Some topics related to metrics and norms, including ultrametrics and ultranorms, 5

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Abstract

These informal notes deal with the strong operator topology for spaces of linear mappings between vector spaces over fields with absolute value functions.

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Part I
Preliminaries

1 q-Semimetrics

Let us begin by reviewing some simple inequalities. If q is a positive real number and a, b are nonnegative real numbers, then

$$\max(a, b) \le (a^q + b^q)^{1/q}. \tag{1.1}$$

If $0 < q_1 \le q_2$ and $a, b \ge 0$, then it follows that

$$\begin{aligned} a^{q_2} + b^{q_2} &\le \max(a, b)^{q_2 - q_1} (a^{q_1} + b^{q_1}) \\ &\le (a^{q_1} + b^{q_1})^{(q_2 - q_1)/q_1 + 1} = (a^{q_1} + b^{q_1})^{q_2/q_1}, \end{aligned} \tag{1.2}$$

using (1.1) in the second step. Equivalently, this means that

$$(a^{q_2} + b^{q_2})^{1/q_2} \le (a^{q_1} + b^{q_1})^{1/q_1} \tag{1.3}$$

for every $a, b \ge 0$ when $q_1 \le q_2$. We also have that

$$(a^q + b^q)^{1/q} \le 2^{1/q} \max(a, b) \tag{1.4}$$

for every $a, b \ge 0$ and $q > 0$, so that

$$\lim_{q \to \infty} (a^q + b^q)^{1/q} = \max(a, b) \tag{1.5}$$

for each $a, b \ge 0$, using (1.1) again too.

Let X be a set, and let $d(x, y)$ be a nonnegative real-valued function defined for $x, y \in X$. Suppose that

$$d(x, x) = 0 \tag{1.6}$$

for every $x \in X$, and that

$$d(x, y) = d(y, x) \tag{1.7}$$

for every $x, y \in X$. We say that $d(x, y)$ is a *q-semimetric* on X for some positive real number q if

$$d(x, z)^q \le d(x, y)^q + d(y, z)^q \tag{1.8}$$

for every $x, y, z \in X$. If this holds with $q = 1$, then we may simply say that $d(x, y)$ is a *semimetric* on X. Thus $d(x, y)$ is a q-semimetric on X for some $q > 0$ if and only if $d(x, y)^q$ is an ordinary semimetric on X.

Of course, (1.8) is the same as saying that

$$d(x, z) \le (d(x, y)^q + d(y, z)^q)^{1/q} \tag{1.9}$$

for every $x, y, z \in X$. Let us say that $d(x, y)$ is a *semi-ultrametric* on X if it satisfies

$$d(x, z) \le \max(d(x, y), d(y, z)) \tag{1.10}$$

for every $x, y, z \in X$, in addition to (1.6) and (1.7). This may be considered as a q-semimetric with $q = \infty$, because of (1.5). If $0 < q_1 \leq q_2 \leq \infty$ and $d(\cdot, \cdot)$ is a q_2-semimetric on X, then $d(\cdot, \cdot)$ is a q_1-semimetric on X as well. This uses (1.3) when $q_2 < \infty$, and (1.1) when $q_2 = \infty$.

A q-semimetric $d(x, y)$ on X is said to be a *q-metric* if

$$d(x, y) > 0 \tag{1.11}$$

for every $x, y \in X$ with $x \neq y$. This is also known as a *metric* on X when $q = 1$. Similarly, a semi-ultrametric $d(x, y)$ on X that satisfies (1.11) is known as an *ultrametric* on X, which corresponds to $q = \infty$. The *discrete metric* on any set X is defined by putting

$$d(x, y) = 1 \tag{1.12}$$

for every $x, y \in X$ with $x \neq y$, and using (1.6) when $x = y$. It is easy to see that the discrete metric on X is an ultrametric on X.

2 q-Absolute value functions

Let k be a field, and let $|\cdot|$ be a nonnegative real-valued function on k. Suppose that

$$|x| = 0 \quad \text{if and only if} \quad x = 0, \tag{2.1}$$

and that

$$|x\, y| = |x|\, |y| \tag{2.2}$$

for every $x, y \in k$. Using this, one can check that

$$|1| = 1, \tag{2.3}$$

where the 1 on the left side of (2.3) refers to the multiplicative identity element in k, and the 1 on the right side of (2.3) is the positive real number. More precisely, (2.3) uses the facts that $|1| > 0$, by (2.1), and $1^2 = 1$ in k. Similarly,

$$|x| = 1 \tag{2.4}$$

for every $x \in k$ that satisfies $x^n = 1$ for some positive integer n, which holds in particular with $n = 2$ when $x = -1$ in k.

Let us say that $|\cdot|$ is a *q-absolute value function* on k for some positive real number q if

$$|x + y|^q \leq |x|^q + |y|^q \tag{2.5}$$

for every $x, y \in k$, in addition to (2.1) and (2.2). If (2.5) holds with $q = 1$, then we say that $|\cdot|$ is an *absolute value function* on k. Note that $|x|$ is a q-absolute value function on k if and only if $|x|^q$ is an absolute value function on k. If $|\cdot|$ is a q-absolute value function on k for some $q > 0$, then

$$d(x, y) = |x - y| \tag{2.6}$$

defines a q-metric on k. This uses (2.1) to get (1.6) and (1.11), and (2.4) with $x = -1$ to get (1.7).

As before, (2.5) is equivalent to asking that

$$(2.7) \qquad |x + y| \le (|x|^q + |y|^q)^{1/q}$$

for every $x, y \in k$. If

$$(2.8) \qquad |x + y| \le \max(|x|, |y|)$$

for every $x, y \in k$, in addition to (2.1) and (2.2), then $|\cdot|$ is said to be an *ultrametric absolute value function* on k. In this case, (2.6) is an ultrametric on $|\cdot|$. An ultrametric absolute value function on k may be considered as a q-absolute value function with $q = \infty$, because of (1.5). If $0 < q_1 \le q_2 \le \infty$ and $|\cdot|$ is a q_2-absolute value function on k, then $|\cdot|$ is a q_1-absolute value function on k too, because of (1.1) and (1.3).

The standard absolute value functions on the fields $\mathbf{R}$, $\mathbf{C}$ are absolute value functions in the sense described in this section. It follows that they are also q-absolute value functions when $0 < q \le 1$, and it is easy to see that they are not q-absolute value functions when $q > 1$. The *trivial absolute value function* is defined on any field k by putting $|0| = 0$ and $|x| = 1$ for every $x \in k$ with $x \ne 0$. This is an ultrametric absolute value function on k, for which the corresponding ultrametric as in (2.6) is the discrete metric on k. Suppose for the moment that $|\cdot|$ is a q-absolute value function on a field k for some $q > 0$, and that $|\cdot|$ is not trivial on k. This means that there is an $x \in k$ such that $x \ne 0$ and $|x| \ne 1$. It follows that there are $y, z \in k$ such that $0 < |y| < 1$ and $|z| > 1$, using x and $1/x$.

3 q-Seminorms

Let k be a field, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. Also let V be a vector space over k, and let N be a nonnegative real-valued function on V that satisfies

$$(3.1) \qquad N(t\,v) = |t|\,N(v)$$

for every $v \in V$ and $t \in k$. Note that this implies that $N(0) = 0$, by taking $t = 0$. We say that N is a *q-seminorm* on V with respect to $|\cdot|$ on k for some positive real number q if

$$(3.2) \qquad N(v + w)^q \le N(v)^q + N(w)^q$$

for every $v, w \in V$. If $q = 1$, then we may simply say that N is a *seminorm* on V.

As usual, (3.2) is the same as saying that

$$(3.3) \qquad N(v + w) \le (N(v)^q + N(w)^q)^{1/q}$$

for every $v, w \in V$. A nonnegative real-valued function N on V that satisfies (3.1) and

$$N(v + w) \le \max(N(v), N(w)) \tag{3.4}$$

for every $v, w \in V$ is said to be a *semi-ultranorm* on V with respect to $|\cdot|$ on k. A semi-ultranorm on V may also be considered as a q-seminorm on V with $q = \infty$, because of (1.5). If $0 < q_1 \le q_2 \le \infty$ and N is a q_2-seminorm on V, then N is a q_1-seminorm on V as well, because of (1.1) and (1.3). If N is a q-seminorm on V for any $q > 0$, then

$$d(v, w) = d_N(v, w) = N(v - w) \tag{3.5}$$

defines a q-semimetric on V.

A q-seminorm N on V for some $q > 0$ is said to be a *q-norm* on V if

$$N(v) > 0 \tag{3.6}$$

for every $v \in V$ with $v \ne 0$. This implies that (3.5) is a q-metric on V. As before, a q-norm on V is also known as a *norm* on V when $q = 1$, and as an *ultranorm* on V when $q = \infty$.

Suppose for the moment that N is a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$, and that (3.6) holds for some $v \in V$. In this case, (3.2) implies (2.5) when $q < \infty$, and (3.4) implies (2.8) when $q = \infty$. Thus $|\cdot|$ should be a q-absolute value function on k when N is a q-seminorm on V with respect to $|\cdot|$ on k, unless N is identically 0 on V.

Let q be a positive real number, and let N be a nonnegative real-valued function on V. Observe that $|\cdot|^q$ is a (q_k/q)-absolute value function on k, since $|\cdot|$ is a q_k-absolute value function on k. Clearly N satisfies (3.1) if and only if N^q satisfies the analogous homogeneity property with respect to $|\cdot|^q$ on k. Similarly, N is a q-seminorm on V with respect to $|\cdot|$ on k if and only if N^q is a seminorm on V with respect to $|\cdot|^q$ on k.

Suppose for the moment again that $|\cdot|$ is the trivial absolute value function on k. In this case, the *trivial ultranorm* is defined by putting $N(0) = 0$ and

$$N(v) = 1 \tag{3.7}$$

for every $v \in V$ with $v \ne 0$. It is easy to see that this defines an ultranorm on V, for which the corresponding ultrametric as in (3.5) is the discrete metric.

Note that k may be considered as a one-dimensional vector space over itself. If $|\cdot|$ is a q_k-absolute value function on k, then $|\cdot|$ may be considered as a q_k-norm on k as a vector space over itself.

4 Associated topologies

Let X be a set, and let $d(\cdot,\cdot)$ be a q-semimetric on X for some $q > 0$. If $x \in X$ and r is a positive real number, then the *open ball* centered at x with radius r corresponding to d is defined as usual by

$$B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}. \tag{4.1}$$

Similarly, the *closed ball* in X centered at $x \in X$ with radius $r \geq 0$ corresponding to d is defined by

$$\overline{B}(x, r) = \overline{B}_d(x, r) = \{y \in X : d(x, y) \leq r\}. \tag{4.2}$$

Suppose for the moment that $q < \infty$, so that $d(\cdot, \cdot)^q$ is a semimetric on X, for which the corresponding open and closed balls in X can be defined in the same way as before. In this case,

$$B_{d^q}(x, r^q) = B_d(x, r) \tag{4.3}$$

for every $x \in X$ and $r > 0$, and

$$\overline{B}_{d^q}(x, r^q) = \overline{B}_d(x, r) \tag{4.4}$$

for every $x \in X$ and $r \geq 0$.

As usual, a subset U of X is said to be an *open set* with respect to $d(\cdot, \cdot)$ if for each $x \in U$ there is an $r > 0$ such that

$$B_d(x, r) \subseteq U. \tag{4.5}$$

It is easy to see that this defines a topology on X. If $q < \infty$, then this is the same as the topology associated to $d(\cdot, \cdot)^q$ on X, because of (4.3). This permits one to reduce to the case of ordinary semimetrics on X, using also the fact that a semi-ultrametric on X is a semimetric on X when $q = \infty$. One can check that open balls in X with respect to d are open sets with respect to this topology, by reducing to the standard argument for ordinary semimetrics, or using an analogous argument directly for q-semimetrics. Similarly, closed balls in X with respect to d are closed sets with respect to this topology. If $d(\cdot, \cdot)$ is a q-metric on X, then this topology is Hausdorff. If $d(\cdot, \cdot)$ is a semi-ultrametric on X, then open balls in X with respect to d are also closed sets with respect to this topology, and closed balls in X with respect to d of positive radius are open sets.

Now let $\mathcal{M}$ be a nonempty collection of q-semimetrics on X. More precisely, we ask that each $d \in \mathcal{M}$ be a q_d-semimetric on X for some $q_d > 0$ that may depend on d. In this situation, a subset U of X is said to be an open set if for each $x \in U$ there are finitely many elements $d_1, \ldots, d_n$ of $\mathcal{M}$ and finitely many positive real numbers $r_1, \ldots, r_n$ such that

$$\bigcap_{j=1}^{n} B_{d_j}(x, r_j) \subseteq U. \tag{4.6}$$

This defines a topology on X, which includes the topology associated to each $d \in \mathcal{M}$. In particular, open balls in X with respect to any $d \in \mathcal{M}$ are open sets with respect to this topology, and the collection of all such open balls forms a sub-base for this topology. Let us say that $\mathcal{M}$ is *nondegenerate* on X if for every $x, y \in X$ with $x \neq y$ there is a $d \in \mathcal{M}$ such that

$$d(x, y) > 0. \tag{4.7}$$

This implies that the topology on X associated to $\mathcal{M}$ is Hausdorff.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. Also let $\mathcal{N}$ be a nonempty collection of q-seminorms on V. As before, we ask more precisely that each $N \in \mathcal{N}$ be a q_N-seminorm on V with respect to $|\cdot|$ on k for some $q_N > 0$ that may depend on N. Thus each $N \in \mathcal{N}$ determines a q_N-semimetric d_N on V as in (3.5), so that

$$\mathcal{M}(\mathcal{N}) = \{d_N : N \in \mathcal{N}\} \tag{4.8}$$

is a nonempty collection of q-semimetrics on V. We say that $\mathcal{N}$ is *nondegenerate* on V if for each $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that

$$N(v) > 0, \tag{4.9}$$

which means that (4.8) is nondegenerate as a collection of q-semimetrics on V.

5 Topological vector spaces

Let k be a field, and let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. This leads to a q_k-metric on k as in (2.6), and hence to a topology on k, as in the previous section. Using standard arguments, one can check that addition and multiplication on k are continuous as mappings from $k \times k$ into k, where $k \times k$ is equipped with the corresponding product topology. Similarly,

$$x \mapsto 1/x \tag{5.1}$$

is continuous as a mapping from $k \setminus \{0\}$ into itself. This uses the topology induced on $k \setminus \{0\}$ by the one on k just mentioned, which is the same as the topology determined by the restriction to $k \setminus \{0\}$ of the q_k-metric associated to $|\cdot|$.

Let V be a vector space over k, and suppose that V is also equipped with a topology. If the vector space operations on V are continuous, then V is said to be a *topological vector space*. More precisely, this means that addition on V should be continuous as a mapping from $V \times V$ into V, using the corresponding product topology on $V \times V$. Similarly, scalar multiplication should be continuous as a mapping from $k \times V$ into V, using the product topology on $k \times V$ associated to the given topology on V and the topology on k determined by the q_k-metric associated to $|\cdot|$ as in the previous paragraph. The condition that $\{0\}$ be a closed set in V is sometimes included in the definition of a topological vector space. This implies that V is Hausdorff as a topological space, by a well-known argument that will be given in the next section. In [19], $|\cdot|$ is required to be nontrivial on k, in the context of topological vector spaces. Although we shall allow $|\cdot|$ to be trivial here, one should be a bit careful about situations in which the nontriviality of $|\cdot|$ may be needed.

Let V be a vector space over k again, and let $\mathcal{N}$ be a nonempty collection of q-seminorms on V with respect to $|\cdot|$ on k. More precisely, the q's here are allowed to depend on the elements of $\mathcal{N}$, as in the previous section. This leads to a collection of q-semimetrics on V, as in (4.8), and hence to a topology on V. One can check that addition and scalar multiplication on V are continuous with respect to this topology, as in the previous paragraph, so that V becomes a topological vector space. If $\mathcal{N}$ is nondegenerate on V, then V is Hausdorff with respect to this topology, as before.

Suppose that V is a topological vector space over k. Continuity of addition on V implies in particular that the translation mappings

$$v \mapsto v + a \tag{5.2}$$

are continuous on V for each $a \in V$. These translation mappings are in fact homeomorphisms on V, since their inverses are of the same type. Similarly, continuity of scalar multiplication implies that the dilation mappings

$$v \mapsto t\, v \tag{5.3}$$

are continuous on V for each $t \in k$. If $t \neq 0$, then this is also a homeomorphism on V, because the inverse mapping is of the same type. Continuity of scalar multiplication also implies that

$$t \mapsto t\, v \tag{5.4}$$

is continuous as a mapping from k into V for each $v \in V$. If $|\cdot|$ is the trivial absolute value function on k, then the corresponding topology on k is the discrete topology, and the continuity of (5.4) is trivial. In this case, continuity of scalar multiplication on V as a mapping from $k \times V$ into V is equivalent to the continuity of (5.3) as a mapping from V into itself for each $t \in k$.

6 Regular topological spaces

A topological space X is said to be *regular* in the strict sense if for each $x \in X$ and closed set $E \subseteq X$ there are disjoint open subsets U_1, U_2 of X such that $x \in U_1$ and $E \subseteq U_2$. This is equivalent to asking that for each $x \in X$ and open set $W \subseteq X$ that contains x there is an open set $U \subseteq X$ such that $x \in U$ and the closure $\overline{U}$ of U in X is contained in W, by a standard argument. If X is regular in the strict sense, and if X satisfies the first or even 0th separation condition, then it is easy to see that X is Hausdorff. In this case, one might say that X is regular in the strong sense, or that X satisfies the *third separation condition.* If the topology on X is determined by a q-semimetric $d(\cdot,\cdot)$ on X for some $q > 0$, then it is easy to see that X is regular in the strict sense. Similarly, if the topology on X is determined by a nonempty collection of q-semimetrics, as in Section 4, then one can check that X is regular in the strict sense. If the topology on X is determined by a q-metric for some $q > 0$, or by a nondegenerate collection of q-semimetrics, then X is regular in the strong sense.

Let k be a field, and let V be a vector space over k. If $a \in V$ and $B \subseteq V$, then we put

$$a + B = B + a = \{a + b : b \in B\}, \tag{6.1}$$

which is the same as the image of B under the translation mapping (5.2). If A is another subset of V, then we put

$$A + B = \{a + b : a \in A,\, b \in B\}, \tag{6.2}$$

so that

$$A + B = \bigcup_{a \in A} (a + B) = \bigcup_{b \in B} (A + b). \tag{6.3}$$

Also put

$$-A = \{-a : a \in A\}, \tag{6.4}$$

which is the same as the image of A under the dilation mapping (5.3) with $t = -1$. Similarly, $a - B$ is defined to be $a + (-B)$, $A - b$ is defined to be $A + (-b)$, and $A - B$ is defined to be $A + (-B)$.

Now let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and suppose that V is a topological vector space over k. If $B \subseteq V$ is an open set, then (6.1) is an open set in V for every $a \in V$, because the translation mapping (5.2) is a homeomorphism on V. If A, B are subsets of V, and either A or B is an open set, then $A + B$ is an open set in V too, because it can be expressed as a union of open sets in V as in (6.3). Similarly, if A is an open subset of V, then $-A$ is an open set as well, because the dilation mapping (5.3) is a homeomorphism on V when $t = -1$. If E is any subset of V, and $W \subseteq V$ is an open set that contains 0, then one can check that

$$\overline{E} \subseteq E + W, \tag{6.5}$$

where $\overline{E}$ is the closure of E in V.

If $U \subseteq V$ is an open set that contains 0, then there are open sets $U_1, U_2 \subseteq V$ that contain 0 and satisfy

$$U_1 + U_2 \subseteq U, \tag{6.6}$$

by continuity of addition on V as a mapping from $V \times V$ into V at $(0, 0)$. This implies that

$$\overline{U_1} \subseteq U, \tag{6.7}$$

as in (6.5), and hence that V is regular as a topological space in the strict sense, using also continuity of translations. If $\{0\}$ is a closed set in V, then $\{a\}$ is a closed set in V for every $a \in V$, by continuity of translations. In this case, it follows that V is Hausdorff as a topological space, because V is regular in the strict sense. Of course, this also means that V is regular in the strong sense as a topological space when $\{0\}$ is a closed set in V.

7 Balanced sets

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. If $t \in k$ and $E \subseteq V$, then we put

$$t\,E = \{t\,v : v \in E\}, \tag{7.1}$$

which is the image of E under the corresponding dilation mapping (5.3). We say that E is *balanced* as a subset of V with respect to $|\cdot|$ on k if

$$t\,E \subseteq E \tag{7.2}$$

for every $t \in k$ with $|t| \le 1$. This implies that $0 \in E$ when $E \ne \emptyset$, by taking $t = 0$ in (7.2). If $E \subseteq V$ is balanced, $t \in k$, and $|t| = 1$, then we have that

$$t\,E = E, \tag{7.3}$$

by applying (7.2) to t and to $1/t$.

Let N be a nonnegative real-valued function on V that satisfies the homogeneity condition (3.1) with respect to $|\cdot|$ on k. Put

$$B_N(0,r) = \{v \in V : N(v) < r\} \tag{7.4}$$

for every $r > 0$, and

$$\overline{B}_N(0,r) = \{v \in V : N(v) \le r\} \tag{7.5}$$

for each $r \ge 0$, which are the open and closed balls in V centered at 0 with radius r with respect to N. It is easy to see that these are balanced subsets of V, because of (3.1). More precisely, if $t \in k$ and $t \ne 0$, then

$$t\,B_N(0,r) = B_N(0,|t|\,r) \tag{7.6}$$

for every $r > 0$, and

$$t\,\overline{B}_N(0,r) = \overline{B}_N(0,|t|\,r) \tag{7.7}$$

for every $r \ge 0$. If N is a q_N-seminorm on V with respect to $|\cdot|$ on k for some $q_N > 0$, then (3.5) defines a corresponding q_N-semimetric d_N on V, and (7.4) and (7.5) are the same as the open and closed balls in V centered at 0 with radius r with respect to d_N, as in (4.1) and (4.2).

Suppose now that V is a topological vector space over k, and that W is an open subset of V that contains 0. Under these conditions, there is an open set $U \subseteq V$ that contains 0 and a positive real number δ such that

$$t\,U \subseteq W \tag{7.8}$$

for every $t \in k$ with $|t| < \delta$. This uses continuity of scalar multiplication on V, as a mapping from $k \times V$ into V. More precisely, this is basically the same as continuity of scalar multiplication on V as a mapping from $k \times V$ into V at $(0,0)$ in $k \times V$. If $|\cdot|$ is the trivial absolute value function on k, then this condition holds automatically with $\delta = 1$.

Suppose for the moment that $|\cdot|$ is not the trivial absolute value function on k, and put

$$U_1 = \bigcup_{0<|t|<\delta} t\,U, \tag{7.9}$$

where more precisely the union is taken over all $t \in k$ such that $0 < |t| < \delta$. The hypothesis that $|\cdot|$ be nontrivial on k implies that there are $t \in k$ with this property, and in particular $0 \in U_1$. Observe that

$$U_1 \subseteq W, \tag{7.10}$$

by (7.8), and that U_1 is an open set in V, because it is a union of open sets. It is easy to see that U_1 is balanced in V, by construction. If the topology on V is determined by a collection $\mathcal{N}$ of q-seminorms on V, then the same conclusion can be obtained more directly using the fact that the corresponding open balls centered at 0 are balanced, which also works when $|\cdot|$ is trivial on k.

8 Absorbing sets

Let k be a field with a q_k-absolute value function $|\cdot|$ for some some $q_k > 0$ again, and let V be a vector space over k. A subset A of V is said to be *absorbing* in V with respect to $|\cdot|$ on k if for each $v \in V$ there is a $t_0(v) \in k$ such that $t_0(v) \neq 0$ and

$$t\, v \in A \tag{8.1}$$

for every $t \in k$ with $|t| \le |t_0(v)|$. Note that this implies that $0 \in A$, so that (8.1) holds automatically for every $v \in V$ when $t = 0$. Equivalently, one can check that $A \subseteq V$ is absorbing in V if and only if for each $v \in V$ there is a $t_1(v) \in k$ such that

$$v \in t\, A \tag{8.2}$$

for every $t \in k$ with $|t| \ge |t_1(v)|$. If $|\cdot|$ is the trivial absolute value function on k, then V is the only absorbing subset of itself in this sense.

Let us suppose from now on in this section that $|\cdot|$ is not trivial on k. As in Section 2, this implies that there are $y, z \in k$ such that $0 < |y| < 1$ and $|z| > 1$. Hence

$$|y^j| = |y|^j \to 0 \quad \text{as } j \to \infty, \tag{8.3}$$

and

$$|z^j| = |z|^j \to +\infty \quad \text{as } j \to \infty. \tag{8.4}$$

It follows that $A \subseteq V$ is absorbing when (8.1) holds for all $t \in k$ such that $|t|$ is sufficiently small, depending on v, or equivalently when (8.2) holds for all $t \in k$ such that $|t|$ is sufficiently large, depending on v. If N is a nonnegative real-valued function on V that satisfies the homogeneity condition (3.1) with respect to $|\cdot|$ on k, then the open and closed balls (7.4) and (7.5) in V centered at 0 with radius r with respect to N are absorbing in V for every $r > 0$.

Suppose for the moment that V is a topological vector space over k, and let W be an open set in V that contains 0. Let $v \in V$ be given, and remember that (5.4) defines a continuous mapping from k into V. Using the continuity of this mapping at $t = 0$, we get that there is a $\delta(v) > 0$ such that

$$t\, v \in W \tag{8.5}$$

for every $t \in k$ with $|t| < \delta(v)$. This implies that W is absorbing in V, using the remarks in the preceding paragraph. More precisely, this corresponds exactly to the continuity of (5.4) at $t = 0$ for each $v \in V$.

Let $\{t_j\}_{j=1}^{\infty}$ be a sequence of elements of k such that $|t_j| \to \infty$ as $j \to \infty$. If $A \subseteq V$ is absorbing, then

$$\bigcup_{j=1}^{\infty} t_j\, A = V, \tag{8.6}$$

by (8.2). If A is a balanced set in V, then

$$t\, A \subseteq t'\, A \tag{8.7}$$

for every $t, t' \in k$ with $|t| \le |t'|$. In this case, (8.6) implies that A is absorbing in V. More precisely, a balanced set $A \subseteq V$ is absorbing in V if for every $v \in V$ there is a $t \in k$ such that $t \neq 0$ and (8.1) holds, or if for every $v \in V$ there is a $t \in k$ that satisfies (8.2).

9 Bounded sets

Let k be a field with a nontrivial q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a topological vector space over k. A subset E of V is said to be *bounded* in V if for each open set $U \subseteq V$ with $0 \in U$ there is a $t \in k$ such that

$$E \subseteq t\, U. \tag{9.1}$$

If U is balanced in V, then it follows that

$$E \subseteq t'\, U \tag{9.2}$$

for every $t' \in k$ such that $|t'| \ge |t|$, as in (8.7). Otherwise, one can first choose a nonempty balanced open set contained in U, as in Section 7. If $E \subseteq V$ is bounded and $U \subseteq V$ is any open set that contains 0, then it follows that (9.1) holds for all $t \in k$ such that $|t|$ is sufficiently large.

If $U \subseteq V$ is an open set that contains 0, then U is absorbing in V, as in the previous section. Using this, it is easy to see that finite subsets of V are bounded. Similarly, one can check that the union of finitely many bounded subsets of V is bounded as well. Note that subsets of bounded sets in V are bounded too. If $E \subseteq V$ is bounded, then one can also check that the closure $\overline{E}$ of E in V is bounded, using the fact that V is regular as a topological space in the strict sense, as in Section 6.

If $\mathcal{B}_0$ is a local base for the topology of V at 0, then it suffices to verify (9.1) for $U \in \mathcal{B}_0$ in order to show that $E \subseteq V$ is bounded. In particular, one can take $\mathcal{B}_0$ to be the collection of nonempty balanced open subsets of V, as in Section 7. If $U \subseteq V$ is nonempty, open, and balanced, then (8.6) and (8.7) hold with $A = U$, because U is absorbing in V, as before. One can use this to check that compact subsets of V are bounded.

Suppose for the moment that the topology on V is determined by a nonempty collection $\mathcal{N}$ of q-seminorms on V with respect to $|\cdot|$ on k. In this case, $E \subseteq V$ is bounded if and only if each $N \in \mathcal{N}$ is bounded on E. To see this, suppose first that E is bounded in V, and let $N \in \mathcal{N}$ be given. Remember that the open unit ball $B_N(0,1)$ in V with respect to N is an open set in V in this situation, as in Section 4. Thus we can apply (9.1) with $U = B_N(0,1)$, since this set obviously contains 0. This implies that N is bounded on E, by (7.6). Conversely, suppose that each $N \in \mathcal{N}$ is bounded on E. This implies that (9.1) holds whenever U is an open ball in V with respect to any element of $\mathcal{N}$ centered at 0 and with positive radius, and $|t|$ is sufficiently large. Similarly, (9.1) holds when U is the intersection of finitely many such balls, and $|t|$ is sufficiently large. It follows that E is bounded in V under these conditions, because finite intersections of such balls form a local base for the topology of V at 0, as in Section 4 again.

If E_1, E_2 are bounded subsets of V, then $E_1 + E_2$ is bounded in V as well. To see this, let $U \subseteq V$ be an open set that contains 0. Continuity of addition on V at 0 implies that there are open sets $U_1, U_2 \subseteq V$ that contain 0 and whose sum $U_1 + U_2$ is contained in U. Thus $E_1 \subseteq t\,U_1$ and $E_2 \subseteq t\,U_2$ for every $t \in k$ with $|t|$ sufficiently large, because E_1 and E_2 are bounded in V. This implies that

$$E_1 + E_2 \subseteq t\,U_1 + t\,U_2 = t\,(U_1 + U_2) \subseteq t\,U \tag{9.3}$$

for every $t \in k$ with $|t|$ sufficiently large, as desired. In particular, it follows that translates of bounded subsets of V are bounded in V, since any subset of V with only one element is bounded. It is easy to see that dilates of bounded subsets of V are bounded too, directly from the definitions.

Let us say that a sequence $\{v_j\}_{j=1}^\infty$ of elements of V is *bounded* in V if the corresponding set of v_j's, $j \ge 1$, is bounded in V. If $\{v_j\}_{j=1}^\infty$ is a sequence of elements of V that converges to some $v \in V$, then the set of v_j's together with v is compact in V. This implies that $\{v_j\}_{j=1}^\infty$ is bounded in V, because compact sets are bounded, as before. Alternatively, if $\{v_j\}_{j=1}^\infty$ converges to 0 in V, then one can check that $\{v_j\}_{j=1}^\infty$ is bounded in V more directly from the definitions. If $\{v_j\}_{j=1}^\infty$ converges to any $v \in V$, then one can reduce to the case where $v = 0$, using the fact that translates of bounded sets in V are bounded, as in the preceding paragraph.

If $\{v_j\}_{j=1}^\infty$ is a bounded sequence in V, and $\{t_j\}_{j=1}^\infty$ is a sequence of elements of k that converges to 0 with respect to $|\cdot|$, then one can verify that $\{t_j\,v_j\}_{j=1}^\infty$ converges to 0 in V. Suppose that $E \subseteq V$ is not bounded in V, so that there is an open set $U \subseteq V$ that contains 0 such that (9.1) does not hold for any $t \in k$. If $\{t_j\}_{j=1}^\infty$ is any sequence of nonzero elements of k, then it follows that there is a sequence $\{v_j\}_{j=1}^\infty$ of elements of E such that

$$t_j\,v_j \not\in U \tag{9.4}$$

for each j. In particular, this means that $\{t_j\,v_j\}_{j=1}^\infty$ does not converge to 0 in V. Of course, one can choose $\{t_j\}_{j=1}^\infty$ so that it converges to 0 with respect to $|\cdot|$ on k, because $|\cdot|$ is supposed to be nontrivial on k.

10 Semimetrization

Let X be a set, let l be a positive integer, and let $d_j(x, y)$ be a q_j-semimetric on X for some $q_j > 0$ and each $j = 1, \ldots, l$. If we put

$$q = \min(q_1, \ldots, q_l), \tag{10.1}$$

then it follows that $d_j(x, y)$ is a q-semimetric on X for each j, as in Section 1. Under these conditions, one can check that

$$d(x, y) = \max_{1 \le j \le l} d_j(x, y) \tag{10.2}$$

defines a q-semimetric on X as well. Observe that

$$B_d(x, r) = \bigcap_{j=1}^{l} B_{d_j}(x, r) \tag{10.3}$$

for every $x \in X$ and $r > 0$, where these open balls are defined as in (4.1). This implies that the topology on X determined by d as in Section 4 is the same as the topology determined by the collection $d_1, \ldots, d_l$.

Now let $d(x, y)$ be any q-semimetric on X, for any $q > 0$. Also let r_0 be a positive real number, and put

$$d'(x, y) = \min(d(x, y), r_0) \tag{10.4}$$

for each $x, y \in X$. It is easy to see that (10.4) defines a q-semimetric on X too. By construction, the open ball in X centered at $x \in X$ with radius $r > 0$ with respect to (10.4) is the same as the corresponding open ball with respect to $d(\cdot, \cdot)$ when $r \le r_0$, and it is the whole space X when $r > r_0$. It follows that (10.4) determines the same topology on X as $d(\cdot, \cdot)$.

Let $d_j(x, y)$ be a q_j-semimetric on X for some $q_j > 0$, for each positive integer j. As in the previous paragraph,

$$d_j'(x, y) = \min(d_j(x, y), 1/j) \tag{10.5}$$

defines a q_j-semimetric on X for each $j \ge 1$, and (10.5) determines the same topology on X as $d_j(x, y)$. Similarly, the collection of $d_j'(x, y)$ with $j \ge 1$ determines the same topology on X as the collection of $d_j(x, y)$ with $j \ge 1$. Put

$$d(x, y) = \max_{j \ge 1} d_j'(x, y) \tag{10.6}$$

for every $x, y \in X$, which is equal to 0 when $d_j'(x, y) = 0$ for each j, and otherwise reduces to the maximum over finitely many j. If $q > 0$ satisfies

$$q_j \ge q \tag{10.7}$$

for each j, then (10.5) is a q-semimetric on X for each $j \ge 1$, as in Section 1. In this case, (10.6) defines a q-semimetric on X too, for basically the same reasons as for (10.2). One can also check that the topology determined on X by (10.6) is the same as the topology determined by the collection of $d'_j(x, y)$ with $j \geq 1$, which is the same as the topology determined by the collection of $d_j(x, y)$ with $j \geq 1$, as before. This uses the fact that the ball in X centered at a point $x \in X$ with radius $r > 0$ with respect to (10.6) is the same as the intersection of the corresponding balls with respect to (10.5) for $j \geq 1$, which reduces to the intersection of the corresponding balls with respect to $d_j(x, y)$ for $j \leq 1/r$.

One can always reduce to the case where the q_j's have a positive lower bound, as in (10.7), as follows. If $q_j < \infty$ for some j, then $d_j(x, y)^{q_j}$ is an ordinary semimetric on X, as in Section 1, and this semimetric determines the same topology on X as $d_j(x, y)$, as in Section 4. Of course, if $q_j \geq 1$, then $d_j(x, y)$ is already an ordinary semimetric on X, as in Section 1. This permits us to get a sequence of ordinary semimetrics on X that determines the same topology on X as the initial sequence of q_j-semimetrics $d_j(x, y)$. Applying the construction described in the preceding paragraph to such a sequence of ordinary semimetrics on X, we get a single ordinary semimetric on X that determines the same topology on X.

11 q-Subadditivity

Let k be a field, and let V be a vector space over k. Let us say that a nonnegative real-valued function N on V is *q-subadditive* for some positive real number q if

$$N(v + w)^q \leq N(v)^q + N(w)^q \tag{11.1}$$

for every $v, w \in V$. If this holds with $q = 1$, then we may simply say that N is *subadditive* on V. As usual, (11.1) is the same as saying that

$$N(v + w) \leq (N(v)^q + N(w)^q)^{1/q} \tag{11.2}$$

for every $v, w \in V$. The ultrametric version of subadditivity is defined by the condition that

$$N(v + w) \leq \max(N(v), N(w)) \tag{11.3}$$

for every $v, w \in V$, which may be considered as q-subadditivity with $q = \infty$, because of (1.5). If $0 < q_1 \leq q_2 \leq \infty$ and N is q_2-subadditive on V, then N is q_1-subadditive on V as well, by (1.1) and (1.3). Note that q-subadditivity could be defined in the same way on any commutative group, instead of a vector space.

Let us say that N is *symmetric* on V if

$$N(-v) = N(v) \tag{11.4}$$

for every $v \in V$, which could also be defined in the same way on any commutative group. This implies that

$$d(v, w) = N(v - w) \tag{11.5}$$

is symmetric in v and w. If a nonnegative real-valued function N on V satisfies $N(0) = 0$, is symmetric, and is q-subadditive for some $q > 0$, then (11.5) is a q-semimetric on V. Observe that

$$d(v + a, w + a) = d(v, w) \tag{11.6}$$

for every $a, v, w \in V$ in this case, so that (11.5) is invariant under translations on V. Similarly,

$$d(v, w) = d(-v, -w) \tag{11.7}$$

for every $v, w \in V$ in this situation, so that (11.5) is invariant under reflection on V.

Suppose now that $|\cdot|$ is a q_k-absolute value function on k for some $q_k > 0$. Let us say that N is *balanced* on V with respect to $|\cdot|$ on k if

$$N(t\, v) \le N(v) \tag{11.8}$$

for every $v \in V$ and $t \in k$ with $|t| \le 1$. This implies that

$$N(t\, v) = N(v) \tag{11.9}$$

for every $v \in V$ and $t \in k$ with $|t| = 1$, by applying (11.8) to t and to $1/t$. In particular, (11.9) implies that N is symmetric, by taking $t = -1$. If N satisfies the homogeneity condition (3.1), then it is easy to see that N is balanced. If $|\cdot|$ is the trivial absolute value function on k, then the homogeneity condition (3.1) reduces to (11.9) and the requirement that $N(0) = 0$. If $|\cdot|$ is any q_k-absolute value function on k and N is balanced on V with respect to $|\cdot|$ on k, then the open and closed balls in V centered at 0 associated to N as in (7.4) and (7.5) are balanced as subsets of V.

Let r_0 be a positive real number, and put

$$N'(v) = \min(N(v), r_0) \tag{11.10}$$

for each $v \in V$. If $N(0) = 0$, N is symmetric on V, or N is balanced on V, then it is easy to see that N' has the same property. Similarly, if N is q-subadditive for some $q > 0$, then N' is q-subadditive too. Let $d(v, w)$ be associated to N as in (11.7), and let $d'(v, w)$ be associated to N' in the same way. Under these conditions, $d'(v, w)$ can be given in terms of $d(v, w)$ as in (10.4).

12 Semimetrization, continued

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. Also let l be a positive integer, and for each $j = 1, \ldots, l$, let N_j be a q_j-seminorm on V for some $q_j > 0$, with respect to $|\cdot|$ on k. Put

$$q = \min(q_1, \ldots, q_l), \tag{12.1}$$

as before, so that N_j is a q-seminorm on V for each j, as in Section 3. Under these conditions, it is easy to see that

$$N(v) = \max_{1 \le j \le l} N_j(v) \tag{12.2}$$

defines a q-seminorm on V too. If $d_j(v, w)$ corresponds to N_j on V as in (3.5) for each j, and $d(v, w)$ corresponds to N in the same way, then $d(v, w)$ can be given in terms of $d_j(v, w)$ as in (10.2).

Suppose now that for each positive integer j, N_j is a q_j-seminorm on V for some $q_j > 0$, with respect to $|\cdot|$ on k. Put

$$N'_j(v) = \min(N_j(v), 1/j) \tag{12.3}$$

for each $j \ge 1$ and $v \in V$, so that $N'_j(0) = 0$ and N'_j is balanced and q_j-subadditive for each j, as in the previous section. Also put

$$N(v) = \max_{j \ge 1} N'_j(v) \tag{12.4}$$

for every $v \in V$, which is equal to 0 when $N_j(v) = 0$ for every j, and otherwise reduces to the maximum over finitely many j. In particular, $N(0) = 0$, and it is easy to see that N is balanced on V. If $q > 0$ and

$$q_j \ge q \tag{12.5}$$

for each $j \ge 1$, then N'_j is q-subadditive for each $j \ge 1$, and one can check that N is q-subadditive on V as well. Let $d(v, w)$ correspond to N as in (11.5), and let $d_j(v, w)$, $d'_j(v, w)$ correspond to N_j, N'_j in the same way for each j. As before, $d'_j(v, w)$ can be given in terms of $d_j(v, w)$ as in (10.5), and $d(v, w)$ can be given in terms of $d'_j(v, w)$ as in (10.6).

If $q_j < \infty$ for some j, then

$$N_j(v)^{q_j} \tag{12.6}$$

is a balanced subadditive nonnegative real-valued function on V that vanishes at 0. Of course, N_j already has these properties when $q_j \ge 1$, and in particular when $q_j = \infty$. Thus we can reduce to the case where $q_j \ge 1$ for every j, using balanced q_j-subadditive nonnegative real-valued functions on V that vanish at 0 instead of q_j-seminorms. It is easy to see that the discussion in the preceding paragraph works as well in this case. Of course, if $d_j(v, w)$ corresponds to N_j as in (11.5), then (12.6) corresponds to

$$d_j(v, w)^{q_j} \tag{12.7}$$

in the same way.

If A is a commutative topological group, then it is well known that there is a collection of translation-invariant semimetrics on A that determines the same topology on A. If there is a local base for the topology of A at 0 with only finitely or countably many elements, then there is a single translation-invariant semimetric on A that determines the same topology. Note that a topological vector space V over k is a commutative topological group with respect to addition in particular. If N is a q-seminorm on V for some $q > 0$, then the corresponding q-semimetric (11.5) is automatically invariant under translations, as in (11.6). If the topology on V is determined by a collection of finitely or countably many q-seminorms on V, then one can use the constructions described in the previous paragraphs to get a single q-seminorm or translation-invariant q-semimetric.

13 Continuous linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k. Also let T be a linear mapping from V into W. If T is continuous at 0, then it is easy to see that T is continuous at every point in V, by continuity of translations. As usual, T is said to be *sequentially continuous* at 0 if for every sequence $\{v_j\}_{j=1}^\infty$ of vectors in V that converges to 0, $\{T(v_j)\}_{j=1}^\infty$ converges to 0 in W. This implies that T is sequentially continuous at every point in V, using continuity of translations again. If T is continuous at 0, then one can check directly that T is sequentially continuous at 0. In the other direction, if there is a local base for the topology of V at 0 with only finitely or countably many elements, and if T is sequentially continuous at 0, then T is continuous at 0, by a standard argument.

Suppose that the topology on W is determined by a nonempty collection $\mathcal{N}_W$ of q-seminorms on W with respect to $|\cdot|$ on k, where $q > 0$ is allowed to depend on the element of $\mathcal{N}_W$, as before. Let $N_W \in \mathcal{N}_W$ and $r > 0$ be given, and remember that

$$\{w \in W : N_W(w) < r\} \tag{13.1}$$

is an open set in W that contains 0, as in Section 4. If T is continuous at 0, then there is an open set $U \subseteq V$ such that $0 \in U$ and $T(U)$ is contained in (13.1), which means that

$$N_W(T(v)) < r \tag{13.2}$$

for every $v \in U$. Conversely, if for each $N_W \in \mathcal{N}_W$ and $r > 0$ there is an open set $U \subseteq V$ with these properties, then T is continuous at 0. This is because subsets of W of the form (13.1) determine a local sub-base for the topology of W at 0 in this situation.

Suppose that the topology on V is also determined by a nonempty collection $\mathcal{N}_V$ of q-seminorms on V with respect to $|\cdot|$ on k, where $q > 0$ may depend on the element of $\mathcal{N}_V$. If $N_{V,1}, \ldots, N_{V,l}$ are finitely many elements of $\mathcal{N}_V$ and $r_1, \ldots, r_l$ are finitely many positive real numbers, then

$$\{v \in V : N_{V,j}(v) < r_j \text{ for every } j = 1, \ldots, l\} \tag{13.3}$$

is an open set in V that contains 0. If $U \subseteq V$ is any open set that contains 0, then there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and positive real numbers $r_1, \ldots, r_l$ such that (13.3) is contained in U. If T is continuous at 0, then it follows that for each $N_W \in \mathcal{N}_W$ and $r > 0$ there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and positive real numbers $r_1, \ldots, r_l$ such that

(13.2) holds for every $v \in V$ that satisfies $N_{V,j}(v) < r_j$ for each $j = 1, \ldots, l$. Conversely, this property implies that T is continuous at 0, by the remarks in the preceding paragraph and the fact that (13.3) is an open set in V that contains 0.

Suppose now that for each $N_W \in \mathcal{N}_W$ there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and a nonnegative real number C such that

$$N_W(T(v)) \le C \max_{1 \le j \le l} N_{V,j}(v) \tag{13.4}$$

for every $v \in V$. This implies that T is continuous at 0, by the criterion mentioned in the previous paragraph. In the other direction, if T is continuous at 0, and if $|\cdot|$ is nontrivial on k, then T has the property just described. To see this, it suffices to take $r = 1$ in the earlier discussion. Note that the $C \ge 0$ in (13.4) may depend on the nontriviality of $|\cdot|$ on k as well as the finitely many positive real numbers $r_1, \ldots, r_l$ obtained from the continuity of T at 0.

14 Bounded linear mappings

Let k be a field with a nontrivial q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k. Thus the notion of bounded subsets of V, W can be defined as in Section 9. A linear mapping T from V into W is said to be *bounded* if for each bounded set $E \subseteq V$ we have that $T(E)$ is a bounded set in W. It is easy to see that continuous linear mappings are bounded, directly from the definitions. More precisely, if T is sequentially continuous at 0, then one can check that T is bounded using the characterization of bounded sets in terms of sequences mentioned in Section 9.

Let us say that T is *strongly bounded* if there is an open set $U \subseteq V$ such that $0 \in U$ and $T(U)$ is bounded in W. It is easy to see that this implies that T is continuous at 0. Of course, if U is a bounded open set in V that contains 0 and T is a bounded linear mapping, then $T(U)$ is bounded in V, so that T is strongly bounded. In particular, if there is a bounded open set in V that contains 0 and T is continuous, then T is strongly bounded. Similarly, if there is a bounded open set in W that contains 0 and T is continuous, then T is strongly bounded.

If there is a local base for the topology of V at 0 with only finitely or countably many elements, then it is well known that every bounded linear mapping T from V into W is continuous. To be more precise, let $\{v_j\}_{j=1}^\infty$ be a sequence of elements of V that converges to 0. Under these conditions on V, it can be shown that there is a sequence $\{t_j\}_{j=1}^\infty$ of nonzero elements of k that converges to 0 with respect to $|\cdot|$, and with the additional property that

$$\{t_j^{-1} \, v_j\}_{j=1}^\infty \tag{14.1}$$

converges to 0 in V. This implies that (14.1) is a bounded sequence in V, as in Section 9. If T is bounded, then it follows that

$$\{T(t_j^{-1} \, v_j)\}_{j=1}^\infty \tag{14.2}$$

is a bounded sequence in W. Because $\{t_j\}_{j=1}^\infty$ converges to 0 in k, we get that

$$T(v_j) = t_j\, T(t_j^{-1}\, v_j) \to 0 \quad \text{as } j \to \infty \tag{14.3}$$

in W, as in Section 9 again. This means that T is sequentially continuous at 0, which implies that T is continuous in this situation.

Alternatively, suppose that T is not continuous at 0. This means that there is an open subset U_W of W such that $0 \in U_W$ and $T^{-1}(U_W)$ does not contain any open subset of V that contains 0. By hypothesis, there is a sequence $U_1, U_2, U_3, \ldots$ of open subsets of V that contain 0 and form a local base for the topology of V at 0. We may also ask that $U_{j+1} \subseteq U_j$ for each $j \geq 1$, since otherwise we can replace U_j with the intersection of $U_1, \ldots, U_j$ for each j. Let t_0 be an element of k such that $|t_0| > 1$, which exists because $|\cdot|$ is nontrivial on k. Thus $t_0^{-j}\, U_j$ is also an open set in V that contains 0 for each positive integer j. It follows that for each positive integer j there is a $u_j \in U_j$ such that

$$t_0^{-j}\, T(u_j) = T(t_0^{-j}\, u_j) \not\in U_W. \tag{14.4}$$

Note that $\{u_j\}_{j=1}^\infty$ converges to 0 in V, because $u_j \in U_j$ for each j. This implies that $\{u_j\}_{j=1}^\infty$ is a bounded sequence in V, as in Section 9. If T is a bounded linear mapping from V into W, then $\{T(u_j)\}_{j=1}^\infty$ should be a bounded sequence in W. This implies that $\{t_0^{-j}\, T(u_j)\}_{j=1}^\infty$ converges to 0 in W, as in Section 9 again, because $\{t_0^{-j}\}_{j=1}^\infty$ converges to 0 in k. This contradicts (14.4), as desired.

15 Bounded linear mappings, continued

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let V, W be vector spaces over k. Also let N_V, N_W be q_V, q_W-seminorms on V, W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. Thus V, W are topological vector spaces over k with respect to the topologies corresponding to N_V, N_W, respectively. A linear mapping T from V into W is said to be *bounded* with respect to N_V, N_W if there is a nonnegative real number C such that

$$N_W(T(v)) \leq C\, N_V(v) \tag{15.1}$$

for every $v \in V$. This is a bit different from the situation considered in the previous section, but the terminology is still basically compatible, as follows.

Suppose for the moment that $|\cdot|$ is nontrivial on k, so that the discussion in the previous section is also applicable. In this situation, a subset of V or W is bounded in the sense of Section 9 if and only if N_V or N_W is bounded on that set, as appropriate. If T satisfies (15.1) for some $C \geq 0$, then it is easy to see that T is bounded in the sense of the previous section. Conversely, suppose that T is bounded in the sense described in the previous section. Note that

$$B_V = \{v \in V : N_V(v) \leq 1\} \tag{15.2}$$

is bounded in V, so that $T(B_V)$ is bounded in W, by hypothesis. Equivalently, this means that there is a $C_1 \geq 0$ such that

$$N_W(T(v)) \leq C_1 \tag{15.3}$$

for every $v \in V$ with $N_V(v) \leq 1$. Using this and the nontriviality of $|\cdot|$ on k, one can check that (15.1) holds for some $C \geq 0$.

As in Section 13, (15.1) implies that T is continuous as a mapping from V into W. More precisely, one can apply the discussion in Section 13 with $\mathcal{N}_V = \{N_V\}$ and $\mathcal{N}_W = \{N_W\}$, so that (15.1) corresponds to (13.4). Similarly, if T is continuous and $|\cdot|$ is nontrivial on k, then (15.1) holds for some $C \geq 0$, as before. Of course, open balls in V and W centered at 0 of any positive radius with respect to N_V, N_W, respectively, are open sets, as in Section 4. They are also bounded sets in the sense of Section 9 when $|\cdot|$ is nontrivial on k.

If T satisfies (15.1) for some $C \geq 0$, then we put

$$\|T\|_{op} = \|T\|_{op,VW} = \inf\{C \geq 0 : (15.1) \text{ holds for this } C\}. \tag{15.4}$$

More precisely, this is the infimum of all nonnegative real numbers C such that (15.1) holds for this C and every $v \in V$. Observe that (15.1) holds with $C = \|T\|_{op}$, which is to say that the infimum is automatically attained.

16 Spaces of linear mappings

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k. The space of all mappings from a nonempty set X into W is a vector space over k as well, with respect to pointwise addition and scalar multiplication. In particular, the space of all linear mappings from V into W may be considered as a linear subspace of the space of all mappings from V into W. Similarly, the space of continuous mappings from a nonempty topological space X into W is a linear subspace of the space of all mappings from X into W, because of the continuity of the vector space operations on W. It follows that the space $\mathcal{CL}(V, W)$ of continuous linear mappings from V into W is a vector space over k too.

Suppose for the moment that $|\cdot|$ is nontrivial on k, so that boundedness of subsets of V, W can be defined as in Section 9, and boundedness of linear mappings can be defined as in Section 14. It is easy to see that the space of bounded linear mappings from V into W is a linear subspace of the space of all linear mappings from V into W. This uses the fact that sums of bounded subsets of W are also bounded in W, as in Section 9, and the observation that scalar multiples of bounded sets are bounded. Similarly, the space of strongly bounded linear mappings from V into W is a linear space. More precisely, if T_1, T_2 are strongly bounded linear mappings from V into W, then there are open subsets U_1, U_2 of V such that $0 \in U_1, U_2$ and $T_1(U_1)$, $T_2(U_2)$ are bounded subsets of W. In this case, $U_1 \cap U_2$ is an open set in V that contains 0, and so one would like to verify that $T_1 + T_2$ maps $U_1 \cap U_2$ to a bounded set in W. This follows from

$$(16.1)\quad (T_1 + T_2)(U_1 \cap U_2) \subseteq T_1(U_1 \cap U_2) + T_2(U_1 \cap U_2) \subseteq T_1(U_1) + T_2(U_2),$$

since $T_1(U_1) + T_2(U_2)$ is a bounded subset of W, as before.

Suppose now that N_V, N_W are q_V, q_W-seminorms on V, W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. Let $\mathcal{BL}(V,W)$ be the space of linear mappings from V into W that are bounded, in the sense of the previous section. One can check that $\mathcal{BL}(V,W)$ is a linear space with respect to pointwise addition and scalar multiplication again. Moreover, (15.4) defines a q_W-seminorm on $\mathcal{BL}(V,W)$, which may be described as the *operator q_W-seminorm* associated to N_V and N_W. If N_W is a q_W-norm on W, then (15.4) defines a q_W-norm on $\mathcal{BL}(V,W)$.

Let Z be another topological vector space over k, and let $T_1 : V \to W$ and $T_2 : W \to Z$ be linear mappings, so that their composition $T_2 \circ T_1$ is a linear mapping from V into Z. Of course, if T_1, T_2 are continuous, then $T_2 \circ T_1$ is continuous as well. If $|\cdot|$ is nontrivial on k, and if T_1, T_2 are bounded in the sense of Section 14, then $T_2 \circ T_1$ is bounded too. Similarly, if T_1 is strongly bounded and T_2 is bounded, then $T_2 \circ T_1$ is strongly bounded. The same conclusion holds when T_1 is continuous and T_2 is strongly bounded. Now let N_V, N_W, and N_Z be q_V, q_W, and q_Z-seminorms on V, W, and Z, respectively, for some $q_V, q_W, q_Z > 0$, as in the preceding paragraph. If T_1 and T_2 are bounded in the sense of the previous section, then one can check that $T_2 \circ T_1$ is bounded, with

$$(16.2)\qquad \|T_2 \circ T_1\|_{op,VZ} \le \|T_1\|_{op,VW}\, \|T_2\|_{op,WZ}.$$

Here the subscripts are used to indicate which spaces and seminorms are used in the particular operator seminorm.

17 Cartesian products

Let I be a nonempty set, let X_j be a topological space for each $j \in I$, and consider the corresponding Cartesian product

$$(17.1)\qquad X = \prod_{j \in I} X_j,$$

equipped with the product topology. Remember that the standard coordinate mappings from X onto the individual X_j's are continuous mappings. A sequence of elements of X converges to another element of X if and only if for each $j \in I$, the corresponding sequence of jth coordinates in X_j converges to the jth coordinate of the limit. If X_j satisfies the first, second, or third separation conditions for each $j \in I$, then X has the same property.

Suppose for the moment that I has only finitely or countably many elements. Suppose also that for each $j \in I$, there is a local base for the topology of X_j at a point $x_j \in X_j$ with only finitely or countably many elements. If x is the element of X whose jth coordinate is x_j for each $j \in I$, then there is a local base for the product topology on X at x with only finitely or countably many elements. Similarly, if there is a base for the topology of X_j with only finitely or countably many elements for each $j \in I$, then there is a base for the product topology on X with only finitely or countably many elements.

Let d_l be a q_{d_l}-semimetric on X_l for some $l \in I$ and $q_{d_l} > 0$. Put

$$\widetilde{d_l}(x, y) = d_l(x_l, y_l) \tag{17.2}$$

for every $x, y \in X$, where x_l, y_l are the lth coordinates of x, y in X_l. It is easy to see that this defines a q_{d_l}-semimetric on X. Suppose that for each $l \in I$, the topology on X_l is determined by a nonempty collection $\mathcal{M}_l$ of q-semimetrics on X_l as in Section 4, where $q > 0$ is allowed to depend on the element of $\mathcal{M}_l$. Let

$$\widetilde{\mathcal{M}_l} = \{\widetilde{d_l} : d_l \in \mathcal{M}_l\} \tag{17.3}$$

be the collection of q-semimetrics on X that correspond to elements of $\mathcal{M}_l$ as in (17.2) for each $l \in I$. Thus

$$\bigcup_{l \in I} \widetilde{\mathcal{M}_l} \tag{17.4}$$

is a nonempty collection of q-semimetrics on X. Under these conditions, one can check that the product topology on X is the same as the topology determined on X by (17.4) as in Section 4.

Of course, $\widetilde{\mathcal{M}_l}$ has the same cardinality as $\mathcal{M}_l$ for each $l \in I$. If I has only finitely many elements, and $\mathcal{M}_l$ has only finitely many elements for each $l \in I$, then it follows that (17.4) has only finitely many elements too. Similarly, if I has only finitely or countably many elements, and if $\mathcal{M}_l$ has only finitely or countably many elements for each $l \in I$, then (17.4) has only finitely or countably many elements as well. The discussion in Section 10 can be applied in these cases.

18 Cartesian products, continued

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let I be a nonempty set, and let V_j be a topological vector space over k for each $j \in I$. Of course, the corresponding Cartesian product

$$V = \prod_{j \in I} V_j. \tag{18.1}$$

is a vector space over k with respect to coordinatewise addition and scalar multiplication. One can also verify that V is a topological vector space over k with respect to the corresponding product topology. Note that the standard coordinate mappings from V onto the individual factors are linear as well as continuous.

If d_l is a q_{d_l}-semimetric on V_l for some $l \in I$ and $q_{d_l} > 0$, then we get a q_{d_l}-semimetric $\widetilde{d_l}$ on V by applying d_l to the lth coordinates of pairs of elements of V, as in (17.2). If d_l is also translation-invariant on V_l, then it is easy to see that $\widetilde{d_l}$ is invariant under translations on V too. Thus a collection $\mathcal{M}_l$ of translation-invariant q-semimetrics on V_l leads to a collection $\widetilde{\mathcal{M}_l}$ of translation-invariant q-semimetrics on V as in (17.3). If the topology on V_l is determined by a nonempty collection $\mathcal{M}_l$ of translation-invariant q-semimetrics for each $l \in I$, then the product topology on V is determined by the corresponding collection of translation-invariant q-semimetrics (17.4).

Similarly, let N_l be a q_{N_l}-seminorm on V_l for some $l \in I$ and $q_{N_l} > 0$, with respect to $|\cdot|$ on k. Put

$$\widetilde{N_l}(v) = N_l(v_l) \tag{18.2}$$

for every $v \in V$, where v_l is the lth coordinate of v in V_l. It is easy to see that this defines a q_{N_l}-seminorm on V. Let

$$d_l(v_l, w_l) = N_l(v_l - w_l) \tag{18.3}$$

be the q_{N_l}-semimetric on V_l associated to N_l in the usual way, and let

$$\widetilde{d_l}(v, w) = \widetilde{N_l}(v - w) \tag{18.4}$$

be the q_{N_l}-semimetric on V associated to $\widetilde{N_l}$ in the same way. Under these conditions, d_l and $\widetilde{d_l}$ are also related to each other as in (17.2), because of (18.2).

Suppose that for each $l \in I$, the topology on V_l is determined by a nonempty collection $\mathcal{N}_l$ of q-seminorms on V_l as in Section 4, where $q > 0$ is allowed to depend on the element of $\mathcal{N}_l$. Let

$$\widetilde{\mathcal{N}_l} = \{\widetilde{N_l} : N_l \in \mathcal{N}_l\} \tag{18.5}$$

be the collection of q-seminorms on V that correspond to elements of $\mathcal{N}_l$ as in (18.2) for each $l \in I$. Thus

$$\bigcup_{l \in I} \widetilde{\mathcal{N}_l} \tag{18.6}$$

is a nonempty collection of q-seminorms on V, and one can check that the product topology on V is the same as the topology determined on V by (18.6) as in Section 4.

More precisely, let

$$\mathcal{M}_l = \mathcal{M}(\mathcal{N}_l) \tag{18.7}$$

be the collection of q-semimetrics on V_l corresponding to elements of $\mathcal{N}_l$ as in (18.3) for each $l \in I$, which is the same as (4.8) in this situation. Similarly, for each $l \in I$, let

$$\widetilde{\mathcal{M}_l} = \mathcal{M}(\widetilde{\mathcal{N}_l}) \tag{18.8}$$

be the collection of q-semimetrics on V that correspond to elements of $\widetilde{\mathcal{N}_l}$ as in (18.4). Equivalently, $\widetilde{\mathcal{M}_l}$ is related to $\mathcal{M}_l$ as in (17.3). Taking the union over $l \in I$, we get that (17.4) consists in this situation of q-semimetrics on V that correspond to elements of (18.6) in the usual way, as in (18.4). Thus the description of the product topology on V in terms of (18.6) in the preceding paragraph corresponds exactly to the analogous statement for collections of q-semimetrics in the previous section.

Suppose that I has only finitely or countably many elements, and that for each $j \in I$, V_j is a topological vector space over k with a local base for its topology at 0 with only finitely or countably many elements. This implies that there is a local base for the product topology on V at 0 with only finitely or countably many elements, as mentioned in the previous section. Similarly, in the context of the preceding paragraphs, if I has only finitely or countably many elements, and if $\mathcal{N}_l$ has only finitely or countably many elements for each $l \in I$, then (18.6) has only finitely or countably many elements. If I has only finitely many elements, and if $\mathcal{N}_l$ has only finitely many elements for each $l \in I$, then (18.6) has only finitely many elements. In these cases, the discussion in Section 12 can be applied.

Suppose now that $|\cdot|$ is nontrivial on k. If E is a bounded subset of V with respect to the product topology, as in Section 9, then the image of E in V_j under the standard coordinate mapping is bounded for each $j \in I$. Conversely, if the image of E in each V_j is bounded, then one can check that E is bounded in V. This is basically the same as saying that if E_j is a bounded subset of V_j for each $j \in I$, then

$$\prod_{j \in I} E_j \tag{18.9}$$

is a bounded set in V. It follows that every bounded subset of V is contained in a product of bounded subsets of the V_j's, as in (18.9).

19 Equicontinuity

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k. Also let $\mathcal{E}$ be a collection of linear mappings from V into W. We say that $\mathcal{E}$ is *equicontinuous* on V if for each open set U_W in W that contains 0 there is an open set U_V in V such that $0 \in U_V$ and

$$T(U_V) \subseteq U_W \tag{19.1}$$

for every $T \in \mathcal{E}$. Of course, this implies that each $T \in \mathcal{E}$ is continuous as a mapping from V into W. If $\mathcal{E}$ has only finitely many elements, and if each element of $\mathcal{E}$ is continuous, then it is easy to see that $\mathcal{E}$ is equicontinuous.

Put

$$\delta_a(v) = a\, v \tag{19.2}$$

for each $a \in k$ and $v \in V$, so that δ_a defines a continuous linear mapping from V into itself for each $a \in k$, by continuity of scalar multiplication. If $|\cdot|$ is nontrivial on k, then

$$\{\delta_a : a \in k,\ |a| \leq 1\} \tag{19.3}$$

is equicontinuous on V. This follows from the fact that nonempty balanced open subsets of V form a local base for the topology of V at 0 in this case, as in Section 7. Using this, one can check that

$$\{\delta_a : a \in k,\ |a| \le r\} \tag{19.4}$$

is equicontinuous on V for each positive real number r. More precisely, one can reduce to showing that this holds when $r = |t|$ for some $t \in k$. One can also look at this more directly in terms of continuity of scalar multiplication on V, as a mapping from $k \times V$ into V. Continuity of this mapping at $(0,0)$ basically corresponds to the equicontinuity of (19.4) for some $r > 0$. If $|\cdot|$ is nontrivial on k, then one can use this and the continuity of δ_t for each $t \in k$ to get that (19.4) is equicontinuous on V for every $r > 0$.

Suppose for the moment that $|\cdot|$ is trivial on k, so that (19.4) contains only δ_0 when $r < 1$, and (19.4) is the same as

$$\{\delta_a : a \in k\} \tag{19.5}$$

when $r \ge 1$. Thus the equicontinuity of (19.4) is trivial when $r < 1$. If there is a local base for the topology of V at 0 consisting of balanced open sets in V, then (19.5) is equicontinuous on V, as before. In particular, this holds when the topology on V is determined by a collection of q-seminorms with respect to the trivial absolute function on k. Conversely, if (19.5) is equicontinuous on V, then one can check that nonempty balanced open subsets of V form a local base for the topology of V at 0, using an argument like the one in Section 7.

Suppose that $|\cdot|$ is nontrivial on k again, and let $\mathcal{E}$ is an equicontinuous collection of linear mappings from V into W. If A is a bounded subset of V, then it is easy to see that

$$\bigcup_{T \in \mathcal{E}} T(A) \tag{19.6}$$

is a bounded subset of W, directly from the definitions. This is a type of uniform boundedness property for $\mathcal{E}$. Now let $\mathcal{E}$ be any collection of linear mappings from V into W, and suppose that there is an open set U_0 in V such that $0 \in U_0$ and

$$\bigcup_{T \in \mathcal{E}} T(U_0) \tag{19.7}$$

is a bounded set in W. This implies that each $T \in \mathcal{E}$ is strongly bounded in the sense defined in Section 14, and indeed this condition may be considered as a uniform version of strong boundedness. It is easy to see that this condition implies that $\mathcal{E}$ is equicontinuous, just as strong boundedness implies continuity. If there is a bounded open set U_0 in V that contains 0, then the uniform boundedness property mentioned earlier implies this condition, and in particular this holds when $\mathcal{E}$ is equicontinuous. Similarly, if there is a bounded open set in W that contains 0, and if $\mathcal{E}$ is equicontinuous, then $\mathcal{E}$ satisfies this condition.

20 Equicontinuity, continued

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, let V, W be topological vector spaces over k, and let $\mathcal{E}$ be a collection of linear mappings from V into W. Suppose that the topology on W is determined by a nonempty collection $\mathcal{N}_W$ of q-seminorms on W with respect to $|\cdot|$ on k, where $q > 0$ is allowed to depend on the element of $\mathcal{N}_W$, as usual. Under these conditions, $\mathcal{E}$ is equicontinuous if and only if for each $N_W \in \mathcal{N}_W$ and $r > 0$ there is an open set $U \subseteq V$ such that $0 \in U$ and

$$T(U) \subseteq \{w \in W : N_W(w) < r\} \tag{20.1}$$

for every $T \in \mathcal{E}$. Equivalently, (20.1) means that

$$N_W(T(v)) < r \tag{20.2}$$

for every $v \in U$ and $T \in \mathcal{E}$. This characterization of equicontinuity is analogous to the corresponding statement for continuous linear mappings in Section 13.

Suppose now that the topology on V is also determined by a nonempty collection $\mathcal{N}_V$ of q-seminorms on V with respect to $|\cdot|$ on k, where $q > 0$ is may depend on the element of $\mathcal{N}_V$. In this case, $\mathcal{E}$ is equicontinuous if and only if for every $N_W \in \mathcal{N}_W$ and $r > 0$ there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and positive real numbers $r_1, \ldots, r_l$ such that (20.1) holds with U equal to

$$\{v \in V : N_{V,j}(v) < r_j \text{ for every } j = 1, \ldots, l\}. \tag{20.3}$$

Equivalently, this means that (20.2) holds for every $T \in \mathcal{E}$ and $v \in V$ such that

$$N_{V,j}(v) < r_j \tag{20.4}$$

for each $j = 1, \ldots, l$. As before, this is very similar to the analogous characterization of continuity in Section 13.

If for each $N_W \in \mathcal{N}_W$ there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and a nonnegative real number C such that

$$N_W(T(v)) \le C \max_{1 \le j \le l} N_{V,j}(v) \tag{20.5}$$

for every $T \in \mathcal{E}$ and $v \in V$, then $\mathcal{E}$ is equicontinuous. This follows from the remarks in the preceding paragraph, since (20.5) implies that (20.2) holds when $v \in V$ satisfies (20.4) and $r_1, \ldots, r_l$ are sufficiently small. In the other direction, if $\mathcal{E}$ is equicontinuous, and $|\cdot|$ is nontrivial on k, then $\mathcal{E}$ satisfies the condition that was just described. This is analogous to another statement for continuous linear mappings in Section 13.

Suppose that the topologies on V, W are determined by single q_V, q_W-seminorms N_V, N_W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. If T is a bounded linear mapping from V into W with respect to N_V and N_W, in the sense of Section 15, then we let $\|T\|_{op}$ be the corresponding operator q_W-seminorm of T, as in (15.4). Let $\mathcal{E}$ be a collection of bounded linear mappings from V into W in this sense, with uniformly bounded operator q_W-seminorms. This means that there is a nonnegative real number C such that

$$\|T\|_{op} \le C \tag{20.6}$$

for every $T \in \mathcal{E}$. This is the same as saying that

$$N_W(T(v)) \le C\, N_V(v) \tag{20.7}$$

for every $T \in \mathcal{E}$ and $v \in V$, as in (15.1). This condition implies that $\mathcal{E}$ is equicontinuous, and indeed it corresponds to (20.5) in this case. In the other direction, if $\mathcal{E}$ is equicontinuous, and if $|\cdot|$ is nontrivial on k, then (20.6) holds for some $C \ge 0$. This follows from the analogous statement in the previous paragraph, since (20.5) reduces to (20.7) in this situation.

21 Cauchy sequences

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a topological vector space over k. A sequence $\{v_j\}_{j=1}^{\infty}$ of elements of V is said to be a *Cauchy sequence* in V if for each open set $U \subseteq V$ with $0 \in U$ there is a positive integer L such that

$$v_j - v_l \in U \tag{21.1}$$

for every $j, l \ge L$. One can check that convergent sequences in V are Cauchy sequences, using continuity of addition on V at 0. Suppose for the moment that the topology on V is determined by a nonempty collection $\mathcal{M}$ of translation-invariant q-semimetrics, where $q > 0$ is allowed to depend on the element of $\mathcal{M}$. In this case, a sequence $\{v_j\}_{j=1}^{\infty}$ of elements of V is a Cauchy sequence in the sense just described if and only if $\{v_j\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to each $d \in \mathcal{M}$ in the usual sense, so that

$$\lim_{j,l \to \infty} d(v_j, v_l) = 0 \tag{21.2}$$

for every $d \in \mathcal{M}$.

Let us say that a topological vector space V over k is *sequentially complete* if every Cauchy sequence of elements of V converges to an element of V. If there is a local base for the topology of V at 0 with only finitely or countably many elements, then one might simply say that V is *complete*. Otherwise, one should normally consider Cauchy nets or filters in V as well. If the topology on V is determined by a translation-invariant metric $d(\cdot,\cdot)$, then it follows that V is sequentially complete as a topological vector space if and only if V is complete as a metric space with respect to $d(\cdot,\cdot)$. In particular, the Baire category theorem holds on V under these conditions. If there is a local base for the topology of V at 0 with only finitely or countably many elements, then there is a translation-invariant semimetric $d(\cdot,\cdot)$ on V that determines the same topology on V, as mentioned in Section 12. If we also ask that $\{0\}$ be a closed set in V, then $d(\cdot, \cdot)$ is a metric on V.

Let I be a nonempty set, and let V_j be a topoogical vector space over k for each $j \in I$. Thus the Cartesian product

$$V = \prod_{j \in I} V_j \tag{21.3}$$

is also a topological vector space over k with respect to the corresponding product topology, as in Section 18. Remember that a sequence of elements of V converges to some element of V with respect to the product topology if and only if the corresponding sequences of jth coordinates converge to the appropriate limit in V_j for each $j \in I$, as in Section 17. Similarly, one can check that a sequence of elements of V is a Cauchy sequence in this situation if and only if the corresponding sequences of jth coordinates are Cauchy sequences in V_j for each $j \in I$. If V_j is sequentially complete as a topological vector space over k for each $j \in I$, then it follows that V is sequentially complete too.

Let V be any topological vector space over k again, and let $\{v_j\}_{j=1}^\infty$ be a Cauchy sequence of elements of V. We would like to verify that the set of v_j's is bounded in V when $|\cdot|$ is nontrivial on k. Let U be an open subset of V that contains 0, and remember that U is absorbing in V, as in Section 8. This implies that for each positive integer j, we have that

$$v_j \in t\,U \tag{21.4}$$

for every $t \in k$ such that $|t|$ is sufficiently large, depending on j. We want to show that if $|t|$ is sufficiently large, then (21.4) holds for every j simultaneously. Let U_1, U_2 be open subsets of V that contain 0 and satisfy

$$U_1 + U_2 \subseteq U, \tag{21.5}$$

which exist by the continuity of addition on V at 0. We may as well ask that U_1, U_2 be balanced in V too, as in Section 7. Because $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence, there is a positive integer L such that

$$v_j \in v_L + U_2 \tag{21.6}$$

for every $j \geq L$. We also have that $v_L \in t\,U_1$ for every $t \in k$ such that $|t|$ is sufficiently large, because U_1 is absorbing in V, as before. It follows that

$$v_j \in v_L + U_2 \subseteq t\,U_1 + U_2 \subseteq t\,U_1 + t\,U_2 \subseteq t\,U \tag{21.7}$$

for every $j \geq L$ when $|t|$ is sufficiently large, using the condition that U_2 be balanced to get the second inclusion. This implies that (21.4) holds simulataneously for every $j \geq 1$ when $|t|$ is sufficiently large, using the fact that U is absorbing to deal with $j < L$.

22 Open subgroups

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, although we are essentially only concerned with $q_k = \infty$ in this section, so that $|\cdot|$ is an ultrametric absolute value function on k. Let V be a vector space over k, which is a commutative group with respect to addition in particular. If N is a semi-ultranorm on V, then it is easy to see that open and closed balls in V centered at 0 with respect to N are subgroups of V as a commutative group with respect to addition. As in Section 4, open and closed balls in V centered at 0 of positive radius are open sets with respect to the topology determined on V by the semi-ultrametric associated to N.

Now let V be a topological vector space over k. If U is an open subset of V that is also a subgroup of V as a commutative group with respect to addition, then it is well known that U is a closed set in V too. This is because the complement of U in V can be expressed as a union of translates of U in V, which are cosets of U in V, so that the complement of U in V is an open set in V as well. Similarly, open subgroups of any topological group are closed sets.

Let A be any subset of V that contains 0, and put

$$A_1 = A \cup (-A). \tag{22.1}$$

Define A_j recursively for each positive integer j by putting

$$A_{j+1} = A_j + A_1, \tag{22.2}$$

which is the same as taking A_j to be the sum of j copies of A_1. It is easy to see that

$$\bigcup_{j=1}^{\infty} A_j \tag{22.3}$$

is the subgroup of V as a commutative group with respect to addition generated by A. If A is an open set in V, then A_1 is an open set in V, which implies that A_j is an open set in V for each $j \geq 1$, and hence that (22.3) is an open set in V. If A is balanced in V, then $A_1 = A$, A_j is balanced in V for each j, and (22.3) is balanced in V as well.

Let W be another topological vector space over k, and let T be a continuous linear mapping from V into W. If U_W is an open set in W, then $T^{-1}(U_W)$ is an open set in V, by continuity. If U_W is a subgroup of W as a commutative group with respect to addition, then $T^{-1}(U_W)$ is a subgroup of V with respect to addition, because linear mappings are group homomorphisms with respect to addition. Thus $T^{-1}(U_W)$ is an open subgroup of V with respect to addition when U_W is an open subgroup in W with respect to addition.

Let $\mathcal{E}$ be an equicontinuous collection of linear mappings from V into W, and let U_W be an open subgroup of W as a commutative group with respect to addition. Thus $0 \in U_W$, and so the equicontinuity of $\mathcal{E}$ implies that there is an open set $U_V \subseteq V$ with $0 \in U_V$ that satisfies (19.1) for every $T \in \mathcal{E}$. Let $\widetilde{U}_V$ be the subgroup of V as a commutative group with respect to addition generated by U_V. Equivalently, if we take $A = U_V$, then $\widetilde{U}_V$ is given as in (22.3), which is an open set in V too, as before. Under these conditions, one can check that

$$T(\widetilde{U}_V) \subseteq U_W \tag{22.4}$$

for every $T \in \mathcal{E}$.

Let U be an open subgroup of V as a commutative group with respect to addition again. If $|\cdot|$ is not the trivial absolute value function on k, then there is a nonempty balanced open set $U_1 \subseteq V$ such that $U_1 \subseteq U$, as in Section 7. Let $\widetilde{U}_1$ be the subgroup of V with respect to addition generated by U_1, which can be given as in (22.3) with $A = U_1$. Thus $\widetilde{U}_1$ is an open set in V, and $\widetilde{U}_1 \subseteq U$, because U is a subgroup of V with respect to addition. We also have that $\widetilde{U}_1$ is balanced in this situation, as mentioned earlier.

Suppose now that $|\cdot|$ is the trivial absolute value function on V. In this case, a balanced subgroup of V with respect to addition is the same as a linear subspace of V. If there is a nonempty balanced open set $U_1 \subseteq V$ such that $U_1 \subseteq U$, then the other remarks in the preceding paragraph still work.

Part II
The strong operator topology

23 Definitions

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k. Remember that the space $\mathcal{CL}(V,W)$ of continuous linear mappings from V into W is also a vector space over k with respect to pointwise addition and scalar multiplication, as in Section 16. Put

$$L_v(T) = T(v) \tag{23.1}$$

for each $v \in V$ and $T \in \mathcal{CL}(V,W)$, which defines a linear mapping from $\mathcal{CL}(V,W)$ into W for each $v \in V$. The *strong operator topology* on $\mathcal{CL}(V,W)$ is defined to be the weakest topology on $\mathcal{CL}(V,W)$ such that L_v is continuous for every $v \in V$.

More precisely, if $v \in V$ and U_W is an open set in W, then

$$L_v^{-1}(U_W) = \{T \in \mathcal{CL}(V,W) : T(v) \in U_W\} \tag{23.2}$$

is an open set in $\mathcal{CL}(V,W)$ with respect to the strong operator topology. The collection of subsets of $\mathcal{CL}(V,W)$ of this form define a sub-base for the strong operator topology on $\mathcal{CL}(V,W)$. Thus $\mathcal{U} \subseteq \mathcal{CL}(V,W)$ is an open set with respect to the strong operator topology if for each $T_0 \in \mathcal{U}$ there are finitely many vectors $v_1, \ldots, v_l$ in V and open sets $U_1, \ldots, U_l$ in W such that $T_0(v_j) \in U_j$ for each $j = 1, \ldots, l$ and

$$\bigcap_{j=1}^{n} L_{v_j}^{-1}(U_j) = \{T \in \mathcal{CL}(V,W) : T(v_j) \in U_j \text{ for each } j = 1, \ldots, l\} \subseteq \mathcal{U}. \tag{23.3}$$

One can start with this as the definition of an open set in $\mathcal{CL}(V, W)$ with respect to the strong operator topology, and check that this defines a topology on $\mathcal{CL}(V, W)$. Using this as the definition, it is easy to see that (23.2) is an open set in $\mathcal{CL}(V, W)$ with respect to the strong operator topology for every $v \in V$ and open set $U \subseteq W$, and that these open sets form a sub-base for the strong operator topology on $\mathcal{CL}(V, W)$.

The strong operator topology on $\mathcal{CL}(V, W)$ can also be described as the topology of pointwise convergence on V. This can be defined on the space of mappings from a nonempty set X into a topological space Y. This space can be identified with the Cartesian product of copies of Y indexed by X, so that the topology of pointwise convergence on X corresponds exactly to the product topology on this Cartesian product, using the given topology on Y on each factor. In particular, the strong operator topology could be defined in the same way on the space of all linear mappings from V into W.

Note that L_v in (23.1) depends linearly on v, because T is linear. Suppose that A is a subset of V whose linear span in V is all of V, so that each element of V can be expressed as a linear combination of finitely many elements of A with coefficients in k. This implies that for every $v \in V$, L_v can be expressed as a linear combination of finitely many mappings of the form L_a with $a \in A$. If L_a is continuous with respect to some topology on $\mathcal{CL}(V, W)$ for every $a \in A$, then it follows that L_v is continuous with respect to the same topology on $\mathcal{CL}(V, W)$ for every $v \in V$. This uses the hypothesis that W be a topological vector space, so that the vector space operations on W are continuous. Thus the strong operator topology on $\mathcal{CL}(V, W)$ can also be described as the weakest topology on $\mathcal{CL}(V, W)$ such that L_a is continuous for every $a \in A$ in this case. Similarly, one can restrict one's attention to $v_j \in A$ in (23.3), and get an equivalent definition of the strong operator topology on $\mathcal{CL}(V, W)$. One can also restrict one's attention to $v \in A$ in (23.2), to get a sub-base for the strong operator topology on $\mathcal{CL}(V, W)$. This would work as well for the space of all linear mappings from V into W, as in the preceding paragraph.

24 Some basic properties

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let W be a topological vector space over k. Let us consider the case where $V = k$, as a one-dimensional vector space over itself, and equipped with the topology determined by the q_k-metric associated to the q_k-absolute value function $|\cdot|$. If $w \in W$, then

$$t \mapsto t\,w \tag{24.1}$$

defines a continuous linear mapping from k into W, by continuity of scalar multiplication on W, as in Section 5. Every linear mapping from k into W is of this form, so that $\mathcal{CL}(k, W)$ can be identified with W. In this situation, the strong operator topology on $\mathcal{CL}(k, W)$ corresponds exactly to the given topology on W with respect to this identification.

Let V be any topological vector space over k again, and suppose that the topology on W is determined by a nonempty collection $\mathcal{M}_W$ of translation-invariant q-semimetrics on W, where $q > 0$ may depend on the element of $\mathcal{M}_W$, as usual. Let d be an element of $\mathcal{M}_W$, so that d is a translation-invariant q_d-semimetric on W for some $q_d > 0$, and put

$$d_v(T_1, T_2) = d(T_1(v), T_2(v)) \tag{24.2}$$

for every $v \in V$ and $T_1, T_2 \in \mathcal{CL}(V, W)$. This defines a translation-invariant q_d-semimetric on $\mathcal{CL}(V, W)$ for each $v \in V$, so that

$$\mathcal{M} = \{d_v : d \in \mathcal{M}_W,\ v \in V\} \tag{24.3}$$

is a nonempty collection of translation-invariant q-semimetrics on $\mathcal{CL}(V, W)$. Under these conditions, the strong operator topology on $\mathcal{CL}(V, W)$ is the same as the topology determined by (24.3). There is an analogous statement for the topology of pointwise convergence on the space of all mappings from V into W, as in Sections 17 and 18. If A is a subset of V whose linear span in V is equal to V, then the subcollection

$$\{d_v : d \in \mathcal{M}_W,\ v \in A\} \tag{24.4}$$

of (24.3) determines the same topology on $\mathcal{CL}(V, W)$. This uses the remarks about the same situation in the previous section, and there is an analogous statement for the space of all linear mappings from V into W.

Similarly, suppose that the topology on W is determined by a nonempty collection $\mathcal{N}_W$ of q-seminorms on W with respect to $|\cdot|$ on k, where $q > 0$ may depend on the element of $\mathcal{N}_W$, as usual. Let N be an element of $\mathcal{N}_W$, so that N is a q_N-seminorm on W for some $q_N > 0$, and put

$$N_v(T) = N(T(v)) \tag{24.5}$$

for every $v \in V$ and $T \in \mathcal{CL}(V, W)$. This defines a q_N-seminorm on $\mathcal{CL}(V, W)$ for each $v \in V$, so that

$$\mathcal{N} = \{N_v : N \in \mathcal{N}_W,\ v \in V\} \tag{24.6}$$

defines a nonempty collection of q-seminorms on $\mathcal{CL}(V, W)$. As before, the strong operator topology on $\mathcal{CL}(V, W)$ is the same as the topology associated to this collection of q-seminorms, and there is an analogous statement for the topology of pointwise convergence on the space of all mappings from V into W. Of course, q-seminorms lead to translation-invariant q-semimetrics in the usual way, and this case is related to the previous one as in Section 18. If A is a subset of V whose linear span is all of V, then the subcollection

$$\{N_v : N \in \mathcal{N}_W,\ v \in A\} \tag{24.7}$$

of (24.6) determines the same topology on $\mathcal{CL}(V, W)$, as in the preceding paragraph. There is an analogous statement for the space of all linear mappings from V into W again.

Suppose now that $W = k$, as a one-dimensional vector space over itself, and with the topology associated to the q_k-absolute value function $|\cdot|$, as before. A linear mapping from V into k is also known as a linear functional on V, so that $\mathcal{CL}(V,k)$ is the dual space of continuous linear functionals on V, which may be denoted V^* or V'. In this case, the strong operator topology on $\mathcal{CL}(V,k)$ is also known as the *weak* topology*. Of course, $|\cdot|$ may be considered as a q_k-norm on k as a one-dimensional vector space over itself. Thus the remarks in the preceding paragraph can be applied to this situation.

25 Equicontinuous sets

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k. If $U_W \subseteq W$ is an open set that contains 0, then there is an open set $\widetilde{U}_W \subseteq W$ that contains 0 and satisfies

$$\widetilde{U}_W + \widetilde{U}_W + \widetilde{U}_W \subseteq U_W, \tag{25.1}$$

because of continuity of addition on W at 0. We may also ask that $\widetilde{U}_W$ be symmetric about 0, in the sense that

$$-\widetilde{U}_W = \widetilde{U}_W, \tag{25.2}$$

because of continuity of $w \mapsto -w$ on W. Suppose that $\mathcal{E}$ is an equicontinuous collection of linear mappings from V into W, so that there is an open set $U_V \subseteq V$ that contains 0 and satisfies

$$T(U_V) \subseteq \widetilde{U}_W \tag{25.3}$$

for every $T \in \mathcal{E}$. If $u, v \in V$ satisfy

$$u - v \in U_V, \tag{25.4}$$

then it follows that

$$T(u) - T(v) = T(u - v) \in \widetilde{U}_W \tag{25.5}$$

for every $T \in \mathcal{E}$.

Let $T_1, T_2 \in \mathcal{E}$ be given, and observe that

$$T_1(u) - T_2(u) = (T_1(u) - T_1(v)) + (T_1(v) - T_2(v)) + (T_2(v) - T_2(u)) \tag{25.6}$$

for every $u, v \in V$. If $u, v \in V$ satisfy (25.4), then we can apply (25.5) to T_1 and T_2, to get that

$$T_1(u) - T_1(v),\ T_2(u) - T_2(v) \in \widetilde{U}_W. \tag{25.7}$$

Combining this with (25.6), we obtain that

$$T_1(u) - T_2(u) \in (T_1(v) - T_2(v)) + \widetilde{U}_W + \widetilde{U}_W \tag{25.8}$$

when (25.4) holds, using (25.2) as well. If we also have that

$$T_1(v) - T_2(v) \in \widetilde{U}_W, \tag{25.9}$$

then it follows that

$$T_1(u) - T_2(u) \in \widetilde{U}_W + \widetilde{U}_W + \widetilde{U}_W \subseteq U_W \tag{25.10}$$

under these conditions, because of (25.1). To summarize, (25.10) holds for every $T_1, T_2 \in \mathcal{E}$ and $u, v \in V$ that satisfy (25.4) and (25.9).

Let A be a nonempty subset of V, and let τ_A be the analogue of the strong operator topology on $\mathcal{CL}(V, W)$ in which we restrict our attention to $v \in A$. This is the weakest topology on $\mathcal{CL}(V, W)$ such that (23.1) is continuous for every $v \in A$, which can be described more precisely as in (23.2) and (23.3), but with the v's in A. This can also be described in terms of the topology of pointwise convergence on A. If the topology on W is determined by a nonempty collection $\mathcal{M}_W$ of translation-invariant q-semimetrics, as in Section 24, then τ_A is determined by the corresponding collection (24.4) of translation-invariant q-semimetrics on $\mathcal{CL}(V, W)$. Similarly, if the topology on W is determined by a nonempty collection $\mathcal{N}_W$ of q-seminorms, as in Section 24, then τ_A is determined by the corresponding collection (24.7) of q-seminorms on $\mathcal{CL}(V, W)$. Of course, τ_A is the same as the strong operator topology on $\mathcal{CL}(V, W)$ when $A = V$, and the same conclusion holds when the linear span of A is equal to V, as in Section 23. If A is any subset of V, then one can check that τ_A is the same as the topology on $\mathcal{CL}(V, W)$ that corresponds to the linear span of A in the same way, as in Section 23.

Let A_1, A_2 be nonempty subsets of V. If $A_1 \subseteq A_2$, then

$$\tau_{A_1} \subseteq \tau_{A_2}, \tag{25.11}$$

and the same conclusion holds when A_1 is contained in the linear span of A_2. Suppose now that

$$A_1 \subseteq \overline{A_2}, \tag{25.12}$$

where $\overline{A_2}$ is the closure of A_2 in V. If $\mathcal{E}$ is an equicontinuous collection of linear mappings from V into W, then the topology induced on $\mathcal{E}$ by τ_{A_2} is at least as strong as the topology induced by τ_{A_1}. To see this, let $u \in A_1$ be given, and let U_W be an open subset of W that contains 0. This leads to a symmetric open subset $\widetilde{U}_W$ of W that contains 0 and satisfies (25.1), and an open subset U_V of V that contains 0 and satisfies (25.3), using the equicontinuity of $\mathcal{E}$ to get U_V. If (25.12) holds, then there is a $v \in A_2$ that satisfies (25.4). Thus (25.9) implies (25.10) for every $T_1, T_2 \in \mathcal{E}$ in this situation, as before. This can be used to verify that every relatively open set in $\mathcal{E}$ with respect to the topology induced by τ_{A_1} is also relatively open with respect to the topology induced by τ_{A_2} when (25.12) holds. As before, the same conclusion holds when A_1 is contained in the closure of the linear span of A_2 in V. In particular, if the closure of the linear span of $A \subseteq V$ is dense in V, and if $\mathcal{E}$ is equicontinuous, then the topology induced on $\mathcal{E}$ by τ_A is the same as the one induced by the strong operator topology on $\mathcal{CL}(V, W)$.

26 Closed sets

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k. Also let v be an element of V, and let L_v be defined as in (23.1). If E is a closed subset of W, then

$$L_v^{-1}(E) = \{T \in \mathcal{CL}(V, W) : T(v) \in E\} \tag{26.1}$$

is a closed set in $\mathcal{CL}(V, W)$ with respect to the strong operator topology. In particular, if $\{0\}$ is a closed set in W, then we can take $E = \{0\}$ in (26.1), to get that

$$\{T \in \mathcal{CL}(V, W) : T(v) = 0\} \tag{26.2}$$

is a closed set in $\mathcal{CL}(V, W)$ with respect to the strong operator topology. This implies that $\{0\}$ is a closed set in $\mathcal{CL}(V, W)$ with respect to the strong operator topology, by taking the intersection of (26.2) over all $v \in V$.

As in Section 23, one can define the topology of pointwise convergence on the space of arbitrary mappings from V into W. If $\{0\}$ is a closed set in W, then it is easy to see that the space of arbitrary linear mappings from V into W is a closed set in the space of arbitrary mappings from V into W with respect to this topology. Suppose that $\mathcal{E}$ is an equicontinuous collection of continuous linear mappings from V into W, and let $\overline{\mathcal{E}}$ be the closure of $\mathcal{E}$ in the space of arbitrary linear mappings from V into W, with respect to the topology of pointwise convergence. Thus $\overline{\mathcal{E}}$ consists of the linear mappings from V into W that can be approximated by elements of $\mathcal{E}$ on finite subsets of V, and using the given topology on W. If $\{0\}$ is a closed set in W, then $\overline{\mathcal{E}}$ is the same as the closure of $\mathcal{E}$ in the space of arbitrary mappings from V into W with respect to the topology of pointwise convergence, by the previous remark.

Let U_W be an open subset of W that contains 0. Because $\mathcal{E}$ is equicontinuous, there is an open set U_V in V such that $0 \in U_V$ and

$$T(U_V) \subseteq U_W \tag{26.3}$$

for every $T \in \mathcal{E}$. Equivalently, this means that

$$T(v) \in U_W \tag{26.4}$$

for every $v \in V$ and $T \in \mathcal{E}$. It follows that

$$T(v) \in \overline{U_W} \tag{26.5}$$

for every $v \in U_V$ and $T \in \overline{\mathcal{E}}$, where $\overline{U_W}$ is the closure of U_W in W. This is the same as saying that

$$T(U_V) \subseteq \overline{U_W} \tag{26.6}$$

for every $T \in \overline{\mathcal{E}}$. This implies that each $T \in \overline{\mathcal{E}}$ is continuous at 0, because W is regular in the strict sense as a topological space. Thus $\overline{\mathcal{E}}$ is contained in $\mathcal{CL}(V, W)$, so that $\overline{\mathcal{E}}$ is the same as the closure of $\mathcal{E}$ in $\mathcal{CL}(V, W)$ with respect to the strong operator topology in this situation. More precisely, (26.6) implies that $\overline{\mathcal{E}}$ is equicontinuous as well, using the regularity of W in the strict sense again. We also get that

$$T(\overline{U_V}) \subseteq \overline{U_W} \tag{26.7}$$

for every $T \in \overline{\mathcal{E}}$, where $\overline{U_V}$ is the closure of U_V in V, because each $T \in \overline{\mathcal{E}}$ is continuous.

Note that a condition like (26.5) defines a closed set of T's with respect to the strong operator topology, by the remarks at the beginning of the section. Hence conditions like (26.6) and (26.7) also define closed sets of T's with respect to the strong operator topology, since they correspond to families of conditions of the previous type. Similarly, families of conditions of this type corresponding to families of open subsets U_W of W that contain 0 define closed sets of T's with respect to the strong operator topology. If one uses a family of U_W's in a local base for the topology of W at 0, and for which the corresponding U_V's are open subsets of V that contain 0, then one gets a closed set of T's with respect to the strong product topology which is equicontinuous as well.

27 Convergence of sequences

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let V, W be topological vector spaces over k. Also let $\{T_j\}_{j=1}^{\infty}$ be a sequence of continuous linear mappings from V into W. If $\{T_j(v)\}_{j=1}^{\infty}$ converges as a sequence of elements of W for every $v \in V$, then we say that $\{T_j\}_{j=1}^{\infty}$ converges *pointwise* on V. If T is a mapping from V into W and $\{T_j(v)\}_{j=1}^{\infty}$ converges to $T(v)$ in W for every $v \in V$, then we may say that $\{T_j\}_{j=1}^{\infty}$ converges to T pointwise on V. It is easy to see that $\{T_j\}_{j=1}^{\infty}$ converges to a continuous linear mapping T from V into W pointwise on V if and only if $\{T_j\}_{j=1}^{\infty}$ converges to T with respect to the strong operator topology on $\mathcal{CL}(V, W)$.

Of course, if $\{0\}$ is a closed set in W, then W is Hausdorff, and the limit of any convergent sequence of elements of W is unique. In this case, if $\{T_j\}_{j=1}^{\infty}$ converges pointwise on V, then we can put

$$T(v) = \lim_{j \to \infty} T_j(v) \tag{27.1}$$

for each $v \in V$. This defines a mapping from V into W, and $\{T_j\}_{j=1}^{\infty}$ converges to T pointwise on V. We also have that T is linear under these conditions, as in the previous section. Otherwise, if $\{0\}$ is not a closed set in W, then we can choose a limit $T(v)$ of $\{T_j(v)\}_{j=1}^{\infty}$ in W for each vector v in a basis for V as a vector space. This defines T as a mapping from the basis for V into W, which can be extended to a linear mapping from V into W. Using this choice of T on V, one can check that $\{T_j(v)\}_{j=1}^{\infty}$ converges to $T(V)$ for every $v \in V$, so that $\{T_j\}_{j=1}^{\infty}$ converges to T pointwise on V.

Let us say that $\{T_j\}_{j=1}^{\infty}$ is equicontinuous on V if the corresponding set of T_j's is equicontinuous on V, as in Section 19. If $\{T_j\}_{j=1}^{\infty}$ is equicontinuous on V and $\{T_j\}_{j=1}^{\infty}$ converges pointwise to a linear mapping T from V into W, then T is continuous as well, as in the previous section again.

Let us say that $\{T_j\}_{j=1}^\infty$ satisfies the *pointwise Cauchy condition* on V if $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in W for every $v \in V$. It is easy to see that this happens if and only if $\{T_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the strong operator topology on $\mathcal{CL}(V, W)$. If $\{T_j\}_{j=1}^\infty$ converges pointwise on V, then $\{T_j\}_{j=1}^\infty$ satisfies the pointwise Cauchy condition on V, since convergent sequences are Cauchy sequences, as in Section 21. If W is sequentially complete and $\{T_j\}_{j=1}^\infty$ satisfies the pointwise Cauchy condition on V, then $\{T_j\}_{j=1}^\infty$ converges pointwise on V.

Let A be a subset of V, and let us say that $\{T_j\}_{j=1}^\infty$ converges pointwise on A if $\{T_j(v)\}_{j=1}^\infty$ converges in W for every $v \in A$. As before, if T is a mapping from A into W, and if $\{T_j(v)\}_{j=1}^\infty$ converges to $T(v)$ in W for every $v \in A$, then we may say that $\{T_j\}_{j=1}^\infty$ converges to T pointwise on A. If $\{T_j\}_{j=1}^\infty$ converges pointwise on A, then $\{T_j\}_{j=1}^\infty$ also converges pointwise on the linear span of A in V. Similarly, if T is a linear mapping from V into W, and if $\{T_j\}_{j=1}^\infty$ converges to T pointwise on A, then $\{T_j\}_{j=1}^\infty$ converges to T pointwise on the linear span of A in V.

If $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in W for each $v \in A$, then we say that $\{T_j\}_{j=1}^\infty$ satisfies the pointwise Cauchy condition on A. This implies that $\{T_j\}_{j=1}^\infty$ satisfies the pointwise Cauchy condition on the linear span of A in V, as before. Of course, pointwise convergence on A implies the pointwise Cauchy condition on A, and the converse holds when W is sequentially complete.

Suppose that T is a continuous mapping from V into W, and that $\{T_j\}_{j=1}^\infty$ converges to T pointwise on a set $A \subseteq V$. If $\{T_j\}_{j=1}^\infty$ is also equicontinuous on V, then one can check that $\{T_j\}_{j=1}^\infty$ converges to T pointwise on the closure $\overline{A}$ of A in V. This uses the same type of argument as in Section 25. If T is linear, then $\{T_j\}_{j=1}^\infty$ converges pointwise to T on the linear span of A in V, as mentioned earlier. If T is linear and continuous, then we get that $\{T_j\}_{j=1}^\infty$ converges to T on the closure of the linear span of A in V when $\{T_j\}_{j=1}^\infty$ is equicontinuous.

Similarly, if $\{T_j\}_{j=1}^\infty$ satisfies the pointwise Cauchy condition on a set $A \subseteq V$, and if $\{T_j\}_{j=1}^\infty$ is equicontinuous on V, then one can verify that $\{T_j\}_{j=1}^\infty$ satisfies the pointwise Cauchy condition on the closure $\overline{A}$ of A in V. As before, this uses arguments like those in Section 25. One might as well pass to the linear span of A in V first, as mentioned earlier. It follows that $\{T_j\}_{j=1}^\infty$ satisfies the pointwise Cauchy condition on the closure of the linear span of A in V under these conditions.

28 Pointwise boundedness

Let k be a field with a nontrivial q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k. Also let $\mathcal{E}$ be a collection of continuous linear mappings from V into W, and put

$$\mathcal{E}_v = \{T(v) : T \in \mathcal{E}\} \tag{28.1}$$

for each $v \in V$. If $\mathcal{E}_v$ is a bounded subset of W for every $v \in V$, then we say that $\mathcal{E}$ is bounded *pointwise* on V. One can check that this happens if and only if $\mathcal{E}$ is bounded with respect to the strong operator topology on $\mathcal{CL}(V, W)$. This basically corresponds to the discussion of bounded subsets of Cartesian products of topological vector spaces over k in Section 18.

If $\mathcal{E}$ is equicontinuous, then $\mathcal{E}$ is uniformly bounded on bounded subsets of V, as in Section 19. Of course, this implies that $\mathcal{E}$ is bounded pointwise on V, because subsets of V with only one element are bounded, as in Section 9. In some situations, it is well known that pointwise bounded subsets of $\mathcal{CL}(V, W)$ are equicontinuous. In particular, the theorem of Banach and Steinhaus implies that this is the case when V is of second category, in the sense of Baire category. More precisely, it suffices to ask that (28.1) be a bounded subset of W for a set of v of second category in V.

A sequence $\{T_j\}_{j=1}^\infty$ of countinuous linear mappings from V into W is said to be bounded pointwise on V if $\{T_j(v)\}_{j=1}^\infty$ is a bounded sequence in W for every $v \in V$. This is the same as saying that the set $\mathcal{E}$ of the T_j's is bounded pointwise on V. As before, this is equivalent to the condition that $\mathcal{E}$ be a bounded subset of $\mathcal{CL}(V, W)$ with respect to the strong operator topology, which is the same as saying that $\{T_j\}_{j=1}^\infty$ is a bounded sequence in $\mathcal{CL}(V, W)$ with respect to the strong product topology. If $\{T_j\}_{j=1}^\infty$ converges pointwise on V, then $\{T_j\}_{j=1}^\infty$ is bounded pointwise on V. This is because convergent sequences in W are bounded, as in Section 9. Similarly, if $\{T_j\}_{j=1}^\infty$ satisfies the pointwise Cauchy condition on V, then $\{T_j\}_{j=1}^\infty$ is bounded pointwise on V. This follows from the fact that Cauchy sequences in W are bounded, as in Section 21.

Suppose that $\{T_j\}_{j=1}^\infty$ is a sequence of continuous linear mappings from V into W that satisfies the pointwise Cauchy condition. Suppose also that W is sequentially complete, so that $\{T_j\}_{j=1}^\infty$ converges pointwise on V. It follows that there is a linear mapping T from V into W such that $\{T_j\}_{j=1}^\infty$ converges to T pointwise on V, as in the previous section. If $\{T_j\}_{j=1}^\infty$ is equicontinuous too, then T is continuous, as in the previous two sections. This type of argument can be used to show that $\mathcal{CL}(V, W)$ is sequentially complete with respect to the strong operator topology in some situations.

29 Bounded linear mappings, revisited

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be vector spaces over k. Also let N_V, N_W be q_V, q_W-seminorms on V, W, respectively, for some $q_V, q_W > 0$, and with respect to $|\cdot|$ on k. As in Section 16, we let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from V into W in the sense of Section 15, equipped with the corresponding operator q_W-seminorm $\|\cdot\|_{op}$. Remember that bounded linear mappings from V into W in this sense are continuous, with respect to the topologies on V, W associated to N_V, N_W, respectively. If $|\cdot|$ is nontrivial on k, then we have seen that the converse holds as well.

If T is a bounded linear mapping from V into W and $v \in V$, then

$$N_W(T(v)) \le \|T\|_{op}\, N_V(v). \tag{29.1}$$

This is because (15.1) holds with $C = \|T\|_{op}$, as mentioned just after (15.4). Hence the topology on $\mathcal{BL}(V, W)$ associated to the operator q_W-seminorm is at least as strong as the one induced by the strong operator topology on $\mathcal{CL}(V, W)$.

Let $\mathcal{E}$ be a subset of $\mathcal{BL}(V, W)$. Suppose that the elements of $\mathcal{E}$ have uniformly bounded operator q_W-seminorms, in the sense that there is a nonnegative real number C such that

$$\|T\|_{op} \leq C \tag{29.2}$$

for every $T \in \mathcal{E}$. Equivalently, this means that

$$N_W(T(v)) \leq C\, N_V(v) \tag{29.3}$$

for every $T \in \mathcal{E}$ and $v \in V$, which implies that $\mathcal{E}$ is equicontinuous, as in Section 20. Thus the remarks in Section 25 can be applied in this situation, and indeed some of the arguments could be simplified. Similarly, let $\overline{\mathcal{E}}$ be the closure of $\mathcal{E}$ in the space of all linear mappings from V into W with respect to the topology of pointwise convergence, as in Section 26. It is easy to see that the elements of $\overline{\mathcal{E}}$ satisfy (29.3) with the same constant C in this situation. It follows that $\overline{\mathcal{E}}$ is contained in $\mathcal{BL}(V, W)$, and that the elements of $\overline{\mathcal{E}}$ satisfy (29.2) as well.

Suppose now that $|\cdot|$ is nontrivial on k, so that $\mathcal{BL}(V, W)$ is the same as $\mathcal{CL}(V, W)$, as before. Let $\mathcal{E}$ be a subset of $\mathcal{BL}(V, W)$ again, and note that the elements of $\mathcal{E}$ have uniformly bounded operator q_W-seminorms if and only if $\mathcal{E}$ is bounded with respect to the topology on $\mathcal{BL}(V, W)$ associated to the operator q_W-seminorm, as in Section 9. If $\mathcal{E}$ is equicontinuous, then the elements of $\mathcal{E}$ have uniformly bounded operator q_W-seminorms in this case, as in Section 20. Pointwise boundedness of $\mathcal{E}$ on V reduces in this situation to the boundedness of

$$\{N_W(T(v)) : T \in \mathcal{E}\} \tag{29.4}$$

as a set of nonnegative real numbers for each $v \in V$. If (29.4) is bounded for a set of v of second category in V, then the Banach–Steinhaus theorem implies that the elements of $\mathcal{E}$ have uniformly bounded operator q_W-seminorms.

30 Continuity of compositions

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W, and Z be topological vector spaces over k. If R is a continuous linear mapping from V into W, and T is a continuous linear mapping from W into Z, then their composition $T \circ R$ is a continuous linear mapping from V into Z. If T is a continuous linear mapping from W into Z, then

$$R \mapsto T \circ R \tag{30.1}$$

defines a linear mapping from $\mathcal{CL}(V, W)$ into $\mathcal{CL}(V, Z)$. It is easy to see that this mapping is continuous with respect to the corresponding strong operator topologies, directly from the definitions. Similarly, if R is a continuous linear mapping from V into W, then

$$T \mapsto T \circ R \tag{30.2}$$

defines a continuous linear mapping from $\mathcal{CL}(W,Z)$ into $\mathcal{CL}(V,Z)$ with respect to the corresponding strong product topologies.

Let us now consider the continuity properties of

$$(R,T) \mapsto T \circ R \tag{30.3}$$

as a mapping from $\mathcal{CL}(V,W)\times\mathcal{CL}(W,Z)$ into $\mathcal{CL}(V,Z)$, using the corresponding strong operator topologies. This amounts to looking at

$$(R,T) \mapsto T(R(v)) \tag{30.4}$$

as a mapping from $\mathcal{CL}(V,W)\times\mathcal{CL}(W,Z)$ into W for each $v \in V$. Let R_0, R be continuous linear mappings from V into W, and let T_0, T be continuous linear mappings from W into Z. Here R_0 and T_0 should be considered as fixed for the moment, and we would like to consider the behavior of (30.3) or (30.4) when R is close to R_0 and T is close to T_0. Basically, we would like to understand when $T(R(v))$ is close to $T_0(R_0(v))$ in Z.

Let $v \in V$ be given, and observe that

$$\begin{aligned} & T(R(v)) - T_0(R_0(v)) \\ & \quad = \quad (T(R(v)) - T(R_0(v))) + (T(R_0(v)) - T_0(R_0(v))). \end{aligned} \tag{30.5}$$

The second part on the right side of (30.5),

$$T(R_0(v)) - T_0(R_0(v)), \tag{30.6}$$

is obviously close to 0 in Z when T is close to T_0 with respect to the strong operator topology on $\mathcal{CL}(W,Z)$, in a suitable sense. The first part on the right side of (30.5) is more complicated, and can be reexpressed as

$$T(R(v) - R_0(v)). \tag{30.7}$$

Of course, one can get $R(v) - R_0(v)$ to be small in W by taking R close to R_0 with respect to the strong operator topology on $\mathcal{CL}(V,W)$. This implies that (30.7) is small in Z when T is fixed, and otherwise one should be more careful.

Let $\mathcal{E}$ be an equicontinuous collection of linear mappings from W into Z. If we restrict our attention to $T \in \mathcal{E}$, then we can say that (30.7) is uniformly small in Z when $R(v) - R_0(v)$ is small in W, which can be arranged by taking R close to R_0 with respect to the strong product topology on $\mathcal{CL}(V,W)$. It follows that (30.5) is small in Z when R is close to R_0 with respect to the strong operator topology on $\mathcal{CL}(V,W)$, T is close to T_0 with respect to the strong operator topology on $\mathcal{CL}(W,Z)$, and $T \in \mathcal{E}$. This implies that for each $v \in V$, (30.4) is continuous as a mapping from $\mathcal{CL}(V,W)\times\mathcal{E}$ into W, with respect to the strong operator topology on $\mathcal{CL}(V,W)$, the topology induced on $\mathcal{E}$ by the strong operator topology on $\mathcal{CL}(W,Z)$, and the associated product topology on their Cartesian product. Thus (30.3) is continuous as a mapping from $\mathcal{CL}(V,W)\times\mathcal{E}$ into $\mathcal{CL}(V,Z)$, with respect to the strong operator topologies on $\mathcal{CL}(V,W)$ and $\mathcal{CL}(V,Z)$, the topology induced on $\mathcal{E}$ by the strong product topology on $\mathcal{CL}(W,Z)$, and the associated product topology on $\mathcal{CL}(V,W)\times\mathcal{E}$.

31 Continuity of inverses

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let V, W be topological vector spaces over k. Also let T be a one-to-one continuous linear mapping from V onto W, whose inverse T^{-1} is continuous as a mapping from W into V. In this section, we would like to consider the continuity of

$$T \mapsto T^{-1} \tag{31.1}$$

for such mappings T, with respect to the appropriate strong operator topologies. As before, this amounts to looking at

$$T \mapsto T^{-1}(w) \tag{31.2}$$

as a mapping into V for each $w \in W$. Let T_0 be a fixed one-to-one continuous linear mapping from V onto W with continuous inverse, and let us consider the continuity properties of (31.1) and (31.2) at T_0.

Observe that

$$T^{-1} - T_0^{-1} = T^{-1} \circ T_0 \circ T_0^{-1} - T^{-1} \circ T \circ T_0^{-1} = T^{-1} \circ (T_0 - T) \circ T_0^{-1} \tag{31.3}$$

as linear mappings from W into V. Let $w_0 \in W$ be given, and put $v_0 = T_0^{-1}(w_0)$, so that

$$T^{-1}(w_0) - T_0^{-1}(w_0) = T^{-1}(T_0(v_0) - T(v_0)) \tag{31.4}$$

by (31.3). It is easy to have $T_0(v_0) - T(v_0)$ be small in W, by taking T close to T_0 with respect to the strong operator topology on $\mathcal{CL}(V, W)$. In order to get (31.4) to be small in V, we shall impose additional restrictions on T, as in the previous section.

Let $\mathcal{E}$ be a collection of one-to-one continuous linear mappings from V onto W with continuous inverses, and let $\mathcal{E}^{-1}$ be the corresponding collection of inverse mappings from W onto V. Suppose that $\mathcal{E}^{-1}$ is equicontinuous as a collection of linear mappings from W into V. If $T \in \mathcal{E}$ and $T_0(v_0) - T(v_0)$ is small in W, then we get that (31.4) is small in V. This implies that (31.2) is continuous as a mapping from $\mathcal{E}$ into V for each $w \in W$, with respect to the topology induced on $\mathcal{E}$ by the strong operator topology on $\mathcal{CL}(V, W)$. It follows that (31.1) is continuous as a mapping from $\mathcal{E}$ into $\mathcal{CL}(W, V)$, with respect to the strong operator topology on $\mathcal{CL}(W, V)$, and the topology induced on $\mathcal{E}$ by the strong operator topology on $\mathcal{CL}(V, W)$.

Let $\mathcal{CL}(V) = \mathcal{CL}(V, V)$ be the space of continuous linear mappings from V into itself. This is an algebra over k, with composition of linear mappings as multiplication. Of course, the identity mapping $I = I_V$ on V is continuous and linear, and is the multiplicative identity element in $\mathcal{CL}(V)$. The collection of one-to-one continuous linear mappings from V onto itself with continuous inverses is a group with respect to composition of mappings. Let $\mathcal{G}$ be a subgroup of this group, and suppose that $\mathcal{G}$ is equicontinuous as a collection of linear mappings from V into itself. Let us also consider $\mathcal{G}$ to be equipped with the topology induced by the strong operator topology on $\mathcal{CL}(V)$. With respect to this topology, multiplication on $\mathcal{G}$ is continuous as a mapping from $\mathcal{G} \times \mathcal{G}$ into $\mathcal{G}$, using the product topology on the domain. This follows from the discussion in the previous section. Similarly, (31.1) is continuous as a mapping from $\mathcal{G}$ into itself, by the remarks in the preceding paragraph. More precisely, this uses the fact that $\mathcal{G}^{-1} = \mathcal{G}$, since $\mathcal{G}$ is a group, so that $\mathcal{G}^{-1}$ is equicontinuous, by hypothesis. This shows that $\mathcal{G}$ is a topological group under these conditions.

32 Some related continuity properties

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W, Z be vector spaces over k. Also let N_V, N_W, and N_Z be q_V, q_W, and q_Z-seminorms on V, W, and Z, respectively, for some $q_V, q_W, q_Z > 0$, and with respect to $|\cdot|$ on k. This leads to spaces $\mathcal{BL}(V,W)$, $\mathcal{BL}(W,Z)$, and $\mathcal{BL}(V,Z)$ of bounded linear mappings between these vector spaces in the sense of Section 15, with the corresponding operator q_W and q_Z-seminorms $\|\cdot\|_{op,VW}$, $\|\cdot\|_{op,WZ}$, and $\|\cdot\|_{op,VZ}$, respectively. Remember that compositions of bounded linear mappings are bounded, as in Section 16. Thus

$$(R,T) \mapsto T \circ R \tag{32.1}$$

defines a mapping from $\mathcal{BL}(V,W) \times \mathcal{BL}(W,Z)$ into $\mathcal{BL}(V,Z)$, as in Section 30. It is easy to see that this mapping is continuous with respect to the topologies on these spaces determined by the corresponding operator seminorms, and using the associated product topology on the Cartesian product in the domain of this mapping. This uses (16.2) and standard computations like (30.5), but is a bit simpler than in Section 30.

Suppose now that V is a topological vector space over k, but where the topology is not necessarily determined by a singe q-seminorm. As before, bounded linear mappings from W into Z are continuous, so that (32.1) also defines a mapping from $\mathcal{CL}(V,W) \times \mathcal{BL}(W,Z)$ into $\mathcal{CL}(V,Z)$. One can check that this mapping is continuous with respect to the strong operator topologies on $\mathcal{CL}(V,W)$ and $\mathcal{CL}(V,Z)$, the topology on $\mathcal{BL}(W,Z)$ determined by $\|\cdot\|_{op,WZ}$, and the corresponding product topology on the Cartesian product. This can be derived from the discussion in Section 30, using the fact that the topology determined on $\mathcal{BL}(W,Z)$ by $\|\cdot\|_{op,WZ}$ is at least as strong as the one induced by the strong operator topology on $\mathcal{CL}(W,Z)$, as in Section 29. In this situation, it is not necessary to restrict one's attention to an equicontinuous collection of mappings from W into Z, because subsets of $\mathcal{BL}(W,Z)$ with bounded operator q_Z-seminorm are equicontinuous.

Let us return to the case where V is equipped with a q_V-seminorm N_V. Let T_0, T be one-to-one bounded linear mappings from V onto W. Suppose that the inverse T_0^{-1} of T_0 is bounded as a linear mapping from W onto V, in the sense of Section 15 again. If $\|T - T_0\|_{op,VW}$ is sufficiently small, then one can verify that T^{-1} is bounded as a linear mapping from W onto V too, with bounded operator q_V-seminorm $\|T^{-1}\|_{op,WV}$. Under suitable conditions, the bijectivity of T can be obtained from this as well. Using this, one can show that

$$T \mapsto T^{-1} \tag{32.2}$$

is continuous for one-to-one bounded linear mappings T from V onto W with bounded inverse, with respect to the topologies corresponding to the appropriate operator seminorms on $\mathcal{BL}(V, W)$ and $\mathcal{BL}(W, V)$. More precisely, one can show that (32.2) is continuous at each such mapping T_0, using (16.2) and (31.3).

As in the previous section, we let $\mathcal{BL}(V) = \mathcal{BL}(V, V)$ be the space of bounded linear mappings from V into itself, in the sense of Section 15. This is an algebra over k, with composition of linear mappings as multiplication, and with the identity mapping $I = I_V$ as the multiplicative identity element. The group of invertible elements in $\mathcal{BL}(V)$ consists of the one-to-one bounded linear mappings from V onto itself with bounded inverse. This is a topological group with respect to the topology induced by the one determined on $\mathcal{BL}(V)$ by the operator q_V-seminorm, by the remarks in this section. Under suitable conditions, this group is an open set in $\mathcal{BL}(V)$ too.

33 Open subgroups, continued

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, although we are essentially only concerned here with $q_k = \infty$, as in Section 22. Also let V, W be topological vector spaces over k, let v be an element of V, and let U_W be a subset of W. If U_W is balanced in W, then it is easy to see that (23.2) is balanced as a subset of $\mathcal{CL}(V, W)$, where $\mathcal{CL}(V, W)$ is considered as a vector space over k. Similarly, if U_W is a subgroup of W as a commutative group with respect to addition, then (23.2) is a subgroup of $\mathcal{CL}(V, W)$, as a commutative group with respect to addition. Of course, if U_W is an open subset of W, then (23.2) is an open subset of $\mathcal{CL}(V, W)$ with respect to the strong operator topology, as in Section 23.

Now let $\mathcal{G}$ be a collection of one-to-one continuous linear mappings from V onto itself with continuous inverses which is a group with respect to composition. Also let U be a subgroup of V as a commutative group with respect to addition, and suppose that

$$T(U) = U \tag{33.1}$$

for every $T \in \mathcal{G}$. Put

$$\mathcal{G}_{v,U} = \{T \in \mathcal{G} : T(v) \in v + U\} \tag{33.2}$$

for each $v \in V$, and let us verify that this defines a subgroup of $\mathcal{G}$. Each element of $\mathcal{G}$ defines an automorphism on V as a commutative group with respect to addition, and V is partitioned by the cosets of U in V. The elements of $\mathcal{G}$ send cosets of U in V onto other cosets of U, because of (33.1). Thus (33.2) is the same as

$$\mathcal{G}_{v,U} = \{T \in \mathcal{G} : T(v + U) = v + U\}, \tag{33.3}$$

which clearly defines a subgroup of $\mathcal{G}$. If U is also an open set in V, then (33.2) is a relatively open subset of $\mathcal{G}$ with respect to the topology induced by the strong operator topology on $\mathcal{CL}(V)$.

Let $\mathcal{G}$ be a group of one-to-one continuous linear mappings from V onto itself again, and let U_0 be a subset of V. Of course,

$$\bigcap_{T \in \mathcal{G}} T(U_0) = \bigcap_{T \in \mathcal{G}} T^{-1}(U_0) \tag{33.4}$$

is automatically invariant under the elements of $\mathcal{G}$, because $\mathcal{G}$ is a group, by hypothesis. Note that (33.4) is contained in U_0, because the identity mapping I on V is an element of $\mathcal{G}$. If U_0 is a subgroup of V as a commutative group with respect to addition, then (33.4) is a subgroup of V with respect to addition as well. Similarly, if U_0 is balanced in V, then (33.4) is balanced in V too.

Observe that the interior of (33.4) is invariant under the elements of $\mathcal{G}$, by continuity. If U_0 is an open set in V that contains 0, and if $\mathcal{G}$ is equicontinuous on V, then 0 is an element of the interior of (33.4). If A is any subgroup of V as commutative group with respect to addition, and if 0 is an element of the interior of A, then it is easy to see that A is an open set in V. If U_0 is a subgroup of V as a commutative group with respect to addition, and if 0 is an element of the interior of (33.4), then it follows that (33.4) is an open subset of V. This gives a way in which to get open subgroups of V with respect to addition that are invariant under $\mathcal{G}$, as in (33.1).

Part III
Multiplication operators

34 k-Valued functions

Let k be a field, and let X be a nonempty set. Also let $c(X, k)$ be the space of all k-valued functions on X, which is a vector space over k with respect to pointwise addition and scalar multiplication. More precisely, $c(X, k)$ is a commutative algebra over k, with respect to pointwise multiplication of functions. The constant function

$$\mathbf{1}_X \tag{34.1}$$

on X equal to the multiplicative identity element 1 in k is the multiplicative identity element in $c(X, k)$. A k-valued function f on X has a multiplicative inverse in $c(X, k)$ if and only if $f(x) \neq 0$ for each $x \in X$, in which case the multiplicative inverse is given by $1/f(x)$.

If $a \in c(X, k)$, then the corresponding multiplication operator M_a on $c(X, k)$ is defined by

$$M_a(f) = a\,f \tag{34.2}$$

for each $f \in c(X, k)$. This defines a linear mapping from $c(X, k)$ into itself for each $a \in c(X, k)$. Thus

$$a \mapsto M_a \tag{34.3}$$

defines a mapping from $c(X, k)$ into the algebra of linear mappings from $c(X, k)$ into itself. It is easy to see that this mapping is injective, linear, and in fact an algebra homomorphism with respect to composition of linear mappings. Note that M_a is the identity mapping on $c(X, k)$ when $a = \mathbf{1}_X$.

Suppose now that $|\cdot|$ is a q_k-absolute value function on k for some $q_k > 0$. Observe that

$$N_x(f) = |f(x)| \tag{34.4}$$

defines a q_k-seminorm on $c(X, k)$ with respect to $|\cdot|$ on k for each $x \in X$. The collection of these seminorms with $x \in X$ determines a topology on $c(X, k)$, as in Section 4, and $c(X, k)$ is a Hausdorff topological space over k with respect to this topology. One can also identify $c(X, k)$ with the Cartesian product of a family of copies of k indexed by X, and the topology on $c(X, k)$ determined by the collection of seminorms (34.4) corresponds to the product topology on the Cartesian product, using the topology on k determined by the q_k-metric associated to $|\cdot|$ on each factor.

Of course, $c(X, k)$ is a topological vector space over k with respect to the topology described in the preceding paragraph, as in Section 5. One can also check that multiplication is continuous on $c(X, k)$, as a mapping from $c(X, k) \times c(X, k)$ into $c(X, k)$, and using the corresponding product topology on the domain. This uses the fact that multiplication on k is continuous as a mapping from $k \times k$ into k, by standard arguments. Similarly,

$$f \mapsto 1/f \tag{34.5}$$

is continuous as a mapping from

$$\{f \in c(X, k) : f(x) \neq 0 \text{ for each } x \in X\} \tag{34.6}$$

into $c(X, k)$, with respect to the topology induced on (34.6) by the topology already defined on $c(X, k)$. This uses the continuity of $t \mapsto 1/t$ on $k \setminus \{0\}$, with respect to the topology induced on $k \setminus \{0\}$ by the one determined on k by the q_k-metric associated to $|\cdot|$.

Continuity of multiplication on $c(X, k)$ implies in particular that the multiplication operator (34.2) is continuous as a mapping from $c(X, k)$ into itself for each $a \in c(X, k)$. Hence (34.3) may be considered as a mapping from $c(X, k)$ into the algebra $\mathcal{CL}(c(X, k))$ of continuous linear mappings from $c(X, k)$ into itself. One can check that (34.3) is continuous as well, with respect to the topology on $c(X, k)$ described earlier, and the corresponding strong operator topology on $\mathcal{CL}(c(X, k))$. More precisely, (34.3) is a homeomorphism from $c(X, k)$ onto its image in $\mathcal{CL}(c(X, k))$, using the topology on its image induced by the strong operator topology on $\mathcal{CL}(c(X, k))$.

Suppose for the moment that $|\cdot|$ is nontrivial on k. Let E be a subset of $c(X, k)$, and put

$$E_x = \{a(x) : a \in E\} \tag{34.7}$$

for each $x \in E$. Observe that E is bounded as a subset of $c(X,k)$ in the sense of Section 9 if and only if E_x is bounded as a subset of k with respect to $|\cdot|$ for every $x \in X$. This condition is the same as saying that the q_k-seminorm (34.4) is bounded on E for each $x \in X$, as in Section 9. One can also look at E as a subset of a Cartesian product, as in Section 18.

Let

$$\mathcal{E} = \{M_a : a \in E\} \tag{34.8}$$

be the image of E under the mapping (34.3), so that $\mathcal{E}$ is a subset of $\mathcal{CL}(c(X,k))$. Thus we can define $\mathcal{E}_f$ as a subset of $c(X,k)$ for each $f \in c(X,k)$ as in (28.1), which reduces in this case to

$$\mathcal{E}_f = \{M_a(f) : a \in E\} = \{a\, f : a \in E\}. \tag{34.9}$$

As in Section 28, $\mathcal{E}$ is bounded as a subset of $\mathcal{CL}(c(X,k))$ with respect to the strong operator topology if and only if (34.9) is bounded in $c(X,k)$ for each $f \in c(X,k)$. In this situation, one can check that this happens if and only if E is bounded as a subset of $c(X,k)$.

If E is bounded in $c(X,k)$, then it is easy to see that (34.8) is equicontinuous as a collection of linear mappings from $c(X,k)$ into itself, directly from the definitions. Conversely, equicontinuous collections of linear mappings are bounded pointwise, as in Section 28. If (34.8) is equicontinuous on $c(X,k)$, then it follows that E is bounded in $c(X,k)$, which can be verified more directly from the definitions as well.

If $|\cdot|$ is the trivial absolute value function on k, then one can check that the collection of all multiplication operators on $c(X,k)$ is equicontinuous. Remember that the corresponding topology on k is discrete in this case, which leads to other simplifications too.

35 Finite support

Let k be a field again, and let X be a nonempty set. The *support* of a k-valued function f on X is defined to be the set of $x \in X$ such that $f(x) \neq 0$. Let $c_{00}(X,k)$ be the space of k-valued functions f on X with finite support, so that $f(x) = 0$ for all but finitely many $x \in X$. This defines an ideal in the commutative algebra $c(X,k)$ defined in the previous section. Of course, if X has only finitely many elements, then every k-valued function on X has finite support, so that $c_0(X,k)$ is the same as $c(X,k)$. If $y \in X$, then let δ_y be the k-valued function on X defined by

$$\begin{aligned} \delta_y(x) &= 1 \quad \text{when } x = y \\ &= 0 \quad \text{when } x \neq y. \end{aligned} \tag{35.1}$$

Thus $\delta_y \in c_{00}(X,k)$ for every $y \in X$, and the collection of δ_y with $y \in X$ is a basis for $c_{00}(X,k)$ as a vector space over k.

If $a \in c(X, k)$, then we can define the corresponding multiplication operator M_a on $c(X, k)$ as in (34.2). Observe that

$$M_a(c_{00}(X, k)) \subseteq c_{00}(X, k), \tag{35.2}$$

since $c_{00}(X, k)$ is an ideal in $c(X, k)$. Let us consider M_a as a linear mapping from $c_{00}(X, k)$ into itself for each $a \in c(X, k)$ in this section. Thus (34.3) defines a mapping from $c(X, k)$ into the algebra of linear mappings from $c_{00}(X, k)$ into itself, and this mapping is an injective algebra homomorphism, as before. Note that

$$M_a(\delta_y) = a(y)\, \delta_y \tag{35.3}$$

for every $a \in c(X, k)$ and $y \in X$.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, which leads to a topology on $c(X, k)$ as in the previous section. It is easy to see that $c_{00}(X, k)$ is dense in $c(X, k)$ with respect to this topology. Of course, for each a in $c(X, k)$, the restriction of M_a to $c_{00}(X, k)$ is continuous continuous with respect to the topology induced on $c_{00}(X, k)$ by the one already defined on $c(X, k)$. Hence (34.3) may be considered as a mapping from $c(X, k)$ into the algebra $\mathcal{CL}(c_{00}(X, k))$ of countinuous linear mappings from $c_{00}(X, k)$ into itself, with respect to the induced topology on $c_{00}(X, k)$. As before, one can check that (34.3) defines a homeomorphism from $c(X, k)$ onto its image in $\mathcal{CL}(c_{00}(X, k))$, with respect to the topology induced on the image by the strong operator topology on $\mathcal{CL}(c_{00}(X, k))$.

Suppose that $|\cdot|$ is nontrivial on k, and let E be a subset of $c(X, k)$. Also let $\mathcal{E}$ be as in (34.8), but considered now as a subset of $\mathcal{CL}(c_{00}(X, k))$. As before, $\mathcal{E}$ is bounded as a subset of $\mathcal{CL}(c_{00}(X, k))$ with respect to the strong operator topology if and only if (34.9) is bounded in $c_{00}(X, k)$ for every $f \in c_{00}(X, k)$. It is easy to see that this happens if and only if E is bounded as a subset of $c(X, k)$, for essentially the same reasons as before. Similarly, $\mathcal{E}$ is equicontinuous on $c_{00}(X, k)$ if and only if E is bounded in $c(X, k)$.

36 Bounded functions

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. As usual, a k-valued function f on X is said to be *bounded* on X if $|f(x)|$ is bounded as a nonnegative real-valued function on X. Let $\ell^\infty(X, k)$ be the space of bounded k-valued functions on X, and put

$$\|f\|_\infty = \|f\|_{\ell^\infty(X,k)} = \sup_{x \in X} |f(x)| \tag{36.1}$$

for each such function f. It is easy to see that $\ell^\infty(X, k)$ is a vector space with respect to pointwise addition and multiplication on X, and in fact a linear subspace of the space $c(X, k)$ of all k-valued functions on X discussed in Section 34. One can also check that (36.1) defines a q_k-norm on $\ell^\infty(X, k)$ with respect to $|\cdot|$ on k, which is known as the *supremum norm* on $\ell^\infty(X, k)$.

The q_k-metric on $\ell^\infty(X, k)$ associated to the supremum norm is known as the *supremum metric*. Of course,

$$\ell^\infty(X, k) \subseteq c(X, k), \tag{36.2}$$

and the topology determined on $\ell^\infty(X, k)$ by the supremum metric is at least as strong as the one induced by the topology defined on $c(X, k)$ in Section 34. If X has only finitely many elements, then every k-valued function on X is bounded, so that $\ell^\infty(X, k)$ is the same as $c(X, k)$. The topology determined on $\ell^\infty(X, k)$ by the supremum metric is the same as the topology on $c(X, k)$ considered in Section 34 in this case.

If $|\cdot|$ is the trivial absolute value function on k, then every k-valued function on X is bounded again, and $\ell^\infty(X, k)$ is the same as $c(X, k)$. In this case, the corresponding supremum norm on $\ell^\infty(X, k)$ is the trivial ultranorm, and the associated supremum metric is the discrete metric, which determines the discrete topology on $\ell^\infty(X, k)$. However, if X has infinitely many elements, then the topology defined on $c(X, k)$ in Section 34 is not the discrete topology.

If f, g are bounded k-valued functions on X, then their product $f\,g$ is bounded on X too, and satisfies

$$\|f\,g\|_\infty \le \|f\|_\infty\,\|g\|_\infty. \tag{36.3}$$

Thus $\ell^\infty(X, k)$ is a commutative algebra with respect to pointwise multiplication of functions, and more precisely a subalgebra of $c(X, k)$. Note that $\mathbf{1}_X$ as in (34.1) is obviously bounded on X, and is the multiplicative identity element in $\ell^\infty(X, k)$. A bounded k-valued function f on X has a multiplicative inverse in $\ell^\infty(X, k)$ if and only if $f(x) \ne 0$ for every $x \in X$ and $1/f$ is bounded on X. This is the same as saying that $|f(x)|$ has a positive lower bound on X.

Using (36.3), one can check that multiplication is continuous on $\ell^\infty(X, k)$, as a mapping from $\ell^\infty(X, k) \times \ell^\infty(X, k)$ into $\ell^\infty(X, k)$. Here we use the topology determined on $\ell^\infty(X, k)$ by the supremum metric, and the corresponding product topology on the Cartesian product. It is easy to see that the set of bounded k-valued functions f on k such that $|f(x)|$ has a positive lower bound on X is an open set in $\ell^\infty(X, k)$ with respect to this topology. One can also verify that

$$f \mapsto 1/f \tag{36.4}$$

is continuous on this set with respect to this topology on $\ell^\infty(X, k)$, by standard arguments. More precisely, this uses the fact that if f, g are bounded k-valued functions on X, if there is a positive lower bound for $|f(x)|$ on X, and if g is sufficiently close to f with respect to the supremum metric, then one can get a uniform lower bound for $|g(x)|$ on X.

If a is a bounded k-valued function on X, then

$$M_a(f) = a\,f \tag{36.5}$$

defines a bounded linear mapping from $\ell^\infty(X, k)$ into itself, in the sense of Section 15. More precisely,

$$\|M_a\|_{op} = \|a\|_\infty \tag{36.6}$$

for every $a \in \ell^\infty(X, k)$, where $\| \cdot \|_{op}$ is the operator q_k-norm on $\mathcal{BL}(\ell^\infty(X, k))$ associated to the supremum norm on $\ell^\infty(X, k)$. Indeed,

$$\|M_a(f)\|_\infty \le \|a\|_\infty \, \|f\|_\infty \tag{36.7}$$

for every $f \in \ell^\infty(X, k)$, by (36.3). This implies that M_a is bounded on $\ell^\infty(X, k)$, with operator q_k-norm less than or equal to $\|a\|_\infty$. To get the opposite inequality, one can use (35.3), and the fact that the supremum norm of δ_y is equal to 1 for every $y \in X$, where δ_y is as in (35.1).

As in Section 34,

$$a \mapsto M_a \tag{36.8}$$

is an injective algebra homomorphism from $\ell^\infty(X, k)$ into $\mathcal{BL}(\ell^\infty(X, k))$. The isometric property (36.6) implies that this mapping is continuous, and in fact bounded with respect to the corresponding norms, in the sense of Section 15. Using (36.6), we also get that (36.8) is a homeomorphism from $\ell^\infty(X, k)$ onto its image in $\mathcal{BL}(\ell^\infty(X, k))$, with respect to the topology induced on the image by the one defined by the operator q_k-norm.

As in Section 29, the topology determined on $\mathcal{BL}(\ell^\infty(X, k))$ by the operator q_k-norm is at least as strong as the one induced by the strong operator topology. In particular, (36.8) is continuous as a mapping from $\ell^\infty(X, k)$ into $\mathcal{BL}(X, k)$, using the topology determined on $\ell^\infty(X, k)$ by the supremum norm and the one induced on $\mathcal{BL}(\ell^\infty(X, k))$ by the strong operator topology. Observe that

$$M_a(\mathbf{1}_X) = a \tag{36.9}$$

for every $a \in \ell^\infty(X, k)$, where $\mathbf{1}_X$ is as in (34.1). This implies that (36.8) is a homeomorphism from $\ell^\infty(X, k)$ onto its image in $\mathcal{BL}(\ell^\infty(X, k))$, with respect to the topology induced on the image by the strong operator topology. One can also use (36.9) to get that the operator q_k-norm of M_a is greater than or equal to $\|a\|_\infty$, since the supremum norm of $\mathbf{1}_X$ is equal to 1.

37 Vanishing at infinity

Let k be a field with a q_k-absolute value function $| \cdot |$ for some $q_k > 0$ again, and let X be a nonempty set. A k-valued function f on X is said to *vanish at infinity* if for each $\epsilon > 0$ we have that

$$|f(x)| < \epsilon \tag{37.1}$$

for all but at most finitely many $x \in X$. Let $c_0(X, k)$ be the space of k-valued functions on X that vanish at infinity. If f has finite support in X, then f obviously vanishes at infinity on X, so that

$$c_{00}(X, k) \subseteq c_0(X, k). \tag{37.2}$$

Similarly, if f vanishes at infinity on X, then it is easy to see that f is bounded on X, so that

$$c_0(X, k) \subseteq \ell^\infty(X, k). \tag{37.3}$$

Of course, if X has only finitely many elements, then every k-valued function f on X vanishes at infinity. If $|\cdot|$ is the trivial absolute value function on k, and if f is a k-valued function on X that vanishes at infinity, then f has finite support in X. Note that $c_0(X, k)$ is a closed set in $\ell^\infty(X, k)$ with respect to the topology determined by the supremum metric, by standard arguments. More precisely, $c_0(X, k)$ is the same as the closure of $c_{00}(X, k)$ in $\ell^\infty(X, k)$ with respect to this topology. Observe also that $c_0(X, k)$ is an ideal in $\ell^\infty(X, k)$ as a commutative algebra with respect to pointwise addition and multiplication.

Let a be a bounded k-valued function on X, so that the corresponding multiplication operator M_a on $\ell^\infty(X, k)$ can be defined as in (36.5). This operator maps $c_0(X, k)$ into itself, because $c_0(X, k)$ is an ideal in $\ell^\infty(X, k)$. Thus, for each $a \in \ell^\infty(X, k)$, we may consider M_a as a bounded linear mapping from $c_0(X, k)$ into itself with respect to the supremum norm, in the sense of Section 15. One can check that

$$\|M_a\|_{op} = \|a\|_\infty \tag{37.4}$$

for every $a \in \ell^\infty(X, k)$, where $\|\cdot\|_{op}$ is the operator q_k-norm on $\mathcal{BL}(c_0(X, k))$ associated to the restriction of the supremum norm to $c_0(X, k)$. This is not quite the same operator norm as in (36.6), because of the restriction of the domain of M_a to $c_0(X, k)$.

As before,

$$a \mapsto M_a \tag{37.5}$$

defines an injective algebra homomorphism from $\ell^\infty(X, k)$ into $\mathcal{BL}(c_0(X, k))$. The isometric property (37.4) implies that this mapping is bounded with respect to the corresponding norms, as in Section 15, and hence is continuous. More precisely, (37.5) defines a homeomorphism from $\ell^\infty(X, k)$ onto its image in $\mathcal{BL}(c_0(X, k))$, with respect to the topology determined by the operator q_k-norm. As in Section 29, the topology determined on $\mathcal{BL}(c_0(X, k))$ by the operator q_k-norm is at least as strong as the one induced by the strong operator topology. It follows that (37.5) is also continuous as a mapping from $\ell^\infty(X, k)$ into $\mathcal{BL}(c_0(X, k))$, using the topology induced on $\mathcal{BL}(c_0(X, k))$ by the strong operator topology.

However, if X has infinitely many elements, then (37.5) is not a homeomorphism from $\ell^\infty(X, k)$ onto its image in $\mathcal{BL}(c_0(X, k))$, with respect to the topology induced on the image by the strong operator topology. The argument for the analogous statement in the previous section using (36.9) does not work in this situation, because $\mathbf{1}_X$ does not vanish at infinity on X when X has infinitely many elements. To be more explicit, put

$$N_f(a) = \|M_a(f)\|_\infty = \|a\, f\|_\infty \tag{37.6}$$

for each $a \in \ell^\infty(X, k)$ and $f \in c_0(X, k)$. This defines a q_k-seminorm on $\ell^\infty(X, k)$ as a function of a for every $f \in c_0(X, k)$. Thus

$$\{N_f : f \in c_0(X, k)\} \tag{37.7}$$

is a nonempty collection of q_k-seminorms on $\ell^\infty(X, k)$, which determines a topology on $\ell^\infty(X, k)$ as in Section 4. By construction, (37.5) is a homeomorphism from $\ell^\infty(X,k)$ onto its image in $\mathcal{BL}(c_0(X,k))$, using the topology determined on $\ell^\infty(X,k)$ by (37.7), and the topology induced on the image by the strong operator topology. Of course, the topology determined on $\ell^\infty(X,k)$ by the supremum norm is at least as strong as the one determined by (37.7), because of (36.3). If X has infinitely many elements, then one can check that the topology determined on $\ell^\infty(X,k)$ by the supremum norm is strictly stronger than the one determined by (37.7).

38 Vanishing at infinity, continued

Let us continue with the same notation and hypotheses as in the previous section. Let E be a subset of $\ell^\infty(X,k)$, and let

$$\mathcal{E} = \{M_a : a \in E\} \tag{38.1}$$

be the image of E under (37.5), so that the elements of $\mathcal{E}$ are considered as bounded linear mappings from $c_0(X,k)$ into itself. Suppose for the moment that the elements of E have bounded supremum norm. This is the same as saying that the elements of $\mathcal{E}$ have bounded operator norm on $c_0(X,k)$, by (37.4). In particular, this implies that $\mathcal{E}$ is equicontinuous on $c_0(X,k)$, as in Section 20.

Consider the topology induced on $\mathcal{E}$ by the strong operator topology. As in Section 25, this is the same as the topology induced on $\mathcal{E}$ by the analogue of the strong operator topology in which we restrict our attention to elements of $c_{00}(X,k)$ instead of $c_0(X,k)$, because $\mathcal{E}$ is equicontinuous on $c_0(X,k)$, and $c_{00}(X,k)$ is dense in $c_0(X,k)$. Similarly, we can restrict our attention to elements of $c_{00}(X,k)$ of the form δ_y as in (35.1) with $y \in X$, since their linear span is all of $c_{00}(X,k)$. This topology on $\mathcal{E}$ corresponds exactly to the topology induced on E by the one defined on $c(X,k)$ in Section 34. More precisely, this means that (37.5) defines a homeomorphism from E onto $\mathcal{E}$ with respect to these topologies.

Let E be any subset of $\ell^\infty(X,k)$ again, and let $\mathcal{E}$ be as in (38.1). Put

$$E_x = \{a(x) : a \in E\} \tag{38.2}$$

for each $x \in X$, and

$$\mathcal{E}_f = \{M_a(f) : a \in E\} = \{a\, f : a \in E\} \tag{38.3}$$

for every $f \in c_0(X,k)$, which corresponds to (28.1) in this situation. If the elements of E have supremum norm bounded by some nonnegative real number C, then the elements of $\mathcal{E}_f$ have supremum norm bounded by C times the supremum norm of f for every $f \in c_0(X,k)$. Let y be any element of X, and let δ_y be as in (35.1) again. The elements of $\mathcal{E}_{\delta_y}$ correspond exactly to multiples of δ_y by elements of E_y, as in (35.3). Thus the elements of $\mathcal{E}_{\delta_y}$ have bounded supremum norms if and only if the elements of E_y have bounded absolute value. If this happens for every $y \in X$, then the elements of $\mathcal{E}_f$ have bounded supremum norm for every $f \in c_{00}(X,k)$.

If the elements of $\mathcal{E}_f$ have bounded supremum norm for every $f \in c_0(X, k)$, then the elements of E have bounded supremum norm too. Of course, this is trivial when $|\cdot|$ is the trivial absolute value function on k, and so we may as well suppose that $|\cdot|$ is nontrivial on k. If k is complete with respect to the q_k-metric associated to $|\cdot|$, then this statement can be derived from the Banach–Steinhaus theorem. This uses the fact that $c_0(X, k)$ is complete with respect to the corresponding supremum metric in this case. More precisely, $\ell^\infty(X, k)$ is complete with respect to the supremum metric when k is complete, which implies that $c_0(X, k)$ is complete too, because $c_0(X, k)$ is a closed set in $\ell^\infty(X, k)$. If k is not already complete, then one can simply pass to a completion. However, in this situation, passing to a completion does not really do much anyway, and one can give a more direct argument without asking that k be complete, as follows. Suppose for the sake of a contradiction that the elements of E do not have bounded supremum norm. This implies that there is a sequence $\{a_j\}_{j=1}^\infty$ of elements of E whose supremum norms tend to $+\infty$ as $j \to \infty$. Hence there is a sequence $\{x_j\}_{j=1}^\infty$ of elements of X such that

$$|a_j(x_j)| \to \infty \quad \text{as } j \to \infty. \tag{38.4}$$

If x is any element of X, then the elements of $\mathcal{E}_{\delta_x}$ have bounded supremum norm, by hypothesis. This implies that the elements of E_x have bounded absolute value, as in the preceding preceding paragraph. Thus x cannot occur in the sequence $\{x_j\}_{j=1}^\infty$ more than finitely many times. Using this, one can reduce to the case where the terms of the sequence $\{x_j\}_{j=1}^\infty$ are distinct elements of X, by passing to a subsequence if necessary. At any rate, one can use such a sequence to get an $f \in c_0(X, k)$ such that the elements of $\mathcal{E}_f$ do not have bounded supremum norms, as desired.

39 r-Summable functions

Let X be a nonempty set, and let h be a nonnegative real-valued function on X. The sum

$$\sum_{x \in X} h(x) \tag{39.1}$$

is defined as a nonnegative extended real number to be the supremum of the collection of sums of $h(x)$ over nonempty finite subsets of X. Thus (39.1) is finite when the collection of sums of $h(x)$ over finite subsets of X has a finite upper bound, in which case h is said to be *summable* on X. If h is summable on X, then it is easy to see that h vanishes at infinity on X, with respect to the standard absolute value function on X. Of course, (39.1) reduces to an ordinary finite sum when h has finite support in X, and in particular when X has only finitely many elements.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let r be a positive real number. A k-valued function f on X is said to be r-*summable* if $|f(x)|^r$ is summable as a nonnegative real-valued function on X, as in the preceding paragraph. Let $\ell^r(X,k)$ be the space of k-valued r-summable functions on X. Note that

$$c_{00}(X,k) \subseteq \ell^r(X,k) \subseteq c_0(X,k), \tag{39.2}$$

by the corresponding remarks in the previous paragraph. If X has only finitely many elements, then every k-valued function on X is r-summable, so that $\ell^r(X,k)$ reduces to $c(X,k)$.

One can verify that $\ell^r(X,k)$ is a vector space with respect to pointwise addition and scalar multiplication for each $r > 0$, by standard arguments. Put

$$\|f\|_r = \|f\|_{\ell^r(X,k)} = \Big(\sum_{x \in X} |f(x)|^r\Big)^{1/r} \tag{39.3}$$

for each $f \in \ell^r(X,k)$, where the right side of (39.3) is defined as a nonnegative real number using the earlier definition of (39.1). If $r = q_k$, then it is easy to see directly that (39.3) defines a r-norm on $\ell^r(X,k)$. The same conclusion holds when $r \le q_k$, because $|\cdot|$ is also an r-absolute value function on k in this case, as in Section 2. If $r \ge q_k$, then one can check that (39.3) defines a q_k-norm on $\ell^r(X,k)$, using Minkowski's inequality for sums corresponding to the exponent $r/q_k \ge 1$. If $f \in \ell^r(X,k)$ for any $r > 0$, then f is bounded on X, and in fact

$$\|f\|_\infty \le \|f\|_r. \tag{39.4}$$

We also have that $f \in \ell^{r_0}(X,k)$ when $r \le r_0 < \infty$, with

$$\|f\|_{r_0} \le \|f\|_r, \tag{39.5}$$

by the same type of computation as in (1.2). It is not too difficult to show that $c_{00}(X,k)$ is dense in $\ell^r(X,k)$ with respect to the q_k or r-metric associated to $\|f\|_r$, because of the way that a sum of the form (39.1) is approximated by finite sums.

If $a \in \ell^\infty(X,k)$ and $f \in \ell^r(X,k)$, then their product is r-summable on X as well, with

$$\|a\,f\|_r \le \|a\|_\infty\,\|f\|_r. \tag{39.6}$$

Thus

$$M_a(f) = a\,f \tag{39.7}$$

defines a bounded linear mapping from $\ell^r(X,k)$ to itself when $a \in \ell^\infty(X,k)$, in the sense of Section 15. As before, we have that

$$\|M_a\|_{op} = \|a\|_\infty \tag{39.8}$$

for every $a \in \ell^\infty(X,k)$, where $\|M_a\|_{op}$ is now the operator q_k or r-norm on $\mathcal{BL}(\ell^r(X,k))$ associated to $\|f\|_r$. More precisely, (39.6) implies that $\|M_a\|_{op}$ is less than or equal to $\|a\|_\infty$, and the opposite inequality can be obtained from (35.3). This uses the fact that

$$\|\delta_y\|_r = 1 \tag{39.9}$$

for every $y \in X$, where δ_y is as in (35.1).

40 r-Summable functions, continued

Let us continue with the same notation and hypotheses as in the preceding section. Put

$$N_{f,r}(a) = \|M_a(f)\|_r = \|a\, f\|_r \tag{40.1}$$

for every $a \in \ell^\infty(X, r)$ and $f \in \ell^r(X, k)$. This defines a q_k or r-seminorm on $\ell^\infty(X, k)$ as a function of a for every $f \in \ell^r(X, k)$, depending on whether $r \geq q_k$ or $r \leq q_k$, as before. Hence

$$\{N_{f,r} : f \in \ell^r(X, k)\} \tag{40.2}$$

is a nonempty collection of q_k or r-seminorms on $\ell^\infty(X, k)$, as appropriate, which leads to a topology on $\ell^\infty(X, k)$ as in Section 4. The topology determined on $\ell^\infty(X, k)$ by the supremum norm is at least as strong as the one corresponding to (40.2), by (39.6).

As in analogous situations discussed earlier,

$$a \mapsto M_a \tag{40.3}$$

defines an injective algebra homomorphism from $\ell^\infty(X, k)$ into $\mathcal{BL}(\ell^r(X, k))$. This mapping is an isometry with respect to the corresponding norms, as in (39.8), and hence a homeomorphism onto its image with respect to the associated topologies. This mapping is also a homeomorphism onto its image with respect to the topology determined on $\ell^\infty(X, k)$ by (40.2) and the topology induced on the image by the strong operator topology. If X has infinitely many elements, then the topology determined on $\ell^\infty(X, k)$ by the supremum norm is strictly stronger than the one corresponding to (40.2). Note that the topology determined on $\ell^\infty(X, k)$ by (40.2) is at least as strong as the one induced by the topology defined on $c(X, k)$ in Section 34, which corresponds to restricting our attention to $f = \delta_y$ as in (35.1) with $y \in X$ in the previous paragraph.

Let E be a subset of $\ell^\infty(X, k)$, and let

$$\mathcal{E} = \{M_a : a \in E\} \tag{40.4}$$

be the image of E under (40.3), so that the elements of $\mathcal{E}$ are considered as bounded linear mappings from $\ell^r(X, k)$ into itself, in the sense of Section 15. Suppose for the moment that the elements of E have bounded supremum norm, which means that the elements of $\mathcal{E}$ have bounded operator norm, by (39.8). Thus $\mathcal{E}$ is equicontinuous on $\ell^r(X, k)$, as in Section 20, and we would like to consider the topology induced on $\mathcal{E}$ by the strong operator topology. Because $c_{00}(X, k)$ is dense in $\ell^r(X, k)$, this is the same as the topology induced on $\mathcal{E}$ by the analogue of the strong operator topology in which we restrict our attention to elements of $c_{00}(X, k)$ instead of $\ell^r(X, k)$, as in Section 25. We can restrict our attention further to elements of $c_{00}(X, k)$ of the form δ_y as in (35.1) with $y \in X$, because their linear span is equal to $c_{00}(X, k)$. Using (40.3), this topology on $\mathcal{E}$ corresponds exactly to the topology on E induced by the one defined on $c(X, k)$

in Section 34. This basically amounts to restricting our attention to $f = \delta_y$ with $y \in X$ in (40.1).

Let E be any subset of $\ell^\infty(X, k)$ again, and put

$$E_x = \{a(x) : a \in E\} \tag{40.5}$$

for every $x \in X$. Also let $\mathcal{E}$ be as in (40.4), and put

$$\mathcal{E}_f = \{M_a(f) : a \in E\} = \{a\, f : a \in E\} \tag{40.6}$$

for each $f \in \ell^r(X, k)$, as in (28.1). In particular,

$$\mathcal{E}_{\delta_y} = \{a(y)\, \delta_y : a \in E\} \tag{40.7}$$

for every $y \in X$, by (35.3). This implies that the elements of $\mathcal{E}_{\delta_y}$ have bounded norm in $\ell^r(X, k)$ if and only if the elements of E_y have bounded absolute value. If the elements of E have bounded supremum norm, then the elements of $\mathcal{E}_f$ have bounded norm in $\ell^r(X, k)$ for every $f \in \ell^r(X, k)$, by (39.6). Conversely, if the elements of $\mathcal{E}_f$ have bounded norm in $\ell^r(X, k)$ for every $f \in \ell^r(X, k)$, then the elements of E have bounded supremum norm. As in Section 38, this is trivial when $|\cdot|$ is the trivial absolute value function on k, and so we may as well suppose that $|\cdot|$ is nontrivial on k. If k is complete with respect to the q_k-metric associated to $|\cdot|$, then one can check that $\ell^r(X, k)$ is complete too, by standard arguments. This permits one to use the Banach–Steinhaus theorem, and otherwise one can pass to a completion of k. One can also argue more directly without asking k to be complete, as in Section 38.

41 Some comparisons

Let us continue with the same notation and hypotheses as in the previous two sections. If $|\cdot|$ is the trivial absolute value function on k, then $c_0(X, k)$ and $\ell^r(X, k)$ are the same as $c_{00}(X, k)$, and $\ell^\infty(X, k)$ is the same as $c(X, k)$. In this case, the topologies determined on $\ell^\infty(X, k)$ by (37.7) and (40.2) are the same as the topology defined on $c(X, k)$ in Section 34. Although we shall be primarily concerned with nontrivial absolute value functions on k, much of the discussion in this section also works when $|\cdot|$ is trivial on k. Note that every positive real number is within a fixed factor of the absolute value of a nonzero element of k when $|\cdot|$ is nontrivial on k.

Let h be a nonnegative real-valued summable function on X, and put

$$\widetilde{N}_{h,r}(a) = \Big(\sum_{x \in X} |a(x)|^r\, h(x)\Big)^{1/r} \tag{41.1}$$

for every $a \in \ell^\infty(X, k)$. Observe that $|a(x)|^r h(x)$ is also summable on X when a is bounded on X, so that the sum on the right side of (41.1) can be defined as a nonnegative real number as in Section 39. More precisely,

$$\widetilde{N}_{h,r}(a) \le \Big(\sum_{x \in X} h(x)\Big)^{1/r} \|a\|_\infty \tag{41.2}$$

for every $a \in \ell^\infty(X, k)$. If $r \le q_k$, then $| \cdot |$ is an r-absolute value function on k, as in Section 2, and one can verify that (41.1) defines an r-seminorm on $\ell^\infty(X, k)$, as in Section 39. Similarly, if $r \ge q_k$, then one can check that (41.1) defines a q_k-seminorm on $\ell^\infty(X, k)$ when $r \ge q_k$, using Minkowski's inequality, as before.

Thus

$$(41.3) \quad \{\widetilde{N}_{h,r} : h \text{ is a nonnegative real-valued summable function on } X\}$$

is a nonempty collection of q_k or r-seminorms on $\ell^\infty(X, k)$, as appropriate, which determines a topology on $\ell^\infty(X, k)$ as in Section 4. If $f \in \ell^r(X, k)$, then

$$(41.4) \qquad h(x) = |f(x)|^r$$

is a nonnegative real-valued summable function on X. It is easy to see that

$$(41.5) \qquad \widetilde{N}_{h,r}(a) = N_{f,r}(a)$$

for every $a \in \ell^\infty(X, k)$ in this case, where $N_{f,r}(a)$ is as in (40.1). This implies that (40.2) is contained in (41.3). It follows that the topology determined on $\ell^\infty(X, k)$ by (41.3) is at least as strong as the one determined by (40.2).

Suppose now that $| \cdot |$ is nontrivial on k, and let h be any nonnegative real-valued summable function on X. We would like to choose a k-valued function f on X such that $|f(x)|^r$ approximates $h(x)$ in a suitable sense. Of course, we may as well take $f(x) = 0$ in k when $h(x) = 0$. Otherwise, if $h(x) > 0$, then we can choose $f(x) \in k$ such that

$$(41.6) \qquad h(x)^{1/r} \le |f(x)| \le C\, h(x)^{1/r},$$

where $C \ge 1$ does not depend on x. This uses the nontriviality of $| \cdot |$ on k, as mentioned at the beginning of the section. Equivalently, we have that

$$(41.7) \qquad h(x) \le |f(x)|^r \le C^r\, h(x)$$

for every $x \in X$, since these inequalities hold automatically when $h(x) = 0$, by construction. The second inequality in (41.7) implies that $f \in \ell^r(X, k)$, because h is summable on X. The first inequality in (41.7) implies that

$$(41.8) \qquad \widetilde{N}_{h,r}(a) \le N_{f,r}(a)$$

for every $a \in \ell^\infty(X, k)$, where $N_{f,r}(a)$ is as in (40.1) again. Combining this with the remarks in the preceding paragraph, we get that the topologies determined on $\ell^\infty(X, k)$ by (40.2) and (41.3) are the same in this case. If $| \cdot |$ is the trivial absolute value function on k, then one can also check that the topologies determined on $\ell^\infty(X, k)$ by (40.2) and (41.3) are the same. This uses the fact that (41.1) can be made arbitrarily small in this situation by requiring a to vanish on suitable finite subsets of X, depending on h.

42 Some comparisons, continued

Let X be a nonempty set, and let g, h be nonnegative real-valued functions on X, with g bounded and h summable. Thus

$$g(x)^r \, h(x) \tag{42.1}$$

is summable on X for every positive real number r. If r_1, r_2 are positive real numbers with $r_1 \le r_2$, then we have that

$$\Big(\sum_{x \in X} g(x)^{r_1} \, h(x)\Big)^{1/r_1} \le \Big(\sum_{x \in X} h(x)\Big)^{(1/r_1)-(1/r_2)} \Big(\sum_{x \in X} g(x)^{r_2} \, h(x)\Big)^{1/r_2}. \tag{42.2}$$

This is simplest when

$$\sum_{x \in X} h(x) = 1, \tag{42.3}$$

so that the first factor on the right side of (42.2) is equal to 1 too. In this case, (42.2) is a well-known consequence of Jensen's inequality. Of course, (42.2) is trivial when $h \equiv 0$ on X, and otherwise it is easy to reduce to the case where (42.3) holds, by dividing h by its sum over X. One can also derive (42.2) from Hölder's inequality.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, as before. Also let h be a nonnegative real-valued summable function on X again, and let a be a bounded k-valued function on X. Thus

$$g(x) = |a(x)| \tag{42.4}$$

is a bounded nonnegative real-valued function on X, to which the remarks in the preceding paragraph can be applied. If r_1, r_2 are positive real numbers with $r_1 \le r_2$, then we get that

$$\widetilde{N}_{h,r_1}(a) \le \Big(\sum_{x \in X} h(x)\Big)^{(1/r_1)-(1/r_2)} \widetilde{N}_{h,r_2}(a), \tag{42.5}$$

where $\widetilde{N}_{h,r}(a)$ is as in (41.1). This is the same as (42.2), reexpressed in this situation.

Let

$$\tau_r \tag{42.6}$$

be the topology determined on $\ell^\infty(X, k)$ by (41.3) for each positive real number r. This is the same as the topology determined on $\ell^\infty(X, k)$ by (40.2), as discussed in the previous section. If r_1, r_2 are positive real numbers with $r_1 \le r_2$, then

$$\tau_{r_1} \subseteq \tau_{r_2}, \tag{42.7}$$

which is to say that τ_{r_2} is at least as strong as τ_{r_1}. This follows from (42.5).

Suppose that $h_1, \ldots, h_n$ are finitely many nonnegative real-valued summable functions on X, and put

$$h(x) = \max_{1 \le j \le n} h_j(x) \tag{42.8}$$

for each $x \in X$. It is easy to see that h is summable on X too, because

$$h(x) \le \sum_{j=1}^{n} h_j(x) \tag{42.9}$$

for each $x \in X$, and hence

$$\sum_{x \in X} h(x) \le \sum_{x \in X} \sum_{j=1}^{n} h_j(x) = \sum_{j=1}^{n} \sum_{x \in X} h_j(x). \tag{42.10}$$

The linearity property of the sum used in the second step in (42.10) is well known, and can be derived directly from the definition of the sum in (39.1). By construction,

$$\max_{1 \le j \le n} \widetilde{N}_{h_j,r}(a) \le \widetilde{N}_{h,r}(a) \tag{42.11}$$

for every $r > 0$ and $a \in \ell^\infty(X, k)$, where $\widetilde{N}_{h,r}(a)$ is as in (41.1) again. This simplifies a bit the way that (41.3) determines a topology on $\ell^\infty(X, k)$, as in Section 4. More precisely, (42.11) implies that open balls in $\ell^\infty(X, k)$ with respect to elements of (41.3) form a base for this topology, and not just a sub-base. Note that if there is an $r > 0$ such that $h_j(x)$ is of the form $|f_j(x)|^r$ for some $f_j \in \ell^r(X, k)$ and each $j = 1, \ldots, n$, then $h(x)$ can be expressed in this way as well.

43 Some more comparisons

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let X be a nonempty set. Also let g be a nonnegative real-valued function on X that vanishes at infinity, and put

$$\widetilde{N}_g(a) = \sup_{x \in X} (|a(x)|\, g(x)) \tag{43.1}$$

for every $a \in \ell^\infty(X, k)$. Note that

$$|a(x)|\, g(x) \tag{43.2}$$

vanishes at infinity as a nonnegative real-valued function on X when a is bounded on X. This implies that the supremum on the right side of (43.1) is attained, because it can be reduced to the maximum over a finite subset of X when (43.2) is not identically 0 on X. Of course, we also have that

$$\widetilde{N}_g(a) \le \Big(\sup_{x \in X} g(x)\Big)\, \|a\|_\infty \tag{43.3}$$

for every $a \in \ell^\infty(X, k)$. Remember that g is bounded on X, since it vanishes at infinity, and indeed its supremum over X is attained for the same reasons as just mentioned. One can check that (43.1) defines a q_k-seminorm on $\ell^\infty(X, k)$, for essentially the same reasons as for the supremum norm.

Thus

$$\{\widetilde{N}_g : g \text{ is a nonnegative real-valued function on } X \text{ that vanishes at infinity}\} \tag{43.4}$$

is a nonempty collection of q_k-seminorms on $\ell^\infty(X, k)$, which determines a topology on $\ell^\infty(X, k)$ as in Section 4. If $f \in c_0(X, k)$, then

$$g(x) = |f(x)| \tag{43.5}$$

defines a nonnegative real-valued function on X that vanishes at infinity. In this case, we have that

$$N_f(a) = \widetilde{N}_g(a) \tag{43.6}$$

for every $a \in \ell^\infty(X, k)$, where $N_f(a)$ is as in (37.6). This implies that (37.7) is contained in (43.4), so that the topology determined on $\ell^\infty(X, k)$ by (43.4) is at least as strong as the one determined by (37.7).

Suppose for the moment that $|\cdot|$ is nontrivial on k, and let g be any nonnegative real-valued function on X that vanishes at infinity again. Because $|\cdot|$ is nontrivial on k, there is a real-number $C \geq 1$ and a k-valued function f on X such that

$$g(x) \leq |f(x)| \leq C\, g(x) \tag{43.7}$$

for every $x \in X$, as in (41.6). The second inequality in (43.7) implies that f vanishes at infinity on X, so that $f \in c_0(X, k)$. Using the first inequality in (43.7), we get that

$$\widetilde{N}_g(a) \leq N_f(a) \tag{43.8}$$

for every $a \in \ell^\infty(X, k)$, where $N_f(a)$ is as in (37.6) again. It follows from this and the remarks in the preceding paragraph that the topology determined on $\ell^\infty(X, k)$ by (43.4) is the same as the one determined by (37.7) in this case. If $|\cdot|$ is the trivial absolute value function on k, then one can also check that the topologies determined on $\ell^\infty(X, k)$ by (37.7) and (43.4) are the same. This is because (43.1) can be made arbitrarily small in this situation by requiring a to vanish on suitable finite subsets of X, depending on g.

If g is any bounded nonnegative real-valued function on X, then (43.2) is a bounded nonnegative real-valued function on X too for every $a \in \ell^\infty(X, k)$. This permits us to define $\widetilde{N}_g(a)$ for every $a \in \ell^\infty(X, k)$ as in (43.1), although the supremum on the right side of (43.1) may not be attained in this situation. As before, $\widetilde{N}_g$ is a q_k-seminorm on $\ell^\infty(X, k)$ that satisfies (43.3). If $g(x) = 1$ for every $x \in X$, then $\widetilde{N}_g$ is the same as the supremum norm on $\ell^\infty(X, k)$. Of course, if X has only finitely many elements, then every nonnegative real-valued function on X automatically vanishes at infinity.

44 Some more comparisons, continued

Let X be a nonempty set, and let g be a nonnegative real-valued function on X that vanishes at infinity. Thus, for each $\epsilon > 0$,

$$\{x \in X : g(x) \geq \epsilon\} \tag{44.1}$$

has only finitely many elements. In particular, this holds when $\epsilon = 1/j$ for some positive integer j. It follows that the support of g has only finitely or countably many elements, by taking the union over all positive integers j. Now let h be a nonnegative real-valued summable function on X. This implies that h vanishes at infinity on X, as mentioned in Section 39. Hence the support of h has only finitely or countable many elements, by the previous remark.

Let g_0, h_0 be nonnegative real-valued functions on X, and put

$$h(x) = g_0(x)\, h_0(x) \tag{44.2}$$

for each $x \in X$. If g_0 is bounded on X, and h_0 is summable on X, then it is easy to see that h is summable on X too. Of course, if g_0 vanishes at infinity on X, then g_0 is bounded on X, as in Section 37. Conversely, let a nonnegative real-valued summable function h on X be given. We would like to check that h can be expressed as in (44.2), where g_0 is a nonnegative real-valued function on X that vanishes at infinity, and h_0 is a nonnegative real-valued summable function on X. We may as well put $g_0(x) = h_0(x) = 0$ for every $x \in X$ such that $h(x) = 0$. If h has finite support in X, then we can put $g_0(x) = 1$ and $h_0(x) = h(x)$ for every $x \in X$ such that $h(x) > 0$. Otherwise, the support of h is countably infinite, as in the preceding paragraph. This permits one to reduce the question to the setting of infinite series with positive terms. In this setting, the question corresponds to Part (b) of Problem 12 on p79-80 at the end of Chapter 3 in [15].

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again. Also let h be a nonnegative real-valued summable function on X, and let r be a positive real number. As in the previous paragraph, there is a nonnegative real-valued summable function h_0 on X and a nonnegative real-valued function g_r on X that vanishes at infinity such that

$$h(x) = g_r(x)^r\, h_0(x) \tag{44.3}$$

for every $x \in X$. This is the same as (44.2), with $g_r(x) = g_0(x)^{1/r}$. If a is any bounded k-valued function on X, then we have that

$$\widetilde{N}_{h,r}(a) = \Big(\sum_{x \in X} |a(x)|^r\, g_r(x)^r\, h_0(x)\Big)^{1/r} \tag{44.4}$$

in this case, where $\widetilde{N}_{h,r}(a)$ is as in (41.1). This implies that

$$\widetilde{N}_{h,r}(a) \leq \Big(\sup_{x \in X} |a(x)|\, |g_r(x)|\Big) \Big(\sum_{x \in X} h_0(x)\Big)^{1/r} \tag{44.5}$$

for every $a \in \ell^\infty(X, k)$. Equivalently, this means that

$$\widetilde{N}_{h,r}(a) \le \Big(\sum_{x \in X} h_0(x)\Big)^{1/r} \widetilde{N}_{g_r}(a) \tag{44.6}$$

for every $a \in \ell^\infty(X, k)$, where $\widetilde{N}_{g_r}(a)$ is as in (43.1).

Let τ_r be the topology determined on $\ell^\infty(X, k)$ by (41.3), as in Section 42. Also let

$$\tau_\infty \tag{44.7}$$

be the topology determined on $\ell^\infty(X, k)$ by (43.4), which is the same as the topology determined on $\ell^\infty(X, k)$ by (37.7), as discussed in the previous section. Using (44.6), we get that

$$\tau_r \subseteq \tau_\infty, \tag{44.8}$$

which is to say that τ_∞ is at least as strong as τ_r. Of course, this may be considered as a continuation of (42.7). Note that the topology determined on $\ell^\infty(X, k)$ by the supremum norm is at least as strong as τ_∞, by (43.3). This is equivalent to a remark in Section 37 too. Similarly, it is easy to see directly that the topology determined on $\ell^\infty(X, k)$ by the supremum norm is at least as strong as τ_r, using (41.2).

45 Continuity of multiplication

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. As in Section 36, multiplication of k-valued functions on X defines a continuous mapping from $\ell^\infty(X, k) \times \ell^\infty(X, k)$ into $\ell^\infty(X, k)$, using the topology determined on $\ell^\infty(X, k)$ by the supremum norm, and the corresponding product topology on the domain of this mapping. Let τ_∞ be the topology determined on $\ell^\infty(X, k)$ by (43.4), as in the previous section. This is the same as the topology determined on $\ell^\infty(X, k)$ by (37.7), as discussed in Section 43. In this section, we would like to look at the continuity of multiplication on $\ell^\infty(X, k)$ with respect to τ_∞.

Let g_1, g_2 be nonnegative real-valued functions on X that vanish at infinity, and put

$$g(x) = g_1(x)\, g_2(x), \tag{45.1}$$

which vanishes at infinity on X as well. It is easy to see that

$$\widetilde{N}_g(a\, b) \le \widetilde{N}_{g_1}(a)\, \widetilde{N}_{g_2}(b) \tag{45.2}$$

for every $a, b \in \ell^\infty(X, k)$, where these q_k-seminorms are as defined in (43.1). More precisely, this can be derived from (36.3). In particular, if $f_1, f_2 \in c_0(X, k)$, and $f(x) = f_1(x)\, f_2(x)$, then $f \in c_0(X, k)$ too, and

$$N_f(a\, b) \le N_{f_1}(a)\, N_{f_2}(b) \tag{45.3}$$

for every $a, b \in \ell^\infty(X, k)$, where these q_k-seminorms are as defined in (37.6). This is the same as (45.2) with $g_1(x) = |f_1(x)|$, $g_2(x) = |f_2(x)|$, and $g(x) = |f(x)|$, as in (43.6).

Now let g be a nonnegative real-valued function on X that vanishes at infinity, and put

$$g_1(x) = g_2(x) = g(x)^{1/2} \tag{45.4}$$

for each $x \in X$. Thus g_1, g_2 vanish at infinity on X and satisfy (45.1), so that (45.2) holds for every $a, b \in \ell^\infty(X, k)$. Using this, one can check that multiplication is continuous as a mapping from $\ell^\infty(X, k) \times \ell^\infty(X, k)$ into $\ell^\infty(X, k)$, using the topology τ_∞ on $\ell^\infty(X, k)$, and the corresponding product topology on the domain of this mapping. If $|\cdot|$ is the trivial absolute value function on k, then $\ell^\infty(X, k)$ is the same as $c(X, k)$, and τ_∞ is the same as the topology defined on $c(X, k)$ in Section 34. In this case, the continuity of multiplication on $\ell^\infty(X, k)$ with respect to τ_∞ just described reduces to the continuity of multiplication on $c(X, k)$ mentioned in Section 34.

Remember that elements of $\ell^\infty(X, k)$ determine bounded multiplication operators on $c_0(X, k)$, as in Section 37. The characterization of τ_∞ as the topology determined on $\ell^\infty(X, k)$ by (37.7) means that τ_∞ corresponds to the topology induced on the collection of these multiplication operators by the associated strong operator topology, as in Section 37 again. Of course, multiplication of elements of $\ell^\infty(X, k)$ corresponds to compositions of the associated multiplication operators. Some continuity properties of compositions of continuous linear mappings related to the strong operator topology were considered in Sections 30 and 32. The continuity of multiplication on $\ell^\infty(X, k)$ with respect to τ_∞ indicated in the preceding paragraph is somewhat stronger than what one would get from the broader discussion in Sections 30 and 32.

46 Continuity of multiplication, continued

Let X be a nonempty set, and let r_1, r_2, and r_3 be positive real numbers such that

$$\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}. \tag{46.1}$$

Also let g_1 and g_2 be bounded nonnegative real-valued functions on X, and let h be a nonnegative real-valued summable function on X. It is well known that

$$\begin{aligned} \Big(\sum_{x \in X} (g_1(x)\, g_2(x))^{r_3}\, h(x)\Big)^{1/r_3} \\ \le \Big(\sum_{x \in X} g_1(x)^{r_1}\, h(x)\Big)^{1/r_1} \Big(\sum_{x \in X} g_2(x)^{r_2}\, h(x)\Big)^{1/r_2}, \end{aligned} \tag{46.2}$$

by Hölder's inequality. More precisely, Hölder's inequality is normally stated with $r_3 = 1$, but it is easy to reduce to that case. Note that each of the three sums in (46.2) are finite under these conditions.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let $\widetilde{N}_{h,r}$ be defined on $\ell^\infty(X,k)$ for each positive real number r as in (41.1). Observe that

$$\widetilde{N}_{h,r_3}(a\,b) \le \widetilde{N}_{h,r_1}(a)\,\widetilde{N}_{h,r_2}(b) \tag{46.3}$$

for every $a, b \in \ell^\infty(X,k)$, by applying (46.2) to $g_1(x) = |a(x)|$ and $g_2(x) = |b(x)|$. Let τ_r be the topology determined on $\ell^\infty(X,k)$ by (41.3) for each positive real number r, as in Section 42. Also let $\ell^\infty(X,k) \times \ell^\infty(X,k)$ be equipped with the product topology associated to τ_{r_1} on the first factor and τ_{r_2} on the second factor. Using (46.3), we get that multiplication is continuous as a mapping from $\ell^\infty(x,k) \times \ell^\infty(X,k)$ into $\ell^\infty(X,k)$, with respect to the product topology on the domain just mentioned, and τ_{r_3} on the range.

If g_1, g_2 are bounded nonnegative real-valued functions on X again, then we have that

$$\Big(\sum_{x \in X} (g_1(x)\,g_2(x))^r\,h(x)\Big)^{1/r} \le \Big(\sup_{x \in X} g_1(x)\Big)\Big(\sum_{x \in X} g_2(x)^r\,h(x)\Big)^{1/r} \tag{46.4}$$

for every $r > 0$. Of course, this is a substitute for (46.2) with $r_1 = \infty$ and $r_2 = r_3 = r$. It follows that

$$\widetilde{N}_{h,r}(a\,b) \le \|a\|_\infty\,\widetilde{N}_{h,r}(b) \tag{46.5}$$

for every $a, b \in \ell^\infty(X,k)$, as before. This implies a continuity property of multiplication on $\ell^\infty(X,k)$ as in the preceding paragraph, with $r_2 = r_3 = r$, and with the topology determined on $\ell^\infty(X,k)$ by the supremum norm in place of τ_{r_1}. We can do a bit better than this, as in the next paragraph.

Let a positive real number r be given. As in Section 44, there are nonnegative real-valued functions g_0, h_0 on X such that g_0 vanishes at infinity, h_0 is summable, and

$$h(x) = g_0(x)^r\,h_0(x) \tag{46.6}$$

for every $x \in X$. This is the same as (44.3), with slightly different notation. If g_1, g_2 are bounded nonnegative real-valued functions on X, then we get that

$$\begin{aligned}
&\Big(\sum_{x \in X} (g_1(x)\,g_2(x))^r\,h(x)\Big)^{1/r} \\
&\quad = \Big(\sum_{x \in X} g_1(x)^r\,g_2(x)^r\,g_0(x)^r\,h_0(x)\Big)^{1/r} \\
&\quad \le \Big(\sup_{x \in X} g_1(x)\,g_0(x)\Big)\Big(\sum_{x \in X} g_2(x)^r\,h_0(x)\Big)^{1/r},
\end{aligned} \tag{46.7}$$

where the second step is basically the same as (46.4). This implies that

$$\widetilde{N}_{h,r}(a\,b) \le \widetilde{N}_{g_0}(a)\,\widetilde{N}_{h_0,r}(b) \tag{46.8}$$

for every $a, b \in \ell^\infty(X,k)$, where $\widetilde{N}_{g_0}(b)$ is as in (43.1). Let τ_∞ be the topology determined on $\ell^\infty(X,k)$ by (43.4), as in Section 44. Using (46.8), we get the same type of continuity property of multiplication on $\ell^\infty(X,k)$ as before, with $r_2 = r_3 = r$, and τ_∞ in place of τ_{r_1}.

47 From $\ell^r(X,k)$ into $c_0(X,k)$

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let X be a nonempty set, and let r be a positive real number. Consider the space

$$\mathcal{BL}(\ell^r(X,k), c_0(X,k)) \tag{47.1}$$

of bounded linear mappings from $\ell^r(X,k)$ into $c_0(X,k)$, using $\|f\|_r$ as in (39.3) on the domain, and the supremum norm on the range. Let

$$\|T\|_{op,r\,\infty} \tag{47.2}$$

be the corresponding operator norm on (47.1). More precisely, this is a q_k-norm on (47.1), because the supremum norm is a q_k-norm on $c_0(X,k)$.

Remember that $\ell^r(X,k)$ is contained in $c_0(X,k)$, and that $\|f\|_r$ is greater than or equal to the supremum norm of f for every $f \in \ell^r(X,k)$, as in Section 39. If T is a bounded linear mapping from $\ell^r(X,k)$ into itself, then it follows that T may be considered as a bounded linear mapping from $\ell^r(X,k)$ into $c_0(X,k)$, using the inclusion of $\ell^r(X,k)$ in $c_0(X,k)$ in the range of T. This leads to a natural inclusion of $\mathcal{BL}(\ell^r(X,k))$ into (47.1). We also have that

$$\|T\|_{op,r\,\infty} \le \|T\|_{op,r\,r} \tag{47.3}$$

for every bounded linear mapping T from $\ell^r(X,k)$ into itself, where $\|T\|_{op,r\,r}$ is the operator norm on $\mathcal{BL}(\ell^r(X,k))$ corresponding to $\|f\|_r$ on the domain and range.

Similarly, if T is a bounded linear mapping from $c_0(X,k)$ into itself, then the restriction of T to $\ell^r(X,k)$ is a bounded linear mapping from $\ell^r(X,k)$ into $c_0(X,k)$. This leads to a natural linear mapping from $\mathcal{BL}(c_0(X,k))$ into (47.1). As before, we have that

$$\|T\|_{op,r\,\infty} \le \|T\|_{op,\infty\,\infty} \tag{47.4}$$

for every bounded linear mapping T from $c_0(X,k)$ into itself, where $\|T\|_{op,\infty\,\infty}$ is the operator norm on $\mathcal{BL}(c_0(X,k))$ corresponding to the supremum norm on the domain and range. More precisely, the left side of (47.4) refers to the operator norm of the restriction of T to $\ell^r(X,k)$.

Let a be a bounded k-valued function on X, and let us consider the corresponding multiplication operator M_a as a bounded linear mapping from $\ell^r(X,k)$ into $c_0(X,k)$. Observe that

$$\|M_a\|_{op,r\,\infty} = \|a\|_\infty. \tag{47.5}$$

More precisely, the fact that M_a defines a bounded linear mapping from $\ell^r(X,k)$ into $c_0(X,k)$ with operator norm less than or equal to $\|a\|_\infty$ can be verified directly from the definitions, and it can also be obtained from (47.3) or (47.4) and the analogous statement for multiplication operators on $\ell^r(X,k)$ or $c_0(X,k)$, respectively. To get the opposite inequality, one can use (35.3), as usual. As in previous situations, $a \mapsto M_a$ defines a linear mapping from $\ell^\infty(X,k)$ into (47.1).

If $f \in \ell^r(X,k)$, then

$$T \mapsto \|T(f)\|_\infty \tag{47.6}$$

defines a q_k-seminorm on (47.1). The topology determined on (47.1) by the collection of q_k-seminorms of the form (47.6) corresponds exactly to the strong operator topology. Of course,

$$\|T(f)\|_\infty \le \|T(f)\|_r \tag{47.7}$$

for every bounded linear mapping T from $\ell^r(X,k)$ into itself and every f in $\ell^r(X,k)$, because of (39.4). This implies that the natural inclusion mapping from $\mathcal{BL}(\ell^r(X,k))$ into (47.1) is also continuous with respect to the associated strong operator topologies.

If $f \in c_0(X,k)$, then (47.6) defines a q_k-seminorm on $\mathcal{BL}(c_0(X,k))$. The topology determined on $\mathcal{BL}(c_0(X,k))$ by this collection of q_k-seminorms corresponds exactly to the strong operator topology. This topology on $\mathcal{BL}(c_0(X,k))$ is at least as strong as the one determined by the collection of q_k-seminorms of the form (47.6) with $f \in \ell^r(X,k)$, because $\ell^r(X,k)$ is contained in $c_0(X,k)$. This implies that the natural mapping from $\mathcal{BL}(c_0(X,k))$ into (47.1) is continuous with respect to their associated strong operator topologies.

As in Section 29, the topology determined on (47.1) by the operator q_k-norm (47.2) is at least as strong as the associated strong operator topology. The mapping from $a \in \ell^\infty(X,k)$ to the multiplication operator M_a in (47.1) is a homeomorphism onto its image with respect to the topology determined on $\ell^\infty(X,k)$ by the supremum norm, and the topology induced on the image by the one determined on (47.1) by (47.2), because of (47.5). It follows that the topology determined on $\ell^\infty(X,k)$ by the supremum norm is at least as strong as the one that corresponds to the strong operator topology on (47.1) via this mapping. The topology on $\ell^\infty(X,k)$ that corresponds to the strong operator topology on (47.1) in this way will be discussed further in the next section.

48 From $\ell^r(X,k)$ into $c_0(X,k)$, continued

Let us continue with the same notation and hypotheses as in the previous section. As in Section 37,

$$N_f(a) = \|a\,f\|_\infty \tag{48.1}$$

defines a q_k-seminorm on $\ell^\infty(X,k)$ as a function of a for every $f \in c_0(X,k)$. Remember that $\ell^r(X,k)$ is contained in $c_0(X,k)$, as in Section 39. Let

$$\tau_{r,\infty} \tag{48.2}$$

be the topology determined on $\ell^\infty(X,k)$ by

$$\{N_f : f \in \ell^r(X,k)\}, \tag{48.3}$$

as in Section 4. By construction, $a \mapsto M_a$ is a homeomorphism from $\ell^\infty(X,k)$ onto its image in (47.1), with respect to $\tau_{r,\infty}$ on $\ell^\infty(X,k)$, and the topology induced on the image by the strong operator topology, as in the preceding section.

Of course, (48.3) is contained in

$$\{N_f : f \in c_0(X, k)\}, \tag{48.4}$$

because $\ell^r(X, k)$ is contained in $c_0(X, k)$. As in Section 44, we let τ_∞ be the topology determined on $\ell^\infty(X, k)$ by (48.4), which is the same as (37.7). Thus

$$\tau_{r,\infty} \subseteq \tau_\infty, \tag{48.5}$$

which is to say that τ_∞ is at least as strong as $\tau_{r,\infty}$, because (48.3) is contained in (48.4). Remember that τ_∞ corresponds to the strong operator topology on $\mathcal{BL}(c_0(X, k))$ via the mapping from $a \in \ell^\infty(X, k)$ to the associated multiplication operator on $c_0(X, k)$, as in Section 37. Using this and the analogous statement for $\tau_{r,\infty}$ mentioned earlier, (48.5) could also be derived from the fact that the natural mapping from $\mathcal{BL}(c_0(X, k))$ into (47.1) is continuous with respect to the associated strong operator topologies, as in the previous section.

Put

$$N_{f,r}(a) = \|a\, f\|_r \tag{48.6}$$

for every $a \in \ell^\infty(X, k)$ and $f \in \ell^r(X, k)$, as in (40.1). This defines a q_k or r-seminorm on $\ell^\infty(X, k)$ as a function of a for every $f \in \ell^r(X, k)$, as in Section 40. Note that

$$N_f(a) \le N_{f,r}(a) \tag{48.7}$$

for every $a \in \ell^\infty(X, k)$ and $f \in \ell^r(X, k)$, because of (39.4). As in Section 42, we let τ_r be the topology determined on $\ell^\infty(X, k)$ by

$$\{N_{f,r} : f \in \ell^r(X, k)\}, \tag{48.8}$$

which is the same as (40.2). Using (48.7), we get that

$$\tau_{r,\infty} \subseteq \tau_r, \tag{48.9}$$

so that τ_r is at least as strong as $\tau_{r,\infty}$. As in Section 40, τ_r corresponds to the strong operator topology on $\mathcal{BL}(\ell^r(X, k))$ by the mapping that sends a in $\ell^\infty(X, k)$ to the associated multiplication operator on $\ell^r(X, k)$. Thus (48.9) could also be obtained from the continuity of the natural inclusion of $\mathcal{BL}(\ell^r(X, k))$ into (47.1) with respect to the corresponding strong operator topologies, as before.

Remember that τ_∞ is at least as strong as τ_r, as in (44.8). Using this, (48.5) could be derived from (48.9), but the earlier argument for (48.5) was more direct. As usual, it is easy to see that $\tau_{r,\infty}$ is at least as strong as the topology induced on $\ell^\infty(X, k)$ by the one defined on $c(X, k)$ in Section 34. If $|\cdot|$ is the trivial absolute value function on k, then $\ell^\infty(X, k)$ is the same as $c(X, k)$, and $\tau_{r,\infty}$ is the same as the topology defined on $c(X, k)$ in Section 34. In this case, τ_r and τ_∞ are the same as this topology as well.

49 Some additional properties

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let X be a nonempty set. Also let r_1, r_2 be positive real numbers with $r_1 \le r_2$. Remember that

$$\ell^{r_1}(X, k) \subseteq \ell^{r_2}(X, k), \tag{49.1}$$

with

$$\|f\|_{r_2} \le \|f\|_{r_1} \tag{49.2}$$

for every $f \in \ell^{r_1}(X, k)$, as in Section 39. If T is a bounded linear mapping from $\ell^{r_2}(X, k)$ into $c_0(X, k)$, then it follows that the restriction of T to $\ell^{r_1}(X, k)$ is a bounded linear mapping from $\ell^{r_1}(X, k)$ into $c_0(X, k)$. We also have that

$$\|T\|_{op, r_1 \infty} \le \|T\|_{op, r_2 \infty}, \tag{49.3}$$

using the notation in (47.2) for the corresponding operator norms, and where the left side of (49.3) refers to the operator norm of the restriction of T to $\ell^{r_1}(X, k)$. The mapping from T to its restriction to $\ell^{r_1}(X, k)$ defines a natural bounded linear mapping from

$$\mathcal{BL}(\ell^{r_2}(X, k), c_0(X, k)) \tag{49.4}$$

into

$$\mathcal{BL}(\ell^{r_1}(X, k), c_0(X, k)) \tag{49.5}$$

with respect to the corresponding operator norms. It is easy to see that this mapping is continuous with respect to the corresponding strong operator topologies as well.

Let $\tau_{r,\infty}$ be the topology determined on $\ell^\infty(X, k)$ by (48.3) for each positive real number r, as in the previous section. Of course, (49.1) implies that

$$\{N_f : f \in \ell^{r_1}(X, k)\} \subseteq \{N_f : f \in \ell^{r_2}(X, k)\}, \tag{49.6}$$

where N_f is as in (48.1). It follows that

$$\tau_{r_1,\infty} \subseteq \tau_{r_2,\infty}, \tag{49.7}$$

so that $\tau_{r_2,\infty}$ is at least as strong as $\tau_{r_1,\infty}$. Remember that $\tau_{r,\infty}$ corresponds to the strong operator topology on (47.1) via the mapping from $a \in \ell^\infty(X, k)$ to the corresponding multiplication operator M_a. Thus (49.7) could also be obtained from the continuity of the natural mapping from (49.4) into (49.5) with respect to the corresponding strong operator topologies, as in the preceding paragraph.

Let r be a positive real number again, and let E be a subset of $\ell^\infty(X, k)$. Remember that $\tau_{r,\infty}$ is at least as strong as the topology induced on $\ell^\infty(X, k)$ by the one defined on $c(X, k)$ in Section 34, as in the previous section. This implies that the topology induced on E by $\tau_{r,\infty}$ is at least as strong as the topology induced on E by the one defined on $c(X, k)$ in Section 34. If the elements of E have bounded supremum norm, then one can check that the topology induced on E by $\tau_{r,\infty}$ is the same as the topology induced on E by the one defined on $c(X,k)$ in Section 34. This is analogous to earlier statements for τ_r and τ_∞, and in fact the present version could be derived from either of the earlier statements using (48.5) or (48.9).

Let E be any subset of $\ell^\infty(X,k)$ again, and let

$$\mathcal{E} = \{M_a : a \in E\} \tag{49.8}$$

be the collection of multiplication operators M_a corresponding to elements of E, considered as bounded linear mappings from $\ell^r(X,k)$ into $c_0(X,k)$. Put

$$\mathcal{E}_f = \{M_a(f) : a \in E\} = \{a\,f : a \in E\} \tag{49.9}$$

for each $f \in \ell^r(X,k)$, as in (28.1). If the elements of E have bounded supremum norm, then the elements of $\mathcal{E}$ have bounded operator norm, and in particular the elements of $\mathcal{E}_f$ have bounded supremum norm for every $f \in \ell^r(X,k)$. Conversely, if the elements of $\mathcal{E}_f$ have bounded supremum norm for every $f \in \ell^r(X,k)$, then the elements of E have bounded supremum norm. This is analogous to the corresponding statements considered in Sections 38 and 40.

50 Some additional properties, continued

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. If g is a nonnegative real-valued function on X that vanishes at infinity, then we put

$$\widetilde{N}_g(a) = \sup_{x \in X} (|a(x)|\,g(x)) \tag{50.1}$$

for every $a \in \ell^\infty(X,k)$, as in (43.1). This defines a q_k-seminorm on $\ell^\infty(X,k)$, as in Section 43. If f is a k-valued function on X that vanishes at infinity and

$$g(x) = |f(x)| \tag{50.2}$$

for each $x \in X$, then $g(x)$ vanishes at infinity on X too, and we have that

$$N_f(a) = \widetilde{N}_g(a) \tag{50.3}$$

for every $a \in \ell^\infty(X,k)$, as in (43.6), where $N_f(a)$ is as in (48.1). As in Section 44, τ_∞ may be defined equivalently as the topology on $\ell^\infty(X,k)$ determined by the collection (43.4) of q_k-seminorms of the form (50.1), where g vanishes at infinity on X.

Let r be a positive real number, and consider the collection

$$\{\widetilde{N}_g : g \text{ is a nonnegative real-valued } r\text{-summable function on } X\} \tag{50.4}$$

of q_k-seminorms on $\ell^\infty(X,k)$. If f is an r-summable k-valued function on X, then (50.2) defines an nonnegative real-valued r-summable on X, so that (48.3) is contained in (50.4), by (50.3). Thus the topology determined on $\ell^\infty(X, k)$ is at least as strong as the topology $\tau_{r,\infty}$ determined on $\ell^\infty(X, k)$ by (48.3). One can check that these two topologies on $\ell^\infty(X, k)$ are the same, using the same type of argument as in Section 43. More precisely, if $|\cdot|$ is nontrivial on k, and if g is any nonnegative real-valued function on X, then there is a k-valued function f on X whose absolute value approximates g in the sense of (43.7). In particular, if g is r-summable on X, then f is r-summable on X as well. In this case, one can check that $\tau_{r,\infty}$ is the same as the topology determined on $\ell^\infty(X, k)$ by (50.4), using the previous statement and (43.8). Otherwise, if $|\cdot|$ is the trivial absolute value function on k, then one can check more directly that $\tau_{r,\infty}$ is the same as the topology determined on $\ell^\infty(X, k)$ by (50.4), as before.

Let r_1, r_2, r_3 be positive real numbers such that

$$\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}. \tag{50.5}$$

If g_1, g_2 are nonnegative real-valued functions on X that are r_1, r_2-summable, respectively, then it is well known that their product $g_1\, g_2$ is r_3-summable on X, with

$$\Big(\sum_{x \in X} (g_1(x)\, g_2(x))^{r_3}\Big)^{1/r_3} \le \Big(\sum_{x \in X} g_1(x)^{r_1}\Big)^{1/r_1} \Big(\sum_{x \in X} g_2(x)^{r_2}\Big)^{1/r_2}. \tag{50.6}$$

This is basically Hölder's inequality for sums, which is normally stated with $r_3 = 1$, and one can easily reduce to that case. If f_1, f_2 are k-valued functions on X that are r_1, r_2-summable on X, respectively, then it follows that their product $f_1\, f_2$ is r_3-summable on X. Now let g be a nonnegative real-valued r_3-summable function on X, and put

$$g_1(x) = g(x)^{r_3/r_1}, \quad g_2(x) = g(x)^{r_3/r_2} \tag{50.7}$$

for each $x \in X$. It is easy to see that g_1 is r_1-summable on X, g_2 is r_2-summable on X, and

$$g(x) = g_1(x)\, g_2(x) \tag{50.8}$$

for every $x \in X$. Consider the mapping from $\ell^\infty(X, k) \times \ell^\infty(X, k)$ into $\ell^\infty(X, k)$ defined by pointwise multiplication of functions. Let us take the domain of this mapping to be equipped with the product topology associated to $\tau_{r_1,\infty}$ and $\tau_{r_2,\infty}$ on the two factors, and let us take the range to be equipped with $\tau_{r_3,\infty}$. One can check that this mapping is continuous under these conditions, using (45.2) with g_1, g_2 as in (50.7).

Let r be a positive real number again, and let g be a nonnegative real-valued r-summable function on X. As in Section 44, g can be expressed as the product of two nonnegative real-valued functions on X, where one of the functions vanishes at infinity, and the other is r-summable. More precisely, this was discussed before with $r = 1$, and it is easy to reduce to that case. Let us now consider multiplication of functions as a mapping from $\ell^\infty(X, k) \times \ell^\infty(X, k)$ into $\ell^\infty(X, k)$ again. Let us take the domain of this mapping to be equipped with the product topology associated to τ_∞ on one factor and $\tau_{r,\infty}$ on the other factor, and let us take the range to be equipped with $\tau_{r,\infty}$. As before, one can check that this mapping is continuous under these conditions, using (45.2) with g_1, g_2 in the factorization of g just mentioned. This is the same type of continuity property of multiplication on $\ell^\infty(X,k)$ as in the preceding paragraph, with $r_2 = r_3 = r$ and τ_∞ in place of $\tau_{r_1,\infty}$. Of course, it would be easier to use the topology on $\ell^\infty(X,k)$ determined by the supremum metric instead of τ_∞.

51 From $\ell^{r_1}(X,k)$ into $\ell^{r_2}(X,k)$

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. Also let r_1 and r_2 be positive real numbers, and consider the space

$$\mathcal{BL}(\ell^{r_1}(X,k), \ell^{r_2}(X,k)) \tag{51.1}$$

of bounded linear mappings from $\ell^{r_1}(X,k)$ into $\ell^{r_2}(X,k)$. If T is such a bounded linear mapping, then we let

$$\|T\|_{op,r_1\,r_2} \tag{51.2}$$

be the corresponding operator norm, with respect to the appropriate ℓ^r norms on the domain and range. More precisely, this is a q_k or r_2-norm on (51.1), because of the corresponding property of the ℓ^{r_2} norm, as in Section 39. Of course, the analogue of (51.1) with $c_0(X,k)$ instead of $\ell^{r_2}(X,k)$ has been discussed in Sections 47 and 49.

Let r_3 be another positive real number, with $r_2 \le r_3$. Thus

$$\ell^{r_2}(X,k) \subseteq \ell^{r_3}(X,k), \tag{51.3}$$

and

$$\|f\|_{r_3} \le \|f\|_{r_2} \tag{51.4}$$

for every $f \in \ell^{r_2}(X,k)$, as in Section 39. This leads to a natural inclusion of (51.1) into

$$\mathcal{BL}(\ell^{r_1}(X,k), \ell^{r_3}(X,k)). \tag{51.5}$$

More precisely, if T is a bounded linear mapping from $\ell^{r_1}(X,k)$ into $\ell^{r_2}(X,k)$, then T may be considered as a bounded linear mapping from $\ell^{r_1}(X,k)$ into $\ell^{r_3}(X,k)$ as well, with

$$\|T\|_{op,r_1\,r_3} \le \|T\|_{op,r_1\,r_2}, \tag{51.6}$$

by (51.4). The natural inclusion mapping from (51.1) into (51.5) is also continuous with respect to the corresponding strong operator topologies, because of (51.4).

Similarly,

$$\ell^{r_2}(X,k) \subseteq c_0(X,k), \tag{51.7}$$

and

$$\|f\|_\infty \le \|f\|_{r_2} \tag{51.8}$$

for every $f \in \ell^{r_2}(X, k)$, as in Section 39. This leads to a natural inclusion of (51.1) into

$$(51.9) \qquad \mathcal{BL}(\ell^{r_1}(X, k), c_0(X, k)).$$

As before, if T is a bounded linear mapping from $\ell^{r_1}(X, k)$ into $\ell^{r_2}(X, k)$, then T may be considered as a bounded linear mapping from $\ell^{r_1}(X, k)$ into $c_0(X, k)$ too, with

$$(51.10) \qquad \|T\|_{op, r_1\, \infty} \le \|T\|_{op, r_1\, r_2}.$$

The natural inclusion mapping from (51.1) into (51.9) is also continuous with respect to the corresponding strong operator topologies. If $r_1 = r_2$, then this reduces to part of the discussion in Section 47.

Now let r_0 be a positive real number with $r_0 \le r_1$, so that

$$(51.11) \qquad \ell^{r_0}(X, k) \subseteq \ell^{r_1}(X, k),$$

and

$$(51.12) \qquad \|f\|_{r_1} \le \|f\|_{r_0}$$

for every $f \in \ell^{r_0}(X, k)$. If T is a bounded linear mapping from $\ell^{r_1}(X, k)$ into $\ell^{r_2}(X, k)$, then the restriction of T to $\ell^{r_0}(X, k)$ is a bounded linear mapping from $\ell^{r_0}(X, k)$ into $\ell^{r_2}(X, k)$, with

$$(51.13) \qquad \|T\|_{op, r_0\, r_2} \le \|T\|_{op, r_1\, r_2},$$

by (51.12). More precisely, the left side of (51.13) refers to the operator norm of the restriction of T to $\ell^{r_0}(X, k)$. This defines a natural linear mapping from (51.1) into

$$(51.14) \qquad \mathcal{BL}(\ell^{r_0}(X, k), \ell^{r_2}(X, k)).$$

This mapping is also continuous with respect to the corresponding strong operator topologies, because of (51.12).

52 $r_1 \le r_2$

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let X be a nonempty set. Also let r_1 and r_2 be positive real numbers, and suppose now that $r_1 \le r_2$. This implies that

$$(52.1) \qquad \ell^{r_1}(X, k) \subseteq \ell^{r_2}(X, k),$$

and that

$$(52.2) \qquad \|f\|_{r_2} \le \|f\|_{r_1}$$

for every $f \in \ell^{r_1}(X, k)$, as in Section 39. If T is a bounded linear mapping from $\ell^{r_1}(X, k)$ into itself, then T may be considered as a bounded linear mapping from $\ell^{r_1}(X, k)$ into $\ell^{r_2}(X, k)$ as well, with

$$(52.3) \qquad \|T\|_{op, r_1\, r_2} \le \|T\|_{op, r_1\, r_1},$$

as in the previous section. Similarly, if T is a bounded linear mapping from $\ell^{r_2}(X,k)$ into itself, then the restriction of T to $\ell^{r_1}(X,k)$ is a bounded linear mapping from $\ell^{r_1}(X,k)$ into $\ell^{r_2}(X,k)$, with

$$\|T\|_{op,r_1\,r_2} \le \|T\|_{op,r_2\,r_2}. \tag{52.4}$$

As usual, the left side of (52.4) refers to the operator norm of the restriction of T to $\ell^{r_1}(X,k)$. Thus we get natural linear mappings from $\mathcal{BL}(\ell^{r_1}(X,k))$ and $\mathcal{BL}(\ell^{r_2}(X,k))$ into (51.1), as before. These mappings are also continuous with respect to the corresponding strong operator topologies.

Let a be a bounded k-valued function on X, and let us now consider the corresponding multiplication operator M_a as a bounded linear mapping from $\ell^{r_1}(X,k)$ into $\ell^{r_2}(X,k)$. Let us check that

$$\|M_a\|_{op,r_1\,r_2} = \|a\|_\infty \tag{52.5}$$

under these conditions. The fact that M_a defines a bounded linear mapping from $\ell^{r_1}(X,k)$ into $\ell^{r_2}(X,k)$ with operator norm less than or equal to $\|a\|_\infty$ can be verified directly from the definitions, using (52.2). This can also be considered as an instance of (52.3) or (52.4), using the analogous statement for multiplication by a as a bounded linear mapping from $\ell^{r_1}(X,k)$ or $\ell^{r_2}(X,k)$ into itself. The opposite inequality can be obtained from (35.3), as usual.

As in Section 40,

$$N_{f,r_2}(a) = \|a\,f\|_{r_2} \tag{52.6}$$

defines a q_k or r_2-seminorm on $\ell^\infty(X,k)$ as a function of a for every f in $\ell^{r_2}(X,k)$. In particular, this holds for every $f \in \ell^{r_1}(X,k)$, because of (52.1). Let

$$\tau_{r_1,r_2} \tag{52.7}$$

be the topology determined on $\ell^\infty(X,k)$ by

$$\{N_{f,r_2} : f \in \ell^{r_1}(X,k)\}, \tag{52.8}$$

as in Section 4. By construction, $a \mapsto M_a$ is a homeomorphism from $\ell^\infty(X,k)$ onto its image in (51.1), with respect to τ_{r_1,r_2} on $\ell^\infty(X,k)$, and the topology induced on the image by the strong operator topology. Note that

$$\tau_r = \tau_{r,r} \tag{52.9}$$

for every positive real number r, where τ_r is as defined in Section 42.

If r_3 is a positive real number with $r_2 \le r_3$, then

$$N_{f,r_3}(a) \le N_{f,r_2}(a) \tag{52.10}$$

for every $a \in \ell^\infty(X,k)$ and $f \in \ell^{r_2}(X,k)$, by (51.4). This implies that

$$\tau_{r_1,r_3} \subseteq \tau_{r_1,r_2} \tag{52.11}$$

when $r_2 \le r_3$, so that τ_{r_1,r_2} is at least as strong as τ_{r_1,r_2}. This could also be derived from the continuity of the natural mapping from (51.1) into (51.5) with respect to the corresponding strong operator topologies, as in the previous section.

Remember that

$$N_f(a) = \|a\, f\|_\infty \tag{52.12}$$

defines a q_k-seminorm on $\ell^\infty(X, k)$ as a function of a for every $f \in c_0(X, k)$, as in Section 37. Moreover,

$$N_f(a) \le N_{f,r_2}(a) \tag{52.13}$$

for every $a \in \ell^\infty(X, k)$ and $f \in \ell^{r_2}(X, k)$, by (51.8). This implies that

$$\tau_{r_1,\infty} \subseteq \tau_{r_1,r_2}, \tag{52.14}$$

where $\tau_{r_1,\infty}$ is as defined in Section 48. This could also be obtained from the continuity of the natural mapping from (51.1) into (51.9) with respect to the corresponding strong operator topologies, as before.

Let r_0 be a positive real number with $r_0 \le r_1$. Observe that

$$\{N_{f,r_2} : f \in \ell^{r_0}(X, k)\} \subseteq \{N_{f,r_2} : f \in \ell^{r_1}(X, k)\}, \tag{52.15}$$

by (51.11). This implies that

$$\tau_{r_0,r_2} \subseteq \tau_{r_1,r_2}, \tag{52.16}$$

so that τ_{r_1,r_2} is at least as strong as τ_{r_0,r_2}. As before, this could also be derived from the continuity of the natural mapping from (51.1) into (51.14) with respect to the corresponding strong operator topologies.

Note that

$$\tau_{r_1,r_2} \subseteq \tau_{r_1,r_1} = \tau_{r_1}. \tag{52.17}$$

This follows from (52.9) and (52.11), with suitable substitutions. Similarly,

$$\tau_{r_1,r_2} \subseteq \tau_{r_2,r_2} = \tau_{r_2}, \tag{52.18}$$

by (52.9) and (52.16), with suitable substitutions again. These inclusions could also be obtained from the continuity of the natural mappings from $\mathcal{BL}(\ell^{r_1}(X, k))$ and $\mathcal{BL}(\ell^{r_2}(X, k))$ into (51.1) with respect to the corresponding strong operator topologies, as mentioned at the beginning of the section.

53 $r_1 \le r_2$, continued

Let us continue with the same notation and hypotheses as in the previous section. If h is a nonnegative real-valued summable function on X, then we put

$$\widetilde{N}_{h,r_2}(a) = \Big(\sum_{x \in X} |a(x)|^{r_2}\, h(x)\Big)^{1/r_2} \tag{53.1}$$

for every $a \in \ell^\infty(X, k)$, as in (41.1). This defines a q_k or r_2-seminorm on $\ell^\infty(X, k)$, as in Section 41. If $f \in \ell^{r_2}(X, k)$, then

$$h(x) = |f(x)|^{r_2} \tag{53.2}$$

is summable on X, and

$$\widetilde{N}_{h,r_2}(a) = N_{f,r_2}(a) \tag{53.3}$$

for every $a \in \ell^\infty(X, k)$. Here $N_{f,r_2}(a)$ is as in (52.6), and (53.3) is the same as (41.5). Of course, if h is r-summable on X for some $r \in (0, 1]$, then h summable on X, as in Section 39. Similarly, if $f \in \ell^{r_1}(X, k)$, then $f \in \ell^{r_2}(X, k)$, because $r_1 \le r_2$, as before. In this case, (53.2) is (r_1/r_2)-summable on X, and $r_1/r_2 \le 1$.

As in Section 4, the collection

$$\{\widetilde{N}_{h,r_2} : h \text{ is a nonnegative real-valued } (r_1/r_2)\text{-summable function on } X\} \tag{53.4}$$

of q_k or r_2-seminorms on $\ell^\infty(X, k)$ determines a topology on $\ell^\infty(X, k)$. Observe that (52.8) is contained in (53.4), by (53.3) and the remarks in the preceding paragraph. This implies that the topology determined on $\ell^\infty(X, k)$ by (53.4) is at least as strong as the topology τ_{r_1,r_2} determined on $\ell^\infty(X, k)$ by (52.8). One can check that these two topologies are actually the same, as in Section 41. More precisely, if $|\cdot|$ is not the trivial absolute value function on k, then one can approximate any nonnegative real-valued function on X by the r_2th power of the absolute value of a k-valued function on X, as in (41.7). This permits one to estimate elements of (53.4) in terms of elements of (52.8), as in (41.8). Otherwise, if $|\cdot|$ is the trivial absolute value function on k, then one can also verify that the topologies are the same, as before.

Let r_0 be a positive real number with $r_0 \le r_1$, so that

$$r_0/r_2 \le r_1/r_2. \tag{53.5}$$

If h is a nonnegative real-valued function on X that is (r_0/r_1)-summable, then it follows that h is (r_1/r_2)-summable on X as well, as in Section 39. This implies that

$$\{\widetilde{N}_{h,r_2} : h \text{ is a nonnegative real-valued } (r_0/r_1)\text{-summable function on } X\} \tag{53.6}$$

is contained in (53.4). This is analogous to (52.15), and gives another way to look at (52.16).

If r is any positive real number, then $\tau_{r,r}$ is the same as the topology τ_r defined on $\ell^\infty(X, k)$ in Section 42, as in (52.9). In this case, the characterization of this topology using (53.4) reduces to the description of τ_r in terms of (41.3), as in Section 42 again. Let us suppose from now on in this section that

$$r_1 < r_2. \tag{53.7}$$

Let h be a nonnegative real-valued (r_1/r_2)-summable function on X, and put

$$g(x) = h(x)^{(r_2 - r_1)/r_2^2} \tag{53.8}$$

for each $x \in X$. The exponent has been chosen so that

$$g(x)^{r_2}\, h(x)^{r_1/r_2} = h(x) \tag{53.9}$$

for every $x \in X$. Thus

$$\widetilde{N}_{h,r_2}(a) = \Big(\sum_{x \in X} |a(x)|^{r_2}\, g(x)^{r_2}\, h(x)^{r_1/r_2}\Big)^{1/r_2} \tag{53.10}$$

for every $a \in \ell^\infty(X, k)$.

Note that

$$g(x)^{(r_1\, r_2)/(r_2 - r_1)} = h(x)^{r_1/r_2} \tag{53.11}$$

for every $x \in X$, so that g is $(r_1\, r_2)/(r_2 - r_1)$-summable on X. Put

$$\widetilde{N}_g(a) = \sup_{x \in X} (|a(x)|\, g(x)) \tag{53.12}$$

for every $a \in \ell^\infty(X, k)$, as in (43.1). It is easy to see that

$$\widetilde{N}_{h,r_2}(a) \le \Big(\sum_{x \in X} h(x)^{r_1/r_2}\Big)^{1/r_2}\, \widetilde{N}_g(a) \tag{53.13}$$

for every $a \in \ell^\infty(X, k)$, using (53.10). Remember that $\tau_{r,\infty}$ is the topology defined on $\ell^\infty(X, k)$ in Section 48 for each $r > 0$, which can also be characterized by (50.4). It is easy to see that

$$\tau_{r_1,r_2} \subseteq \tau_{r,\infty} \quad \text{with } r = (r_1\, r_2)/(r_2 - r_1), \tag{53.14}$$

using (53.13) and the characterization of τ_{r_1,r_2} in terms of (53.4).

Let r_3 be a real number with $r_2 \le r_3$, and observe that

$$\begin{aligned} &\widetilde{N}_{h,r_2}(a) \\ &\quad \le \Big(\sum_{x \in X} h(x)^{r_1/r_2}\Big)^{1/r_2 - 1/r_3} \Big(\sum_{x \in X} |a(x)|^{r_3}\, g(x)^{r_3}\, h(x)^{r_1/r_2}\Big)^{1/r_3} \end{aligned} \tag{53.15}$$

for every $a \in \ell^\infty(X, k)$. This is basically the same as (42.2), using (53.10), and with suitable substitutions. If we put

$$h'(x) = g(x)^{r_3}\, h(x)^{r_1/r_2} \tag{53.16}$$

for each $x \in X$, then (53.15) can be reexpressed as saying that

$$\widetilde{N}_{h,r_2}(a) \le \Big(\sum_{x \in X} h(x)^{r_1/r_2}\Big)^{1/r_2 - 1/r_3}\, \widetilde{N}_{h',r_3}(a) \tag{53.17}$$

for every $a \in \ell^\infty(X, k)$, where $\widetilde{N}_{h', r_3}(a)$ is defined as in (53.1).

Using the definition (53.8) of g, we get that

$$h'(x) = h(x)^{r'}, \tag{53.18}$$

where

$$r' = (r_2 - r_1)\, r_3 / r_2^2 + r_1 / r_2, \tag{53.19}$$

It follows that h' is $(r_1/(r_2\, r'))$-summable on X, because h is (r_1/r_2)-summable on X, by hypothesis. Put

$$r_1' = (r_1/(r_2\, r'))\, r_3, \tag{53.20}$$

so that $r_1'/r_3 = r_1/(r_2\, r')$, and hence h' is (r_1'/r_3)-summable on X. Thus (53.17) implies that

$$\tau_{r_1, r_2} \subseteq \tau_{r_1', r_3}, \tag{53.21}$$

using the characterization of τ_{r_1, r_2} in terms of (53.4), and similarly for τ_{r_1', r_3}. Note that

$$r_2\, r'/r_3 = (r_2 - r_1)/r_2 + r_1/r_3, \tag{53.22}$$

which gives another way to look at (53.20).

54 Another perspective

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. Also let r be a positive real number, and let g, h be nonnegative real-valued functions on X, with g bounded and h summable. Thus

$$g(x)^r\, h(x) \tag{54.1}$$

is summable on X too, and we put

$$\widetilde{N}_{g,h,r}(a) = \Big(\sum_{x \in X} |a(x)|^r\, g(x)^r\, h(x) \Big)^{1/r} \tag{54.2}$$

for every $a \in \ell^\infty(X, k)$. This is the same as $\widetilde{N}_{g^r\, h, r}(a)$ in the notation of (41.1) and (53.1). In particular, $\widetilde{N}_{g,h,r}$ defines a q_k or r-seminorm on $\ell^\infty(X, k)$, as in Section 41.

Clearly

$$\widetilde{N}_{g,h,r}(a) \le \Big(\sup_{x \in X} (|a(x)|\, g(x)) \Big) \Big(\sum_{x \in X} h(x) \Big)^{1/r} \tag{54.3}$$

for every $a \in \ell^\infty(X, k)$. This is the same as saying that

$$\widetilde{N}_{g,h,r}(a) \le \Big(\sum_{x \in X} h(x) \Big)^{1/r} \widetilde{N}_g(a) \tag{54.4}$$

for every $a \in \ell^\infty(X, k)$, where $\widetilde{N}_g(a)$ is as in (53.12). If r_1, r_2 are positive real numbers with $r_1 \le r_2$, then we have that

$$\widetilde{N}_{g,h,r_1}(a) \le \Big(\sum_{x \in X} h(x) \Big)^{1/r_1 - 1/r_2} \widetilde{N}_{g,h,r_2}(a) \tag{54.5}$$

for every $a \in \ell^\infty(X, k)$. This follows from (42.2), with suitable substitutions.

Suppose that g is ρ-summable on X for some positive real number ρ, so that $g(x)^r$ is (ρ/r)-summable on X. If ρ_0 is the positive real number that satisfies

$$\frac{1}{\rho_0} = \frac{r}{\rho} + 1, \tag{54.6}$$

then (54.1) is ρ_0-summable on X, by Hölder's inequality. Note that $\rho_0 < 1$, because $r, \rho > 0$.

Now let ρ_0 be any positive real number strictly less than 1, and let ρ be the positive real number that satisfies (54.6), with r given as before. Let h_0 be a nonnegative real-valued ρ_0-summable function on X, so that

$$h(x) = h_0(x)^{\rho_0} \tag{54.7}$$

defines a summable function on X. If we put

$$g(x) = h_0(x)^{(1-\rho_0)/r} \tag{54.8}$$

for each $x \in X$, then $h_0(x)$ is equal to (54.1) for every $x \in X$, by construction. One can also verify that g is ρ-summable on X under these conditions, using (54.6) and the ρ_0-summability of h_0 on X.

Let ρ be a positive real number again, and let

$$\tau'_{\rho,r} \tag{54.9}$$

be the topology determined on $\ell^\infty(X, k)$ by

$$\begin{aligned} \{\widetilde{N}_{g,h,r} : {} & g, h \text{ are nonnegative real-valued functions on } X, \\ & g \text{ is } \rho\text{-summable on } X, \text{ and } h \text{ is summable on } X\}, \end{aligned} \tag{54.10}$$

as in Section 4. If ρ_1, ρ_2 are positive real numbers with $\rho_1 \le \rho_2$, then it is easy to see that

$$\tau'_{\rho_1,r} \subseteq \tau'_{\rho_2,r}. \tag{54.11}$$

This uses the fact that if g is ρ_1-summable on X, then g is ρ_2-summable on X, as in Section 39. Similarly, if r_1, r_2 are positive real numbers with $r_1 \le r_2$, then we have that

$$\tau'_{\rho,r_1} \subseteq \tau'_{\rho,r_2} \tag{54.12}$$

for every $\rho > 0$, because of (54.5).

Let $\tau_{r,\infty}$ be the topology defined on $\ell^\infty(X,k)$ for each positive real number r as in Section 48. This is the same as the topology determined on $\ell^\infty(X,k)$ by (50.4), as in Section 50. It is easy to see that

$$\tau'_{\rho,r} \subseteq \tau_{\rho,\infty} \tag{54.13}$$

for every $\rho, r > 0$, using (54.4).

Let $\tau_{r_1 . r_2}$ be the topology defined on $\ell^\infty(X,k)$ as in Section 52 for positive real numbers r_1, r_2 with $r_1 \leq r_2$. This topology can be described equivalently by (53.4), as in the previous section. Suppose that $r_1 < r_2$, and put

$$\rho_0 = r_1/r_2 < 1. \tag{54.14}$$

Let $\rho > 0$ be as in (54.6), with $r = r_2$, and this choice of ρ_0. Under these conditions, one can check that

$$\tau_{r_1,r_2} - \tau'_{\rho,r_2}, \tag{54.15}$$

where τ'_{ρ,r_2} is as defined earlier. More precisely, (53.4) is basically the same as (54.10) in this situation. This is because (54.1) is ρ_0-summable on X when g is ρ-summable and h is summable, and every nonnegative real-valued ρ_0-summable function on X is of this form, as before.

Of course, we can also start with positive real numbers ρ, r, and let ρ_0 be as in (54.6), so that $0 < \rho_0 < 1$. If we put

$$r_1 = \rho_0\, r \quad \text{and} \quad r_2 = r, \tag{54.16}$$

then we have that $0 < r_1 = \rho_0\, r_2 < r_2$ and $\rho_0 = r_1/r_2$, and (54.15) holds, as before.

References

[1] J. Cassels, *Local Fields*, Cambridge University Press, 1986.

[2] R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Mathematics **242**, Springer-Verlag, 1971.

[3] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bulletin of the American Mathematical Society **83** (1977), 569–645.

[4] G. David and S. Semmes, *Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure*, Oxford University Press, 1997.

[5] G. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.

[6] G. Folland, *Real Analysis*, 2nd edition, Wiley, 1999.

[7] F. Gouvêa, *p-Adic Numbers: An Introduction*, 2nd edition, Springer-Verlag, 1997.

[8] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, 2001.

[9] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Volumes I, II, 1970, 1979.

[10] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, 1975.

[11] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, 1974.

[12] J. Kelley, *General Topology*, Springer-Verlag, 1975.

[13] J. Kelley, I. Namioka, et al., *Linear Topological Spaces*, Springer-Verlag, 1976.

[14] J. Kelley and T. Srinivasan, *Measure and Integral*, Springer-Verlag, 1988.

[15] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976.

[16] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.

[17] W. Rudin, *Fourier Analysis on Groups*, Wiley, 1990.

[18] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, 1991.

[19] H. Schaefer and M. Wolff, *Topological Vector Spaces*, 2nd edition, Springer-Verlag, 1999.

[20] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[21] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, with the assistance of T. Murphy, Princeton University Press, 1993.

[22] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[23] M. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, 1975.

[24] F. Trèves, *Topological Vector Spaces, Distributions, and Kernels*, Dover, 2006.

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