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Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm

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Abstract

We define an infinite sequence of new invariants, δ_n, of a group G that measure the size of the successive quotients of the derived series of G. In the case that G is the fundamental group of a 3-manifold, we obtain new 3-manifold invariants. These invariants are closely related to the topology of the 3-manifold. They give lower bounds for the Thurston norm which provide better estimates than the bound established by McMullen using the Alexander norm. We also show that the δ_n give obstructions to a 3-manifold fibering over S^1 and to a 3-manifold being Seifert fibered. Moreover, we show that the δ_n give computable algebraic obstructions to a 4-manifold of the form $X \times S^1$ admitting a symplectic structure even when the obstructions given by the Seiberg–Witten invariants fail. There are also applications to the minimal ropelength and genera of knots and links in S^3.

Keywords: Alexander norm; Thurston norm; Derived series

1. Introduction

1.1. Summary of results

In this paper, we define new 3-manifold invariants and show that they give new information about the topology of the 3-manifold. Given a 3-dimensional manifold X and a cohomology class $\psi \in H^1(X; \mathbb{Z})$

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we define a sequence of invariants $\delta_n(\psi)$ which arise as degrees of "higher-order Alexander polynomials". These integers measure the "size" of the successive quotients, $G_r^{(n+1)}/G_r^{(n+2)}$, of the terms of the (rational) derived series of $G=\pi_1(X)$. Loosely speaking, $\delta_n(\psi)$ is the degree of a polynomial that kills the elements of the first homology of the regular $G/G_r^{(n+1)}$-cover of X. The precise definitions are given in Section 5. In the case of knot exteriors and zero surgery on knots, these covering spaces were studied by Cochran [3] and Cochran et al. [4]. They defined similar generalized Alexander modules and were able to obtain important new results on knot concordance.

Although these invariants are defined algebraically, they have many exciting topological applications. We show that the degree δ_n of each of our family of polynomials gives a lower bound for the Thurston norm of a class in $H_2(X, \partial X; \mathbb{Z}) \cong H^1(X; \mathbb{Z})$ of a 3-dimensional manifold. We show that these invariants can give much more precise estimates of the Thurston norm than previously known computable invariants. We also show that the δ_n give obstructions to a 3-manifold fibering over a circle and to a 3-manifold being Seifert fibered. Moreover, we show that the δ_n give computable algebraic obstructions to a 4-manifold of the form $X \times S^1$ admitting a symplectic structure even when the obstructions given by the Seiberg–Witten invariants fail. Some other applications are to the minimal ropelength and genera of knots and links in S^3.

Note that $G_r^{(n+1)}/G_r^{(n+2)}$ is a module over $\mathbb{Z}[G/G_r^{(n+1)}]$. When $n=0$, $G_r^{(1)}/G_r^{(2)}$ is the classical Alexander module. Since $G/G_r^{(1)}$ is the (torsion-free) abelianization of G, $G_r^{(1)}/G_r^{(2)}$ is a module over the commutative polynomial ring in several variables $\mathbb{Z}[G/G_r^{(1)}]$. These modules have been studied thoroughly and with much success. For general n, however, these "higher-order Alexander modules" are modules over non-commutative rings. Very little was previously known in this case due to the difficulty of classifying such modules.

Let X be a compact, connected, orientable 3-manifold and let $\psi \in H^1(X; \mathbb{Z})$. There is a Poincaré duality isomorphism $H^1(X; \mathbb{Z}) \cong H_2(X, \partial X; \mathbb{Z})$. If an oriented surface F in X represents a class $[F] \in H_2(X, \partial X; \mathbb{Z})$ that corresponds to ψ under this isomorphism, we say that F is dual to ψ and vice versa. We measure the complexity of X via the Thurston norm which is defined in [33] as follows. If F is any compact connected surface, let $\chi(X)$ be its Euler characteristic and let $\chi\_(F) = |\chi(F)|$ if $\chi(F) \leqslant 0$ and equal 0 otherwise. For a surface $F = \amalg F_i$ with multiple components, let $\chi\_(F) = \Sigma \chi\_(F_i)$. Note that $-\chi(F) \leqslant \chi\_(F)$ in all cases. The *Thurston norm* of $\psi \in H^1(X; \mathbb{Z})$ is

$$\|\psi\|_T = \inf\{\chi\_(F) | F \text{ is a properly embedded oriented surface dual to } \psi\}.$$

This norm extends continuously to all of $H^1(X; \mathbb{R})$. This norm is difficult to compute except for in the simplest of examples because it is a minimum over an unknown set.

Thurston showed that the unit ball of the norm is a finite sided polyhedron and that the set of classes of $H_2(X, \partial X; \mathbb{R})$ representable by a fiber of a fibration over S^1 corresponds to lattice points lying in the cone of the union of some open faces of this polyhedron [33]. This norm has been useful in the resolution of many open questions in 3-manifolds. Gabai used the Thurston norm to show the existence of taut, finite-depth codimension one foliations 3-manifolds (see [11–13]). In particular, he shows that if X is a compact, connected, irreducible and oriented 3-manifold and F is any norm minimizing surface then there is a taut foliation of finite depth containing F as a compact leaf. Corollaries of Gabai's existence theorems are that the Property R and Poenaru conjectures are true.

In a recent paper, McMullen defined the Alexander norm of a cohomology class of a 3-manifold via the Alexander polynomial and proved that it is a lower bound for the Thurston norm [26]. This theorem has also been recently proved by Vidussi [35] using Seiberg–Witten theory and the work of Kronheimer [22,23], Kronheimer and Mrowka [24] and Meng and Taubes [27]. We prove in Section 10 that the (unrefined) higher-order degrees $\bar{\delta}_n$ also give lower bounds for the Thurston norm. When $n=0$, $\bar{\delta}_0(\psi)=\|\psi\|_A$ hence this gives another proof of McMullen's theorem.

Theorem 10.1. *Let X be a compact, orientable 3-manifold (whose boundary if any is a union of tori). For all $\psi \in H^1(X;\mathbb{Z})$ and $n \geqslant 0$*

$$\bar{\delta}_n(\psi) \leqslant \|\psi\|_T$$

except for the case when $\beta_1(X)=1$, $n=0$, $X \ncong S^1 \times S^2$, and $X \ncong S^1 \times D^2$. In this case, $\bar{\delta}_0(\psi) \leqslant \|\psi\|_T + 1 + \beta_3(X)$ whenever ψ is a generator of $H^1(X;\mathbb{Z}) \cong \mathbb{Z}$. Moreover, equality holds in all cases when $\psi : \pi_1(X) \twoheadrightarrow \mathbb{Z}$ can be represented by a fibration $X \to S^1$.

This theorem generalizes the classical result that for a knot complement, the degree of the Alexander polynomial is less than or equal to twice the genus of the knot. It also generalizes McMullen's theorem. We remark that $\delta_n = \bar{\delta}_n$ except for some cases where $\bar{\delta}_n = 0$. In fact, for most of the cases that we are interested in, the $\bar{\delta}_n$ in Theorem 10.1 can be replaced with δ_n.

Not only do the δ_n give lower bounds for the Thurston norm, but we construct 3-manifolds for which δ_n give much sharper bounds for the Thurston norm than bounds given by the Alexander norm. In Theorem 11.2, we start with a 3-manifold X and subtly alter it to obtain a new 3-manifold X'. The resulting X' cannot be distinguished from X using the ith-order Alexander modules for $i<n$ but the nth-order degrees of X' are strictly greater than those of X. We alter a fibered 3-manifold in this manner to obtain the following result.

Theorem 11.1. *For each $m \geqslant 1$ and $\mu \geqslant 2$ there exists a 3-manifold X with $\beta_1(X)=\mu$ such that*

$$\|\psi\|_A = \delta_0(\psi) < \delta_1(\psi) < \cdots < \delta_m(\psi) \leqslant \|\psi\|_T$$

for all $\psi \in H^1(X;\mathbb{Z})$. Moreover, X can be chosen so that it is closed, irreducible and has the same classical Alexander module as a 3-manifold that fibers over S^1.

An interesting application of Theorem 10.1 is to show that the δ_n give new obstructions to a 3-manifold fibering over S^1. The previously known algebraic obstructions to a 3-manifold fibering over S^1 are that the Alexander module is finitely generated and (when $\beta_1(X)=1$) the Alexander polynomial is monic. For $i, j, n \geqslant 0$ let $d_{ij} = \delta_i - \delta_j$ and let $r_n(X)$ be the nth-order rank of the module $G_r^{(n+1)}/G_r^{(n+2)}$ (see Section 5).

Theorem 12.1. *Let X be a compact, connected, orientable 3-manifold. If at least one of the following conditions is satisfied then X does not fiber over S^1.*

(1) *$r_n(X) \neq 0$ for some $n \geqslant 0$,*
(2) *$\beta_1(X) \geqslant 2$ and there exists $i, j \geqslant 0$ such that $d_{ij}(\psi) \neq 0$ for all $\psi \in H^1(X;\mathbb{Z})$,*
(3) *$\beta_1(X)=1$ and $d_{ij}(\psi) \neq 0$ for some $i, j \geqslant 1$ and $\psi \in H^1(X;\mathbb{Z})$,*
(4) *$\beta_1(X)=1$, $X \ncong S^1 \times S^2$, $X \ncong S^1 \times D^2$ and $d_{0j}(\psi) \neq 1+\beta_3(X)$ for some $j \geqslant 1$ where ψ is a generator of $H^1(X;\mathbb{Z})$.*

As a corollary, we see that the examples in Theorem 11.1 cannot fiber over S^1 but have the same classical Alexander module and polynomial as a fibered 3-manifold.

Corollary 12.2. *For each* $\mu \geqslant 1$, *Theorem* 11.1 *gives an infinite family of closed irreducible* 3-*manifolds* X *where* $\beta_1(X) = \mu$, X *does not fiber over* S^1, *and* X *cannot be distinguished from a fibered* 3-*manifold using the classical Alexander module*.

A second application of Theorem 10.1 is to show that the $\bar{\delta}_n$ give obstructions to a 4-manifold of the form $X \times S^1$ admitting a symplectic structure. Recently, Vidussi has extended the work of Kronheimer to show that if a 4-manifold of the form $X \times S^1$ (X irreducible) admits a symplectic structure then there is a face of the Thurston norm ball of X that is contained in a face of the Alexander norm ball of X. We use his work to prove the following.

Theorem 12.5. *Let* X *be a closed irreducible* 3-*manifold. If at least one of the following conditions is satisfied then* $X \times S^1$ *does not admit a symplectic structure.*

(1) $\beta_1(X) \geqslant 2$ *and there exists an* $n \geqslant 1$ *such that* $\bar{\delta}_n(\psi) > \bar{\delta}_0(\psi)$ *for all* $\psi \in H^1(X; \mathbb{Z})$.
(2) $\beta_1(X) = 1$, ψ *is a generator of* $H^1(X; \mathbb{Z})$, *and* $\bar{\delta}_n(\psi) > \bar{\delta}_0(\psi) - 2$ *for some* $n \geqslant 1$.

Hence if X is one of the examples in Theorem 11.1, then $X \times S^1$ cannot admit a symplectic structure. We note that X has the same Alexander module as a fibered 3-manifold hence $X \times S^1$ cannot be distinguished from a symplectic 4-manifold using the Seiberg–Witten invariants.

Corollary 12.6. *For each* $\mu \geqslant 1$, *Theorem* 11.1 *gives an infinite family of* 4-*manifolds* $X \times S^1$ *where* $\beta_1(X) = \mu$, $X \times S^1$ *does not admit a symplectic structure, and* X *cannot be distinguished from fibered* 3-*manifold using the classical Alexander module.*

Another application of Theorem 10.1 is to give computable lower bounds for the *ropelength* of knots and links. The ropelength of a link is the quotient of its length by it's thickness. In [2] Cantarella et al. show that the minimal ropelength $R(L_i)$ of the ith component of a link $L = \coprod L_i$ is bounded from below by $2\pi(1 + \sqrt{\|\psi_i\|_T})$. Here ψ_i is the cohomology class that evaluates to 1 on the meridian of L_i and 0 on the meridian of every other component of L. In Example 8.3, we use Corollary 10.5 to estimate ropelength for a specific link from [2, Fig. 11, p. 278].

Corollary 10.5. *Let* $X = S^3 - L$ *and* ψ_i *be as defined above. For each* $n \geqslant 0$,

$$R(L_i) \geqslant 2\pi(1 + \sqrt{\delta_n(\psi_i) - 1}).$$

Moreover, if $\beta_1(X) \geqslant 2$ *or* $n \geqslant 1$ (*or both*) *then*

$$R(L_i) \geqslant 2\pi(1 + \sqrt{\bar{\delta}_n(\psi_i)}).$$

Lastly, we remark that the higher-order degrees give obstructions to a 3-manifold admitting a Seifert fibration.

Proposition 8.5. *Let X be a compact, orientable Seifert fibered manifold that does not fiber over S^1. If $\beta_1(X) \geqslant 2$ or $n \geqslant 1$ then*

$$\bar{\delta}_n(\psi) = 0$$

for all $\psi \in H^1(X; \mathbb{Z})$.

1.2. Outline of paper

In Section 2, we review the classical Alexander module, the multivariable Alexander polynomial, and the Alexander norm of a 3-manifold. In Section 3 we define the rational derived series of a group. This series is a slight modification of the derived series so that successive quotients are torsion free. This series will be used to define the higher-order covers of a 3-manifold, the first homology of which will be the chief object of study in this paper.

In Section 4 we define certain skew Laurent polynomial rings $\mathbb{K}_n[t^{\pm 1}]$ which contain $\mathbb{Z}\Gamma_n$ and depend on a class in the first cohomology of the 3-manifold. Here, Γ_n is the group of deck translations of the higher-order covers. These will be extremely important in our investigations. Of particular importance is the fact that they are non-commutative (left and right) principal ideal domains. Similar rings were used in the work of [4], where it was essential that the rings were PIDs.

In Section 5 we define the new higher-order invariants. If X is any topological space, we define the higher-order Alexander module and rank of X. Finally, if $\psi \in H^1(X; \mathbb{Z})$ we define the higher-order degrees $\delta_n(\psi)$ and $\bar{\delta}_n(\psi)$.

Section 6 is devoted to the computation of these invariants using Fox's Free Calculus. That is, the higher-order invariants can be computed directly from a finite presentation of $\pi_1(X)$. The reader familiar with Fox's Free Calculus should be aware that the classical definitions must be slightly altered since we are using right instead of left modules.

In Section 7, we give a finite presentation of the homology of X with coefficients in $\mathbb{K}_n[t^{\pm 1}]$. This will be crucial to prove that $\bar{\delta}_n$ is bounded above by the Thurston norm. In Section 8, we compute the higher-order invariants of some well known 3-manifolds and give some topological properties of the invariants. The most important computation in this section is the computation of the higher-order degrees and ranks for 3-manifolds that fiber over S^1.

Section 9 contains the algebra concerning the rank of a torsion module over a skew (Laurent) polynomial ring. Proposition 9.1 will be used in the proof of Theorem 10.1. In Section 10, we show that the higher-degrees are lower bounds for the Thurston norm. We also prove a theorem relating higher-order degrees of a cohomology class ψ to the first Betti number of a surface dual to ψ, and prove a result for links in S^3.

In Section 11 we prove the Realization Theorem and construct examples of 3-manifolds whose higher degrees increase. We finish the paper by investigating the applications of Theorem 10.1 to 3-manifolds that fiber over S^1 and symplectic 4-manifolds of the form $X \times S^1$ in Section 12.

2. The Alexander polynomial

In this section, we define the Alexander polynomial, the Alexander module, and the Alexander norm of a 3-manifold. For more information about the Alexander polynomial we refer the reader to [8,18,21,29].

Let G be a finitely presented group and let X be a finite CW complex with $\pi_1(X, x_0)$ isomorphic to G. We can assume that X has one 0-cell, x_0. Let X_0 be the universal torsion free abelian cover of X and $\tilde{x}_0$ be the inverse image of x_0 in X_0. That is, X_0 is the cover induced by the homomorphism from G onto $\mathrm{ab}(G)$. Here, $\mathrm{ab}(G) = (G/[G, G])/\{\mathbb{Z}\text{-torsion}\}$ which is isomorphic to $\mathbb{Z}^\mu$ where $\mu = \beta_1(X)$ is the first Betti number of X. (The reason for the "0" in X_0 will become apparent later in the paper.)

The *Alexander module* of X is defined to be

$$A_X = H_1(X_0, \tilde{x}_0; \mathbb{Z})$$

considered as a $\mathbb{Z}[\mathrm{ab}(G)]$-module. After choosing a basis $\{x_1, \dots, x_\mu\}$ for $H_1(X)$ the ring $\mathbb{Z}[\mathrm{ab}(G)]$ can be identified with the ring of Laurent polynomials in several variables $x_1, \dots, x_\mu$ with integral coefficients. The ring $\mathbb{Z}[\mathrm{ab}(G)]$ has no zero divisors and is in fact a unique factorization domain. We note that A_X is finitely presented as

$$\mathbb{Z}[\mathrm{ab}(G)]^r \xrightarrow{\tilde{\partial}_2} \mathbb{Z}[\mathrm{ab}(G)]^s \to A_X,$$

where r is the number of relations and s is the number of generators of a presentation of G. This presentation is obtained by lifting each cell of X to $\mathrm{ab}(G)$ cells of the torsion free abelian cover, X_0.

Let Λ be a finitely generated free abelian group. For any finitely presented $\mathbb{Z}[\Lambda]$-module M with presentation

$$\mathbb{Z}[\Lambda]^r \xrightarrow{P} \mathbb{Z}[\Lambda]^s \to M$$

we define the ith elementary ideal $E_i(M) \subseteq \mathbb{Z}[F]$ to be the ideal generated by the $(s-i) \times (s-i)$ minors of the matrix P. This ideal is independent of the presentation of M. The *Alexander ideal* is $I(X) = E_1(A_X)$, the first elementary ideal of A_X. The *Alexander polynomial* Δ_X of X is the greatest common divisor of the elements of the Alexander ideal. Equivalently, we could have defined Δ_X to be a generator of the smallest principal ideal containing $I(X)$. Note that $\Delta_X \in \mathbb{Z}[\mathrm{ab}(G)]$ and is well-defined up to units in $\mathbb{Z}[\mathrm{ab}(G)]$. We point out the necessity that $\mathbb{Z}[\mathrm{ab}(G)]$ be a UFD in the definition of Δ_X.

Now let $\psi \in H^1(X; \mathbb{Z})$. Let $\Delta_X = \sum_{i=1}^m a_i g_i$ for $a_i \in \mathbb{Z}\backslash\{0\}$ and $g_i \in \mathrm{ab}(G)$. The *Alexander norm* of $\psi \in H^1(X; \mathbb{R})$ is defined to be

$$\|\psi\|_A = \sup_{i,j} \psi(g_i - g_j),$$

where ψ is a homomorphism from G to $\mathbb{Z}$. In this paper, we view $\mathbb{Z}$ as the multiplicative group generated by t. Hence the Alexander norm is equal to the degree of the one-variable polynomial $\sum_{i=1}^m \psi(g_i)$ corresponding to ψ.

We note that the Alexander (as well as the Thurston) norm is actually semi-norms since it can be zero on a non-zero vector of $H^1(X; \mathbb{R})$.

3. Rational derived series

This paper investigates the homology of the covering spaces of a 3-manifold corresponding to the rational derived series of a group. We begin by defining the rational derived series of G and proving some of the properties of the quotient $G/G_r^{(n+1)}$. The most important for our purposes will be that $G/G_r^{(n+1)}$ is solvable and its successive quotients $G_r^{(i)}/G_r^{(i+1)}$ are $\mathbb{Z}$-torsion-free and abelian.

Definition 3.1. Let $G_r^{(0)}=G$. For $n\geqslant 1$ define $G_r^{(n)}=[G_r^{(n-1)},G_r^{(n-1)}]P_{n-1}$ where $P_{n-1}=\{g\in G_r^{(n-1)}\mid g^k\in[G_r^{(n-1)},G_r^{(n-1)}]$ for some $k\in Z-\{0\}\}$ to be the nth *term of the rational derived series* of G.

We denote by Γ_n the quotient $G/G_r^{(n+1)}$ and by ϕ_n the quotient map $G\twoheadrightarrow\Gamma_n$. By the following lemma, Γ_n is a group. Note that if G is a finite group then $G_r^{(n)}=G$ hence $\Gamma_n=\{1\}$ for all $n\geqslant 0$. Hence, in this paper we will only be interested in groups with $\beta_1(G)\geqslant 1$.

Lemma 3.2. *$G_r^{(n)}$ is a normal subgroup of $G_r^{(i)}$ for $0\leqslant i\leqslant n$.*

Proof. We show that $[G_r^{(n-1)},G_r^{(n-1)}]$ and P_{n-1} are both normal subgroups of G. Since $G_r^{(i+1)}\subseteq G_r^{(i)}$ for all $i\geqslant 0$, $[G_r^{(n-1)},G_r^{(n-1)}]\subseteq G_r^{(n-1)}$, and $P_{n-1}\subseteq G_r^{(n-1)}$ it follows that $[G_r^{(n-1)},G_r^{(n-1)}]$ and P_{n-1} are normal in $G_r^{(i)}$ for $0\leqslant i\leqslant n$. Therefore $G_r^{(n)}$ is a normal subgroup of $G_r^{(i)}$ for $0\leqslant i\leqslant n$. Let N be a normal subgroup of G. Then $[N,N]$ is normal in G since $g(\prod[n_1,n_2])g^{-1}=\prod[gn_1g^{-1},gn_2g^{-1}]$. Therefore $[G_r^{(n-1)},G_r^{(n-1)}]$ is normal in G by induction on n. Now we show that P_{n-1} is a closed under multiplication. Let $p_1,p_2\in P_{n-1}$ then for some $k_1,k_2\neq 0$, $p_1^{k_1},p_2^{k_2}\in[G_r^{(n-1)},G_r^{(n-1)}]$. Now for any two elements $w_1,w_2\in G_r^{(n-1)}$, we have $w_1w_2=w_2w_1c$ where $c=[w_1^{-1},w_2^{-1}]\in[G_r^{(n-1)},G_r^{(n-1)}]$.Hence $(p_1p_2)^{k_1k_2}=(p_1^{k_1})^{k_2}(p_2^{k_2})^{k_1}\prod c_i$ where $c_i\in[G_r^{(n-1)},G_r^{(n-1)}]$ so $p_1p_2\in P_{n-1}$, which shows that P_{n-1} is a subgroup of G. Now if $g\in G$ then $(gp_1g^{-1})^{k_1}=gp_1^{k_1}g^{-1}\in[G_r^{(n-1)},G_r^{(n-1)}]$ since $[G_r^{(n-1)},G_r^{(n-1)}]$ is normal in G. Therefore P_{n-1} is a normal subgroup of G. □

Definition 3.3. A group Γ is poly-torsion-free-abelian (PTFA) if it admits a normal series $\{1\}=G_0\lhd G_1\lhd\cdots\lhd G_n=\Gamma$ such that each of the factors G_{i+1}/G_i is torsion-free abelian. (In the group theory literature only a subnormal series is required.)

Remark 3.4. If $A\lhd G$ is torsion-free-abelian and G/A is PTFA then G is PTFA. Any PTFA group is torsion-free and solvable (the converse is not true). Any subgroup of a PTFA group is a PTFA group [28, Lemma 2.4, p. 421].

We show that the successive quotients of the rational derived series are torsion-free abelian. In fact, the following lemma implies that if N is a normal subgroup of $G_r^{(i)}$ with $G_r^{(i)}/N$ torsion-free-abelian then $G_r^{(i+1)}\subseteq N$.

Lemma 3.5. *$G_r^{(i)}/G_r^{(i+1)}$ is isomorphic to $(G_r^{(i)}/[G_r^{(i)},G_r^{(i)}])/\{\mathbb{Z}$-torsion$\}$ for all $i\geqslant 0$.*

Proof. Since $[G_r^{(i)},G_r^{(i)}]\subseteq G_r^{(i+1)}$, we can extend the natural projection $p: G_r^{(i)}\twoheadrightarrow G_r^{(i)}/G_r^{(i+1)}$ to a surjective map $p_1: G_r^{(i)}/[G_r^{(i)},G_r^{(i)}]\twoheadrightarrow G_r^{(i)}/G_r^{(i+1)}$. If $[g]$ is a torsion element in $G_r^{(i)}/[G_r^{(i)},G_r^{(i)}]$ then $[g]^k=[g^k]=1$ so $g\in P_i\subseteq G_r^{(i+1)}$. Hence we can extend p_1 to $p_2: (G_r^{(i)}/[G_r^{(i)},G_r^{(i)}])/T\twoheadrightarrow G_r^{(i)}/G_r^{(i+1)}$ where T is the torsion subgroup of $G_r^{(i)}/[G_r^{(i)},G_r^{(i)}]$. We show that p_2 is injective hence is an isomorphism. Suppose $p_2(g_2)=1$ then $p(g)=1$ for any g such that $q_2(q_1(g))=g_2$. Hence $g=fh$ where $f\in$

$[G_r^{(i)}, G_r^{(i)}]$ and $h^k \in [G_r^{(i)}, G_r^{(i)}]$ for some $k \neq 0$. Therefore $(g_2)^k = (q_2(q_1(fh)))^k = (q_2(q_1(f)q_1(h)))^k = (q_2(q_1(h)))^k = q_2(q_1(h^k)) = q_2(1) = 1$ which implies that $g_2 = 1$. $\square$

If $G = \pi_1(X)$ this shows that $G_r^{(n)}/G_r^{(n+1)} \cong H_1(X_{\Gamma_{n-1}})/\{\mathbb{Z}\text{-torsion}\}$ where $X_{\Gamma_{n-1}}$ is the regular Γ_{n-1} cover of X. When $n = 0$, note that $G/G_r^{(1)} = G_r^{(0)}/G_r^{(1)} \cong H_1(X)/\{\mathbb{Z}\text{-torsion}\} \cong \mathbb{Z}^{\beta_1(X)}$.

Corollary 3.6. *Γ_n is a PTFA group.*

Proof. Consider the subnormal series

$$1 = \frac{G_r^{(n+1)}}{G_r^{(n+1)}} \lhd \frac{G_r^{(n)}}{G_r^{(n+1)}} \lhd \cdots \lhd \frac{G_r^{(i)}}{G_r^{(n+1)}} \lhd \cdots \lhd \frac{G_r^{(1)}}{G_r^{(n+1)}} \lhd \frac{G_r^{(0)}}{G_r^{(n+1)}} = \Gamma_n.$$

$G_r^{(i)}$ is a normal subgroup of $G_r^{(j)}$ for $0 \leqslant j \leqslant i$ hence $G_r^{(i)}/G_r^{(n+1)}$ is a normal subgroup of $G_r^{(j)}/G_r^{(n+1)}$. From the lemma above, $(\frac{G_r^{(i)}}{G_r^{(n+1)}})/(\frac{G_r^{(i+1)}}{G_r^{(n+1)}}) = G_r^{(i)}/G_r^{(i+1)}$ is isomorphic to $(G_r^{(i)}/[G_r^{(i)}, G_r^{(i)}])/\{\mathbb{Z}\text{-torsion}\}$ hence is torsion free and abelian. $\square$

We next show that if the successive quotients of the derived series of G are torsion-free then the rational derived series agrees with the derived series. In general we only know that $G^{(i)} \subseteq G_r^{(i)}$ for all $i \geqslant 0$.

Corollary 3.7. *If $G^{(i)}/G^{(i+1)}$ is $\mathbb{Z}$-torsion-free for all i then $G_r^{(i)} = G^{(i)}$ for all i.*

Proof. We prove this by induction on i. First, we know that $G_r^{(0)} = G^{(0)} = G$. Now assume that $G_r^{(i)} = G^{(i)}$, then by assumption

$$G_r^{(i)}/[G_r^{(i)}, G_r^{(i)}] = G^{(i)}/G^{(i+1)}$$

is $\mathbb{Z}$-torsion-free. Hence Lemma 3.5 gives us $G_r^{(i)}/G_r^{(i+1)} = G_r^{(i)}/[G_r^{(i)}, G_r^{(i)}]$ hence $G_r^{(i+1)} = [G_r^{(i)}, G_r^{(i)}] = G^{(i+1)}$. $\square$

Strebel showed that if G is the fundamental group of a (classical) knot exterior then the quotients of successive terms of the derived series are torsion-free abelian [32]. Hence for knot exteriors we have $G_r^{(i)} = G^{(i)}$. This is also well known to be true for free groups. Since any non-compact surface has free fundamental group, this also holds for all orientable surface groups.

4. Skew Laurent polynomial rings

In this section, we define some skew Laurent polynomial rings, $\mathbb{K}_n[t^{\pm 1}]$, which are obtained from $\mathbb{Z}\Gamma_n$ by inverting elements of the ring that are “independent” of $\psi \in H^1(G; \mathbb{Z})$. Very similar rings were used in the work of Cochran et al. [4, Definition 3.1]. Skew polynomial rings with coefficients in a (skew) field are known to be left and right principal ideal domains as is discussed herein.

Let Γ be a PTFA group as defined in the previous section. A crucial property of $\mathbb{Z}\Gamma$ is that is has a (skew) quotient field. Recall that if R is a commutative integral domain then R embeds in its field of quotients.

However, if R is non-commutative domain then this is no longer always possible (and is certainly not as trivial if it does exist). We discuss conditions which guarantee the existence of such a (skew) field.

Let R be a ring and S be a subset of R. S is a *right divisor set* of R if the following properties hold.

(1) $0 \notin S$, $1 \in S$.
(2) S is multiplicatively closed.
(3) Given $r \in R$, $s \in S$ there exists $r_1 \in R$, $s_1 \in S$ with $rs_1 = sr_1$.

It is known that if $S \subseteq R$ is a right divisor set then the *right quotient ring* RS^{-1} exists ([28, p. 146] or [31, p. 52]). By RS^{-1} we mean a ring containing R with the property that

(1) Every element of S has an inverse in RS^{-1}.
(2) Every element of RS^{-1} is of the form rs^{-1} with $r \in R$, $s \in S$.

If R has no zero-divisors and $S = R - \{0\}$ is a right divisor set then R is called an *Ore domain*. If R is an Ore domain, RS^{-1} is a skew field, called the *classical right ring of quotients* of R (see [31]). It is observed in [4, Proposition 2.5] that the group ring of a PTFA group has a right ring of quotients.

Proposition 4.1 (*Passman [28], pp. 591–592,611*). *If Γ is PTFA then $\mathbb{Q}\Gamma$ is a right* (*and left*) *Ore domain*; *i.e. $\mathbb{Q}\Gamma$ embeds in its classical right ring of quotients $\mathcal{K}$, which is a skew field.*

If $\mathcal{K}$ is the (right) ring of quotients of $\mathbb{Z}\Gamma$, it is a $\mathcal{K}$-bimodule and a $\mathbb{Z}\Gamma$-bimodule. Note that $\mathcal{K} = \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1}$ as above. We list a some properties of $\mathcal{K}$.

Remark 4.2. If R is an Ore domain and S is a right divisor set then RS^{-1} is flat as a left R-module [31, Proposition II.3.5]. In particular, $\mathcal{K}$ is a flat left $\mathbb{Z}\Gamma$-module, i.e. $\cdot \otimes_{\mathbb{Z}\Gamma} \mathcal{K}$ is exact.

Remark 4.3. Every module over $\mathcal{K}$ is a free module [31, Proposistion I.2.3] and such modules have a well defined rank $\mathrm{rk}_{\mathcal{K}}$ which is additive on short exact sequences [5, p. 48].

If M is a right R-module with R an Ore domain then the *rank of* M is defined as rank $M = rk_{\mathcal{K}}(M \otimes_R \mathcal{K})$. Combining Remarks 4.2 and 4.3 we have the following

Remark 4.4. If $\mathscr{C}$ is a non-negative finite chain complex of finitely generated free right $\mathbb{Z}\Gamma$-modules then the Euler characteristic $\chi(\mathscr{C}) = \sum_{i=0}^{\infty} (-1)^i \mathrm{rank}\, C_i$ is defined and is equal to $\sum_{i=0}^{\infty} (-1)^i \mathrm{rank}\, H_i(\mathscr{C})$.

The rest of this section will be devoted to the rings $\mathbb{K}_n^{\psi}[t^{\pm 1}]$. Consider the group $\Gamma_n = G/G_r^{(n+1)}$ for $n \geqslant 0$. Since Γ_n is PTFA (Corollary 3.6), $\mathbb{Z}\Gamma_n$ embeds in its right ring of quotients, which we denote by $\mathcal{K}_n$. Let $\psi \in H^1(G;\mathbb{Z})$ be primitive. Since $H^1(G;\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{Z})$, ψ can be considered as an epimorphism from G to $\mathbb{Z}$. In particular, ψ is trivial on $G_r^{(n+1)}$ so it induces a well defined homomorphism $\overline{\psi}: \Gamma_n \twoheadrightarrow \mathbb{Z}$. Let Γ_n' be the kernel of $\overline{\psi}$. Since Γ_n' is a subgroup of Γ_n, Γ_n' is PTFA by Remark 3.4. Therefore $\mathbb{Z}\Gamma_n'$ is an Ore domain and $S_n = \mathbb{Z}\Gamma_n' - \{0\}$ is a right divisor set of $\mathbb{Z}\Gamma_n$ [28, p. 609]. Let $\mathbb{K}_n = (\mathbb{Z}\Gamma_n')S_n^{-1}$ be the right ring of quotients of $\mathbb{Z}\Gamma_n'$, $g_\psi: \mathbb{Z}\Gamma_n' \to \mathbb{K}_n$ be the embedding of $\mathbb{Z}\Gamma_n'$ into $\mathbb{K}_n$, and $R_n = (\mathbb{Z}\Gamma_n)S_n^{-1}$.

We will show that R_n is isomorphic to a certain skew Laurent polynomial ring $\mathbb{K}_n[t^{\pm 1}]$ (defined below) which is a non-commutative principal right and left ideal domain by Cohn [6, 2.1.1. p. 49]. That is, $\mathbb{K}_n[t^{\pm 1}]$ has no zero divisors and every right and left ideal is principal.

We recall the definition of a skew Laurent polynomial ring. If K is a skew field, α is an automorphism of K and t is an indeterminate, the *skew (Laurent) polynomial ring in t over K* associated with α is the ring consisting of all expressions of the form

$$t^{-m}a_{-m} + \cdots + t^{-1}a_{-1} + a_0 + ta_1 + \cdots + t^n a_n,$$

where $a_i \in K$. The operations are coordinate-wise addition and multiplication defined by the usual multiplication for polynomials and the rule $at = t\alpha(a)$ [5, p. 54].

Consider the split short exact sequence

$$0 \longrightarrow \ker(\overline{\psi}) \longrightarrow \Gamma_n \xrightarrow{\overline{\phi}} \mathbb{Z} \longrightarrow 0.$$

Choose a splitting $\xi\colon \mathbb{Z} \to \Gamma_n$. Then ξ induces an automorphism of $\Gamma'_n = \ker(\overline{\psi})$ by $g \mapsto \xi(t)^{-1} g \xi(t)$. This induces a ring automorphism of $\mathbb{Z}\Gamma'_n$ and hence a field automorphism α of $\mathbb{K}_n$ by $\alpha(rs^{-1}) = \alpha(r)\alpha(s)^{-1}$. This defines $\mathbb{K}_n[t^{\pm 1}]$ as above.

Proposition 4.5. *The embedding $g_\psi\colon \mathbb{Z}\Gamma'_n \to \mathbb{K}_n$ extends to an isomorphism $R_n \to \mathbb{K}_n[t^{\pm 1}]$.*

Proof. Any element of Γ_n can be written uniquely as $\xi(t)^m a_m$ for some $m \in \mathbb{Z}$ and $a_m \in \mathbb{Z}\Gamma'_n$. It follows that $\mathbb{Z}\Gamma_n$ is isomorphic to the skew (Laurent) polynomial ring $\mathbb{Z}\Gamma'_n[x^{\pm 1}]$ by sending $\xi(t)^m a_m$ to $x^m a_m$. The automorphism of $\mathbb{Z}\Gamma'_n$ is induced by conjugation, $a \to x^{-1}ax$ since $a\xi(t) = \xi(t)(\xi(t)^{-1}a\xi(t))$.

Hence there is an obvious ring homomorphism of $\mathbb{Z}\Gamma_n \to \mathbb{K}_n[t^{\pm 1}]$ extending g_ψ. Note that the automorphism $g \mapsto \xi(t)^{-1}g\xi(t)$ defining $\mathbb{K}_n[t^{\pm 1}]$ agrees with conjugation in Γ so this map is a ring homomorphism. The non-zero elements of $\mathbb{Z}\Gamma'_n$ map to invertible elements in $\mathbb{K}_n[t^{\pm 1}]$. It is then easy to show that $R_n \cong \mathbb{K}_n[t^{\pm 1}]$. $\square$

We note that $(\mathbb{Z}\Gamma_n)S^{-1}$ depends on the (primitive) class $\psi \in H^1(G;\mathbb{Z})$. Moreover, the isomorphism of $(\mathbb{Z}\Gamma_n)S^{-1}$ with $\mathbb{K}_n[t^{\pm 1}]$ depends on the splitting $\xi\colon \mathbb{Z} \to \Gamma_n$. For any $\psi \in H^1(X;\mathbb{Z})$ we have

$$\mathbb{Z}\Gamma_n \hookrightarrow \mathbb{K}_n[t^{\pm 1}] \hookrightarrow \mathscr{K}_n.$$

One should note that the first and last rings only depend on the group G while the middle ring $\mathbb{K}_n[t^{\pm 1}]$ depends on the homomorphism $\psi\colon G \to \mathbb{Z}$ and splitting $\xi\colon \mathbb{Z} \to \Gamma_n$. Often we write $\mathbb{K}_n^\psi[t^{\pm 1}]$ to emphasize the class ψ on which $\mathbb{K}_n^\psi[t^{\pm 1}]$ is dependent. From Remark 4.2 we have the following.

Remark 4.6. $\mathbb{K}_n^\psi[t^{\pm 1}]$ and $\mathscr{K}_n$ are flat left $\mathbb{Z}\Gamma_n$-modules.

5. Definition of invariants

Suppose X is a connected CW-complex with $A \subseteq X$ and $x_0 \in A$ a basepoint. Let $\phi\colon \pi_1(X) \to \Gamma$ be a homomorphism and $X_\Gamma \xrightarrow{p} X$ denote the regular Γ-cover of X associated to ϕ. That is, X_Γ is defined to be the pullback of the universal cover of $K(\Gamma, 1)$. We note that there is an induced coefficient system on

A, $\phi \circ i_* : \pi_1(A) \to \Gamma$ where i is the inclusion map of A into X. Thus, we have a regular covering map of the pair $(X_\Gamma, A_\Gamma) \xrightarrow{\mathrm{p}} (X, A)$. If ϕ is not surjective then X_Γ is the disjoint union of $\Gamma/\mathrm{Im}\,(\phi)$ copies of the regular $\mathrm{Im}(\phi)$-cover corresponding to $\phi : \pi_1(X) \twoheadrightarrow \mathrm{Im}(\phi)$.

There is a natural homomorphism $\tau : \Gamma \to G(X_\Gamma)$ where $G(X_\Gamma)$ is the group of deck transformations of X_Γ (see [14] or [25] for more details). We note that τ is an isomorphism when ϕ is surjective. This map is defined by sending γ to the deck transformation τ_γ that takes x_0 to $\tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the unique lift of γ starting at x_0. τ gives us a left Γ action on X_Γ by $\tilde{x}\gamma = \tau_\gamma(\tilde{x})$. We make this into a right action by defining $\tilde{x}\gamma = \gamma^{-1}\tilde{x}$. Hence $\tilde{x}\gamma = \tau_{\gamma^{-1}}(\tilde{x})$.

The right action of Γ on X_Γ induces a right action on the group $C_\star(X_\Gamma)$ of singular n-chains on X_Γ, by sending a singular n-simplex $\sigma : \Delta^n \to X_\Gamma$ to the composition $\Delta^n \to X_\Gamma \xrightarrow{\gamma} X_\Gamma$. The action of Γ on $C_\star(X_\Gamma)$ makes $C_\star(X_\Gamma)$ a right $\mathbb{Z}\Gamma$-module.

Let $\mathscr{M}$ be a $\mathbb{Z}\Gamma$-bimodule. The equivariant homology of X and (X, A) are defined as follows.

Definition 5.1. Given $X, A, \phi, \mathscr{M}$ as above, let

$$H_\star(X; \mathscr{M}) \equiv H_\star(C_\star(X_\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathscr{M})$$

and

$$H_\star(X, A; \mathscr{M}) \equiv H_\star(C_\star(X_\Gamma, A_\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathscr{M})$$

as right $\mathbb{Z}\Gamma$-modules.

These are well-known to be isomorphic to the homology of X and (X, A) with coefficient system induced by ϕ [36, Theorem VI 3.4].

We now restrict to the case when Γ is PTFA. We state the following useful proposition. A proof of this can be found in [3, Section 3]. We remark that the finiteness condition in Proposition 5.2 is necessary.

Proposition 5.2. Suppose $\pi_1(X)$ *is finitely generated and* $\phi : \pi_1(X) \to \Gamma$ *is non-trivial. Then*

$$\operatorname{rank} H_1(X; \mathbb{Z}\Gamma) \leqslant \beta_1(X) - 1.$$

Let $G = \pi_1(X, x_0)$. Define the nth-order cover $X_n \xrightarrow{\mathrm{p}_n} X$ of X to be the regular Γ_n-cover corresponding to the coefficient system $\phi_n : G \twoheadrightarrow \Gamma_n$ where $\Gamma_n = G/G_r^{(n+1)}$ is as defined in Section 3. Recall that $\mathbb{Z}\Gamma_n$ has a (skew) field of quotients $\mathscr{K}_n$. If R is any ring with $\mathbb{Z}\Gamma_n \subseteq R \subseteq \mathscr{K}_n$ then R is a $\mathbb{Z}\Gamma_n$-bimodule. Moreover, $H_*(X; R)$ can be considered as a right R-module. We will be interested in the cases when R is $\mathbb{Z}\Gamma_n$, $\mathscr{K}_n$ and $\mathbb{K}_n[t^{\pm 1}]$ where $\mathbb{K}_n[t^{\pm 1}]$ is as described in the Section 4. If M is a right (left) R-module where R is an Ore domain then we let $T_R M$ be the R-torsion submodule of M. When there is no confusion we suppress the R and just write TM.

We define the higher-order modules. The integral invariants that we can extract from these modules will be our chief interest for the rest of this paper.

Definition 5.3. The nth-order Alexander module of a CW-complex X is

$$\mathscr{A}_n(X) \equiv T_{\mathbb{Z}\Gamma_n} H_1(X; \mathbb{Z}\Gamma_n)$$

considered as right $\mathbb{Z}\Gamma_n$-module. Similarly, we define

$$\bar{\mathscr{A}}_n(X) \equiv H_1(X; \mathbb{Z}\Gamma_n)$$

considered as right $\mathbb{Z}\Gamma_n$-module.

Let X and Y be 3-manifolds with $G = \pi_1(X)$ and $H = \pi_1(Y)$. Suppose that G is isomorphic to H. We would like for their higher-order Alexander modules to be "the same". However, they are modules over different (albeit isomorphic) rings. We remedy this dilemma with the following definition. It is easy to verify that the following defines an equivalence relation.

Definition 5.4. Let M and N be right (left) R and S-modules, respectively, and $f : R \to S$ be an isomorphism. N can be made into a right (left) R-module via f. We say that M is equivalent to N provided N is isomorphic to M as a right (left) R-module.

Let X be a topological space with $G = \pi_1(X)$. The higher-order Alexander modules can be defined group theoretically. Define a right $\mathbb{Z}[G/G_r^{(n+1)}]$-module structure on $G_r^{(n+1)}/[G_r^{(n+1)}, G_r^{(n+1)}]$ by $[h][g] = [g^{-1}hg]$ for $h \in G_r^{(n+1)}$ and $g \in G$. We see that

$$\bar{\mathscr{A}}_n(X) \cong \frac{G_r^{(n+1)}}{[G_r^{(n+1)}, G_r^{(n+1)}]}$$

as a right $\mathbb{Z}[\frac{G}{G_r^{(n+1)}}]$-module. We also note that

$$\bar{\mathscr{A}}_n(X)/\{\mathbb{Z}\text{-torsion}\} \cong G_r^{(n+1)}/G_r^{(n+2)}$$

by Proposition 3.5. Suppose that Y is homeomorphic to X, then $\pi_1(Y)$ is isomorphic to G. It is easy to verify that the isomorphism of groups leads to an equivalence of $\bar{\mathscr{A}}_n(X)$ and $\bar{\mathscr{A}}_n(Y)$. Therefore the equivalence classes of the higher-order Alexander modules are topological invariants. Similarly, one can easily verify that the rest of the definitions in this section are invariants of X or a pair (X, ψ).

Definition 5.5. The nth-order rank of X is

$$r_n(X) = \mathrm{rk}_{\mathscr{K}_n} H_1(X; \mathscr{K}_n).$$

In the literature, the classical Alexander module of a 3-manifold is often defined as $H_1(X, x_0; \mathbb{Z}\Gamma_0)$ (see Section 2) and $\alpha(X) = \mathrm{rk}\, H_1(X, x_0; \mathbb{Z}\Gamma_0)$ is called the nullity of X [18]. We will now show that $H_1(X; \mathbb{Z}\Gamma_n)$ and $H_1(X, x_0; \mathbb{Z}\Gamma_n)$ are related by $r_n(X) = \mathrm{rk}_{\mathscr{K}_n} H_1(X, x_0; \mathscr{K}_n) - 1$ and $T_{\mathbb{Z}\Gamma_n} H_1(X; \mathbb{Z}\Gamma_n) = T_{\mathbb{Z}\Gamma_n} H_1(X, x_0; \mathbb{Z}\Gamma_n)$. Hence, we could have defined $\mathscr{A}_n(X)$ and $r_n(X)$ using homology rel basepoint as well.

Proposition 5.6. *Let Γ be PTFA, and $\phi : \pi_1(X, x_0) \to \Gamma$ be non-trivial. Then*

$$\mathrm{rk}_{\mathscr{K}} H_1(X; \mathscr{K}) = \mathrm{rk}_{\mathscr{K}} H_1(X, x_0; \mathscr{K}) - 1 \tag{5.1}$$

and

$$T_R H_1(X;\, R) \cong T_R H_1(X, x_0;\, R) \tag{5.2}$$

for any ring R such that $\mathbb{Z}\Gamma \subseteq R \subseteq \mathscr{K}$, *where* $\mathscr{K}$ *is the* (*skew*) *field of quotients of* $\mathbb{Z}\Gamma$.

Proof. To prove (5.1), consider the long exact sequence the pair (X, x_0),

$$0 \to H_1(X;\, \mathscr{K}) \xrightarrow{\rho} H_1(X, x_0;\, \mathscr{K}) \xrightarrow{\partial} H_0(x_0;\, \mathscr{K}) \to H_0(X;\, \mathscr{K}).$$

Since $\phi\colon \pi_1(X) \to \Gamma$ is non-trivial, $H_0(X;\, \mathscr{K}) = 0$ by the following Lemma. The first result follows since $H_0(x_0;\, \mathscr{K}) \cong \mathscr{K}$.

Lemma 5.7. *Suppose X is a connected CW complex. If* $\phi\colon \pi_1(X) \to \Gamma$ *is a non-trivial coefficient system and* Γ *is PTFA then* $H_0(X;\, \mathscr{K}) = 0$.

Proof. By [36, p. 275] and [1, p. 34], $H_0(X;\, \mathscr{K})$ is isomorphic to the cofixed set $\mathscr{K}/\mathscr{K}I$ where I is the augmentation ideal of $\mathbb{Z}[\pi_1(X)]$ acting via $\mathbb{Z}[\pi_1(X)] \to \mathbb{Z}\Gamma \to \mathscr{K}$. If ϕ is non-trivial then the composition is non-trivial. Thus I contains an element that is a unit hence $\mathscr{K}I = \mathscr{K}$. □

We show that the map $\rho\colon H_1(X;\, R) \to H_1(X, x_0;\, R)$ restricts to an isomorphism from $TH_1(X;\, R)$ onto $TH_1(X, x_0;\, R)$. Certainly $\rho\colon TH_1(X;\, R) \to TH_1(X, x_0;\, R)$ is a monomorphism. Let $\sigma \in TH_1(X, x_0;\, R)$ with $\sigma r = 0$ where $r \neq 0$. Since $H_0(x_0;\, R) \cong R$ is R-torsion-free, $\partial(\sigma) = 0$ so there exists $\theta \in H_1(X;\, R)$ with $\rho(\theta) = \sigma$. We see that θ is R-torsion since $\rho(\theta r) = \rho(\theta) r = \sigma r = 0$ and ρ is a monomorphism. Therefore $\rho\colon TH_1(X;\, R) \to TH_1(X, x_0;\, R)$ is surjective. □

For any primitive class $\psi \in H^1(X;\, \mathbb{Z})$ and splitting $\xi\colon \mathbb{Z} \to \Gamma_n$ we consider the skew Laurent polynomial ring $\mathbb{K}_n[t^{\pm 1}]$. We note that $TH_1(X;\, \mathbb{K}_n[t^{\pm 1}])$ is a finitely generated right $\mathbb{K}_n$-module. Moreover, any module over $\mathbb{K}_n$ has a well defined rank which is additive on short exact sequences by Remark 4.3.

Definition 5.8. Let X be a finite CW-complex. For each primitive $\psi \in H^1(X;\, \mathbb{Z})$ and $n \geqslant 0$ we define the refined nth-order *Alexander module corresponding* to ψ to be $\mathscr{A}_n^{\psi}(X) = T_{\mathbb{K}_n[t^{\pm 1}]} H_1(X;\, \mathbb{K}_n[t^{\pm 1}])$ viewed as a right $\mathbb{K}_n[t^{\pm 1}]$-module.

Since $\mathscr{A}_n^{\psi}(X)$ is a finitely generated module over the principal ideal domain $\mathbb{K}_n[t^{\pm 1}]$,

$$\mathscr{A}_n^{\psi}(X) \cong \bigoplus_{i=1}^{m} \frac{\mathbb{K}_n[t^{\pm 1}]}{p_i(t)\mathbb{K}_n[t^{\pm 1}]}$$

for some non-zero $p_i(t) \in \mathbb{K}_n[t^{\pm 1}]$ [20, Theorem 16, p. 43]. We define the refined nth-order degree of ψ to be the degree of the polynomial $\prod p_i(t)$. One can verify that this is equal to the rank of $\mathscr{A}_n^{\psi}(X)$ as a $\mathbb{K}_n$-module. Note that while the degree of $\prod p_i(t)$ is well-defined and independent of ψ, the polynomial $\prod p_i(t)$ is not well-defined.

Definition 5.9. Let X be a finite CW-complex. For each primitive $\psi \in H^1(X;\mathbb{Z})$ and $n \geqslant 0$ we define the refined nth-order degree of ψ to be

$$\delta_n(\psi) = \mathrm{rk}_{\mathbb{K}_n} \mathscr{A}_n^{\psi}(X).$$

We extend by linearity to define $\delta_n(\psi)$ for non-primitive classes ψ.

Similarly we define the unrefined higher-order Alexander modules and degrees.

Definition 5.10. Let X is a finite CW-complex. For each primitive $\psi \in H^1(X;\mathbb{Z})$ and $n \geqslant 0$ we define the *unrefined* nth-order *Alexander module corresponding to* ψ to be $\bar{\mathscr{A}}_n^{\psi}(X) = H_1(X;\mathbb{K}_n[t^{\pm 1}])$ viewed as a right $\mathbb{K}_n[t^{\pm 1}]$-module. The *unrefined* nth-order degree of ψ is

$$\bar{\delta}_n(\psi) = \mathrm{rk}_{\mathbb{K}_n} \bar{\mathscr{A}}_n^{\psi}(X)$$

if $\mathrm{rk}_{\mathbb{K}_n} \bar{\mathscr{A}}_n^{\psi}(X)$ is finite and 0 otherwise. We extend by linearity to define $\bar{\delta}_n(\psi)$ for non-primitive classes ψ.

We note that

$$\bar{\mathscr{A}}_n^{\psi}(X) \cong \left(\bigoplus_{i=1}^{m} \frac{\mathbb{K}_n[t^{\pm 1}]}{p_i(t)\mathbb{K}_n[t^{\pm 1}]} \right) \bigoplus \mathbb{K}_n[t^{\pm 1}]^{r_n(X)}.$$

Hence $\mathrm{rk}_{\mathbb{K}_n} \bar{\mathscr{A}}_n^{\psi}(X)$ is finite if and only if $r_n(X) = 0$.

Remark 5.11. If $r_n(X) = 0$ then $\bar{\delta}_n(\psi) = \delta_n(\psi)$ otherwise $0 = \bar{\delta}_n(\psi) \leqslant \delta_n(\psi)$.

We now show that $\bar{\delta}_0(\psi)$ is equal to the Alexander norm of ψ hence $\bar{\delta}_0(\psi)$ is a convex function.

Proposition 5.12. $\bar{\delta}_0(\psi) = \|\psi\|_A$ *for all* $\psi \in H^1(X;\mathbb{Z})$.

Proof. Recall that $\Gamma_0 = \mathbb{Z}^{\beta_1(X)}$ hence $\mathbb{Z}\Gamma_0$ is isomorphic to the polynomial ring in several variables. Let $v\colon \mathbb{Z}\Gamma_0 \hookrightarrow \mathbb{K}_0[t^{\pm 1}]$ be the embedding of $\mathbb{Z}\Gamma_0$ into the principal ideal domain $\mathbb{K}_0[t^{\pm 1}]$ and $\Delta_X = \sum n_g g$ be the Alexander polynomial of X. We begin by showing that $\|\psi\|_A = \deg\, v(\Delta_X)$. For all j consider the polynomial $\Delta_X^j = \sum_{\psi(g)=t^j} n_g g$. Note that any such g can be written (using the splitting ξ) as $h_g \tau^j$ where $\psi(\tau) = t$. We see that $v(\Delta_X^j) = v(\sum_{\psi(g)=t^j} n_g g) = (\sum n_g h_g) t^j$ where $c_j \equiv \sum n_g h_g \in \mathbb{Z}[\ker \overline{\psi}_0]$. Since v is a monomorphism we have $c_j \neq 0$ unless $n_g = 0$ for all g with $\psi(g) = t^j$. It follows that $\deg\, v(\Delta_X) = \deg\, v(\sum \Delta_X^j) = \deg \sum c_j t^j = \|\psi\|_A$. After choosing a group presentation for G, Fox's Free Calculus (Section 6) gives us a presentation matrix M for $H_1(X, x_0; \mathbb{Z}[\Gamma_0]) = H_1(\widetilde{X}, \widetilde{x}_0)$ where $\widetilde{X}$ is the torsion-free abelian cover of X. Moreover a presentation of $H_1(X, x_0; \mathbb{K}_0[t^{\pm 1}])$ is also given by M^v, that is we consider each entry in as an element of $\mathbb{K}_0[t^{\pm 1}]$. If s is the number of generators in the presentation of G then $\Delta_X = \gcd(E_1(H_1(X, x_0; \mathbb{Z}\Gamma_0))) = \gcd\{d_1, \ldots, d_k\}$ where $\{d_1, \ldots, d_k\}$ is the set of determinants of the $(s-1) \times (s-1)$ minors of M (Section 2). Note that Γ_0 is free abelian so $\mathbb{K}_0$ is a commutative field and hence and $\mathbb{Z}\Gamma_0$ and $\mathbb{K}_0[t^{\pm 1}]$ are unique factorization domains, since any principal ideal domain is a unique factorization domain. We compute $\gcd(E_1(H_1(X, x_0; \mathbb{K}_0[t^{\pm 1}]))) = \gcd\{v(d_1), \ldots, v(d_k)\}$.

Since v is an embedding, one can check that the degrees of $v(\gcd\{d_1,\ldots,d_k\})$ and $\gcd\{v(d_1),\ldots,v(d_k)\}$ are equal. It follows that

$$\begin{aligned}\|\psi\|_A &= \deg\, v(\Delta_X)\\ &= \deg\, v(\gcd\{d_1,\ldots,d_k\})\\ &= \deg\, \gcd\{v(d_1),\ldots,v(d_k)\}\\ &= \deg\, \gcd\{E_1(H_1(X,x_0;\mathbb{K}_0[t^{\pm 1}]))\}\end{aligned}$$

so to complete the proof it suffices to show that $\deg(\gcd\{v(d_1),\ldots,v(d_k)\}) = \bar{\delta}_0(\psi)$. Since $\mathbb{K}_0[t^{\pm 1}]$ is a principal ideal domain, $H_1(X,x_0;\mathbb{K}_0[t^{\pm 1}])$ is isomorphic to a direct sum of cyclic $\mathbb{K}_0[t^{\pm 1}]$-modules. That is, M^v is equivalent to a matrix of the form

$$\begin{pmatrix} p_1(t) & & \\ & \ddots & \\ & & p_{s-1}(t)\\ 0 & \cdots & 0\end{pmatrix},$$

where $p_i(t)$ is zero for some i if and only if $r_0(X)>0$. We note that the last row of the matrix can be assumed to be zero since $\mathrm{rk}_{\mathscr{K}_0} H_1(X,x_0;\mathscr{K}_0) = \mathrm{rk}_{\mathscr{K}_0} H_1(X;\mathscr{K}_0)+1$. Hence if $r_0(X)=0$, $H_1(X;\mathbb{K}_0[t^{\pm 1}]) = TH_1(X;\mathbb{K}_0[t^{\pm 1}]) = TH_1(X,x_0;\mathbb{Z}\Gamma_0)$ so $\bar{\delta}_0(\psi) = \deg(p_1(t)\cdots p_{s-1}(t))$. Otherwise $p_i(t)=0$ for some i so we have $\bar{\delta}_0(\psi)=0=\deg(p_1(t)\cdots p_{s-1}(t))$. Using the latter presentation of $H_1(X,x_0;\mathbb{K}_0[t^{\pm 1}])$ we compute $\gcd(E_1(H_1(X,x_0;\mathbb{K}_0[t^{\pm 1}]))) = p_1(t)\cdots p_{s-1}(t)$ so

$$\begin{aligned}\|\psi\|_A &= \deg\, \gcd\{E_1(H_1(X,x_0;\mathbb{K}_0[t^{\pm 1}]))\}\\ &= \deg(p_1(t)\cdots p_{s-1}(t))\\ &= \bar{\delta}_0(\psi). \qquad \square\end{aligned}$$

6. Computing δ_i and $\mathscr{A}_n^{\psi}$ via Fox's Free Calculus

We will describe a method of computing the higher-order invariants using Fox's Free Calculus. We remark that this is slightly different than the classically defined free derivatives because we are working with right (instead of the usual left) modules. We refer the reader to [9,10,7,17] for more on the Free Calculus (for left modules).

Let G be any finitely presented group with presentation

$$P = \langle x_1,\ldots,x_l \,|\, r_1,\ldots,r_m\rangle,$$

$F=\langle x_1,\ldots,x_l\rangle$ be the free group on l generators and $\chi: F\twoheadrightarrow G$. For each x_i there is a mapping $\frac{\partial}{\partial x_i}: F\twoheadrightarrow \mathbb{Z}F$ called the ith free derivative. This map is determined by the two conditions

$$\begin{aligned}&\frac{\partial x_j}{\partial x_i} = \delta_{i,j},\\ &\frac{\partial (uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u\,\frac{\partial v}{\partial x_i}.\end{aligned}$$

From these, one can prove that

$$\frac{\partial u^{-1}}{\partial x_i} = -u^{-1}\frac{\partial u}{\partial x_i}.$$

The map $\chi\colon F \twoheadrightarrow G$ extends by linearity to a map $\chi\colon \mathbb{Z}F \twoheadrightarrow \mathbb{Z}G$. The matrix

$$\left(\frac{\partial r_j}{\partial x_i}\right)^{\chi} = \begin{pmatrix} \chi\left(\frac{\partial r_1}{\partial x_1}\right) & \cdots & \chi\left(\frac{\partial r_n}{\partial x_1}\right) \\ \vdots & \ddots & \vdots \\ \chi\left(\frac{\partial r_1}{\partial x_m}\right) & \cdots & \chi\left(\frac{\partial r_n}{\partial x_m}\right) \end{pmatrix}$$

with entries in $\mathbb{Z}G$ is called the Jacobian of the presentation P. We note that this matrix is dependent on the presentation.

Suppose X be a finite CW-complex, $G \cong \pi_1(X, x_0)$ and $\phi\colon G \to \Gamma$. We can assume that X has one 0-cell, x_0. Hence the chain complex of $(X_\Gamma, \widetilde{x}_0)$ is

$$\cdots \to \mathbb{Z}\Gamma^m \xrightarrow{\widetilde{\partial}_2} \mathbb{Z}\Gamma^l \to 0,$$

where l and m are the number of one and two cells of X, respectively. We define an involution on the group ring $\mathbb{Z}F$ by

$$\overline{\sum m_i f_i} = \sum m_i f^{-1}$$

and extend $\phi\colon G \twoheadrightarrow \Gamma$ to $\phi\colon \mathbb{Z}G \twoheadrightarrow \mathbb{Z}\Gamma$ by linearity. It is straightforward to verify that $\widetilde{\partial}_2 = (\overline{\frac{\partial r_j}{\partial x_i}})^{\chi\phi}$. Hence $H_1(X, x_0; \mathbb{Z}\Gamma)$ is finitely presented as $(\overline{\frac{\partial r_j}{\partial x_i}})^{\chi\phi}$. We remark that the existence of the involution in the presentation of H_1 is necessary since we chose to work with right rather than left modules. In the case that Γ is abelian, the involution is not necessary.

Let $\iota\colon \mathbb{Z}\Gamma \to R$ be a ring homomorphism. Then R is a $\mathbb{Z}\Gamma$-R-bimodule and we can consider the right R-module $H_1(X, x_0; R)$. The chain complex for $(X, x_0; R)$ is

$$\cdots \to \mathbb{Z}\Gamma^n \otimes_{\mathbb{Z}\Gamma} R \xrightarrow{\widetilde{\partial}_2 \otimes id_M} \mathbb{Z}\Gamma^m \otimes_{\mathbb{Z}\Gamma} R \to 0.$$

Since $\mathbb{Z}\Gamma^k \otimes_{\mathbb{Z}\Gamma} R \cong R^k$, it follows that $H_1(X, x_0; R)$ is finitely presented as

$$\left(\overline{\frac{\partial r_j}{\partial x_i}}\right)^{\chi\phi\iota}. \tag{6.1}$$

Now let $\phi = \phi_n\colon G \twoheadrightarrow \Gamma_n$ be as defined in Section 3 and $\psi\colon G \twoheadrightarrow \mathbb{Z}$. Choose a splitting $\xi\colon \mathbb{Z} \to \Gamma_n$ and let $R = \mathbb{K}_n[t^{\pm 1}]$. We can use (6.1) to show that $H_1(X, x_0; \mathbb{K}_n[t^{\pm 1}])$ is finitely presented as $(\overline{\frac{\partial r_j}{\partial x_i}})^{\chi\phi_n\iota_\xi}$ where $\iota_\xi\colon \mathbb{Z}\Gamma_n \hookrightarrow \mathbb{K}_n[t^{\pm 1}]$ is the embedding of $\mathbb{Z}\Gamma_n$ into $\mathbb{K}_n[t^{\pm 1}]$.

Moreover, we compute the $\delta_n(\psi)$ as follows. Since $\mathbb{K}_n[t^{\pm 1}]$ is a principal ideal domain, $(\overline{\frac{\partial r_j}{\partial x_i}})^{\chi\phi_n\iota_\xi}$ is equivalent to a diagonal presentation matrix of the form $\{p_1(t), \ldots, p_\lambda(t), 0_{(r,s)}\}$ [20, Theorem 16, p. 43] and $0_{(r,s)}$ is a $r \times s$ size matrix of zeros. Proposition 5.6 implies that $r_n(X) = \mathrm{rk}_{K_n} H_1(X, x_0; K_n) - 1$ hence

$$r_n(X) = r - 1.$$

The above presentation implies that $TH_1(X, x_0; \mathbb{K}_n[t^{\pm 1}])$ has a diagonal presentation matrix of the form $\{p_1(t), \ldots, p_\lambda(t)\}$. Moreover, Proposition 5.6 gives $TH_1(X; \mathbb{K}_n[t^{\pm 1}]) \cong TH_1(X, x_0; \mathbb{K}_n[t^{\pm 1}])$. Thus we have used Fox's Free Calculus to derive a presentation matrix for $\mathscr{A}_n^\psi$ and we have shown that

$$\delta_n(\psi) = \deg \prod_{1 \leqslant i \leqslant \lambda} p_i(t).$$

7. A presentation of $\bar{\mathscr{A}}_n^\psi$ in terms of a surface dual to ψ

In the previous section, we used Fox's Free Calculus to find a presentation matrix of the higher-order Alexander module, $\mathscr{A}_n^\psi(X)$. When X is a 3-manifold, we will show that the localized modules $\bar{\mathscr{A}}_n^\psi(X)$ are finitely presented and that the presentation matrix has topological significance. The matrix will depend on the surface dual to a cohomology class. The presentation will be the higher-order analog of the presentation obtained from a Seifert matrix for knot complements. The presentation obtained will be the main tool that we use in Section 10 to prove that the higher-order degrees give lower bounds for the Thurston norm.

Let X^3 be a compact, orientable 3-manifold (possibly with boundary), $G = \pi_1(X, x_0)$ and $\psi \in H^1(X; \mathbb{Z})$. Let $\phi\colon G \to \Gamma$ be a non-trivial coefficient system and $X_\Gamma \xrightarrow{p} X$ be the regular Γ cover of X. For any ψ as above, there exists a properly embedded surface F in X such that the class $[F] \in H_2(X, \partial X; \mathbb{Z})$ is Poincare dual to ψ. We say that F is dual to ψ.

Let F be a surface dual to ψ, $Y = X - (F \times (0, 1))$, $F_+ = F \times \{1\}$ (see Fig. 1 for an example), and x_0 be a point of $F = F \times \{0\} \subset Y$. Let R be a ring and $\tau\colon \mathbb{Z}\Gamma \to R$ be a ring homomorphism defining R as a $\mathbb{Z}\Gamma$-bimodule. We will exhibit a presentation of $H_1(X; R)$ in terms of $H_1(F; R)$ and $H_1(Y; R)$. First we remark that it makes sense to speak of the homology of F with coefficients in R. By this, we mean the homology corresponding to the coefficient system $\pi_1(F, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{\phi} \Gamma$. Similarly, we have coefficient systems for $\pi_1(Y, x_0)$ and the other terms that are involved in Proposition 7.1 below.

Before we state Proposition 7.1, we need the following notation. Let c be a path in Y with initial point $c(0) = (x_0, 1)$ and endpoint $c(1) = (x_0, 0)$. Let $c_+(s) = (x_0, s)$ and α be the closed curve $c_+ \cdot c$ based at

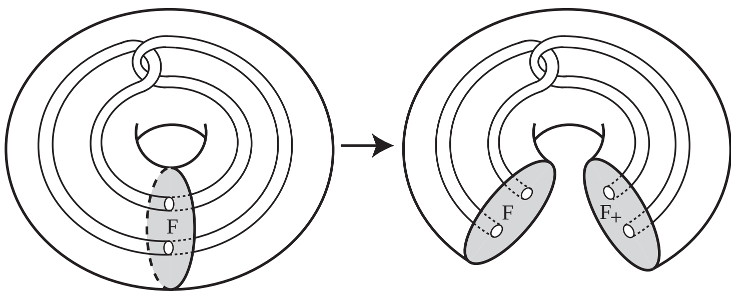

Fig. 1. The Whitehead manifold cut open along F.

x_0. Let $i_\pm : F \to Y$ include F into Y by $i_-(f) = (f, 0)$ and $i_+(f) = (f, 1)$. Finally let $j : Y \to X$ be the inclusion of Y into X.

Proposition 7.1. *Suppose $\gamma = \phi(\alpha)$ is a non-zero element of Γ and either some element of the augmentation ideal of $\mathbb{Z}[\pi_1(F)]$ is invertible (under $\tau \circ \phi \circ i_*$) in R or $\pi_1(F) = 1$. Then the sequence*

$$H_1(F; R) \xrightarrow{\eta} H_1(Y; R) \xrightarrow{j_*} H_1(X; R) \to H_0(F \cup F_+ \cup c; R)$$

is exact where $\eta = (i_+)_* - (i_-)_*\gamma$.

Proof. For convenience, we will omit the R in $H_1(-; R)$ in this proof. Let $U = (F \times I) \cup \alpha$ where $I = [0, 1]$. Then $X = U \cup Y$ and $U \cap Y = F \cup F_+ \cup c$. Consider the homology Mayer–Vietoris sequence for (U, Y) with coefficients in R [36],

$$H_1(F \cup F_+ \cup c) \to H_1((F \times I) \cup \alpha) \oplus H_1(Y) \to H_1(X) \to H_0(F \cup F_+ \cup c) \to . \tag{7.1}$$

We examine the H_1 terms involving F in (7.1). We will compute the homology of these term using the Mayer–Vietoris sequences for $(F, F_+ \cup c)$,

$$0 \to H_1(F) \oplus H_1(F_+ \cup c) \to H_1(F \cup F_+ \cup c) \to H_0(x_0) \to H_0(F) \oplus H_0(F_+ \cup c) \tag{7.2}$$

and $(F \times I, \alpha)$,

$$0 \to H_1(F \times I) \oplus H_1(\alpha) \to H_1(F \times I \cup \alpha) \to H_0(x_0) \to H_0(F \times I) \oplus H_0(\alpha). \tag{7.3}$$

The ideas behind the rest of the proof in both of the cases (stated in the hypothesis) are similar however the proof when $\pi_1(F) \neq 1$ is more technical. Hence we will first consider the special case when $\pi_1(F) = 1$. Since $\pi_1(F)$ is trivial, both $H_0(x_0) \to H_0(F) \oplus H_0(F_+ \cup c)$ and $H_0(x_0) \to H_0(F \times I) \oplus H_0(\alpha)$ are injective. Hence, $H_1(F \cup F_+ \cup c; \mathscr{M}) \cong H_1(F; \mathscr{M}) \oplus H_1(F_+ \cup c; \mathscr{M})$ and $H_1((F \times I) \cup \alpha) \cong H_1((F \times I)) \oplus H_1(\alpha)$.

Since γ is non-trivial in Γ, the curve α does not lift to the Γ-cover of X. Therefore $H_1(\alpha) = 0$ and hence $H_1((F \times I \cup \alpha)) \cong H_1(F \times I)$. Moreover, $H_1((F \times I)) \cong H_1(F)$ where the isomorphism is induced by the map which sends (f, s) to $(f, 0)$.

We analyze the first term in the sequence. The isomorphism $\pi_1(F, x_0) \to \pi_1(F_+ \cup c, x_0)$ given by $[\beta] \mapsto [c \cdot i_+(\beta) \cdot \bar{c}]$ induces an isomorphism $g : H_1(F) \to H_1(F_+ \cup c)$. By $\bar{c}$ we mean the curve defined by $\bar{c}(s) = c(1-s)$. Note that $[c \cdot i_+(\beta) \cdot \bar{c}] = \alpha^{-1}[\beta]\alpha$ in $\pi_1(F \times I, x_0)$. Therefore $H_1(F \cup F_+ \cup c) \cong H_1(F) \oplus H_1(F)$. We note that the composition

$$H_1(F) \to H_1(F_+ \cup c) \to H_1(F \times I \cup \alpha) \to H_1(F)$$

sends σ to $\sigma\gamma$ and the composition $H_1(F) \to H_1(F \times I \cup \alpha) \to H_1(F)$ is the identity.

Using the isomorphisms above, we rewrite (7.1) as

$$H_1(F) \oplus H_1(F) \xrightarrow{(f_F, f_Y),} H_1(F) \oplus H_1(Y) \to H_1(X) \to H_0(F \cup F_+ \cup c) \to ,$$

where $f_F(\sigma_1, \sigma_2) = \sigma_1 + \sigma_2\gamma$ and $f_Y(\sigma_1, \sigma_2) = -((i_-)_*(\sigma_1) + (i_+)_*(\sigma_2))$. It follows from Lemma 7.2 that the sequence

$$H_1(F) \xrightarrow{\eta} H_1(Y) \xrightarrow{j_*} H_1(X) \to H_0(F_- \cup F_+)$$

is exact with $\eta(\sigma) = -((i_-)_*(\sigma\gamma) + (i_+)_*(-\sigma)) = ((i_+)_* - (i_-)_*\gamma)(\sigma)$. The proof of Lemma 7.2 is straightforward hence omitted.

Lemma 7.2. *Suppose $A \oplus A \xrightarrow{(f_A, f_B)} A \oplus B \xrightarrow{g} C \xrightarrow{h} D$ is an exact sequence of right R-modules with $f_A(a_1, a_2) = a_1 + a_2 r$ where $r \in R$, then $A \xrightarrow{f'} B \xrightarrow{g'} C \xrightarrow{h} D$ is an exact sequence of right R-modules with $f'(a) \equiv f_B(ar, -ar)$ and $g'(b) \equiv g(0, b)$.*

Now we assume that some element of the augmentation ideal of $\mathbb{Z}[\pi_1(F)]$ is invertible in R. By [36, p. 275] and [1, p. 34], $H_0(F)$ is isomorphic to the cofixed set R/RJ where J is the augmentation ideal of $\mathbb{Z}[\pi_1(F)]$. Therefore $H_0(F) = H_0(F_+ \cup c) = 0$. We note that $H_0(x_0)$ is the free R-module of rank one generated by $[x_0]$. Choose a splitting ξ_1 for short exact sequence in (7.2) to get

$$H_1(F \cup F_+ \cup c) \cong M \oplus H_1(F) \oplus H_1(F_+ \cup c), \tag{7.4}$$

where M is the free R-module of rank one generated by $\xi_1([x_0])$. Let β be a curve in $F \cup F_+ \cup c$ representing $\xi_1([x_0])$.

Consider the sequence in (7.3). We note that $H_0(\alpha) = R/\langle \tau(\gamma) - 1 \rangle$. Since γ is non-trivial, $\mathrm{Im}(H_1(F \times I \cup \alpha) \to H_0(x_0))$ is the free R-module of rank one generated by $(\tau(\gamma) - 1)[x_0]$. Moreover, $[\beta] \mapsto u(\tau(\gamma) - 1)[x_0]$ where u is a unit of R under the boundary homomorphism. We choose the splitting $\xi_2 : u(\tau(\gamma) - 1)[x_0] \mapsto [\beta]$ to get

$$H_1(F \times I \cup \alpha) \cong N \oplus H_1(F \times I) \oplus H_1(\alpha), \tag{7.5}$$

where N is the free R-module of rank one generated by $[\beta]$.

Using the isomorphisms in (7.4) and (7.5), we can rewrite (7.1) as

$$\begin{aligned} M \oplus H_1(F) \oplus H_1(F_+ \cup c) \to N \oplus H_1(F \times I) \oplus H_1(\alpha) \oplus H_1(Y) \to H_1(X) \to \\ H_0(F \cup F_+ \cup c) \to . \end{aligned}$$

We use Lemma 7.3 to get an exact sequence without the M and N terms as in the case when $\pi_1(F) = 1$. To complete the proof of the proposition, we use the same argument as in the case when $\pi_1(F) = 1$. □

Lemma 7.3. *Let M and N be free right R-modules of rank one generated by m and n, respectively. Suppose $M \oplus A \xrightarrow{f} N \oplus B \xrightarrow{g} C \xrightarrow{h'} D$ is an exact sequence of right R-modules with $f(rm, a) = (rn, f_2(rm, a))$ for some $f_2 : M \oplus A \to B$. Let $\eta : 0 \oplus B \to B$ be the isomorphism defined by $(0, b) \mapsto b$. Then $A \xrightarrow{f'} B \xrightarrow{g'} C \xrightarrow{h'} D$ is an exact sequence of right R-modules where f' and g' are defined by $f'(a) = \eta(f(0, a))$ and $g'(b) = (g(0, b))$.*

Proof. The proof is straightforward hence omitted. □

Now we consider the presentation of $H_1(X; \mathbb{K}_n[t^{\pm 1}])$ where $\mathbb{K}_n$ is the skew field of fractions of $\mathbb{Z}\Gamma'_n$ as defined before. Let F and Y be as defined above. Since $\pi_1(F, x_0)$ and $\pi_1(Y, x_0)$ are contained in the kernel of ψ, we can consider the homology of F and Y with coefficients in $\mathbb{Z}\Gamma'$ and $\mathbb{K}_n$. Since F and Y are finite CW-complexes, $H_1(F; \mathbb{K}_n)$ and $H_1(Y; \mathbb{K}_n)$ are finitely generated free modules hence are isomorphic to $\mathbb{K}_n^l$ and $\mathbb{K}_n^m$, respectively. Thus the $\mathbb{K}_n$-module homomorphisms $i_\pm : H_1(F; \mathbb{K}_n) \to H_1(Y; \mathbb{K}_n)$ can be represented by $l \times m$ matrices $V_{\pm}$ with coefficients in $\mathbb{K}$. We will show that the higher-order module corresponding to ψ is presented by $V_+ - V_- t$.

Proposition 7.4. $\mathrm{Im}(j_*)$ *is finitely presented as* $\mathbb{K}_n[t^{\pm 1}]^l \xrightarrow{P} \mathbb{K}_n[t^{\pm 1}]^m \overset{j_*}{\twoheadrightarrow} \mathrm{Im}(j_*)$ *where* $P = V_+ - V_- t$. *Moreover, if* $\psi|_{\pi_1(F^j)}$ *is non-trivial for each component* F^j *of* F *then* $\bar{\mathscr{A}}_n^{\psi}(X)$ *is finitely presented as*

$$\mathbb{K}_n[t^{\pm 1}]^l \xrightarrow{P} \mathbb{K}_n[t^{\pm 1}]^m \overset{j_*}{\twoheadrightarrow} \bar{\mathscr{A}}_n^{\psi}(X).$$

Proof. Let $\gamma = \phi_n(\alpha)$ be as in Proposition 7.1. We note that $\psi(\gamma) = t$. Choose the splitting $\xi : t \mapsto \gamma$. Since $\pi_1(F, x_0) \subset \Gamma_n'$ we have

$$C_*(F_{\Gamma_n}) \otimes_{\mathbb{Z}\Gamma_n} \mathbb{K}_n[t^{\pm 1}] \cong (C_*(F_{\Gamma_n'}) \otimes_{\mathbb{Z}\Gamma_n'} \mathbb{K}_n) \otimes_{\mathbb{K}_n} \mathbb{K}_n[t^{\pm 1}].$$

Moreover, $\mathbb{K}_n[t^{\pm 1}]$ is a direct sum of free $\mathbb{K}_n$-modules ($\mathbb{K}_n[t^{\pm 1}] \cong \oplus_{i=-\infty}^{\infty} \mathbb{K}_n$). Therefore $\mathbb{K}_n[t^{\pm 1}]$ is a flat left $\mathbb{K}_n$-module. Thus

$$\begin{aligned} H_1(F; \mathbb{K}_n[t^{\pm 1}]) &\cong H_1(F; \mathbb{K}_n) \otimes_{\mathbb{K}_n} \mathbb{K}_n[t^{\pm 1}] \\ &\cong \mathbb{K}_n^l \otimes_{\mathbb{K}_n} \mathbb{K}_n[t^{\pm 1}] \\ &\cong \mathbb{K}_n[t^{\pm 1}]^l. \end{aligned}$$

Similarly, we have $H_1(F; \mathbb{K}_n[t^{\pm 1}]) \cong \mathbb{K}_n[t^{\pm 1}]^m$. The first result follows from Proposition 7.1. If $\psi|_{\pi_1(F^j)}$ is non-trivial for all j then $H_0(F \cup F_+ \cup c; \mathbb{K}_n[t^{\pm 1}]) = 0$ so $\mathrm{Im}(j_*) = \bar{\mathscr{A}}_n^{\psi}(X)$. $\square$

We use the following lemma to show that it suffices to use the presentation matrix $V_+ - V_- t$ when computing $\mathscr{A}_n^{\psi}(X)$.

Lemma 7.5. *Suppose* $B \xrightarrow{g} C \xrightarrow{h} D$ *is an exact sequence of right* R*-modules where* D *is* R*-torsion free and* R *is an Ore domain then* $T_R C = T_R \,\mathrm{Im}(g)$.

Proof. Since $\mathrm{Im}(g) \subseteq C$, it is easy to verify $T\,\mathrm{Im}(g) \subseteq TC$. Let $c \in TC$ then there exists a non-zero $r \in R$ such that $cr = 0$. This says that $h(c)r = h(cr) = 0$ so that $h(c)$ is R-torsion in D hence $h(c) = 0$. By exactness at C we see that $c \in \mathrm{Im}(g)$ and $cr = 0$ so $TC \subseteq T\,\mathrm{Im}(g)$. $\square$

Proposition 7.6. $\mathscr{A}_n^{\psi}(X)$ *is isomorphic to the* $\mathbb{K}_n[t^{\pm 1}]$*-torsion submodule of* $\mathrm{cok}(V_+ - V_- t)$.

Proof. Recall that $\mathscr{A}_n^{\psi}(X) \cong T_{\mathbb{K}_n[t^{\pm 1}]} \bar{\mathscr{A}}_n^{\psi}(X)$. The result follows immediately from Lemma 7.5 and Proposition 7.4. $\square$

8. Examples

In this section we will compute r_n, δ_n, and $\bar{\delta}_n$ for some well known 3-manifolds and relate their values to those given by the Thurston norm. In each of the examples we denote the fundamental group of X

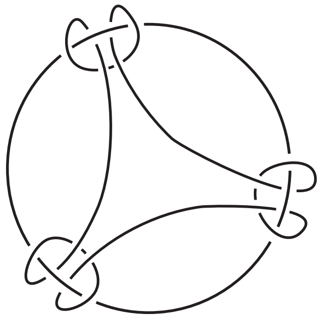

Fig. 2. Each component of L has minimal ropelength at least $2\pi(1+\sqrt{3})$.

by G. Of particular importance will be the 3-manifolds which fiber over S^1 and those which are Seifert fibered. We start with the standard examples.

Example 8.1. 3-torus.

Let $X = S^1 \times S^1 \times S^1$ then $G_r^{(1)} = \{1\}$ hence $r_n(X) = \delta_n(\psi) = 0$ for all ψ and $n \geqslant 0$. Note that since $H_2(X;\mathbb{Z})$ is generated by tori, the Thurston norm is zero for all $\psi \in H^1(X,;\mathbb{Z})$. More generally, if G is any finitely generated abelian group (with $\beta_1(G) \geqslant 1$) then $G_r^{(n)} = T$ where T is the torsion subgroup of G. Hence $r_n(X) = \delta_n(\psi) = 0$.

Example 8.2. $X = \mathop{\#}\limits_{i=1}^{m} S^2 \times S^1$.

Let $X = \#_{i=1}^m S^2 \times S^1$ for $m \geqslant 1$ then $G = F_m$, the free group on m generators. Since $H_2(X;\mathbb{Z})$ is generated by spheres $\|\psi\|_T = 0$ for all ψ. Moreover, every class in $H_2(X;\mathbb{Z})$ can be represented by a disjoint union of embedded spheres. Hence there exists a surface F dual to ψ such that $H_1(F;\mathbb{K}_n[t^{\pm 1}]) = 0$. By Proposition 7.6 we have $\mathscr{A}_n^{\psi}(X) = T\ \mathrm{cok}(\mathbf{0})$. Therefore $\delta_n(\psi) = 0$ for all ψ and $n \geqslant 0$.

Since r_n only depends on the group G we can assume that X is a wedge of m circles. By Remark 4.4, $1 - m = \chi(X) = \sum_{i=0}^{1} rk_{\mathscr{K}_n} H_i(X;\mathscr{K}_n)$. Since ϕ_n is a non-trivial homomorphism, by Lemma 5.7, $H_0(X;\mathscr{K}_n) = 0$. Therefore $r_n(X) = m - 1$.

There is a large class of 3-manifolds for which $r_0(X) \geqslant 1$. Recall that a *boundary link* is a link L in S^3 such that the components admit mutually disjoint Seifert surfaces. It is easy to see that each of these surfaces lifts to the universal abelian cover of $X = S^3 - L$. By Proposition 10.6, $r_0(X) \geqslant 1$. Hence the Alexander norm for X is always trivial. It is often true that the refined Alexander norm, δ_0, is non-trivial. We compute an example below where this is the case.

Let L be the link pictured in Fig. 2. Let F be a Seifert surface of one of the components of L as in figure representing the minimal $\chi\_$ and ψ be dual to F. We will show in Example 8.3 that $\delta_0(\psi) = 4$. Moreover, each component bounds a once punctured genus two surface hence $\|\psi\|_T \leqslant 3$. Hence by Corollary 10.4, we get

$$\delta_0 \leqslant \|\psi\|_T + 1.$$

We conclude that $\|\psi\|_T = 3$. Hence, even for $n = 0$, δ_n gives a sharper bound for the Thurston norm that the Alexander norm. This also shows that the minimal ropelength of each of the components of L is at least $2\pi(1+\sqrt{3})$. For more information on the ropelength of knots and links, see [2].

Example 8.3. Let L be the link in Fig. 2 and $X = S^3 - L$. We use the techniques of Section 6 to compute $\delta_0(\psi)$. Using a Wirtinger presentation, we present $G = \pi_1(X)$ as

$$\begin{aligned}&\langle a, b, c, d, e, f, g, h, i, j, k, l \mid bg^{-1}ic^{-1}i^{-1}g, cj^{-1}la^{-1}l^{-1}j, fe^{-1}hg^{-1}h^{-1}e,\\ &\quad ih^{-1}kj^{-1}k^{-1}h, lk^{-1}ed^{-1}e^{-1}k, da^{-1}e^{-1}a, ebf^{-1}b^{-1},\\ &\quad gb^{-1}h^{-1}b, hci^{-1}c^{-1}, jc^{-1}k^{-1}c, kal^{-1}a^{-1}\rangle.\end{aligned}$$

Using Fox's Free Calculus we obtain a presentation matrix M for $H_1(X_0, \tilde{x}_0)$ (see below). Here, x is the abelianization of a and y is the abelianization of d. Since we used a Wirtinger presentation for G, x and y represent the meridians of L.

$$\begin{pmatrix}
0 & -y & 0 & 0 & 0 & 1-y & 0 & 0 & 0 & 0 & y-1\\
y & 0 & 0 & 0 & 0 & 0 & y-1 & 1-y & 0 & 0 & 0\\
-y & y & 0 & 0 & 0 & 0 & 0 & 0 & y-1 & 1-y & 0\\
0 & 0 & 0 & 0 & -y & x & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1-y & 0 & y-1 & -1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & y & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0\\
1-x & 0 & -y & 0 & 0 & 0 & 0 & x & 0 & 0 & 0\\
0 & 0 & y-1 & 1-y & 0 & 0 & 0 & -1 & 1 & 0 & 0\\
x-1 & 0 & 0 & y & 0 & 0 & 0 & 0 & -x & 0 & 0\\
0 & 1-x & 0 & -y & 0 & 0 & 0 & 0 & 0 & x & 0\\
0 & 0 & 0 & y-1 & 1-y & 0 & 0 & 0 & 0 & -1 & 1\\
0 & x-1 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & -x
\end{pmatrix}.$$

This is equivalent (using the moves in Lemma 9.2) to the matrix

$$\begin{pmatrix}
1-x-y & 0 & 0 & 0\\
0 & 1-x-y & 0 & 0\\
0 & 0 & xy-x-y & 0\\
0 & 0 & 0 & xy-x-y\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{pmatrix}.$$

Hence $r_0(X) = 1$ and

$$\mathscr{A}_0(X) = \frac{\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]}{\langle 1-x-y\rangle} \oplus \frac{\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]}{\langle 1-x-y\rangle} \oplus \frac{\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]}{\langle xy-x-y\rangle} \oplus \frac{\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]}{\langle xy-x-y\rangle}.$$

Let ψ be dual to a Seifert surface for one of the components of L. Then ψ maps one of the generators of $H_1(X)$ to t and the other to 1. The link is symmetric so either choice will suffice. Say $x \mapsto t$ and $y \mapsto 1$. Choose the splitting $t \mapsto x$. Each of the polynomials in the latter matrix has degree 1 in $\mathbb{K}_0[t^{\pm 1}]$ since $1-x-y \mapsto (1-y)+t$ and $xy-x-y \mapsto t(y-1)-y$. Therefore $\delta_0(\psi) = 4$ as desired. In fact, if ψ

maps $x \mapsto t^m$ and $y \mapsto t^n$ then

$$\begin{aligned}\delta_0(\psi) = {} & \deg(t^{2n} + t^{3m+n} + t^{2m+2n} + t^{2m+n} + t^{m+2*n} + t^{m+n} + t^{2m} + t^{3m} + t^{3n} + t^{4m} \\ & + t^{4n} + t^{3m+2n} + t^{2m+3n} + t^{m+3n} + t^{4m+2n} + t^{4m+n} + t^{3m+3n} + t^{4n+2m} + t^{4n+m}),\end{aligned}$$

whereas $\bar{\delta}_0(\psi) = 0$.

Although the invariants are defined algebraically, they respect much of the topology of the 3-manifold. We begin by considering those 3-manifolds which fiber over S^1. In this case the higher-order invariants behave in a very special manner.

Proposition 8.4. *If X is a compact, orientable* 3*-manifold that fibers over S^1 then*

$$r_n(X) = 0.$$

Let ψ be dual to a fibered surface. If $n=0$, $\beta_1(X)=1$, $X \ncong S^1 \times S^2$ and $X \ncong S^1 \times D^2$ then $\delta_n(\psi) = \|\psi\|_T + 1 + \beta_3(X)$. Otherwise,

$$\delta_n(\psi) = \|\psi\|_T.$$

Proof. Consider $X \to S^1$ with fiber surface F and ψ be the element of $H^1(X;\mathbb{Z})$ which is dual to F. The Γ_n-cover of X factors through the infinite cyclic cover corresponding to ψ with total space $F \times \mathbb{R}$. Hence X_n is homeomorphic to $F_n \times \mathbb{R}$ where F_n is a regular cover of F. It follows that $H_*(X;\mathbb{K}_n[t^{\pm 1}])$ is isomorphic to $H_*(F;\mathbb{K}_n)$ as a $\mathbb{K}_n$-module. In particular, $\delta_n(\psi) = \mathrm{rk}_{\mathbb{K}_n} H_1(F;\mathbb{K}_n)$. Moreover, since $H_1(F;\mathbb{K}_n)$ is a finitely generated $\mathbb{K}_n$-module, $r_n(X) = 0$ for all n. That is, $H_1(X;\mathbb{K}_n[t^{\pm 1}])$ is a torsion module for all $n \geqslant 0$.

We restrict to the case that $n=0$ and $\beta_1(X)=1$. We note that $F_0 = F$ and $\mathbb{K}_0 = \mathbb{Q}$ so that $\mathrm{rk}_{\mathbb{K}_0} H_1(F;\mathbb{K}_0) \cong \beta_1(F)$. Thus $\delta_0(\psi) = \beta_1(F) = -\chi(F) + 1 + \beta_3(X)$. As long as $X \ncong S^1 \times S^2$ and $X \ncong S^1 \times D^2$ the Euler characteristic of F is non-positive hence $\|\psi\|_T = -\chi(F)$. Therefore $\delta_0(\psi) = \|\psi\|_T + 1 + \beta_3(X)$.

Note that if the Euler characteristic of F is ever positive then $\pi_1(F) = 1$. Thus we have $H_1(F;\mathbb{K}_n) = 0$ for all $n \geqslant 0$. Therefore both $\delta_n(\psi)$ and $\|\psi\|$ are zero and hence equal for all $n \geqslant 0$.

Otherwise, F_n factors through a non-trivial free abelian cover of F. By Lemma 5.7, $H_0(F;\mathbb{K}_n) = 0$. Since F_n is non-compact, $H_2(F;\mathbb{K}_n) = 0$. It follows that $\delta_n(\psi) = \mathrm{rk}_{\mathbb{K}_n} H_1(F;\mathbb{K}_n) = -\chi(F)$. Moreover, F has non-positive Euler characteristic so $-\chi(F) = \|\psi\|_T$. □

As with the first two examples in this section, there is a large class of 3-manifolds which have vanishing (unrefined) higher-order degrees. This is the class of Seifert fibered manifolds that do not fiber over S^1. We remark that the condition that X not fiber over S^1 is necessary by the previous proposition. Some good references on Seifert fibered manifolds are [15, Chapter 2, 16, Chapter 12, 19, Chapter VI].

Proposition 8.5. *Let X be a compact, orientable Seifert fibered manifold that does not fiber over S^1. If $\beta_1(X) \geqslant 2$ or $n \geqslant 1$ then*

$$\bar{\delta}_n(\psi) = 0$$

for all $\psi \in H^1(X;\mathbb{Z})$.

Proof. This is most easily proven using Theorem 10.1 and some well known results about Seifert fibered 3-manifolds. By Theorem VI.34 of [19], we see that any two-sided incompressible surface in X must be a disc, annulus, or a torus. Therefore the Thurston norm of X is trivial. Theorem 10.1 implies that $\bar{\delta}_n(\psi) \leqslant \|\psi\|_T = 0$ whenever $\beta_1(X)$ or $n \geqslant 1$. $\square$

We end this section by showing that under the connected sum of 3-manifolds, the degrees are additive and the ranks plus 1 are additive. The following is not at all obvious because the fundamental groups of the spaces involved are completely different!

Proposition 8.6. *Suppose $X = X_1 \# X_2$, $\beta_1(X_i) \geqslant 1$ and $\psi \in H^1(X;\mathbb{Z})$. Then*

$$r_n(X) = r_n(X_1) + r_n(X_2) + 1$$

and

$$\delta_n(\psi) = \delta_n(\psi_1) + \delta_n(\psi_2)$$

where $\psi = \psi_1 \oplus \psi_2$.

Proof. We begin by showing that $r_n(X) = r_n(X_1) + r_n(X_2) + 1$. Consider the following Mayer–Vietoris sequence of R-modules for any ring R with $\mathbb{Z}\Gamma_n \subseteq R \subseteq \mathcal{K}_n$. By Γ_n we mean the quotient of $G = \pi_1(X)$ by the $(n+1)^{\text{st}}$ term of the rational derived series of G.

$$0 \to H_1(X_1; R) \oplus H_1(X_2; R) \xrightarrow{v} H_1(X; R) \xrightarrow{\partial_1} H_0(S^2; R) \to \tag{8.1}$$

We note that $H_0(S^2; R) \cong R$ since S^2 is simply connected. For $j = 1, 2$ let $i_j : G_j \to G$ be the inclusion map, $\mathrm{pr}_j : G \to G_j$ be the projection onto the jth factor, and $\Gamma_n^j = (G_j)/(G_j)_r^{(n+1)}$ where $G_j = \pi_1(X_j)$. Since $G = G_1 * G_2$, $\mathrm{pr}_j \circ i_j = id_{G_j}$. Hence the induced maps $\Gamma_n^j \xrightarrow{\bar{i}_j} \Gamma_n \xrightarrow{\overline{\mathrm{pr}}_j} \Gamma_n^j$ are also the identity making $\bar{i}_j$ a monomorphism. Thus the Γ_n^j cover of X_j, X_{jn}, can be constructed as the regular cover corresponding to the map $\phi_n \circ i_j : G_j \twoheadrightarrow \mathrm{Im}(\phi_n \circ i_j)$. We extend $\bar{i}_j$ to a ring monomorphism $\bar{i}_j : \mathbb{Z}\Gamma_n^j \to \mathbb{Z}\Gamma_n$.

The map $G_j \to G/G_r^{(n+1)}$ is the zero map if and only if $G_j/(G_j)_r^{(n+1)} = 0$. We assumed $\beta_1(X_j) > 0$ hence $G_j/(G_j)_r^{(n+1)} \neq 0$. By Lemma 5.7, $H_0(X_j; \mathcal{K}_n) = 0$. Replacing R by $\mathcal{K}_n$ in (8.1) we have

$$r_n(X) = \mathrm{rk}_{\mathcal{K}_n} H_1(X_1; \mathcal{K}_n) + \mathrm{rk}_{\mathcal{K}_n} H_1(X_2; \mathcal{K}_n) + 1.$$

We will show that $\mathrm{rk}_{\mathcal{K}_n} H_1(X_1; \mathcal{K}_n) = \mathrm{rk}_{\mathcal{K}_n^j} H_1(X_1; \mathcal{K}_n^j)$ hence

$$r_n(X) = r_n(X_1) + r_n(X_2) + 1.$$

Let $\widetilde{X}_{jn}$ be the cover of X_j corresponding to $G_j \to \Gamma_n$. Then X'_{jn} is a disjoint union of Γ_n/Γ_n^j copies of X_{jn}. The extension of $\overline{\mathrm{pr}}_j$ to a ring homomorphism $\overline{\mathrm{pr}}_j : \mathbb{Z}\Gamma_n \to \mathbb{Z}\Gamma_n^j$ gives $\mathbb{Z}\Gamma_n^j$ the structure as a $\mathbb{Z}\Gamma_n$-bimodule. Moreover, since $\overline{\mathrm{pr}}_j \circ \bar{i}_j = id_{\Gamma_n^j}$, $\cdot \otimes_{\mathbb{Z}\Gamma_n^j} (\mathbb{Z}\Gamma_n \otimes_{\mathbb{Z}\Gamma_n} \mathbb{Z}\Gamma_n^j)$ acts trivially on any right $\mathbb{Z}\Gamma_n^j$-module.

Therefore

$$\begin{aligned} C_*(\widetilde{X}_{jn})\otimes_{\mathbb{Z}\Gamma_n}\mathbb{Z}\Gamma_n^j &\cong (C_*(X_{jn})\otimes_{\mathbb{Z}\Gamma_n^j}\mathbb{Z}\Gamma_n)\otimes_{\mathbb{Z}\Gamma_n}\mathbb{Z}\Gamma_n^j \\ &\cong C_*(X_{jn})\otimes_{\mathbb{Z}\Gamma_n^j}(\mathbb{Z}\Gamma_n\otimes_{\mathbb{Z}\Gamma_n}\mathbb{Z}\Gamma_n^j) \\ &\cong C_*(X_{jn}). \end{aligned} \tag{8.2}$$

$\bar{i}_j : \mathbb{Z}\Gamma_n^j \to \mathbb{Z}\Gamma_n$ is a monomorphism, hence we can extend $\bar{i}_j$ to the right ring of quotients of $\mathbb{Z}\Gamma_n^j$ and $\mathbb{Z}\Gamma_n$, $\bar{i}_j : \mathscr{K}_n^j \to \mathscr{K}_n$. Therefore $\mathscr{K}_n$ is a flat left $\mathscr{K}_n^j$-module by the following lemma.

Lemma 8.7. *Suppose that R is a right and left principal ideal domain, S has no zero divisors, and $f : R \hookrightarrow S$ is a ring monomorphism. Then S is a flat left R-module.*

Proof. Let $s \in S$ and $r \in R$ with $s \neq 0$. Suppose that $f(r)s = 0$. S has no zero-divisors hence $f(r) = 0$. Moreover, f is a monomorphism so $r = 0$. Therefore S is R-torsion-free. Since R is a PID, every finitely generated torsion-free R-module is free hence flat. Every module is the direct limit of its finitely generated submodules. Hence S is the direct limit of flat modules. Thus, by [31, Proposition 10.3], S is flat. □

We apply $-\otimes_{\mathbb{Z}\Gamma_n^j}\mathscr{K}_n$ to (8.2) to get

$$C_*(\widetilde{X}_{jn})\otimes_{\mathbb{Z}\Gamma_n}\mathscr{K}_n \cong C_*(X_{jn})\otimes_{\mathbb{Z}\Gamma_n^j}\mathscr{K}_n.$$

Since $C_*(X_{jn})\otimes_{\mathbb{Z}\Gamma_n^j}\mathscr{K}_n \cong C_*(X_{jn})\otimes_{\mathbb{Z}\Gamma_n^j}\mathscr{K}_n^j\otimes_{\mathscr{K}_n^j}\mathscr{K}_n$ and $\mathscr{K}_n$ is a flat left $\mathscr{K}_n^j$-module,

$$H_*(C_*(\widetilde{X}_{jn})\otimes_{\mathbb{Z}\Gamma_n}\mathscr{K}_n) \cong H_*(C_*(X_{jn})\otimes_{\mathbb{Z}\Gamma_n^j}\mathscr{K}_n^j)\otimes_{\mathscr{K}_n^j}\mathscr{K}_n.$$

Thus $\operatorname{rk}_{\mathscr{K}_n^j} H_1(X_1; \mathscr{K}_n^j) = \operatorname{rk}_{\mathscr{K}_n} H_1(X_1; \mathscr{K}_n)$ as desired.

Now we show that

$$T_R H_1(X; R) \cong T_R H_1(X_1; R) \oplus T_R H_1(X_2; R). \tag{8.3}$$

First we note that $T(H_1(X_1; R) \oplus H_1(X_2; R)) \cong TH_1(X_1; R) \oplus TH_1(X_2; R)$. Consider the restriction of v in (8.1) to the torsion submodule of $H_1(X_1; R) \oplus H_1(X_2; R)$,

$$v_T : T(H_1(X_1; R) \oplus H_1(X_2; R)) \to TH_1(X; R).$$

We show that v_T is an isomorphism. It is immediate that v_T is a monomorphism since v is a monomorphism. To show that v_T is surjective, let $x \in TH_1(X; R)$ and $0 \neq r \in R$ with $xr = 0$. Since $H_0(S^2; R)$ is R-torsion free, $\partial_1(x) = 0$ hence there exists $y \in H_1(X_1; R) \oplus H_1(X_2; R)$ such that $v_T(y) = x$. Moreover, $v_T(yr) = v_T(y)r = xr = 0$. Since v_T is a monomorphism $yr = 0$. Hence $y \in T(H_1(X_1; R) \oplus H_1(X_2; R))$.

Since $H^1(X; \mathbb{Z}) \cong H^1(X_1; \mathbb{Z}) \oplus H^1(X_2; \mathbb{Z})$, ψ can be uniquely written as $\psi_1 \oplus \psi_2$ where $\psi_j \in H^1(X_j; \mathbb{Z})$. Note that ψ_j need not be a primitive class in $H^1(X)$. For each j, let d_j be the largest divisor of ψ_j. Hence, there exist ψ'_j primitive with $d_j\psi'_j = \psi$. Recall that $\ker \psi'_j = \ker \psi_j$ and $\delta_n(\psi_j) = d_j\delta_n(\psi'_j)$.

Substitute $R = \mathbb{Z}\Gamma_n(\mathbb{Z} \ker \psi)^{-1}$ into (8.3). Then

$$\delta_n(\psi) = \operatorname{rk}_{\mathbb{K}_n} TH_1(X_1; \mathbb{Z}\Gamma_n(\mathbb{Z} \ker \psi)^{-1}) \oplus \operatorname{rk}_{\mathbb{K}_n} TH_1(X_2; \mathbb{Z}\Gamma_n(\mathbb{Z} \ker \psi)^{-1}),$$

where $\mathbb{K}_n=\mathbb{Z}\ker\psi(\mathbb{Z}\ker\psi-\{0\})^{-1}$ is the right ring of quotients of $\mathbb{Z}\ker\psi$. Recall that if S is a right divisor set then RS^{-1} exists and is the ring obtained by inverting all of the elements in S. Let $R_n=\mathbb{Z}\Gamma_n(\mathbb{Z}\ker\psi-\{0\})^{-1}$, $R_n^j=\mathbb{Z}\Gamma_n^j(\mathbb{Z}\ker\psi'_j-\{0\})^{-1}$, and $\mathbb{K}_n^j=\mathbb{Z}\ker\psi'_j(\mathbb{Z}\ker\psi'_j-\{0\})^{-1}$. To complete the proof we must show that

$$\mathrm{rk}_{\mathbb{K}_n}T_{R_n}H_1(X_j;R_n)\cong\mathrm{rk}_{\mathbb{K}_n^j}T_{R_n^j}H_1(X_1;R_n^j).$$

Since $\psi=\psi_j\circ\bar{i}_j$, $\bar{i}_j(\ker\psi_j)\subset\ker\psi$, we can extend $\bar{i}_j$ to $\bar{i}_j:R_n^j\to R_n$. By Lemma 8.7, R_n is a flat left R_n^j-module. Therefore

$$H_*(C_*(\widetilde{X}_{jn})\otimes_{\mathbb{Z}\Gamma_n}R_n)\cong H_*(C_*(X_{jn})\otimes_{\mathbb{Z}\Gamma_n^j}R_n^j)\otimes_{R_n^j}R_n. \tag{8.4}$$

Let $M=H_1(X_j;R_n^j)$ then $M\cong(R_n^j)^m\oplus T_{R_n^j}M$ since M is finitely generated and R_n^j is a principal ideal domain. It is straightforward to show that

$$T_{R_n}(M\otimes_{R_n^j}R_n)\cong T_{R_n^j}M\otimes_{R_n^j}R_n.$$

Moreover, $T_{R_n^j}M\cong\frac{R_n^j}{\langle r_1\rangle}\oplus\cdots\oplus\frac{R_n^j}{\langle r_k\rangle}$ so it suffices to show that $\mathrm{rk}_{\mathbb{K}_n}\frac{R_n}{\langle\bar{i}_j(r)\rangle}=d_j\mathrm{rk}_{\mathbb{K}_n^j}\frac{R_n^j}{\langle r\rangle}$ for any non-zero $r\in R_n^j$. Note that this would imply

$$\begin{aligned}\mathrm{rk}_{\mathbb{K}_n}(T_{R_n^j}M\otimes_{R_n^j}R_n)&=\mathrm{rk}_{\mathbb{K}_n}\left(\frac{R_n}{\langle\bar{i}_j(r_1)\rangle}\oplus\cdots\oplus\frac{R_n}{\langle\bar{i}_j(r_k)\rangle}\right)\\&=\mathrm{rk}_{\mathbb{K}_n}\frac{R_n}{\langle\bar{i}_j(r_1)\rangle}+\cdots+\mathrm{rk}_{\mathbb{K}_n}\frac{R_n}{\langle\bar{i}_j(r_k)\rangle}\\&=d_j\mathrm{rk}_{\mathbb{K}_n^j}\frac{R_n^j}{\langle r_1\rangle}+\cdots+d_j\mathrm{rk}_{\mathbb{K}_n^j}\frac{R_n^j}{\langle r_k\rangle}.\end{aligned}$$

Let $T\in\Gamma_n^j$ such that $\psi_j(T)=t^{d_j}$. We can write any element in Γ_n^j as $T^m\kappa$ where $\kappa\in\ker\psi_j$. Hence r can be written as a non-constant (Laurent) polynomial in T with coefficients in $\mathbb{K}_j$. We can assume that $r=a_0+Ta_1+\cdots+T^qa_q$ with $a_0\neq0$.

Since ψ is surjective, there is an $S\in\Gamma_n$ such that $\psi(S)=t$. We can write any element in Γ_n as S^pf where $f\in\ker\psi$. In particular, any element of Γ_n that maps t^{d_j} under ψ can be written as S_j^df. Since $\psi(\bar{i}_j(T))=\psi(T)=t_j^d$, $\bar{i}_j(T)=S_j^df$ for some $f\in\ker\Gamma_n$. Hence

$$\begin{aligned}\bar{i}_j(r)&=\bar{i}_j(a_0)+\bar{i}_j(T)\bar{i}_j(a_1)+\cdots+\bar{i}_j(T)^q\bar{i}_j(a_q)\\&=\bar{i}_j(a_0)+S_j^df\bar{i}_j(a_1)+\cdots+(S_j^df)^q\bar{i}_j(a_q)\\&=\bar{i}_j(a_0)+S_j^dg_1\bar{i}_j(a_1)+\cdots+S^{d_jq}g_k\bar{i}_j(a_q)\end{aligned}$$

for some $g_i\in\ker\Gamma_n$. We note that $\bar{i}_j(a_i)\in\ker\Gamma_n$ which gives us our desired result. This completes the proof that $\delta_n(\psi)=\delta_n(\psi_1)+\delta_n(\psi_2)$. $\square$

An immediate consequence is that $r_n(X)\geqslant 1$ whenever the hypotheses in Proposition 8.6 are satisfied. In particular, we have $\bar{\delta}_n(\psi)=0$ for all $\psi\in H^1(X;\mathbb{Z})$. We note that if $\beta_1(X_1)=0$ then $H^1(X_2;\mathbb{Z})\cong H^1(X;\mathbb{Z})$ is an isomorphism, $\delta_n^{X_2}(\psi)=\delta_n^X(\psi)$ and $r_n(X)=r_n(X_2)$ (similarly if $\beta_1(X_2)=0$).

Corollary 8.8. *Let X be a compact, orientable 3-manifold with $r_k(X)=0$ for some $k\geqslant 0$. Suppose that $G=\pi_1(X)$ does not satisfy both $\frac{G}{G_r^{(1)}}\cong\mathbb{Z}$ and $\frac{G_r^{(1)}}{G_r^{(2)}}=0$. Then there exists an irreducible 3-manifold Y with $H=\pi_1(Y)$ such that*

$$\frac{G}{G_r^{(n+1)}}=\frac{H}{H_r^{(n+1)}}$$

for all $n\geqslant 0$.

Proof. We assume that $\beta_1(X)\geqslant 1$. We can factor X as $X=X_1\#\cdots\#X_l$ where each X_i is prime [16]. Since $r_k(X)=0$, there is exactly one i such that $\beta_1(X_i)\neq 0$ by Proposition 8.6. Let Y be the aforementioned factor and $H=\pi_1(Y)$. It is easy to verify that $\frac{G}{G_r^{(n+1)}}=\frac{H}{H_r^{(n+1)}}$. Moreover, the hypothesis on G guarantees that $Y\neq S^2\times S^1$. Therefore Y is irreducible. $\square$

9. Rank of torsion modules over skew polynomial rings

In this section we will show that the rank of a (torsion) module presented by an $m\times m$ matrix of the form $A+tB$ (where A, B have coefficients in $\mathbb{K}$) has rank at most m as a $\mathbb{K}$ vector space. This is well known when $\mathbb{K}$ is a commutative field. In this case the rank of M over $\mathbb{K}$ is the degree of the determinant of $A+tB$ which is a polynomial with degree less than or equal to m. We will use this result in the proof that the higher-order degrees give lower bounds for the Thurston norm in the next section. For a first read, the reader may wish to only read the statements in Proposition 9.1 and Lemma 9.2 before proceeding to the next section.

Let M be a right $\mathbb{K}[t^{\pm 1}]$-module with presentation matrix of the form $A+tB$ where A and B are $l\times m$ matrices with coefficients in $\mathbb{K}$, l is the number of generators of M and $\mathbb{K}$ is a (skew) field. We denote by TM the $\mathbb{K}[t^{\pm 1}]$-torsion submodule of M. Using the embedding $\mathbb{K}\to\mathbb{K}[t^{\pm 1}]$ by $k\mapsto k\cdot 1$ we consider TM as a module over $\mathbb{K}$.

Proposition 9.1. *If M is a right $\mathbb{K}[t^{\pm 1}]$-module with presentation matrix of the form $A+tB$ where A and B are $l\times m$ matrices with coefficients in $\mathbb{K}$ and $\mathbb{K}$ is a division ring then*

$$\mathrm{rk}_{\mathbb{K}}TM\leqslant\min\{l,m\}. \tag{9.1}$$

We begin by stating when two presentation matrices of a finitely presented right R-module H are equivalent.

Lemma 9.2. [37, pp. 117–120]. *Two presentation matrices of H are related by a finite sequence of the following operations.*

(1) *Interchange two rows or two columns.*

(2) *Multiply a row* (*on left*) *or column* (*on right*) *by a unit of R.*
(3) *Add to any row a R-linear combination of other rows* (*multiplying a row by unit of R on left*) *or to any column a R-linear combination of other columns* (*multiplying a column by a unit of R on right*).
(4) $P \to (P \ \ *)$, *where* $*$ *is a R-linear combination of columns of P.*
(5) $P \to \begin{pmatrix} P & * \\ 0 & 0 \end{pmatrix}$, *where* $*$ *is an arbitrary column.*

We will find the following lemmas useful in the proof of Proposition 9.1.

Lemma 9.3. *A presentation matrix of the form* $\begin{pmatrix} A_1+tI_s & A_2 \\ A_3 & A_4 \end{pmatrix}_{l\times m}$ *where* A_i *has entries in* $\mathbb{K}$ (*a non-commutative division ring*) *is related* (*in the sense of Lemma* 9.2) *to a matrix of the form* $\begin{pmatrix} A_1'+tI_s & A_2' \\ A_3' & 0 \end{pmatrix}_{(l-r)\times(m-r)}$ *for some* $r \geqslant 0$.

Proof. Since A_4 is a matrix over $\mathbb{K}$ [20, Corollary to Theorem 16, p. 43] there are C and D such that

$$CA_4D = \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix}.$$

Here, C and D are units in the rings of $l \times l$ and $m \times m$ matrices with entries in $\mathbb{K}$, respectively. Hence we can get the new presentation matrix

$$\begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} A_1+tI_s & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} 0 & I \\ D & 0 \end{pmatrix} = \begin{pmatrix} A_1+tI_s & A_2D \\ CA_3 & CA_4D \end{pmatrix}$$
$$= \begin{pmatrix} A_1+tI_s & A_2D \\ CA_3 & \begin{matrix} 0 & 0 \\ 0 & I_r \end{matrix} \end{pmatrix}.$$

Now we can make the last r rows of the matrix of the form $(0 \ \ I_r)$ by adding $(\text{column}(m-(l-i)))\cdot(-a_{i,j})$ to column j for each non-zero entry $a_{i,j}$ in the last r rows of CA_3. In general this will change A_1+tI_s to $A_1'+tI_s$ for some A_1' whose entries lie in $\mathbb{K}$. Using operation 5, we delete the last r rows and columns to obtain our desired result. □

Lemma 9.4. *If* $A_3 \neq 0$ *then the presentation matrix* $\begin{pmatrix} A_1+tI_s & A_2 \\ A_3 & 0 \end{pmatrix}$ *of size* $l \times m$ *is related to one of the form* $\begin{pmatrix} A_1'+tI_{s-1} & A_2' \\ A_3' & 0 \end{pmatrix}$ *of size* $(l-r)\times(m-r)$ *where* $r \geqslant 1$.

Proof. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & 0 \end{pmatrix}$$

and $a_{k,i}$ be the (k,i) entry of A. By permuting rows in A_3 we can assume that the last row has a non-zero element. Suppose that the first non-zero element in this row occurs in the ith column. We can assume that this element is 1. Now if $a_{l,j}$ is any other non-zero entry in the last row $(i<j\leqslant s)$ we add $(\text{column } i)\cdot(-a_{l,j})$ to $(\text{column } j)$ to get a presentation with a zero in column j of the last row. However, this changes the $(i,\ j)$ the entry of our matrix to $(a_{i,j}-a_{i,i}a_{l,j})-ta_{l,j}$ which does not lie in $\mathbb{K}$. To remedy this, we add $(ta_{l,j}t^{-1})\cdot(\text{ row} j)$ to $(\text{row } i)$. Performing these two steps for all non-zero $a_{l,j}$ gives us a

matrix whose last row is of the form $(0, \dots, 0, 1, 0, \dots, 0)$. By cyclically permuting columns i through m (so that the ith column becomes the mth column) and using the operation of type 5 in Lemma 9.2, we see that this matrix is related to the matrix obtained by deleting column i and row l. We note that all the entries in row i lie in $\mathbb{K}$. For the final step we cyclically permute rows i through s (so that the ith row becomes the sth row) and use Lemma 9.3 to get our desired result. $\square$

Proof of Proposition 9.1. Let $P = A + tB$ be a presentation matrix of M. As in the proof of Lemma 9.3 there are $C_{l\times l}$ and $D_{m\times m}$ (units in the rings of matrices over $\mathbb{K}$) such that

$$CBD = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix},$$

where $s \leqslant \min\{l, m\}$. Now if $C^t \equiv tCt^{-1}$ we have

$$C^t PD = C^t(A + tB)D = C^t AD + tCBD.$$

Hence M has a presentation matrix of the form

$$A + t\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}_{l\times m} = \begin{pmatrix} a_{1,1}+t & \cdots & a_{1,s} & \\ \vdots & \ddots & \vdots & * \\ a_{s,1} & \cdots & a_{s,s}+t & \\ & * & & * \end{pmatrix}, \tag{9.2}$$

where by A is now $C^t AD$, a matrix with entries in $\mathbb{K}$. We can now use Lemma 9.3 and Lemma 9.4 to get a new presentation matrix

$$\begin{pmatrix} A_1' + tI_{s'} & A_2' \\ 0 & 0 \end{pmatrix},$$

where $s' \leqslant s$. It follows that TM has a presentation matrix

$$(A_1' + tI_{s'} \;\; A_2'), \tag{9.3}$$

where A_i' are matrices in $\mathbb{K}$. Let σ_i $(1 \leqslant i \leqslant s')$ be the generators of TM corresponding to (9.3). We show that these generate TM as a $\mathbb{K}$-module. Let $a_{i,j}$ be the (i, j) the entry of A_1' then we have the relations $\sigma_1 a_{1,j} + \cdots + \sigma_j(a_{j,j} + t) + \cdots + \sigma_{s'} a_{s',j} = 0$ for $j \leqslant s'$. Hence $\sigma_i t = -\sum \sigma_k a_{k,i}$ is in the span of $\{\sigma_i\}$. We prove by induction on n that $\sigma_i t^n$ is in the span of $\{\sigma_i\}$. Suppose $\sigma_i t^n = \sum \sigma_k b_{k,i}$ where $b_{k,i} \in \mathbb{K}$ then

$$\begin{aligned}
\sigma_i t^{n+1} &= \left(\sum_k \sigma_k b_{k,i}\right) t \\
&= \sum_k \sigma_k t(t^{-1} b_{k,i} t) \\
&= \sum_k \left(\sum_l \sigma_l b_{l,k}\right)(t^{-1} b_{k,i} t) \\
&= \sum_l \sigma_l \left(\sum_k b_{l,k}(t^{-1} b_{k,i} t)\right)
\end{aligned}$$

for all $i \leqslant s'$. Therefore any element $\sum \sigma_i p_i(t)$ with $p_i(t) \in \mathbb{K}[t^{\pm 1}]$ can be written as a linear combination of σ_i with coefficients in $\mathbb{K}$. It follows that $\operatorname{rk}_{\mathbb{K}} TM \leqslant s' \leqslant s \leqslant \min\{l, m\}$. $\square$

10. Relationships of δ_n and $\bar{\delta}_n$ to the Thurston norm

In this section, we will prove one of the main theorems of this paper. We show that the higher-order degrees of a 3-manifold give lower bounds for the Thurston norm. The result when X is a knot complement appears in [3] although it uses some of our work.

Theorem 10.1. *Let X be a compact, orientable 3-manifold (whose boundary if any is a union of tori). For all $\psi \in H^1(X; \mathbb{Z})$ and $n \geqslant 0$*

$$\bar{\delta}_n(\psi) \leqslant \|\psi\|_T$$

except for the case when $\beta_1(X) = 1$, $n = 0$, and $X \ncong S^1 \times D^2$. In this case, $\bar{\delta}_0(\psi) \leqslant \|\psi\|_T + 1 + \beta_3(X)$ whenever ψ is a generator of $H^1(X; \mathbb{Z}) \cong \mathbb{Z}$. Moreover, equality holds in all cases when $\psi : \pi_1(X) \twoheadrightarrow \mathbb{Z}$ can be represented by a fibration $X \to S^1$.

The proof of this theorem will follow almost directly from Propositions 7.4 and 9.1. However, because of some technical details we postpone the proof until after Corollary 10.7. We will begin the section by proving a more generalized (but less applicable) version of Theorem 10.1. We first introduce some notation.

Let X be a 3-manifold, $\psi \in H^1(X; \mathbb{Z})$, $G = \pi_1(X)$, $\Gamma_n = G/G_r^{(n+1)}$. Recall that if F is an embedded surface dual to ψ, we can consider the homology of F with coefficients in $\mathbb{K}_n$, where $\mathbb{K}_n$ is the field of fractions of $\mathbb{Z}\Gamma_n'$. Define the higher-order Betti numbers of F to be

$$b_i^n(F) = \operatorname{rk}_{\mathbb{K}_n} H_i(F; \mathbb{K}_n).$$

By Remark 4.4 we see that the Euler characteristic of F can be computed using b_i^n,

$$\chi(F) = \sum (-1)^i b_i^n(F) \tag{10.1}$$

for any $n \geqslant 0$.

Now we consider the collection of Thurston norm minimizing surfaces dual to ψ, $\mathscr{F}_\psi$. It is very possible that a surface in $\mathscr{F}_\psi$ is highly disconnected. One could ask, "What is the minimal number of components of a surface in $\mathscr{F}_\psi$?" For our purposes, it will turn out to be important to compute the number of components of surface in $\mathscr{F}_\psi$ that lift to the nth order cover of X. To be precise we make the following definitions.

Let $F = \coprod F^i$ be a (possibly disconnected) surface. We define $N_n(F)$ to be the number of components of F with $i_*(\pi_1(F^i)) \subseteq G_r^{(n+1)}$ and $N_n^c(F)$ to be the number of closed components of F with $i_*(\pi_1(F^i)) \subseteq G_r^{(n+1)}$. Finally, we define

$$\mathscr{N}_n(\psi) = \min_{F \in \mathscr{F}_\psi} \{N_n(F) + N_n^c(F)\}.$$

Theorem 10.2. *Let X be a compact, orientable* 3*-manifold* (*possibly with boundary*). *For all $\psi \in H^1(X;\mathbb{Z})$ and $n \geqslant 0$*

$$\delta_n(\psi) \leqslant \|\psi\|_T + \mathcal{N}_n(\psi).$$

Proof. Let F be a Thurston norm minimizing surface dual to ψ that minimizes $N_n(F)+N_n^c(F)$. We remark that a connected surface has $b_0^n(F)=0$ if and only if the coefficient system $\phi \circ i_* : \pi_1(F) \to G/G_r^{(n)}$ is non-trivial by Lemma 5.7. Therefore $N_n(F)=b_0^n(F)$. Similarly, we have $N_n^c(F)=b_2^n(F)$. By (10.1),

$$\begin{aligned} b_1^n(F) &= -\chi(F) + N_n(F) + N_n^c(F) \\ &\leqslant \|\psi\|_T + \mathcal{N}_n(\psi). \end{aligned}$$

To complete the proof, we show that $\delta_n(\psi) \leqslant b_1^n(F)$. By Proposition 7.6, $\mathcal{A}_n^{\psi}(X)$ has a presentation matrix of the form $A+tB$ of size $(b_1^n(F) \times m)$ where $m = rk_{\mathbb{K}_n} H_1(Y;\mathbb{K}_n)$. Thus, by Proposition 9.1 we have

$$\delta_n(\psi) = \mathrm{rk}_{\mathbb{K}_n} \mathcal{A}_n^{\psi}(X) \leqslant \min\{b_1^n(F), m\} \leqslant b_1^n(F). \qquad \square$$

We note that the term $\mathcal{N}_n(\psi)$ is an invariant of the pair (X,ψ). However, in a general, $\mathcal{N}_n(\psi)$ may be difficult to compute. Fortunately, in some cases, we may be able bound this term by a constant.

Suppose that we are interested in the genera of knots or links. More generally, suppose we are only interested in the connected surfaces embedded in a 3-manifold. Then it is reasonable to measure the complexity of the surface by its first Betti number. Using the proof of Theorem 10.2, we can find a lower bound for the first Betti number of F that has no “extra term”.

Corollary 10.3. *If F is any surface dual to ψ then $\delta_n(\psi) \leqslant \beta_1(F)$.*

Proof. Since $N_n(F) \leqslant \beta_0(F)$ and $N_n^c(F) \leqslant \beta_2(F)$,

$$\begin{aligned} b_1^n(F) &= -\chi(F) + N_n(F) + N_n^c(F) \\ &= \beta_1(F) + (N_n(F) - \beta_0(F)) + (N_n^c(F) - \beta_2(F)) \\ &\leqslant \beta_1(F). \end{aligned}$$

Therefore, $\delta_n(\psi) \leqslant b_1^n(F) \leqslant \beta_1(F)$. $\square$

We consider the case when X is the complement of a link L in S^3. If L has m components then $H_1(X;\mathbb{Z}) \cong \mathbb{Z}^m$ generated by the m meridians μ_i. Let ψ_i be defined by $\psi_i(\mu_j) = t^{\delta_{ij}}$. That is, ψ_i is dual to any surface that algebraically intersects the ith meridian once and the jth meridian zero times for $j \neq i$. We will show that a Thurston norm minimizing surface dual to ψ_i can be chosen to be connected and hence we can bound the term $\mathcal{N}_n(\psi_i)$ by 1.

Corollary 10.4. *Let $X = S^3 - L$ and ψ_i be as defined above. Then*

$$\delta_n(\psi_i) \leqslant \|\psi_i\|_T + 1$$

for all $n \geqslant 0$.

Proof. We show that for all $i \in \{1, \ldots, m\}$ there exists a Thurston minimizing surface F_i which is connected and has non-trivial boundary. Hence $\mathcal{N}_n(\psi_i) \leqslant 1$ and $\mathcal{N}_n^c(\psi_i) = 0$. The result follows from Theorem 10.2.

Let $F = \coprod F^j$ be a Thurston norm minimizing surface dual to ψ_i. Let $\{\alpha_k\}$ be the set of boundary components of F. Suppose that α_k and α_l are parallel and have opposite orientation. Then we can glue an annulus along α_k and α_l to get a new surface whose relative homology class and $\chi\_$ are unchanged. Altering our surface in this way, we can assume that there is exactly one k_0 such that $\alpha_{k_0} \cdot \mu_i = 1$ and $\alpha_k \cdot \mu_j = 0$ for all k whenever $j \neq i$. Secondly, we can assume that all the components of F have boundary since every closed surface is zero in $H_2(X, \partial X; \mathbb{Z})$. Now, let F_i be the connected component of F having α_{k_0} as one of its boundary components. Then F_i represents the same relative homology class as F and $\chi\_(F_i) \leqslant \chi\_(F)$. Thus F_i is a Thurston norm minimizing surface dual to ψ which is connected and has non-trivial boundary. $\square$

The *ropelength* of a link is the quotient of its length by it's thickness. In [2, Corollary 22], Cantarella et al. show that the *minimal ropelength* $R(L_i)$ of the ith component of a link $L = \coprod L_i$ is bounded from below by $2\pi(1 + \sqrt{\|\psi_i\|_T})$. Hence the higher-order degrees give computable lower bounds for the ropelength of knots and links.

Corollary 10.5. *Let $X = S^3 - L$ and ψ_i be as defined above. For each $n \geqslant 0$,*

$$R(L_i) \geqslant 2\pi(1 + \sqrt{\delta_n(\psi_i) - 1}).$$

Moreover, if $\beta_1(X) \geqslant 2$ or $n \geqslant 1$ (or both) then

$$R(L_i) \geqslant 2\pi(1 + \sqrt{\bar{\delta}_n(\psi_i)}).$$

Proof. The first (respectively second) statement follows from the bound given in Corollary 22 [2] and Corollary 10.4 (respectively Theorem 10.1). $\square$

Although it seems that the second statement in the Corollary is “stronger”, in practice the first statement is often more useful. That is, $\bar{\delta}_n = 0$ whenever the rank is positive hence gives no new information. We exemplify this phenomena in Example 8.3.

We would like to determine conditions that will guarantee that a surface will not lift to the nth-order cover of X. We show that if $\beta_1(X) \geqslant 2$ then $r_n(X) = 0$ guarantees that no homologically essential surface can lift to the nth-order cover of X. In particular, if $r_0(X) = 0$ then $i_*\pi_1(F) \nsubseteq G_r^{(1)}$ so that $i_*\pi_1(F) \nsubseteq G_r^{(n+1)}$ for all $n \geqslant 0$. If $\beta_1(X) = 1$ a surface representing the generator of $H_2(X, \partial X; \mathbb{Z})$ can only lift if the rational derived series stabilizes at the first step, i.e. $G_r^{(1)} = G_r^{(2)} = \cdots = G_r^{(n+1)}$.

Proposition 10.6. *If there exists a compact, connected, orientable, two-sided properly embedded surface $F \subseteq X$ with $\beta_1(X) \geqslant 2$ such that $0 \neq [F] \in H_2(X, \partial X; \mathbb{Z})$ and $i_*\pi_1(F) \subseteq G_r^{(n+1)}$ then $r_n(X) \geqslant 1$.*

Proof. Let $Y = \overline{X \backslash (F \times I)}$, since $[F] \neq 0$ F does not separate X. Hence Y is connected. Let γ be a oriented simple closed curve that intersects F exactly once, then $G = \pi_1(X) = \langle \pi_1(Y), \gamma |$ relations from $\pi_1(F)\rangle$. If $\pi_1(Y) \subseteq G_r^{(1)}$ then $G/G_r^{(1)} = \langle \gamma \rangle$ which contradicts $\beta_1(X) \geqslant 2$. This implies that $\pi_1(Y) \nsubseteq G_r^{(1)}$ hence

$\pi_1(Y)\nsubseteq G_r^{(n+1)}$ for all $n\geqslant 0$. Now we consider the Mayer–Vietoris sequence

$$0 \to \mathrm{Im}(i_* \oplus j_*) \to H_1(X; \mathscr{K}_n) \to H_0(F_- \amalg F_+; \mathscr{K}_n) \to \\ H_0(F \times I; \mathscr{K}_n) \oplus H_0(Y; \mathscr{K}_n) \to H_0(X; \mathscr{K}_n) \to 0.$$

Since $\pi_1(Y)\nsubseteq G_r^{(n+1)}$, $\pi_1(Y) \to G \to \Gamma_n$ is a non-trivial coefficient system. Therefore we have $H_0(Y; \mathscr{K}_n) = 0$ and $H_0(X; \mathscr{K}_n) = 0$ by Lemma 5.7. We note that $\mathrm{rk}_{\mathscr{K}_n} H_0(F; \mathscr{K}_n) = 1$ since $\pi_1(F) \subseteq G_r^{(n+1)}$. It follows that

$$\begin{aligned} r_n(X) &= \mathrm{rk}_{\mathscr{K}_n} H_1(X; \mathscr{K}_n) \\ &= \mathrm{rk}_{\mathscr{K}_n} H_0(F; \mathscr{K}_n) + \mathrm{rk}_{\mathscr{K}_n} \mathrm{Im}(i_* \oplus j_*) \\ &\geqslant 1. \qquad \square \end{aligned}$$

In particular, if there is a non-trivial surface that lifts to the nth cover then $r_n(X)\geqslant 1$.

Corollary 10.7. *If there exists a compact, connected, orientable, two-sided properly embedded surface $F \subseteq X$ with $\beta_1(X)\geqslant 2$ such that $0 \neq [F] \in H_2(X, \partial X; \mathbb{Z})$ and $i_*\pi_1(F) \subseteq G_r^{(n+1)}$ then $\bar{\delta}_n(\psi) = 0$ for all $\psi \in H^1(X; \mathbb{Z})$.*

Proof. $r_n(X)\geqslant 1$ implies that $\bar{\delta}_n(\psi) = 0$ for all $\psi \in H^1(X; \mathbb{Z})$. $\square$

We now prove the main theorem of this section.

Proof of Theorem 10.1. We break the proof up into two cases.

Case 1: Let X be a 3-manifold with $\beta_1(X)\geqslant 2$. Let $F = \cup F_i$ be a surface dual to ψ that is minimal with respect to $\|\cdot\|_T$. We can assume that $[F_i] \neq 0$ for all i. If any component of F, say F_j lifts to the nth rational derived cover of X, i.e. $\pi_1(F_j) \subset G_r^{(n+1)}$ then $\bar{\delta}_n(\psi) = 0$ by Corollary 10.7. Otherwise $\mathscr{N}_n(\psi) = 0$ so by Theorem 10.2 we have $\bar{\delta}_n(\psi) \leqslant \delta_n(\psi) \leqslant \|\psi\|_T$.

Case 2: Let X be a 3-manifold with $\beta_1(X) = 1$ and ψ be a generator of $H^1(X; \mathbb{Z})$. Let $F = \cup F_i$ be a surface dual to ψ that is minimal with respect to $\|\cdot\|_T$. Since $\beta_1(\ker\, \psi) < \infty$ and the boundary (if any) is a union of tori, we can assume that F is a connected surface with $\beta_2(F) = \beta_3(X)$ [26, Proposition 6.1]. Therefore $\mathscr{N}_0(\psi) \leqslant 1 + \beta_3(X)$ so by Theorem 10.2 we have $\bar{\delta}_0(\psi) = \delta_0(\psi) \leqslant \|\psi\|_T + 1 + \beta_3(X)$. Now suppose $n\geqslant 1$. If $\pi_1(F)\nsubseteq G_r^{(2)}$ (hence $\pi_1(F)\nsubseteq G_r^{(n+1)}$) then $\mathscr{N}_n(\psi) = 0$ so the result follows from Theorem 10.2. Otherwise, by Proposition 10.9, $\bar{\delta}_n(\psi) = \delta_n(\psi) = 0$. We remark that if the higher-order degrees of $S^1 \times S^2$ and $S^1 \times D^2$ are zero.

The last sentence in the theorem follows from the calculations in Proposition 8.4. Note that $r_n(X) = 0$ for fibered 3-manifolds so that $\bar{\delta}_n(\psi) = \delta_n(\psi)$ for all n. $\square$

To complete the proof of Theorem 10.1, we need prove Proposition 10.9 which states that if a homologically essential surface dual to ψ lifts to the nth order cover then $\mathscr{A}_i^{\psi}(X) = 0$ for $i < n$. This will be our main objective for the rest of this section.

We begin by showing that $\mathscr{A}_n^{\psi}(X)$ is generated by $H_1(F; \mathbb{K}_n[t^{\pm 1}])$. The idea behind the proof is simple. If $\alpha \neq 0$ is $\mathbb{K}_n[t^{\pm 1}]$-torsion then there exists a $p(t) \in \mathbb{K}[t^{\pm 1}]$ such that $\alpha p(t) = 0$. Moreover, since $\mathbb{K}$ is a (skew) field, we can assume that $p(t) = 1 + ta_1 + \cdots + t^m a_m$ where $a_m \neq 0$. Thus α and $\alpha t a_1 + \cdots + \alpha t^m a_m$

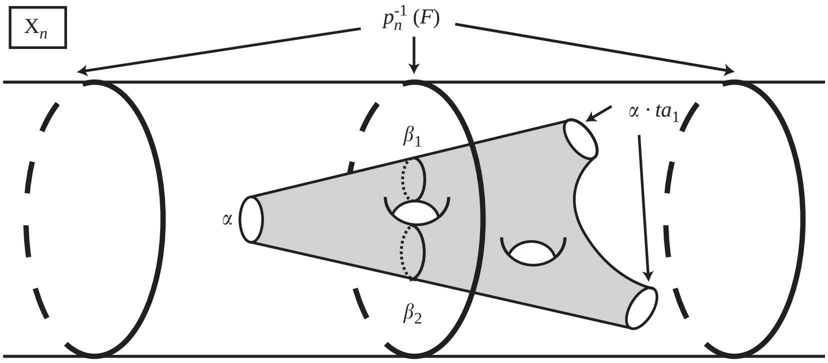

Fig. 3. $\mathscr{A}_n^{\psi}(X)$ is generated by $H_1(F;\mathbb{K}_n[t^{\pm 1}])$.

cobound a surface, S in X_n (see Fig. 3). Since the power of t on each term of the latter sum is positive, S must intersect a lift of the surface F. Hence α is homologous to the intersection of S with the lift of F. Note that in Fig. 3, α is homologous to $\beta_1+\beta_2$.

Lemma 10.8. $\mathscr{A}_n^{\psi}(X) \subseteq \mathrm{Im}(i_*)$ *where*

$$i_*\colon H_1(F;\mathbb{K}_n[t^{\pm 1}]) \to H_1(X;\mathbb{K}_n[t^{\pm 1}]).$$

Proof. By Proposition 7.6, $\mathscr{A}_n^{\psi}(X) \subseteq T\,\mathrm{Im}(j_*)$ where $j_*\colon H_1(Y;\mathbb{K}_n[t^{\pm 1}]) \xrightarrow{j_*} H_1(X;\mathbb{K}_n[t^{\pm 1}])$. We will show that $T\,\mathrm{Im}(j_*) \subseteq \mathrm{Im}(i_*)$ which completes the proof.

Let $\sigma_X \in T\,\mathrm{Im}(j_*)$ with $j_*(\sigma_Y)=\sigma_X$. Since σ_X is $\mathbb{K}_n[t^{\pm 1}]$-torsion, there exists $p(t) \in \mathbb{K}_n[t^{\pm 1}]$ such that $\sigma_X\,p(t)=0$. We have $j_*(\sigma_Y\,p(t)) = j_*(\sigma_Y)\,p(t) = \sigma_X\,p(t) = 0$ so there exists $\sigma_F \in H_1(F;\mathbb{K}_n[t^{\pm 1}])$ such that $\eta(\sigma_F) = \sigma_Y\,p(t)$. We can assume that $p(t) = 1 + tc_1 + \cdots + t^m c_m$ since $\sigma_X\,p(t)=0$ if and only if $\sigma_X\,p(t)u=0$ for any unit $u \in \mathbb{K}_n[t^{\pm 1}]$. Now $H_1(F;\mathbb{K}_n[t^{\pm 1}]) \simeq H_1(F;\mathbb{K}_n)\otimes_{\mathbb{K}_n}\mathbb{K}_n[t^{\pm 1}]$ and $H_1(Y;\mathbb{K}_n[t^{\pm 1}]) \simeq H_1(Y;\mathbb{K}_n)\otimes_{\mathbb{K}_n}\mathbb{K}_n[t^{\pm 1}]$ so every element in $H_1(F;\mathbb{K}_n[t^{\pm 1}])$ (resp. $H_1(Y;\mathbb{K}_n[t^{\pm 1}])$) has the form $\sum_{i=-\infty}^{\infty}\alpha_i \otimes t^i$(resp. $\sum_{i=-\infty}^{\infty}\beta_i \otimes t^i$) such that $\alpha_i \in H_1(F;\mathbb{K}_n)$ (resp. $\beta_i \in H_1(Y;\mathbb{K}_n)$) and there are only finitely many non-zero α_i (resp. β_i). We write $\sigma_F = \sum_{i=-\infty}^{\infty}\alpha_i \otimes t^i$ and (as with $p(t)$) we may write $\sigma_Y = \sum_{i=0}^{k}\beta_i \otimes t^i$. Using this notation we now have

$$\begin{aligned}
\sigma_Y\,p(t) &= \sum_{i=0}^{k}(\beta_i \otimes t^i)\sum_{j=0}^{m} t^j c_j \\
&= \sum_{l=0}^{k+m}\sum_{i+j=l}(\beta_i (c_j)^t \otimes t^{i+j}) \\
&= \sum_{l=0}^{k+m}\left(\sum_{i+j=l}\beta_i (c_j)^t\right)\otimes t^l
\end{aligned}$$

and

$$\begin{aligned}\eta(\sigma_F) &= \sum_{i=-\infty}^{\infty} \eta(\alpha_i \otimes t^i)\\ &= \sum_{i=-\infty}^{\infty} (i_-)_*(\alpha_i \otimes t^i) - (i_+)_*(\alpha_i \otimes t^i)t\\ &= \sum_{i=-\infty}^{\infty} (i_-)_*(\alpha_i) \otimes t^i - (i_+)_*(\alpha_i) \otimes t^{i+1}\\ &= \sum_{i=-\infty}^{\infty} (i_-)_*(\alpha_i) \otimes t^i - (i_+)_*(\alpha_{i-1}) \otimes t^i\\ &= \sum_{i=-\infty}^{\infty} ((i_-)_*(\alpha_i) - (i_+)_*(\alpha_{i-1})) \otimes t^i.\end{aligned}$$

Recall that $\eta(\sigma_F)=\sigma_Y p(t)$ which implies that $\sum_{i+j=l} \beta_i (c_j)^t=(i_-)_*(\alpha_l)-(i_+)_*(\alpha_{l-1})$ for all $0\leqslant l\leqslant k+m$. In particular $c_0 = 1$ so when $l\leqslant k$ we can write β_l as a combination of β_i and $(i_-)_*(\alpha_l) + (i_+)_*(\alpha_{l-1})$ with $i < l$. That is,

$$\beta_l \otimes t^l = (i_-)_*(\alpha_l) - (i_+)_*(\alpha_{l-1}) \otimes t^l - \sum_{i+j=l,i<l} \beta_i (c_j)^t \otimes t^l.$$

We will prove by induction that $j_*(\beta_l \otimes t^l) \in \operatorname{Im}(i_*)$ for each l implying that $\sigma_X = j_*(\sigma_Y) = j_*(\sum_{0\leqslant l\leqslant k} \beta_l \otimes t^l) \in \operatorname{Im}(i_*)$ which completes the proof. We first note that

$$\begin{aligned}j_*((i_-)_*(\alpha_l) + (i_+)_*(\alpha_{l-1}) \otimes t^l) &= j_*((i_-)_*(\alpha_l \otimes t^l) + (i_+)_*(\alpha_{l-1} \otimes t^l))\\ &= i_*(\alpha_l \otimes t^l - \alpha_{l-1} \otimes t^l)\\ &= i_*(\alpha_l - \alpha_{l-1} \otimes t^l).\end{aligned}$$

It follows that $\beta_0 \otimes 1 = (i_-)_*(\alpha_0) - (i_+)_*(\alpha_{-1}) \otimes 1 = i_*(\alpha_0 - \alpha_{-1} \otimes 1) \in \operatorname{Im}(i_*)$. Now we assume that $\beta_i \otimes t^i = i_*(\gamma_p)$ for all $i \leqslant l-1$ so that

$$\begin{aligned}j_*(\beta_l \otimes t^l) &= j_*((i_-)_*(\alpha_l) - (i_+)_*(\alpha_{l-1}) \otimes t^l) - \sum_{i+j=l,i<l} j_*(\beta_i (c_j)^t \otimes t^l)\\ &= i_*(\alpha_l - \alpha_{l-1} \otimes t^l) - \sum_{i+j=l,i<l} j_*(\beta_i \otimes t^l) c_j\\ &= i_*(\alpha_l - \alpha_{l-1} \otimes t^l) - \sum_{i+j=l,i<l} i_*(\gamma_i) c_j\\ &\in \operatorname{Im}(i_*). \qquad \square\end{aligned}$$

We can use this to show that if ψ is dual to a union of surfaces in X whose fundamental groups all include into the $(n+1)^{\text{st}}$ rational derived subgroup of $G = \pi_1(X)$ then $\delta_i(\psi) = 0$ for $i \leqslant n-1$.

Proposition 10.9. *If there exists a union of properly embedded surfaces $F = \cup F_j$ in X with $[F] \in H_2(X, \partial X; \mathbb{Z})$ dual to $\psi \in H^1(X;\mathbb{Z})$ such that for all j, $\pi_1(F_j) \subseteq G_r^{(n+1)}$ then $\mathscr{A}_i^{\psi}(X) = 0$ whenever* $0 \leqslant i \leqslant n-1$.

Proof. Consider the following diagram of abelian groups:

$$\begin{tikzcd} H_1(F_{\Gamma_i}) \arrow[r, "i_*"] \arrow[d, "\rho_F"'] & H_1(X_{\Gamma_i}) \arrow[d, "\rho_X"'] \\ H_1(F;\mathbb{K}_i[t^{\pm 1}]) \arrow[r, "i_*"] & H_1(X;\mathbb{K}_i[t^{\pm 1}]), \end{tikzcd} \tag{10.2}$$

where i_* is induced by the inclusion map $i: F \to X$ and $\rho_F([\sigma]) = [\sigma \otimes 1]$ (similarly for ρ_X). First we observe that if $i \leqslant n-1$, every class in $\pi_1(F_{\Gamma_i})$ gets mapped into $G_r^{(n+1)} \subseteq G_r^{(i+2)}$ by $i_* \circ p_*$ hence is zero in $G_r^{(i+1)}/G_r^{(i+2)} = H_1(X_{\Gamma_i})/\{\mathbb{Z}\text{-torsion}\}$. Therefore $i_*: H_1(F_{\Gamma_i}) \to H_1(X_{\Gamma_i})$ maps $H_1(F_{\Gamma_i})$ to the $\mathbb{Z}$-torsion subgroup of $H_1(X_{\Gamma_i})$. Since $H_1(X; K_i[t^{\pm 1}])$ is $\mathbb{Z}$-torsion free, $\rho_X \circ i_* = 0$.

$$\begin{tikzcd} F_{\Gamma_n} \arrow[r] \arrow[d, "p"'] & X_{\Gamma_n} \arrow[d, "p"'] \\ F \arrow[r] & X. \end{tikzcd}$$

Since (10.2) commutes, the image of ρ_F goes to zero under i_*. Therefore if $[\sigma \otimes p(t)]$ is an element of $H_1(F;\mathbb{K}_i[t^{\pm 1}])$, $i_*([\sigma \otimes p(t)]) = i_*([\sigma \otimes 1])p(t) = 0p(t) = 0$ (i_* is a $\mathbb{K}_i[t^{\pm 1}]$-module homomorphism). By Lemma 10.8, $\mathscr{A}_i^{\psi}(X)$ is generated by $\mathrm{Im}(i_*) = 0$ hence $\mathscr{A}_i^{\psi}(X) = 0$. □

Corollary 10.10. *Let X be a 3-manifold with $\beta_1(X) = 1$ and F a surface dual to a generator of $H^1(X;\mathbb{Z})$. If $\pi_1(F) \subseteq G_r^{(2)}$ then $G_r^{(i)} = G_r^{(i+1)}$ for all $i \geqslant 1$.*

Proof. Since $\beta_1(X) = 1$, $r_n(X) = 0$ by Proposition 5.2. That is, $H_1(X;\mathbb{K}_i[t^{\pm 1}])$ is a torsion module. When $i = 0$, $\mathbb{K}_0 = \mathbb{Q}$ so that

$$TH_1(X;\mathbb{K}_0[t^{\pm 1}]) = H_1(X;\mathbb{Q}[t^{\pm 1}]) = H_1(X_{\Gamma_0}) \otimes \mathbb{Q} = G_r^{(1)}/G_r^{(2)} \otimes \mathbb{Q}.$$

If $\pi_1(F) \subset G_r^{(2)}$, Proposition 10.9 implies $TH_1(X;\mathbb{K}_0[t^{\pm 1}]) = 0$. Since $G_r^{(1)}/G_r^{(2)}$ is $\mathbb{Z}$-torsion free, $G_r^{(1)}/G_r^{(2)} \to G_r^{(1)}/G_r^{(2)} \otimes \mathbb{Q}$ sending $g \mapsto g \otimes 1$ is a monomorphism. Therefore $G_r^{(i)} = G_r^{(i+1)}$ for all $i \geqslant 1$. □

11. Realization theorem

We are ready to prove that the invariants δ_n give much more information than the classical invariants. In fact, we subtly alter 3-manifolds to obtain new 3-manifolds with striking behavior. Cochran proves this result when $\beta_1(X) = 1$ [3].

Theorem 11.1. *For each $m \geqslant 1$ and $\mu \geqslant 2$ there exists a 3-manifold X with $\beta_1(X) = \mu$ such that*

$$\|\psi\|_A = \delta_0(\psi) < \delta_1(\psi) < \cdots < \delta_m(\psi) \leqslant \|\psi\|_T$$

for all $\psi \in H^1(X; \mathbb{Z})$. Moreover, X can be chosen so that it is closed, irreducible and has the same classical Alexander module as a 3-manifold that fibers over S^1.

The proof of this will be an application of the following more technical theorem. We will postpone the proof until later in the section. Theorem 11.2 is a tool that will allow us to subtly alter 3-manifolds in order to construct new 3-manifolds whose degrees are unchanged up to the nth stage but increase at the nth stage.

Theorem 11.2 (*Realization Theorem*). *Let X be a compact, orientable 3-manifold with $G = \pi_1(X)$ and $G_r^{(n)}/G_r^{(n+1)} \neq 0$ for some $n \geqslant 0$. Let $[x]$ be a primitive class in $H_1(X; \mathbb{Z})$. Then for any positive integer k, there exists a 3-manifold $X(n, k)$ homology cobordant to X such that*

(1)

$$\frac{G}{G_r^{(i+1)}} \cong \frac{H}{H_r^{(i+1)}} \quad \text{for } 0 \leqslant i \leqslant n-1,$$

where $H = \pi_1(X(n, k))$ and

(2)

$$\delta_n^{X(n,k)}(\psi) \geqslant \delta_n^X(\psi) + k|p|.$$

for any $\psi \in H^1(X(n, k); \mathbb{Z})$ with $\psi(x) = t^p$.

Proof. Let X be a compact 3-manifold with $G_r^{(n)}/G_r^{(n+1)} \neq 0$. $G = G_r^{(1)}$ implies that $G_r^{(n)} = G_r^{(n+1)}$ hence our hypothesis guarantees that $\beta_1(X) \geqslant 1$. Since $[x]$ is a primitive class in $H_1(X; \mathbb{Z})$, we can present G as

$$G \cong \langle x_1, \ldots, x_\mu, y_1, \ldots, y_l | R_1, \ldots, R_m \rangle,$$

where $y_i \in G_r^{(1)}$, $x_1 = x$, and $\{[x_1], \ldots, [x_\mu]\}$ is a basis for $G/G_r^{(1)}$.

Begin by adding a 1-handle to $X \times I$ to obtain a 4-manifold V with boundary $(X \sqcup X') \cup (\partial X \times I)$ where X' is obtained from X by taking the connect sum with $S^1 \times S^2$. Then $\pi_1(V) \cong \pi_1(X') \cong G * \langle z \rangle$ where z is the generator of $\pi_1(S^1 \times S^2)$. Choose a non-trivial element $B \in G_r^{(n)} - G_r^{(n+1)}$, and let $w = zx^{-1}$ and $\alpha = [A_k, B]$ where A_k is defined inductively as

$$A_1 = w$$
$$A_k = [A_{k-1}, x] \quad \text{for } k \geqslant 2.$$

Now add a 2-handle to V along a curve c (any framing) embedded in X' representing $w[x, \alpha]$ to obtain a 4-manifold W with boundary $(X \sqcup -X(n, k)) \cup (\partial X \times I)$. Let $E = \pi_1(W)$, $H = \pi_1(X(n, k))$ and denote by i and j the inclusion maps of X and $X(n, k)$ into W, respectively.

Adding the 2-handle to X' kills the element $w[x, \alpha]$ in $G * \langle z \rangle \cong G * \langle w \rangle$ so $E \cong \langle G, w | w[x, \alpha] \rangle = \langle x_1, \ldots, x_\mu, y_1, \ldots, y_l, w | R_1, \ldots, R_m, w[x, \alpha] \rangle$. We see that $X(n, k)$ is the 3-manifold obtained by performing Dehn surgery (with integer surgery coefficient corresponding to the framing of the 2-handle) along the curve c. Let γ be the meridian curve to c in $X(n,k)$. The dual handle decomposition of W rel $X(n,k)$ is obtained by adding to $X(n,k)$ a 0-framed 2-handle along γ and 3-handle. This gives us $E \cong \langle H|\gamma\rangle$.

We show that

$$\frac{G}{[G_r^{(i)}, G_r^{(i)}]} \xrightarrow{\cong} \frac{E}{[E_r^{(i)}, E_r^{(i)}]} \xleftarrow{\cong} \frac{H}{[H_r^{(i)}, H_r^{(i)}]} \tag{11.1}$$

for $0 \leqslant i \leqslant n$. Using Lemma 3.5, this will imply that

$$\frac{G}{G_r^{(i+1)}} \xrightarrow{\cong} \frac{E}{E_r^{(i+1)}} \xleftarrow{\cong} \frac{H}{H_r^{(i+1)}}. \tag{11.2}$$

There is a surjective map $\mathrm{pr}: \langle G, w|w[x,\alpha]\rangle \twoheadrightarrow G$ defined by killing w so that $\mathrm{pr} \circ i_* = id_G$. Consider the induced maps

$$\frac{G}{[G_r^{(i)}, G_r^{(i)}]} \xrightarrow{\bar{i}_*} \frac{E}{[E_r^{(i)}, E_r^{(i)}]} \overset{\overline{\mathrm{pr}}}{\twoheadrightarrow} \frac{G}{[G_r^{(i)}, G_r^{(i)}]}.$$

We will show by induction that $w \in [E_r^{(i)}, E_r^{(i)}]$ for $0 \leqslant i \leqslant n$. Since $w = [[A_k, B], x]$, it is clear that $w \in [E_r^{(0)}, E_r^{(0)}]$. Now suppose that $w \in [E_r^{(i-1)}, E_r^{(i-1)}]$ for some $i \leqslant n$. Since $A_k = [A_{k-1}, x]$ and $A_1 = w$, we have $A_k \in [E_r^{(i-1)}, E_r^{(i-1)}] \subseteq E_r^{(i)}$. Moreover, since $B \in G_r^{(n)}$, we have $B \in E_r^{(n)} \subseteq E_r^{(i)}$ for $i \leqslant n$. Therefore $[A_k, B] \in [E_r^{(i)}, E_r^{(i)}]$ for $i \leqslant n$ and hence $w \in [E_r^{(i)}, E_r^{(i)}] \subseteq E_r^{(i+1)}$. It follows that $\overline{\mathrm{pr}}$ is an isomorphism. Since $\overline{\mathrm{pr}} \circ \bar{i}_*$ is an isomorphism, $\bar{i}_*$ is an isomorphism for $0 \leqslant i \leqslant n$.

Now consider the maps

$$\frac{H}{[H_r^{(i)}, H_r^{(i)}]} \xrightarrow{\bar{j}_*} \frac{E}{[E_r^{(i)}, E_r^{(i)}]},$$

where now we are considering E as the group $\langle H|\gamma\rangle$. We will show that $\gamma \in [H_r^{(i)}, H_r^{(i)}]$ for $0 \leqslant i \leqslant n$ hence the above map will be an isomorphism. Recall that $X(n,k)$ can be obtained from X by first doing Dehn surgery on a 0-framed unlinked trivial knot in X to get the manifold $X' = X \# (S^1 \times S^2)$ and then performing Dehn surgery along a curve c representing $w[x,\alpha]$ in X'. Let $Y = X' - N(c)$ be the 3-manifold obtained by removing a regular neighborhood of c in X'. We use the notations $P = \pi_1(Y)$, $K = \pi_1(X') \cong G * \langle z\rangle$, and $l: Y \to X'$ be the inclusion map. Let γ be the meridian of c based at x_0 as in Fig. 4. We show that $\gamma \in [P_r^{(i)}, P_r^{(i)}]$ for $0 \leqslant i \leqslant n$ which implies that $\gamma \in [H_r^{(i)}, H_r^{(i)}]$ since $P \twoheadrightarrow H$.

To begin, we will show that

$$\gamma = [\gamma, u_1]^{v_1} \cdots [\gamma, u_{2^k}]^{v_{2^k}} [\lambda_1, \lambda_2]^{w_1} \cdots [\lambda_{m-1}, \lambda_m]^{w_{m/2}}, \tag{11.3}$$

where $l_*(u_j) \in K_r^{(n)}$ and $\lambda_j \in Ncl\langle\gamma\rangle = \ker(l_*: P \twoheadrightarrow K)$. Using this, we will show that the induced map $l_*: P_r^{(i)} \twoheadrightarrow K_r^{(i)}$ is surjective for $0 \leqslant i \leqslant n+1$. Assuming these two statements to be true for now, we shall prove by induction on i that $\gamma \in [P_r^{(i)}, P_r^{(i)}]$ for $0 \leqslant i \leqslant n$ as desired. It is clear from (11.3) that $\gamma \in [P_r^{(0)}, P_r^{(0)}]$. Now suppose that $\gamma \in [P_r^{(i-1)}, P_r^{(i-1)}] \subseteq P_r^{(i)}$ for some $i \leqslant n$. Then $\lambda_j \in P_r^{(i)}$ for all j. Since $l_*: P_r^{(i)} \twoheadrightarrow K_r^{(i)}$ is surjective and $l_*(u_j) \in K_r^{(n)}$, it follows that $u_j = p_j \Lambda_j$ where $p_j \in P_r^{(i)}$ and $\Lambda_j \in Ncl\langle\gamma\rangle \subseteq P_r^{(i)}$.Thus by (11.3), $\gamma \in [P_r^{(i)}, P_r^{(i)}]$ for $0 \leqslant i \leqslant n$.

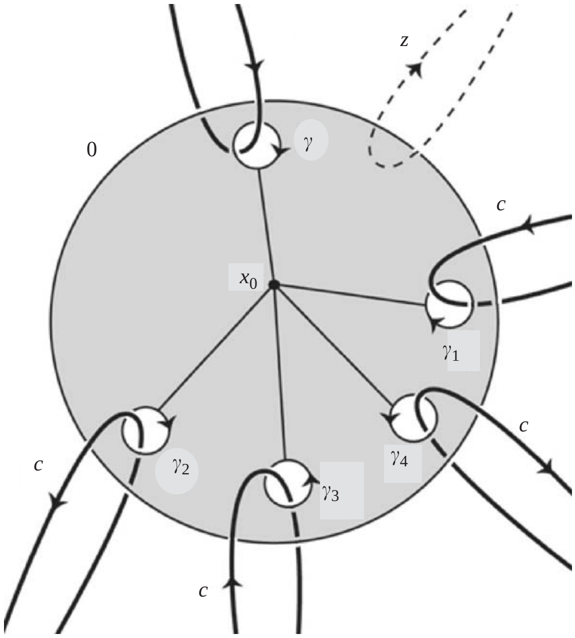

Fig. 4. γ and the $\gamma_i^{\pm 1}$s cobound a punctured sphere.

Let c be a curve representing the element

$$\begin{aligned} w[x,\alpha] &= z\alpha x^{-1}\alpha^{-1} \\ &= zA_kBA_k^{-1}B^{-1}x^{-1}BA_kB^{-1}A_k^{-1}. \end{aligned}$$

For simplicity, we prove the case when c intersects the cosphere (belt sphere) of the 1-handle attached to $X \times I$ exactly $1+2^{k+1}$ times. The proof can be modified for the case where c intersects the cosphere more than $1+2^{k+1}$ times. The 2^{k+1} intersections are a result of the 2^{k-1} occurrences of z and z^{-1} in A_k and the first intersection is a result the first z that occurs in $z\alpha x^{-1}\alpha^{-1}$. Note that the only occurrences of z in $\alpha x^{-1}\alpha^{-1}$ show up in A_k since B is an element of G. Let γ_j be the meridian of c based at x_0 corresponding to the jth occurrence of z or z^{-1} in $\alpha x^{-1}\alpha^{-1}$ as shown in Fig. 4.

Before proceeding, we sketch the idea of the next part of the proof. First we note that γ is a product of $\gamma_j^{\pm 1}$. We can pair each γ_j with γ_{2^k+1-j} since they bound an annulus as in Fig. 5. Thus they are related by $\gamma_{2^k+1-j} = u^{-1}\gamma_j u$ and hence $\gamma_j\gamma_{2^k+1-j}^{-1} = [\gamma_j, u^{-1}]$. We show that $l_*(u)$ is in the nth term of the derived series of K. Since γ_j is a conjugate of γ, γ is a product of commutators which can be written as $[\gamma, u]^v$ with $l_*(u) \in K_r^{(n)}$.

Let $a^b = b^{-1}ab$. Since the longitude of the unknot is trivial in P, we see that γ is equal to a product of the $\gamma_j^{\pm 1}$ as in Fig. 4. Moreover, we can order the γ_j as we choose since switching γ_i and γ_j only changes

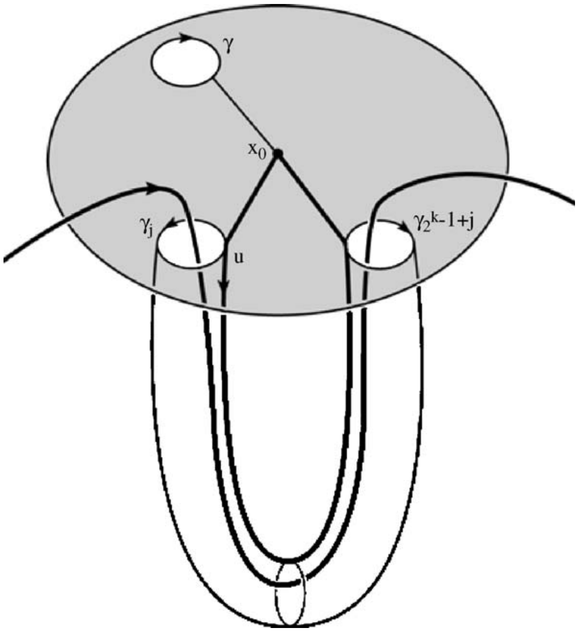

Fig. 5. γ_j and γ_{2^k+1-j} differ by a conjugation.

the element by a commutator of elements of $Ncl\langle\gamma\rangle$. For example,

$$\begin{aligned}\lambda_1\gamma_i\gamma_j\lambda_2 &= \lambda_1\gamma_j\gamma_i[\gamma_i^{-1},\gamma_j^{-1}]\lambda_2\\ &= \lambda_1\gamma_j\gamma_i\lambda_2[\gamma_i^{-1},\gamma_j^{-1}]^{\lambda_2}.\end{aligned}$$

Hence we see that

$$\gamma = \left(\prod_{j=1}^{2^{k-1}} \gamma_j^{\pm 1}\gamma_{2^k+1-j}^{\pm 1}\right)\left(\prod_{j=1}^{2^{k-1}} \gamma_{2^k+j}^{\pm 1}\gamma_{2^{k+1}+1-j}^{\pm 1}\right)[\lambda_1,\lambda_2]^{w_1}\cdots[\lambda_{m-1},\lambda_m]^{w_{m/2}},$$

where $\lambda_j \in Ncl\langle\gamma\rangle$. The chosen ordering will become clear in the next paragraph.

If the jth occurrence of a $z^{\pm 1}$ is a z then $\gamma_j = \gamma^{p_j}$ where $l_*(p_j) = z\omega_j$ and ω_j is the word that occurs in $\alpha x^{-1}\alpha^{-1}$ up to but not including the jth z. Whereas, if the jth occurrence of a $z^{\pm 1}$ is a z^{-1} then $\gamma_j = \gamma^{p_j}$ where $l_*(p_j) = z\omega_j z^{-1}$ where ω_j is the word that occurs in $\alpha x^{-1}\alpha^{-1}$ up to but not including the jth z^{-1}. Now we consider the case $1 \leqslant j \leqslant 2^{k-1}$. The jth occurrence of $z^{\pm 1}$ in $\alpha x^{-1}\alpha^{-1} = A_k B A_k^{-1} B^{-1} x^{-1} B A_k B^{-1} A_k^{-1}$ occurs in the first A_k and the $(2^k - j + 1)$th occurrence of $z^{\pm 1}$ occurs in the first A_k^{-1} as the opposite power as the jth occurrence. Hence if $p_j = z\omega_j$ then $p_{2^k+1-j} = zA_k B A_k^{-1}\omega_j$ and if $p_j = z\omega_j z^{-1}$ then $p_{2^k+1-j} = zA_k B A_k^{-1}\omega_j z^{-1}$. Moreover, the term $\gamma_j^{\pm 1}\gamma_{2^k+1-j}^{\pm 1}$ in the formula above will always be of the form $(\gamma_j\gamma_{2^k+1-j}^{-1})^{\pm 1}$. Similarly, the $(2^k + j)$th occurrence of $z^{\pm 1}$ in $\alpha x^{-1}\alpha^{-1}$ occurs in the second A_k and the $(2^{k+1} + 1 - j)$th occurrence of $z^{\pm 1}$ occurs in the second A_k^{-1} as the opposite power as the $(2^k + j)$th occurrence. Thus, if $p_{2^k+j} = z\alpha x^{-1} B\omega_j$ then $p_{2^{k+1}+1-j} = z\alpha x^{-1}\alpha^{-1}\omega_j$, if

$p_{2^k+j} = z\alpha x^{-1} B\omega_j z^{-1}$ then $p_{2^{k+1}+1-j} = z\alpha x^{-1}\alpha^{-1}\omega_j z^{-1}$, and $\gamma^{\pm 1}_{2^k+j}\gamma^{\pm 1}_{2^{k+1}+1-j} = (\gamma_{2^k+j}\gamma^{-1}_{2^{k+1}+1-j})^{\pm 1}$.In either case we see that for $1 \leqslant j \leqslant 2^{k-1}$

$$\begin{aligned}\gamma_j \gamma^{-1}_{2^k+1-j} &= \gamma^{p_j}(\gamma^{-1})^{p_{2^k+1-j}}\\ &= [\gamma, p_j p^{-1}_{2^k+1-j}]^{p_j}\end{aligned}$$

and

$$\begin{aligned}\gamma_{2^k+j}\gamma^{-1}_{2^{k+1}+1-j} &= \gamma^{p_{2^k+j}}(\gamma^{-1})^{p_{2^{k+1}+1-j}}\\ &= [\gamma, p_{2^k+j} p^{-1}_{2^{k+1}+1-j}]^{p_{2^k+j}},\end{aligned}$$

where $l_*(p_j p^{-1}_{2^k+1-j}) = l_*((B^{-1})^{(zA_k)^{-1}}) \in K_r^{(n)}$ and $l_*(p_{2^k+j} p^{-1}_{2^{k+1}+1-j}) = l_*((B\alpha)^{(z\alpha x^{-1})^{-1}}) \in K_r^{(n)}$. Therefore

$$\gamma = [\gamma, u_1]^{v_1} \cdots [\gamma, u_{2^k}]^{v_{2^k}} [\lambda_1, \lambda_2]^{w_1} \cdots [\lambda_{m-1}, \lambda_m]^{w_{m/2}},$$

where $l_*(u_j) \in K_r^{(n)}$ and $\lambda_j \in Ncl\langle\gamma\rangle$ as desired.

Before proceeding, we note that γ can be simplified to the form

$$\gamma = [\gamma, u_1]^{v_1} \cdots [\gamma, u_{2^k+m}]^{v_{2^k+m}}, \tag{11.4}$$

where $l_*(u_j) \in K_r^{(n)}$, since if $\lambda_1, \lambda_2 \in Ncl\langle\gamma\rangle$ then $[\lambda_1, \lambda_2]$ is a product of elements of the form $[\gamma, u]^v$ where $l_*(u) = 1$. This is easily verified using the relation

$$[ab, c] = [b, c]^a [a, c].$$

We prove by induction that $l_* : P_r^{(i)} \to K_r^{(i)}$ is surjective for $0 \leqslant i \leqslant n+1$. It is clear that $l_* : P_r^{(0)} \to K_r^{(0)}$ is surjective. Now assume that $l_* : P_r^{(i)} \twoheadrightarrow K_r^{(i)}$ for $i \leqslant n$ and let $g \in K_r^{(i+1)}$. We note that if $G \twoheadrightarrow H$ is surjective then $[G, G] \twoheadrightarrow [H, H]$ is surjective. Therefore it suffices to consider g such that $g^k \in [K_r^{(i)}, K_r^{(i)}]$ for some $k \neq 0$. Since $P_r^{(i)} \twoheadrightarrow K_r^{(i)}$ is surjective, $l_* : [P_r^{(i)}, P_r^{(i)}] \twoheadrightarrow [K_r^{(i)}, K_r^{(i)}]$ is surjective and hence there exists an $f \in [P_r^{(i)}, P_r^{(i)}]$ such that $l_*(f) = g^k$. Moreover, since $P \twoheadrightarrow K$ there exists a $p \in P$ such that $l_*(p) = g$. It follows that $l_*(f) = l_*(p^k)$ and hence $p^k = f\lambda$ where $\lambda \in Ncl\langle\gamma\rangle$.Since $i \leqslant n$, $l_*(u_j) \in K_r^{(i)}$. Hence, by the induction hypothesis, there exist $q_j \in P_r^{(i)}$ with $u_j = q_j \Lambda_j$ for $\Lambda_j \in Ncl\langle\gamma\rangle$. Using (11.4) we have

$$\gamma = [\gamma, q_1\Lambda_1]^{v_1} \cdots [\gamma, q_{2^k+m}\Lambda_{2^k+m}]^{v_{2^k+m}}$$

hence $\gamma \in [P_r^{(i)}, P_r^{(i)}]$. Since $\lambda \in Ncl\langle\gamma\rangle$, $\lambda \in [P_r^{(i)}, P_r^{(i)}]$ so that $p^k = f\lambda \in [P_r^{(i)}, P_r^{(i)}]$. Thus $p \in P_r^{(i+1)}$ and $l_*(p) = g$ which implies that $l_* : P_r^{(i+1)} \to K_r^{(i+1)}$ is surjective for $i \leqslant n$. This concludes the proof of (11.2).

The isomorphisms in (11.1) and (11.2) imply the following three statements. First, we can obtain the $G/G_r^{(n+1)}$ and $H/H_r^{(n+1)}$-regular covers of X and $X(n,k)$, respectively by restricting to the boundary of the $E/E_r^{(n+1)}$-regular cover of W. Secondly, when $i = 0$ the inclusion maps i and j induce isomorphisms on $H_1(-,\mathbb{Z})$ hence on $H^1(-,\mathbb{Z})$. In fact, i and j induce isomorphisms on all integral homology groups. Thus, we can consider the homomorphism ψ as a homomorphism on E and H as well as G. Lastly,

$$\frac{G_r^{(i)}}{[G_r^{(i)},G_r^{(i)}]} \xrightarrow{\cong} \frac{E_r^{(i)}}{[E_r^{(i)},E_r^{(i)}]} \xleftarrow{\cong} \frac{H_r^{(i)}}{[H_r^{(i)},H_r^{(i)}]}$$

for $0 \leqslant i \leqslant n-1$. In particular this implies that for any $\psi \in H^1(X(n,k),\mathbb{Z})$ and $0 \leqslant i \leqslant n-1$, $\delta_i^{X(n,k)}(\psi) = \delta_i^W(\psi) = \delta_i^X(\psi)$.

We use the presentation given by the Fox Free Calculus (Section 6) to compute $\delta_i^W(\psi)$. Let $F = F\langle x_1,\dots,x_\mu,y_1,\dots,y_l,w\rangle$, $\chi\colon F \twoheadrightarrow G$. Recall that

$$\frac{\partial[C,D]}{\partial w} = (1-[C,D]D)\,\frac{\partial C}{\partial w} + (C-[C,D])\,\frac{\partial D}{\partial w}$$

for any $C, D \in F$. We compute $\frac{\partial A_k}{\partial w} = (1-A_kx)\frac{\partial A_{k-1}}{\partial w}$ and $\frac{\partial A_1}{\partial w} = 1$ so

$$\frac{\partial A_k}{\partial w} = (1-A_kx)\cdots(1-A_2x).$$

It follows that

$$\begin{aligned}
\frac{\partial w[x,\alpha]}{\partial w} &= 1 + w(x-[x,\alpha])\,\frac{\partial \alpha}{\partial w}\\
&= 1 + w(x-[x,\alpha])\left((1-\alpha B)\,\frac{\partial A_k}{\partial w} + (A_k-\alpha)\,\frac{\partial B}{\partial w}\right)\\
&= 1 + w(x-[x,\alpha])\left((1-\alpha B)(1-A_kx)\cdots(1-A_2x) + (A_k-\alpha)\,\frac{\partial B}{\partial w}\right).
\end{aligned}$$

Similarly we compute $\frac{\partial w[x,\alpha]}{\partial x}$ and $\frac{\partial w[x,\alpha]}{\partial v}$ when $v \in \{x_2,\dots,x_\mu,y_1,\dots,y_l\}$:

$$\begin{aligned}
\frac{\partial w[x,\alpha]}{\partial x} &= w\left[(1-[x,\alpha]) + (x-[x,\alpha])\left((1-\alpha B)\,\frac{\partial A_k}{\partial x} + (A_k-\alpha)\,\frac{\partial B}{\partial x}\right)\right],\\
\frac{\partial w[x,\alpha]}{\partial v} &= w(x-[x,\alpha])\left((1-\alpha B)\,\frac{\partial A_k}{\partial v} + (A_k-\alpha)\,\frac{\partial B}{\partial v}\right).
\end{aligned}$$

We note that $\frac{\partial A_k}{\partial v} = 0$ since A_k does not involve v. Moreover $\frac{\partial A_k}{\partial x}^{\chi i_* \phi_n^E} = 0$ since $\frac{\partial A_1}{\partial x} = 0$ and

$$\frac{\partial A_k}{\partial x} = (1-[A_{k-1},x])\,\frac{\partial A_{k-1}}{\partial x} + (A_{k-1}-[A_{k-1},x]).$$

Using the involution and projecting to $\mathbb{Z}[E/E_r^{(n+1)}]$ we get

$$\begin{aligned}
\overline{\frac{\partial w[x,\alpha]}{\partial w}}^{\chi i_* \phi_n^E} &= \overline{1+(x-1)(1-B)(1-x)^{k-1}}\\
\overline{\frac{\partial w[x,\alpha]}{\partial x_i}}^{\chi i_* \phi_n^E} &= \overline{\frac{\partial w[x,\alpha]}{\partial y_i}}^{\chi i_* \phi_n^E} = 0.
\end{aligned}$$

Thus

$$\left(\begin{array}{cc}\left(\overline{\dfrac{\partial R_j}{\partial x_i}}\right)^{\chi i_*\phi_n^E} & 0 \\ 0 & \overline{1+(x-1)(1-B)(1-x)^{k-1}}\end{array}\right)\otimes_{\mathbb{Z}[E/E_r^{(n+1)}]} id_{\mathbb{K}_n^E[t^{\pm 1}]} \tag{11.5}$$

is a presentation of $H_1(W,*;\mathbb{K}_n^E[t^{\pm 1}])$.

Let ψ be a primitive class in $H^1(W;\mathbb{Z})$ with $\psi(x)=t^p$ and let ξ be a splitting of $\overline{\psi}_n: E/E_r \twoheadrightarrow \mathbb{Z}$. We rewrite $a=1+(x-1)(1-B)(1-x)^{k-1}$ as a polynomial in t. The lowest and highest degree terms of a are B and

$$t^{kp}(\xi(t)^{-kp}x^k x^{1-k}(1-B)x^{k-1}),$$

respectively. Our assumption that $B\notin G_r^{(n+1)}$ (hence $B\notin E_r^{(n+1)}$ guarantees that $B-1$ is a unit in $\mathbb{K}_n^E[t^{\pm 1}]$). Therefore the degree of a is k. Moreover, deg $\overline{\sum t^i a_i}=$ deg $\sum t^i a_i$ hence

$$\deg \overline{1+(x-1)(1-B)(1-x)^{k-1}}=kp.$$

Lastly, $H_1(X,*;\mathbb{K}_n^G[t^{\pm 1}])$ is presented as $(\overline{\frac{\partial R_j}{\partial x_i}})^{\chi i\phi_n^G}\otimes_{\mathbb{Z}[G/G_r^{(n+1)}]} id_{\mathbb{K}_n^G[t^{\pm 1}]}$ therefore

$$\delta_n^W(\psi)=\delta_n^X(\psi)+kp.$$

To finish the proof we will show that $\delta_n^{X(n,k)}(\psi)\geqslant\delta_n^W(\psi)$ for any ψ. Since $(W,X(n,k))$ has only 2 and 3-handles $H_1(W,X(n,k);R)=0$. By Lemma 11.5, $H_2(W,X(n,k);R)$ is R-torsion. We have the following long exact sequence of pairs:

$$\to TH_2(W,X(n,k);R)\xrightarrow{\partial} H_1(X(n,k);R)\xrightarrow{j_*} H_1(W;R)\to 0.$$

Since $j_*(TH_1(X(n,k);R))\subseteq TH_1(W;R)$ we can consider the homomorphism

$$TH_1(X(n,k);R)\xrightarrow{j_*} TH_1(W;R).$$

We show that this map is surjective. Let $\sigma\in TH_1(W;R)$ and $\theta\in H_1(X(n,k);R)$ such that $j_*(\theta)=\sigma$. There exists $r\in R$ such that $\sigma r=0$ so $j_*(\theta r)=j_*(\theta)r=\sigma r=0$. By exactness, this implies that $\theta r\in \operatorname{Im}\partial$. Hence there exists $\upsilon\in TH_2(W,X(n,k);R)$ with $\partial(\upsilon)=\theta r$ and $\upsilon s=0$ for some non-zero $s\in R$. Therefore $\theta(rs)=(\theta r)s=\partial(\upsilon)s=\partial(\upsilon s)=\partial(0)=0$ which implies $\theta\in TH_1(X(n,k);R)$ Lastly, let $R=\mathbb{K}_n[t^{\pm 1}]$ then $TH_1(X(n,k);\mathbb{K}_n[t^{\pm 1}])$ and $TH_1(W;\mathbb{K}_n[t^{\pm 1}])$ can be considered as free $\mathbb{K}_n$-modules with finite rank. We have j_* surjective so

$$\delta_n^{X(n,k)}(\psi)=\mathrm{rk}_{\mathbb{K}_n}TH_1(X(n,k);\mathbb{K}_n[t^{\pm 1}])\geqslant \mathrm{rk}_{\mathbb{K}_n}TH_1(W;\mathbb{K}_n[t^{\pm 1}])=\delta_n^W(\psi). \qquad \square$$

Before proceeding, we will construct a specific example. We will begin with zero surgery on the trivial link with 2 components and subtly alter the manifold to increase δ_1.

Example 11.3. A specific example of $X(1,1)$ when $X=S^1\times S^2\# S^1\times S^2$. Let X be 0-surgery on the 2-component trivial link. Then $X=S^1\times S^2\# S^1\times S^2$ with π_1 generated by x and y (Fig. 6). Let

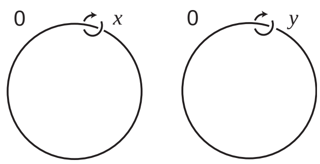

Fig. 6. $X = S^1 \times S^2 \# S^1 \times S^2$.

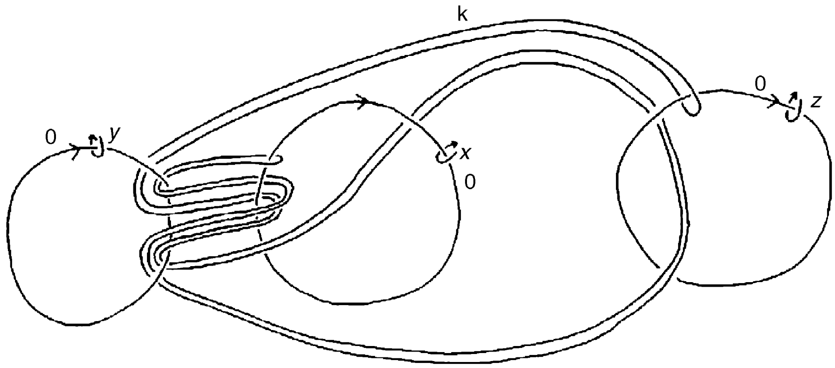

Fig. 7. $X(1, 1)$ is our resulting manifold.

$B = [x, y]$ and construct $X(1, 1)$ as in the theorem by doing k-framed surgery on c (see Fig. 7). For all $\psi \in H^1(X(1, 1))$,

$$\delta_0^{X(1,1)}(\psi) = \delta_0^X(\psi) = 0.$$

Moreover, we have

$$\delta_1^{X(1,1)}(\psi_x) \geqslant \delta_1^X(\psi_x) + 1 = 1.$$

We can assume that the manifolds that we have constructed to be irreducible.

Proposition 11.4. *If X is irreducible and ∂-irreducible then $X(n, k)$ is irreducible and ∂-irreducible.*

Proof. Recall that $X(n, k)$ can be constructed from X by first taking a connected sum with $S^1 \times S^2$ and then doing integer surgery on a curve c. Recall that the first homology class of c was equal to $x^{-1}z$ where x was a generator of $H_1(X)$ and z was the generator of $S^1 \times S^2$. Let $M = (X \# S^1 \times S^2)$. We will show that $M - c$ is irreducible and ∂-irreducible. A theorem of M. Scharlemann [30, p. 481] implies that $X(n, k)$ is irreducible. It is clear that $X(n, k)$ is ∂-irreducible.

First we show that $M - c$ is ∂-irreducible. Since M is ∂-irreducible, it suffices to show that $\ker i_* : \pi_1(\partial c) \to \pi_1(M)$ is trivial. Any curve on the boundary of c that is parallel to $m \neq 0$ copies of c is non-trivial in $\pi_1(M)$ since it is non-zero in homology. Any other curve on ∂c is homotopic to the meridian of c which we showed to be non-trivial in the proof of Theorem 11.2.

Let S be a non-separating 2-sphere in M that represents the class $\{pt\} \times S^2$ and $N = M - S$. Choose S so that it minimizes $\#(S \cap c)$. Note that $N = M - (B_1 \sqcup B_2)$ where B_1 and B_2 are disjoint 3-balls in M. After

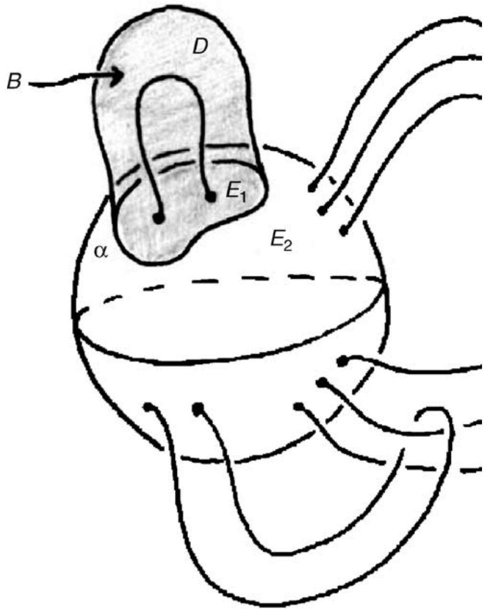

Fig. 8. D is a compression disc.

isotoping c to make it transverse to S, let P be the punctured 2-sphere in $M - c$ obtained by puncturing S at each intersection point with c. Let M' be the manifold obtained by cutting $M - c$ along P. M' has two copies of P in it's boundary, denote these punctured 2-spheres P_1 and P_2 (so that $P_i \subset \partial B_i$). We will show that M' is irreducible and P_1 is incompressible in M'. It will follow that $M - c$ is irreducible.

Suppose that $M - c$ is reducible and let Σ be a 2-sphere in $M - c$ that does not bound a 3-ball and minimizes $\#(\Sigma \cap P)$. Since M' is irreducible, we have $\#(\Sigma \cap P) \geqslant 1$. Consider the intersection of Σ and P and let α be an innermost circle on Σ. Then α bounds a disc D in M'. Since P_1 is incompressible in M', α bounds a disc E in P. $D \cup E$ is an embedded 2-sphere in M' so it bounds a 3-ball B in M'. We use B to isotope Σ in $M - c$ to get rid of the intersection α. This contradicts the minimality of $\#(\Sigma \cap P)$. Thus $M - c$ is irreducible.

We note that M' is homeomorphic to $M - f(W)$ where W is a wedge of spheres and $f : W \to M$ is an embedding of W into M. Suppose that M' is reducible and let Σ be an embedded 2-sphere in M' that does not bound a 3-ball. Since M is irreducible, Σ bounds a 3-ball B in M. Hence $M = B \cup_\Sigma V$. $f(W)$ is connected and $f(W) \cap \Sigma = \emptyset$ so either $f(W) \subset B$ or $f(W) \subset V$. However, the homology class of c is equal to $x^{-1}z$ hence $f(W) \nsubseteq B$. Therefore $f(W) \subset V$ hence Σ must bound a ball in M', a contradiction. Thus M' is irreducible.

Suppose that P_1 is compressible in M'. Let α be an curve on P_1 that bounds an embedded disc D in M'. α bounds the discs E_1 and E_2 on S. Since M is irreducible, $D \cup E_1$ bound a 3-ball B in M. If $B \cap B_2 = \emptyset$ then either $D \cup E_1$ or $D \cup E_2$ bounds a 3-ball B' in N. We can use B' to isotope c and reduce the number of intersections of c with E_1 or E_2 (see Fig. 8). This contradicts the minimality of $\#(S \cap c)$.

Now suppose that $B \cap B_2 \neq \emptyset$. Using B we can assume that either $D \cup E_1$ or $D \cup E_2$ bounds a 3-ball B' in $M - B_1$. Let $S' = \partial B'$. We note that $\#(S' \cap c) < \#(S \cap c)$ since c intersects E_1 and E_2. Moreover,

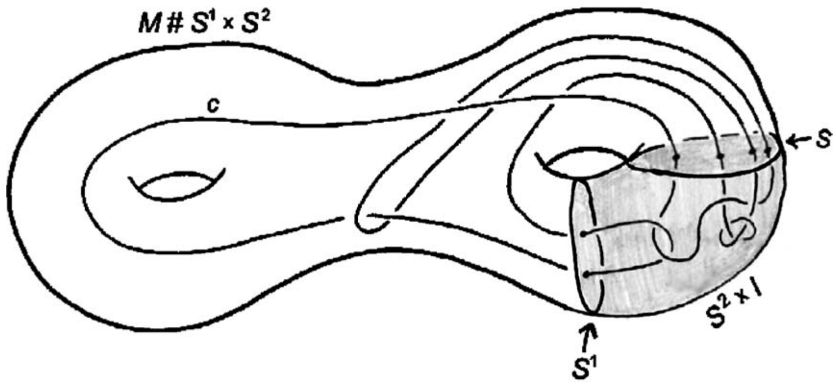

Fig. 9. S and S' cobound an $S^2 \times I$.

S' and ∂B_2 cobound an embedded $S^2 \times I$ in N. Therefore S' and S cobound an embedded $S^2 \times I$ in M (see Fig. 9) which can be used to isotope the curve c to reduce the number of intersections with S. This contradicts our minimality condition hence P_1 is incompressible in M'. This completes the proof that $M - c$ is irreducible. □

Lemma 11.5. *The manifold $X(n,k)$ is $\mathcal{K}_i$-homology cobordant to X for $i \leqslant n$. That is, $i: X \to W$ and $j: X(n,k) \to W$ induce isomorphisms on homology with $\mathcal{K}_i$ coefficients where W is the cobordism between X and $X(n,k)$.*

Proof. Consider the relative chain complex $C_*(W, X(n,k))$. Since W is obtained from $X(n,k)$ by adding only 2 and 3-handles, W is homotopy equivalent to a cell complex obtained by adding a single 2 and 3-cell. Hence we can assume $C_2(W, X(n,k)) = C_3(W, X(n,k)) \cong \mathbb{Z}$ and $C_j(W, X(n,k)) = 0$ for all other j. For all $i \leqslant n$ we lift the cells of $(W, X(n,k))$ to form the chain complex of $(\widetilde{W}, \widetilde{X}(n,k))$

$$0 \to C_3(\widetilde{W}, \widetilde{X}(n,k)) \otimes \mathcal{K}_i \overset{\widetilde{\partial}_3 \otimes id}{\to} C_2(\widetilde{W}, \widetilde{X}(n,k)) \otimes \mathcal{K}_i \to 0,$$

where $(\widetilde{W}, \widetilde{X}(n,k))$ are the regular $E/E_r^{(i+1)}$-cover of $(W, X(n,k))$. Since $\widetilde{\partial}_3 \otimes id : \mathcal{K}_i \to \mathcal{K}_i$, $\widetilde{\partial}_3 \otimes id$ is an isomorphism if and only if $\partial_3 \otimes id \neq 0$ if and only if $\partial_3(\sigma) \neq 0$ for some σ. But since $H_*(W, X(n,k)) = 0$, $\partial_3 : C_3(W, X(n,k)) \to C_2(W, X(n,k))$ is not the zero map, hence $\widetilde{\partial}_3$ is not the zero map. Therefore $H_*(W, X(n,k); \mathcal{K}_i) = 0$ which gives us $j_* : H_*(X(n,k); \mathcal{K}_i) \overset{\cong}{\to} H_*(W; \mathcal{K}_i)$. The proof that $i_* : H_*(X; \mathcal{K}_i) \overset{\cong}{\to} H_*(W; \mathcal{K}_i)$ follows almost verbatim except that (W, X) has only cells in dimensions 1 and 2. □

Lemma 11.6. *For each $n \geqslant 1$, if $\delta_{n-1}(\psi) \neq 0$ for some $\psi \in H^1(X; \mathbb{Z})$ then $G_r^{(n)}/G_r^{(n+1)} \neq 0$.*

Proof. If $\delta_{n-1}(\psi) \neq 0$ for some ψ then the rank of $H_1(X_{n-1})$ as an abelian group is at least 1, hence $H_1(X_{n-1})/\{\mathbb{Z} - torsion\} \neq 0$. But, Lemma 3.5 gives $G_r^{(n)}/G_r^{(n+1)} = H_1(X_{n-1})/\{\mathbb{Z}\text{-torsion}\}$ so $G_r^{(n)}/G_r^{(n+1)} \neq 0$. □

We now proceed with the proof of Theorem 11.1.

Proof of Theorem 11.1. Let X_0 be a 3-manifold with $r_0(X_0)=0$, $\beta_1(X_0)$ and whose universal torsion-free abelian cover has non-trivial β_1 and let $G=\pi_1(X_0)$. For example, it is well-known that for each $\mu \geqslant 1$ there exist 3-manifolds X (with and without boundary) which fibers over S^1 with fiber a surface of genus $g \geqslant 2$ and such that $\beta_1(X)=\mu$. Each of these would satisfy the necessary conditions on X_0 mentioned above (see Proposition 8.4). Let $\{x_1, \ldots, x_\mu\}$ be a basis of $H_1(X_0;\mathbb{Z})/\{\mathbb{Z}\text{-torsion}\}$ and $\{\psi_{x_1}, \ldots, \psi_{x_\mu}\}$ the (Hom) dual basis of $H^1(X_0;\mathbb{Z})$. Since $\beta_1((X_0)_{\Gamma_0})>0$, $G_r^{(1)}/G_r^{(2)} \neq 0$ we can use Theorem 11.2 with $k=1$ to construct a new manifold X_1 with $\delta_0^{X_0}(\psi)=\delta_0^{X_1}(\psi)$ and $\delta_1^{X_0}(\psi)<\delta_1^{X_1}(\psi)$for all $\psi \in H^1(X_1;\mathbb{Z})$. We do this by first constructing X_1^1 from X_0 to accomplish $\delta_0^{X_0}(\psi)=\delta_0^{X_1^1}(\psi)$ for all $\psi \in H^1(X_1^1;\mathbb{Z})$ and $\delta_1^{X_0}(\psi_{x_1})<\delta_1^{X_0}(\psi_{x_1})+1 \leqslant \delta_1^{X_1^1}(\psi_{x_1})$. We note that Theorem 11.2 guarantees $\delta_1^{X_1^1}(\psi) \geqslant \delta_1^{X_0}(\psi)$ for all other ψ. Now we continue this for all other basis elements $\psi_{x_2}, \ldots, \psi_{x_\mu}$ to get a 3-manifold $X_1=X_1^\mu$ with $\delta_0^{X_0}(\psi)=\delta_0^{X_1}(\psi)$ and $\delta_1^{X_0}(\psi)<\delta_1^{X_1}(\psi)$ for all $\psi \in H^1(X_1;\mathbb{Z})$.

In particular, $\delta_1^{X_1}(\psi)>0$ so by Lemma 11.6 $G_r^{(2)}/G_r^{(3)} \neq 0$. Hence we can construct X_2 with $\delta_i^{X_2}(\psi)=\delta_i^{X_1}(\psi)$ when $i \leqslant 1$ and $\delta_2^{X_1}(\psi)<\delta_2^{X_2}(\psi)$ for all $\psi \in H^1(X_2;\mathbb{Z})$. We continue this process until we obtain a 3-manifold $X=X_m$ with $\delta_0^X(\psi)<\delta_1^X(\psi)<\cdots<\delta_m^X(\psi)$ for all $\psi \in H^1(X;\mathbb{Z})$. Since $r_0(X_0)=0$, Lemma 11.5 guarantees that $r_0(X)=0$ hence $\|\psi\|_A=\delta_0(\psi)$ and $\delta_m(\psi) \leqslant \|\psi\|_T$.

If we choose X_0 to be closed, then X will be closed. Finally, to guarantee that X is irreducible, it suffices to choose X_0 irreducible by Proposition 11.4. □

We note that since $r_i(X)=0$, $\delta_i(\psi)=\bar{\delta}_i(\psi)$ hence we could have stated this theorem in terms of $\bar{\delta}_i$ as well as δ_i.

12. Applications

We show that the higher-order degrees give new computable algebraic obstructions to 3-manifolds fibering over S^1 even when the classical Alexander module fails. Moreover, using the work of Kronheimer, Mrowka, and Vidussi we are able to show that the higher-order degrees give new computable algebraic obstructions a 4-manifold of the form $X \times S^1$ admitting a symplectic structure, even when the Seiberg–Witten invariants fail.

12.1. Fibered 3-manifolds

Recall that if X is a compact, orientable 3-manifold that fibers over S^1 then by Proposition 8.4, the higher-order ranks must be zero. Moreover, if $\beta_1(X) \geqslant 2$ and ψ is dual to a fibered surface then $\delta_n(\psi)$ must be equal to the Thurston norm for all n and hence are constant as a function of n. We define the following function of ψ. Let $d_{ij}: H^1(X;\mathbb{Z}) \to \mathbb{Z}$ be defined by $d_{ij}=\delta_i-\delta_j$ for $i, j \geqslant 0$. Note that $d_{ij}=0$ if and only if $\delta_i=\delta_j$ for all $\psi \in H^1(X;\mathbb{Z})$.

Theorem 12.1. *Let X be a compact, connected, orientable 3-manifold. If at least one of the following conditions is satisfied then X does not fiber over S^1.*

(1) $r_n(X) \neq 0$ *for some* $n \geqslant 0$,
(2) $\beta_1(X) \geqslant 2$ *and there exists* $i, j \geqslant 0$ *such that* $d_{ij}(\psi) \neq 0$ *for all* $\psi \in H^1(X;\mathbb{Z})$,
(3) $\beta_1(X) = 1$ *and* $d_{ij}(\psi) \neq 0$ *for some* $i, j \geqslant 1$ *and* $\psi \in H^1(X;\mathbb{Z})$,
(4) $\beta_1(X) = 1$, $X \ncong S^1 \times S^2$, $X \ncong S^1 \times D^2$ *and* $d_{0j}(\psi) \neq 1 + \beta_3(X)$ *for some* $j \geqslant 1$ *where* ψ *is a generator of* $H^1(X;\mathbb{Z})$.

Proof. We consider each of the cases separately.

(1) This follow immediately from Proposition 8.4.

To prove that each of last three conditions implies that X does not fiber over S^1 we can assume that $r_n(X)=0$ for all $n \geqslant 0$. Otherwise the conclusion would be (vacuously) true since X would satisfy condition (1). Hence $\delta_n = \bar{\delta}_n$ for all $n \geqslant 0$ by Remark 5.11.

(2) If X fibers over S^1 and $\beta_1(X) \geqslant 2$ then for all $n \geqslant 0$, $\delta_n(\psi) = \|\psi\|_T$ for some $\psi \in H^1(X;\mathbb{Z})$ by Proposition 8.4. Hence $\delta_n(\psi)$ is a constant function of n. In particular, $d_{ij}(\psi) = 0$ for all $i, j \geqslant 0$ which contradicts our hypothesis.

(3) If X fibers over S^1 and $\beta_1(X) = 1$ then for all $n \geqslant 1$ and $\psi \in H^1(X;\mathbb{Z})$, $\delta_n(\psi) = \|\psi\|_T$ by Proposition 8.4. Hence $\delta_n(\psi)$ is a constant function of n for $n \geqslant 1$. In particular, $d_{ij}(\psi) = 0$ for all $i, j \geqslant 1$ and $\psi \in H^1(X;\mathbb{Z})$ which contradicts our hypothesis.

(4) If X fibers over S^1, $\beta_1(X) = 1$, $X \ncong S^1 \times S^2$, $X \ncong S^1 \times D^2$ and ψ is a generator of $H^1(X;\mathbb{Z})$ then by Proposition 8.4, $\delta_0(\psi) = \|\psi\|_T + 1 + \beta_3(X)$. The rest of the proof is similar to the previous two cases. □

The previously known algebraic obstructions to a 3-manifold fibering over S^1 are that the Alexander module $H_1(X;\mathbb{Z}\Gamma_0)$ is finitely generated and (when $\beta_1(X) = 1$) the Alexander polynomial is monic. If $\beta_1(X) \geqslant 2$, the Alexander module being finitely generated implies that $r_0(X) = 0$.

Consider the 3-manifolds in Theorem 11.1. We note that $\delta_1 > \delta_0$ hence they cannot fiber over S^1. Moreover, they can be chosen to have the same Alexander module as those of a 3-manifold that fibers over S^1 as remarked in the first paragraph in the proof of Theorem 11.1.

Corollary 12.2. *For each* $\mu \geqslant 1$, *Theorem* 11.1 *gives an infinite family of closed irreducible* 3-*manifolds* X *where* $\beta_1(X) = \mu$, X *does not fiber over* S^1, *and* X *cannot be distinguished from a fibered* 3-*manifold using the classical Alexander module.*

12.2. Symplectic 4-manifolds of the form $X \times S^1$

We now turn our attention to symplectic 4-manifolds of the form $X \times S^1$. It is well known that if X is a closed 3-manifold that fibers over S^1 then $X \times S^1$ admits a symplectic structure. Taubes conjectures the converse to be true.

Conjecture 12.3 (*Taubes*). *Let X be a* 3-*manifold such that* $X \times S^1$ *admits a symplectic structure. Then X admits a fibration over* S^1.

Using the work of Meng–Taubes and Kronheimer–Mrowka, Vidussi [35] has recently given a proof of McMullen's inequality using Seiberg–Witten theory. This generalizes the work of Kronheimer [23] who dealt with the case that X is the 0-surgery on a knot. Moreover, Vidussi shows that if $X \times S^1$ admits a symplectic structure (and $\beta_1(X) \geqslant 2$) then the Alexander and Thurston norms of X coincide on a cone over a face of the Thurston norm ball of X, supporting the conjecture of Taubes.

Theorem 12.4 (*Kronheimer [23] and Vidussi [34,35]*). *Let X be an closed, irreducible 3-manifold such that $X \times S^1$ admits a symplectic structure. If $\beta_1(X) \geqslant 2$ there exists a $\psi \in H^1(X;\mathbb{Z})$ such that $\|\psi\|_A = \|\psi\|_T$. If $\beta_1(X) = 1$ then for any generator ψ of $H^1(X;\mathbb{Z})$, $\|\psi\|_A = \|\psi\|_T + 2$.*

Consequently, we show that the higher-order degrees of a 3-manifold X give new computable algebraic obstructions to a 4-manifold of the form $X \times S^1$ admitting a symplectic structure.

Theorem 12.5. *Let X be a closed irreducible 3-manifold. If at least one of the following conditions is satisfied then $X \times S^1$ does not admit a symplectic structure.*

(1) $\beta_1(X) \geqslant 2$ *and there exists an $n \geqslant 1$ such that $\bar{\delta}_n(\psi) > \bar{\delta}_0(\psi)$ for all $\psi \in H^1(X;\mathbb{Z})$.*
(2) $\beta_1(X) = 1$, *ψ is a generator of $H^1(X;\mathbb{Z})$, and $\bar{\delta}_n(\psi) > \bar{\delta}_0(\psi) - 2$ for some $n \geqslant 1$.*

Proof. If $\beta_2(X) \geqslant 2$, $n \geqslant 1$, and $X \times S^1$ admits a symplectic structure then by Theorems 10.1, 12.4, and Proposition 5.12, $\bar{\delta}_n(\psi) \leqslant \|\psi\|_T = \bar{\delta}_0(\psi)$ for some $\psi \in H^1(X;\mathbb{Z})$. If $\beta_2(X) = 1$, $n \geqslant 1$, ψ is a generator of $H^1(X;\mathbb{Z})$ and $X \times S^1$ admits a symplectic structure then by Theorems 10.1 and 12.4, $\bar{\delta}_n(\psi) \leqslant \|\psi\|_T = \bar{\delta}_0(\psi) - 2$. $\square$

Thus, Theorem 11.1 gives examples of 4-manifolds of the form $X \times S^1$ which do not admit a symplectic structure but cannot be distinguished from a symplectic 4-manifold using the invariants of Seiberg–Witten theory.

Corollary 12.6. *For each $\mu \geqslant 1$, Theorem 11.1 gives an infinite family of 4-manifolds $X \times S^1$ where $\beta_1(X) = \mu$, $X \times S^1$ does not admit a symplectic structure, and X cannot be distinguished from fibered 3-manifold using the classical Alexander module.*

We note that the conditions in Theorem 12.5 are (strictly) stronger that the conditions in Theorem 12.1. The cause of this discrepancy is our lack of knowledge of the behavior of higher-order degrees when $X \times S^1$ admits a symplectic structure. We make the following conjecture, supporting the conjecture of Taubes.

Conjecture 12.7. *If X is a closed, orientable, irreducible 3-manifold such that $X \times S^1$ admits a symplectic structure then there exists a $\psi \in H^1(X;\mathbb{Z})$ such that $\bar{\delta}_n(\psi) = \|\psi\|_T$ for all $n \geqslant 1$.*

More interesting would be the possibility of finding an a symplectic 4-manifold of the form $X \times S^1$ such that $\bar{\delta}_1(\psi) < \bar{\delta}_0(\psi)$ for all $\psi \in H^1(X;\mathbb{Z})$; giving a counterexample to the conjecture of Taubes 12.3.

We conclude with the remark that Conjecture 12.7 is true when X is a knot complement in S^3, [3, Theorem 9.5]. The proof of this relies on the fact that the higher-order degrees are non-decreasing in n [3, Theorem 5.4] and are bounded by the Thurston norm. More precisely, Cochran proves that $\delta_0(\psi) - 1 \leqslant \delta_1(\psi) \leqslant \cdots \leqslant \delta_n(\psi) \cdots$ whenever $X = S^3 - K$ and ψ is a generator $H^1(X;\mathbb{Z})$. Moreover, the proof of Theorem 5.4 in [3] can be modified to prove that higher-order degrees are non-decreasing (in n) when X is any finite CW-complex homotopy equivalent to a 2-complex with Euler characteristic zero. Hence Conjecture 12.7 is also true for any 3-manifold with **non-empty** toroidal boundary.

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