Some topics related to metrics and norms, including ultrametrics and ultranorms, 6

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Abstract

In these notes, some basic classes of examples of vector spaces over fields with absolute value functions are considered.

Contents

Part I
Basic notions and examples

1 k-Valued functions

Let k be a field, and let X be a nonempty set. Consider the space $c(X, k)$ of k-valued functions on k. This is a vector space over k with respect to pointwise addition and scalar multiplication. More precisely, $c(X, k)$ is also a commutative algebra with respect to pointwise multiplication of functions. Later we shall consider absolute value functions on k, and related topologies on $c(X, k)$. Let

$$\mathbf{1}_X \tag{1.1}$$

be the function on X equal to the multiplicative identity element 1 in k at every point. This is the multiplicative identity element in $c(X, k)$.

The *support* of a k-valued function f on X is defined as usual by

$$\operatorname{supp} f = \{x \in X : f(x) \neq 0\}. \tag{1.2}$$

Let $c_{00}(X,k)$ be the space of k-valued functions f on X such that $\operatorname{supp} f$ has only finitely many elements. This is a linear subspace of $c(X,k)$, and in fact an ideal in $c(X,k)$ as a commutative algebra. We shall consider some topologies on $c_{00}(X,k)$ later as well. Of course, if X has only finitely many elements, then $c_{00}(X,k)$ is the same as $c(X,k)$. If $y \in X$, then we let δ_y be the k-valued function on X defined by

$$\begin{aligned} \delta_y(x) &= 1 \quad \text{when } x = y \\ &= 0 \quad \text{when } x \neq y. \end{aligned} \tag{1.3}$$

Thus $\delta_y \in c_{00}(X,k)$ for each $y \in X$, and

$$\{\delta_y : y \in X\} \tag{1.4}$$

is a basis for $c_{00}(X,k)$ as a vector space over k.

If $f \in c_{00}(X,k)$, then

$$\sum_{x \in k} f(x) \tag{1.5}$$

may be defined as an element of k, by reducing to the sum over any finite subset of X that contains the support of f. Of course,

$$f \mapsto \sum_{x \in X} f(x) \tag{1.6}$$

defines a linear mapping from $c_{00}(X,k)$ into k, which is to say a linear functional on $c_{00}(X,k)$. Similarly, if a is any k-valued function on X, then

$$f \mapsto \sum_{x \in X} a(x)\, f(x) \tag{1.7}$$

defines a linear functional on $c_{00}(X,k)$. It is easy to see that every linear functional on $c_{00}(X,k)$ is of this form, using the basis (1.4). If a has finite support in X, then (1.7) also defines a linear functional on $c(X,k)$.

If a is any k-valued function on X again, then

$$M_a(f) = a\, f \tag{1.8}$$

defines a linear mapping from $c(X,k)$ into itself, which is the *multiplication operator* associated to a. Note that this operator also maps $c_{00}(X,k)$ into itself. In particular,

$$M_a(\delta_y) = a(y)\, \delta_y \tag{1.9}$$

for every $y \in X$. Similarly, if a has finite support in X, then M_a maps $c(X,k)$ into $c_{00}(X,k)$. We shall consider continuity properties of these and other linear mappings later too.

2 Some inequalities

Let X be a nonempty set again, and let f be a nonnegative real-valued function on X with finite support. Put

$$\|f\|_r = \Big(\sum_{x \in X} f(x)^r\Big)^{1/r} \tag{2.1}$$

for every positive real number r, and

$$\|f\|_\infty = \max_{x \in X} f(x). \tag{2.2}$$

Observe that

$$\|f\|_\infty \le \|f\|_r \tag{2.3}$$

for every $r > 0$. If $0 < r_1 \le r_2 < \infty$, then we have that

$$\|f\|_{r_2}^{r_2} = \sum_{x \in X} f(x)^{r_2} \le \|f\|_\infty^{r_2 - r_1} \sum_{x \in X} f(x)^{r_1} \le \|f\|_{r_1}^{r_2 - r_1} \, \|f\|_{r_1}^{r_1} = \|f\|_{r_1}^{r_2}, \tag{2.4}$$

using (2.3) with r replaced by r_1 in the second inequality. This implies that

$$\|f\|_{r_2} \le \|f\|_{r_1} \tag{2.5}$$

under these conditions. Note that

$$\|f\|_r \le (\#\operatorname{supp} f)^{1/r} \, \|f\|_\infty \tag{2.6}$$

for every $r > 0$, where $\#\operatorname{supp} f$ is the number of elements in the support of f. Combining this with (2.3), we get that

$$\lim_{r \to \infty} \|f\|_r = \|f\|_\infty, \tag{2.7}$$

because $a^{1/r} \to 1$ as $r \to \infty$ for every positive real number a.

If a, b are any two nonnegative real numbers, then

$$\max(a, b) \le (a^r + b^r)^{1/r} \tag{2.8}$$

for every $r > 0$, which is the same as (2.3) when $\operatorname{supp} f$ has at most two elements. If $0 < r_1 \le r_2 < \infty$, then

$$(a^{r_2} + b^{r_2})^{1/r_2} \le (a^{r_1} + b^{r_1})^{1/r_1} \tag{2.9}$$

for every $a, b \ge 0$, which is the same as (2.5) when $\#\operatorname{supp} f \le 2$. In particular,

$$(a + b)^r \le a^r + b^r \tag{2.10}$$

for every $a, b \ge 0$ when $0 < r \le 1$, by taking $r_1 = r$ and $r_2 = 1$ in (2.9). Observe that

$$(a^r + b^r)^{1/r} \le 2^{1/r} \, \max(a, b) \tag{2.11}$$

for every $a, b \geq 0$ and $r > 0$, which corresponds to (2.6) when $\# \operatorname{supp} f \leq 2$. It follows that

$$\lim_{r \to \infty} (a^r + b^r)^{1/r} = \max(a, b) \tag{2.12}$$

for every $a, b \geq 0$, as in (2.7).

Let g be another nonnegative real-valued function on X with finite support, so that $f + g$ is a nonnegative real-valued function on X with finite support as well. It is well known that

$$\|f + g\|_r \leq \|f\|_r + \|g\|_r \tag{2.13}$$

when $1 \leq r \leq \infty$, by *Minkowski's inequality* for sums. Of course, equality holds trivially in (2.13) when $r = 1$, and it is easy to verify (2.13) directly when $r = \infty$. If $0 < r \leq 1$, then we have that

$$\|f + g\|_r^r = \sum_{x \in X} (f(x) + g(x))^r \leq \sum_{x \in X} f(x)^r + \sum_{x \in X} g(x)^r = \|f\|_r^r + \|g\|_r^r, \tag{2.14}$$

using (2.10) in the second step. If

$$(\operatorname{supp} f) \cap (\operatorname{supp} g) = \emptyset, \tag{2.15}$$

then

$$\|f + g\|_r^r = \sum_{x \in X} (f(x) + g(x))^r = \sum_{x \in X} f(x)^r + \sum_{x \in X} g(x)^r = \|f\|_r^r + \|g\|_r^r \tag{2.16}$$

when $0 < r < \infty$, and

$$\|f + g\|_\infty = \max(\|f\|_\infty, \|g\|_\infty). \tag{2.17}$$

Put

$$h(x) = \max(f(x), g(x)) \tag{2.18}$$

for every $x \in X$, which defines another nonnegative real-valued function on X with finite support. If r is any positive real number, then

$$h(x)^r \leq f(x)^r + g(x)^r \tag{2.19}$$

for every $x \in X$, which is basically the same as (2.8). It follows that

$$\|h\|_r^r = \sum_{x \in X} h(x)^r \leq \sum_{x \in X} f(x)^r + \sum_{x \in X} g(x)^r = \|f\|_r^r + \|g\|_r^r \tag{2.20}$$

when $0 < r < \infty$. Similarly, it is easy to see that

$$\|h\|_\infty = \max(\|f\|_\infty, \|g\|_\infty). \tag{2.21}$$

If f, g satisfy (2.15), then equality holds in (2.19) and (2.20), which basically corresponds to (2.16).

3 q-Semimetrics

Let X be a set, and let q be a positive real number. A nonnegative real-valued function $d(x, y)$ defined for $x, y \in X$ is said to be a *q-semimetric* on X if it satisfies the following three conditions. First,

$$d(x, x) = 0 \quad \text{for every } x \in X. \tag{3.1}$$

Second,

$$d(x, y) = d(y, x) \quad \text{for every } x, y \in X. \tag{3.2}$$

Third,

$$d(x, z)^q \le d(x, y)^q + d(y, z)^q \quad \text{for every } x, y, z \in X. \tag{3.3}$$

If we also have that

$$d(x, y) > 0 \quad \text{for every } x, y \in X \text{ with } x \ne y, \tag{3.4}$$

then $d(\cdot, \cdot)$ is said to be a *q-metric* on X. If $d(x, y)$ is a q-semimetric or a q-metric on X with $q = 1$, then we may simply say that $d(x, y)$ is a *semimetric* or a *metric* on X, as appropriate.

Let $d(x, y)$ be a nonnegative real-valued function defined for $x, y \in X$ again. Note that $d(x, y)$ is a q-semimetric or a q-metric on X for some $q > 0$ if and only if $d(x, y)^q$ is an ordinary semimetric or metric on X, respectively. Clearly (3.3) holds if and only if

$$d(x, z) \le (d(x, y)^q + d(y, z)^q)^{1/q} \tag{3.5}$$

for every $x, y, z \in X$. The right side of (3.5) decreases monotonically in q, as in (2.9). If $0 < q_1 \le q_2 < \infty$ and $d(x, y)$ is a q_2-semimetric or q_2-metric on X, then it follows that $d(x, y)$ is a q_1-semimetric or q_1-metric on X as well, as appropriate.

If $d(x, y)$ satisfies (3.1), (3.2), and

$$d(x, z) \le \max(d(x, y), d(y, z)) \quad \text{for every } x, y, z \in X, \tag{3.6}$$

then $d(x, y)$ is said to be a *semi-ultrametric* on X. If (3.4) holds too, then $d(x, y)$ is said to be an *ultrametric* on X. Observe that (3.6) implies (3.5) for every $q > 0$, by (2.8), so that a semi-ultrametric or an ultrametric on X is a q-semimetric or q-metric on X for every $q > 0$, as appropriate. We shall often consider semi-ultrametrics and ultrametrics on X to be q-semimetrics and q-metrics on X with $q = \infty$, respectively, because of (2.12). Remember that the *discrete metric* is defined on any set X by putting $d(x, y)$ equal to 1 when $x \ne y$ and to 0 when $x = y$, which is an ultrametric on X.

Let $d(x, y)$ be a q-semimetric or q-metric on a set X for some $q > 0$. If a is any positive real number, then it is easy to see that

$$d(x, y)^a \tag{3.7}$$

defines a (q/a)-semimetric or (q/a)-metric on X, as appropriate. This also works when $q = \infty$, with q/a interpreted as being ∞ too. This means that if $d(x, y)$ is a semi-ultrametric or an ultrametric on X, then (3.7) has the same property for every $a > 0$.

4 q-Absolute value functions

Let k be a field, and let q be a positive real number. A nonnegative real-valued function $|x|$ defined on k is said to be a *q-absolute value function* on k if it satisfies the following three conditions. First,

$$|x| = 0 \quad \text{if and only if} \quad x = 0. \tag{4.1}$$

Second,

$$|x\,y| = |x|\,|y| \quad \text{for every } x, y \in k. \tag{4.2}$$

Third,

$$|x + y|^q \le |x|^q + |y|^q \quad \text{for every } x, y \in k. \tag{4.3}$$

If $|x|$ is a q-absolute value function on k with $q = 1$, then $|x|$ is also simply said to be an *absolute value function* on k. Thus $|x|$ is an absolute value function on k for some $q > 0$ if and only if $|x|^q$ is an ordinary absolute value function on k.

As before, (4.3) is the same as saying that

$$|x + y| \le (|x|^q + |y|^q)^{1/q} \tag{4.4}$$

for every $x, y \in k$. The right side of (4.4) decreases monotonically in q, by (2.9). If $0 < q_1 \le q_2 < \infty$ and $|x|$ is a q_2-absolute value function on k, then we get that $|x|$ is a q_1-absolute value function on k as well. If a nonnegative real-valued function $|x|$ on k satisfies (4.1), (4.2), and

$$|x + y| \le \max(|x|, |y|) \quad \text{for every } x, y \in k, \tag{4.5}$$

then $|x|$ is said to be an *ultrametric absolute value function* on k. If $|x|$ is an ultrametric absolute value function on k, then $|x|$ is a q-absolute value function on k for every $q > 0$, by (2.8), and we may consider $|x|$ as a q-absolute value function with $q = \infty$, because of (2.12).

The standard absolute value functions on the fields $\mathbf{R}$ of real numbers and $\mathbf{C}$ of complex numbers are absolute value functions in this sense, and they are not q-absolute value functions for any $q > 1$. The *trivial absolute value function* is defined on any field k by putting $|x|$ equal to 1 when $x \ne 0$ and equal to 0 when $x = 0$, and is an ultrametric absolute value function on k. If $|x|$ is a q-absolute value function on a field k for some $q > 0$, then it is easy to see that

$$|x|^a \tag{4.6}$$

defines a (q/a)-absolute value function on k for every positive real number a. This also works when $q = \infty$, so that (4.6) is an ultrametric absolute value function on k for every $a > 0$ when $|x|$ is an ultrametric absolute value function on k.

Let $|x|$ be a nonnegative real-valued function on k that satisfies (4.1) and (4.2). It is easy to see that $|1| = 1$, where the first 1 is the multiplicative identity element in k, and the second 1 is the corresponding real number. This implies that

$$|x| = 1 \tag{4.7}$$

for every $x \in k$ such that $x^n = 1$ for some positive integer n, and in particular when $x = -1$. If $|x|$ is a q-absolute value function on k for some q, $0 < q \leq \infty$, then it follows that

$$d(x, y) = |x - y| \tag{4.8}$$

is a q-metric on k. Note that (4.8) is the standard Euclidean metric on $\mathbf{R}$ or $\mathbf{C}$ when $|x|$ is the standard absolute value function, and that (4.8) is the discrete metric on a field k when $|x|$ is the trivial absolute value function on k.

5 q-Seminorms

Let k be a field, and let V be a vector space over k. Also let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. A nonnegative real-valued function N on V is said to be a *q-seminorm* on V for some positive real number q with respect to $|\cdot|$ on k if N satisfies the following two conditions. First,

$$N(t\,v) = |t|\,N(v) \quad \text{for every } t \in k \text{ and } v \in V. \tag{5.1}$$

Second,

$$N(v + w)^q \leq N(v)^q + N(w)^q \quad \text{for every } v, w \in V. \tag{5.2}$$

Note that (5.1) implies that $N(0) = 0$. If we also have that

$$N(v) > 0 \quad \text{for every } v \in V \text{ with } v \neq 0, \tag{5.3}$$

then N is said to be a *q-norm* on V. If N is a q-seminorm or a q-norm on V with $q = 1$, then we may simply say that N is a *seminorm* or a *norm* on V, as appropriate.

As usual, (5.2) is the same as saying that

$$N(v + w) \leq (N(v)^q + N(w)^q)^{1/q} \tag{5.4}$$

for every $v, w \in V$. The right side of (5.4) is monotonically decreasing in q, by (2.9). If $0 < q_1 \leq q_2 < \infty$ and N is a q_2-seminorm or a q_2-norm on V, then it follows that N is also a q_1-seminorm or a q_1-norm on V, as appropriate. If a nonnegative real-valued function N on V satisfies (5.1) and

$$N(v + w) \leq \max(N(v), N(w)) \quad \text{for every } v, w \in V, \tag{5.5}$$

then N is said to be a *semi-ultranorm* on V with respect to $|\cdot|$ on k. If N satisfies (5.3) as well, then N is said to be an *ultranorm* on V with respect to $|\cdot|$ on k. If N is a semi-ultranorm or an ultranorm on V, then it is easy to see that N is a q-seminorm or a q-norm on V for every $q > 0$, as appropriate, using (2.8). We shall consider semi-ultranorms and ultranorms as q-seminorms and q-norms with $q = \infty$, respectively, because of (2.12). If N is a q-seminorm or q-norm on V for some $q > 0$, then

$$d(v, w) = d_N(v, w) = N(v - w) \tag{5.6}$$

defines a q-semimetric or q-metric on V, as appropriate.

Suppose that N is a q-seminorm on V with respect to $|\cdot|$ on k for some $q > 0$, and that there is a $v \in V$ such that $N(v) > 0$. If $q < \infty$, then one can use (5.1) and (5.2) to get that $|\cdot|$ satisfies (4.3) on k. Similarly, if $q = \infty$, then one can use (5.1) and (5.5) to get that $|\cdot|$ satisfies (4.5) on k. In both cases, we get that $|\cdot|$ is a q-absolute value function on k under these conditions. This means that we should normally have $q \le q_k$, unless N is identically equal to 0 on V, or we can replace q_k with a larger value.

Let a be a positive real number, so that $|t|^a$ is a q_k/a-absolute value function on k, as in the previous section. If N is a nonnegative real-valued function on V that satisfies (5.1) with respect to $|\cdot|$ on k, then

$$N(v)^a \tag{5.7}$$

satisfies the analogous condition with respect to $|t|^a$ on k. If N is a q-seminorm or a q-norm on V with respect to $|t|$ on k for some $q > 0$, then one can check that (5.7) is a (q/a)-seminorm or a (q/a)-norm on V with respect to $|t|^a$ on k, as appropriate.

6 Some examples

Let k be a field, and let V be a vector space over k. The *trivial ultranorm* on V is defined by putting $N(v)$ equal to 1 when $v \ne 0$ and equal to 0 when $v = 0$. It is easy to see that this is an ultranorm on V with respect to the trivial absolute value function on k, for which the corresponding ultrametric on V as in (5.6) is the discrete metric.

Now let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and let X be a nonempty set. Also let $c(X,k)$ be the vector space of k-valued functions on X, as in Section 1. If $x \in X$, then

$$N_x(f) = |f(x)| \tag{6.1}$$

defines a q_k-seminorm on $c(X,k)$ with respect to $|\cdot|$ on k.

Let $c_{00}(X,k)$ be the linear subspace of $c(X,k)$ consisting of functions with finite support, as in Section 1 again. Put

$$\|f\|_\infty = \max_{x \in X} |f(x)| \tag{6.2}$$

for each $f \in c_{00}(X,k)$. It is not difficult to check that this defines a q_k-norm on $c_{00}(X,k)$ with respect to $|\cdot|$ on k. Note that (6.2) is the same as (2.2) applied to $|f(x)|$ as a nonnegative real-valued function on X with finite support. If $|\cdot|$ is the trivial absolute value function on k, then (6.2) is the trivial ultranorm on $c_{00}(X,k)$.

Let r be a positive real number, and put

$$\|f\|_r = \Big(\sum_{x \in X} |f(x)|^r\Big)^{1/r} \tag{6.3}$$

for every $f \in c_{00}(X, k)$. This is the same as (2.1) applied to $|f(x)|$ as a nonnegative real-valued function on X with finite support. It is easy to see that

$$\|t\, f\|_r = |t|\, \|f\|_r \tag{6.4}$$

for every $t \in k$ and $f \in c_{00}(X, k)$. If $q_k \le r$, then one can check that (6.3) defines a q_k-norm on $c_{00}(X, k)$ with respect to $|\cdot|$ on k, using Minkowski's inequality for sums. More precisely, this uses (2.13) with r replaced by $r/q_k \ge 1$. Similarly, if $r \le q_k$, then (6.3) defines an r-norm on $c_{00}(X, k)$ with respect to $|\cdot|$ on k. In this case, $|\cdot|$ may be considered as an r-absolute value function on k, as in Section 4. This permits the r-norm version of the triangle inequality to be obtained directly from the definitions.

7 q-Subadditivity

Let k be a field, and let V be a vector space over k. Also let N be a nonnegative real-valued function on V, and let q be a positive real number. We say that N is *q-subadditive* on V if

$$N(v + w)^q \le N(v)^q + N(w)^q \tag{7.1}$$

for every $v, w \in V$. We may also simply say that N is *subadditive* on V when this holds with $q = 1$. As usual, (7.1) is equivalent to asking that

$$N(v + w) \le (N(v)^q + N(w)^q)^{1/q} \tag{7.2}$$

for every $v, w \in V$. Similarly, let us say that N is q-subadditive on V with $q = \infty$ if

$$N(v + w) \le \max(N(v), N(w)) \tag{7.3}$$

for every $v, w \in V$. If $0 < q_1 \le q_2 \le \infty$ and N is q_2-subadditive on V, then it is easy to see that N is q_1-subadditive on V as well, using (2.8) and (2.9). If N is q-subadditive on V for some $q > 0$ and a is a positive real number, then

$$N(v)^a \tag{7.4}$$

is (q/a)-subadditive on V.

Let us say that N is *symmetric* on V if

$$N(-v) = N(v) \tag{7.5}$$

for every $v \in V$. If N is symmetric and q-subadditive on V for some $q > 0$, and if $N(0) = 0$, then

$$d(v, w) = d_N(v, w) = N(v - w) \tag{7.6}$$

defines a q-semimetric on V. This q-semimetric is automatically invariant under translations on V, in the sense that

$$d(v + u, w + u) = d(v, w) \tag{7.7}$$

for every $u, v, w \in V$. Similarly,

(7.8) $$d(-v, -w) = d(v, w)$$

for every $v, w \in V$. Of course, if $N(v) > 0$ for every $v \in V$ with $v \neq 0$, then (7.6) is a q-metric on V.

Suppose now that $|\cdot|$ is a q_k-absolute value function on k for some $q_k > 0$. Let us say that N is *balanced* on V if

(7.9) $$N(t\,v) \le N(v)$$

for every $v \in V$ and $t \in k$ with $|t| \le 1$. If $|t| = 1$, then the previous condition can be applied to t and to $1/t$, to get that

(7.10) $$N(t\,v) = N(v)$$

for every $v \in V$. In particular, this implies that N is symmetric on V. If N satisfies the homogeneity condition (5.1), then it is easy to see that N is balanced on V. If $|\cdot|$ is the trivial absolute value function on k, then the homogeneity condition (5.1) is the same as (7.10) and the condition that $N(0) = 0$. Note that the balanced property is preserved by taking positive powers of N on V, as well as taking positive powers of $|\cdot|$ on k.

8 Open and closed balls

Let X be a set, and let $d(x, y)$ be a q-semimetric on X for some $q > 0$. The *open ball* in X centered at a point x with radius $r > 0$ associated to $d(\cdot, \cdot)$ is defined as usual by

(8.1) $$B(x, r) = B_d(x, r) = \{y \in X : d(x, y) < r\}.$$

Similarly, the *closed ball* in X centered at $x \in X$ with radius $r \ge 0$ associated to $d(\cdot, \cdot)$ is defined by

(8.2) $$\overline{B}(x, r) = \overline{B}_d(x, r) = \{y \in X : d(x, y) \le r\}.$$

If a is a positive real number, then $d(x, y)^a$ defines a (q/a)-semimetric on X, as in Section 3. It is easy to see that

(8.3) $$B_{d^a}(x, r^a) = B_d(x, r)$$

for every $x \in X$ and $r > 0$, and that

(8.4) $$\overline{B}_{d^a}(x, r^a) = \overline{B}_d(x, r)$$

for every $x \in X$ and $r \ge 0$.

A subset U of X is said to be an *open set* with respect to $d(\cdot, \cdot)$ if for each $x \in U$ there is an $r > 0$ such that

(8.5) $$B_d(x, r) \subseteq U.$$

It is easy to see that this defines a topology on X. Note that $d(x,y)^a$ determines the same topology on X for each $a > 0$, because of (8.4). This permits one to reduce to the case of ordinary semimetrics on X, by taking $a = q$ when $q < 1$. In particular, open balls in X with respect to $d(\cdot,\cdot)$ are open sets with respect to this topology, as in the case of ordinary semimetrics. Similarly, closed balls in X with respect to $d(\cdot,\cdot)$ are closed sets with respect to this topology. If $d(x,y)$ is a q-metric on X, then X is Hausdorff with respect to this topology.

Suppose for the moment that $d(\cdot,\cdot)$ is a semi-ultrametric on X. If $x, y \in X$ satisfy $d(x,y) < r$ for some $r > 0$, then it is easy to see that

$$B_d(x,r) \subseteq B_d(y,r). \tag{8.6}$$

This implies that

$$B_d(x,r) = B_d(y,r) \tag{8.7}$$

under these conditions, by interchanging the roles of x and y. Similarly, if $x, y \in X$ satisfy $d(x,y) \le r$ for some $r \ge 0$, then one can verify that

$$\overline{B}_d(x,r) = \overline{B}_d(y,r). \tag{8.8}$$

It follows that closed balls in X of positive radius are open sets with respect to the topology determined by $d(\cdot,\cdot)$, and one can also check that open balls in X are closed sets with respect to this topology in this situation.

Let k be a field, and let V be a vector space over k. Also let N be a nonnegative real-valued function on V that is symmetric, q-subadditive for some $q > 0$, and satisfies $N(0) = 0$. Thus (7.6) defines a q-semimetric on V, to which the previous remarks can be applied. In this situation, we may also use the notation

$$B(v,r) = B_N(v,r) = \{w \in V : N(v-w) < r\} \tag{8.9}$$

for $v \in V$ and $r > 0$, in place of (8.1). Similarly, we may use the notation

$$\overline{B}(v,r) = \overline{B}_N(v,r) = \{w \in V : N(v-w) \le r\} \tag{8.10}$$

for $v \in V$ and $r \ge 0$, in place of (8.2).

9 Finitely many q-semimetrics

Let l be a positive integer, and for each $j = 1, \ldots, l$, let $d_j(x,y)$ be a q_j-semimetric on X for some $q_j > 0$. Put

$$q = \min_{1 \le j \le l} q_j > 0, \tag{9.1}$$

and note that d_j is a q-semimetric on X for each j, as in Section 3. One can check that

$$d(x,y) = \max_{1 \le j \le l} d_j(x,y) \tag{9.2}$$

also defines a q-semimetric on X under these conditions. Observe that

$$B_d(x,r) = \bigcap_{j=1}^{l} B_{d_j}(x,r) \tag{9.3}$$

for each $x \in X$ and $r > 0$, where these open balls are as defined in (8.1).

Let V be a vector space over a field k, and let l be a positive integer again. Suppose that for each $j = 1, \ldots, l$, N_j is a nonnnegative real-valued function on V which is q_j-subadditive for some $q_j > 0$. If q is defined as in (9.1), then N_j is q-subadditive on V for each j, as in Section 7. As before, one can check that

$$N(v) = \max_{1 \le j \le l} N_j(v) \tag{9.4}$$

is also q-subadditive on V under these conditions. Of course, if N_j is symmetric on V for each j, then N is symmetric too. If $N_j(0) = 0$ for each j, then $N(0) = 0$. If N_j is symmetric and q_j-subadditive on V, and if $N_j(0) = 0$, then

$$d_j(v,w) = N_j(v-w) \tag{9.5}$$

is a q_j-semimetric on V, as in Section 7. If these conditions hold for each $j = 1, \ldots, l$, and d is defined on V as in (9.2), then we have that

$$d(v,w) = \max_{1 \le j \le l} N_j(v-w) = N(v-w) \tag{9.6}$$

for every $v, w \in V$. In this case, (9.3) is the same as saying that

$$B_N(v,r) = \bigcap_{j=1}^{l} B_{N_j}(v,r) \tag{9.7}$$

for every $v \in V$ and $r > 0$, using the notation in (8.9).

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. If N_j is balanced on V with respect to $|\cdot|$ on k for each $j = 1, \ldots, l$, as in Section 7, then N is balanced on V with respect to $|\cdot|$ on k too. Similarly, if N_j satisfies the homogeneity condition (5.1) for each j, then N satisfies (5.1) as well. If N_j is a q_j-seminorm on V with respect to $|\cdot|$ on k for each j, then it follows that N is a q-seminorm on V with respect to $|\cdot|$ on k, where q is as in (9.1).

Of course, one can take positive powers of q_j-semimetrics or q_j-subadditive functions, to adjust finite values of q_j. In the context of q_j-seminorms, one should take the same power of $|\cdot|$ on k. If $|\cdot|$ is nontrivial on k, then different powers of $|\cdot|$ lead to different functions on k. Thus different powers of q_j-seminorms can satisfy different homogeneity conditions, so that their maximum is no longer homogeneous. An advantage of the balanced property is that it is preserved by taking positive powers, without changing $|\cdot|$ on k.

10 Bounded q-semimetrics

Let X be a set, and let $d(x, y)$ be a q-semimetric on X for some $q > 0$. Also let t be a positive real number, and put

$$d_t(x, y) = \min(d(x, y), t) \tag{10.1}$$

for each $x, y \in X$. It is easy to see that this is a q-semimetric on X as well. If $x \in X$ and $r > 0$, then

$$B_{d_t}(x, r) = B_d(x, r) \tag{10.2}$$

when $r \le t$, and

$$B_{d_t}(x, r) = X \tag{10.3}$$

when $r > t$, where these open balls in X are defined as in (8.1). This implies that the topology determined on X by $d_t(x, y)$ is the same as the one determined by $d(x, y)$.

Let V be a vector space over a field k, and let N be a nonnegative real-valued function on V. Let t be a positive real number again too, and put

$$N_t(v) = \min(N(v), t) \tag{10.4}$$

for each $v \in V$. If N is q-subadditive on V for some $q > 0$, then N_t is also q-subadditive on V. Of course, N_t is symmetric on V when N is symmetric on V, and $N_t(0) = 0$ when $N(0) = 0$. Suppose that N has all three of these properties, and let $d(v, w)$ be the corresponding q-semimetric on V, as in (7.6). If $d_t(v, w)$ is defined on V as in (10.1), then

$$d_t(v, w) = \min(N(v - w), t) = N_t(v - w) \tag{10.5}$$

corresponds to N_t in the same way. Using the notation in (8.9), we have that

$$B_{N_t}(v, r) = B_N(v, r) \tag{10.6}$$

for every $v \in V$ when $0 < r \le t$, and

$$B_{N_t}(v, r) = V \tag{10.7}$$

for every $v \in V$ when $r > t$, as in (10.2) and (10.3).

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. If N is balanced on V with respect to $|\cdot|$ on k, then N_t is clearly balanced on V with respect to $|\cdot|$ on k too. However, if $|\cdot|$ is not the trivial absolute value function on k, then the only way that N_t can satisfy the homogeneity condition (5.1) is if $N(v) = 0$ for every $v \in V$.

11 Collections of q-semimetrics

Let X be a set, and let $\mathcal{M}$ be a nonempty collection of q-semimetrics on X. More precisely, each $d \in \mathcal{M}$ should be a q_d-semimetric on X for some $q_d > 0$

that may depend on d. Let us say that $U \subseteq X$ is an *open set* with respect to $\mathcal{M}$ if for each $x \in U$ there are finitely many elements $d_1, \ldots, d_l$ of $\mathcal{M}$ and positive real numbers $r_1, \ldots, r_l$ such that

$$\bigcap_{j=1}^{l} B_{d_j}(x, r_j) \subseteq U, \tag{11.1}$$

where these open balls are defined as in (8.1). This defines a topology on X, which contains the topologies associated to the elements of $\mathcal{M}$. In particular, open balls in X with respect to elements of $\mathcal{M}$ are open sets with respect to $\mathcal{M}$, and form a sub-base for this topology. Let us say that $\mathcal{M}$ is *nondegenerate* on X if for each $x, y \in X$ with $x \neq y$ there is a $d \in \mathcal{M}$ such that

$$d(x, y) > 0. \tag{11.2}$$

This implies that X is Hausdorff with respect to the topology determined by $\mathcal{M}$.

Let k be a field, and let V be a vector space over k. Also let $\mathcal{N}$ be a nonempty collection of nonnegative real-valued functions N on V that are symmetric on V and satisfy $N(0) = 0$. Suppose that each $N \in \mathcal{N}$ is q_N-subadditive for some $q_N > 0$. Thus each $N \in \mathcal{N}$ leads to a q_N-semimetric d_N on V as in (7.6). Put

$$\mathcal{M}(\mathcal{N}) = \{d_N : N \in \mathcal{N}\}. \tag{11.3}$$

This defines a nonempty collection of q-semimetrics on V, which leads to a topology on V, as in the previous paragraph. Let us say that $\mathcal{N}$ is *nondegenerate* on V if for each $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that

$$N(v) > 0, \tag{11.4}$$

which implies that (11.3) is nondegenerate as a collection of q-semimetrics on V. If k is equipped with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, then we may ask that each $N \in \mathcal{N}$ be a q_N-seminorm on V with respect to $|\cdot|$ on k for some $q_N > 0$.

As before, open balls in V with respect to elements of $\mathcal{N}$ form a sub-base for the topology determined on V by (11.3). Similarly, consider the collection of subsets of V of the form

$$\bigcap_{j=1}^{l} B_{N_j}(0, r_j), \tag{11.5}$$

where $N_1, \ldots, N_l$ are finitely many elments of $\mathcal{N}$, $r_1, \ldots, r_l$ are positive real numbers, and $B_{N_j}(0, r_j)$ is as in (8.9). This collection is a local base for the topology on V determined by (11.3) at 0. Of course, one can get local bases for the topology on V at other points in the same way, and there is an analogous statement for topologies determined by arbitrary collections of q-semimetrics, as discussed at the beginning of the section. Because the elements of (11.3) are invariant under translations on V, one might as well focus on local bases for the topology of V at 0 in this situation.

As a basic class of examples, let X be a nonempty set, and let $c(X, k)$ be the space of k-valued functions on X, as in Section 1. Remember that

$$N_x(f) = |f(x)| \tag{11.6}$$

defines a q_k-seminorm on $c(X, k)$ for each $x \in X$, so that

$$\mathcal{N} = \{N_x : x \in X\} \tag{11.7}$$

is a nonempty collection of q_k-seminorms on $c(X, k)$, which is also nondegenerate on $c(X, k)$. Note that $c(X, k)$ can be identified with the Cartesian product of a family of copies of k indexed by X. Of course, the q_k-metric on k associated to $|\cdot|$ as in (4.8) determines a topology on k as in the previous section. It is easy to see that the topology determined on $c(X, k)$ by (11.7) is the same as the product topology corresponding to the topology on k just mentioned.

12 Sequences of q-semimetrics

Let X be a set, and suppose that for each positive integer j, $d_j(x, y)$ is a q_j-semimetric on X for some $q_j > 0$. Put

$$\widetilde{d}_j(x, y) = \min(d_j(x, y), 1/j) \tag{12.1}$$

for every $x, y \in X$ and $j \geq 1$, which corresponds to (10.1) with $t = 1/j$. As in Section 10, (12.1) is a q_j-semimetric on X that determines the same topology on X as $d_j(x, y)$ for each $j \geq 1$. Put

$$d(x, y) = \max_{j \geq 1} \widetilde{d}_j(x, y) \tag{12.2}$$

for each $x, y \in X$. More precisely, this is equal to 0 when $\widetilde{d}_j(x, y) = 0$ for every j, and otherwise (12.2) reduces to the maximum over finitely many j, since (12.1) is automatically less than or equal to $1/j$. Suppose that $q > 0$ satisfies

$$q_j \geq q \tag{12.3}$$

for each $j \geq 1$. In this case, one can check that (12.2) defines a q-semimetric on X. Note that one can always ensure that (12.3) holds with $q = 1$, for instance, by replacing $d_j(x, y)$ with $d_j(x, y)^{q_j}$ when $q_j < 1$.

By construction,

$$B_d(x, r) = \bigcap_{j=1}^{\infty} B_{\widetilde{d}_j}(x, r) \tag{12.4}$$

for every $x \in X$ and $r > 0$, where these open balls are defined as in (8.1). Remember that

$$B_{\widetilde{d}_j}(x, r) = B_{d_j}(x, r) \tag{12.5}$$

when $r \le 1/j$, as in (10.2), and that

$$B_{\widetilde{d}_j}(x, r) = X \tag{12.6}$$

when $r > 1/j$, as in (10.3). Combining this with (12.4), we get that

$$B_d(x, r) = \bigcap_{j=1}^{[1/r]} B_{d_j}(x, r) \tag{12.7}$$

when $r \le 1$, where $[1/r]$ is the integer part of $1/r$. Similarly, (12.4) is equal to X when $r > 1$. Using (12.7), one can verify that the topology determined on X by (12.2) is the same as the topology determined on X by the collection of d_j's with $j \ge 1$ as in the previous section.

Now let V be a vector space over a field k, and let N_j be a nonnegative real-valued function on V for each positive integer j. Put

$$\widetilde{N}_j(v) = \min(N_j(v), 1/j) \tag{12.8}$$

for each $v \in V$ and $j \ge 1$, which corresponds to (10.4) with $t = 1/j$. Suppose that for each $j \ge 1$, N_j is q_j-subadditive on V for some $q_j > 0$. This implies that $\widetilde{N}_j$ is q_j-subadditive on V for each $j \ge 1$ too, as in Section 10. Put

$$N(v) = \max_{j \ge 1} \widetilde{N}_j(v) \tag{12.9}$$

for each $v \in V$. As before, this is equal to 0 when $\widetilde{N}_j(v) = 0$ for every j, and otherwise (12.9) reduces to the maximum over finitely many j. If there is a $q > 0$ such that $q_j \ge q$ for every $j \ge 1$, then one can verify that N is q-subadditive on V. One can always reduce to the case where this holds with $q = 1$, for instance, by replacing N_j with $N_j(v)^{q_j}$ when $q_j < 1$.

If N_j is symmetric on V for each j, then $\widetilde{N}_j$ is symmetric on V for each j too, and hence N is symmetric on V as well. Similarly, if $N_j(0) = 0$ for every j, then $\widetilde{N}_j(0) = 0$ for every j, and so $N(0) = 0$. Under these conditions,

$$d_j(v, w) = N_j(v - w) \tag{12.10}$$

defines a q_j-semimetric on V for each $j \ge 1$. In this situation, (12.1) corresponds to (12.8) in the same way for each j, and (12.2) corresponds to (12.9). Suppose now that $|\cdot|$ is a q_k-absolute value function on k for some $q_k > 0$. If N_j is balanced on V with respect to $|\cdot|$ on k for each j, then $\widetilde{N}_j$ is balanced on V for each j, as in Section 10. This implies that N is balanced on V as well.

13 Symmetric and balanced sets

Let V be a vector space over a field k. If $a \in V$ and $B \subseteq V$, then we put

$$a + B = B + a = \{a + v : v \in B\} \tag{13.1}$$

and

$$-B = \{-v : v \in B\}, \tag{13.2}$$

where $-v$ is the additive inverse of v in B. Similarly, put

$$A + B = \bigcup_{a \in A} (a + B) = \bigcup_{b \in B} (A + b) = \{a + b : a \in A,\ b \in B\} \tag{13.3}$$

for every $A, B \subseteq V$. We can define $a - B$ and $A - B$ analogously, which are the same as $a + (-B)$ and $A + (-B)$, respectively. If $t \in k$ and $B \subseteq V$, then we put

$$t\,B = \{t\,v : v \in B\}, \tag{13.4}$$

which is the same as $-B$ when $t = -1$.

Let us say that $B \subseteq V$ is *symmetric* in V (around 0) if

$$-B = B. \tag{13.5}$$

Note that $A \cap (-A)$ and $A \cup (-A)$ are symmetric in V for any $A \subseteq V$. Let N be a nonnegative real-valued function on V, and let the corresponding open and closed balls $B_N(v, r)$ and $\overline{B}_N(v, r)$ in V be defined as in (8.9) and (8.10) for $v \in V$ and $r > 0$ and $r \geq 0$, respectively. If N is symmetric on V, as in (7.5), then $B_N(0, r)$ is symmetric in V for every $r > 0$, and $\overline{B}_N(0, r)$ is symmetric in V for every $r \geq 0$.

Suppose now that $|\cdot|$ is a q_k-absolute value function on k for some $q_k > 0$. A set $E \subseteq V$ is said to be *balanced* in V if

$$t\,E \subseteq E \tag{13.6}$$

for every $t \in k$ with $|t| \leq 1$. In particular, this implies that

$$t\,E = E \tag{13.7}$$

for every $t \in k$ with $|t| = 1$, since we can apply (13.6) to t and to $1/t$. It follows that balanced sets are symmetric, since we can take $t = -1$ in (13.7). If $A \subseteq V$, then

$$\bigcup \{t\,A : t \in k,\ |t| \leq 1\} \tag{13.8}$$

is the smallest balanced subset of V that contains A, which may be described as the *balanced hull* of A in V.

Let N be a nonnegative real-valued function on V that is balanced, as in Section 7. It is easy to see that $B_N(0, r)$ is balanced as a subset of V for every $r > 0$, and similarly that $\overline{B}_N(0, r)$ is balanced in V for every $r \geq 0$. Suppose now that N satisfies the homogeneity condition (5.1). In this case, we have that

$$t\,B_N(0, r) = B_N(0, |t|\,r) \tag{13.9}$$

for every $r > 0$ and $t \in k$ with $t \neq 0$. Similarly,

$$t\,\overline{B}_N(0, r) = \overline{B}_N(0, |t|\,r) \tag{13.10}$$

for every $r \geq 0$ and $t \in k$ with $t \neq 0$. If $t = 0$, then the left side of (13.10) is equal to $\{0\}$, and the right side of (13.10) is equal to $\overline{B}_N(0, 0)$. Thus (13.10) still holds when $t = 0$ and $N(v) > 0$ for every $v \in V$ with $v \neq 0$.

14 Topological vector spaces

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. We say that V is a *topological vector space* over k with respect to $|\cdot|$ on k if the vector space operations are continuous on V. More precisely, this means that addition should be continuous as a mapping from $V \times V$ into V, using the corresponding product topology on $V \times V$. Similarly, scalar multiplication should be continuous as a mapping from $k \times V$ into V. This uses the topology determined on k by the q_k-metric associated to $|\cdot|$ as in (4.8), and the corresponding product topology on $k \times V$. In [11], the requirement that $|\cdot|$ be nontrivial on k is included in the definition of a topological vector space. We shall not include this condition in the definition here, but it is needed for some basic properties, as we shall see. Sometimes the requirement that $\{0\}$ be a closed set in V is included in the definition of a topological vector space, which implies that V is Hausdorff. We shall also not include this condition in the definition here, although one might be primarily concerned with situations where it holds.

Continuity of addition on V implies in particular that for each $a \in V$, the translation mapping

$$v \mapsto v + a \tag{14.1}$$

is continuous as a mapping from V into itself. More precisely, such a translation mapping is a homeomorphism from V onto itself, since the inverse mapping is given by translation by $-a$. Continuity of addition on V as a mapping from $V \times V$ into V at $(0, 0)$ means that for each open set $W \subseteq V$ with $0 \in W$ there are open sets $U_1, U_2 \subseteq V$ such that $0 \in U_1$, $0 \in U_2$, and

$$U_1 + U_2 \subseteq W. \tag{14.2}$$

One can also take $U_1 = U_2$, by replacing U_1 and U_2 by their intersection. If $A, B \subseteq V$, and either A or B is an open set in V, then it is easy to see that $A + B$ is an open set in V too, using (13.3) and continuity of translations.

Continuity of scalar multiplication on V implies in particular that for each $t \in k$,

$$v \mapsto t\, v \tag{14.3}$$

is continuous as a mapping from V into itself. Of course, the continuity of (14.3) is trivial when $t = 0$. Otherwise, (14.3) should be a homeomorphism from V onto itself when $t \neq 0$, because the inverse mapping is given by multiplication by $1/t$. If $|\cdot|$ is the trivial absolute value function on k, then the corresponding topology on k is discrete. In this case, the continuity of scalar multiplication on V as a mapping from $k \times V$ into V with respect to the corresponding product topology on $k \times V$ reduces to the continuity of (14.3) on V for every $t \in k$.

If N is a q-seminorm on V for some $q > 0$, then V is a topological vector space over k with respect to the topology determined by the associated q-semimetric, as in (5.6). Similarly, if $\mathcal{N}$ is a nonempty collection of q-seminorms on V, then V is a topological vector space over k with respect to the topology determined by $\mathcal{N}$ as in Section 11. In particular, if X is a nonempty set, and $c(X, k)$ is the space of k-valued functions on X, then $c(X, k)$ is a topological vector space over k with respect to the topology determined by the collection (11.7) of q_k-seminorms.

15 Regular topological spaces

A topological space X is said to be *regular* in the strict sense if for each $x \in X$ and closed set $E \subseteq X$ with $x \not\in E$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $E \subseteq V$. Equivalently, this means that for each $x \in X$ and open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$ and the closure $\overline{U}$ of U in X is contained in W. Sometimes the first or 0th separation condition is included in the definition of regularity. Otherwise, one may say that X satisfies the *third separation condition* when X is regular in the strict sense and satisfies the first or 0th separation condition. It is easy to see that X is Hausdorff in this case. If Y is a subset of X equipped with the induced topology and X satisfies the 0th, first, or second separation condition, then it is easy to see that Y has the same property. Similarly, if X is regular in the strict sense, then Y is regular in te strict sense too.

If the topology on X is determined by a q-semimetric $d(x, y)$ for some $q > 0$, then X is regular in the strict sense. This follows from the fact that open and closed balls in X with respect to $d(\cdot, \cdot)$ are open and closed as subsets of X with respect to the topology determined on X by $d(\cdot, \cdot)$, respectively. One can also use the fact that

$$V(x, r) = \{y \in X : d(x, y) > r\} = X \setminus \overline{B}(x, r) \tag{15.1}$$

is an open set in X for every $x \in X$ and $r \geq 0$, which is equivalent to saying that $\overline{B}(x, r)$ is a closed set in X. Similarly, if $\mathcal{M}$ is a nonempty collection of q-semimetrics on X, then X is regular with respect to the topology determined by $\mathcal{M}$ as in Section 11.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a topological vector space over k. If E is any subset of V, and if $U \subseteq V$ is an open set that contains 0, then one can check that

$$\overline{E} \subseteq E + U, \tag{15.2}$$

where $\overline{E}$ is the closure of E in V. More precisely, $\overline{E}$ is equal to the intersection of all sets of the form $E + U$, where $U \subseteq V$ is an open set that contains 0. If $W \subseteq V$ is an open set that contains 0, then we have seen that there are open sets $U_1, U_2 \subseteq V$ that contain 0 and satisfy (14.2), by continuity of addition on V at 0. It follows that

$$\overline{U_1} \subseteq U_1 + U_2 \subseteq W, \tag{15.3}$$

using (15.2) in the first step. If v is any element of V, then any open set in V that contains v can be expressed as $v + W$, where $W \subseteq V$ is an open set that contains 0, by continuity of translations. If U_1 is as in the previous remark, then $v + U_1$ is an open set in V that contains v and whose closure is contained in $v + W$, using continuity of translations on V again. This implies that V is regular as a topological space in the strict sense.

If $\{0\}$ is a closed set in V, then $\{v\}$ is a closed set in V for every $v \in V$, because of continuity of translations. This means that V satisfies the first separation condition as a topological space. It follows that V is Hausdorff, because V is regular in the strict sense, as in the preceding paragraph.

16 Balanced open sets

Let k be a field with a q_k-absolute value function for some $q_k > 0$ again, and let V be a topological vector space over k. If W is an open set in V that contains 0, then there should be an open set $U \subseteq V$ with $0 \in U$ and a $\delta > 0$ such that

$$t\,U \subseteq W \tag{16.1}$$

for every $t \in k$ with $|t| < \delta$. This follows from continuity of scalar multiplication on V as a mapping from $k \times V$ into V at $(0,0)$. If $|\cdot|$ is the trivial absolute value function on k, then this condition is vacuous. In this case, $|t| < \delta$ implies that $t = 0$ when $\delta \leq 1$, and (16.1) holds automatically when $t = 0$.

Suppose that $|\cdot|$ is not trivial on k, and let W, U, and δ be as in the previous paragraph. Put

$$U_1 = \bigcup \{t\,U : t \in k,\ 0 < |t| < \delta\}. \tag{16.2}$$

The nontriviality of $|\cdot|$ on k ensures that there are $t \in k$ with $0 < |t| < \delta$, so that the right side of (16.2) is the union of a nonempty collection of subsets of V. Of course, $0 \in U_1$, because $0 \in U$. It is easy to see that U_1 is an open set in V, because $t\,U$ is an open set in V for every $t \in k$ with $t \neq 0$. We also have that

$$U_1 \subseteq W, \tag{16.3}$$

by (16.1). Observe that U_1 is balanced in V, by construction. It follows that the collection of nonempty balanced open subsets of V is a local base for the topology of V at 0 when $|\cdot|$ is nontrivial on k.

Suppose that the topology on V is determined by a nonempty collection $\mathcal{N}$ of q-seminorms on V. If $N \in \mathcal{N}$, $r > 0$, and $B_N(0,r)$ is as in (8.9), then $B_N(0,r)$ is a balanced open set in V. Similarly, if $N_1, \ldots, N_l$ are finitely many elements of $\mathcal{N}$, and $r_1, \ldots, r_l$ are finitely many positive real numbers, then

$$\bigcap_{j=1}^{l} B_{N_j}(0, r_j) \tag{16.4}$$

is a balanced open set in V. Remember that the collection of subsets of V of the form (11.5) is a local base for the topology of V at 0 in this situation, as in Section 11. This works whether or not $|\cdot|$ is trivial on k.

Continuity of scalar multiplication on V also implies that for each $v \in V$, $t \mapsto t\,v$ is a continuous mapping from k into V. As usual, we use the topology determined on k by the q-metric (4.8) associated to $|\cdot|$ here. If $|\cdot|$ is trivial on k, then k is equipped with the discrete topology, and this condition is vacuous. Of course, this continuity condition is trivial when $v = 0$.

17 Product topologies

Let I be a nonempty set, let X_j be a topological space for each $j \in I$, and consider the corresponding Cartesian product

$$X = \prod_{j \in I} X_j. \tag{17.1}$$

If $x \in X$ and $j \in I$, then we let x_j denote the jth coordinate of x in X_j, as usual. A subset W of X is said to be an open set in X with respect to the *strong product topology* if for each $x \in W$ there is an open set U_j in X_j for every $j \in I$ such that $x_j \in U_j$ for every $j \in I$ and

$$\prod_{j \in I} U_j \subseteq W. \tag{17.2}$$

It is easy to see that this defines a topology on X. If U_j is an open set in X_j for each $j \in I$, then

$$U = \prod_{j \in I} U_j \tag{17.3}$$

is an open set in X with respect to the strong product topology, and the collection of these open sets forms a base for the strong product topology on X.

Of course, the ordinary *product topology* on X is defined in the same way, but with the additional condition that $U_j = X_j$ for all but finitely many $j \in I$ in the definition of an open set in X. If U_j is an open set in X for each $j \in I$, and $U_j = X_j$ for all but finitely many $j \in I$, then (17.3) is an open set in X with respect to the product topology, and the collection of these open sets forms a base for the product topology on X. Every open set in X with respect to the product topology is also an open set with respect to the strong product topology, and the two topologies on X are the same when I has only finitely many elements. If X_j is equipped with the discrete topology for every $j \in I$, then the strong product topology on X is the same as the discrete topology on X.

If E_j is a closed set in X_j for each $j \in I$, then one can check that

$$E = \prod_{j \in I} E_j \tag{17.4}$$

is a closed set in X with respect to the product topology. This implies that E is also a closed set in X with respect to the strong product topology. Similarly, let A_j be a subset of X_j for each $j \in I$, and put

$$A = \prod_{j \in I} A_j. \tag{17.5}$$

If $\overline{A_j}$ is the closure of A_j in X_j for each $j \in I$, then one can check that

$$\prod_{j \in I} \overline{A_j} \tag{17.6}$$

is the same as the closure of A in X with respect to both the product and strong product topologies. Note that the closure of any subset of X with respect to the strong product topology is contained in the closure of that set with respect to the product topology.

If X_j satisfies the 0th, first, or second separation condition for every $j \in I$, then it is easy to see that X has the same property with respect to the product topology. This implies that X has the same property with respect to the strong product topology as well, since open sets in X with respect to the product topology are open with respect to the strong product topology too. If X_j is regular in the strict sense for each $j \in I$, then one can check that X is regular in the strict sense with respect to both the product topology and the strong product topology. More precisely, let $x \in X$ be given, and let $W \subseteq X$ be an open set with respect to the product topology or the strong product topology that contains x. This means that there is an open set U_j in X_j for each $j \in I$ that satisfies $x_j \in U_j$ for every $j \in I$ and (17.2), and with $U_j = X_j$ for all but finitely many $j \in I$ in the case of the ordinary product topology. In both cases, for each $j \in I$, the regularity of X_j in the strict sense implies that there is an open set V_j in X_j such that $x_j \in V_j$ and the closure $\overline{V_j}$ of V_j in X_j is contained in U_j. We can also take $V_j = X_j$ when $U_j = X_j$, so that $V_j = X_j$ for all but finitely many $j \in I$ in the case of the ordinary product topology. Thus

$$V = \prod_{j \in I} V_j \tag{17.7}$$

is an open set in X with respect to the product topology or strong product topology, as appropriate, and $x \in V$. The closure of V in X with respect to either the product topology or strong product topology is equal to

$$\prod_{j \in I} \overline{V_j}, \tag{17.8}$$

as in the previous paragraph. By construction, (17.8) is contained in (17.3), and hence (17.8) is contained in W, as desired.

18 Product topologies, continued

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. As in Section 1, we let $c(X, k)$ be the space of k-valued functions on X. This can be identified with the Cartesian product of a family of copies of k indexed by X, and the topology determined on $c(X, k)$ by (11.7) corresponds exactly to the product topology. This uses the topology determined on k by the q_k-metric (4.8) associated to $|\cdot|$ on each factor. Let us now consider the topology on $c(X, k)$ that corresponds to the strong product topology, as in the previous section.

If $f \in c(X, k)$ and ρ is a positive real-valued function defined on X, then put

$$(18.1) \qquad U_\rho(f) = \{g \in c(X, k) : |f(x) - g(x)| < \rho(x)\}.$$

This corresponds to the Cartesian product of a family of open disks in k indexed by X. The topology on $c(X, k)$ that corresponds to the strong product topology can be described equivalently by saying that $W \subseteq c(X, k)$ is an open set if for each $f \in W$ there is a positive real-valued function ρ on X such that

$$(18.2) \qquad U_\rho(f) \subseteq W.$$

It is easy to check directly that this defines a topology on $c(X, k)$. If X has only finitely many elements, then this is the same as the topology on $c(X, k)$ determined by (11.7).

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k. This implies that

$$(18.3) \qquad U_\rho(f) = \{f\}$$

for every $f \in c(X, k)$ when $\rho(x) \leq 1$ for every $x \in X$. It follows that the topology on $c(X, k)$ described in the preceding paragraph is the same as the discrete topology in this case. Of course, the trivial absolute value function on k corresponds to the discrete metric on k, which determines the discrete topology on k. As in the previous section, the strong product topology reduces to the discrete topology when the individual factors are equipped with the discrete topology.

One can check that (18.1) is an open set in $c(X, k)$ with respect to the topology just defined for every $f \in c(X, k)$ and positive real-valued function ρ on X. If one thinks of this topology as being the strong product topology, as in the previous section, then it suffices to observe that (18.1) is the Cartesian product of a family of open subsets of k. One can also verify that (18.1) is an open set more directly in terms of the definition given in this section, using the fact that open balls in k are open sets. Note that the collection of subsets of $c(X, k)$ of the form (18.1) is a base for this topology. Similarly, if $f \in c(X, k)$ is fixed, then the collection of subsets of $c(X, k)$ of the form (18.1) is a local base for this topology on $c(X, k)$ at f.

If $f \in c(X, k)$ and ρ is a nonnegative real-valued function on X, then put

$$(18.4) \qquad E_\rho(f) = \{g \in c(X, k) : |f(x) - g(x)| \leq \rho(x)\}.$$

This corresponds to the Cartesian product of a family of closed disks in k indexed by X. This is a closed set in $c(X, k)$ with respect to the topology determined by (11.7), which corresponds to the ordinary product topology, as before. It follows that (18.4) is also a closed set in $c(X, k)$ with respect to the topology considered in this section, corresponding to the strong product topology. If $q_k = \infty$, then open disks in k of positive radius are closed sets, so that (18.1) may be identified with the Cartesian product of a family of closed sets in k

too. This implies that (18.1) is a closed set in $c(X,k)$ with respect to these topologies as well. Similarly, closed disks in k of positive radius are open sets when $q_k = \infty$, so that (18.4) corresponds to the Cartesian product of a family of open sets of k when $\rho(x) > 0$ for every $x \in X$. Under these conditions, (18.4) is also an open set in $c(X,k)$ with respect to the topology considered in this section.

It is easy to see that addition of functions on X defines a continuous mapping from $c(X,k) \times c(X,k)$ into $c(X,k)$ with respect to this topology on $c(X,k)$, and the associated product topology on $c(X,k) \times c(X,k)$. This basically follows from continuity of addition on k. More precisely,

$$U_\rho(f + f_0) = U_\rho(f) + f_0 \tag{18.5}$$

for every $f_0, f \in c(X,k)$ and positive real-valued function ρ on X. This implies that translations are continuous on $c(X,k)$ with respect to this topology. If $q_k = \infty$, then

$$U_\rho(0) + U_\rho(0) \subseteq U_\rho(0) \tag{18.6}$$

for every positive real-valued function ρ on X. Suppose for the moment that $q_k < \infty$, and that ρ_1, ρ_2, and ρ_3 are positive real-valued functions on X such that

$$\rho_1(x)^{q_k} + \rho_2(x)^{q_k} \le \rho_3(x)^{q_k} \tag{18.7}$$

for every $x \in X$. Under these conditions, it is easy to see that

$$U_{\rho_1}(0) + U_{\rho_2}(0) \subseteq U_{\rho_3}(0). \tag{18.8}$$

Using (18.6) and (18.8), one can check that addition of functions is continuous as a map from $c(X,k) \times c(X,k)$ into $c(X,k)$ for any $q_k > 0$.

If $f \in c(X,k)$ and ρ_1, ρ_2 are positive real-valued functions on X such that

$$\rho_1(x) \le \rho_2(x) \tag{18.9}$$

for every $x \in X$, then

$$U_{\rho_1}(f) \subseteq U_{\rho_2}(f). \tag{18.10}$$

Let $t \in k$ with $t \neq 0$ be given, as well as a positive real-valued function ρ on X. Thus $|t|\,\rho$ is a positive real-valued function on X too, and it is easy to see that

$$t\,U_\rho(0) = U_{|t|\,\rho}(0). \tag{18.11}$$

In particular, $U_\rho(0)$ is balanced in $c(X,k)$. We also get that

$$f \mapsto t\,f \tag{18.12}$$

defines a continuous mapping from $c(X,k)$ into itself with respect to the strong product topology for every $t \in k$, and a homeomorphism from $c(X,k)$ onto itself when $t \neq 0$.

19 Functions with finite support

Let us continue with the same notation and hypotheses as in the previous section. In particular, let us consider $c(X, k)$ equipped with the topology described in the previous section, which corresponds to the strong product topology. If X has only finitely many elements, then this topology on $c(X, k)$ is the same as the topology determined by (11.7), and $c(X, k)$ is a topological vector space over k with respect to this topology. If X is any nonempty set and $|\cdot|$ is the trivial absolute value function on k, then we have seen that this topology on $c(X, k)$ is the same as the discrete topology, which implies that $c(X, k)$ is a topological vector space over k in this situation too. Otherwise, suppose that X has infinitely many elements, and that $|\cdot|$ is not the trivial absolute value function on k. If f is a k-valued function on X such that $f(x) \neq 0$ for infinitely many $x \in X$, then one can check that

$$t \mapsto t\, f \tag{19.1}$$

is not continuous as a mapping from k into $c(X, k)$ with respect to this topology. This implies that $c(X, k)$ is not a topological vector space over k with respect to this topology.

Remember that $c_{00}(X, k)$ is the linear subspace of $c(X, k)$ consisting of functions with finite support in X, as in Section 1. If $f \in c_{00}(X, k)$, then it is easy to see that (19.1) defines a continuous mapping from k into $c(X, k)$, with respect to this topology on $c(X, k)$. Let us now consider $c_{00}(X, k)$ to be equipped with the topology induced by this topology on $c(X, k)$. Of course, if X has only finitely many elements, then $c_{00}(X, k)$ is the same as $c(X, k)$, and this topology is the same as the one determined by (11.7). If X is any nonempty set and $|\cdot|$ is the trivial absolute value function on k, then we have seen that this topology on $c(X, k)$ is the discrete topology, so that the induced topology on $c_{00}(X, k)$ is discrete too.

One can check that $c_{00}(X, k)$ is a topological vector space over k with respect to this topology. More precisely, we have already seen that addition of functions defines a continuous mapping from $c(X, k) \times c(X, k)$ into $c(X, k)$ with respect to this topology on $c(X, k)$, and using the associated product topology on the domain of this mapping. This implies the analogous continuity property of addition on $c_{00}(X, k)$ with respect to the induced topology. To get continuity of scalar multiplication on $c_{00}(X, k)$, one can use (18.11) and the continuity of (19.1) when $f \in c_{00}(X, k)$. In fact, scalar multiplication is continuous as a mapping from $k \times c(X, k)$ into $c(X, k)$ along $k \times c_{00}(X, k)$.

Let $f \in c(X, k)$ be given, and put

$$W(f) = \{g \in c(X, k) : g(x) \neq 0 \text{ for every } x \in X \text{ such that } f(x) \neq 0\}. \tag{19.2}$$

This corresponds to the Cartesian product of the family of subsets of k indexed by $x \in X$ defined by taking k when $f(x) = 0$ and $k \setminus \{0\}$ when $f(x) \neq 0$. Thus $W(f)$ is an open set in $c(X, k)$ with respect to the topology corresponding to the strong product topology, since k and $k \setminus \{0\}$ are open sets in k. By construction, $f \in W(f)$, and

$$\operatorname{supp} f \subseteq \operatorname{supp} g \tag{19.3}$$

for every $g \in W(f)$, using the notation for the support of a k-valued function on X in (1.2). In particular, if the support of f has infinitely many elements, then the support of each $g \in W(f)$ has infinitely many elements as well. This implies that $c_{00}(X, k)$ is a closed set in $c(X, k)$ with respect to this topology. Note that $c_{00}(X, k)$ is dense in $c(X, k)$ with respect to the topology determined by (11.7).

20 Weighted maximum seminorms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. Also let $c_{00}(X, k)$ be the space of k-valued functions on X with finite support, as in Section 1. If $f \in c_{00}(X, k)$ and w is a nonnegative real-valued function on X, then put

$$\|f\|_{\infty, w} = \max_{x \in X} (w(x)\, |f(x)|). \tag{20.1}$$

This is the same as $\|f\|_\infty$ defined in (6.2) when $w(x) = 1$ for every $x \in X$. As before, one can check that (20.1) is a q_k-seminorm on $c_{00}(X, k)$ with respect to $|\cdot|$ on k. If $w(x) > 0$ for every $x \in X$, then (20.1) is positive when f is not identically equal to 0 on X, so that (20.1) defines a q_k-norm on $c_{00}(X, k)$. If w_1, w_2 are nonnegative real-valued functions on X such that

$$w_1(x) \le w_2(x) \tag{20.2}$$

for every $x \in X$, then we have that

$$\|f\|_{\infty, w_1} \le \|f\|_{\infty, w_2} \tag{20.3}$$

for every $f \in c_{00}(X, k)$.

Let $f \in c_{00}(X, k)$ and a nonnegative real-valued function w on X be given. Observe that

$$\begin{aligned} &\{g \in c_{00}(X, k) : \|f - g\|_{\infty, w} < r\} \\ &\quad = \{g \in c_{00}(X, k) : w(x)\, |f(x) - g(x)| < r \text{ for every } x \in X\} \end{aligned} \tag{20.4}$$

for each $r > 0$. This is the same as the open ball in $c_{00}(X, k)$ centered at f with radius r with respect to (20.1), as in (8.9). Similarly,

$$\begin{aligned} &\{g \in c_{00}(X, k) : \|f - g\|_{\infty, w} \le r\} \\ &\quad = \{g \in c_{00}(X, k) : w(x)\, |f(x) - g(x)| \le r \text{ for every } x \in X\} \end{aligned} \tag{20.5}$$

for each $r \ge 0$. This is the closed ball in $c_{00}(X, k)$ centered at f with radius r with respect to (20.1), as in (8.10).

Suppose for the moment that $w(x) > 0$ for every $x \in X$. Let $r > 0$ be given, and put

$$\rho(x) = r/w(x) \tag{20.6}$$

for every $x \in X$, so that ρ is a positive real-valued function on X. If $U_\rho(f)$ is as in (18.1), then (20.4) is the same as

$$U_\rho(f) \cap c_{00}(X, k). \tag{20.7}$$

Similarly, if r is a nonnegative real number, then (20.6) defines a nonnegative real-valued function on X. If $E_\rho(f)$ is as in (18.4), then (20.5) is the same as

$$E_\rho(f) \cap c_{00}(X, k). \tag{20.8}$$

Of course,

$$\{\|\cdot\|_{\infty,w} : w \text{ is a positive real-valued function on } X\} \tag{20.9}$$

is a nonempty collection of q_k-norms on $c_{00}(X, k)$. Thus (20.9) determines a topology on $c_{00}(X, k)$, as in Section 11. It is easy to see that this topology is the same as the one induced on $c_{00}(X, k)$ by the topology on $c(X, k)$ corresponding to the strong product topology. This uses the description of this topology on $c(X, k)$ in Section 18, and the fact that (20.4) is the same as (20.7) when ρ is as in (20.6).

Equivalently, one could use the collection

$$\{\|\cdot\|_{\infty,w} : w \text{ is a nonnegative real-valued function on } X\} \tag{20.10}$$

of q_k-seminorms on $c_{00}(X, k)$, which contains (20.9). The additional elements in (20.10) do not affect the topology on $c_{00}(X, k)$, because every nonnegative real-valued function on X is less than or equal to a positive real-valued function on X. If w is a nonnegative real-valued function on X and r is a positive real number, then one can interpret (20.6) as a function on X with values in the positive extended real numbers, which is equal to $+\infty$ when $w(x) = 0$. If one extends the definition of $U_\rho(f)$ in (18.1) to this case, then (20.4) is equal to (20.7) when w is a nonnegative real-valued function on X and ρ is as in (20.6). Note that $U_\rho(f)$ is an open set in $c(X, k)$ with respect to the topology corresponding to the strong product topology when $0 < \rho \le \infty$ on X, so that (20.7) is an open set in $c_{00}(X, k)$ with respect to the induced topology.

Similarly, one can extend the definition of $E_\rho(f)$ in (18.4) to the case where $0 \le \rho \le \infty$ on X. It is easy to see that $E_\rho(f)$ is a closed set in $c(X, k)$ with respect to the topology that corresponds to the strong product topology in this case, for essentially the same reasons as before. Thus (20.8) is a closed set in $c_{00}(X, k)$ with respect to the induced topology in this situation as well. If w is a nonnegative real-valued function on X and r is a nonnegative real number, then one can define ρ on X as in (20.6), with $\rho(x) = +\infty$ when $w(x) = 0$, even if $r = 0$ too. Using this convention, we get that (20.5) is equal to (20.8), as before.

If $|\cdot|$ is the trivial absolute value function on k, then $\|\cdot\|_\infty$ is the trivial ultranorm on $c_{00}(X,k)$. This is the same as $\|\cdot\|_{\infty,w}$ with $w \equiv 1$ on X, and hence is an element of (20.9). In particular, it follows that the topology determined on $c_{00}(X,k)$ by (20.9) is the discrete topology in this case. Of course, we have already seen that the topology on $c(X,k)$ corresponding to the strong product topology is discrete in this case, so that the induced topology on $c_{00}(X,k)$ is discrete as well.

If X has only finitely many elements, then $c_{00}(X,k)$ is the same as $c(X,k)$, and the corresponding product and strong product topologies are the same too. Equivalently, this means that the topologies determined on $c_{00}(X,k)$ by (11.7) and (20.9) are the same in this case. More precisely, each element of (20.9) determines the same topology on $c_{00}(X,k)$ in this situation.

21 q-Absolute value functions, continued

Let k be a field, and let $|\cdot|$ be a q-absolute value function on k for some $q > 0$. Observe that

$$\{|x| : x \in k \setminus \{0\}\} \tag{21.1}$$

is a subgroup of the multiplicative group $\mathbf{R}_+$ of positive real numbers. In particular, (21.1) is the trivial subgroup $\{1\}$ of $\mathbf{R}_+$ if and only if $|\cdot|$ is the trivial absolute value function on k. If 1 is not a limit point of (21.1) in $\mathbf{R}_+$ with respect to the standard topology on $\mathbf{R}$, then $|\cdot|$ is said to be *discrete* on k. If $|\cdot|$ is nontrivial and discrete on k, then one can show that (21.1) consists of the integer powers of single positive real number different from 1. Otherwise, if $|\cdot|$ is not discrete on k, then (21.1) is dense in $\mathbf{R}_+$ with respect to the standard topology on $\mathbf{R}$. Of course, 0 is a limit point of (21.1) with respect to the standard topology on $\mathbf{R}$ when $|\cdot|$ is nontrivial on k.

If $x \in k$ and n is a positive integer, then we let $n \cdot x$ denote the sum of n x's in k. We say that $|\cdot|$ is *archimedian* on k if there are positive integers n such that $|n \cdot 1|$ is arbitrarily large. Otherwise, if there is a finite upper bound for $|n \cdot 1|$ with n in the set $\mathbf{Z}_+$ of positive integers, then $|\cdot|$ is said to be *non-archimedian* on k. In particular, if a field k has positive characteristic, then there are only finitely many elements of k of the form $n \cdot 1$ for some $n \in \mathbf{Z}_+$, and so every q-absolute value function on k is non-archimedian. If $|\cdot|$ is an ultrametric absolute value function on any field k, then it is easy to see that

$$|n \cdot 1| \le 1 \tag{21.2}$$

for every $n \in \mathbf{Z}_+$. If $|\cdot|$ is any q-absolute value function on a field k, then one can check that

$$|n^j \cdot 1| = |(n \cdot 1)^j| = |n \cdot 1|^j \tag{21.3}$$

for every $j, n \in \mathbf{Z}_+$. If $|n \cdot 1| > 1$ for some $n \in \mathbf{Z}_+$, then it follows that $|\cdot|$ is archimedian on k. Equivalently, if $|\cdot|$ is non-archimedian on k, then (21.2) holds for every $n \in \mathbf{Z}_+$. In fact, it is well known that every non-archimedian q-absolute function is an ultrametric absolute value function.

Let $|\cdot|_1$ and $|\cdot|_2$ be q_1, q_2-absolute value functions on k for some $q_1, q_2 > 0$. If there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{21.4}$$

for every $x \in k$, then $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* on k. Of course, this implies an analogous relationship between the q_1, q_2-metrics on k associated to $|\cdot|_1$, $|\cdot|_2$, respectively, as in (4.8). It follows that the corresponding q_1, q_2-metrics on k determine the same topology on k, as in Section 8. Conversely, if the q_1, q_2-metrics on k associated to $|\cdot|_1$, $|\cdot|_2$ determine the same topology on k, then it is well known that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k.

Let $|\cdot|$ be an archimedian q-absolute value function on a field k for some $q > 0$. Thus k has characteristic 0, as before, so that there is a natural embedding of the field $\mathbf{Q}$ of rational numbers into k. This leads to an induced q-absolute value function on $\mathbf{Q}$, which is also archimedian. A famous theorem of Ostrowski implies that this induced q-absolute value function on $\mathbf{Q}$ is equivalent to the standard absolute value function on $\mathbf{Q}$. In particular, the induced absolute value function on $\mathbf{Q}$ is not discrete. This implies that $|\cdot|$ is not discrete on k. If $|\cdot|$ is a discrete q-absolute value function on a field k for some $q > 0$, then it follows that $|\cdot|$ is non-archimedian on k, and hence that $|\cdot|$ is an ultrametric absolute value function on k.

22 Absorbing sets

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. A set $A \subseteq V$ is said to be *absorbing* in V if for each $v \in V$ there is a $t_0(v) \in k$ such that

$$v \in t\, A \tag{22.1}$$

for every $t \in k$ with $|t| \geq |t_0(v)|$. Of course, V is automatically absorbing as a subset of itself. If $|\cdot|$ is nontrivial on k, then there are elements of k with arbitrarily large absolute value, and it is enough to ask that (22.1) hold for every $t \in k$ such that $|t|$ is sufficiently large. Otherwise, if $|\cdot|$ is the trivial absolute value function on k, and if A is absorbing in V, then (22.1) holds with $t = 1$ for every $v \in V$. It follows that V is the only absorbing subset of itself when $|\cdot|$ is trivial on k. Note that $0 \in A$ for any absorbing set A in V.

Equivalently, A is absorbing in V if for each $v \in V$ there is a $t_1(v) \in k$ such that $t_1(v) \neq 0$ and

$$t'\, v \in A \tag{22.2}$$

for every $t' \in k$ with $|t'| \leq |t_1(v)|$. More precisely, this condition also implies that $0 \in A$, so that (22.2) holds for every $v \in V$ when $t' = 0$. If $t' \neq 0$, then (22.2) corresponds to (22.1) with $t = 1/t'$. Thus $t_0(v)$ basically corresponds to $1/t_1(v)$, except that one is allowed to take $t_0(v) = 0$ in the previous formulation. However, this is only possible when $v = 0$, in which case one might as well take $t_0(0) = 1$. If $|\cdot|$ is nontrivial on k, then there are nonzero elements whose absolute value is arbitrarily small. In this situation, it is enough to ask that (22.2) hold for all $t' \in k$ such that $|t'|$ is sufficiently small.

If A is balanced in V, then it suffices to verify that for each $v \in V$ there be a $t \in k$ such that (22.1) holds, in order to check that A is absorbing in V. Equivalently, this means that for each $v \in V$ there is a $t' \in k$ such that $t' \neq 0$ and (22.2) holds. Let N be a nonnegative real-valued function on V that satisfies the homogeneity condition (5.1) with respect to $|\cdot|$ on k. If $|\cdot|$ is nontrivial on k, then it is easy to see that open and closed balls in V with respect to N centered at 0 and with positive radius are absorbing in V, where these balls are defined as in (8.9) and (8.10). This can be derived directly from the definitions, or using (13.9) and (13.10).

Let V be a topological vector space over k with respect to $|\cdot|$ on k, and let U be an open subset of V that contains 0. Continuity of scalar multiplication on V implies that for each $v \in V$,

$$t \mapsto t\,v \tag{22.3}$$

is continuous as a mapping from k into V, with respect to the topology determined on k by the q_k-metric associated to $|\cdot|$ as in (4.8). Continuity of this mapping at $t = 0$ implies that for each $v \in V$ there be a $\delta(v) > 0$ such that

$$t\,v \in U \tag{22.4}$$

for every $t \in k$ with $|t| < \delta(v)$. If $|\cdot|$ is nontrivial on k, then this implies that U is absorbing in V, as before.

23 Bounded sets

Let k be a field with a nontrivial q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a topological vector space over k with respect to $|\cdot|$ on k. A subset E of V is said to be *bounded* in V if for each open set $U \subseteq V$ with $0 \in U$ there is a $t_0 \in k$ such that

$$E \subseteq t_0\,U. \tag{23.1}$$

If U is balanced in V, then (23.1) implies that

$$E \subseteq t\,U \tag{23.2}$$

for every $t \in k$ such that $|t| \geq |t_0|$. Remember that the collection of balanced nonempty open subsets of V is a local base for the topology of V at 0, because $|\cdot|$ is nontrivial on k, as in Section 16. If E is bounded in V, then it follows that for each open set $U \subseteq V$ with $0 \in U$ there is a $t_0 \in k$ such that (23.2) holds for every $t \in k$ with $|t| \geq |t_0|$, since we can reduce to the case where U is balanced in V. Similarly, if $\mathcal{B}_0$ is a local base for the topology of V at 0, then it suffices to verify that for each $U \in \mathcal{B}_0$ there be a $t_0 \in k$ such that (23.1) holds, in order to show that E is bounded. In particular, one can take $\mathcal{B}_0$ to be the collection of balanced nonempty open subsets of V.

Remember that an open set $U \subseteq V$ that contains 0 is absorbing in V when $|\cdot|$ is nontrivial on k, as in the previous section. This implies that finite subsets of V are bounded. One can check that the union of finitely many bounded subsets of V is bounded too, using the formulation of boundedness in terms of (23.2). If $E \subseteq V$ is bounded, then it is easy to see that the closure $\overline{E}$ of E is bounded in V as well. This uses the regularity of V as a topological space in the strict sense, as in Section 15, to get that every open set in V that contains 0 contains the closure of another open set in V that contains 0. This also uses the continuity of scalar multiplication on V, to get that the closure behaves well with respect to scalar multiplication. Note that the balanced hull of a bounded subset of V is bounded in V as well, where the balanced hull is as defined in (13.8).

Suppose that $E_1, E_2 \subseteq V$ are bounded sets, and let us verify that $E_1 + E_2$ is bounded in V, where $E_1 + E_2$ is as defined in (13.3). Let $U \subseteq V$ be an open set that contains 0, and let $U_1, U_2 \subseteq V$ be open sets that contain 0 and satisfy

$$U_1 + U_2 \subseteq U, \tag{23.3}$$

as in (14.2). The boundedness of E_1, E_2 in V imply that

$$E_1 \subseteq t\,U_1 \quad \text{and} \quad E_2 \subseteq t\,U_2 \tag{23.4}$$

for every $t \in k$ such that $|t|$ is sufficiently large. It follows that

$$E_1 + E_2 \subseteq t\,U_1 + t\,U_2 = t\,(U_1 + U_2) \subseteq t\,U \tag{23.5}$$

when $|t|$ is sufficiently large, as desired. In particular, this implies that translates of bounded subsets of V are bounded, because subsets of V with only one element are bounded, as before.

Suppose that $E \subseteq V$ is compact, and let us check that E is bounded in V. Let $U \subseteq V$ be an open set that contains 0, which we may as well take to be balanced in V, as in Section 16. Let t be an element of k such that $|t| > 1$, which exists because $|\cdot|$ is nontrivial on k. Note that $t^j\,U$ is an open set in V for each j, by continuity of scalar multiplication on V, and that

$$t^j\,U \subseteq t^{j+1}\,U \tag{23.6}$$

for each j, because U is balanced. We also have that

$$\bigcup_{j=1}^{\infty} t^j\,U = V, \tag{23.7}$$

because U is absorbing in V, as in the previous section. If E is compact, then E is contained in the union of finitely many sets of the form $t^j\,U$ with $j \in \mathbf{Z}_+$. This implies that E is contained in $t^j\,U$ for a single $j \in \mathbf{Z}_+$, as desired, because of (23.6).

Suppose now that the topology on V is determined by a nonempty collection $\mathcal{N}$ of q-seminorms on V, as in Section 11. More precisely, each $N \in \mathcal{N}$ should be a q_N-seminorm on V for some $q_N > 0$, and with respect to $|\cdot|$ on k. In this situation, a subset E of V is bounded if and only if each $N \in \mathcal{N}$ is bounded on E. The "only if" part of this statement uses the fact that open balls in V with respect to elements of $\mathcal{N}$ centered at 0 with positive radius are open sets in V. The "if" part uses the fact that intersections of finitely many such open balls form a local base for the topology of V at 0, as in (11.5).

Let X be a nonempty set, and let $c(X,k)$ be the vector space of k-valued functions on X, as in Section 1. Also let E be a subset of $c(X,k)$, and put

$$E_x = \{f(x) : f \in E\} \tag{23.8}$$

for each $x \in X$, which is a subset of k. Let us say that E is bounded *pointwise* on X if E_x is bounded in k with respect to $|\cdot|$ for each $x \in X$. Remember that $c(X,k)$ is a topological vector space over k with respect to the topology determined by the collection (11.7) of q_k-seminorms on $c(X,k)$. It is easy to see that E is bounded in $c(X,k)$ with respect to this topology if and only if E is bounded pointwise on X, by the remarks in the preceding paragraph.

Let $c_{00}(X,k)$ be the linear subspace of $c(X,k)$ consisting of functions with finite support in X, as in Section 1 again. Remember that $c_{00}(X,k)$ is a topological vector space over k with respect to the topology induced by the topology on $c(X,k)$ that corresponds to the strong product topology, as in Section 19. Equivalently, this is the topology determined on $c_{00}(X,k)$ by the collection (20.9) of weighted maximum norms. Let E be a subset of $c_{00}(X,k)$, and put

$$X_E = \{x \in X : E_x \neq \{0\}\}, \tag{23.9}$$

which is the set of $x \in X$ for which there is an $f \in E$ such that $f(x) \neq 0$. If E is bounded in $c_{00}(X,k)$ with respect to the topology just mentioned, then we would like to check that X_E has only finitely many elements. Suppose for the sake of a contradiction that X_E has infinitely many elements, and let $\{x_j\}_{j=1}^\infty$ be a sequence of distinct elements of X_E. By construction, for each $j \geq 1$, there is an $f_j \in E$ such that $f_j(x_j) \neq 0$. Using this, it is easy to see that there is a positive real-valued function w on X such that the corresponding maximum norm (20.1) is not bounded on E. This implies that E is not bounded with respect to this topology on $c_{00}(X,k)$. Alternatively, if ρ is a positive real-valued function on X, then

$$U_\rho(0) \cap c_{00}(X,k) \tag{23.10}$$

is an open set in $c_{00}(X,k)$ that contains 0, where $U_\rho(0)$ is as defined in (18.1). In order to verify that E is not bounded in $c_{00}(X,k)$, it suffices to choose $\rho > 0$ such that E is not contained in any dilate of (23.10) by an element of k.

Equivalently,

$$X_E = \bigcup_{f \in E} \operatorname{supp} f, \tag{23.11}$$

where $\operatorname{supp} f$ is the support of $f \in c(X,k)$, as in (1.2). If E is bounded in $c_{00}(X,k)$, then it is easy to see that E also has to be bounded pointwise on X. Conversely, if E is bounded pointwise on X, and if X_E has only finitely many elements, then one can check that E is bounded in $c_{00}(X,k)$.

24 Bounded sequences

Let k be a field with a nontrivial q_k-abolute value function $|\cdot|$ for some $q_k > 0$, and let V be a topological vector space over k with respect to $|\cdot|$ on k. A sequence $\{v_j\}_{j=1}^\infty$ of elements of V is said to be *bounded* in V if the set of v_j's with $j \in \mathbf{Z}_+$ is bounded as a subset of V, as in the previous section. If E is any bounded subset of V, then it is easy to see that every subset of E is bounded in V too. In particular, this implies that every sequence of elements of E is bounded in V. Conversely, suppose that $E \subseteq V$ is not bounded in V. This means that there is an open set $U \subseteq V$ such that $0 \in U$ and E is not contained in any dilate of U. Let t be an element of k such that $|t| > 1$, which exists because $|\cdot|$ is nontrivial on k. Thus

$$E \not\subseteq t^j\, U \tag{24.1}$$

for any $j \in \mathbf{Z}_+$, which implies that there is a sequence $\{v_j\}_{j=1}^\infty$ of elements of E such that

$$v_j \not\in t^j\, U \tag{24.2}$$

for each $j \geq 1$. It follows that $\{v_j\}_{j=1}^\infty$ is not bounded in V.

If $\{v_j\}_{j=1}^\infty$ is a sequence of elements of V that converges to some $v \in V$, then $\{v_j\}_{j=1}^\infty$ is bounded in V. One way to see this is to observe that the subset of V consisting of the v_j's with $j \in \mathbf{Z}_+$ together with v is compact in V, which implies boundedness, as in the previous section. Alternatively, if $\{v_j\}_{j=1}^\infty$ converges to 0 in V, and if $U \subseteq V$ is an open set that contains 0, then we already have that $v_j \in U$ for all but finitely many j. If U is also balanced in V, then it follows that $v_j \in t\,U$ for every j when $|t|$ is sufficiently large, because U is absorbing in V, as in Section 22. If $\{v_j\}_{j=1}^\infty$ converges to any $v \in V$, then one can reduce to the case where $v = 0$, using the fact that translates of bounded subsets of V are bounded in V, as in the previous section.

Let X be a nonempty set, and let $c(X, k)$ be the vector space of k-valued functions on X, as in Section 1. A sequence $\{f_j\}_{j=1}^\infty$ of elements of $c(X, k)$ is said to be bounded *pointwise* on X if for each $x \in X$, $\{f_j(x)\}_{j=1}^\infty$ is bounded as a sequence of elements of k with respect to $|\cdot|$ on k. This is equivalent to asking that the set of f_j's be bounded pointwise on X, as in the previous section. Let us say that $\{f_j\}_{j=1}^\infty$ converges *pointwise* on X to some $f \in c(X, k)$ if $\{f_j(x)\}_{j=1}^\infty$ converges to $f(x)$ in k for every $x \in X$, with respect to the q_k-metric on k associated to $|\cdot|$ as in (4.8). If $\{f_j\}_{j=1}^\infty$ converges pointwise on X, then $\{f_j\}_{j=1}^\infty$ is pointwise bounded on X, because convergent sequences in k are bounded with respect to $|\cdot|$. Remember that $c(X, k)$ is a topological vector space with respect to the topology determined by the collection (11.7) of q_k-seminorms on X, and that boundedness in $c(X, k)$ with respect to this topology is equivalent to pointwise boundedness. Similarly, a sequence $\{f_j\}_{j=1}^\infty$ of elements of $c(X, k)$ converges to $f \in c(X, k)$ with respect to this topology if and only if $\{f_j\}_{j=1}^\infty$ converges to f pointwise on X.

Let $c_{00}(X, k)$ be the linear subspace of $c(X, k)$ consisting of functions on X with finite support, as usual, equipped with the topology induced by the topology on $c(X,k)$ that corresponds to the strong product topology, as in Section 19. A sequence $\{f_j\}_{j=1}^\infty$ of elements of $c_{00}(X,k)$ is bounded with respect to this topology if and only if $\{f_j\}_{j=1}^\infty$ is bounded pointwise on X and

$$\bigcup_{j=1}^{\infty} \operatorname{supp} f_j \tag{24.3}$$

has only finitely many elements, by the discussion in the previous section. Similarly, $\{f_j\}_{j=1}^\infty$ converges to $f \in c_{00}(X,k)$ with respect to this topology if and only if $\{f_j\}_{j=1}^\infty$ converges to f pointwise on X and (24.3) has only finitely many elements. More precisely, the "only if" part of this statement uses the fact that convergent sequences are bounded, as before, to get that (24.3) has only finitely many elements. The rest of this characterization of convergence of sequences in $c_{00}(X,k)$ can be verified directly from the definitions.

25 Countable local bases

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a topological vector space over k with respect to $|\cdot|$ on k. Of course, if there is a q-semimetric on V for some $q > 0$ that determines the same topology on V, then there is a local base for the topology of V at 0 with only finitely or countably many elements. Conversely, if there is a local base for the topology of V at 0 with only finitely or countably many elements, then it is well known that there is a translation-invariant semimetric on V that determines the same topology on V. More precisely, this works for commutative topological groups with a countable local base at 0, and topological vector spaces are commutative topological groups with respect to addition.

Suppose that the topology on V is determined by a nonempty collection $\mathcal{N}$ of q-seminorms on V with respect to $|\cdot|$ on k, as in Section 11. If $\mathcal{N}$ has only finitely or countably many elements, then it is easy to see that there is a local base for the topology of V at 0 with only finitely or countably many elements. If $\mathcal{N}$ has only finitely many elements, then one can get a single q-seminorm on V that determines the same topology on V as in Section 9. Similarly, if $\mathcal{N}$ is countably infinite, then one can get a single translation-invariant q-semimetric on V that determines the same topology on V as in Section 12. More precisely, if $|\cdot|$ is trivial on k and $\mathcal{N}$ is countably infinite, then the discussion in Section 12 leads to a q-seminorm on V that determines the same topology on V.

Let X be a nonempty set, and let $c(X,k)$ be the vector space of k-valued functions on X, as in Section 1. As usual, $c(X,k)$ is a topological vector space over k with respect to the collection (11.7) of q_k-seminorms on $c(X,k)$. If X has only finitely many elements, then the same topology on $c(X,k)$ is determined by the q_k-norm (6.2). If X is countably infinite, then one can get a translation-invariant q_k-metric on $c(X,k)$ that determines the same topology as in Section 12. If X is countably infinite and $|\cdot|$ is trivial on k, then one can get a q_k-norm on $c(X,k)$ that determines the same topology as in Section 12. However, if X

has infinitely many elements and $|\cdot|$ is not trivial on k, then this topology on $c(X, k)$ cannot be described by a single q-norm for any $q > 0$. One way to see this is to observe that there is no bounded open subset of $c(X, k)$ in this case. If X is uncountable, then there is no local base for this topology on $c(X, k)$ at 0 with only finitely or countably many elements.

Let $c_{00}(X, k)$ be the linear subspace of $c(X, k)$ consisting of functions with finite support, equipped with the topology induced by the topology on $c(X, k)$ that corresponds to the strong product topology, as in Section 19. If X has only finitely many elements, then $c_{00}(X, k)$ is the same as $c(X, k)$ with the topology considered in the preceding paragraph. If X is any nonempty set again and $|\cdot|$ is trivial on k, then this topology on $c_{00}(X, k)$ is the same as the discrete topology, which can be described by the trivial ultranorm. Otherwise, if X has infinitely many elements and $|\cdot|$ is not trivial on k, then one can check that $c_{00}(X, k)$ does not have a local base for its topology at 0 with only finitely or countably many elements. In particular, this happens already when X is countably infinite.

26 Weighted ℓ^r seminorms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. Also let $c_{00}(X, k)$ be the vector space of k-valued functions on X with finite support, as in Section 1. If $f \in c_{00}(X, k)$, r is a positive real number, and w is a nonnegative real-valued function on X, then put

$$\|f\|_{r,w} = \Big(\sum_{x \in X} w(x)^r \, |f(x)|^r\Big)^{1/r}. \tag{26.1}$$

This is the same as (2.1) applied to $w(x)\,|f(x)|$ as a nonngetaive real-valued function on X with finite support. If $w(x) = 1$ for every $x \in X$, then (26.1) is also the same as $\|f\|_r$ defined in (6.3). One can check that (26.1) defines a q_k-seminorm on $c_{00}(X, k)$ when $r \geq q_k$, and that (26.1) defines an r-seminorm on $c_{00}(X, k)$ when $r \leq q_k$, for essentially the same reasons as before. If $w(x) > 0$ for every $x \in X$, then (26.1) defines a q_k or r-norm on $c_{00}(X, k)$, as appropriate.

Remember that $\|f\|_{\infty,w}$ was defined for $f \in c_{00}(X, k)$ and $w \geq 0$ on X in (20.1). If $0 < r_1 \leq r_2 \leq \infty$, then

$$\|f\|_{r_2,w} \leq \|f\|_{r_1,w} \tag{26.2}$$

for every $f \in c_{00}(X, k)$ and nonnegative real-valued function w on X. This follows from (2.5), applied to $w(x)\,|f(x)|$. If w_1, w_2 are nonnegative real-valued functions on X such that

$$w_1(x) \leq w_2(x) \tag{26.3}$$

for every $x \in X$, then we have that

$$\|f\|_{r,w_1} \leq \|f\|_{r,w_2} \tag{26.4}$$

for every $f \in c_{00}(X, k)$ and $r > 0$. This includes the case where $r = \infty$, which was already mentioned in (20.3).

Let N be a q-seminorm on $c_{00}(X,k)$ with respect to $|\cdot|$ on k for some $q > 0$. Put

$$w_N(y) = N(\delta_y) \tag{26.5}$$

for each $y \in X$, where δ_y is as defined in (1.3). Thus w_N is a nonnegative real-valued function on X, and one can check that

$$N(f) \le \|f\|_{q,w_N} \tag{26.6}$$

for every $f \in c_{00}(X,k)$. More precisely, one can express f as a linear combination of δ_y's, and use the q-seminorm version of the triangle inequality to get (26.6). Note that $q = \infty$ is permitted here.

If X has only finitely many elements, then

$$\|f\|_{r,w} \le (\#X)^{1/r}\,\|f\|_{\infty,w} \le (\#X)^{1/r}\left(\max_{x\in X} w(x)\right)\|f\|_\infty \tag{26.7}$$

for every $f \in c_{00}(X,k)$, $r > 0$, and nonnegative real-valued function w on X. Here $\#X$ denotes the number of elements in X, and $(\#X)^{1/r}$ may be interpreted as being equal to 1 when $r = \infty$. Suppose now that X is countably infinite, and let r be a positive real number. Also let w_r be a positive real-valued function on X such that

$$\sum_{x\in X} w_r(x)^{-r} < \infty, \tag{26.8}$$

where the sum on the left side is defined to be the supremum of the sums of $w_r(x)^{-r}$ over all nonempty finite subsets of X. Equivalently, if $\{x_j\}_{j=1}^\infty$ is a sequence of elements of X in which every element of X occurs exactly once, then the sum on the left side of (26.8) can be given by the infinite series

$$\sum_{j=1}^\infty w_r(x_j)^{-r}, \tag{26.9}$$

and (26.8) means that this series converges. If $f \in c_{00}(X,k)$ and w is a nonnegative real-valued function on X, then we get that

$$\begin{aligned} \sum_{x\in X} w(x)^r\,|f(x)|^r &= \sum_{x\in X} (w(x)^r\,w_r(x)^r\,|f(x)|^r)\,w_r(x)^{-r} \\ &\le \Big(\sum_{x\in X} w_r(x)^{-r}\Big)\max_{x\in X}(w(x)^r\,w_r(x)^r\,|f(x)|^r), \end{aligned} \tag{26.10}$$

and hence

$$\|f\|_{r,w} \le \Big(\sum_{x\in X} w_r(x)^{-r}\Big)^{1/r}\|f\|_{\infty,w\,w_r}. \tag{26.11}$$

Note that the first step in (26.7) is basically the same as (26.11) with $w_r(x) = 1$ for every $x \in X$, when X has only finitely many elements.

27 Balanced q-convexity

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. Also let E be a balanced subset of V, and let q be a positive real number. Suppose that $v_1, \ldots, v_n$ are finitely many elements of E, and that $t_1, \ldots, t_n$ are finitely many elements of k such that

$$\sum_{j=1}^{n} |t_j|^q \le 1. \tag{27.1}$$

As a basic form of *q-convexity*, one might ask that

$$\sum_{j=1}^{n} t_j\, v_j \in E \tag{27.2}$$

under these conditions.

Note that (27.1) is equivalent to

$$\Big(\sum_{j=1}^{n} |t_j|^q\Big)^{1/q} \le 1. \tag{27.3}$$

The left side of (27.3) decreases monotonically in q, as in (2.5). Thus (27.3) becomes less restrictive as q increases, which means that this type of q-convexity condition becomes more restrictive as q increases.

Of course, the q-convexity condition described earlier is trivial when $n = 1$, because E is supposed to be balanced in V. Sometimes one might ask that this condition hold only when $n = 2$, and then try to get the analogous condition when $n > 2$ using the $n = 2$ case repeatedly. In particular, this works when every positive real number can be expressed as $|t|$ for some $t \in k$.

Similarly, consider the weaker q-convexity condition in which (27.1) is replaced by a strict inequality. Suppose that $|\cdot|$ is not discrete on k, so that the positive values of $|\cdot|$ on k are dense in the set of positive real numbers with respect to the standard topology, as in Section 21. In this case, one can also obtain this weaker version of q-convexity from its analogue with only $n = 2$.

As usual, the analogue of (27.3) with $q = \infty$ is

$$\max(|t_1|, \ldots, |t_n|) \le 1, \tag{27.4}$$

as in (2.7). Thus one can include $q = \infty$ in the q-convexity condition mentioned at the beginning of the section by replacing (27.1) with (27.4). Of course, (27.1) implies (27.4) for any positive real number q, so that q-convexity with $q = \infty$ implies q-convexity for every finite q. Because E is supposed to be balanced in V, q-convexity with $q = \infty$ is the same as saying that E is closed under addition. A nice feature of q-convexity with $q = \infty$ is that the analogous condition with only $n = 2$ can be repeated easily to get all $n \in \mathbf{Z}_+$.

If N is a q-seminorm on V with respect to $|\cdot|$ on k, then open and closed balls in V with respect to N centered at 0 and with positive radius are balanced in V, and it is easy to see that they satisfy the q-convexity condition described earlier. The closed ball centered at 0 with radius 0 with respect to N is a linear subspace of V, and linear subspaces of V satisfy this q-convexity condition with $q = \infty$.

If $|\cdot|$ is the trivial absolute value function on k, then (27.1) implies that $t_j = 0$ for all but at most one j. In this case, any balanced set $E \subseteq V$ satisfies the earlier q-convexity condition when $0 < q < \infty$. However, (27.4) is vacuous in this situation, so that only linear subspaces of V satisfy the analogous condition with $q = \infty$.

28 Some sums of sets

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a topological vector space over k. Also let $U_0 \subseteq V$ be an open set that contains 0, and let $U_1 \subseteq V$ be an open set that contains 0 and satisfies

$$U_1 + U_1 \subseteq U_0, \tag{28.1}$$

as in (14.2). Repeating the process, we obtain a sequence of open subsets $U_1, U_2, U_3, \ldots$ of V such that $0 \in U_j$ and

$$U_j + U_j \subseteq U_{j-1} \tag{28.2}$$

for each $j \in \mathbf{Z}_+$. Using induction, one can check that

$$U_1 + U_2 + \cdots + U_{n-1} + U_n + U_n \subseteq U_0 \tag{28.3}$$

for each positive integer n, where only the last term U_n is repeated on the left side of (28.3). It follows in particular that

$$U_1 + U_2 + \cdots + U_{n-1} + U_n \subseteq U_0 \tag{28.4}$$

for each $n \in \mathbf{Z}_+$, where the last term U_n on the left side of (28.4) is no longer repeated, since $0 \in U_n$. Of course, we can also choose the U_j's to belong to any given local base for the topology of V at 0. In particular, if $|\cdot|$ is nontrivial on k, then we can choose the U_j's to be balanced in V for each j.

If U_0 happens to be closed under addition, then we can simply take $U_j = U_0$ for each $j \in \mathbf{Z}_+$. In particular, this holds when U_0 satisfies the q-convexity condition discussed in the previous section with $q = \infty$.

Suppose now that U_0 is balanced and satisfies the q-convexity condition discussed in the previous section for some positive real number q. This says exactly that if $t_1, \ldots, t_n$ are finitely many elements of k that satisfy (27.1), then

$$t_1\, U_0 + t_2\, U_0 + \cdots + t_n\, U_0 \subseteq U_0. \tag{28.5}$$

Suppose that $t_1, t_2, t_3, \ldots$ is an infinite sequence of elements of k such that

$$\sum_{j=1}^{\infty} |t_j|^q \leq 1. \tag{28.6}$$

Thus (27.1) holds for each positive integer n, so that (28.5) holds for each $n \in \mathbf{Z}_+$ too. If $t_j \neq 0$ for each $j \in \mathbf{Z}_+$, then $t_j\, U_0$ is an open set in V for each j as well. If $|\cdot|$ is nontrivial on k, then it is easy to see that there are sequences of nonzero elements of k that satisfy (28.6). In this case, we have strict inequality in (27.1) for each n, so that the corresponding weaker version of q-convexity would be sufficient to get (28.5).

Part II

Continuous linear mappings

29 Continuity conditions

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k with respect to $|\cdot|$ on k. A linear mapping T from V into W is continuous at 0 if for each open set U_W in W that contains 0 there is an open set U_V in V that contains 0 and satisfies

$$T(U_V) \subseteq U_W. \tag{29.1}$$

In this case, we have that

$$T(v + U_V) = T(v) + T(U_V) \subseteq T(v) + U_W \tag{29.2}$$

for every $v \in V$, so that T is continuous everywhere on V. In order to check that T is continuous, it suffices to consider U_W in a local sub-base for the topology of W at 0. Similarly, if T is continuous, then one can take U_V to be in a local base for the topology of V at 0.

Suppose that the topology on W is determined by a nonempty collection $\mathcal{N}_W$ of q-seminorms on W with respect to $|\cdot|$ on k, as in Section 11, and where $q > 0$ is allowed to depend on the element of $\mathcal{N}_W$, as usual. In this case, it suffices to take U_W to be an open ball in W with respect to an element of $\mathcal{N}_W$ centered at 0, since these open balls form a local sub-base for the topology of W at 0, by hypothesis. Thus a linear mapping T from V into W is continuous if and only if for each $N_W \in \mathcal{N}_W$ and $r > 0$ there is an open set $U_V \subseteq V$ such that $0 \in U_V$ and

$$N_W(T(u)) < r \tag{29.3}$$

for every $u \in U_V$. Of course, this implies that

$$N_W(T(t\,u)) = |t|\, N_W(T(u)) < |t|\, r \tag{29.4}$$

for every $u \in U_V$ and $t \in k$ with $t \neq 0$. Equivalently, this means that

$$N_W(T(v)) < |t|\, r \tag{29.5}$$

for every $v \in t\, U_V$ and $t \in k$ with $t \neq 0$.

Suppose now that the topology on V is also determined by a nonempty collection $\mathcal{N}_V$ of q-seminorms on V with respect to $|\cdot|$ on k, where $q > 0$ may depend on the element of $\mathcal{N}_V$. As in Section 11, intersections of finitely many open balls in V with respect to elements of $\mathcal{N}_V$ form a local base for the topology of V at 0. Thus a linear mapping T from V into W is continuous if and only if for each $N_W \in \mathcal{N}_W$ and $r > 0$ there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and positive real numbers $r_1, \ldots, r_l$ such that (29.3) holds for every $u \in V$ that satisfies

$$N_{V,j}(u) < r_j \tag{29.6}$$

for each $j = 1, \ldots, l$. As before, this implies that (29.4) holds for every $u \in V$ that satisfies (29.6) for each $j = 1, \ldots, l$ and every $t \in k$ with $t \neq 0$. It follows that (29.5) holds for every $v \in V$ that satisfies

$$N_{V,j}(v) < |t|\, r_j \tag{29.7}$$

for each $j = 1, \ldots, l$, where $t \in k$ and $t \neq 0$.

Let $N_W \in \mathcal{N}_W$ be given, and suppose that there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and a positive real number C such that

$$N_W(T(u)) \le C \max_{1 \le j \le l} N_{V,j}(u) \tag{29.8}$$

for every $u \in V$. Under these conditions, (29.3) holds for every $u \in V$ that satisfies (29.6) with $r_j = r/C$ for each j. In the other direction, let N_W in $\mathcal{N}_W$ and $r > 0$ be given, and suppose that there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and positive real numbers $r_1, \ldots, r_l$ such that (29.4) holds for every $u \in V$ that satisfies (29.6) for each $j = 1, \ldots, l$. Thus (29.5) holds for every $v \in V$ that satisfies (29.7) for each $j = 1, \ldots, l$, where $t \in k$ and $t \neq 0$, as before. If $|\cdot|$ is nontrivial on k, then one can check that this implies that (29.8) holds for some $C > 0$ and every $u \in V$.

30 Continuous linear functionals

Let k be a field, and let V be a vector space over k. Of course, k may be considered as a 1-dimensional vector space over itself, and a linear mapping from V into k is known as a *linear functional* on V. Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and suppose that V is a topological vector space over k with respect to $|\cdot|$ on k. We may also consider $|\cdot|$ as a q-norm on k, and k as a topological vector space over itself with respect to the topology determined by this q-norm, which is the same as the topology determined by the q-metric (4.8) associated to $|\cdot|$ on k. Thus a *continuous linear functional* on V is a continuous linear mapping from V into k, where k is considered as a topological vector space over itself in this way.

If λ is a continuous linear functional on V, then there is an open set $U \subseteq V$ such that $0 \in U$ and

$$|\lambda(u)| < 1 \tag{30.1}$$

for every $u \in U$. Conversely, let λ be a linear functional on V, and suppose that there is an open set $U \subseteq V$ that contains 0 and satisfies (30.1). This implies that

$$|\lambda(t\,u)| = |t|\,|\lambda(u)| < |t| \tag{30.2}$$

for every $u \in U$ and $t \in k$ with $t \neq 0$, which is the same as saying that

$$|\lambda(v)| < |t| \tag{30.3}$$

for every $v \in t\,U$ and $t \in k$ with $t \neq 0$. If $|\cdot|$ is nontrivial on k, then it follows that λ is continuous on V under these conditions. Otherwise, if $|\cdot|$ is the trivial absolute value function on k, then (30.1) implies that

$$\lambda(u) = 0 \tag{30.4}$$

for every $u \in U$, so that λ is continuous in this case as well.

Let X be a nonempty set, and let $c(X, k)$ be the vector space of k-valued functions on X, as in Section 1. Also let g be a k-valued function on X with finite support. If f is any k-valued function on X, then $f\,g$ has finite support in X, which is contained in the support of g. Put

$$\lambda_g(f) = \sum_{x \in X} f(x)\,g(x), \tag{30.5}$$

where the sum on the right side of (30.5) can be defined as an element of k by reducing to the sum over any finite subset of X that contains the support of $f\,g$. In this situation, one can reduce to the sum over any finite subset of X that contains the support of g, and which need not depend on f. In particular, (30.5) defines a linear functional on $c(X, k)$. If $c(X, k)$ is equipped with the topology determined by the collection (11.7) of q_k-seminorms, then it is easy to see that λ_g is a continuous linear functional on $c(X, k)$.

Of course, if X has only finitely many elements, then every linear functional on $c(X, k)$ is of the form (30.5) for some k-valued function g on X. Similarly, let X_0 be a finite subset of X, and let λ be a linear functional on $c(X, k)$ such that

$$\lambda(f) = 0 \tag{30.6}$$

for every $f \in c(X, k)$ such that $f(x) = 0$ for each $x \in X_0$. Under these conditions, λ can be expressed as (30.5) for some k-valued function g on X with support contained in X_0. Now let λ be any continuous linear functional on $c(X, k)$ with respect to the topology determined by (11.7). As before, there is an open set U in $c(X, k)$ such that $0 \in U$ and $\lambda(U)$ is contained in the open unit ball in k. Because of the way that the topology is defined on $c(X, k)$, there is a finite subset X_0 of X such that

$$\{f \in c(X, k) : f(x) = 0 \text{ for every } x \in X_0\} \subseteq U. \tag{30.7}$$

This implies that (30.6) holds for every $f \in c(X, k)$ such that $f(x) = 0$ for each $x \in X_0$. More precisely, this works whether or not $|\cdot|$ is trivial on k. It follows that λ can be expressed as (30.5) for some k-valued function g on X with support contained in X_0, as mentioned earlier. In particular, g has finite support in X.

31 Mappings on $c_{00}(X,k)$

Let k be a field, let X be a nonempty set, and let $c_{00}(X,k)$ be the vector space of k-valued functions on X with finite support, as in Section 1. Also let V be a vector space over k, and let T be a linear mapping from $c_{00}(X,k)$ into V. Thus

$$T(f) = \sum_{y \in X} f(y)\, T(\delta_y) \tag{31.1}$$

for every $f \in c_{00}(X,k)$, where δ_y is as in (1.3). More precisely, $f(y) = 0$ for all but finitely many $y \in X$, so that the sum on the right side of (31.1) can be reduced to a finite sum in V. By hypothesis, $T(\delta_y) \in V$ for every $y \in X$, and any choice of elements of V for each $y \in X$ leads to a linear mapping from $c_{00}(X,k)$ into V in this way.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$, and suppose that V is a topological vector space over k with respect to $|\cdot|$ on k. Let us also take $c_{00}(X,k)$ to be equipped with the topology induced by the topology on $c(X,k)$ that corresponds to the strong product topology, as in Section 19. If $|\cdot|$ is the trivial absolute value function on k, then this topology on $c_{00}(X,k)$ is the same as the discrete topology, so that every mapping from $c_{00}(X,k)$ into any topological space is continuous. Thus we suppose from now on in this section that $|\cdot|$ is nontrivial on k. If X has only finitely many elements, then it is easy to see that every linear mapping from $c_{00}(X,k)$ into V is continuous.

Let us check that every linear mapping T from $c_{00}(X,k)$ into V is continuous when X is countably infinite. To do this, we may as well suppose that $X = \mathbf{Z}_+$. Let U be any open set in V that contains 0. As in Section 28, there is a sequence $U_1, U_2, U_3, \ldots$ of open sets in V that contain 0 and satisfy

$$U_1 + U_2 + \cdots + U_n \subseteq U \tag{31.2}$$

for each positive integer n. We may also ask that U_j be balanced in V for each j, since $|\cdot|$ is nontrivial on k. As in Section 22, U_j is absorbing in V for each j too, using the nontriviality of $|\cdot|$ on k again. Let T be a linear mapping from $c_{00}(\mathbf{Z}_+, k)$ into V, so that $T(\delta_j) \in V$ for each $j \in \mathbf{Z}_+$, where $\delta_j \in c_{00}(\mathbf{Z}_+, k)$ is as in (1.3) with $X = \mathbf{Z}_+$. If j is any positive integer, then there is a positive real number $\rho(j)$ such that

$$t\, T(\delta_j) \in U_j \tag{31.3}$$

for every $t \in k$ such that $|t| < \rho(j)$, because U_j is absorbing in V.

Let f be an element of $c_{00}(X,k)$, and let n be a positive integer such that $f(j) = 0$ when $j > n$, which exists because f has finite support. If $|f(j)| < \rho(j)$ for each j, then we get that

$$T(f) = \sum_{j=1}^{n} f(j)\, T(\delta_j) \in U_1 + U_2 + \cdots + U_n \subseteq U. \tag{31.4}$$

More precisely, the first step in (31.4) is basically the same as (31.1), since $f(j) = 0$ when $j > n$. The second step in (31.4) uses (31.3), and the third step uses (31.2). It follows that T is continuous as a mapping from $c_{00}(\mathbf{Z}_+, k)$ into V, as desired, because (31.4) says exactly that $T(f) \in U$ for every f in a suitable neighborhood of 0 in $c_{00}(\mathbf{Z}_+, k)$ with respect to the topology under consideration.

32 Some related topologies

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let X be a nonempty set, and let $c_{00}(X, k)$ be the vector space of k-valued functions on X with finite support, as in Section 1. If $f \in c_{00}(X, k)$ and w is a nonnegative real-valued function on X, then we put

$$\|f\|_{r,w} = \Big(\sum_{x \in X} w(x)^r \, |f(x)|^r\Big)^{1/r} \tag{32.1}$$

for every positive real number r, as in (26.1). Similarly, put

$$\|f\|_{\infty,w} = \max_{x \in X}(w(x)\, |f(x)|), \tag{32.2}$$

as in (20.1). Remember that $\|f\|_{r,w}$ defines a q_k-seminorm on $c_{00}(X, k)$ when $q_k \le r$, and an r-seminorm on $c_{00}(X, k)$ when $r \le q_k$, as in Sections 20 and 26. If $w(x) > 0$ for every $x \in X$, then $\|f\|_{r,w}$ defines a q_k or r-norm on $c_{00}(X, k)$, as appropriate.

If $0 < r \le \infty$, then we let

$$\tau_r \tag{32.3}$$

be the topology determined on $c_{00}(X, k)$ by

$$\{\|\cdot\|_{r,w} : w \text{ is a positive real-valued function on } X\}, \tag{32.4}$$

as in Section 11. Equivalently, one can allow all nonnegative real-valued functions on X here, and get the same topologies on $c_{00}(X, k)$. As in Section 20, τ_∞ is the same as the topology on induced on $c_{00}(X, k)$ by the topology on $c(X, k)$ that corresponds to the strong product topology. It is easy to see that

$$\tau_{r_2} \subseteq \tau_{r_1} \tag{32.5}$$

when $0 < r_1 \le r_2 \le \infty$, using (26.2). If X has only finitely many elements, then

$$\tau_r = \tau_\infty \tag{32.6}$$

for every $r > 0$, because of (26.7). Similarly, (32.6) holds for every $r > 0$ when X is countably infinite, because of (26.11). If $|\cdot|$ is the trivial absolute value function on X, then τ_∞ is the discrete topology on $c(X, k)$, and hence τ_r is the discrete topology on X for every $r > 0$.

Let us suppose from now on in this section that $|\cdot|$ is nontrivial on k. Let V be a topological vector space over k, and let $U \subseteq V$ be an open set that contains

0. Thus U is absorbing in V, as in Section 22, because $|\cdot|$ is nontrivial on k. Let T be a linear mapping from $c_{00}(X,k)$, and let δ_y be as in (1.3) for each $y \in X$. If $y \in X$, then $T(\delta_y) \in V$, and there is a $t(y) \in k$ such that $t(y) \neq 0$ and

$$t\,T(\delta_y) \in U \tag{32.7}$$

for every $t \in k$ with $|t| \le |t(y)|$, because U is absorbing in V. Let $f \in c_{00}(X,k)$ be given, and let A be a finite subset of X that contains the support of f. Thus

$$T(f) = \sum_{y \in A} f(y)\,T(\delta_y), \tag{32.8}$$

as in (31.1). Equivalently,

$$T(f) = \sum_{y \in A} (t(y)^{-1}\,f(y))\,(t(y)\,T(\delta_y)), \tag{32.9}$$

and $t(y)\,T(\delta_y) \in U$ for each y, by (32.7). Put

$$w(y) = 1/|t(y)| \tag{32.10}$$

for each $y \in X$, so that w is a positive real-valued function on X.

By construction,

$$\|f\|_{r,w}^r = \sum_{y \in A} w(y)^r\,|f(y)|^r = \sum_{y \in A} |t(y)^{-1}\,f(y)|^r \tag{32.11}$$

when $0 < r < \infty$, and

$$\|f\|_{\infty,w} = \max_{y \in A}(w(y)\,|f(y)|) = \max_{y \in A} |t(y)^{-1}\,f(y)|. \tag{32.12}$$

Suppose that U is balanced in V, and that U satisfies the r-convexity condition discussed in Section 27 for some $r > 0$. If

$$\|f\|_{r,w} < 1, \tag{32.13}$$

then it is easy to see that

$$T(f) \in U \tag{32.14}$$

under these conditions, using (32.9). More precisely, it suffices to ask that U satisfy the weaker version of r-convexity in which (27.1) is replaced by a strict inequality. If there is a local base for the topology of V consisting of balanced open sets that satisfy this r-convexity property, then it follows that every linear mapping T from $c_{00}(X,k)$ into V is continuous with respect to the topology τ_r on $c_{00}(X,k)$.

33 Sequential continuity

Let X and Y be topological spaces, and let f be a mapping from X into Y. As usual, f is said to be *sequentially continuous* at a point $x \in X$ if for each sequence $\{x_j\}_{j=1}^\infty$ of elements of X that converges to x we have that $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in Y. Of course, if f is continuous at x, then f is sequentially continuous at x. If f is not continuous at x, then there is an open set $V \subseteq Y$ that contains $f(x)$ with the property that if $U \subseteq X$ is an open set that contains x, then $f(U) \not\subseteq V$. If there is a local base for the topology of X at x with only finitely or countably many elements, then one can use this to get a sequence $\{x_j\}_{j=1}^\infty$ of elements of X that converges to x such that $\{f(x_j)\}_{j=1}^\infty$ does not converge to $f(x)$ in Y. If $f : X \to Y$ is sequentially continuous at every $x \in X$, then we may simply say that f is sequentially continuous as a mapping from X into Y. Thus continuous mappings are sequentially continuous in this sense, and the converse holds when X satisfies the first countability condition, so that for each $x \in X$ there is a local base for the topology of X at x with only finitely or countably many elements.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k with respect to $|\cdot|$ on k. If a linear mapping T from V into W is sequentially continuous at any point in V, then T is sequentially continuous at every point in V, because of continuity of translations. Thus one might normally simply say that T is sequentially continuous when T is sequentially continuous at 0. Of course, if there is a local base for the topology of V at any point with only finitely or countably many elements, then V has the same property at every point, because of continuity of translations again. In this case, sequential continuity at 0 implies continuity at 0, as before.

Let X be a nonempty set, and let $c_{00}(X, k)$ be the space of k-valued functions on X with finite support, as in Section 1. Remember that $c_{00}(X, k)$ is a topological vector space over k with respect to the topology induced by the topology on $c(X, k)$ that corresponds to the strong product topology, as in Section 19. If $\{f_j\}_{j=1}^\infty$ is a sequence of elements of $c_{00}(X, k)$ that converges to another element f of $c_{00}(X, k)$ with respect to this topology, then $\{f_j\}_{j=1}^\infty$ converges to f pointwise on X, and the supports of the f_j's are contained in a finite subset of X, as in Section 24. Let V be another topological vector space over k with respect to $|\cdot|$ on k, and let T be a linear mapping from $c_{00}(X, k)$ into V. It is easy to see that T is sequentially continuous, using the properties of convergent sequences in $c_{00}(X, k)$ just mentioned.

Let V be a topological vector space over k again, and suppose that there is a local base for the topology of V at 0 with only finitely or countably many elements. This implies that there is a sequence $U_1, U_2, U_3, \ldots$ of open subsets of V that contain 0 such that every open set in V that contains 0 also contains U_j for some j. We may ask that $U_{j+1} \subseteq U_j$ for each j too, since otherwise we can replace U_j with $U_1 \cap \cdots \cap U_j$. Suppose that $|\cdot|$ is nontrivial on k, and let t_0 be a nonzero element of k with $|t_0| < 1$. Let W be another topological vector space over k, and let T be a linear mapping from V into W. If T is not continuous at

0, then there is an open set $U_W \subseteq W$ that contains 0 and satisfies

$$T(t_0^j\, U_j) \not\subseteq U_W \tag{33.1}$$

for each j. This implies that there is a sequence $\{v_j\}_{j=1}^{\infty}$ of elements of V such that $v_j \in U_j$ and

$$T(t_0^j\, v_j) \not\in U_W \tag{33.2}$$

for each j. In particular, $\{v_j\}_{j=1}^{\infty}$ converges to 0 in V under these conditions, because $v_j \in U_j$ for each j.

34 Bounded linear mappings

Let k be a field with a nontrivial q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k with respect to $|\cdot|$ on k. Remember that boundedness of subsets of V or W was defined in Section 23. A linear mapping T from V into W is said to be *bounded* if for each bounded set $E \subseteq V$ we have that $T(E)$ is bounded in W. If T is continuous, then it is easy to see that T is bounded, directly from the definitions. Similarly, let us check that T is bounded when T is sequentially continuous. Suppose that $E \subseteq V$ is bounded, and that $T(E)$ is not bounded in W. Let t be an element of k such that $|t| > 1$, which exists because $|\cdot|$ is nontrivial on k. If $T(E)$ is not bounded in W, then there is a sequence $\{v_j\}_{j=1}^{\infty}$ of elements of E such that $\{t^{-j}\, T(v_j)\}_{j=1}^{\infty}$ does not converge to 0 in W, by the same type of argument as in (24.1) and (24.2). However, it is easy to see that $\{t^{-j}\, v_j\}_{j=1}^{\infty}$ converges to 0 in V under these conditions, because $\{v_j\}_{j=1}^{\infty}$ is bounded in V and $\{t^{-j}\}_{j=1}^{\infty}$ converges to 0 in k. If T is sequentially continuous, then it follows that

$$t^{-j}\, T(v_j) = T(t^{-j}\, v_j) \to 0 \quad \text{as } j \to \infty \tag{34.1}$$

in W, which is a contradiction, as desired.

Suppose for the moment that there is a local base for the topology of V at 0 with only finitely or countably many elements. Also let t_0 be a nonzero element of k such that $|t_0| < 1$, which exists because $|\cdot|$ is nontrivial on k. If T is not continuous at 0, then there is an open set $U_W \subseteq W$ that contains 0 and a sequence $\{v_j\}_{j=0}^{\infty}$ of elements of V that converges to 0 such that (33.2) holds for each j, as in the previous section. Note that $\{v_j\}_{j=1}^{\infty}$ is a bounded sequence in V, since it converges to 0 in V, as in Section 24. If T is a bounded linear mapping from V into W, then it follows that $\{T(v_j)\}_{j=1}^{\infty}$ is a bounded sequence in W. Using this, it is easy to see that

$$T(t_0^j\, v_j) = t_0^j\, T(v_j) \to 0 \quad \text{as } j \to \infty \tag{34.2}$$

in W, because $\{t_0^j\}_{j=1}^{\infty}$ converges to 0 in k. This contradicts (33.2), so that bounded linear mappings from V into W are continuous in this situation.

Let X be a nonempty set, and let $c_{00}(X, k)$ be the space of k-valued functions on X with finite support, as in Section 1. As usual, this is a topological vector space over k with respect to the topology induced by the topology on $c(X,k)$ that corresponds to the strong product topology, as in Section 19. If T is any linear mapping from $c_{00}(X,k)$ into a topological vector space V over k, then T is bounded as a linear mapping from $c_{00}(X,k)$ into V. One way to see this is to use the fact that T is sequentially continuous, as in the previous section. Alternatively, if $E \subseteq c_{00}(X,k)$ is bounded with respect to this topology on $c_{00}(X,k)$, then E is bounded pointwise on X, and the supports of the elements of E are contained in a finite subset of X, as in Section 23. Using this, one can check more directly that $T(E)$ is bounded in V. One can also look at this in terms of the continuity of T on finite-dimensional linear subspaces of $c_{00}(X,k)$.

Let V, W be topological vector spaces over k again, and let T_1, T_2 be bounded linear mappings from V into W. We would like to check that their sum $T_1 + T_2$ also defines a bounded linear mapping from V into W. If E is any subset of V, then it is easy to see that

$$(T_1 + T_2)(E) \subseteq T_1(E) + T_2(E). \tag{34.3}$$

If E is bounded in V, then $T_1(E)$ and $T_2(E)$ are bounded in W, and hence $T_1(E)+T_2(E)$ is bounded in W, as in Section 23. This implies that $(T_1+T_2)(E)$ is bounded in W, using (34.3), and the fact that subsets of bounded sets are bounded as well.

35 Strongly bounded linear mappings

Let k be a field with a nontrivial q_k-absolute value function $|\cdot|$ for some $q_k > 0$ again, and let V, W be topological vector spaces over k. Let us say that a linear mapping T from V into W is *strongly bounded* if there is an open set $U_V \subseteq V$ such that $0 \in U_V$ and $T(U_V)$ is bounded in W. This implies that T is continuous, and in particular that T is bounded. In the other direction, if T is continuous, and if there is a nonempty bounded open set in W, then T is strongly bounded. Similarly, if T is bounded, and if there is a nonempty bounded open set in V, then T is strongly bounded.

Let X be a nonempty set, let $c(X,k)$ be the vector space of all k-valued functions on X, and let $c_{00}(X,k)$ be the linear subspace of $c(X,k)$ consisting of functions with finite support in X, as in Section 1. Remember that $c_{00}(X,k)$ is a topological vector space with respect to the topology induced by the topology on $c(X,k)$ that corresponds to the strong product topology, as in Section 19. Let V be a topological vector space over k again, and let T be a strongly bounded linear mapping from V into $c_{00}(X,k)$, with respect to the topology on $c_{00}(X,k)$ just mentioned. Thus there is an open set $U_V \subseteq V$ such that $0 \in U_V$ and $T(U_V)$ is bounded in $c_{00}(X,k)$. In this situation, it follows that the supports of the elements of $T(U_V)$ are contained in a finite subset of X, as in Section 23. This implies that the support of every element of $T(V)$ is contained in the same finite subset of X, because U_V is absorbing in V, as in Section 22. This is basically the same as saying that $T(V)$ has finite dimension in this case.

Let us now take $c(X, k)$ to be equipped with the topology determined by (11.7). Let W be a topological vector space over k, and let T be a strongly bounded linear mapping from $c(X, k)$ into W, with respect to this topology on $c(X, k)$. Hence there is an open set U in $c(X, k)$ such that $0 \in U$ and $T(U)$ is bounded in W. Because of the way that this topology on $c(X, k)$ is defined, there is a finite set $X_0 \subseteq X$ such that

$$\{f \in c(X, k) : f(x) = 0 \text{ for every } x \in X_0\} \tag{35.1}$$

is contained in U. This implies that T maps (35.1) onto a bounded subset of W, because $T(U)$ is bounded in W. If $\{0\}$ is a closed set in W, then one can check that $\{0\}$ is the only bounded linear subspace of W. It follows that T is equal to 0 on (35.1) under these conditions, so that $T(f)$ only depends on the restriction of f to X_0 for each $f \in c(X, k)$.

Let V, W be topological vector spaces over k again, and suppose that T_1, T_2 are strongly bounded linear mappings from V into W. Thus there are open sets $U_1, U_2 \subseteq V$ such that $0 \in U_1, U_2$ and $T_1(U_1)$, $T_2(U_2)$ are bounded subsets of W. Hence $U_1 \cap U_2$ is an open set in V that contains 0, and $T_1(U_1 \cap U_2)$, $T_2(U_1 \cap U_2)$ are bounded subsets of W. As in (34.3),

$$(T_1 + T_2)(U_1 \cap U_2) \subseteq T_1(U_1 \cap U_2) + T_2(U_1 \cap U_2), \tag{35.2}$$

which implies that $(T_1+T_2)(U_1 \cap U_2)$ is bounded in W, because sums of bounded sets are bounded too. This shows that $T_1 + T_2$ is strongly bounded as a linear mapping from V into W as well.

36 Linear functionals on $c_{00}(X, k)$

Let k be a field, and let X be a nonempty set. As in Section 1, $c(X, k)$ denotes the space of k-valued functions on X, and $c_{00}(X, k)$ is the linear subspace of $c(X, k)$ consisting of functions with finite support in X. If $f \in c_{00}(X, k)$ and $g \in c(X, k)$, then $f\,g \in c_{00}(X, k)$ too, and we put

$$\lambda_g(f) = \sum_{x \in X} f(x)\, g(x), \tag{36.1}$$

which reduces to a finite sum in k, as usual. This defines a linear functional on $c_{00}(X, k)$ for each $g \in c(X, k)$, and every linear functional on $c_{00}(X, k)$ is of this form.

Let $|\cdot|$ be a q_k-absolute value function on k for some $q_k > 0$. Observe that

$$|\lambda_g(f)| \le \|f\|_{q_k, |g|} \tag{36.2}$$

for every $f \in c_{00}(X, k)$ and $g \in c(X, k)$, where the right side of (36.2) is as in (20.1) or (26.1), depending on whether $q_k = \infty$ or $q_k < \infty$, respectively. In both cases, we are taking

$$w(x) = |g(x)| \tag{36.3}$$

for each $x \in X$, which defines a nonnegative real-valued function on X.

Note that

$$N_g(f) = |\lambda_g(f)| \tag{36.4}$$

defines a q_k-seminorm on $c_{00}(X, k)$ for each $g \in c(X, k)$. Using this, (36.2) corresponds to (26.6) applied to (36.4). Similarly, (36.3) corresponds to (26.5) applied to (36.4).

It follows from (36.2) that λ_g is continuous on $c_{00}(X, k)$ with respect to the topology τ_{q_k} defined in Section 32, and using the topology on k determined by the q_k-metric associated to $|\cdot|$, as usual. This could also be derived from the discussion in Section 32, but it is easier to use (36.2).

If X has only finitely many elements, then $c_{00}(X, k)$ is the same as $c(X, k)$, and the topologies τ_r on $c_{00}(X, k)$ defined in Section 32 are the same as the topology determined on $c(X, k)$ by (11.7). If X is countably infinite, then all of the topologies τ_r on $c_{00}(X, k)$ defined in Section 32 are the same as the topology induced on $c_{00}(X, k)$ by the strong product topology on $c(X, k)$ as in Section 19. In this situation, the continuity of λ_g for any $g \in c(X, k)$ with respect to this topology on $c_{00}(X, k)$ can be obtained from the discussion in Section 31. This can also be derived from (36.2), using (26.11) with $r = q_k$ when $q_k < \infty$.

Let X be any nonempty set again. If $g \in c_{00}(X, k)$, then λ_g can be defined in the same way as a linear functional on $c(X, k)$, and this linear functional is continuous with respect to the topology determined on $c(X, k)$ by (11.7), as in Section 30. In particular, the restriction of λ_g to $c_{00}(X, k)$ is continuous with respect to the topology induced by the topology determined on $c(X, k)$ by (11.7), which is the same as the topology determined by (11.7) as a collection of q_k-seminorms on $c_{00}(X, k)$. Conversely, if λ is any linear functional on $c_{00}(X, k)$ that is continuous with respect to this topology on $c_{00}(X, k)$, then λ is of the form λ_g with $g \in c_{00}(X, k)$. This is analogous to the argument in Section 30, with some simplifications.

Suppose for the moment that g is a k-valued function on X whose support has only finitely or countably many elements. Under these conditions, λ_g is continuous with respect to the topology induced on $c_{00}(X, k)$ be the strong product topology on $c(X, k)$, as in Section 19. This can be verified using arguments like those for sets X with only finitely or countably many elements. Alternatively, if X_1 is any nonempty subset of X, then there is a natural linear mapping from $c_{00}(X, k)$ onto $c_{00}(X_1, k)$, which is defined by restricting k-valued functions f on X to X_1. It is easy to see that this mapping is continuous with respect to the topologies that correspond to the strong product topology as before. If X_1 contains the support of g, then λ_g on $c_{00}(X, k)$ is the same as the analogous linear functional on $c_{00}(X_1, k)$ composed with the restriction mapping from $c_{00}(X, k)$ onto $c_{00}(X_1, k)$. In particular, if the support of g has only finitely or countably many elements, then one can take X_1 to be a subset of X with only finitely or countably many elements that contains the support of g. This permits one to reduce to the earlier discussion applied to X_1.

37 The case where $k = \mathbf{R}, \mathbf{C}$

Suppose now that $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function, so that we can take $q_k = 1$. Let X be any nonempty set again, and let $c_{00}(X, k)$ be the space of k-valued functions on X with finite support, as in Section 1. If g is any k-valued function on X, then λ_g defines a continuous linear functional on $c_{00}(X, k)$ with respect to the topology τ_1 defined in Section 32, as in the previous section. Suppose instead that λ_g is continuous on $c_{00}(X, k)$ with respect to the topology τ_r defined in Section 32 for some $r > 1$. This implies that there is a positive real-valued function w on X such that

$$|\lambda_g(f)| \le \|f\|_{r,w} \tag{37.1}$$

for every $f \in c_{00}(X, k)$, where $\|f\|_{r,w}$ is as in (20.1) or (26.1), depending on whether $r = \infty$ or $r < \infty$. More precisely, the continuity of λ_g with respect to τ_r on $c_{00}(X, k)$ implies a condition like (29.8). In this situation, this corresponds to having a constant times the maximum of finitely many weighted ℓ^r norms applied to f on the right side of (37.1). In order to get (37.1), one can take w to be the same constant times the maximum of the finitely many weights on X just mentioned.

If $f \in c_{00}(X, k)$, then there is an $\widetilde{f} \in c_{00}(X, k)$ such that

$$|\widetilde{f}(x)| = |f(x)| \tag{37.2}$$

and

$$\widetilde{f}(x)\, g(x) = |f(x)|\, |g(x)| \tag{37.3}$$

for every $x \in X$. Applying (37.1) to $\widetilde{f}$, we get that

$$\sum_{x \in X} |f(x)|\, |g(x)| = \lambda_g(\widetilde{f}) \le \|\widetilde{f}\|_{r,w} = \|f\|_{r,w}. \tag{37.4}$$

This implies that

$$\sum_{x \in X} (w(x)\, |f(x)|)\, (|g(x)|\, w(x)^{-1}) = \sum_{x \in X} |f(x)|\, |g(x)| \le \|f\|_{r,w} \tag{37.5}$$

for every $f \in c_{00}(X, k)$. If $f \in c_{00}(X, k)$, then $w^{-1} f \in c_{00}(X, k)$ too, and we can apply (37.5) to $w^{-1} f$ to get that

$$\sum_{x \in X} |f(x)|\, |g(x)|\, w(x)^{-1} \le \|w^{-1} f\|_{r,w} = \|f\|_r. \tag{37.6}$$

The second step in (37.6) uses the definitions (20.1), (26.1) of $\|\cdot\|_{r,w}$, and $\|f\|_r$ is the corresponding unweighted ℓ^r norm of f, as in Section 6.

If A is any nonempty finite subset of X, then we can apply (37.6) to the function f equal to 1 on A and to 0 on $X \setminus A$, to get that

$$\sum_{x \in A} |g(x)|\, w(x)^{-1} \le (\#A)^{1/r}, \tag{37.7}$$

where $\#A$ denotes the number of elements of A. Let $\epsilon > 0$ be given, and suppose that

$$|g(x)|\, w(x)^{-1} \ge \epsilon \tag{37.8}$$

for each $x \in A$. Combining this with (37.7), we get that

$$\epsilon\, (\#A) \le (\#A)^{1/r}, \tag{37.9}$$

and hence

$$\#A \le \epsilon^{-r/(r-1)}. \tag{37.10}$$

Remember that $r > 1$, by hypothesis. It follows that the number of $x \in X$ that satisfy (37.8) is bounded by the right side of (37.10), and is finite in particular. Applying this to $\epsilon = 1/n$ with $n \in \mathbf{Z}_+$, we get that the support of $|g(x)|\, w(x)^{-1}$ in X has only finitely or countably many elements. Equivalently, this means that the support of g in X has only finitely or countably many elements.

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q > 0$, and suppose that $|\cdot|$ is archimedian on k, as in Section 21. Suppose also that k is complete with respect to the q-metric (4.8) associated to $|\cdot|$, in the usual sense that Cauchy sequences in k with respect to this q-metric converge to elements of k. As usual, one can reduce to the case of ordinary absolute value functions and metrics, by replacing $|x|$ on k by $|x|^q$ when $q < 1$. Under these conditions, a famous theorem of Ostrowski implies that k is isomorphic to $\mathbf{R}$ or $\mathbf{C}$, in such a way that $|\cdot|$ corresponds to a q-absolute value function on $\mathbf{R}$ or $\mathbf{C}$ that is equivalent to the standard absolute value function. Remember that equivalence of q-absolute value functions on a field was defined in Section 21.

38 Weak topologies

Let k be a field with a q_k-absolute-value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k. If λ is a linear functional on V, then

$$N_\lambda(v) = |\lambda(v)| \tag{38.1}$$

defines a q_k-seminorm on V. Let Λ be a nonempty collection of linear functionals on V, so that

$$\mathcal{N}(\Lambda) = \{N_\lambda : \lambda \in \Lambda\} \tag{38.2}$$

is a nonempty collection of q_k-seminorms on V. The topology determined on V by $\mathcal{N}(\Lambda)$ as in Section 11 is known as the *weak topology* determined on V by Λ. By construction, each element of Λ is a continuous linear functional on V with respect to this topology. This implies that linear combinations of elements of Λ are also continuous on V with respect to this topology, because of continuity of addition and multiplication on k. Equivalently, this is the weakest topology on V with respect to which the elements of Λ are continuous. Let us say that Λ is *nondegenerate* on V if for each $v \in V$ with $v \ne 0$ there is a $\lambda \in \Lambda$ such that

$$\lambda(v) \ne 0. \tag{38.3}$$

This is the same as saying that Λ separates points in V, and it implies that $\mathcal{N}(\Lambda)$ is nondegenerate on V, as defined in Section 11.

Let U be an open subset of V with respect to the weak topology determined by Λ, and suppose that $0 \in U$. This implies that there are finitely many elements $\lambda_1, \ldots, \lambda_l$ of Λ and positive real numbers $r_1, \ldots, r_l$ such that

$$\{u \in V : |\lambda_j(u)| < r_j \text{ for every } j = 1, \ldots, l\} \subseteq U. \tag{38.4}$$

In particular, this means that

$$\{u \in V : \lambda_j(u) = 0 \text{ for every } j = 1, \ldots, l\} \subseteq U. \tag{38.5}$$

We may also ask that the λ_j's be linearly independent as linear functionals on V, by discarding ones that can be expressed as linear combinations of the others.

Let μ be a linear functional on V, and suppose that μ is continuous with respect to the weak topology determined on V by Λ. This implies that there is an open set $U \subseteq V$ with respect to this topology such that $0 \in U$ and

$$|\mu(u)| < 1 \tag{38.6}$$

for every $u \in U$, as in Section 30. Let $\lambda_1, \ldots, \lambda_l$ be finitely many elements of Λ that satisfy (38.5). Under these conditions, we have that

$$\{u \in V : \lambda_j(u) = 0 \text{ for every } j = 1, \ldots l\} \subseteq \{u \in V : \mu(u) = 0\}. \tag{38.7}$$

More precisely, if $|\cdot|$ is trivial on k, then (38.7) follows directly from (38.5), because (38.6) implies that $\mu(u) = 0$. Otherwise, suppose that $|\cdot|$ is not trivial on k, and that $\lambda_j(u) = 0$ for each $j = 1, \ldots, l$. If $t \in k$, then we have that $\lambda_j(t\,u) = t\,\lambda_j(u) = 0$ for each $j = 1, \ldots, l$, so that $t\,u \in U$, by (38.5). This implies that

$$|t|\,|\mu(u)| = |\mu(t\,u)| < 1, \tag{38.8}$$

as in (38.6). It follows that $\mu(u) = 0$, as desired, because (38.8) holds for every $t \in k$, and $|\cdot|$ is supposed to be nontrivial on k. This shows that (38.7) holds in both cases. Using (38.7), one can check that μ can be expressed as a linear combination of $\lambda_1, \ldots, \lambda_l$.

Let X be a nonempty set, and let $c(X, k)$ be the vector space of k-valued functions on X, as in Section 1. If $x \in X$, then

$$\lambda_x(f) = f(x) \tag{38.9}$$

defines a linear functional on $c(X, k)$. The q_k-seminorm on $c(X, k)$ corresponding to (38.9) as in (38.1) is the same as (11.6) for each $x \in X$. Thus the topology determined on $c(X, k)$ by (11.7) is the same as the weak topology corresponding to the collection of linear functionals on $c(X, k)$ of the form (38.9) for some $x \in X$.

39 Dual norms

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a vector space over k with a q-norm N for some $q > 0$, with respect to $|\cdot|$ on k. A linear functional λ on V is said to be *bounded* if there is a nonnegative real number C such that

$$|\lambda(v)| \le C\,N(v) \tag{39.1}$$

for every $v \in V$. Suppose for the moment that $|\cdot|$ is not trivial on k, so that boundedness of subsets of V can be defined as in Section 23, which can be used to define boundedness of linear mappings from V into other topological vector spaces over k as in Section 34. In this case, a subset of V is bounded in the sense of Section 23 if and only if it is bounded with respect to N in the usual sense, which means that it is contained in a ball centered at 0 with respect to N. A linear functional λ on V is bounded in the sense of Section 34 if and only if λ is bounded on bounded subsets of V, which is equivalent to asking that λ be bounded on every ball in V centered at 0 with respect to N in this situation. Clearly (39.1) implies that λ is bounded in the sense of Section 34. Conversely, if λ is bounded on any ball in V centered at 0 with positive radius with respect to N, then it is easy to see that λ satisfies a condition like (39.1), using the nontriviality of $|\cdot|$ on k.

If λ satisfies a condition like (39.1), then λ is continuous on V, as in Section 29. Conversely, suppose that λ is continuous at 0 on V, which implies that there is a positive real number r such that

$$|\lambda(v)| < 1 \tag{39.2}$$

for every $v \in V$ with $N(v) < r$. If $|\cdot|$ is nontrivial on k, then this implies a condition like (39.1), as in the previous paragraph. If $|\cdot|$ is trivial on k, then (39.2) implies that $\lambda(v) = 0$, and we also get a condition like (39.1). Thus boundedness of a linear functional on V in the sense of (39.1) is equivalent to continuity in both cases.

Let V' be the *dual space* of continuous linear functionals on V. It is easy to see that V' is also a vector space over k with respect to pointwise addition and scalar multiplication. If $\lambda \in V'$, then put

$$N'(\lambda) = \inf\{C \ge 0 : (39.1)\text{ holds}\}, \tag{39.3}$$

which is more precisely the infimum of the set of nonnegative real numbers C for which (39.1) holds. This set is nonempty, as in the preceding paragraph. One can check that the infimum is always attained, so that

$$|\lambda(v)| \le N'(\lambda)\,N(v) \tag{39.4}$$

for every $v \in V$.

Of course, (39.1) holds with $C = 0$ if and only if $\lambda(v) = 0$ for every $v \in V$. Using this, one can verify that N' defines a q_k-norm on V' with respect to $|\cdot|$ on k, which is the *dual q_k-norm* on V' associated to N on V. In particular, the q_k-norm version of the triangle inequality for N' follows from the corresponding property of $|\cdot|$ on k.

40 Dual spaces

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a topological vector space over k with respect to $|\cdot|$ on k. Also let V' be the *dual space* of continuous linear functionals on V. As before, one can verify that V' is also a vector space over k with respect to pointwise addition and scalar multiplication. More precisely, V' may be described as the *topological dual* of V, to emphasize the role of the continuity condition.

Suppose for the moment that $|\cdot|$ is nontrivial on k, so that boundedness of subsets of V can be defined as in Section 23. Let E be a nonempty bounded subset of V. If $\lambda \in V'$, then $\lambda(E)$ is a bounded subset of k, as in Section 34. Put

$$N'_E(\lambda) = \sup_{v \in E} |\lambda(v)| \tag{40.1}$$

for each $\lambda \in V'$. One can check that this defines a q_k-seminorm on V' with respect to $|\cdot|$ on k. Thus

$$\mathcal{N}' = \{N'_E : E \text{ is a nonempty bounded subset of } V\} \tag{40.2}$$

is a collection of q_k-seminorms on V'. This collection is nonempty, and in fact nondegenerate, because finite subsets of V are bounded in V. This leads to a topology on V', as in Section 11. If the topology on V is determined by a single q-norm N for some $q > 0$, then the topology determined on V' by (40.2) is the same as the topology determined by the dual q_k-norm N' defined in (39.3). More precisely, if E is any nonempty bounded subset of V, then (40.1) is bounded by a constant multiple of $N'(\lambda)$ in this case. In the other direction, if E is a ball in V centered at 0 with positive radius with respect to N, then E is a nonempty bounded subset of V, and N' is bounded by a constant multiple of N'_E.

Let $|\cdot|$ be any q_k-absolute value function on k again. If $v \in V$, then

$$L_v(\lambda) = \lambda(v) \tag{40.3}$$

defines a linear functional on V'. The collection of these linear functionals L_v with $v \in V$ leads to a topology on V' as in Section 38, which is known as the *weak* topology* on V'. This collection automatically separates points in V', so that V' is always Hausdorff with respect to the weak* topology. Suppose now that $|\cdot|$ is nontrivial on k, so that (40.2) also determines a topology on V', as before. If $v \in V$, then $E_v = \{v\}$ is a bounded set in V, and

$$N'_{E_v}(\lambda) = |\lambda(v)| = |L_v(\lambda)| \tag{40.4}$$

for every $\lambda \in V'$, where N'_{E_v} is as in (40.1). This implies that the topology determined on V' by (40.2) is at least as strong as the weak* topology on V'.

Let X be a nonempty set, and let $c_{00}(X, k)$ be the space of k-valued functions on X with finite support, as in Section 1. Remember that every linear functional on $c_{00}(X, k)$ can be expressed as λ_g as in (36.1) for some k-valued function g on X. These linear functionals are also continuous on $c_{00}(X, k)$ with respect to the topology τ_{q_k} defined in Section 32, as in Section 36. Thus we can identify the topological dual of $c_{00}(X,k)$ as a topological vector space over k with respect to the topology τ_{q_k} with the space $c(X,k)$ of all k-valued functions on X. The corresponding weak* topology on the topological dual of $c_{00}(X,k)$ with respect to the topology τ_{q_k} corresponds exactly to the topology determined on $c(X,k)$ by (11.7). Remember that τ_{q_k} contains the topology τ_∞ on $c_{00}(X,k)$, which is the same as the topology induced on $c_{00}(X,k)$ by the strong product topology on $c(X,k)$, as in Section 19. If E is a bounded subset of $c_{00}(X,k)$ with respect to τ_{q_k}, then it follows that E is bounded with respect to the topology induced on $c_{00}(X,k)$ by the strong product topology on $c(X,k)$, which were characterized in Section 23. Using this characterization, it is easy to see that the topology determined on the topological dual of $c_{00}(X,k)$ with respect to τ_{q_k} by (40.2) is the same as the weak* topology.

41 The dual of $c(X,k)$

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let X be a nonempty set, and let $c(X,k)$ be the space of k-valued functions on X, as in Section 1. Remember that $c(X,k)$ is a topological vector space over k with respect to the topology determined by (11.7). If g is a k-valued function on X with finite support, then

$$\lambda_g(f) = \sum_{x \in X} f(x)\, g(x) \tag{41.1}$$

defines a continuous linear functional on $c(X,k)$ with respect to this topology, as in Section 30. We have also seen that every continuous linear functional on $c(X,k)$ is of this form. Thus the topological dual $c(X,k)'$ of $c(X,k)$ with respect to this topology can be identified as a vector space over k with the space $c_{00}(X,k)$ of k-valued functions on X with finite support.

Suppose that $|\cdot|$ is nontrivial on k, so that boundedness of subsets of $c(X,k)$ can be defined as in Section 23, with respect to the topology determined by (11.7). As in Section 23, a subset E of $c(X,k)$ is bounded with respect to this topology if and only if E is bounded pointwise on X. In particular, if w is a nonnegative real-valued function on X, then

$$E_w = \{f \in c(X,k) : |f(x)| \le w(x) \text{ for every } x \in X\} \tag{41.2}$$

is a bounded subset of $c(X,k)$. Every bounded subset of $c(X,k)$ is contained in a set of this form, and one can also restrict one's attention to positive real-valued functions w on X. It follows that the topology determined on $c(X,k)'$ by (40.2) with $V = c(X,k)$ is the same as the topology obtained by restricting one's attention to bounded subsets of $c(X,k)$ of the form (41.2).

Let w be a nonnegative real-valued function on X again, and put

$$w_0(x) = \sup\{|t| : t \in k, |t| \le w(x)\} \tag{41.3}$$

for each $x \in X$. Thus w_0 is a nonnegative real-valued function on X such that

$$w_0(x) \le w(x) \tag{41.4}$$

for every $x \in X$. By construction,

$$E_{w_0} = E_w, \tag{41.5}$$

where E_w and E_{w_0} are as in (41.2). If $|\cdot|$ is not discrete on k, as in Section 21, then

$$w_0(x) = w(x) \tag{41.6}$$

for every $x \in X$. Otherwise, if $|\cdot|$ is discrete on k, then the nontriviality of $|\cdot|$ on k implies that w_0 is greater than or equal to a positive constant multiple of w on X. In this case, one can also simply restrict one's attention to nonnegative real-valued functions w on X whose values are values of $|\cdot|$ on k, so that (41.6) holds on X. Note that the values of w_0 on X are values of $|\cdot|$ on k when $|\cdot|$ is discrete on k.

Suppose for the moment that $|\cdot|$ is an ultrametric absolute value function on k, so that we can take $q_k = \infty$. If $g \in c_{00}(X, k)$ and w is a nonnegative real-valued function on X, then

$$\begin{aligned} N'_{E_w}(\lambda_g) = \sup_{f \in E_w} |\lambda_g(f)| &= \sup_{f \in E_w} \Big|\sum_{x \in X} f(x)\, g(x)\Big| \\ &\le \sup_{f \in E_w} \Big(\max_{x \in X}(|f(x)|\,|g(x)|)\Big). \end{aligned} \tag{41.7}$$

The first step in (41.7) corresponds to (40.1), the second step uses (41.1), and the third step uses the ultrametric version of the triangle inequality. It follows that

$$N'_{E_w}(\lambda_g) \le \max_{x \in X}(w(x)\,|g(x)|) = \|g\|_{\infty, w}, \tag{41.8}$$

using the definition (41.2) in the first step, and the definition (20.1) of $\|\cdot\|_{\infty,w}$ in the second step. If w_0 is as in (41.3), then we get that

$$N'_{E_w}(\lambda_g) = N'_{E_{w_0}}(\lambda_g) \le \|g\|_{\infty, w_0} \tag{41.9}$$

for every $g \in c_{00}(X, k)$, using (41.5) in the first step, and the analogue of (41.8) with w replaced by w_0 in the second step.

In the other direction, we have that

$$N'_{E_w}(\lambda_g) \ge \max_{x \in X}(w_0(x)\,|g(x)|) = \|g\|_{\infty, w_0} \tag{41.10}$$

for every $g \in c_{00}(X, k)$, by considering $f \in E_w$ supported at a single point in X. Combining this with (41.9), we get that

$$N'_{E_w}(\lambda_g) = \|g\|_{\infty, w_0} \tag{41.11}$$

for every $g \in c_{00}(X, k)$. If (41.6) holds, then it follows that

$$N'_{E_w}(\lambda_g) = \|g\|_{\infty, w} \tag{41.12}$$

for every $g \in c_{00}(X, k)$. In particular, (41.12) holds when $|\cdot|$ is not discrete on k, as before. Otherwise, if $|\cdot|$ is discrete on k, then we have seen that w_0 is greater than or equal to a positive constant multiple of w on X, because $|\cdot|$ is nontrivial on k. This implies that $\|g\|_{\infty, w_0}$ is greater than or equal to the same positive constant times $\|g\|_{\infty, w}$ for each $g \in c_{00}(X, k)$, by the definition (20.1) of $\|g\|_{\infty, w}$. Combining this with (41.10), we get that $N'_{E_w}(\lambda_g)$ is greater than or equal to the same positive constant times $\|g\|_{\infty, w}$ for every $g \in c_{00}(X, k)$.

To summarize a bit, we are supposing for the moment that $|\cdot|$ is a nontrivial ultrametric absolute value function on k. As before, the topology determined on $c(X, k)'$ by (40.2) with $V = c(X, k)$ is the same as the topology obtained by restricting one's attention to subsets of $c(X, k)$ of the form (41.2). The discussion in the preceding paragraph implies that this topology on $c(X, k)'$ corresponds exactly to the topology induced on $c_{00}(X, k)$ by the strong product topology on $c(X, k)$, as in Section 19. This also uses the description of this topology on $c_{00}(X, k)$ in Section 20. This correspondence between topologies is a bit simpler when $|\cdot|$ is not discrete on k, so that (41.12) holds for every nonnegative real-valued function w on X. Otherwise, if $|\cdot|$ is discrete on k, then we have (41.8) and an analogous inequality in the other direction with an extra constant factor, which is sufficient for the earlier statement about topologies on $c_{00}(X, k)$. In this case, one can also simply restrict one's attention to nonnegative real-valued functions w on X whose values are values of $|\cdot|$ on k, so that we have (41.6), and hence (41.12).

Now let k be $\mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function, so that we can take $q_k = 1$. If $g \in c_{00}(X, k)$ and w is a nonnegative real-valued function on X, then it is easy to see that

$$N'_{E_w}(\lambda_g) \le \sum_{x \in X} w(x)\, |g(x)| = \|g\|_{1, w}, \tag{41.13}$$

using the definition (26.1) of $\|g\|_{1, w}$ in the second step. If f is a k-valued function on X such that

$$|f(x)| = w(x) \quad \text{and} \quad f(x)\, g(x) = |g(x)|\, w(x) \tag{41.14}$$

for every $x \in X$, then $f \in E_w$ and

$$\lambda_g(f) = \sum_{x \in X} f(x)\, g(x) = \sum_{x \in X} w(x)\, |g(x)| = \|g\|_{1, w}. \tag{41.15}$$

This implies that

$$N'_{E_w}(\lambda_g) \ge |\lambda_g(f)| = \|g\|_{1, w}, \tag{41.16}$$

and hence that

$$N'_{E_w}(\lambda_g) = \|g\|_{1, w}, \tag{41.17}$$

by combining (41.13) and (41.16). It follows that the topology determined on $c(X, k)'$ by (40.2) with $V = c(X, k)$ corresponds exactly to the topology τ_1 defined on $c_{00}(X, k)$ in Section 32 in this case.

42 Equicontinuity

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k with respect to $|\cdot|$ on k. A collection $\mathcal{E}$ of linear mappings from V into W is said to be *equicontinuous* if for each open set $U_W \subseteq W$ that contains 0 there is an open set $U_V \subseteq V$ such that $0 \in U_V$ and

$$T(U_V) \subseteq U_W. \tag{42.1}$$

More precisely, this is really an equicontinuity condition at 0, which implies a uniform version of equicontinuity at every point, because of linearity, and as in (29.2). In particular, equicontinuity of $\mathcal{E}$ implies that each element of $\mathcal{E}$ is continuous. If $\mathcal{E}$ consists of finitely many continuous linear mappings from V into W, then it is easy to see that $\mathcal{E}$ is equicontinuous. In order to check that a collection $\mathcal{E}$ of linear mappings from V into W is equicontinuous, it suffices to verify that the previous condition holds for every U_W in a local sub-base for the topology of W at 0. Similarly, if $\mathcal{E}$ is equicontinuous, then one can take U_V to be in a local base for the topology of V at 0.

If V is a topological vector space over k, then

$$v \mapsto t\, v \tag{42.2}$$

defines a continuous linear mapping from V into itself for each $t \in k$, as in Section 14. Let r be a positive real number, and let $\mathcal{E}_r$ be the collection of linear mappings on V of the form (42.2) with $|t| \le r$. Thus $\mathcal{E}_r$ is equicontinuous on V if and only if for each open set $U \subseteq V$ that contains 0 there is another open set $\widetilde{U} \subseteq V$ such that $0 \in \widetilde{U}$ and

$$t\, \widetilde{U} \subseteq U \tag{42.3}$$

for every $t \in k$ with $|t| \le r$. The definition of a topological vector space implies that this holds for some $r > 0$, as mentioned at the beginning of Section 16. If $|\cdot|$ is not the trivial absolute value function on k, then one can use this and the continuity of (42.2) for each $t \in k$ to get that $\mathcal{E}_r$ is equicontinuous on V for every $r > 0$. However, if $|\cdot|$ is the trivial absolute value function on k, then $\mathcal{E}_r$ consists of only multiplication by 0 when $r < 1$, so that $\mathcal{E}_r$ is trivially equicontinuous. In this case, $\mathcal{E}_1$ consists of all linear mappings on V of the form (42.3) for some $t \in k$. If there is a local base for the topology of V at 0 consisting of balanced open sets, then it is easy to see that $\mathcal{E}_1$ is equicontinuous on V. The converse also holds, by the same type of argument as in Section 16.

Let V, W be topological vector spaces over k again, and suppose that the topology on W is determined by a nonempty collection $\mathcal{N}_W$ of q-seminorms on W with respect to $|\cdot|$ on k, as in Section 11. As usual, $q > 0$ is allowed to depend on the element of $\mathcal{N}_W$ here. In this situation, a collection $\mathcal{E}$ of linear mappings from V into W is equicontinuous if and only if for each $N_W \in \mathcal{N}_W$ and $r > 0$ there is an open set $U_V \subseteq V$ such that $0 \in U_V$ and

$$N_W(T(u)) < r \tag{42.4}$$

for every $u \in U_V$ and $T \in \mathcal{E}$. This uses the fact that open balls in W with respect to elements of $\mathcal{N}_W$ centered at 0 and with positive radius form a local sub-base for the topology of W at 0, as in Section 11. If $t \in k$ and $t \neq 0$, then (42.4) implies that

$$N_W(T(v)) < |t|\, r \tag{42.5}$$

for every $v \in t\, U_V$ and $T \in \mathcal{E}$, as in Section 29.

Now suppose that the topology on V is also determined by a nonempty collection $\mathcal{N}_V$ of q-seminorms on V with respect to $|\cdot|$ on k, where $q > 0$ is allowed to depend on the element of $\mathcal{N}_V$. In this case, a collection $\mathcal{E}$ of linear mappings from V into W is equicontinuous if and only if for each $N_W \in \mathcal{N}_W$ and $r > 0$ there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and positive real numbers $r_1, \ldots, r_l$ such that (42.4) holds for every $u \in V$ that satisfies

$$N_{V,j}(u) < r_j \tag{42.6}$$

for each $j = 1, \ldots, l$, and for every $T \in \mathcal{E}$. In particular, if for each $N_W \in \mathcal{N}_W$ there are finitely many elements $N_{V,1}, \ldots, N_{V,l}$ of $\mathcal{N}_V$ and a nonnegative real number C such that

$$N_W(T(u)) \le C \max_{1 \le j \le l} N_{V,j}(u) \tag{42.7}$$

for every $v \in V$ and $T \in \mathcal{E}$, then $\mathcal{E}$ satisfies the preceding characterization of equicontinuity. In the other direction, if $\mathcal{E}$ is equicontinuous, and if $|\cdot|$ is nontrivial on k, then one can verify that $\mathcal{E}$ satisfies this second condition. This is similar to the analogous statement for continuous linear mappings in Section 29.

43 Uniform boundedness

Let k be a field with a nontrivial q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V, W be topological vector spaces over k with respect to $|\cdot|$ on k. Thus boundedness of subsets of V and W can be defined as in Section 23. Let us say that a collection $\mathcal{E}$ of linear mappings from V into W is *uniformly bounded* on a set $E \subseteq V$ if

$$\bigcup_{T \in \mathcal{E}} T(E) \tag{43.1}$$

is bounded in W. If this holds for every bounded set $E \subseteq V$, then we say that $\mathcal{E}$ is uniformly bounded on bounded subsets of V. If $\mathcal{E}$ is uniformly bounded on bounded subsets of V, then every element of $\mathcal{E}$ should be a bounded linear mapping from V into W, as in Section 34. If $\mathcal{E}$ is a collection of finitely many bounded linear mappings from V into W, then $\mathcal{E}$ is uniformly bounded on bounded subsets of V. This uses the fact that the union of finitely many bounded subsets of W is also bounded in W, as in Section 23.

Let $\mathcal{E}$ be an equicontinuous collection of linear mappings from V into W, and let us verify that $\mathcal{E}$ is uniformly bounded on bounded subsets of V. This is analogous to the fact that continuous linear mappings are bounded, as in

Section 34. Let E be a bounded subset of V, and let U_W be an open set in W that contains 0. Thus there is an open set $U_V \subseteq V$ that contains 0 and satisfies (42.1), by equicontinuity. Because E is bounded in V, there is a $t_0 \in k$ such that

$$E \subseteq t\, U_V \tag{43.2}$$

for every $t \in k$ with $|t| \geq |t_0|$. This implies that

$$T(E) \subseteq t\, T(U_V) \subseteq t\, U_W \tag{43.3}$$

for every $T \in \mathcal{E}$ and $t \in k$ with $|t| \geq |t_0|$, by (42.1). It follows that (43.1) is contained in $t\, U_W$ for every $t \in k$ with $|t| \geq |t_0|$, so that $\mathcal{E}$ is uniformly bounded on E, as desired.

Let $\mathcal{E}$ be a collection of linear mappings from V into W again, and put

$$\mathcal{E}(v) = \{T(v) : T \in \mathcal{E}\} \tag{43.4}$$

for each $v \in V$. If $\mathcal{E}(v)$ is a bounded subset of W for each $v \in V$, then $\mathcal{E}$ is said to be bounded *pointwise* on V. This is the same as saying that $\mathcal{E}$ is uniformly bounded on $E = \{v\}$ for each $v \in V$, which holds in particular when $\mathcal{E}$ is uniformly bounded on bounded subsets of V. If $\mathcal{E}$ is a collection of continuous linear mappings from V into W, then the Banach–Steinhaus theorem gives a criterion for the equicontinuity of $\mathcal{E}$ in terms of pointwise boundedness and Baire category. More precisely, it suffices to ask that the set of $v \in V$ such that $\mathcal{E}(v)$ is bounded in W be of second category in W.

Let us say that a collection $\mathcal{E}$ of linear mappings from V into W is *uniformly strongly bounded* if there is an open set $U \subseteq V$ such that $0 \in U$ and $\mathcal{E}$ is uniformly bounded on U. In particular, this implies that every element of $\mathcal{E}$ is strongly bounded, as in Section 35. If $\mathcal{E}$ is a collection of finitely many strongly bounded linear mappings from V into W, then it is easy to see that $\mathcal{E}$ is uniformly strongly bounded. If $\mathcal{E}$ is any uniformly strongly bounded collection of linear mappings from V into W, then $\mathcal{E}$ is equicontinuous. If $\mathcal{E}$ is an equicontinuous collection of linear mappings from V into W and if there is a nonempty bounded open subset of W, then $\mathcal{E}$ is uniformly strongly bounded. More precisely, if there is a nonempty bounded open subset of W, then a suitable translate of it will contain 0. Similarly, if a collection $\mathcal{E}$ of linear mappings from V into W is uniformly bounded on bounded subsets of V, and if there is a nonempty bounded open subset of V, then $\mathcal{E}$ is uniformly strongly bounded.

44 Mappings on $c_{00}(X, k)$, continued

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, let X be a nonempty set, and let $c_{00}(X, k)$ be the space of k-valued functions on X with finite support, as in Section 1. Also let V be a topological vector space over k with respect to $|\cdot|$ on k, and let $\mathcal{E}$ be a collection of linear mappings from $c_{00}(X, k)$ into V. If $|\cdot|$ is trivial on k, then the topology induced on $c_{00}(X, k)$ by the strong product topology on $c(X, k)$ as in Section 19 is the same as the discrete topology on $c_{00}(X, k)$. In this case, $\mathcal{E}$ is automatically equicontinuous with respect to this topology on $c_{00}(X, k)$. Let us suppose from now on in this section that $|\cdot|$ is nontrivial on k.

Let $\delta_y \in c_{00}(X, k)$ be as defined in (1.3) for each $y \in X$. Put

$$\mathcal{E}(\delta_y) = \{T(\delta_y) : T \in \mathcal{E}\} \tag{44.1}$$

for each $y \in X$, as in (43.4), and which is a subset of V in this situation. Of course, if $\mathcal{E}$ is bounded pointwise on $c_{00}(X, k)$, as in the previous section, then $\mathcal{E}(\delta_y)$ is a bounded subset of V for every $y \in X$. Conversely, if $\mathcal{E}(\delta_y)$ is bounded in V for each $y \in X$, then it is easy to see that $\mathcal{E}$ is bounded pointwise on $c_{00}(X, k)$, because the δ_y's form a basis for $c_{00}(X, k)$ as a vector space over k. This also implies that $\mathcal{E}$ is uniformly bounded on bounded subsets of $c_{00}(X, k)$ with respect to the topology induced on $c_{00}(X, k)$ by the strong product topology on $c(X, k)$, because of the characterization of bounded subsets of $c_{00}(X, k)$ with respect to this topology in Section 23.

If X has only finitely or countably many elements, then we have seen that every linear mapping from $c_{00}(X, k)$ into V is continuous with respect to the topology induced on $c_{00}(X, k)$ by the strong product topology on $c_{00}(X, k)$, as in Section 31. Similarly, if X has only finitely or countably many elements, and if $\mathcal{E}(\delta_y)$ is bounded in V for every $y \in X$, then $\mathcal{E}$ is equicontinuous with respect to this topology on $c_{00}(X, k)$. This can be verified directly when X has only finitely many elements. If X is countably infinite, then the argument is almost the same as the one for continuity of linear mappings from $c_{00}(X, k)$ into V, as in Section 31.

Let $0 < r \leq \infty$ be given, and let τ_r be the corresponding topology defined on $c_{00}(X, k)$ as in Section 32. The topology induced on $c_{00}(X, k)$ by the strong product topology on $c(X, k)$ is the same as τ_∞, which is contained in τ_r, as in (32.5). In particular, this implies that every bounded set in $c_{00}(X, k)$ with respect to τ_r is bounded with respect to the topology induced on $c_{00}(X, k)$ by the strong product topology on $c(X, k)$. If $\mathcal{E}(\delta_y)$ is a bounded subset of V for each $y \in X$, then it follows that $\mathcal{E}$ is uniformly bounded on bounded subsets of $c_{00}(X, k)$ with respect to τ_r, because of the analogous statement for the topology induced on $c_{00}(X, k)$ by the strong product topology on $c(X, k)$ mentioned earlier. Note that every bounded set in $c_{00}(X, k)$ with respect to the topology induced by the strong product topology on $c(X, k)$ is also bounded with respect to τ_r, because of the characterization of these bounded sets in Section 23. Suppose now that there is a local base for the topology of V at 0 consisting of balanced open sets that satisfy the r-convexity condition discussed in Section 27. If $\mathcal{E}(\delta_y)$ is a bounded set in V for each $y \in X$, then one can check that $\mathcal{E}$ is equicontinuous with respect to τ_r on $c_{00}(X, k)$. This is very similar to the argument in Section 32 for the continuity of linear mappings from $c_{00}(X, k)$ into V with respect to τ_r on $c_{00}(X, k)$ under these conditions.

45 Equicontinuous linear functionals

Let k be a field with a q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let V be a topological vector space over k with respect to $|\cdot|$ on k. As usual, we may consider k as a one-dimensional topological vector space over itself, with respect to the topology determined by the q_k-metric (4.8) associated to $|\cdot|$ on k. Let $\mathcal{E}$ be a collection of linear functionals on V. If $\mathcal{E}$ is equicontinuous on V, then there is an open set $U \subseteq V$ such that $0 \in U$ and

$$|\lambda(u)| < 1 \tag{45.1}$$

for every $\lambda \in \mathcal{E}$ and $u \in U$. Conversely, suppose that there is an open set $U \subseteq V$ such that $0 \in U$ and (45.1) holds for every $\lambda \in \mathcal{E}$ and $u \in U$. If $|\cdot|$ is trivial on k, then (45.1) implies that

$$\lambda(u) = 0, \tag{45.2}$$

and it is easy to see that $\mathcal{E}$ is equicontinuous on V. If $|\cdot|$ is nontrivial on k, then one can also check that the previous condition implies that $\mathcal{E}$ is equicontinuous on V, using (30.3). In this case, $\mathcal{E}$ is uniformly strongly bounded on V, as in Section 43.

Let X be a nonempty set, let $c(X, k)$ be the space of k-valued functions on X, and let $c_{00}(X, k)$ be the linear subspace of $c(X, k)$ consisting of functions with finite support on X, as in Section 1. Remember that $c(X, k)$ is a topological vector space over k with respect to the topology determined by (11.7). If g is an element of $c_{00}(X, k)$, then

$$\lambda_g(f) = \sum_{x \in X} f(x)\, g(x) \tag{45.3}$$

defines a continuous linear functional on $c(X, k)$. Every continuous linear functional on $c(X, k)$ is of this form, as in Section 30. Note that

$$\lambda_g(\delta_y) = g(y) \tag{45.4}$$

for every $g \in c_{00}(X, k)$ and $y \in X$, where $\delta_y \in c_{00}(X, k)$ is as defined in (1.3). Let G be a subset of $c_{00}(X, k)$, and put

$$\mathcal{E}_G = \{\lambda_g : g \in G\}. \tag{45.5}$$

Thus $\mathcal{E}_G$ is a collection of continuous linear functionals on $c(X, k)$, and every collection of continuous linear functionals on $c(X, k)$ is of this form.

Suppose for the moment that $|\cdot|$ is trivial on k. If there is a finite set $X_0 \subseteq X$ such that

$$\operatorname{supp} g \subseteq X_0 \tag{45.6}$$

for every $g \in G$, then it is easy to see that $\mathcal{E}_G$ is equicontinuous on $c(X, k)$. Conversely, suppose that $\mathcal{E}_G$ is equicontinuous on $c(X, k)$. This implies that there is an open set U in $c(X, k)$ such that $0 \in U$ and

$$\lambda_g(f) = 0 \tag{45.7}$$

for every $g \in G$ and $f \in U$, as before. In this situation, we get that there is a finite set $X_0 \subseteq X$ such that

$$\{f \in c(X,k) : f(x) = 0 \text{ for every } x \in X_0\} \subseteq U, \tag{45.8}$$

as in (30.7). In particular, if $y \in X \setminus X_0$, then (45.8) implies that $\delta_y \in U$. It follows that

$$g(y) = \lambda_g(\delta_y) = 0 \tag{45.9}$$

for every $g \in G$ and $y \in X \setminus X_0$, using (45.4) in the first step, and (45.7) in the second step. This implies that (45.6) holds for every $g \in G$.

Suppose now that $|\cdot|$ is nontrivial on k. If $\mathcal{E}_G$ is equicontinuous on $c(X,k)$, then there is an open set U in $c(X,k)$ such that $0 \in U$ and

$$|\lambda_g(f)| < 1 \tag{45.10}$$

for every $g \in G$ and $f \in U$. As before, there is a finite set $X_0 \subseteq X$ such that (45.8) holds, because of the way that the topology is defined on $c(X,k)$. If f is an element of the left side of (45.8), then $t\,f$ is an element of the left side of (45.8) for every $t \in k$, so that $t\,f \in U$ for every $t \in k$. This implies that

$$|t|\,|\lambda_g(f)| = |\lambda_g(t\,f)| < 1 \tag{45.11}$$

for every $g \in G$ and $t \in k$, and hence that (45.7) holds for every $g \in G$, because $|\cdot|$ is nontrivial on k. It follows that (45.6) holds for every $g \in G$, using (45.9), as before. The equicontinuity of $\mathcal{E}_G$ on $c(X,k)$ also implies that $\mathcal{E}_G$ is bounded pointwise on $c(X,k)$, as in Section 43. This implies in turn that G is bounded pointwise on X, by (45.4). Conversely, if there is a finite set $X_0 \subseteq X$ such that (45.6) holds for every $g \in G$, and if G is bounded pointwise on X, then one can check that $\mathcal{E}_G$ is equicontinuous on $c(X,k)$. Of course, the pointwise boundedness of G on X is the same as the pointwise boundedness of G on X_0 when (45.6) holds for every $g \in G$.

46 Back to $c_{00}(X,k)$

Let k be a field with a nontrivial q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. As in Section 1, $c(X,k)$ denotes the space of k-valued functions on X, and $c_{00}(X,k)$ is the linear subspace of $c(X,k)$ consisting of functions with finite support in X. Let $\delta_y \in c_{00}(X,k)$ be as in (1.3) for each $y \in X$, and remember that the collection of δ_y with $y \in X$ is a basis for $c_{00}(X,k)$ as a vector space over k. If $g \in c(X,k)$, then

$$\lambda_g(f) = \sum_{x \in X} f(x)\,g(x) \tag{46.1}$$

defines a linear functional on $c_{00}(X,k)$, and every linear functional on $c_{00}(X,k)$ is of this form, as in Section 36. Note that

$$\lambda_g(\delta_y) = g(y) \tag{46.2}$$

for every $g \in c(X,k)$ and $y \in X$, as in (45.4), although the setting here is a bit different. If G is a subset of $c(X,k)$, then

$$\mathcal{E}_G = \{\lambda_g : g \in G\} \tag{46.3}$$

is a collection of linear functionals on $c_{00}(X,k)$, and every collection of linear functionals on $c_{00}(X,k)$ is of this form.

If $\mathcal{E}_G$ is bounded pointwise on $c_{00}(X,k)$, then G is bounded pointwise on X, by (46.2). Conversely, suppose that G is bounded pointwise on X. This means that there is a nonnegative real-valued function w on X such that

$$|g(x)| \le w(x) \tag{46.4}$$

for every $g \in G$ and $x \in X$. Using this, one can check that

$$|\lambda_g(f)| \le \|f\|_{q_k,w} \tag{46.5}$$

for every $g \in G$ and $f \in c_{00}(X,k)$, where the right side of (46.5) is as in (20.1) when $q_k = \infty$, and as in (26.1) when $q_k < \infty$. This simple estimate is similar to (36.2), and one can also derive (46.5) from (36.2).

It follows from (46.5) that $\mathcal{E}_G$ is equicontinuous with respect to the topology τ_{q_k} defined on $c_{00}(X,k)$ as in Section 32. This could also be obtained from the discussion in Section 44, but it is easier to use (46.5). Note that (46.5) implies immediately that $\mathcal{E}_G$ is bounded pointwise on $c_{00}(X,k)$. Similarly, let E be a subset of $c_{00}(X,k)$ that is bounded pointwise on X, and for which the supports of the elements of E are contained in a finite subset of X. It is easy to see that $\|f\|_{q_k,w}$ is bounded on E, so that $\mathcal{E}_G$ is uniformly bounded on E, by (46.5). If $E \subseteq c_{00}(X,k)$ is bounded with respect to the topology induced by the product topology on $c(X,k)$, as in Section 19, then we have seen in Section 23 that E has the properties just mentioned. In particular, if E is bounded in $c_{00}(X,k)$ with respect to τ_{q_k}, then E has these properties, because τ_{q_k} contains the topology induced on $c_{00}(X,k)$ by the strong product topology on $c(X,k)$, as in Section 32.

Suppose now that $q_k < \infty$, and let X_1 be a subset of X with only finitely or countably many elements. Of course, if X has only finitely or countably many elements, then one can simply take $X_1 = X$. Suppose also that

$$\operatorname{supp} g \subseteq X_1 \tag{46.6}$$

for every $g \in G$, so that we can take w in (46.4) to satisfy

$$\operatorname{supp} w \subseteq X_1 \tag{46.7}$$

as well. Under these conditions, there is another nonnegative real-valued function $\widetilde{w}$ on X supported in X_1 such that

$$\|f\|_{q_k,w} \le \|f\|_{\infty,\widetilde{w}} \tag{46.8}$$

for every $f \in c_{00}(X, k)$. To see this, one can basically reduce to the case where $X = X_1$, because it suffices to verify (46.8) when f is supported in X_1. Thus one can get (46.8) as in (26.7) when X_1 has finitely many elements, and as in (26.11) when X_1 is countably infinite. It follows that

$$(46.9) \qquad |\lambda_g(f)| \le \|f\|_{\infty, \widetilde{w}}$$

for every $g \in G$ and $f \in c_{00}(X, k)$, by combining (46.5) and (46.9). This implies that (46.9) that $\mathcal{E}_G$ is equicontinuous on $c_{00}(X, k)$ with respect to the topology τ_∞ defined in Section 32, which is the same as the topology induced on $c_{00}(X, k)$ by the strong product topology on $c(X, k)$. This could also be derived from remarks in Section 44 when X has only finitely or countably many elements, and otherwise one can reduce to that situation. As in Section 36, one can treat λ_g on $c_{00}(X, k)$ as the composition of the analogous linear functional on $c_{00}(X_1, k)$ with the natural restriction mapping from $c_{00}(X, k)$ onto $c_{00}(X_1, k)$. This permits the equicontinuity of $\mathcal{E}_G$ on $c_{00}(X, k)$ to be obtained from the equicontinuity of the analogous collection of linear functionals on $c_{00}(X_1, k)$.

47 Back to $c_{00}(X, k)$, continued

In this section, we take $k = \mathbf{R}$ or $\mathbf{C}$ with the standard absolute value function, so that we can take $q_k = 1$. Let X be a nonempty set again, and let $c(X, k)$ and $c_{00}(X, k)$ be as in Section 1, as usual. Each $g \in c(X, k)$ determines a linear functional λ_g on $c_{00}(X, k)$ as in (46.1), and a subset G of $c(X, k)$ determines a collection of linear functionals on $c_{00}(X, k)$ as in (46.3). If G is bounded pointwise on X, then there is a nonnegative real-valued function w on X that satisfies (46.4), which implies that (46.5) holds with $q_k = 1$. In particular, this implies that $\mathcal{E}_G$ is equicontinuous with respect to the topology τ_1 defined on $c_{00}(X, k)$ in Section 32.

Now let $1 < r \le \infty$ be given, and suppose that G is a subset of $c(X, k)$ such that $\mathcal{E}_G$ is equicontinuous on $c_{00}(X, k)$ with respect to the topology τ_r defined in Section 32. This implies that there is a positive real-valued function w on X such that

$$(47.1) \qquad |\lambda_g(f)| \le \|f\|_{r,w}$$

for every $g \in G$ and $f \in c_{00}(X, k)$, where $\|f\|_{r,w}$ is as in (20.1) or (26.1). More precisely, the equicontinuity of $\mathcal{E}_G$ with respect to τ_r on $c_{00}(X, k)$ implies a condition like (42.7), which corresponds in this situation to having a constant times the maximum of finitely many weighted ℓ^r norms on the right side of (47.1). As in (37.1), one can get (47.1) by taking w to be the same constant times the maximum of the finitely many weights on X just mentioned. Using (47.1), one can check that

$$(47.2) \qquad \sum_{x \in X} |f(x)| \, |g(x)| \, w(x)^{-1} \le \|f\|_r$$

for every $g \in G$ and $f \in c_{00}(X, k)$, as in (37.6), and where $\|f\|_r$ is the unweighted ℓ^r norm of f, as in Section 6.

Let $1 \leq r' < \infty$ be the exponent conjugate to r, in the sense that

$$1/r + 1/r' = 1. \tag{47.3}$$

In particular, $r' = 1$ when $r = \infty$, and $r' < \infty$ because $r > 1$. If A is any nonempty finite subset of X, then it is well known that (47.2) implies that

$$\Big(\sum_{x \in A} |g(x)|^{r'} \, w(x)^{-r'}\Big)^{1/r'} \leq 1 \tag{47.4}$$

for every $g \in G$, using suitable choices of functions f supported on A. Equivalently, this means that

$$\Big(\sum_{x \in X} |g(x)|^{r'} \, w(x)^{-r'}\Big)^{1/r'} \leq 1 \tag{47.5}$$

for every $g \in G$, where the sum over X on the left side of (47.5) is defined to be the supremum of the corresponding subsums over nonempty finite subsets A of X. In particular, this implies that the support of each $g \in G$ has only finitely or countably many elements, as in Section 37.

Observe that

$$|\lambda_g(f)| \leq \sum_{x \in X} |f(x)| \, |g(x)| = \sum_{x \in X} (w(x) \, |f(x)|) \, (|g(x)| \, w(x)^{-1}) \tag{47.6}$$

for every $f \in c_{00}(X, k)$ and $g \in c(X, k)$. If A is a nonempty finite subset of X that contains the support of f, then we get that

$$|\lambda_g(f)| \leq \sum_{x \in A} (w(x) \, |f(x)|) \, (|g(x)| \, w(x)^{-1}). \tag{47.7}$$

This implies that

$$|\lambda_g(f)| \leq \Big(\sum_{x \in A} |g(x)|^{r'} \, w(x)^{-r'}\Big)^{1/r'} \, \|f\|_{r,w} \tag{47.8}$$

when $\operatorname{supp} f \subseteq A$, by Hölder's inequality. If (47.5) holds for every $g \in G$, then it follows that (47.1) holds for every $g \in G$ and $f \in c_{00}(X, k)$. Of course, this implies that $\mathcal{E}_G$ is equicontinuous on $c_{00}(X, k)$ with respect to τ_r, as before.

48 Linear functionals on $c(X, k)$

Let k be a field with a nontrivial q_k-absolute value function $|\cdot|$ for some $q_k > 0$, and let X be a nonempty set. Also let $c(X, k)$ be the space of k-valued functions on X, and let $c_{00}(X, k)$ be the subspace of $c(X, k)$ consisting of functions with finite support in X, as in Section 1. If $g \in c_{00}(X, k)$, then we can define a linear functional λ_g on $c(X, k)$ as in (45.3). As in Section 30, λ_g is a continuous linear functional on $c(X, k)$ with respect to the topology determined on $c(X, k)$ by (11.7) for each $g \in c_{00}(X, k)$, and every continuous linear functional on $c(X, k)$ with respect to this topology is of this form. Let G be a subset of $c_{00}(X, k)$, and let $\mathcal{E}_G$ be the collection of linear functionals λ_g on $c(X, k)$ with $g \in G$, as in (45.5). Suppose that $\mathcal{E}_G$ is bounded pointwise on $c(X, k)$, as in Section 43, so that

$$\mathcal{E}_G(f) = \{\lambda_g(f) : g \in G\} \tag{48.1}$$

is a bounded subset of k for every $f \in c(X, k)$. Note that

$$\mathcal{E}_G(\delta_y) = \{g(y) : g \in G\} \tag{48.2}$$

for each $y \in X$, by (45.4), where $\delta_y \in c_{00}(X, k)$ is as in (1.3). Thus the pointwise boundedness of $\mathcal{E}_G$ on $c(X, k)$ implies that G is bounded pointwise on X in particular. We would like to show that the pointwise boundedness of $\mathcal{E}_G$ on $c(X, k)$ also implies that the supports of the elements of G are contained in a finite subset of X, as in (45.6). Of course, this is trivial when X has only finitely many elements.

Suppose for the sake of a contradiction that there is no finite subset of X that contains the supports of the elements of G. In this case, there is a sequence $\{g_j\}_{j=1}^\infty$ of nonzero elements of G such that

$$\operatorname{supp} g_{l+1} \not\subseteq \bigcup_{j=1}^{l} \operatorname{supp} g_j \tag{48.3}$$

for each positive integer l. This implies that there is a sequence $\{x_j\}_{j=1}^\infty$ of distinct elements of X such that x_l is in the support of g_l for each $l \in \mathbf{Z}_+$, and x_l is not in the support of g_j when $j < l$. Equivalently, this means that $g_l(x_l) \neq 0$ for each l, and that $g_j(x_l) = 0$ when $j < l$. If f is a k-valued function on X that is supported on the set of x_j's, $j \in \mathbf{Z}_+$, then we get that

$$\lambda_{g_l}(f) = \sum_{j=1}^{l} f(x_j)\, g_l(x_j) \tag{48.4}$$

for each $l \in \mathbf{Z}_+$. Using this, one can choose f such that $|\lambda_{g_l}(f)| \to \infty$ as $l \to \infty$, contradicting the hypothesis that $\mathcal{E}_G$ be bounded pointwise on $c(X, k)$. Thus the pointwise boundedness of $\mathcal{E}_G$ on $c(X, k)$ implies that the supports of the elements of G are contained in a finite subset of X, as desired. We have also seen that G is bounded pointwise on X in this situation, so that $\mathcal{E}_G$ is equicontinuous on $c(X, k)$, as in Section 45.

Suppose for the moment that X has only finitely or countably many elements. This implies that the topology determined on $c(X, k)$ by (11.7) is also determined by a translation-invariant q_k-metric, as in Sections 9 and 12. If k is complete with respect to the q_k-metric (4.8) associated to $|\cdot|$, then one can check that $c(X, k)$ is complete with respect to any translation-invariant q_k-metric that determines the same topology on $c(X, k)$. Under these conditions, the Baire category theorem implies that $c(X, k)$ is of second category. Thus the equicontinuity of $\mathcal{E}_G$ could be obtained from the pointwise boundedness of $\mathcal{E}_G$ on $c(X, k)$ using the Banach–Steinhaus theorem in this situation.

If k is not complete, then one can pass to a completion. It is easy to see that the pointwise boundedness of $\mathcal{E}_G$ on $c(X, k)$ would be maintained in passing to a completion of k. If X is uncountable, then one can consider the restrictions of the elements of $\mathcal{E}_G$ to the subspace of $c(X, k)$ consisting of functions with support contained in a countable set $X_1 \subseteq X$. This subspace can be identified with the space $c(X_1, k)$ of k-valued functions on X_1, and the equicontinuity of the restrictions of the elements of $\mathcal{E}_G$ to this subspace implies that the supports of the restrictions of the elements of G to X_1 are contained in a finite subset of X_1, as before. If this holds for every countable set $X_1 \subseteq X$, then the supports of the elements of G should be contained in a finite subset of X.

49 Continuity of multiplication

Let k be a field with a q_k-absolute value function $|\cdot|$, and let X be a nonempty set. As usual, we let $c(X, k)$ be the space of k-valued functions on X, equipped with the topology determined by (11.7). This makes $c(X, k)$ into a topological vector space over k with respect to $|\cdot|$ on k, and $c(X, k)$ is also a commutative algebra over k with respect to pointwise multiplication of functions. It is easy to see that

$$(f, g) \mapsto f\,g \tag{49.1}$$

defines a continuous mapping from

$$c(X, k) \times c(X, k) \tag{49.2}$$

into $c(X, k)$, using this topology on $c(X, k)$, and the corresponding product topology on (49.2). Thus $c(X, k)$ may be considered as a commutative topological algebra with respect to this topology.

As in Section 1, the subspace $c_{00}(X, k)$ of $c(X, k)$ consisting of functions with finite support in X is an ideal in $c(X, k)$ as a commutative algebra. Thus (49.1) also leads to a mapping from $c(X, k) \times c_{00}(X, k)$ into $c_{00}(X, k)$. Let $0 < r \leq \infty$ be given, and let τ_r be the corresponding topology defined on $c_{00}(X, k)$ as in Section 32. Let us take

$$c(X, k) \times c_{00}(X, k) \tag{49.3}$$

to be equipped with the product topology associated to the topology on $c(X, k)$ mentioned in the preceding paragraph and τ_r on $c_{00}(X, k)$. If X has infinitely many elements, and if $|\cdot|$ is nontrivial on k, then one can check that (49.1) is not continuous as a mapping from (49.3) into $c_{00}(X, k)$ with respect to these topologies. However, if X has only finitely many elements, then $c_{00}(X, k)$ is the same as $c(X, k)$, τ_r is the same as the topology on $c(X, k)$ mentioned in the previous paragraph, and the continuity of this mapping is the same as before. If $|\cdot|$ is trivial on k, then τ_r is the same as the discrete topology on $c_{00}(X, k)$

for every $r > 0$, and one can verify that (49.1) is continuous as a mapping from (49.3) into $c_{00}(X, k)$, even when X has infinitely many elements.

Let $0 < r_1, r_2, r_3 \leq \infty$ be given, and suppose that

$$1/r_3 = 1/r_1 + 1/r_2, \tag{49.4}$$

with the usual convention that $1/\infty = 0$. Remember that $\|f\|_r$ is defined for nonnegative real-valued functions on X with finite support and $r > 0$ in (2.1) and (2.2). If f_1, f_2 are nonnegative real-valued functions on X with finite supprt, then it is well known that

$$\|f_1\, f_2\|_{r_3} \leq \|f_1\|_{r_1}\, \|f_2\|_{r_2}, \tag{49.5}$$

by Hölder's inequality. Remember too that $\|f\|_{r,w}$ is defined for $f \in c_{00}(X, k)$, $r > 0$, and nonnegative real-valued functions w on X as in (20.1) and (26.1). Let w_1, w_2 be nonnegative real-valued functions on X, and put

$$w_3(x) = w_1(x)\, w_2(x) \tag{49.6}$$

for each $x \in X$. If $f_1, f_2 \in c_{00}(X, k)$, then

$$\|f_1\, f_2\|_{r_3,w_3} \leq \|f_1\|_{r_1,w_1}\, \|f_2\|_{r_2,w_2}. \tag{49.7}$$

This can be obtained from (49.5) applied to $|f_1(x)|\, w_1(x)$ and $|f_2(x)|\, w_2(x)$.

Let us now consider (49.1) as a mapping from

$$c_{00}(X, k) \times c_{00}(X, k) \tag{49.8}$$

into $c_{00}(X, k)$. More precisely, let $r_1, r_2 > 0$ be given, and let us take (49.8) to be equipped with the product topology corresponding to τ_{r_1} and τ_{r_2} on the two factors of $c_{00}(X, k)$, where τ_r is as in Section 32. Similarly, let $r_3 > 0$ be given, and let us take $c_{00}(X, k)$ in the range of (49.1) to be equipped with τ_{r_3}. If (49.4) holds, then one can use (49.7) to show that (49.1) is continuous with respect to these topologies. In this argument, one should begin with a positive real-valued function w_3 on X, and choose positive real-valued functions w_1, w_2 on X so that (49.6) holds.

50 Multiplication operators

Let us continue with the same basic notation and hypotheses as in the previous section. If $a \in c(X, k)$, then

$$M_a(f) = a\, f \tag{50.1}$$

defines a linear mapping from $c(X, k)$ into itself, as in (1.8). It is easy to see that this mapping is continuous with respect to the topology determined on $c(X, k)$ by (11.7). Let A be a subset of $c(X, k)$, and consider

$$\mathcal{E}_A = \{M_a : a \in A\} \tag{50.2}$$

as a collection of linear mappings from $c(X, k)$ into itself. If $|\cdot|$ is trivial on k, then one can check that (50.2) is equicontinuous on $c(X, k)$, even when $A = c(X, k)$. Otherwise, if $|\cdot|$ is not trivial on k, and if A is bounded pointwise on X, then one can verify that $\mathcal{E}_A$ is equicontinuous on $c(X, k)$. In the other direction, if $|\cdot|$ is not trivial on k, and if $\mathcal{E}_A$ is bounded pointwise on $c(X, k)$, then A is bounded pointwise on X. One way to look at this is to use the fact that

$$M_a(\mathbf{1}_X) = a \tag{50.3}$$

for every $a \in c(X, k)$, where $\mathbf{1}_X$ is the k-valued function on X equal to 1 at every point, as in (1.1).

We can also consider (50.1) as a linear mapping from $c_{00}(X, k)$ into itself for each $a \in c(X, k)$. Let $0 < r \leq \infty$ be given, and let w be a nonnegative real-valued function on X. Thus $\|f\|_{r,w}$ can be defined for $f \in c_{00}(X, k)$ as in (20.1) and (26.1). Observe that

$$\|M_a(f)\|_{r,w} = \|a\, f\|_{r,w} = \|f\|_{r,|a|\, w} \tag{50.4}$$

for each $a \in c(X, k)$ and $f \in c_{00}(X, k)$, using $|a(x)|\, w(x)$ as a nonnegative real-valued function on X in the last step. It follows from (50.4) that for each $a \in c(X, k)$, M_a is continuous as a linear mapping from $c_{00}(X, k)$ into itself, with respect to the topology τ_r defined in Section 32 on both the domain and range of this mapping.

Let A be a subset of $c(X, k)$ again, and let us now consider (50.2) as a collection of linear mappings on $c_{00}(X, k)$. If $|\cdot|$ is trivial on k, then the topology τ_r defined on $c_{00}(X, k)$ as in Section 32 reduces to the discrete topology for each $r > 0$. Thus $\mathcal{E}_A$ is automatically equicontinuous on $c_{00}(X, k)$ with respect to τ_r for each $r > 0$ in this case. Otherwise, suppose for the moment that $|\cdot|$ is not trivial on k. If $\mathcal{E}_A$ is bounded pointwise on $c_{00}(X, k)$, then it is easy to see that A is bounded pointwise on X, using (1.9). In the other direction, suppose that A is bounded pointwise on X, so that there is a nonnegative real-valued function w_A on X such that

$$|a(x)| \leq w_A(x) \tag{50.5}$$

for every $a \in A$ and $x \in X$. If w is any nonnegative real-valued function on X and $0 < r \leq \infty$, then we can combine (50.4) and (50.5) to get that

$$\|M_a(f)\|_{r,w} \leq \|f\|_{r,w_A\, w} \tag{50.6}$$

for each $f \in c_{00}(X, k)$. This implies that for each $r > 0$, $\mathcal{E}_A$ is equicontinuous on $c_{00}(X, k)$ with respect to τ_r on both the domain and range under these conditions.

If $a \in c_{00}(X, k)$, then (50.1) may be considered as a linear mapping from $c(X, k)$ into $c_{00}(X, k)$. One can check that this mapping is continuous with respect to the topology determined on $c(X, k)$ by (11.7) and the topology τ_r defined on $c_{00}(X, k)$ in Section 32 for any $r > 0$. Let A be a subset of $c_{00}(X, k)$, and let us consider (50.2) as a collection of linear mappings from $c(X, k)$ into

$c_{00}(X, k)$. If $|\cdot|$ is trivial on k, then τ_r is the discrete topology on $c_{00}(X, k)$ for each $r > 0$, as before. In this case, one can verify that $\mathcal{E}_A$ is equicontinuous with respect to the topology determined on $c(X, k)$ by (11.7) if and only if the supports of the elements of A are contained in a finite subset of X. Suppose now that $|\cdot|$ is not trivial on k. If $\mathcal{E}_A$ is bounded pointwise on $c(X, k)$ with respect to τ_r on $c_{00}(X, k)$ for some $r > 0$, then

$$\mathcal{E}_A(\mathbf{1}_X) = \{M_a(\mathbf{1}_X) : a \in A\} = A \tag{50.7}$$

is a bounded set in $c_{00}(X, k)$ with respect to τ_r. This implies that A is bounded pointwise on X, and that the supports of the elements of A are contained in a finite subset of X. Conversely, if A has these two properties, then it is easy to see that $\mathcal{E}_A$ is equicontinuous with respect to the topology determined on $c(X, k)$ by (11.7) and τ_r on $c_{00}(X, k)$ for any $r > 0$.

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