# Math 541 - Spring 2019
# $L^2$ Invariants in Topology and Geometry

Instructor: Shelly Harvey

In algebraic topology, one considers invariants of finite CW-complexes like Euler characteristic, signature, and Betti numbers. However, often this doesn't tell you much about the space. Instead of studying the space $X$, one can study the its universal cover $\widetilde{X}$ and consider the algebraic invariants of $\widetilde{X}$. If the fundamental group of $X$, $G = \pi_1(X)$, is finite, then $\widetilde{X}$ is still a finite CW-complex and everything works as usual. However, it is often the case that $G$ is infinite in which these invariants don't make sense (or are infinite). For example, let $X = S^1 \times S^2$ be a knot complement. Then $H_1(\widetilde{X}) \cong \mathbb{Z}^\infty$. Since $G \cong \mathbb{Z}$ is isomorphic to the group of deck translation of $\widetilde{X} \to X$, there is a left $\mathbb{Z} = \langle t \rangle$ action on $\widetilde{X}$ making $H_1(\widetilde{X})$ into a left $\mathbb{Z}[t^{\pm 1}]$-module and we can easily verify that $H_1(\tilde{X}) \cong \mathbb{Z}[t^{\pm 1}]$ is a free module of rank 1. More generally, if $X$ is a finite CW-complex with $G = \mathbb{Z}^m$ then $H_1(\tilde{X})$ is a finitely generated module over the Laurent polynomial ring in $m$ variables hence it has a well-defined and finite rank. We could define the $L^2$ Betti number of this space to be the rank of this module. For general $G$, $H_1(\widetilde{X})$ is a left model over the group ring $\mathbb{Z}[G]$. In most cases, this ring is quite difficult to work with. For example, it is not Noetherian and finitely generated modules over it don't have a nice well-defined notion of "rank." The remedy to this is to pass to the $L^2$ completion to obtain Hilbert spaces and use powerful tools from functional analysis to define $L^2$-invariants. These invariants have many connections to group theory, differential geometry, ergodic theory, K-theory, and low-dimensional topology.

We will start by discussing the basics of group von Neumann algebras of countable discrete groups including canonical trace, the Fuglede-Kadison determinant, and the Ore localization. We will uses these to to define $L^2$ algebraic topological invariants including $L^2$-Betti and $L^2$-torsion, and discuss their applications to geometry and topology. If we have time, we will define $L^2$-signatures and discuss their applications. We will loosely be following the survey article "$L^2$-invariants of regular coverings of compact manifolds and CW-complexes" Wolfgang Lück, found in pages 735–817 of the Handbook of Geometric Topology from 2002.

I will assume that each student is familiar with fundamental groups, covering spaces, finitely presented groups, homology, cohomology, basic homological algebra, Poincare duality, and modules over general rings. I will assume no knowledge of functional analysis and cover the necessary material in class. Depending on your background, you may need to do some background reading outside of class to fill in details.

## Course Information

Instructor: Shelly Harvey
Time: MWF 10-11am
Text: "$L^2$-invariants of regular coverings of compact manifolds and CW-complexes" Wolfgang Lück, found in pages 735–817 of the Handbook of Geometric Topology from 2002
Location: TBA
Prerequisites: Math 444/539, Math 445/540, and Math 463/563.