \in \overline{D}(r)$, as in Sections 22 and 31. If A is a finite subset of $(\mathbf{Z}_+ \cup \{0\})^n$ again and $t \in \overline{D}(r)$, then

$$\begin{aligned} N(f(t) - f^A(t)) &\le \max_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} (N(f_\alpha)\, |t^\alpha|) \\ (33.9) \qquad &\le \max_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} (N(f_\alpha)\, r^\alpha) = \|f(T) - f^A(T)\|_{\infty, r}, \end{aligned}$$

as in (31.7). Remember that $N(f_\alpha)\, r^\alpha$ vanishes at infinity as a function of $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, by hypothesis. Let $\epsilon > 0$ be given, and let $A(\epsilon)$ be a finite subset of $(\mathbf{Z}_+ \cup \{0\})^n$ such that

$$(33.10) \qquad N(f_\alpha)\, r^\alpha < \epsilon$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A(\epsilon)$. If $A(\epsilon) \subseteq A$ and $t \in \overline{D}(r)$, then

$$(33.11) \qquad \begin{aligned} N(f(t) - f^A(t)) &\le \max_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A} (N(f_\alpha)\, r^\alpha) \\ &\le \max_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n \setminus A(\epsilon)} (N(f_\alpha)\, r^\alpha) < \epsilon, \end{aligned}$$

by (33.9) and (33.10). This shows that $f(t)$ can be approximated uniformly by the $f^A(t)$'s on $\overline{D}(r)$ in this situation as well.

34 Continuity conditions

Let k be a field with an absolute value function $|\cdot|$, and let n be a positive integer. Of course, k^n may be considered as a vector space over k, with respect to coordinatewise addition and scalar multiplication. Put

$$(34.1) \qquad \|t\|_\infty = \max_{1 \le j \le n} |t_j|$$

for each $t \in k$, which defines a norm on k^n with respect to $|\cdot|$ on k. If $|\cdot|$ is an ultrametric absolute value function on k, then (34.1) is an ultranorm on k^n. The topology determined on k^n by the metric associated to (34.1) is the same as the product topology corresponding to the topology determined on k by the metric associated to $|\cdot|$.

If α is any multi-index, then t^α defines a continuous k-valued function on k^n, by standard arguments. The restriction of t^α to any bounded subset of k^n with respect to (34.1) is bounded and uniformly continuous on that set, with respect to the restriction of the metric associated to (34.1) to the bounded set in k^n, and the metric associated to $|\cdot|$ on k. Let V be a vector space over k, and let N be a norm on V with respect to $|\cdot|$ on k. If $f(T)$ is a formal polynomial in $T_1, \ldots, T_n$ with coefficients in V and $t \in k^n$, then $f(t)$ can be defined as an element of V, as in Section 2. This defines a mapping from k^n into V associated to $f(T)$. It is easy to see that this mapping is continuous, with respect to the topology determined on V by N. The restriction of this mapping to any bounded subset of k^n is bounded and uniformly continuous on that set, with respect to the restriction of the metric associated to (34.1) to the bounded set in k^n, and the metric associated to N on V.

Let $r = (r_1, \ldots, r_n)$ be an n-tuple of positive real numbers, and let $f(T)$ be an element of the space $V_r^1[[T_1, \ldots, T_n]]$ defined in Section 30. Suppose that V is complete with respect to the metric associated to N, and let $\overline{D}(r)$ be the closed polydisk in k^n associated to r as in (33.1). Thus $f(t)$ can be defined as an element of V for each $t \in \overline{D}(r)$, as before. This defines a mapping from $\overline{D}(r)$ into V associated to $f(T)$. Note that $\overline{D}(r)$ is bounded in k^n with respect to (34.1). The mapping on $\overline{D}(r)$ associated to $f(T)$ is bounded with respect to N on V, as in Section 30. This mapping can be approximated uniformly on $\overline{D}(r)$ by polynomial mappings, as in the previous section. The restrictions of these polynomial mappings to $\overline{D}(r)$ are uniformly continuous on $\overline{D}(r)$, with respect to the restriction of the metric associated to (34.1) to $\overline{D}(r)$ and the metric associated to N on V, as in the preceding paragraph. It follows that the mapping from $\overline{D}(r)$ into V associated to $f(T)$ is uniformly continuous, with respect to the restriction of the metric associated to (34.1) to $\overline{D}(r)$ and the metric associated to N on V.

Suppose now that N is an ultranorm on V, and let $f(T)$ be an element of the space $V_{0,r}[[T_1, \ldots, T_n]]$ defined in Section 31. As before, $f(t)$ can be defined as an element of V for each $t \in \overline{D}(r)$, because V is complete. This defines a mapping from $\overline{D}(r)$ into V associated to $f(T)$, and this mapping is bounded with respect to N on V, as in Section 31. This mapping can be approximated uniformly on $\overline{D}(r)$ by polynomial mappings, as in the previous section again. Hence this mapping is uniformly continuous with respect to the restriction of the metric associated to (34.1) to $\overline{D}(r)$ and the metric associated to N on V, because of the corresponding uniform continuity properties of the restrictions of polynomial mappings to $\overline{D}(r)$, as before.

35 Open polydisks

Let k be a field with an absolute value function $|\cdot|$, and let n be a positive integer. Also let $R = (R_1, \ldots, R_n)$ be an n-tuple of positive extended real numbers, so that $0 < R_j \le \infty$ for each $j = 1, \ldots, n$. Put

$$D(R) = \{t \in k^n : |t_j| < R_j \text{ for each } j = 1, \ldots, n\}, \tag{35.1}$$

which is the open polydisk in k^n centered at 0 associated to R. Note that (35.1) is an open set in k^n, with respect to the product topology corresponding to the topology determined on k by the metric associated to $|\cdot|$. Similarly, closed polydisks in k^n are closed sets with respect to this product topology.

Let us use $\mathbf{R}_+$ to denote the set of positive real numbers, so that $\mathbf{R}_+^n$ is the set of n-tuples of positive real numbers. Let V be a vector space over k with a norm N with respect to $|\cdot|$ on k, and suppose that V is complete with respect to the metric associated to N. Also let $T_1, \ldots, T_n$ be commuting indeterminates, and let $f(T)$ be a formal power series in $T_1, \ldots, T_n$ with coefficients in V. Suppose that for every $r \in \mathbf{R}_+^n$ with

$$r_j < R_j \tag{35.2}$$

for each $j = 1, \ldots, n$, we have that

$$f(T) \in V_r^1[[T_1, \ldots, T_n]], \tag{35.3}$$

where $V_r^1[[T_1, \ldots, T_n]]$ is as in Section 30. Let $t \in D(R)$ be given, and suppose that $r \in \mathbf{R}_+^n$ satisfies

$$|t_j| \le r_j < R_j \tag{35.4}$$

for each $j = 1, \ldots, n$. Under these conditions, $f(t)$ can be defined as an element of V, as in Section 30. More precisely, (35.3) implies that the sum used to define $f(t)$ is the sum of a summable V-valued function on $(\mathbf{Z}_+ \cup \{0\})^n$, because of (35.4). This sum does not depend on r, but different r's may be used to get the summability condition being used.

This defines a mapping from $D(R)$ into V associated to $f(T)$. If $r \in \mathbf{R}_+$ satisfies (35.2), then the restriction of this mapping to $\overline{D}(r)$ is bounded and uniformly continuous, as in the previous section. Let us check that this mapping is continuous on $D(R)$. Let $t_0 = (t_{0,1}, \ldots, t_{0,n}) \in D(R)$ be given, and let us verify that this mapping is continuous at t_0. Because $t_0 \in D(R)$, there is an $r \in \mathbf{R}_+^n$ such that

$$|t_{0,j}| < r_j < R_j \tag{35.5}$$

for each $j = 1, \ldots, n$. As before, the restriction of the mapping to $D(r) \subseteq \overline{D}(r)$ is continuous, and in particular it is continuous at t_0. The continuity of the restriction of this mapping to $D(r)$ at t_0 implies that the mapping on $D(R)$ is continuous at t_0, because $t_0 \in D(r)$ and $D(r)$ is an open set in k^n.

If $|\cdot|$ is an ultrametric absolute value function on k, then closed balls in k of positive radius are open sets, as in Section 13. If $r \in \mathbf{R}_+^n$, then it follows that the closed polydisk $\overline{D}(r)$ is an open set in k^n.

36 Submultiplicativity conditions

Let k be a field with an absolute value function $|\cdot|$, and let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k. Also let

$T_1, \ldots, T_n$ be commuting indeterminates for some $n \in \mathbf{Z}_+$, and let $f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, T^\alpha$ and $g(T) = \sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} g_\beta \, T^\beta$ be formal power series in $T_1, \ldots, T_n$ with coefficients in $\mathcal{A}$. If $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$, then put

$$E_\gamma = \{(\alpha, \beta) \in (\mathbf{Z}_+ \cup \{0\})^n \times (\mathbf{Z}_+ \cup \{0\})^n : \alpha + \beta = \gamma\}, \tag{36.1}$$

as in Section 24. Remember that the product of $f(T)$ and $g(T)$ is defined by $f(T)\, g(T) = h(T) = \sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} h_\gamma \, T^\gamma$, where

$$h_\gamma = \sum_{(\alpha,\beta) \in E_\gamma} f_\alpha \, g_\beta \tag{36.2}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$, as in Section 4. Thus

$$N(h_\gamma) \le \sum_{(\alpha,\beta) \in E_\gamma} N(f_\alpha \, g_\beta) \le \sum_{(\alpha,\beta) \in E_\gamma} N(f_\alpha) \, N(g_\beta) \tag{36.3}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$, using the triangle inequality in the first step, and the submultiplicativity of N in the second step.

Let $r \in \mathbf{R}_+^n$ be given, and observe that

$$N(h_\gamma) \, r^\gamma \le \sum_{(\alpha,\beta) \in E_\gamma} N(f_\alpha) \, N(g_\beta) \, r^\gamma = \sum_{(\alpha,\beta) \in E_\gamma} (N(f_\alpha) \, r^\alpha) \, (N(g_\beta) \, r^\beta) \tag{36.4}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$. Suppose that $f(T)$ and $g(T)$ are elements of the space $\mathcal{A}^1_r[[T_1, \ldots, T_n]]$ defined in Section 30. Using (36.4), we get that

$$\begin{aligned} \|h(T)\|_{1,r} &= \sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} N(h_\gamma) \, r^\gamma \\ &\le \sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} \Big(\sum_{(\alpha,\beta) \in E_\gamma} (N(f_\alpha) \, r^\alpha) \, (N(g_\beta) \, r^\beta) \Big), \end{aligned} \tag{36.5}$$

where $\|h(T)\|_{1,r}$ is as defined in Section 30. The right side of (36.5) is equal to

$$\Big(\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N(f_\alpha) \, r^\alpha \Big) \Big(\sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} N(g_\beta) \, r^\beta \Big) \tag{36.6}$$

as in Section 24. This implies that $h(T) \in \mathcal{A}^1_r[[T_1, \ldots, T_n]]$, with

$$\|h(T)\|_{1,r} \le \|f(T)\|_{1,r} \, \|g(T)\|_{1,r}. \tag{36.7}$$

Suppose that $\mathcal{A}$ is complete with respect to the metric associated to N, and let t be an element of the closed polydisk $\overline{D}(r)$ in k^n associated to r. In this case, $f(t)$, $g(t)$, and $h(t)$ can be defined as elements of $\mathcal{A}$, as in Section 30. Note that

$$h_\gamma \, t^\gamma = \sum_{(\alpha,\beta) \in E_\gamma} f_\alpha \, g_\beta \, t^\gamma = \sum_{(\alpha,\beta) \in E_\gamma} (f_\alpha \, t^\alpha) \, (g_\beta \, t^\beta) \tag{36.8}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$, using (36.2) in the first step. Under these conditions, we have that

$$h(t) = f(t)\, g(t), \tag{36.9}$$

as in Section 24.

Suppose now that N is an ultranorm on $\mathcal{A}$. Observe that

$$N(h_\gamma) \le \max_{(\alpha,\beta)\in E_\gamma} N(f_\alpha\, g_\beta) \le \max_{(\alpha,\beta)\in E_\gamma} (N(f_\alpha)\, N(g_\beta)) \tag{36.10}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$, using (36.2) in the first step, and the submultiplicativity of N in the second step. Let $r \in \mathbf{R}_+^n$ be given again, and let us use (36.10) to get that

$$\begin{aligned} N(h_\gamma)\, r^\gamma &\le \max_{(\alpha,\beta)\in E_\gamma} (N(f_\alpha)\, N(g_\beta))\, r^\gamma \\ &= \max_{(\alpha,\beta)\in E_\gamma} ((N(f_\alpha)\, r^\alpha)\, (N(g_\beta)\, r^\beta)) \end{aligned} \tag{36.11}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$. Suppose that $f(T)$ and $g(T)$ are elements of the space $\mathcal{A}_r^\infty[[T_1, \ldots, T_n]]$ defined in Section 31, so that

$$N(h_\gamma)\, r^\gamma \le \|f(T)\|_{r,\infty}\, \|g\|_{r,\infty} \tag{36.12}$$

for every $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$. This means that $h(T) \in \mathcal{A}_r^\infty[[T_1, \ldots, T_n]]$ too, with

$$\|h(T)\|_{r,\infty} \le \|f(T)\|_{r,\infty}\, \|g(T)\|_{r,\infty}. \tag{36.13}$$

If $f(T)$ and $g(T)$ are elements of the space $\mathcal{A}_{0,r}[[T_1, \ldots, T_n]]$ defined in Section 31, then one can check that $h(T)$ is an element of $\mathcal{A}_{0,r}[[T_1, \ldots, T_n]]$ as well, using (36.11). Suppose that $\mathcal{A}$ is complete with respect to the metric associated to N again, and that $t \in \overline{D}(r)$. Under these conditions, $f(t)$, $g(t)$, and $h(t)$ can be defined as elements of $\mathcal{A}$, as in Section 31. One can verify that (36.9) also holds in this situation. This uses (36.8) and the remarks in Section 24, as before.

37 More sums in algebras

Let k be a field with an absolute value function $|\cdot|$, and let $\mathcal{A}_0$ be an algebra over k with a submultiplicative norm N_0 with respect to $|\cdot|$ on k. Suppose that $\mathcal{A}_0$ has a multiplicative identity element e_0, with $N_0(e_0) = 1$. Let $n \in \mathbf{Z}_+$ and $x \in \mathcal{A}_0^n$ be given, with

$$x_j\, x_l = x_l\, x_j \tag{37.1}$$

for each $j, l = 1, \ldots, n$. Put

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \tag{37.2}$$

for each $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, as in Section 3, where $x_j^{\alpha_j}$ is interpreted as being equal to e_0 when $\alpha_j = 0$. Thus

$$N_0(x^\alpha) \le N_0(x_1^{\alpha_1}) \cdots N_0(x_n^{\alpha_n}) \le N_0(x_1)^{\alpha_1} \cdots N_0(x_n)^{\alpha_n} \tag{37.3}$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, where $N_0(x_j)^{\alpha_j}$ is interpreted as being equal to 1 when $\alpha_j = 0$, as usual. Note that

$$x^{\alpha+\beta} = x^\alpha \, x^\beta \tag{37.4}$$

for every $\alpha, \beta \in (\mathbf{Z}_+ \cup \{0\})^n$, by (37.1).

Let $T_1, \ldots, T_n$ be commuting indeterminates, and let $r \in \mathbf{R}_+^n$ be given. Also let $f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, T^\alpha$ be a formal power series in $T_1, \ldots, T_n$ with coefficients in k such that

$$\|f(T)\|_{1,r} = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} |f_\alpha| \, r^\alpha \tag{37.5}$$

is finite. This is the same type of condition as in Section 30, where k is considered as a one-dimensional vector space over itself, and $|\cdot|$ is considered as a norm on k. Suppose that

$$N_0(x_j) \le r_j \tag{37.6}$$

for each $j = 1, \ldots, n$, so that

$$N_0(x^\alpha) \le r^\alpha \tag{37.7}$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, by (37.3). This implies that

$$\sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} |f_\alpha| \, N_0(x^\alpha) \le \|f(T)\|_{1,r}. \tag{37.8}$$

In particular, this means that $f_\alpha \, x^\alpha$ is a summable $\mathcal{A}_0$-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^n$ with respect to N_0. Suppose that $\mathcal{A}_0$ is complete with respect to the metric associated to N_0. Put

$$f(x) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha \, x^\alpha, \tag{37.9}$$

where the sum on the right is defined as an element of $\mathcal{A}_0$ as in Section 22. Of course,

$$N_0(f(x)) \le \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} |f_\alpha| \, N_0(x^\alpha) \le \|f(T)\|_{1,r}. \tag{37.10}$$

Let $g(T) = \sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} g_\beta \, T^\beta$ be another formal power series in $T_1, \ldots, T_n$ with coefficients in k such that $\|g(T)\|_{1,r}$ is finite. As before, the coefficients of the product $h(T) = \sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} h_\gamma \, T^\gamma$ of $f(T)$ and $g(T)$ are given as in (36.2), and $\|h(T)\|_{1,r}$ is less than or equal to the product of $\|f(T)\|_{1,r}$ and $\|g(T)\|_{1,r}$, as in (36.7). Thus $g(x)$ and $h(x)$ can be defined as elements of $\mathcal{A}_0$ too, as in Section 22. If $\gamma \in (\mathbf{Z}_+ \cup \{0\})^n$ and E_γ is as in (36.1), then

$$h_\gamma \, x^\gamma = \sum_{(\alpha,\beta) \in E_\gamma} f_\alpha \, g_\beta \, x^\gamma = \sum_{(\alpha,\beta) \in E_\gamma} (f_\alpha \, x^\alpha) \, (g_\beta \, x^\beta), \tag{37.11}$$

using (36.2) in the first step, and (37.4) in the second step. It follows that

$$h(x) = f(x)\, g(x), \tag{37.12}$$

as in Section 24.

Suppose now that N_0 is an ultranorm on $\mathcal{A}_0$. In this case, we let $f(T) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} f_\alpha\, T^\alpha$ be a formal power series in $T_1, \ldots, T_n$ with coefficients in k such that

$$|f_\alpha|\, r^\alpha \tag{37.13}$$

vanishes at infinity as a nonnegative real-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^n$. This type of condition was discussed in Section 31, where k is considered as a one-dimensional vector space over itself again, and $|\cdot|$ is considered as a norm on k. Put

$$\|f(T)\|_{\infty,r} = \sup_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} (|f_\alpha|\, r^\alpha), \tag{37.14}$$

as before. Suppose that (37.6) holds for each $j = 1, \ldots, n$ again, so that (37.7) holds for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$. Thus

$$N_0(f_\alpha\, x^\alpha) = |f_\alpha|\, N_0(x^\alpha) \le |f_\alpha|\, r^\alpha \tag{37.15}$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, which implies that

$$f_\alpha\, x^\alpha \tag{37.16}$$

vanishes at infinity as an $\mathcal{A}_0$-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^n$ with respect to N_0. As before, we ask that $\mathcal{A}_0$ be complete with respect to the metric associated to N_0. Under these conditions, $f(x)$ can be defined as an element of $\mathcal{A}$ as in (37.9), where the sum on the right side of (37.9) is defined as in Section 22. We also have that

$$N_0(f(x)) \le \max_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^n} N_0(f_\alpha\, x^\alpha) \le \|f(T)\|_{\infty,r} \tag{37.17}$$

in this situation.

Let $g(T) = \sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} g_\beta\, T^\beta$ be another formal power series in $T_1, \ldots, T_n$ with coefficients in k such that

$$|g_\beta|\, r^\beta \tag{37.18}$$

vanishes at infinity as a nonnegative real-valued function of β on $(\mathbf{Z}_+ \cup \{0\})^n$. Remember that the coefficients of the product $h(T) = \sum_{\gamma \in (\mathbf{Z}_+ \cup \{0\})^n} h_\gamma\, T^\gamma$ of $f(T)$ and $g(T)$ are given as in (36.2). As in the previous section,

$$|h_\gamma|\, r^\gamma \tag{37.19}$$

vanishes at infinity as a nonnegative real-valued function of γ on $(\mathbf{Z}_+ \cup \{0\})^n$. It follows that $g(x)$ and $h(x)$ can be defined as elements of $\mathcal{A}_0$ as well, as in Section 22. Under these conditions, we have that (37.12) holds, as in Section 24, and using (37.11), as before.

38 Some more continuity conditions

Let k be a field with an absolute value function $|\cdot|$ again, and let $\mathcal{A}_0$ be an algebra over k with a submultiplicative norm N_0. As before, we ask that $\mathcal{A}_0$ have a multiplicative identity element e, with $N_0(e) = 1$, and that $\mathcal{A}_0$ be complete with respect to the metric associated to N_0. Let r be a positive real number, and let

$$f(T) = \sum_{j=0}^{\infty} f_j\, T^j \tag{38.1}$$

be a formal power series in an indeterminate T with coefficients in k such that

$$\|f(T)\|_{1,r} = \sum_{j=0}^{\infty} |f_j|\, r^j \tag{38.2}$$

is finite. If $x \in \mathcal{A}_0$ satisfies $N_0(x) \le r$, then

$$N_0(f_j\, x^j) = |f_j|\, N_0(x^j) \le |f_j|\, N_0(x)^j \le |f_j|\, r^j \tag{38.3}$$

for every nonnegative integer j. This implies that

$$f(x) = \sum_{j=0}^{\infty} f_j\, x^j \tag{38.4}$$

is defined as an element of $\mathcal{A}_0$, because the series on the right converges absolutely with respect to N_0. We also have that

$$N_0(f(x)) \le \sum_{j=0}^{\infty} N_0(f_j\, x^j) \le \sum_{j=0}^{\infty} |f_j|\, N_0(x)^j \le \|f(T)\|_{1,r} \tag{38.5}$$

in this case. Of course, this corresponds to taking $n = 1$ in the previous section.

Similarly, if l is a nonnegative integer, then

$$\begin{aligned} N_0\Big(f(x) - \sum_{j=0}^{l} f_j\, x^j\Big) = N_0\Big(\sum_{j=l+1}^{\infty} f_j\, x^j\Big) &\le \sum_{j=l+1}^{\infty} |f_j|\, N_0(x)^j \\ &\le \sum_{j=l+1}^{\infty} |f_j|\, r^j \end{aligned} \tag{38.6}$$

when $N_0(x) \le r$. Let

$$\overline{B}_{\mathcal{A}_0}(0, r) = \{x \in \mathcal{A}_0 : N_0(x) \le r\} \tag{38.7}$$

be the closed ball in $\mathcal{A}_0$ centered at 0 with radius r with respect to N_0. It follows from (38.6) that the partial sums

$$\sum_{j=0}^{l} f_j\, x^j \tag{38.8}$$

converge to $f(x)$ uniformly on (38.7) as $l \to \infty$, with respect to the metric on $\mathcal{A}_0$ associated to N_0. One can check that (38.8) is uniformly continuous on (38.7) for each $l \geq 0$, with respect to the metric on $\mathcal{A}_0$ associated to N_0, and its restriction to (38.7). This implies that $f(x)$ is uniformly continuous as a mapping from (38.7) into $\mathcal{A}_0$ too, by standard arguments.

Let R be a positive real number, and let

$$B_{\mathcal{A}_0}(0, R) = \{x \in \mathcal{A}_0 : N_0(x) < R\} \tag{38.9}$$

be the open ball in $\mathcal{A}_0$ centered at 0 with radius R with respect to N_0. Suppose that (38.2) is finite for every positive real number r with $r < R$, so that $f(x)$ is defined as an element of $\mathcal{A}_0$ for every x in (38.9). Under these conditions, $f(x)$ is continuous as a mapping from (38.9) into $\mathcal{A}_0$, with respect to the metric on $\mathcal{A}_0$ associated to N_0, and its restriction to (38.9). This can be obtained from the continuity of $f(x)$ on the smaller closed balls, as before. If (38.2) is finite for every $r > 0$, then $f(x)$ is defined an continuous on all of $\mathcal{A}_0$.

Suppose now that N_0 is an ultranorm on $\mathcal{A}_0$. Let $r > 0$ be given again, and suppose that

$$\lim_{j \to \infty} |f_j| \, r^j = 0. \tag{38.10}$$

Put

$$\|f(T)\|_{\infty, r} = \sup_{j \geq 0} (|f_j| \, r^j), \tag{38.11}$$

as usual. If $x \in \mathcal{A}_0$ and $N_0(x) \leq r$, then

$$\lim_{j \to \infty} N_0(f_j \, x^j) = 0, \tag{38.12}$$

by (38.3). This implies that the series on the right side of (38.4) converges in $\mathcal{A}_0$, as in Section 16. Note that

$$N_0(f(x)) \leq \max_{j \geq 0} N_0(f_j \, x^j) \leq \max_{j \geq 0} (|f_j| \, N_0(x)^j) \leq \|f(T)\|_{\infty, r} \tag{38.13}$$

in this situation.

If l is a nonnegative integer, then we get that

$$\begin{aligned} N_0\Big(f(x) - \sum_{j=0}^{l} f_j \, x^j\Big) = N_0\Big(\sum_{j=l+1}^{\infty} f_j \, x^j\Big) \quad &\leq \quad \max_{j \geq l+1} (|f_j| \, N(x)^j) \\ &\leq \quad \max_{j \geq l+1} (|f_j| \, r^j) \end{aligned} \tag{38.14}$$

when $N_0(x) \leq r$. This implies that the partial sums (38.8) converge uniformly to $f(x)$ on (38.7) as $l \to \infty$, because of (38.10). It follows that $f(x)$ is uniformly continuous as a mapping from (38.7) into $\mathcal{A}_0$, with respect to the metric on $\mathcal{A}_0$ associated to N_0 and its restriction to (38.7), as before.

39 Combining indeterminates again

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let m and n be positive integers, and let $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be $m+n$ commuting indeterminates. As before, we can identify $(\mathbf{Z}_+ \cup \{0\})^{m+n}$ with the Cartesian product of $(\mathbf{Z}_+ \cup \{0\})^m$ and $(\mathbf{Z}_+ \cup \{0\})^n$ in the obvious way. A formal power series in $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$ with coefficients in V can be expressed as

$$f(X,Y) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^m} \sum_{\beta \in (\mathbf{Z}_+ \cup \{0\})^n} f_{\alpha,\beta} \, X^\alpha \, Y^\beta, \tag{39.1}$$

where $f_{\alpha,\beta} \in V$ for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^m$ and $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$. More precisely, $f_{\alpha,\beta}$ corresponds to a V-valued function on $(\mathbf{Z}_+ \cup \{0\})^{m+n}$, which is being expressed as a function of α and β. If $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$, then

$$f_\beta(X) = \sum_{\alpha \in (\mathbf{Z}_+ \cup \{0\})^m} f_{\alpha,\beta} \, X^\alpha \tag{39.2}$$

defines a formal power series in $X_1, \ldots, X_m$ with coefficients in V, which is to say an element of $V[[X_1, \ldots, X_m]]$. Thus

$$\sum_{\beta \in (\mathbf{Z}_+ \cup \{0\}) n} f_\beta(X) \, Y^\beta \tag{39.3}$$

defines a formal power series in $Y_1, \ldots, Y_n$ with coefficients in $V[[X_1, \ldots, X_m]]$, which is to say an element of

$$(V[[X_1, \ldots, X_m]])[[Y_1, \ldots, Y_n]]. \tag{39.4}$$

This defines a one-to-one correspondence between (39.4) and the space

$$V[[X_1, \ldots, X_m, Y_1, \ldots, Y_n]] \tag{39.5}$$

of formal power series in $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$ with coefficients in V, as in Section 5.

Let $r = (r_1, \ldots, r_{m+n}) \in \mathbf{R}_+^{m+n}$ be an $(m+n)$-tuple of positive real numbers, and let $r_X = (r_{X,1}, \ldots, r_{X,m}) \in \mathbf{R}_+^m$ and $r_Y = (r_{Y,1}, \ldots, r_{Y,n}) \in \mathbf{R}_+^n$ be defined by

$$r_{X,j} = r_j \text{ for } j = 1, \ldots, m, \quad r_{Y,l} = r_{m+l} \text{ for } l = 1, \ldots, n. \tag{39.6}$$

If we identify $\mathbf{R}_+^{m+n}$ with the Cartesian product of $\mathbf{R}_+^m$ and $\mathbf{R}_+^n$ in the usual way, then

$$r \text{ corresponds to } (r_X, r_Y). \tag{39.7}$$

Put

$$w_{r_X}^X(\alpha) = r_X^\alpha = r_{X,1}^{\alpha_1} \cdots r_{X,m}^{\alpha_m} = r_1^{\alpha_1} \cdots r_m^{\alpha_m} \tag{39.8}$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, and

$$w_{r_Y}^Y(\beta) = r_Y^\beta = r_{Y,1}^{\beta_1} \cdots r_{Y,n}^{\beta_n} = r_{m+1}^{\beta_1} \cdots r_{m+n}^{\beta_n} \tag{39.9}$$

for every $\beta \in (\mathbf{Z}_+ \cup \{0\})^n$. Thus

$$w_r(\alpha, \beta) = w^X_{r_X}(\alpha)\, w^Y_{r_Y}(\beta) = r_1^{\alpha_1} \cdots r_m^{\alpha_m}\, r_{m+1}^{\beta_1} \cdots r_{m+n}^{\beta_n} \tag{39.10}$$

is the analogous function associated to r on $(\mathbf{Z}_+ \cup \{0\})^{m+n}$. These functions correspond to r_X, r_Y, and r as in (29.1).

We can define $V^1_{r_X}[[X_1, \ldots, X_m]]$ as in Section 30, with its corresponding norm. Using this, we can define

$$(V^1_{r_X}[[X_1, \ldots, X_n]])^1_{r_Y}[[Y_1, \ldots, Y_n]] \tag{39.11}$$

in the same way, with its corresponding norm. Similarly, we can define

$$V^1_r[[X_1, \ldots, X_m, Y_1, \ldots, Y_n]], \tag{39.12}$$

with its corresponding norm. The one-to-one correspondence between (39.4) and (39.5) mentioned earlier leads to an isometric linear mapping from (39.11) onto (39.12). This follows from the remarks about weighted ℓ^1 spaces associated to Cartesian products of nonempty sets in Section 26.

We can define $V^\infty_{r_X}[[X_1, \ldots, X_m]]$ as in Section 31, with its corresponding norm. Using this, we can define

$$(V^\infty_{r_X}[[X_1, \ldots, X_m]])^\infty_{r_Y}[[Y_1, \ldots, Y_n]], \tag{39.13}$$

with its corresponding norm. We can also define

$$V^\infty_r[[X_1, \ldots, X_m, Y_1, \ldots, Y_n]], \tag{39.14}$$

with its corresponding norm. As before, the one-to-one correspondence between (39.4) and (39.5) mentioned earlier leads to an isometric linear mapping from (39.13) onto (39.14). This uses the remarks about weighted ℓ^∞ spaces associated to Cartesian product of nonempty sets in Section 26.

We can define $V_{0,r_X}[[X_1, \ldots, X_m]]$ as in Section 31 as well, and we take it to be equipped with the restriction of the norm from $V^\infty_{r_X}[[X_1, \ldots, X_m]]$, as usual. Using this, we can define

$$(V_{0,r_X}[[X_1, \ldots, X_m]])_{0,r_Y}[[Y_1, \ldots, Y_n]] \tag{39.15}$$

as in Section 31. Similarly, we can define

$$V_{0,r}[[X_1, \ldots, X_m, Y_1, \ldots, Y_n]]. \tag{39.16}$$

As in the previous situations, the one-to-one correspondence between (39.4) and (39.5) mentioned earlier maps (39.15) onto (39.16). This uses the remarks in Section 26 about functions that vanish at infinity with respect to a weight.

40 Logarithmic convexity

Let n be a positive integer, and remember that a subset A of $\mathbf{R}^n$ is said to be *convex* if for every $x, y \in A$ and $t \in \mathbf{R}$ with $0 \le t \le 1$ we have that

$$t\,x + (1-t)\,y \in A. \tag{40.1}$$

Let E be a subset of $\mathbf{R}^n_+$, and let $\log E$ be the subset of $\mathbf{R}^n$ consisting of points of the form

$$(\log r_1, \ldots, \log r_n), \tag{40.2}$$

where $r = (r_1, \ldots, r_n) \in E$. If $\log E$ is convex in $\mathbf{R}^n$, then E is said to be *logarithmically convex* in $\mathbf{R}^n_+$. Equivalently, let $r(0) = (r_1(0), \ldots, r_n(0))$ and $r(1) = (r_1(1), \ldots, r_n(1))$ be elements of E, and let $r(t) = (r_1(t), \ldots, r_n(t))$ be defined for $t \in \mathbf{R}$ by

$$r_j(t) = r_j(0)^t\, r_j(1)^{1-t}, \tag{40.3}$$

for $j = 1, \ldots, n$. The condition that E be logarithmically convex means that $r(t) \in E$ when $0 \le t \le 1$.

If $r \in \mathbf{R}^n_+$ and $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, then we put

$$w_r(\alpha) = r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n}, \tag{40.4}$$

as in (29.1). Let $r(0), r(1) \in \mathbf{R}^n_+$ and $t \in \mathbf{R}$ be given, and let $r(t) \in \mathbf{R}^n_+$ be as in (40.3). Observe that

$$w_{r(t)}(\alpha) = w_{r(0)}(\alpha)^t\, w_{r(1)}(\alpha)^{1-t} \tag{40.5}$$

for every $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$. This corresponds to (27.7), with $X = (\mathbf{Z}_+ \cup \{0\})^n$, $w_0 = w_{r(0)}$, $w_1 = w_{r(1)}$, and $w_t = w_{r(t)}$.

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let $T_1, \ldots, T_n$ be commuting indeterminates, and let $f(T)$ be a formal power series in $T_1, \ldots, T_n$ with coefficients in V. Put

$$E^1(f(T)) = \{r \in \mathbf{R}^n_+ : f(T) \in V^1_r[[T_1, \ldots, T_n]]\}, \tag{40.6}$$

where $V^1_r[[T_1, \ldots, T_n]]$ is as in Section 30. Equivalently, this is the set of $r \in \mathbf{R}^n_+$ such that the coefficients f_α of $f(T)$ are summable with respect to N on V and w_r on $(\mathbf{Z}_+ \cup \{0\})^n$, as in Section 25. Using the remarks in Section 27, we get that $E^1(f(T))$ is logarithmically convex. More precisely, let $r(0), r(1) \in E^1(f(T))$ and $t \in \mathbf{R}$ be given, with $0 \le t \le 1$, and let $r(t) \in \mathbf{R}^n_+$ be as in (40.3). By hypothesis, $N(f_\alpha)$ is summable with respect to $w_{r(0)}$ and $w_{r(1)}$ as a nonnegative real-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^n$. This implies that $N(f_\alpha)$ is summable with respect to $w_{r(t)}$ as a function of $\alpha \in (\mathbf{Z}_+ \cup \{0\})^n$, because of (40.5) and the remarks in Section 27. Hence $r(t) \in E^1(f(T))$, as desired.

Similarly, put

$$E^\infty(f(T)) = \{r \in \mathbf{R}^n_+ : f(T) \in V^\infty_r[[T_1, \ldots, T_n]]\}, \tag{40.7}$$

where $V_r^\infty[[T_1, \ldots, T_n]]$ is as in Section 31. This is the same as the set of $r \in \mathbf{R}_+^n$ such that the coefficients f_α of $f(T)$ are bounded with respect to N on V and w_r on $(\mathbf{Z}_+ \cup \{0\})^n$, as in Section 25. It is easy to see that $E^\infty(f(T))$ is logarithmically convex, using the remarks in Section 27. Indeed, suppose that $r(0), r(1) \in E^\infty(f(T))$, so that $N(f_\alpha)$ is bounded with respect to $w_{r(0)}$ and $w_{r(1)}$ as a nonnegative real-valued function of α on $(\mathbf{Z}_+ \cup \{0\})^n$. This implies that $N(f_\alpha)$ is bounded with respect to $w_{r(t)}$ when $0 \le t \le 1$, as in Section 27.

Let us also put

$$E_0(f(T)) = \{r \in \mathbf{R}_+^n : f(T) \in V_{0,r}[[T_1, \ldots, T_n]]\}, \tag{40.8}$$

where $V_{0,r}[[T_1, \ldots, T_n]]$ is as in Section 31. This is the set of $r \in \mathbf{R}_+^n$ such that f_α vanishes at infinity with respect to N on V and w_r on $(\mathbf{Z}_+ \cup \{0\})^n$, as in Section 25. Note that

$$E^1(f(T)) \subseteq E_0(f(T)) \subseteq E^\infty(f(T)), \tag{40.9}$$

by (31.3). One can check that $E_0(f(T))$ is logarithmically convex, using the remarks in Section 27 again. More precisely, suppose that $r(0) \in E_0(f(T))$ and $r(1) \in E^\infty(f(T))$, so that $N(f_\alpha)$ vanishes at infinity with respect to $w_{r(0)}$, and $N(f_\alpha)$ is bounded with respect to $w_{r(1)}$. If $0 < t < 1$, then it follows that $N(f_\alpha)$ vanishes at infinity with respect to $w_{r(t)}$, as in Section 27. In particular, this holds when both $r(0)$ and $r(1)$ are elements of $E_0(f(T))$, because of the second inclusion in (40.9).

41 Multiplicative ultranorms

Let k be a field with an absolute value function $|\cdot|$, and let $\mathcal{A}$ be an algebra over k with a norm N with respect to $|\cdot|$ on k. If

$$N(a\,b) = N(a)\,N(b) \tag{41.1}$$

for every $a, b \in \mathcal{A}$, then N is said to be *multiplicative* on $\mathcal{A}$. Let us suppose from now on in this section that N is a multiplicative ultranorm on $\mathcal{A}$.

Let T be an indeterminate, and let $f(T) = \sum_{j=0}^\infty f_j\,T^j$, $g(T) = \sum_{l=0}^\infty g_l\,T^l$ be formal power series in T with coefficients in $\mathcal{A}$. Their product $h(T) = f(T)\,g(T)$ is given by $h(T) = \sum_{n=0}^\infty h_n\,T^n$, where

$$h_n = \sum_{j=0}^n f_j\,g_{n-j} \tag{41.2}$$

for each nonnegative integer n. Let r be a positive real number, and suppose that $f(T)$, $g(T)$ are elements of the space $\mathcal{A}_{0,r}[[T]]$ defined in Section 31. This implies that $h(T) \in \mathcal{A}_{0,r}[[T]]$ too, as in Section 36. Put

$$\|f(T)\|_{\infty,r} = \sup_{j \ge 0}(N(f_j)\,r^j), \tag{41.3}$$

where the supremum is taken over all nonnegative integers j, and similarly for $g(T)$, $h(T)$, as in Section 31.

We would like to verify that

$$\|h(T)\|_{\infty,r} = \|f(T)\|_{\infty,r} \, \|g(T)\|_{\infty,r} \tag{41.4}$$

under these conditions. It suffices to check that

$$\|f(T)\|_{\infty,r} \, \|g(T)\|_{\infty,r} \le \|h(T)\|_{\infty,r}, \tag{41.5}$$

because the opposite inequality was already given in Section 36. This is trivial when $f(T) = 0$ and when $g(T) = 0$, and so we may suppose that $f(T), g(T) \neq 0$. The hypothesis that $f(T) \in \mathcal{A}_{0,r}[[T]]$ means that

$$\lim_{j \to \infty} (N(f_j) \, r^j) = 0. \tag{41.6}$$

In particular, this implies that the supremum in (41.3) is attained. Let j_0 be the smallest nonnegative integer such that

$$N(f_{j_0}) \, r^{j_0} = \|f(T)\|_{\infty,r}. \tag{41.7}$$

Similarly, let l_0 be the smallest nonnegative integer such that

$$N(g_{l_0}) \, r^{l_0} = \|g(T)\|_{\infty,r}. \tag{41.8}$$

In order to get (41.5), we shall look at $h_{j_0+l_0}$, as usual. Observe that

$$f_{j_0} \, g_{l_0} = h_{j_0+l_0} - \sum_{j=0}^{j_0-1} f_j \, g_{j_0+l_0-j} - \sum_{j=j_0+1}^{j_0+l_0} f_j \, g_{j_0+l_0-j}, \tag{41.9}$$

by (41.2) with $n = j_0 + l_0$. More precisely, the first sum on the right side of (41.9) should be interpreted as being equal to 0 when $j_0 = 0$, and the second sum on the right side of (41.9) is interpreted as being equal to 0 when $l_0 = 0$. It follows that

$$\begin{aligned} &N(f_{j_0} \, g_{l_0}) \\ &\quad \le \max\Big(N(h_{j_0+l_0}), \max_{0 \le j \le j_0-1} N(f_j \, g_{j_0+l_0-j}), \max_{j_0+1 \le j \le j_0+l_0} N(f_j \, g_{j_0+l_0-j})\Big), \end{aligned} \tag{41.10}$$

by the ultranorm version of the triangle inequality. As before, the maximum over $0 \le j \le j_0 - 1$ is interpreted as being equal to 0 when $j_0 = 0$, and the maximum over $j_0 + 1 \le j \le j_0 + l_0$ is interpreted as being equal to 0 when $l_0 = 0$. We also have that

$$\|f(T)\|_{\infty,r} \, \|g(T)\|_{\infty,r} = N(f_{j_0}) \, r^{j_0} \, N(g_{l_0}) \, r^{l_0} = N(f_{j_0} \, g_{l_0}) \, r^{j_0+l_0}, \tag{41.11}$$

using (41.7) and (41.8) in the first step, and (41.1) in the second step. Combining this with (41.10), we obtain that

$$\begin{aligned} &\|f(T)\|_{\infty,r} \, \|g(T)\|_{\infty,r} \\ &\quad \le \max\Big(N(h_{j_0+l_0}) \, r^{j_0+l_0}, \max_{0 \le j \le j_0-1} N(f_j \, g_{j_0+l_0-j}) \, r^{j_0+l_0}, \\ &\qquad\qquad \max_{j_0+1 \le j \le j_0+l_0} N(f_j \, g_{j_0+l_0-j}) \, r^{j_0+l_0}\Big), \end{aligned} \tag{41.12}$$

with the same interpretations for the right side as before.

Of course,

$$N(h_{j_0+l_0})\, r^{j_0+l_0} \le \|h(T)\|_{\infty,r}. \tag{41.13}$$

If $0 \le j \le j_0 - 1$, then

$$N(f_j)\, r^j < \|f(T)\|_{\infty,r}, \tag{41.14}$$

by the definition of j_0. This implies that

$$\begin{aligned} N(f_j\, g_{j_0+l_0-j})\, r^{j_0+l_0} &= N(f_j)\, r^j\, N(g_{j_0+l_0-j})\, r^{j_0+l_0-j} \\ &< \|f(T)\|_{\infty,r}\, \|g(T)\|_{\infty,r} \end{aligned} \tag{41.15}$$

when $0 \le j \le j_0 - 1$. Similarly, if $j_0 + 1 \le j \le j_0 + l_0$, then $0 \le j_0 + l_0 - j \le l_0 - 1$, and hence

$$N(g_{j_0+l_0-j})\, r^{j_0+l_0-j} < \|g(T)\|_{\infty,r}, \tag{41.16}$$

by the definition of l_0. It follows that (41.15) holds in this case as well. Thus the second and third expressions in the maximum on the right side of (41.12) are strictly less than the left side of (41.12). Note that this also holds when either of these expressions is interpreted as being equal to 0, as in the preceding paragraph. This means that the left side of (41.12) is less than or equal to the first expression in the maximum on the right side of (41.12). This shows that (41.5) holds, as desired, because of (41.13).

42 Some limits in r

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let T be an indeterminate, and let $f(T) = \sum_{j=0}^{\infty} f_j\, T^j$ be a formal power series in T with coefficients in V. If r is a positive real number, then

$$\|f(T)\|_{1,r} = \sum_{j=0}^{\infty} N(f_j)\, r^j \tag{42.1}$$

can be defined as a nonnegative extended real number, as in Section 30. It is easy to see that this sum increases monotonically in r, as in Section 32. This uses the fact that r^j increases monotonically in r for each $j \ge 0$.

Let r_0 be a positive real number, and suppose that (42.1) is finite when $0 < r < r_0$. We would like to check that

$$\|f(T)\|_{1,r_0} = \sup_{0<r<r_0} \|f(T)\|_{1,r}, \tag{42.2}$$

where the supremum on the right side is defined as a nonnegative extended real number. This is the same as saying that

$$\|f(T)\|_{1,r} \to \|f(T)\|_{1,r_0} \quad \text{as } r \to r_0-, \tag{42.3}$$

because (42.1) increases monotonically in r. Of course, the right side of (42.2) is less than or equal to the left side, by monotonicity in r. Thus it is enough to verify that

$$\|f(T)\|_{1,r_0} \le \sup_{0<r<r_0} \|f(T)\|_{1,r}. \tag{42.4}$$

If n is a nonnegative integer, then

$$\sum_{j=0}^{n} N(f_j)\, r_0^j = \lim_{r \to r_0-} \sum_{j=0}^{n} N(f_j)\, r^j \le \sup_{0<r<r_0} \sum_{j=0}^{\infty} N(f_j)\, r^j. \tag{42.5}$$

This implies (42.4), by taking the supremum over $n \ge 0$ of the sum on the left.

Similarly, if r is a positive real number, then

$$\|f(T)\|_{\infty,r} = \sup_{j\ge 0}(N(f_j)\, r^j) \tag{42.6}$$

can be defined as a nonnegative extended real number. Note that this increases monotonically in r. Let r_0 be a positive real number again, and suppose that (41.3) is finite when $0 < r < r_0$. Let us check that

$$\|f(T)\|_{\infty,r_0} = \sup_{0<r<r_0} \|f(T)\|_{\infty,r}, \tag{42.7}$$

where the supremum on the right is defined as a nonnegative extended real number. As before, this is the same as saying that

$$\|f(T)\|_{\infty,r} \to \|f(T)\|_{\infty,r_0} \quad \text{as } r \to r_0-, \tag{42.8}$$

because of monotonicity. The right side of (42.7) is automatically less than or equal to the left side, by monotonicity, and so it suffices to verify that

$$\|f(T)\|_{\infty,r_0} \le \sup_{0<r<r_0} \|f(T)\|_{\infty,r}. \tag{42.9}$$

If j is a nonnegative integer, then

$$N(f_j)\, r_0^j = \lim_{r \to r_0} (N(f_j)\, r^j) \le \sup_{0<r<r_0} \|f(T)\|_{\infty,r}. \tag{42.10}$$

This implies (42.9), as desired.

Now let $\mathcal{A}$ be an algebra over k, and let N be a multiplicative ultranorm on $\mathcal{A}$ with respect to $|\cdot|$ on k. Let r_0 be a positive real number, and suppose that $f(T)$, $g(T)$ are elements of the space $\mathcal{A}^{\infty}_{r_0}[[T]]$ defined in Section 31. This means that $h(T) = f(T)\, g(T) \in \mathcal{A}^{\infty}_{r_0}[[T]]$ as well, as in Section 36. Under these conditions, we have that

$$\|h(T)\|_{\infty,r_0} = \|f(T)\|_{\infty,r_0}\, \|g(T)\|_{\infty,r_0}. \tag{42.11}$$

Indeed, if $0 < r < r_0$, then $f(T), g(T) \in \mathcal{A}_{0,r}[[T]]$, as in Section 32. Thus (41.4) holds when $0 < r < r_0$, as before. To get (42.11), one can take the limit as $r \to r_0-$ on both sides of (42.11), using (42.8).

Part IV

Some additional properties

43 More on metrics and ultrametrics

Let a be a positive real number, with $a \le 1$. If r, t are nonnegative real numbers, then it is well known that

$$(r+t)^a \le r^a + t^a. \tag{43.1}$$

To see this, observe first that

$$\max(r,t) \le (r^a + t^a)^{1/a}. \tag{43.2}$$

It follows that

$$r + t \le (r^a + t^a) \max(r,t)^{1-a} \le (r^a + t^a)^{1+(1-a)/a} = (r^a + t^a)^{1/a}, \tag{43.3}$$

using (43.2) in the second step. This implies (43.1), as desired.

Let X be a set, and let $d(x,y)$ be a metric on X. If $0 < a \le 1$, then $d(x,y)^a$ defines a metric on X too. More precisely, one can check that $d(x,y)^a$ satisfies the triangle inequality on X, using (43.1) and the triangle inequality for $d(x,y)$. If $d(x,y)$ is an ultrametric on X, then one can verify that $d(x,y)^a$ is an ultrametric on X for every $a > 0$.

Suppose that $d(x,y)^a$ is a metric on X for some $a > 0$, which includes the cases mentioned in the preceding paragraph. Let $B_d(x,r)$ and $B_{d^a}(x,r)$ denote the open balls in X centered at $x \in X$ with radius $r > 0$ with respect to $d(\cdot,\cdot)$ and $d(\cdot,\cdot)^a$, respectively, as in Section 13. It is easy to see that

$$B_{d^a}(x, r^a) = B_d(x,r) \tag{43.4}$$

for every $x \in X$ and $r > 0$. Similarly, let $\overline{B}_d(x,r)$ and $\overline{B}_{d^a}(x,r)$ be the closed balls in X centered at $x \in X$ with radius $r \ge 0$ with respect to $d(\cdot,\cdot)$ and $d(\cdot,\cdot)^a$, respectively. As before,

$$\overline{B}_{d^a}(x, r^a) = \overline{B}_d(x,r) \tag{43.5}$$

for every $x \in X$ and $r \ge 0$.

It follows from (43.4) that $d(\cdot,\cdot)$ and $d(\cdot,\cdot)^a$ determine the same topologies on X. More precisely, the identity mapping on X is uniformly continuous as a mapping from X equipped with $d(\cdot,\cdot)$ into X equipped with $d(\cdot,\cdot)^a$. The identity mapping on X is also uniformly continuous as a mapping from X equipped with $d(\cdot,\cdot)^a$ into X equipped with $d(\cdot,\cdot)$. In particular, a sequence $\{x_j\}_{j=1}^\infty$ of elements of X is a Cauchy sequence with respect to $d(\cdot,\cdot)$ if and only if $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $d(\cdot,\cdot)$. It follows that X is complete with respect to $d(\cdot,\cdot)$ if and only if X is complete with respect to $d(\cdot,\cdot)^a$.

44 More on absolute value functions

Let k be a field, and let $|\cdot|$ be an absolute value function on k. If a is a positive real number with $a \le 1$, then $|x|^a$ defines an absolute value function on k as well. Indeed, the triangle inequality for $|x|^a$ can be obtained from (43.1) and the triangle inequality for $|x|$. If $|\cdot|$ is an ultrametric absolute value function on k, then $|x|^a$ is an ultrametric absolute value function on k for every $a > 0$.

Let $|x|_1$ and $|x|_2$ be two absolute value functions on k. If there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{44.1}$$

for every $x \in k$, then $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* on k. Of course, this implies that

$$|x - y|_2 = |x - y|_1^a \tag{44.2}$$

for every $x, y \in k$. It follows that the topologies determined on k by the metrics associated to $|\cdot|_1$ and $|\cdot|_2$ are the same in this case, as in the previous section. Conversely, if the topologies determined on k be the metrics associated to $|\cdot|_1$ and $|\cdot|_2$ are the same, then it is well known that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k in this sense.

If $x \in k$ and $n \in \mathbf{Z}_+$, then $n \cdot x$ denotes the sum of n x's in k, as before. Let $|\cdot|$ be an absolute value function on k again. If there are positive integers n such that $|n \cdot 1|$ can be arbitrarily large, where 1 is the multiplicative identity element in k, then $|\cdot|$ is said to be *archimedean* on k. Otherwise, $|\cdot|$ is said to be *non-archimedean* on k. Observe that

$$|n^j \cdot 1| = |(n \cdot 1)^j| = |n \cdot 1|^j \tag{44.3}$$

for every $j, n \in \mathbf{Z}_+$. If $|n \cdot 1| > 1$ for some $n \in \mathbf{Z}_+$, then (44.3) tends to $+\infty$ as $j \to \infty$, so that $|\cdot|$ is archimedean on k. If $|\cdot|$ is non-archimedean on k, then it follows that

$$|n \cdot 1| \le 1 \tag{44.4}$$

for every $n \in \mathbf{Z}_+$. If $|\cdot|$ is an ultrametric absolute value function on k, then it is easy to see that (44.4) holds for every $n \in \mathbf{Z}_+$, so that $|\cdot|$ is non-archimedean on k. Conversely, it is well known that non-archimedean absolute value functions are ultrametric absolute value functions. Note that every absolute value function on k is non-archimedean when k has positive characteristic.

Let $|\cdot|$ be an absolute value function on the field $\mathbf{Q}$ of rational numbers. A famous theorem of Ostrowski that $|\cdot|$ is either equivalent to the standard Euclidean absolute value function on $\mathbf{Q}$, or $|\cdot|$ is the trivial absolute value function on $\mathbf{Q}$, or $|\cdot|$ is equivalent to the p-adic absolute value function on $\mathbf{Q}$ for some prime number p. More precisely, the first case occurs when $|\cdot|$ is archimedean on $\mathbf{Q}$. If $|n| = 1$ for every $n \in \mathbf{Z}_+$, then $|\cdot|$ is trivial on $\mathbf{Q}$. The third case occurs when $|\cdot|$ is non-archimedean on $\mathbf{Q}$, and $|n| < 1$ for some $n \in \mathbf{Z}_+$.

Suppose that $|\cdot|$ is an archimedean absolute value function on a field k. In particular, this implies that k has characteristic 0, as before. This means that there is a natural embedding of $\mathbf{Q}$ into k. The absolute value function induced on $\mathbf{Q}$ by $|\cdot|$ on k is also archimedean, and hence is equivalent to the standard Euclidean absolute value function on $\mathbf{Q}$, as in the previous paragraph. If k is complete with respect to the metric associated to $|\cdot|$, then another famous theorem of Ostrowski implies that k is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, and that $|\cdot|$ corresponds to an absolute value function that is equivalent to the standard Euclidean absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate, using this isomorphism.

45 p-Adic integers

Let p be a prime number, and remember that $\mathbf{Q}_p$ is the field of p-adic numbers, with the p-adic absolute value function $|\cdot|_p$. The ultrametric on $\mathbf{Q}_p$ associated to $|\cdot|_p$ is known as the *p-adic metric*. Put

$$\mathbf{Z}_p = \{x \in \mathbf{Q}_p : |x|_p \le 1\}, \tag{45.1}$$

which is the set of *p-adic integers*. This is the same as the closed unit ball in $\mathbf{Q}_p$ with respect to the p-adic metric. In particular, $\mathbf{Z}_p$ is both open and closed in $\mathbf{Q}_p$, with respect to the topology determined by the p-adic metric, as in Section 13. Note that $\mathbf{Z} \subseteq \mathbf{Z}_p$, by definition of $|\cdot|_p$. Thus $\mathbf{Z}_p$ contains the closure of $\mathbf{Z}$ in $\mathbf{Q_p}$, with respect to the p-adic metric. Let us check that $\mathbf{Z}_p$ is equal to the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$.

Let $l \in \mathbf{Z}$ be given, and observe that

$$|p\,l|_p = |p|_p\,|l|_p \le 1/p < 1. \tag{45.2}$$

This implies that

$$\sum_{j=0}^{n} (p\,l)^j = \frac{1-(p\,l)^{n+1}}{1-p\,l} \to \frac{1}{1-p\,l} \tag{45.3}$$

as $n \to \infty$, with respect to the p-adic metric. Of course, $\sum_{j=0}^{n}(p\,l)^j \in \mathbf{Z}$ for each $n \ge 0$. It follows that $1/(1-p\,l)$ can be approximated by integers with respect to the p-adic metric.

Suppose that $x \in \mathbf{Q}$ satisfies $|x|_p \le 1$. This means that x can be expressed as a/b, where $a, b \in \mathbf{Z}$, $b \ne 0$, and in fact b is not a multiple of p. Let c be an integer such that $b\,c \equiv 1$ modulo p, which is to say that there is an integer l with $b\,c = 1 - p\,l$. Thus

$$x = \frac{a}{b} = \frac{a\,c}{b\,c} = \frac{a\,c}{1-p\,l}. \tag{45.4}$$

This implies that x can be approximated by integers with respect to the p-adic metric, by the remarks in the preceding paragraph.

Remember that $\mathbf{Q}$ is dense in $\mathbf{Q}_p$, by construction. If $y \in \mathbf{Z}_p$, $x \in \mathbf{Q}$, and $|y-x|_p \le 1$, then $|x|_p \le 1$ too, by the ultrametric version of the triangle inequality. Hence y can be approximated by $x \in \mathbf{Q}$ with $|x|_p \le 1$ with respect to the p-adic metric. These x's can be approximated by integers with respect to the p-adic metric, as in the previous paragraph. This shows that every $y \in \mathbf{Z}_p$ can be approximated by integers with respect to the p-adic metric, so that $\mathbf{Z}_p$ is the same as the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$.

Note that the set $\mathbf{Z}_+$ is dense in $\mathbf{Z}$ with respect to the p-adic metric. It follows that $\mathbf{Z}_+$ is also dense in $\mathbf{Z}_p$ with respect to the p-adic metric.

46 Some basic polynomials

Let k be a field, and let X, Y be commuting indeterminates. Of course, X and Y may be considered as elements of the algebra $k[X, Y]$ of formal polynomials in X and Y with coefficients in k, using the multiplicative identity element 1 in k. If j is a nonnegative integer, then $(X+Y)^j$ may be considered as an element of $k[X, Y]$ too. This polynomial can be expressed as

$$(X+Y)^j = \sum_{l=0}^{j} \left(\binom{j}{l} \cdot 1 \right) X^l \, Y^{j-l}, \tag{46.1}$$

as in the binomial theorem. More precisely, the binomial coefficient $\binom{j}{l}$ is a positive integer, so that $\binom{j}{l} \cdot 1$ is defined as an element of k.

Let $|\cdot|$ be an absolute value function on k, and let $\rho = (\rho_1, \rho_2)$ be an ordered pair of positive real numbers. Of course, k may be considered as a one-dimensional vector space over itself, and $|\cdot|$ may be considered as a norm on k. Observe that

$$\|(X+Y)^j\|_{1,\rho} = \sum_{l=0}^{j} \left| \binom{j}{l} \cdot 1 \right| \rho_1^l \, \rho_2^{j-l} \tag{46.2}$$

for each nonnegative integer j, where the left side is as in (29.3). It is easy to see that

$$|n \cdot 1| \le n \, |1| = n \tag{46.3}$$

for every $n \in \mathbf{Z}_+$, using the triangle inequality, so that

$$\left| \binom{j}{l} \cdot 1 \right| \le \binom{j}{l} \tag{46.4}$$

for all nonnegative integers j, l with $l \le j$. This implies that

$$\|(X+Y)^j\|_{1,\rho} \le \sum_{l=0}^{j} \binom{j}{l} \rho_1^l \, \rho_2^{j-l} = (\rho_1 + \rho_2)^j \tag{46.5}$$

for each $j \ge 0$, using the binomial theorem in the second step.

Suppose that k has characteristic 0, so that there is a natural embedding of $\mathbf{Q}$ into k. This leads to an absolute value function on $\mathbf{Q}$, induced by $|\cdot|$ on k. Suppose that this induced absolute value function on $\mathbf{Q}$ is the same as the standard absolute value function on k, so that equality holds in the first step in

(46.3). In particular, equality holds in (46.4) for every $0 \le l \le j$. This means that equality holds in (46.5), so that

$$\|(X+Y)^j\|_{1,\rho} = (\rho_1 + \rho_2)^j \tag{46.6}$$

for every $j \ge 0$.

Let k be any field again, and note that

$$\|(X+Y)^j\|_{\infty,\rho} = \max_{0 \le l \le j} \left(\left| \binom{j}{l} \cdot 1 \right| \rho_1^l \, \rho_2^{j-l} \right) \tag{46.7}$$

for every nonnegative integer j, where the left side is as in (29.4). Suppose now that $|\cdot|$ is an ultrametric absolute value function on k. This implies that

$$\left| \binom{j}{l} \cdot 1 \right| \le 1 \tag{46.8}$$

when j, l are nonnegative integers with $l \le j$, as in (44.4). Hence

$$\|(X+Y)^j\|_{\infty,\rho} \le \max(\rho_1, \rho_2)^j \tag{46.9}$$

for each $j \ge 0$, by (46.7). In fact,

$$\|(X+Y)^j\|_{\infty,\rho} = \max(\rho_1, \rho_2)^j \tag{46.10}$$

for every $j \ge 0$, because $\binom{j}{l} = 1$ when $l = 0$ or j.

47 Adding indeterminates again

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. Also let $f(T) = \sum_{j=0}^\infty f_j \, T^j$ be a formal power series in an indeterminate T with coefficients in V, and let X, Y be commuting indeterminates. As before,

$$f(X+Y) = \sum_{j=0}^\infty f_j \, (X+Y)^j = \sum_{j=0}^\infty \sum_{l=0}^j \binom{j}{l} \cdot f_j \, X^l \, Y^{j-l} \tag{47.1}$$

defines a formal power series in X and Y with coefficients in V. Let $\rho = (\rho_1, \rho_2)$ be an ordered pair of positive real numbers, and observe that

$$\|f(X+Y)\|_{1,\rho} = \sum_{j=0}^\infty \sum_{l=0}^j N\Big(\binom{j}{l} \cdot f_j \Big) \, \rho_1^l \, \rho_2^{j-l}, \tag{47.2}$$

where the left side is as in Section 30. More precisely, the sum on the right can be arranged in this way using remarks in Section 20. Remember that

$$\binom{j}{l} \cdot f_j = \Big(\binom{j}{l} \cdot 1 \Big) \, f_j \tag{47.3}$$

for all nonnegative integers j, l with $l \le j$, where 1 is the multiplicative identity element in k. Thus

$$N\Big(\binom{j}{l} \cdot f_j\Big) = \Big|\binom{j}{l} \cdot 1\Big| \, N(f_j) \le \binom{j}{l} \, N(f_j) \tag{47.4}$$

when $0 \le l \le j$, using (46.4) in the second step. It follows that

$$\begin{aligned} \sum_{l=0}^{j} N\Big(\binom{j}{l} \cdot f_j\Big) \, \rho_1^l \, \rho_2^{j-l} &= N(f_j) \sum_{l=0}^{j} \Big|\binom{j}{l} \cdot 1\Big| \, \rho_1^l \, \rho_2^{j-l} \\ &\le N(f_j) \sum_{l=0}^{j} \binom{j}{l} \, \rho_1^l \, \rho_2^{j-l} = N(f_j) \, (\rho_1 + \rho_2)^j \end{aligned} \tag{47.5}$$

for every nonnegative integer j, using the binomial theorem in the third step. Combining this with (47.2), we get that

$$\begin{aligned} \|f(X+Y)\|_{1,\rho} &= \sum_{j=0}^{\infty} N(f_j) \sum_{l=0}^{j} \Big|\binom{j}{l} \cdot 1\Big| \, \rho_1^l \, \rho_2^{j-l} \\ &\le \sum_{j=0}^{\infty} N(f_j) \, (\rho_1 + \rho_2)^j = \|f(T)\|_{1,\rho_1+\rho_2}, \end{aligned} \tag{47.6}$$

where the right side is as in Section 30. In particular, if $f(T)$ is an element of the space $V^1_{\rho_1+\rho_2}[[T]]$ defined in Section 30, then $f(X+Y)$ is an element of the space $V^1_\rho[[X,Y]]$ defined in Section 30.

Suppose that k has characteristic 0, and that the absolute value function on $\mathbf{Q}$ induced by $|\cdot|$ on k and the natural embedding of $\mathbf{Q}$ in k is the standard absolute value function. In this case, the inequalities in the second steps of each of (47.4), (47.5), and (47.6) are equalities, as in the previous section. Hence

$$\|f(X+Y)\|_{1,\rho} = \|f(T)\|_{1,\rho_1+\rho_2} \tag{47.7}$$

in this situation.

Let k be any field again, and observe that

$$\|f(X+Y)\|_{\infty,\rho} = \sup_{0 \le l \le j} \Big(N\Big(\binom{j}{l} \cdot f_j\Big) \, \rho_1^l \, \rho_2^{j-l}\Big), \tag{47.8}$$

where the left side is as in Section 31. More precisely, the supremum on the right is taken over all nonnegative integers j, l with $l \le j$. Equivalently,

$$\|f(X+Y)\|_{\infty,\rho} = \sup_{0 \le l \le j} \Big(N(f_j) \, \Big|\binom{j}{l} \cdot 1\Big| \, \rho_1^l \, \rho_2^{j-l}\Big) \tag{47.9}$$

by the first step in (47.4). Suppose from now on in this section that $|\cdot|$ is an ultrametric absolute value function on k, so that (46.8) holds. Using this, it is easy to see that

$$\|f(X+Y)\|_{\infty,\rho} \le \sup_{j \ge 0} \Big(N(f_j) \, \max(\rho_1, \rho_2)^j\Big) = \|f(T)\|_{\infty,\max(\rho_1,\rho_2)}, \tag{47.10}$$

where the supremum in the middle is taken over all nonnegative integers j, and the right side is as in Section 31. We also have that

$$N(f_j)\,\rho_1^j,\ N(f_j)\,\rho_2^j \le \|f(X+Y)\|_{\infty,\rho} \tag{47.11}$$

for every $j \ge 0$, by taking $l = j$ or 0 in the right side of (47.9). Thus

$$\|f(X+Y)\|_{\infty,\rho} = \sup_{j\ge 0} \Big(N(f_j)\,\max(\rho_1,\rho_2)^j\Big) = \|f(T)\|_{\infty,\max(\rho_1,\rho_2)}. \tag{47.12}$$

Note that $f(T)$ is in the space $V_{0,\max(\rho_1,\rho_2)}[[T]]$ defined in Section 31 when

$$\lim_{j\to\infty} \Big(N(f_j)\,\max(\rho_1,\rho_2)^j\Big) = 0. \tag{47.13}$$

Similarly, $f(X+Y)$ is in the space $V_{0,\rho}[[X,Y]]$ defined in Section 31 when

$$N\Big(\binom{j}{l}\cdot f_j\Big)\,\rho_1^l\,\rho_2^{j-l} \tag{47.14}$$

vanishes at infinity on the set of ordered pairs (j,l) of nonnegative integers with $l \le j$. More precisely, in the definition of $V_{0,\rho}[[X,Y]]$, (47.14) was considered as a function of $(l, j-l)$, on the set $(\mathbf{Z}_+ \cup \{0\})^2$ of ordered pairs of nonnegative integers. It is easy to see that (47.14) vanishes at infinity as a function of $(l, j-l)$ if and only if it vanishes at infinity as a function of (j,l). Of course, (47.14) is the same as

$$N(f_j)\,\bigg|\binom{j}{l}\cdot 1\bigg|\,\rho_1^l\,\rho_2^{j-l}, \tag{47.15}$$

as in the first step in (47.4). Thus $f(X+Y) \in V_{0,\rho}[[X,Y]]$ exactly when (47.15) vanishes at infinity on the set of ordered pairs (j,l) of nonnegative integers with $l \le j$. Remember that $|\cdot|$ is supposed to be an ultrametric absolute value function on k, so that (46.8) holds. If (47.13) holds, then one can use this to verify that (47.15) vanishes at infinity as a function of (j,l). Conversely, if (47.15) vanishes at infinity as a function of (j,l), then one can get (47.13), by taking $l = j$ or 0 in (47.15). This shows that $f(T) \in V_{0,\max(\rho_1,\rho_2)}[[T]]$ if and only if $f(X+Y) \in V_{0,\rho}[[X,Y]]$.

48 Adding arguments

Let k be a field with an absolute value function $|\cdot|$ again, let V be a vector space over k with a norm N with respect to $|\cdot|$ on k, and let $f(T) = \sum_{j=0}^\infty f_j\,T^j$ be a formal power series in an indeterminate T with coefficients in V. Also let X, Y be commuting indeterminates, and let

$$F(X,Y) = f(X+Y) = \sum_{j=0}^\infty f_j\,(X+Y)^j = \sum_{j=0}^\infty \sum_{l=0}^j \binom{j}{l}\cdot f_j\,X^l\,Y^{j-l} \tag{48.1}$$

be the formal power series in X and Y with coefficients in V in (47.1). Let r be a positive real number, and suppose for the moment that $f(T)$ is an element of the space $V_r^1[[T]]$ defined in Section 30. If $\rho = (\rho_1, \rho_2)$ is an ordered pair of positive real numbers such that

$$\rho_1 + \rho_2 \le r, \tag{48.2}$$

then it follows that $F(X, Y)$ is an element of the space $V_\rho^1[[X, Y]]$ defined in Section 30. This uses the remarks in Section 32 and the previous section. Suppose that V is complete with respect to the metric associated to N, and let $x, y \in k$ be given, with

$$|x| \le \rho_1, \ |y| \le \rho_2. \tag{48.3}$$

Thus

$$|x + y| \le |x| + |y| \le \rho_1 + \rho_2 \le r. \tag{48.4}$$

Under these conditions, we have that

$$f(x + y) = F(x, y), \tag{48.5}$$

where both sides of the equation are defined as elements of V as in Section 30. More precisely,

$$f(x + y) = \sum_{j=0}^{\infty} f_j\, (x + y)^j, \tag{48.6}$$

where the right side converges absolutely in V. Of course,

$$\sum_{j=0}^{\infty} f_j\, (x + y)^j = \sum_{j=0}^{\infty} \sum_{l=0}^{j} \binom{j}{l} \cdot f_j\, x^l\, y^{j-l}, \tag{48.7}$$

by the binomial theorem. By definition, $F(x, y)$ can be expressed as the sum of $\binom{j}{l} \cdot f_j\, x^l\, y^{j-l}$ over nonnegative integers j, l with $l \le j$. This sum can also be obtained using the iterated sum on the right side of (48.7), as in Section 23.

Suppose now that $| \cdot |$ is an ultrametric absolute value function on k, and that N is an ultranorm on V. Let $r > 0$ be given again, and suppose that $f(T)$ is an element of the space $V_{0,r}[[T]]$ defined in Section 31. If $\rho \in \mathbf{R}_+^2$ satisfies

$$\rho_1, \rho_2 \le r, \tag{48.8}$$

then $F(X, Y)$ is an element of the space $V_{0,\rho}[[X, Y]]$ defined in Section 31. As before, this uses the remarks in Section 32 and the previous section. Suppose that V is complete with respect to the metric associated to N again, and let x, y be elements of k that satisfy (48.3). This implies that

$$|x + y| \le \max(|x|, |y|) \le \max(\rho_1, \rho_2) \le r, \tag{48.9}$$

by the ultrametric version of the triangle inequality. We would like to verify that (48.5) holds, where both sides of the equation are defined as elements of

V as in Section 31. In particular, $f(x+y)$ can be expressed as in (48.6), where the series on the right side of the equation converges in V. This series can be reexpressed as in (48.7), using the binomial theorem. The sum used to define $F(x,y)$ is equal to the iterated sum on the right side of (48.7), as in Section 23.

Let $|\cdot|$ be any absolute value function on k again, and suppose that $f(T)$ is a formal power series in T with coefficients in k. In this case, $F(X,Y)$ is defined as a formal power series in X and Y with coefficients in k. Let $r>0$ be given, and suppose that $f(T)$ is an element of the space $k_r^1[[T]]$ defined in Section 30. Here we take $V=k$, considered as a one-dimensional vector space over itself, with $|\cdot|$ as the norm on V. Let $\rho\in\mathbf{R}_+^2$ be given such that (48.2) holds, and observe that $F(X,Y)$ is an element of the space $k_\rho^1[[X,Y]]$ defined in Section 30. This uses the remarks in Section 32 and the previous section, as usual. Let $\mathcal{A}_0$ be an algebra over k with a submultiplicative norm N_0 with respect to $|\cdot|$ on k, and suppose that $\mathcal{A}_0$ has a mutliplicative identity element e_0 with $N_0(e_0)=1$. Let $x,y\in\mathcal{A}_0$ be given, with

$$x\,y=y\,x \tag{48.10}$$

and

$$N_0(x)\le\rho_1,\ N_0(y)\le\rho_2. \tag{48.11}$$

Hence

$$N_0(x+y)\le N_0(x)+N_0(y)\le\rho_1+\rho_2\le r. \tag{48.12}$$

Suppose that $\mathcal{A}_0$ is complete with respect to the metric associated to N_0, so that $f(x+y)$ and $F(x,y)$ can be defined as elements of $\mathcal{A}_0$ as in Section 37. One can check that (48.5) holds in this situation as well. This uses the hypothesis (48.10) that x and y commute, to get (48.7).

Suppose that $|\cdot|$ is an ultrametric absolute value function on k, and that N_0 is an ultranorm on $\mathcal{A}_0$. Let $r>0$ be given, and suppose that $f(T)$ is an element of the space $k_{0,r}[[T]]$ defined in Section 31, with $V=k$. Let $\rho\in\mathbf{R}_+^2$ be given such that (48.8) holds, so that $F(X,Y)$ is an element of the space $k_{0,\rho}[[X,Y]]$ defined in Section 31, with $V=k$. This uses the remarks in Section 32 and the previous section again. Let x, y be elements of $\mathcal{A}_0$ that satisfy (48.10) and (48.11), so that

$$N_0(x+y)\le\max(N_0(x),N_0(y))\le\max(\rho_1,\rho_2)\le r, \tag{48.13}$$

by the ultrametric version of the triangle inequality. If $\mathcal{A}_0$ is complete with respect to the metric associated to N_0, then $f(x+y)$ and $F(x,y)$ can be defined as elements of $\mathcal{A}_0$, as in Section 37. It is easy to see that (48.5) holds in this case too, using the same types of arguments as before.

49 The exponential function

If z is a complex number, then the exponential of z is defined as usual by

$$\exp(z)=\sum_{j=0}^\infty(1/j!)\,z^j. \tag{49.1}$$

More precisely, it is well known and easy to see that the series on the right converges absolutely with respect to the standard absolute value function on $\mathbf{C}$, using the ratio test, for instance. This implies that the series converges in $\mathbf{C}$, so that $exp(z)$ is defined as a complex number. Of course, if z is a real number, then $\exp(z) \in \mathbf{R}$ too.

Let k be a field of characteristic 0, so that there is a natural embedding of $\mathbf{Q}$ into k. If T is an indeterminate, then the exponential function can be defined as a formal power series in T with coefficients in k by

$$\exp(T) = \sum_{j=0}^{\infty} (1/j!)\, T^j, \tag{49.2}$$

as in Section 9. Let $|\cdot|$ be an absolute value function on k, and suppose for the rest of the section that the absolute value function induced on $\mathbf{Q}$ by $|\cdot|$ on k and the natural embedding of $\mathbf{Q}$ into k is the same as the standard Euclidean absolute value function on $\mathbf{Q}$. If r is a positive real number, then

$$\|\exp(T)\|_{1,r} = \sum_{j=0}^{\infty} |1/j!|\, r^j = \exp(r), \tag{49.3}$$

where the left side is defined as in Section 30, with $V = k$ and $N = |\cdot|$. The right side of the equation is defined as a positive real number, as in the previous paragraph.

Let X and Y be commuting indeterminates, so that

$$\exp(X+Y) = \sum_{j=0}^{\infty} (1/j!)\, (X+Y)^j = \sum_{j=0}^{\infty} \sum_{l=0}^{j} (1/l!)\, (1/(j-l)!)\, X^l\, Y^{j-l} \tag{49.4}$$

defines a formal power series in X and Y with coefficients in k. Of course,

$$\exp(X+Y) = \exp(X)\, \exp(Y), \tag{49.5}$$

as in Section 9. Let $\rho = (\rho_1, \rho_2)$ be an ordered pair of positive real numbers, so that

$$\|\exp(X+Y)\|_{1,\rho} = \sum_{j=0}^{\infty} \sum_{l=0}^{j} (1/l!)\, (1/(j-l)!)\, \rho_1^l\, \rho_2^{j-l}, \tag{49.6}$$

where the left side is defined as in Section 30 again, with $V = k$ and $N = |\cdot|$. Thus

$$\|\exp(X+Y)\|_{1,\rho} = \sum_{j=0}^{\infty} (1/j!)\, (\rho_1 + \rho_2)^j = \exp(\rho_1 + \rho_2), \tag{49.7}$$

using the binomial theorem in the first step. This can also be seen in terms of the remarks in Section 47.

Let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k. Suppose that $\mathcal{A}$ has a multiplicative identity element e with $N(e) = 1$, and that $\mathcal{A}$ is complete with respect to the metric associated to N. If $x \in \mathcal{A}$, then

$$\exp(x) = \sum_{j=0}^{\infty} (1/j!)\, x^j \tag{49.8}$$

is defined as an element of $\mathcal{A}$, as in Section 37. More precisely, the series on the right converges absolutely with respect to N, and

$$N(\exp(x)) \le \sum_{j=0}^{\infty} (1/j!)\, N(x^j) \le \sum_{j=0}^{\infty} (1/j!)\, N(x)^j = \exp(N(x)). \tag{49.9}$$

If $y \in \mathcal{A}$ commutes with x, then

$$\begin{aligned} \exp(x+y) &= \sum_{j=0}^{\infty} (1/j!)\, (x+y)^j \\ &= \sum_{j=0}^{\infty} \sum_{l=0}^{j} (1/l!)\, (1/(j-l)!)\, x^l\, y^{j-l} = \exp(x)\, \exp(y). \end{aligned} \tag{49.10}$$

The second step uses the binomial theorem, as usual. In the third step, we use the fact that the double sum corresponds to the Cauchy product of the series defining $\exp(x)$ and $\exp(y)$.

50 Another case

Let k be a field of characteristic 0 with an absolute value function $|\cdot|$. In this section, we suppose that the absolute value function induced on $\mathbf{Q}$ by the natural embedding of $\mathbf{Q}$ into k and $|\cdot|$ on k is the trivial absolute value function on $\mathbf{Q}$. In particular, this implies that $|\cdot|$ is non-archimedean on k, so that $|\cdot|$ is an ultrametric absolute value function on k, as in Section 44.

Let T be an indeterminate, so that $\exp(T)$ can be defined as a formal power series in T with coefficients in k as in (49.2). In this situation,

$$\|\exp(T)\|_{\infty,1} = 1, \tag{50.1}$$

where the left side is as defined in Section 31, with $V = k$ and $N = |\cdot|$. Similarly, if X and Y are commuting indeterminates, then $\exp(X+Y)$ is defined as a formal power series in X and Y with coefficients in k as in (49.4). It is easy to see that

$$\|\exp(X+Y)\|_{\infty,(1,1)} = 1, \tag{50.2}$$

where the left side is as defined as in Section 31 again. More precisely, this corresponds to taking $r = (1,1)$ as an ordered pair of positive real numbers in Section 31.

Of course,

$$\exp(T) - 1 = \sum_{j=1}^{\infty} (1/j!)\, T^j. \tag{50.3}$$

If r is a positive real number with $r \le 1$, then

$$\|\exp(T) - 1\|_{\infty,r} = r, \tag{50.4}$$

where the left side is as defined in Section 31 again.

Let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k and a multiplicative identity element e with $N(e) = 1$. If $x \in \mathcal{A}$, then

$$N((1/j!)\, x^j) = N(x^j) \le N(x)^j \tag{50.5}$$

for every nonnegative integer j. Suppose that $\mathcal{A}$ is complete with respect to the metric associated to N, and that

$$N(x) < 1. \tag{50.6}$$

Under these conditions, $\exp(x)$ can be defined as an element of $\mathcal{A}$, as in (49.8) and Section 37. This uses the fact that $\exp(T)$ is in the space $k_r^1[[T]]$ defined in Section 30 with $V = k$ when $0 < r < 1$, as in Section 32.

Suppose from now on in this section that N is an ultranorm on $\mathcal{A}$. In this case, we have that

$$N(\exp(x)) \le \max_{j \ge 0} N((1/j!)\, x^j) = 1, \tag{50.7}$$

using (50.5) in the second step. Similarly,

$$N(\exp(x) - e) = N\Big(\sum_{j=1}^{\infty} (1/j!)\, x^j\Big) \le \max_{j \ge 1} N((1/j!)\, x^j) = N(x), \tag{50.8}$$

using (50.5) in the third step. Suppose that $y \in \mathcal{A}$ also satisfies

$$N(y) < 1, \tag{50.9}$$

so that $\exp(y)$ is defined as an element of $\mathcal{A}$ too. Note that

$$N(x + y) \le \max(N(x), N(y)) < 1, \tag{50.10}$$

by the ultrametric version of the triangle inequality for N. Thus $\exp(x + y)$ is defined as an element of $\mathcal{A}$ as well. If $x\, y = y\, x$, then

$$\exp(x + y) = \exp(x)\, \exp(y), \tag{50.11}$$

as in (49.10).

51 Some basic estimates

Let p be a prime number, and let $|\cdot|_p$ be the p-adic absolute value function on $\mathbf{Q}$, as in Section 14. Also let j be a nonnegative integer, and let us review some basic estimates for

$$|1/j!|_p = 1/|j!|_p, \tag{51.1}$$

as in [1, 3]. If r is a nonnegative real number, then let $[r]$ be the integer part of r, which is the largest nonnegative integer less than or equal to r. Observe that for each $n \in \mathbf{Z}_+$,

$$[j/n] \tag{51.2}$$

is the same as the number of positive integer multiples of n that are less than or equal to j. Thus, for each $l \in \mathbf{Z}_+$,

$$[j/p^l] \tag{51.3}$$

is the same as the number of positive integer multiples of p^l less than or equal to j. Using this, one can check that

$$\sum_{l=1}^{\infty} [j/p^l] \tag{51.4}$$

is the same as the total number of factors of p in $j!$. Of course, (51.3) is equal to 0 when $j < p^l$, so that all but finitely many terms in (51.4) are equal to 0. It follows that (51.1) is equal to p raised to the power (51.4).

If $j \ge 1$, then

$$\sum_{l=1}^{\infty} [j/p^l] < \sum_{l=1}^{\infty} j/p^l = (j/p) \sum_{l=0}^{\infty} p^{-l} = (j/p)\,(1 - (1/p))^{-1} = j/(p-1). \tag{51.5}$$

This implies that

$$(p-1) \sum_{l=1}^{\infty} [j/p^l] < j. \tag{51.6}$$

Because the left side of the inequality is an integer, we get that

$$(p-1) \sum_{l=1}^{\infty} [j/p^l] \le j - 1 \tag{51.7}$$

when $j \ge 1$. Equivalently,

$$\sum_{l=1}^{\infty} [j/p^l] \le (j-1)/(p-1) \tag{51.8}$$

when $j \ge 1$. Thus

$$|1/j!|_p \le p^{(j-1)/(p-1)} \tag{51.9}$$

when $j \ge 1$, because the left side is equal to p to the power (51.4), as before.

Suppose for the moment that $j = p^n$ for some positive integer n. In this case,

$$\sum_{l=1}^{\infty} [j/p^l] = \sum_{l=1}^{n} p^{n-l} = \sum_{l=0}^{n-1} p^l = (p^n - 1)/(p-1). \tag{51.10}$$

This also works when $n = 0$, with the second and third sums interpreted as being equal to 0. This shows that equality holds in (51.8) in this situation. It follows that equality holds in (51.9) in this situation as well.

Put

$$r_p = p^{-1/(p-1)}, \tag{51.11}$$

which is a positive real number less than 1. Using (51.9), we get that

$$|1/j!|_p \, r_p^{j-1} \le 1 \tag{51.12}$$

when $j \ge 1$. If $j = p^n$ for some nonnegative integer n, then equality holds in (51.12), as in the preceding paragraph.

52 Some consequences

Let k be a field of characteristic 0 with an absolute value function $|\cdot|$ again. Suppose that the absolute value function induced on $\mathbf{Q}$ by the natural embedding of $\mathbf{Q}$ into k and $|\cdot|$ on k is the same as the p-adic absolute value function $|\cdot|$ on $\mathbf{Q}$ for some prime number p. Note that $|\cdot|$ is non-archimedian on k, and hence $|\cdot|$ is an ultrametric absolute value function on k, as in Section 44.

Let T be an indeterminate, and remember that $\exp(T)$ can be defined as a formal power series in T with coefficients in k as in (49.2). If r is a positive real number, then

$$\|\exp(T)\|_{\infty,r} = \sup_{j \ge 0}(|1/j!|_p \, r^j), \tag{52.1}$$

where the left side is as defined in Section 31, with $V = k$ and $N = |\cdot|$. Let r_p be as in (51.11), and remember that $0 < r_p < 1$. It is easy to see that

$$\|\exp(T)\|_{\infty,r_p} = 1, \tag{52.2}$$

using (51.12) and the usual convention that $0! = 1$. If X and Y are commuting indeterminates, then $\exp(X + Y)$ can be defined as a formal power series in X and Y with coefficients in k as in (49.4). Note that

$$\|\exp(X + Y)\|_{\infty,(r_p,r_p)} = 1, \tag{52.3}$$

where the left side is also defined as in Section 31. This can be verified directly from the definitions, using (51.12) again, or from the remarks in Section 47.

Similarly,

$$\|\exp(T) - 1\|_{\infty,r} = \sup_{j \ge 1}(|1/j!|_p \, r^j) \tag{52.4}$$

for every $r > 0$, where the left side is defined as in Section 31 again. If $r \le r_p$, then one can check that

$$\|\exp(T) - 1\|_{\infty,r} = r, \tag{52.5}$$

using (51.12).

Let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k and a multiplicative identity element e such that $N(e) = 1$, as before. In this case, if $x \in \mathcal{A}$, then

$$N((1/j!)x^j) = |1/j!|_p \, N(x^j) \le |1/j!|_p \, N(x)^j \tag{52.6}$$

for every nonnegative integer j. This implies that

$$N((1/j!)\, x^j) \le r_p^{1-j} \, N(x)^j \tag{52.7}$$

when $j \ge 1$, by (51.9). Suppose that $\mathcal{A}$ is complete with respect to the metric associated to N. If

$$N(x) < r_p, \tag{52.8}$$

then $\exp(x)$ can be defined as an element of $\mathcal{A}$, as in (49.8) and Section 37.

Suppose that N is an ultranorm on $\mathcal{A}$, so that

$$N(\exp(x)) \le \max_{j \ge 0} N((1/j!)\, x^j) \le \max_{j \ge 0}(|1/j!|_p \, N(x)^j) = 1, \tag{52.9}$$

using the inequalities in the previous paragraph. Moreover,

$$\begin{aligned} N(exp(x) - e) = N\Big(\sum_{j=1}^{\infty} (1/j!)\, x^j\Big) &\le \max_{j \ge 1} N((1/j!)\, x^j) \\ &\le \max_{j \ge 1}(r_p^{1-j} \, N(x)^j) = N(x). \end{aligned} \tag{52.10}$$

If $y \in \mathcal{A}$ satisfies

$$N(y) < r_p \tag{52.11}$$

too, then $\exp(y)$ can be defined as an element of $\mathcal{A}$ as well. In this situation,

$$N(x + y) \le \max(N(x), N(y)) < r_p, \tag{52.12}$$

so that $\exp(x+y)$ is also defined as an element of $\mathcal{A}$. If x and y commute, then $\exp(x + y)$ is equal to the product of $\exp(x)$ and $\exp(y)$, as before.

53 More on $\mathbf{Z}_p$

Let p be a prime number, and let $\mathbf{Z}_p$ be the set of p-adic integers, as in Section 45. If $x, y \in \mathbf{Z}_p$, then one can check that $x + y$ and $x\,y$ are elements of $\mathbf{Z}_p$ too, so that $\mathbf{Z}_p$ is a subring of $\mathbf{Q}_p$. Put

$$p^j \, \mathbf{Z}_p = \{p^j \, x : x \in \mathbf{Z}_p\} = \{y \in \mathbf{Q}_p : |y|_p \le p^{-j}\} \tag{53.1}$$

for each integer j, which is the same as $\mathbf{Z}_p$ when $j = 0$. If $j \ge 1$, then it is easy to see that $p^j \, \mathbf{Z}_p$ is an ideal in $\mathbf{Z}_p$. Of course,

$$p^j \, \mathbf{Z} = \{p^j \, x : x \in \mathbf{Z}\} \tag{53.2}$$

is an ideal in $\mathbf{Z}$ for each nonnegative integer j.

Let j be a positive integer, so that the quotient

$$\mathbf{Z}_p / p^j \, \mathbf{Z}_p \tag{53.3}$$

is defined as a commutative ring. The inclusion of $\mathbf{Z}$ in $\mathbf{Z}_p$ leads to a natural ring homomorphism from $\mathbf{Z}$ into (53.3). The kernel of this homomorphism is equal to

$$\mathbf{Z} \cap (p^j \, \mathbf{Z}_p) = p^j \, \mathbf{Z}. \tag{53.4}$$

Thus we get an injective ring homomorphism from

$$\mathbf{Z}/p^j \, \mathbf{Z} \tag{53.5}$$

into (53.3). One can verify that this homomorphism is also surjective, using the fact that $\mathbf{Z}$ is dense in $\mathbf{Z}_p$.

This shows that (53.3) is isomorphic to (53.5) as a ring. In particular, $\mathbf{Z}_p$ can be expressed as the union of p^j translates of $p^j \, \mathbf{Z}_p$. Each translate of $p^j \, \mathbf{Z}_p$ in $\mathbf{Q}_p$ is a closed ball in $\mathbf{Q}_p$ of radius p^{-j} with respect to the p-adic metric. Remember that a subset E of a metric space X is said to be *totally bounded* in X if for each $\epsilon > 0$, E can be covered by finitely many balls of radius ϵ. It follows that $\mathbf{Z}_p$ is totally bounded as a subset of $\mathbf{Q}_p$ with respect to the p-adic metric, because $j \in \mathbf{Z}_+$ is arbitrary.

It is well known that a subset E of a complete metric space X is compact in X if and only if E is closed and totally bounded in X. Of course, $\mathbf{Q}_p$ is complete with respect to the p-adic metric, by construction. Thus $\mathbf{Z}_p$ is compact in $\mathbf{Q}_p$, because $\mathbf{Z}_p$ is closed and totally bounded. Note that $p^l \, \mathbf{Z}_p$ is also compact in $\mathbf{Q}_p$ for every $l \in \mathbf{Z}$, because $x \mapsto p^l \, x$ is a continuous mapping on $\mathbf{Q}_p$. If $E \subseteq \mathbf{Q}_p$ is closed and bounded, then it follows that E is compact, because $E \subseteq p^l \, \mathbf{Z}_p$ when $-l$ is sufficiently large.

Part V
Binomial expansions

54 Binomial coefficient polynomials

Let k be a field of characteristic 0, and let A be an indeterminate. Put

$$b_j(A) = \binom{A}{j} = \frac{A\,(A-1)\cdots(A-j+1)}{j!} \tag{54.1}$$

for each positive integer j, as in Section 10. More precisely, this defines a formal polynomial in A with coefficients in k, using the natural embedding of $\mathbf{Q}$ into k. As before, (54.1) is interpreted as being the constant polynomial corresponding to 1 when $j = 0$.

Let $|\cdot|$ be an absolute value function on k. If r is a positive real number, then

$$\|\cdot\|_{1,r,k[A]} = \|\cdot\|_{1,r} \tag{54.2}$$

can be defined on the algebra $k[A]$ of formal polynomials in A with coefficients in k as in Section 29, with $V = k$ and $N = |\cdot|$. More precisely, this is a norm on $k[A]$ with respect to $|\cdot|$ on k. Suppose for the rest of the section that the absolute value function induced on $\mathbf{Q}$ by $|\cdot|$ on k and the natural embedding of $\mathbf{Q}$ into k is the standard Euclidean absolute value function on $\mathbf{Q}$. Observe that

$$\|b_j(A)\|_{1,r,k[A]} \le \frac{r\,(r+1)\cdots(r+j-1)}{j!} \tag{54.3}$$

for every positive integer j and $r > 0$. This follows from submultiplicativity of the norm (54.2) on $k[A]$, as in Section 36. If $j = 0$, then the left side of (54.3) is equal to 1, and the right side of (54.3) can be interpreted as being equal to 1 as well.

Put

$$B_j(r) = \frac{r\,(r+1)\cdots(r+j-1)}{j!} \tag{54.4}$$

for each $r > 0$ and $j \in \mathbf{Z}_+$, which is the same as the right side of (54.3). As before, we can interpret this as being equal to 1 for every $r > 0$ when $j = 0$. Note that

$$B_{j+1}(r) = B_j(r)\,(r+j)/(j+1) \tag{54.5}$$

for every $j \ge 0$ and $r > 0$. Thus

$$B_{j+1}(r)/B_j(r) = (r+j)/(j+1) \to 1 \quad \text{as } j \to \infty \tag{54.6}$$

for every $r > 0$. If t is a positive real number with $t < 1$, then it follows that

$$\sum_{j=0}^{\infty} B_j(r)\,t^j < \infty \tag{54.7}$$

for every $r > 0$, by the ratio test.

Let X be another indeterminate, which we can take to commute with A. As in Section 10,

$$B(A, X) = \sum_{j=0}^{\infty} \binom{A}{j} X^j = \sum_{j=0}^{\infty} b_j(A)\,X^j \tag{54.8}$$

may be considered as a formal power series in X with coefficients in $k[A]$. Let r be a positive real number, so that (54.2) defines a norm on $k[A]$, as before. If t is a positive real number, then

$$\|\cdot\|_{1,t,(k[A])[[X]]} = \|\cdot\|_{1,t} \tag{54.9}$$

can be defined on the space $(k[A])[[X]]$ of formal power series in X with coefficients in $k[A]$ as in Section 30, with $V = k[A]$ and N taken to be the norm

(54.2) on $k[A]$. Note that

$$\|B(A, X)\|_{1,t,(k[A])[[X]]} = \sum_{j=0}^{\infty} \|b_j(A)\|_{1,r,k[A]} \, t^j \le \sum_{j=0}^{\infty} B_j(r) \, t^j \tag{54.10}$$

for every $t > 0$, using the definition of (54.9) in the first step, and (54.3) in the second step. In particular, the right side is finite when $t < 1$, as in (54.7). This means that (54.8) is an element of the space $(k[A])^1_t[[X]]$ defined in Section 30 when $t < 1$.

Let $\mathcal{A}$ be an algebra over k with a submultiplicative norm N with respect to $|\cdot|$ on k, and a multiplicative identity element e with $N(e) = 1$. Also let $a \in \mathcal{A}$ be given, so that

$$b_j(a) = \binom{a}{j} = \frac{a\,(a-e)\,\cdots(a-(j-1)\,e)}{j!} \tag{54.11}$$

is defined as an element of $\mathcal{A}$ for each positive integer j. This is interpreted as being equal to e when $j = 0$, as usual. Note that

$$N(b_j(a)) \le B_j(N(a)) \tag{54.12}$$

for every $j \ge 0$, by (54.3) and the remarks in Section 29. As in Section 10,

$$B(a, X) = \sum_{j=0}^{\infty} \binom{a}{j} X^j = \sum_{j=0}^{\infty} b_j(a)\, X^j \tag{54.13}$$

may be considered as a formal power series in X with coefficients in $\mathcal{A}$. If $t \in \mathbf{R}_+$, then

$$\|\cdot\|_{1,t,\mathcal{A}[[X]]} = \|\cdot\|_{1,t} \tag{54.14}$$

can be defined on the space $\mathcal{A}[[X]]$ of formal power series in X with coefficients in $\mathcal{A}$, as in Section 30, and using N on $\mathcal{A}$. Using (54.12), we get that

$$\|B(a, X)\|_{1,t,\mathcal{A}[[X]]} = \sum_{j=0}^{\infty} N(b_j(a))\, t^j \le \sum_{j=0}^{\infty} B_j(N(a))\, t^j \tag{54.15}$$

for every $t > 0$. The sum on the right is finite when $t < 1$, as in (54.7). Thus (54.13) is an element of the space $\mathcal{A}^1_t[[X]]$ defined in Section 30 when $t < 1$.

55 An easy case

Let k be a field of characteristic 0, and let $|\cdot|$ be an absolute value function on k. In this section, we suppose that the absolute value function induced on $\mathbf{Q}$ by $|\cdot|$ on k and the natural embedding of $\mathbf{Q}$ into k is the trivial absolute value function on $\mathbf{Q}$. This implies that $|\cdot|$ is non-archimedean on k, so that $|\cdot|$ is an ultrametric absolute value function on k, as in Section 44.

Let A be an indeterminate, and let $b_j(A)$ be as in the previous section for each nonnegative integer j. Also let r be a positive real number, so that

$$\|\cdot\|_{\infty,r,k[A]} = \|\cdot\|_{\infty,r} \tag{55.1}$$

can be defined on the algebra $k[A]$ of formal polynomials in A with coefficients in k as in Section 29, with $V = k$ and $N = |\cdot|$. Note that (55.1) is an ultranorm on $k[A]$ with respect to $|\cdot|$ on k, because $|\cdot|$ is an ultrametric absolute value function on k. If $j \in \mathbf{Z}_+$, then

$$\|b_j(A)\|_{\infty,r,k[A]} = \prod_{l=0}^{j-1} \|A - l\|_{\infty,r,k[A]}. \tag{55.2}$$

This follows from the definition (54.1) of $b_j(A)$, and the multiplicativity of (55.1) on $k[A]$, as in Section 41. If $l \in \mathbf{Z}_+$, then

$$\|A - l\|_{\infty,r,k[A]} = \max(r, 1), \tag{55.3}$$

by the definition of (55.1), and the hypothesis on $|\cdot|$ in this section. Of course,

$$\|A\|_{\infty,r,k[A]} = r. \tag{55.4}$$

Thus

$$\|b_j(A)\|_{\infty,r,k[A]} = r\,\max(r, 1)^{j-1} \tag{55.5}$$

for each $j \in \mathbf{Z}_+$. If $j = 0$, then $b_j(A)$ is the constant polynomial corresponding to 1, so that the left side of (55.5) is equal to 1.

Let X be another indeterminate, which we take to commute with A, and let $B(A, X)$ be as in (54.8). This may be considered as a formal power series in X with coefficients in $k[A]$, as before. If $t \in \mathbf{R}_+$, then

$$\|\cdot\|_{\infty,t,(k[A])[[X]]} = \|\cdot\|_{\infty,t} \tag{55.6}$$

can be defined on the space $(k[A])[[X]]$ of formal power series in X with coefficients on $k[A]$ as in Section 31, with $V = k[A]$ and N equal to the ultranorm (55.1) on $k[A]$. By definition of (55.6),

$$\|B(A, X)\|_{\infty,t,(k[A])[[X]]} = \sup_{j \ge 0}(\|b_j(A)\|_{\infty,r,k[A]}\, t^j) \tag{55.7}$$

for every $t > 0$. Similarly,

$$B(A, X) - 1 = \sum_{j=1}^{\infty} \binom{A}{j} X^j = \sum_{j=1}^{\infty} b_j(A)\, X^j, \tag{55.8}$$

so that

$$\|B(A, X) - 1\|_{\infty,t,(k[A])[[X]]} = \sup_{j \ge 1}(\|b_j(A)\|_{\infty,r,k[A]}\, t^j) \tag{55.9}$$

for each $t > 0$.

Suppose for the moment that $r \ge 1$, so that

$$\|b_j(A)\|_{\infty,r,k[A]}\, t^j = r^j\, t^j = (r\, t)^j \tag{55.10}$$

for every $t > 0$ and $j \ge 0$, using (55.5) in the first step. If $t \le 1/r$, then it follows that

$$\|B(A,X)\|_{\infty,t,(k[A])[[X]]} = 1, \tag{55.11}$$

by (55.7). In particular, this means that $B(A,X)$ is an element of the space $(k[A])^{\infty}_t[[X]]$ defined in Section 31 when $t \le 1/r$. If $t < 1/r$, then $B(A,X)$ is an element of the space $(k[A])_{0,t}[[X]]$ defined in Section 31. We also get that

$$\|B(A,X) - 1\|_{\infty,t,(k[A])[[X]]} = r\, t \tag{55.12}$$

when $t \le 1/r$, using (55.9) and (55.10).

Suppose now that $r \le 1$. If $t > 0$, then

$$\begin{aligned} \|b_j(A)\|_{\infty,r,k[A]}\, t^j &= r\, t^j \quad \text{when } j \ge 1 \\ &= 1 \qquad \text{when } j = 0, \end{aligned} \tag{55.13}$$

by (55.5). This implies that (55.11) holds when $t \le 1$, and in particular that $B(A,X)$ is an element of $(k[A])^{\infty}_t[[X]]$. If $t < 1$, then $B(A,X)$ is an element of $(k[A])_{0,t}[[X]]$. Using (55.9) and (55.13), we obtain that (55.12) holds when $t \le 1$ as well.

56 The next case

Let k be a field of characteristic 0 again, and let $|\cdot|$ be an absolute value function on k. Suppose that the absolute value function induced on $\mathbf{Q}$ by $|\cdot|$ on k and the natural embedding of $\mathbf{Q}$ into k is the p-adic absolute value function $|\cdot|_p$ on $\mathbf{Q}$ for some prime number p. In particular, this means that $|\cdot|$ is non-archimedean on k, and hence is an ultrametric absolute value function on k, as in Section 44.

Let A be an indeterminate, and let $b_j(A)$ be as in Section 54 for each non-negative integer j. Also let $r \in \mathbf{R}_+$ be given, so that

$$\|\cdot\|_{\infty,r,k[A]} = \|\cdot\|_{\infty,r} \tag{56.1}$$

can be defined on the algebra $k[A]$ of formal polynomials in A with coefficients in k as in Section 29, with $V = k$ and $N = |\cdot|$. More precisely, this is an ultranorm on $k[A]$ with respect to $|\cdot|$ on k, because $|\cdot|$ is an ultrametric absolute value function on k. If $j \in \mathbf{Z}_+$, then

$$\|b_j(A)\|_{\infty,r,k[A]} = (1/|j!|_p) \prod_{l=0}^{j-1} \|A - l\|_{\infty,r,k[A]}. \tag{56.2}$$

This uses the definition (54.1) of $b_j(A)$, and the fact that (56.1) is multiplicative on $k[A]$, as in Section 41. Note that

$$\|A - l\|_{\infty,r,k[A]} = \max(r, |l|_p) \tag{56.3}$$

for every nonnegative integer l, by the definition of $\|\cdot\|_{\infty,r,k[A]}$. It follows that

$$\|b_j(A)\|_{\infty,r,k[A]} = (1/|j!|_p) \prod_{l=0}^{j-1} \max(r,|l|_p) \tag{56.4}$$

for every $j \in \mathbf{Z}_+$. As before, the left side of (56.4) is equal to 1 when $j = 0$.

Let X be another indeterminate again, which we take to commute with A, and let $B(A,X)$ be as in (54.8). This may be considered as a formal power series in X with coefficients in $k[A]$, as usual. If $t \in \mathbf{R}_+$, then

$$\|\cdot\|_{\infty,t,(k[A])[[X]]} = \|\cdot\|_{\infty,t} \tag{56.5}$$

can be defined on the space $(k[A])[[X]]$ of formal power series in X with coefficients in $k[A]$ as in Section 31, with $V = k[A]$ and N taken to be $\|\cdot\|_{\infty,r,k[A]}$ on $k[A]$. As before,

$$\|B(A,X)\|_{\infty,t,(k[A])[[X]]} = \sup_{j\geq 0}(\|b_j(A)\|_{\infty,r,k[A]}\, t^j) \tag{56.6}$$

for every $t > 0$. Similarly,

$$\|B(A,X)-1\|_{\infty,t,(k[A])[[X]]} = \sup_{j\geq 1}(\|b_j(A)\|_{\infty,r,k[A]}\, t^j) \tag{56.7}$$

for every $t > 0$.

Suppose that $r \geq 1$, so that

$$\|b_j(A)\|_{\infty,r,k[A]} = r^j/|j!|_p \tag{56.8}$$

for every nonnegative integer j, by (56.4). This implies that

$$\|B(A,X)\|_{\infty,t,(k[A])[[X]]} = \sup_{j\geq 0}((r\,t)^j/|j!|_p) \tag{56.9}$$

for every $t > 0$, by (56.6). Remember that

$$1/|j!|_p \leq p^{(j-1)/(p-1)} \leq p^{j/(p-1)} \tag{56.10}$$

for every $j \in \mathbf{Z}_+$, as in Section 51. If $t \leq p^{-1/(p-1)}/r$, then it follows that

$$\|B(A,X)\|_{\infty,t,(k[A])[[X]]} = 1, \tag{56.11}$$

and in particular that $B(A,X)$ is an element of the space $(k[A])^\infty_t[[X]]$ defined in Section 31. If $t < p^{-1/(p-1)}/r$, then $B(A,X)$ is an element of the space $(k[A])_{0,t}[[X]]$ defined in Section 31.

Using (56.7) and (56.8), we get that

$$\|B(A,X)-1\|_{\infty,t,(k[A])[[X]]} = \sup_{j\geq 1}((r\,t)^j/|j!|_p) = r\,t \sup_{j\geq 1}((r\,t)^{j-1}/|j!|_p) \tag{56.12}$$

for every $t > 0$. If $t \leq p^{-1/(p-1)}/r$, then the first inequality in (56.10) implies that

$$\|B(A,X)-1\|_{\infty,t,(k[A])[[X]]} = r\,t. \tag{56.13}$$

This also uses the fact that equality holds in the first step in (56.10) when $j = 1$.

57 A related situation

Let p be a prime number, and let us take $k = \mathbf{Q}_p$, with the p-adic absolute value $|\cdot|_p$. If $a \in \mathbf{Q}_p$ and $j \in \mathbf{Z}_+$, then

$$b_j(a) = \binom{a}{j} = \frac{a\,(a-1)\cdots(a-j+1)}{j!} \tag{57.1}$$

defines an element of $\mathbf{Q}_p$. As before, this is interpreted as being equal to 1 for every $a \in \mathbf{Q}_p$ when $j = 0$. Note that $b_j(a)$ defines a continuous mapping from $\mathbf{Q}_p$ into itself for each $j \geq 0$, with respect to the p-adic metric on $\mathbf{Q}_p$. Of course, if a is a nonnegative integer, then it is well known that (57.1) is a nonnegative integer as well.

Remember that $\mathbf{Z}_p$ consists of $x \in \mathbf{Q}_p$ with $|x|_p \leq 1$, as in Section 45. Equivalently, $\mathbf{Z}_p$ is the closure of $\mathbf{Z}$ in $\mathbf{Q}_p$ with respect to the p-adic metric, as before. More precisely, we have seen that $\mathbf{Z}_p$ is the same as the closure of $\mathbf{Z}_+$ in $\mathbf{Q}_p$. If $a \in \mathbf{Z}_p$, then it follows that

$$b_j(a) \in \mathbf{Z}_p \tag{57.2}$$

for every $j \geq 0$. This uses the continuity of b_j on $\mathbf{Q}_p$, as in the previous paragraph.

Let X be an indeterminate, so that

$$B(a, X) = \sum_{j=0}^{\infty} \binom{a}{j} X^j = \sum_{j=0}^{\infty} b_j(a)\, X^j \tag{57.3}$$

defines a formal power series in X with coefficients in $\mathbf{Q}_p$ for every $a \in \mathbf{Q}_p$. If r is a positive real number, then let

$$\|\cdot\|_{\infty, r, \mathbf{Q}_p[[X]]} = \|\cdot\|_{\infty, r} \tag{57.4}$$

be defined on the space $\mathbf{Q}_p[[X]]$ of formal power series in X with coefficients in $\mathbf{Q}_p$ as in Section 31, with $V = \mathbf{Q}_p$ and $N = |\cdot|_p$. If $a \in \mathbf{Z}_p$, then it is easy to see that

$$\|B(a, X)\|_{\infty, 1, \mathbf{Q}_p[[X]]} = 1, \tag{57.5}$$

using (57.2). In particular, (57.3) is an element of the space $(\mathbf{Q}_p)_1^\infty[[X]]$ defined in Section 31 when $a \in \mathbf{Z}_p$. Similarly, if $a \in \mathbf{Z}_p$ and $0 < r \leq 1$, then we have that

$$\|B(a, X) - 1\|_{\infty, r, \mathbf{Q}_p[[X]]} \leq r. \tag{57.6}$$

Let $\mathcal{A}$ be an algebra over $\mathbf{Q}_p$ with a submultiplicative norm N with respect to $|\cdot|_p$ and a multiplicative identity element e such that $N(e) = 1$. If $a \in \mathbf{Q}_p$ and $x \in \mathcal{A}$, then

$$N(b_j(a)\, x^j) = |b_j(a)|_p\, N(x^j) \leq |b_j(a)|_p\, N(x)^j \tag{57.7}$$

for every nonnegative integer j. Thus

$$N(b_j(a)\, x^j) \le N(x)^j \tag{57.8}$$

for every $j \ge 0$ when $a \in \mathbf{Z}_p$. Suppose that $\mathcal{A}$ is complete with respect to the metric associated to N, and let $x \in \mathcal{A}$ be given, with $N(x) < 1$. If $a \in \mathbf{Z}_p$, then

$$B(a, x) = \sum_{j=0}^{\infty} \binom{a}{j} x^j = \sum_{j=0}^{\infty} b_j(a)\, x^j \tag{57.9}$$

defines an element of $\mathcal{A}$, as in Section 37. This also uses the remarks in Section 32 to get that (57.3) is an element of the space $(\mathbf{Q}_p)^1_r[[X]]$ defined in Section 30 with $V = \mathbf{Q}_p$ when $0 < r < 1$. If N is an ultranorm on $\mathcal{A}$, then

$$N(B(a, x) - e) = N\Big(\sum_{j=1}^{\infty} b_j(a)\, x^j\Big) \le \max_{j \ge 1} N(b_j(a)\, x^j) \le N(x), \tag{57.10}$$

using (57.8) in the third step.

58 Continous functions as coefficients

Let p be a prime number, and let $C(\mathbf{Z}_p, \mathbf{Q}_p)$ be the space of continuous mappings from $\mathbf{Z}_p$ into $\mathbf{Q}_p$, as before. Of course, this uses the p-adic metric on $\mathbf{Q}_p$, and its restriction to $\mathbf{Z}_p$. Remember that $\mathbf{Z}_p$ is compact, as in Section 53, so that continuous mappings from $\mathbf{Z}_p$ into $\mathbf{Q}_p$ are automatically bounded. Thus the supremum norm

$$\|f\|_{sup} = \|f\|_{sup, \mathbf{Z}_p} = \|f\|_{C(\mathbf{Z}_p, \mathbf{Q}_p)} \tag{58.1}$$

can be defined on $C(\mathbf{Z}_p, \mathbf{Q}_p)$ in the usual way, using the p-adic absolute value function $|\cdot|_p$ on $\mathbf{Q}_p$. Note that $C(\mathbf{Z}_p, \mathbf{Q}_p)$ is a commutative algebra over $\mathbf{Q}_p$ with respect to pointwise multiplication of functions.

As in the previous section, (57.1) defines a continuous mapping from $\mathbf{Q}_p$ into itself for each nonnegative integer j. In this section, it is convenient to consider the restriction of this mapping to $\mathbf{Z}_p$, which defines an element b_j of $C(\mathbf{Z}_p, \mathbf{Q}_p)$. It is easy to see that

$$\|b_j\|_{C(\mathbf{Z}_p, \mathbf{Q}_p)} = 1 \tag{58.2}$$

for every $j \ge 0$, because of (57.2) and the fact that $b_j(j) = 1$. Of course, b_0 is interpreted as the constant function equal to 1 on $\mathbf{Z}_p$. This is the same as the multiplicative identity element in $C(\mathbf{Z}_p, \mathbf{Q}_p)$.

Let X be an indeterminate, and let us consider

$$B(\cdot, X) = \sum_{j=0}^{\infty} b_j(\cdot)\, X^j \tag{58.3}$$

as a formal power series in X with coefficients in $C(\mathbf{Z}_p, \mathbf{Q}_p)$. If r is a positive real number, then let

$$\|\cdot\|_{\infty, r, C(\mathbf{Z}_p, \mathbf{Q}_p)[[X]]} = \|\cdot\|_{\infty, r} \tag{58.4}$$

be defined on the space $C(\mathbf{Z}_p, \mathbf{Q}_p)[[X]]$ of formal power series in X with coefficients in $C(\mathbf{Z}_p, \mathbf{Q}_p)$ as in Section 31, with $V = C(\mathbf{Z}_p, \mathbf{Q}_p)$ and N taken to be the supremum norm (58.1). Clearly

$$\|B(\cdot, X)\|_{\infty,1,C(\mathbf{Z}_p,\mathbf{Q}_p)} = 1, \tag{58.5}$$

by (58.2). Thus (58.3) is an element of the space $C(\mathbf{Z}_p, \mathbf{Q}_p)_1^\infty[[X]]$ defined in Section 31. If $0 < r \le 1$, then

$$\|B(\cdot, X) - 1\|_{\infty,r,C(\mathbf{Z}_p,\mathbf{Q}_p)} = r. \tag{58.6}$$

Let $\mathcal{A}$ be an algebra over $\mathbf{Q}_p$ with a submultiplicative norm N with respect to $|\cdot|_p$ and a multiplicative identity element e with $N(e) = 1$. Consider the space $C(\mathbf{Z}_p, \mathcal{A})$ of continuous mappings from $\mathbf{Z}_p$ into $\mathcal{A}$. This uses the restriction of the p-adic metric to $\mathbf{Z}_p$, and the metric on $\mathcal{A}$ associated to N. As before, continuous mappings from $\mathbf{Z}_p$ into $\mathcal{A}$ are automatically bounded, because $\mathbf{Z}_p$ is compact. Let

$$\|f\|_{sup} = \|f\|_{sup,\mathbf{Z}_p} = \|f\|_{C(\mathbf{Z}_p,\mathcal{A})} \tag{58.7}$$

be the supremum norm on $C(\mathbf{Z}_p, \mathcal{A})$, corresponding to N on $\mathcal{A}$. If j is a nonnegative integer and $x \in \mathcal{A}$, then $b_j\, x^j$ defines an element of $C(\mathbf{Z}_p, \mathcal{A})$. It is easy to see that

$$\|b_j\, x^j\|_{C(\mathbf{Z}_p,\mathcal{A})} = \|b_j\|_{C(\mathbf{Z}_p,\mathbf{Q}_p)}\, N(x^j) \le N(x)^j \tag{58.8}$$

for every $j \ge 0$.

Suppose that $\mathcal{A}$ is complete with respect to the metric associated to N. This implies that $C(\mathbf{Z}_p, \mathcal{A})$ is complete with respect to the metric associated to the supremum norm, as usual. Let $x \in \mathcal{A}$ be given, with $N(x) < 1$, and observe that

$$B(\cdot, x) = \sum_{j=0}^{\infty} b_j\, x^j \tag{58.9}$$

defines an element of $C(\mathbf{Z}_p, \mathcal{A})$, because the series on the right converges absolutely with respect to the supremum norm, by (58.8). More precisely, the right side of (58.9) converges in $C(\mathbf{Z}_p, \mathcal{A})$ with respect to the supremum norm, which means that the partial sums converge uniformly on $\mathbf{Z}_p$. If $a \in \mathbf{Z}_p$, then let $B(a, x)$ be the value of (58.9) at a, as an $\mathcal{A}$-valued function on $\mathbf{Z}_p$. This is equivalent to the definition of $B(a, x)$ in the previous section, because uniform convergence implies pointwise convergence. If N is an ultranorm on $\mathcal{A}$, then the corresponding supremum norm on $C(\mathbf{Z}_p, \mathbf{Q}_p)$ is an ultranorm as well. In this case, we have that

$$\begin{aligned} \|B(\cdot, x) - e\|_{C(\mathbf{Z}_p,\mathcal{A})} &= \Big\|\sum_{j=1}^{\infty} b_j\, x^j\Big\|_{C(\mathbf{Z}_p,\mathcal{A})} \\ &\le \max_{j\ge 1} \|b_j\, x^j\|_{C(\mathbf{Z}_p,\mathcal{A})} = N(x), \end{aligned} \tag{58.10}$$

by (58.8).

59 Adding exponents

Let k be a field of characteristic 0, and let A_1, A_2 be commuting indeterminates. If j is a positive integer, then

$$\begin{aligned}(59.1)\quad b_j(A_1 + A_1) &= \binom{A_1 + A_2}{j} \\ &= \frac{(A_1 + A_1)\,(A_1 + A_2 - 1)\cdots(A_1 + A_2 - j + 1)}{j!}\end{aligned}$$

may be considered as a formal polynomial in A_1 and A_2 with coefficients in k, using the natural embedding of $\mathbf{Q}$ into k. This is interpreted as being the constant polynomial corresponding to 1 when $j = 0$, as usual.

Let $|\cdot|$ be an absolute value function on k, and let $\rho = (\rho_1, \rho_2)$ be an ordered pair of positive real numbers. Consider the norm

$$\|\cdot\|_{1,\rho,k[A_1,A_2]} = \|\cdot\|_{1,\rho} \tag{59.2}$$

defined on the algebra $k[A_1, A_2]$ of formal polynomials in A_1 and A_2 with coefficients in k as in Section 29, with $V = k$ and $N = |\cdot|$. Suppose for the rest of the section that the absolute value function induced on $\mathbf{Q}$ by $|\cdot|$ on k and the natural embedding of $\mathbf{Q}$ into k is the standard absolute value function on $\mathbf{Q}$. Put

$$B_j(r) = \frac{r\,(r+1)\cdots(r+j-1)}{j!} \tag{59.3}$$

for every $r \in \mathbf{R}_+$ and $j \in \mathbf{Z}_+$, as in Section 54. This is interpreted as being equal to 1 for every $r > 0$ when $j = 0$, as before. It is easy to see that

$$\|b_j(A_1 + A_2)\|_{1,\rho,k[A_1,A_2]} \le B_j(\rho_1 + \rho_2) \tag{59.4}$$

for every $j \ge 0$. This uses the submultiplicativity of (59.2) on $k[A_1, A_2]$, as in Section 36. Alternatively, if A is another indeterminate, then

$$\|b_j(A_1 + A_2)\|_{1,\rho,k[A_1,A_2]} = \|b_j(A)\|_{1,\rho_1+\rho_2,k[A]} \tag{59.5}$$

for every $j \ge 0$, as in Section 47. One can use this to get (59.4) from the analogous statement for $b_j(A)$ in Section 54.

Let X be another indeterminate, which we can take to commute with A_1 and A_2. We may consider

$$B(A_1 + A_2, X) = \sum_{j=0}^{\infty} \binom{A_1 + A_2}{j} = \sum_{j=0}^{\infty} b_j(A_1 + A_2)\, X^j \tag{59.6}$$

as a formal power series in X with coefficients in $k[A_1, A_2]$. Of course, $B(A_1, X)$ and $B(A_2, X)$ can be defined as formal power series in X with coefficients in $k[A_1]$ and $k[A_2]$, respectively, as in Sections 10 and 54. In particular, they can both be considered as formal power series in X with coefficients in $k[A_1, A_2]$. Remember that

$$B(A_1 + A_2, X) = B(A_1, X)\, B(A_2, X) \tag{59.7}$$

as formal power series with coefficients in $k[A_1, A_2]$, as in Section 11.

Let t be a positive real number, and let

$$\| \cdot \|_{1,t,(k[A_1,A_2])[[X]]} = \| \cdot \|_{1,t} \tag{59.8}$$

be defined on the space $(k[A_1, A_2])[[X]]$ of formal power series in X with coefficients in $k[A_1, A_2]$ as in Section 30, with $V = k[A_1, A_2]$ and N taken to be (59.2). Observe that

$$\begin{aligned} \|B(A_1 + A_2, X)\|_{1,t,(k[A_1+A_2])[[X]]} &= \sum_{j=0}^{\infty} \|b_j(A_1 + A_2)\|_{1,\rho,k[A_1,A_2]}\, t^j \\ &\le \sum_{j=0}^{\infty} B_j(\rho_1 + \rho_2)\, t^j, \end{aligned} \tag{59.9}$$

using the definition of (59.8) in the first step, and (59.4) in the second step. If $t < 1$, then the right side of (59.9) is finite, by (54.7). Thus (59.6) is an element of the space $(k[A_1, A_2])^1_t[[X]]$ defined in Section 30 when $t < 1$.

Let $\mathcal{A}$ be an algebra over k with a multiplicative identity element e. If $a \in \mathcal{A}$, then $b_j(a)$ can be defined as an element of $\mathcal{A}$ for every nonnegative integer j, as in Sections 10 and 54. Similarly, $B(a, X)$ is defined as a formal power series in X with coefficients in $\mathcal{A}$, as before. If a_1, a_2 are commuting elements of $\mathcal{A}$, then

$$B(a_1 + a_2, X) = B(a_1, X)\, B(a_2, X), \tag{59.10}$$

as in Section 11. If N is a submultiplicative norm on $\mathcal{A}$ and $0 < t < 1$, then $B(a, X)$ is an element of the corresponding space $\mathcal{A}^1_t[[X]]$ defined in Section 30 for every $a \in \mathcal{A}$, as in Section 54.

60 Other cases

Let k be a field of characteristic 0, and let $|\cdot|$ be an absolute value function on k. Suppose that $|\cdot|$ is non-archimedean on k, so that $|\cdot|$ is an ultrametric absolute value function on k, as in Section 44. Let A_1 and A_2 be commuting indeterminates, and let $r \in \mathbf{R}_+$ be given. The norm

$$\| \cdot \|_{\infty,(r,r),k[A_1,A_2]} = \| \cdot \|_{\infty,(r,r)} \tag{60.1}$$

can be defined on the algebra $k[A_1, A_2]$ of formal polynomials in A_1 and A_2 with coefficients in k as in Section 29, with $V = k$, $N = |\cdot|$, and using (r, r) as the ordered pair of real numbers. More precisely, (60.1) is an ultranorm on $k[A_1, A_2]$, because $|\cdot|$ is non-archmedean on k.

Let A be another indeterminate, so that

$$\| \cdot \|_{\infty,r,k[A]} = \| \cdot \|_{\infty,r} \tag{60.2}$$

can be defined on the algebra $k[A]$ of formal polynomials in A with coefficients in k as in Section 29 too, with $V = k$ and $N = |\cdot|$. If j is a nonnegative integer, then $b_j(A)$ and $b_j(A_1 + A_2)$ can be defined as formal polynomials in A and in A_1, A_2, respectively, with coefficients in k, as before. Under these conditions,

$$\|b_j(A_1 + A_2)\|_{\infty,(r,r),k[A_1,A_2]} = \|b_j(A)\|_{\infty,r,k[A]} \tag{60.3}$$

for every $j \geq 0$, as in Section 47. Let X be another indeterminate, which we can take to commute with A_1, A_2, and A, and let $t \in \mathbf{R}_+$ be given. As before,

$$\| \cdot \|_{\infty,t,(k[A])[[X]]} = \| \cdot \|_{\infty,t} \tag{60.4}$$

can be defined on the space $(k[A])[[X]]$ of formal power series in X with coefficients in $k[A]$ as in Section 31, with $V = k[A]$ and N equal to (60.2). Similarly,

$$\| \cdot \|_{\infty,t,(k[A_1,A_2])[[X]]} = \| \cdot \|_{\infty,t} \tag{60.5}$$

can be defined on the space $(k[A_1, A_2])[[X]]$ of formal power series in X with coefficients in $k[A_1, A_2]$ as in Section 31, with $V = k[A_1, A_2]$ and N equal to (60.1). Remember that $B(A, X)$ is defined as a formal power series in X with coefficients in $k[A]$ as in Sections 10 and 54, and that $B(A_1 + A_2, X)$ can be defined as a formal power series in X with coefficients in $k[A_1, A_2]$, as in the previous section. It is easy to see that

$$\|B(A_1 + A_2, X)\|_{\infty,t,(k[A_1,A_2])[[X]]} = \|B(A, X)\|_{\infty,t,(k[A])[[X]]}, \tag{60.6}$$

using (60.3) and the definitions of these norms.

Suppose for the moment that the absolute value function induced on $\mathbf{Q}$ by $|\cdot|$ on k and the natural embedding of $\mathbf{Q}$ into k is the trivial absolute value function on $\mathbf{Q}$, as in Section 55. If $r \geq 1$ and $t \leq 1/r$, then

$$\|B(A_1 + A_2)\|_{\infty,t,(k[A_1,A_2])[[X]]} = 1, \tag{60.7}$$

by (60.6) and the analogous statement for $B(A, X)$ in Section 55. Similarly, (60.7) holds when $r, t \leq 1$, because of the analogous statement for $B(A, X)$ in Section 55.

Suppose now that the absolute value function induced on $\mathbf{Q}$ by $|\cdot|$ on k and the natural embedding of $\mathbf{Q}$ into k is the p-adic absolute value function $|\cdot|_p$ on $\mathbf{Q}$ for some prime number p. If $r \geq 1$ and $t \leq p^{-1/(p-1)}/r$, then (60.7) holds, because of (60.6) and the analogous statement for $B(A, X)$ in Section 56.

Let us now simply take $k = \mathbf{Q}_p$ for some prime number p, with the p-adic absolute value $|\cdot|_p$. If $a \in \mathbf{Q}_p$, then $B(a, X)$ may be considered as a formal power series in X with coefficients in $\mathbf{Q}_p$, as before. In particular, if $a \in \mathbf{Z}_p$, then $B(a, X)$ is a formal power series in X with coefficients in $\mathbf{Z}_p$, as in Section 57. Remember that

$$B(a_1 + a_2, X) = B(a_1, X)\, B(a_2, X) \tag{60.8}$$

for every $a_1, a_2 \in \mathbf{Q}_p$, as in Section 11. Let $\mathcal{A}$ be an algebra over $\mathbf{Q}_p$ with a submultiplicative norm N with respect to $|\cdot|_p$ and a multiplcative identity element e with $N(e)$, and suppose that $\mathcal{A}$ is complete with respect to the metric associated to N. Let $x \in \mathcal{A}$ be given, with $N(x) < 1$, so that $B(a, x)$ can be defined as an element of $\mathcal{A}$ when $a \in \mathbf{Z}_p$, as in Section 57. Also let $a_1, a_2 \in \mathbf{Z}_p$ be given, so that $a_1 + a_2 \in \mathbf{Z}_p$, as in Section 53. Under these conditions, we have that

$$B(a_1 + a_2, x) = B(a_1, x)\, B(a_2, x), \tag{60.9}$$

as in Section 24.

61 Commutativity conditions

Let k be a field of characteristic 0 again, and let $|\cdot|$ be an absolute value function on k. Also let $\mathcal{A}$ be an algebra over k with a submultiplicative N and a multiplicative identity element e, and suppose that $\mathcal{A}$ is complete with respect to the metric associated to N. If $a, x \in \mathcal{A}$, then one might like to define

$$B(a, x) = \sum_{j=0}^{\infty} \binom{a}{j} x^j = \sum_{j=0}^{\infty} b_j(a)\, x^j \tag{61.1}$$

as an element of $\mathcal{A}$, under suitable conditions. In particular, we have seen situations in which the convergence of this series can be obtained by estimating $N(b_j(a))$ and asking that $N(x)$ satisfy a corresponding restriction. However, basic properties of the sum can involve additional commutativity conditions.

If $a_1, a_2, x \in \mathcal{A}$, then

$$\begin{aligned} B(a_1, x)\, B(a_2, x) &= \Big(\sum_{j=0}^{\infty} b_j(a_1)\, x^j\Big) \Big(\sum_{l=0}^{\infty} b_l(a_2)\, x^l\Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} b_j(a_1)\, x^j\, b_{n-j}(a_2)\, x^{n-j}\Big) \end{aligned} \tag{61.2}$$

formally, where the right side is the Cauchy product of the series corresponding to$B(a_1, x)$ and $B(a_2, x)$. Remember that the second step in (61.2) holds under suitable conditions, as in Section 24. If a_2 commutes with x, then $b_l(a_2)$ commutes with x^j for all $j, l \geq 0$, so that the right side of (61.2) reduces to

$$\sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} b_j(a_1)\, b_{n-j}(a_2)\Big)\, x^n. \tag{61.3}$$

If a_1 commutes with a_2 as well, then this is the same as

$$\sum_{n=0}^{\infty} b_n(a_1 + a_2)\, x^n, \tag{61.4}$$

as in Section 11. Of course, this sum corresponds to $B(a_1 + a_2, x)$, as in (61.1).

In particular, if a_1 and a_2 are multiples of e by elements of k, then a_1 and a_2 commute with each other, and with every element of $\mathcal{A}$. In this case, one might as well take a_1 and a_2 to be elements of k, and consider these power series as having coefficients in k.

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