Moduli spaces of sheaves on a K3 surface and Galois representations

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Outline

Overview

- Given a K3 surface S defined over an arbitrary field k, we can form various moduli spaces of sheaves $\mathcal{M}/k$
- Consider the base change $\overline{\mathcal{M}} := \mathcal{M} \times_k \bar{k}$, which has a natural action of $\mathrm{Gal}(\bar{k}/k)$
- For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, we can study the induced action on cohomology:

$$\sigma^* \colon H^i(\overline{\mathcal{M}}, \mathbb{Q}_\ell) \to H^i(\overline{\mathcal{M}}, \mathbb{Q}_\ell)$$

Question: Given two moduli spaces $\mathcal{M}_1$, $\mathcal{M}_2$, how are the resulting Galois representations related?

Background

Definition

A **K3 surface** S/k is a smooth projective variety of dimension 2 such that $\omega_S = \mathscr{O}_S$ and $H^1(S, \mathscr{O}_S) = 0$.

Examples:

1. $S = \left\{(x : y : z : w) \in \mathbb{P}^3_k : x^4 + y^4 + z^4 + w^4 = 0\right\}$ for $\operatorname{char} k \neq 2$
2. Any smooth quartic $S = V(f) \subset \mathbb{P}^3_k$

Remark: Abelian varieties also have $\omega_X = \mathscr{O}_X$, so you can think of K3 surfaces as 2-dimensional generalizations of elliptic curves.

Fact: $/\mathbb{C}$,

$$H^i(S, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^{22} & i = 2 \\ \mathbb{Z} & i = 4 \end{cases}$$

Outline

The Hilbert scheme of points

Definition

The Hilbert scheme of points on S, $\mathrm{Hilb}^n S = S^{[n]}$ parameterizes 0-dimensional subschemes $Z \subset S$ of length n, i.e. $\dim H^0(Z, \mathscr{O}_Z) = n$.

Example: Points of $S^{[2]}$ are of the following form:

- If $k = \bar{k}$:
  - pairs of points $p_1, p_2 \in S$
  - a point $p \in S$ with a tangent direction
- If $k \neq \bar{k}$, there are more points:
  e.g. if $k = \mathbb{F}_q$, also have $p \in S(\mathbb{F}_{q^2}) \setminus S(\mathbb{F}_q)$

Question: What is $H^*(S^{[n]})$?

Answer: (Göttsche, '90) Use the Weil Conjectures

Zeta functions and the Weil Conjectures

Definition

For X a smooth projective variety over $\mathbb{F}_q$, the **zeta function of X** is

$$Z(X,t) := \exp\left(\sum_{r\geq 1} \#X(\mathbb{F}_{q^r})\frac{t^r}{r}\right)$$

The Weil conjectures state that $Z(X,t)$ is a rational function, it satisfies a functional equation, it has prescribed zeros, and gives a comparison to singular cohomology

Aside: Can also define $\zeta_X(s) := \prod_{x\in X \text{closed}} \frac{1}{1-|k(x)|^{-s}}$, and then

- $\zeta_X(s) = Z(X, q^{-s})$
- If $X = \operatorname{Spec}\mathbb{Z}$, then $\zeta_X(s) = \zeta(s)$, the Riemann zeta function

Connection to cohomology

Let $F : \overline{X} \to \overline{X}$ be the absolute Frobenius morphism (q^{th} power map on the structure sheaf).

By the Lefschetz fixed point theorem,

$$\#X(\mathbb{F}_{q^r}) = \text{fixed points of } F^r = \sum_{i \geq 0} (-1)^i \operatorname{tr}\left(F^{r*}|_{H^i(\overline{X}, \mathbb{Q}_\ell)}\right)$$

This can be plugged into $Z(X,t) = \exp\left(\sum_{r \geq 1} \#X(\mathbb{F}_{q^r}) \frac{t^r}{r}\right)$

Example: $S^{[2]}$

Let $\#S(\mathbb{F}_{q^r}) = N_r = 1 + \sum_{i=1}^{22} \alpha_i^r + q^{2r}$ where $|\alpha_i| = q$. Then

$$\#S^{[2]}(\mathbb{F}_{q^r}) = \binom{N_r}{2} + N_r(q^r + 1) + \frac{N_{2r} - N_r}{2}$$

$$Z(S^{[2]}, t) = [(1-t)\prod_{i=1}^{22}(1-\alpha_i t)(1-qt) \prod_{1 \le i \le j \le 22} (1-\alpha_i\alpha_j t)\prod_{i=1}^{22}(1-\alpha_i qt)$$
$$\cdot (1-q^2 t)\prod_{i=1}^{22}(1-\alpha_i q^2 t)(1-q^3 t)(1-q^4 t)]^{-1}$$

Conclusion:

$$H^i(S^{[2]}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & i = 0, 8 \\ \mathbb{Q}_\ell^{23} & i = 2, 6 \\ \mathbb{Q}_\ell^{276} & i = 4 \end{cases}$$

Generalizations

Fact: $S^{[n]}$ parameterizes rank 1 sheaves on S:
To a 0-dimensional subscheme $Z \subset S$ we can associate the ideal sheaf $\mathcal{I}_Z \subset \mathscr{O}_S$

Generalize this:

1. For other moduli spaces of sheaves on S, what is $Z(\mathcal{M}, t)$?
2. Consider S defined over an arbitrary field k, and study the Galois action in place of the Frobenius action

Outline

Definitions

Fix a K3 surface S defined over an arbitrary field k.

Definition

For a coherent sheaf $\mathscr{F}$ on S, the **Mukai vector** of $\mathscr{F}$ is

$$\begin{aligned} v(\mathscr{F}) &= \mathrm{ch}(\mathscr{F})\sqrt{\mathrm{td}\, S} \\ &= (\mathrm{rk}\,\mathscr{F}, c_1(\mathscr{F}), \chi(\mathscr{F}) - \mathrm{rk}\,\mathscr{F}) \end{aligned}$$

in $H^0(S) \oplus H^2(S) \oplus H^4(S)$.

Definition

For $v \in H^*(S)$, the **moduli space of stable sheaves on S**, $\mathcal{M} = \mathcal{M}(v)$ parameterizes isomorphism classes of pure sheaves $\mathscr{F}$ on S with $v(\mathscr{F}) = v$, satisfying a stability condition.

Background

Examples:

1. $\mathcal{M}(1, 0, 1-n) \cong S^{[n]}$
2. $\mathcal{M}(0, 1, d+1-g)$: the general element is a degree d line bundle on a genus g curve in S

Facts:

1. For v geometrically primitive, $\mathcal{M}(v)$ is a smooth projective variety
2. If $\dim \mathcal{M}(v) = 2$, then $\mathcal{M}(v)$ is again a K3 surface
3. If $\dim \mathcal{M}(v) = 2n$, then $\mathcal{M}(v)$ is deformation equivalent to $S^{[n]}$, but it need not be birational to it

Results

Theorem (F., '18)

Let S be a K3 surface defined over a finite field. Let $\mathcal{M}(v_1)$ and $\mathcal{M}(v_2)$ be moduli spaces of stable sheaves on S with v_1, v_2 geometrically primitive such that $\dim \mathcal{M}(v_1) = \dim \mathcal{M}(v_2)$. Then

$$Z(\mathcal{M}(v_1), t) = Z(\mathcal{M}(v_2), t).$$

Theorem (F., '18)

Let S be a K3 surface defined over an arbitrary field. Let $\mathcal{M}(v_1)$ and $\mathcal{M}(v_2)$ be moduli spaces of stable sheaves on S with v_1, v_2 geometrically primitive such that $\dim \mathcal{M}(v_1) = \dim \mathcal{M}(v_2)$. Then $H^i(\overline{\mathcal{M}(v_1)}, \mathbb{Q}_\ell) \cong H^i(\overline{\mathcal{M}(v_2)}, \mathbb{Q}_\ell)$ as Galois representations for all $i \geq 0$.

A key tool in the proof: Lifting to characteristic zero

Definition

For k perfect with char $k = p > 0$, the **ring of Witt vectors** $W(k)$ is a complete discrete valuation ring of characteristic zero with residue field k.

Examples:

- If $k = \mathbb{F}_p$, then $W(k) = \mathbb{Z}_p$
- If $k = \overline{\mathbb{F}}_p$, then $W(k)$ is the ring of integers in $\mathrm{Frac}(\widehat{\mathbb{Q}_p^{un}})$, the completion of the maximal unramified extension of $\mathbb{Q}_p$

A key tool in the proof: Lifting to characteristic zero

Proposition (Charles, '16)

Let S/k be a K3 surface over an algebraically closed field with $\operatorname{char} k = p > 0$, and let $L_1, \ldots, L_r$ be line bundles on S with L_1 ample. If $r \leq 10$, then there exists a complete DVR $W(k) \subset W'$, finite over $W(k)$, and a smooth projective relative K3 surface $\mathcal{S} \to \operatorname{Spec} W'$ such that

- $\mathcal{S}_0 \cong S$, *and*
- *The image of the specialization map $Pic(\mathcal{S}) \to Pic(S)$ contains $L_1, \ldots, L_r$.*

Lifting to characteristic zero

For S/k a K3 surface with char $k = p > 0$, we can **lift $\overline{S}$ to characteristic zero**:

Let η be the generic point of Spec W'. Then we have

$$\begin{tikzcd}
\mathcal{S}_\eta \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] & \mathcal{S}_0 \cong \overline{S} \arrow[l] \arrow[d] \\
\operatorname{Spec} \operatorname{Frac} W' \arrow[r] & \operatorname{Spec} W' & \operatorname{Spec} \bar{k} \arrow[l]
\end{tikzcd}$$

Upshot: $\mathcal{S}_\eta$ is a K3 surface defined over a field of characteristic zero, at which point Hodge theory and results over $\mathbb{C}$ become accessible.

Lifting the moduli space as well

- Get a lift of an ample class from $\overline{S}$ to $\mathcal{S}$, which is needed for the stability condition
- Get a lift of the Mukai vector from $\overline{S}$ to $\mathcal{S}$, so can form the **relative moduli space** of stable sheaves on $\mathcal{S} \to \operatorname{Spec} W'$, whose generic fiber is again defined in characteristic zero
- These moduli spaces have been studied extensively over $\mathbb{C}$, as they are primary examples of holomorphic symplectic manifolds

References

- Charles, François. "Birational boundedness for holomorphic symplectic varieties, Zarhin's trick for $K3$ surfaces, and the Tate conjecture." *Ann. of Math. (2)*, 184(2):487-526, 2016.
- Göttsche, Lothar. "The Betti numbers of the Hilbert scheme of points on a smooth projective surface." *Math. Ann.*, 286(1-3):193-207, 1990.
- Eyal Markman. "Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces." *Journal fur die reine und angewandte Mathematik*, pages 61-82, 2002.
- Eyal Markman. "On the monodromy of moduli spaces of sheaves on K3 surfaces." *J. Algebraic Geom.*, 17(1):29-99, 2008.