

Monotonicity of degrees of generalized Alexander polynomials of groups and 3-manifolds

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Abstract

We investigate the behavior of the higher-order degrees, $\bar{\delta}_n$, of a finitely presented group G . These $\bar{\delta}_n$ are functions from $H^1(G; \mathbb{Z})$ to \mathbb{Z} whose values are the degrees of certain higher-order Alexander polynomials. We show that if $\text{def}(G) \geq 1$ or G is the fundamental group of a compact, orientable 3-manifold then $\bar{\delta}_n$ is a monotonically increasing function of n for $n \geq 1$. This is false for general groups. As a consequence, we show that if a 4-manifold of the form $X \times S^1$ admits a symplectic structure then X “looks algebraically like” a 3-manifold that fibers over S^1 , supporting a positive answer to a question of Taubes. This generalizes a theorem of S. Vidussi and is an improvement on previous results of the author. We also find new conditions on a 3-manifold X that will guarantee that the Thurston norm of $f^*(\psi)$, for $\psi \in H^1(X; \mathbb{Z})$ and $f: Y \rightarrow X$ a surjective map on π_1 , will be at least as large as the Thurston norm of ψ . When X and Y are knot complements, this gives a partial answer to a question of J. Simon.

More generally, we define Γ -degrees, $\bar{\delta}_\Gamma$, corresponding to a surjective map $G \twoheadrightarrow \Gamma$ for which Γ is poly-torsion-free-abelian. Under certain conditions, we show they satisfy a monotonicity condition if one varies the group Γ . As a result, we show that these generalized degrees give obstructions to the deficiency of a group being positive and obstructions to a finitely presented group being the fundamental group of a compact, orientable 3-manifold.

0. Introduction

In [Ha1], we defined some new invariants $\bar{\delta}_n$ for a finite CW-complex X . These invariants depend only on the fundamental group of X and measure the “size” of the successive quotients of the rational derived series of $\pi_1(X)$. Given X and a cohomology class $\psi \in H^1(X)$, $\bar{\delta}_n(\psi)$ is defined to be the degree of a “higher-order Alexander polynomial”. Although defined algebraically, these degrees have many topological applications in the case that X is a 3-manifold. In this case, we showed that the $\bar{\delta}_n$ give new estimates for the Thurston norm of a 3-manifold generalizing a theorem of C. McMullen [McM]. Recall that the **Thurston norm** of a class $\psi \in H^1(X; \mathbb{Z})$,

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$\|\psi\|_T$, is defined to be the minimum negative euler characteristic of all (possibly disconnected) surfaces F whose homology class $[F] \in H_2(X, \partial X; \mathbb{Z})$ is Poincaré dual to ψ and such that each component of F is non-positively curved. The $\bar{\delta}_n$ also give new algebraic obstructions to a 3-manifold fibering over S^1 , to a 4-manifold of the form $X \times S^1$ admitting a symplectic structure, and to a 3-manifold being Seifert fibered. They were also shown to have applications to minimal ropelength and genera of knots and links in S^3 . Related work has been done by T. Cochran, K. Orr, P. Teichner, for knots and knot concordance in [Co] and [COT]. Recently, V. Turaev [T] has generalized some of the results in [Ha1].

Since $\bar{\delta}_n$ only depends on the fundamental group, we can consider $\bar{\delta}_n$ as an invariant of a general group G , $\bar{\delta}_n: H^1(G; \mathbb{Z}) \rightarrow \mathbb{Z}$. In this paper, we continue to investigate the special behavior of the $\bar{\delta}_n$ when G is the fundamental group of a 3-manifold (with empty or toroidal boundary) or a group with deficiency at least 1. The results give new algebraic information about the topology of a symplectic 4-manifold of the form $X \times S^1$. They give obstructions to a finitely presented group having positive deficiency or being the fundamental group of a compact, orientable 3-manifold (with or without boundary). They also give new information about the behavior of the Thurston norm under a map between 3-manifolds which is surjective on π_1 . We state some of our main theorems and their applications below.

In [Ha1], we constructed examples of 3-manifolds for which $\bar{\delta}_n$ is a strictly increasing function of n for $n \geq 0$. Moreover, it was conjectured that the $\bar{\delta}_n$ are always a monotonically increasing function of n for $n \geq 1$. We show that this conjecture is true. This implies that the $\bar{\delta}_n$ always give estimates for the Thurston norm that are at least as good as (and sometimes better than) McMullen’s estimate given by the Alexander norm. By $\bar{\delta}_n \leq \bar{\delta}_{n+1}$ (respectively $\bar{\delta}_n = 0$) we mean that $\bar{\delta}_n(\psi) \leq \bar{\delta}_{n+1}(\psi)$ (respectively $\bar{\delta}_n(\psi) = 0$) for all $\psi \in H^1(X)$.

COROLLARY 2.10. *Let X be a closed, orientable, connected 3-manifold. If $\beta_1(X) \geq 2$ then*

$$\bar{\delta}_0 \leq \bar{\delta}_1 \leq \dots \leq \bar{\delta}_n \leq \dots$$

If $\beta_1(X) = 1$ and ψ is a generator of $H^1(X)$ then

$$\bar{\delta}_0(\psi) - 2 \leq \bar{\delta}_1(\psi) \leq \dots \leq \bar{\delta}_n(\psi) \leq \dots$$

As a consequence of Corollary 2.10, we show (in Theorem 3.8) that if 4-manifold of the form $X \times S^1$ admits a symplectic structure then X “looks algebraically like” a 3-manifold which fibers over S^1 , thus further supporting a conjecture of Taubes. The proof of Theorem 3.8 uses a theorem of Vidussi [Vi], who proves this theorem in the case $n = 0$.

THEOREM 3.8. *Let X be an closed, irreducible 3-manifold such that $X \times S^1$ admits a symplectic structure. If $\beta_1(X) \geq 2$ there exists a $\psi \in H^1(X; \mathbb{Z})$ such that*

$$\bar{\delta}_0(\psi) = \bar{\delta}_1(\psi) = \dots = \bar{\delta}_n(\psi) = \dots = \|\psi\|_T.$$

If $\beta_1(X) = 1$ then for any generator ψ of $H^1(X; \mathbb{Z})$,

$$\bar{\delta}_0(\psi) - 2 = \bar{\delta}_1(\psi) = \dots = \bar{\delta}_n(\psi) = \dots = \|\psi\|_T.$$

More generally, we define $\bar{\delta}_\Gamma(\psi)$ for any group G and any “admissible pair” $(\phi_\Gamma: G \rightarrow \Gamma, \psi: G \rightarrow \mathbb{Z})$ of G . When G is a finitely presented group with $\text{def}(G) \geq 1$, we show

that the $\bar{\delta}_n$ satisfy a monotonicity condition. We also prove a similar theorem when G is the fundamental group of a closed, orientable 3-manifold (see Theorem 2.9).

THEOREM 2.2. *Let G be a finitely presented group with $\text{def}(G) \geq 1$ and $(\phi_\Lambda, \phi_\Gamma, \psi)$ be an admissible triple for G . If $(\phi_\Lambda, \phi_\Gamma, \psi)$ is not initial then*

$$\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi) \tag{1}$$

otherwise

$$\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi) - 1. \tag{2}$$

As a consequence of the monotonicity theorems, we see that the $\bar{\delta}_\Gamma$ give obstructions to the deficiency of a group being positive or being the fundamental group of a compact, orientable 3-manifold. These obstructions are non-trivial even when the groups Γ and Λ are abelian. For example, we can easily recover the (known) result that \mathbb{Z}^m cannot be the fundamental group of a compact 3-manifold when $m \geq 4$ (see below, or Example 3.2 for more details).

PROPOSITION 3.1. *Let G be a finitely presented group and $(\phi_\Lambda, \phi_\Gamma, \psi)$ be an admissible triple for G .*

- (i) *Suppose $(\phi_\Lambda, \phi_\Gamma, \psi)$ is not initial. If $\bar{\delta}_\Lambda(\psi) < \bar{\delta}_\Gamma(\psi)$ then $\text{def}(G) \leq 0$ and G cannot be the fundamental group of a compact, orientable 3-manifold (with or without boundary).*
- (ii) *Suppose $(\phi_\Lambda, \phi_\Gamma, \psi)$ is initial. If $\bar{\delta}_\Lambda(\psi) < \bar{\delta}_\Gamma(\psi) - 1$ then $\text{def}(G) \leq 0$ and G cannot be the fundamental group of a compact, orientable 3-manifold with at least one boundary component which is not a 2-sphere. In addition, if $\bar{\delta}_\Lambda(\psi) < \bar{\delta}_\Gamma(\psi) - 2$ then G cannot be the fundamental group of a compact, orientable 3-manifold (with or without boundary).*

Let us consider the simplest case when Λ is the abelianization (modulo torsion) of G and $\Gamma = \mathbb{Z}$. In this case, $\bar{\delta}_\mathbb{Z}(\psi)$ is equal to the rank of $H_1((X_G)_\psi; \mathbb{Z})$ as an abelian group where $(X_G)_\psi$ is the infinite cyclic cover of X_G , a finite CW-complex with $\pi_1(X_G) = G$, corresponding to ψ (as long as this number is finite). Moreover, $\bar{\delta}_\Lambda(\psi)$ is equal the Alexander norm of ψ which depends only on ψ and the multivariable Alexander polynomial of G . For example, the Alexander polynomial of \mathbb{Z}^m is 1 so $\bar{\delta}_{\mathbb{Z}^m}(\psi) = 0$ for any ψ . Moreover, the first homology of any infinite cyclic cover of the m -torus is \mathbb{Z}^{m-1} so $\bar{\delta}_\mathbb{Z}(\psi) = m - 1$. Thus, as mentioned above, we see that \mathbb{Z}^m cannot be the fundamental group of a compact 3-manifold.

Recall that the i th-order degree of a group $\bar{\delta}_i(\psi)$ is a specific example of one of the $\bar{\delta}_\Gamma(\psi)$. We give examples of finite 2-complexes $X_{n,g}$ with $\beta_1(X_{n,g}) = 1$ for $n, g \geq 1$ such that the i th-order degrees for $0 \leq i \leq n - 1$ of $X_{n,g}$ are “large” but the n th-order degree is 0. Thus the fundamental group of these spaces cannot have positive deficiency nor can they be the fundamental group of a compact, orientable 3-manifold (see Proposition 2.5 and Example 3.4).

Theorem 2.9 also has applications to the study of the behavior of the genus of a knot under a surjective map on π_1 . The following question was asked by J. Simon (see R. Kirby’s Problem List [Ki, question 1.12(b)]).

Question 1.12(b) of [Ki] (J. Simon). If J and K are knots in S^3 and $f: S^3 \setminus L \rightarrow S^3 \setminus K$ is surjective on π_1 , is $g(L) \geq g(K)$?

The answer to the above question is known to be “yes” when $\bar{\delta}_0(K) = 2g(K)$. We strengthen this result to the case when $\bar{\delta}_n(K) = 2g(K) - 1$.

COROLLARY 3.12. *Suppose J and K are knots in S^3 such that there exists a surjective homomorphism $\rho: \pi_1(S^3 \setminus L) \rightarrow \pi_1(S^3 \setminus K)$. If $\bar{\delta}_0(K) = 2g(K)$ or $\bar{\delta}_n(K) = 2g(K) - 1$ for some $n \geq 1$ then $g(L) \geq g(K)$.*

We also prove this is the case if we replace the genus of a knot by the Thurston norm. The following corollary is a generalization of the result due to Gabai [Ga] that a degree one map $f: X \rightarrow Y$ between three manifolds gives the inequality $\|f^*(\psi)\|_T \geq \|\psi\|_T$ for all $\psi \in H^1(Y; \mathbb{Z})$. For simplicity, we state only the case when $\beta_1(Y) \geq 2$.

COROLLARY 3.11. *Suppose there exists an epimorphism $\rho: \pi_1(X) \rightarrow \pi_1(Y)$, where X and Y are compact, orientable 3-manifolds, with toroidal or empty boundaries, such that $\beta_1(X) = \beta_1(Y) \geq 2$ and $r_0(X) = 0$. Let $\psi \in H^1(\pi_1(Y); \mathbb{Z})$. If $\bar{\delta}_n(\psi) = \|\psi\|_T$ for some $n \geq 0$ then*

$$\|\rho^*(\psi)\|_T \geq \|\psi\|_T.$$

1. Definitions

We will define the higher-order degrees $\bar{\delta}_\Gamma$ and ranks r_r of a group G and surjective homomorphism $\phi_\Gamma: G \rightarrow \Gamma$. This definition will agree with the definition of $\bar{\delta}_n$ given for a CW-complex X (as defined in section 3 of [Ha1]) when $G = \pi_1(X)$, $\Gamma = G/G_r^{(n+1)}$ and $\phi_\Gamma = \phi_n: G \rightarrow G/G_r^{(n+1)}$ is the quotient map. For more details see [Ha1, sections 3, 4, 5] and [Co, sections 2, 3, 5].

We recall the definition of a poly-torsion-free-abelian group.

Definition 1.1. A group Γ is poly-torsion-free-abelian (PTFA) if it admits a normal series $\{1\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = \Gamma$ such that each of the factors G_{i+1}/G_i is torsion-free abelian.

Remark 1.2. Recall that if $A \triangleleft G$ is torsion-free-abelian and G/A is PTFA then G is PTFA. Any PTFA group is torsion-free and solvable (the converse is not true). Also, any subgroup of a PTFA group is a PTFA group [Pa, lemma 2.4, p. 421].

Some examples of interesting series associated to a group G are the rational lower central series of G (see Stallings [Sta]), the rational lower central series of the rational commutator subgroup of G , the rational derived series $G_r^{(n)}$ of G (defined below), and the torsion-free derived series $G_H^{(n)}$ of G (see [CH]). In this paper, our examples and applications will use the rational derived series of a group (defined below). We point out that the torsion-free derived series is very interesting since it gives new concordance invariants of links in S^3 (see [CH] or [Ha3]). For any of the subgroups N in the above mentioned series, G/N is a PTFA group. In particular, for each $n \geq 0$, $G/G_r^{(n+1)}$ is PTFA by Corollary 3.6 of [Ha1]. We recall the definition of $G_r^{(n)}$.

Definition 1.3. Let G be a group and $G_r^{(0)} = G$. For $n \geq 1$ define

$$G_r^{(n)} = \{g \in G_r^{(n-1)} \mid g^k \in [G_r^{(n-1)}, G_r^{(n-1)}] \text{ for some } k \in \mathbb{Z} - \{0\}\}$$

to be the n th term of the rational derived series of G .

R. Strebel showed that if G is the fundamental group of a (classical) knot exterior then the quotients of successive terms of the derived series are torsion-free abelian [Str]. Hence for knot exteriors we have $G_r^{(i)} = G^{(i)}$. This is also well known to be true for free groups. Since any non-compact surface has free fundamental group, this also holds for all orientable surface groups.

We make some remarks about PTFA groups. Recall that if Γ is PTFA then $\mathbb{Z}\Gamma$ is an Ore domain and hence $\mathbb{Z}\Gamma$ embeds in its right ring of quotients $\mathcal{K}_\Gamma := \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1}$ which is a skew field. More generally, if $S \subseteq R$ is a right divisor set of a ring R then the right quotient ring RS^{-1} exists ([Pa, p. 146] or [Ste, p. 52]). By RS^{-1} we mean a ring containing R with the property that:

- (1) every element of S has an inverse in RS^{-1} ;
- (2) every element of RS^{-1} is of the form rs^{-1} with $r \in R, s \in S$.

If R is an Ore domain and S is a right divisor set then RS^{-1} is flat as a left R -module [Ste, proposition II.3.5]. In particular, \mathcal{K}_Γ is a flat left $\mathbb{Z}\Gamma$ -module. Moreover, every finitely generated right module over a skew field is free and such modules have a well defined rank, $\text{rank}_{\mathcal{K}_\Gamma}$, which is additive on short exact sequences [Co1, p. 48]. Thus, if C is a non-negative finite chain complex of finitely generated free right $\mathbb{Z}\Gamma$ -modules then the Euler characteristic $\chi(C) = \sum_{i=0}^\infty (-1)^i \text{rank } C_i$ is defined and is equal to $\sum_{i=0}^\infty (-1)^i \text{rank}_{\mathcal{K}_\Gamma} H_i(C; \mathcal{K}_\Gamma)$. In this paper, we will repeatedly use this fact about the Euler characteristic.

Let $\psi: G \rightarrow \mathbb{Z}$ be a surjective homomorphism. Note that we will always be considering \mathbb{Z} as the multiplicative group $\mathbb{Z} = \langle t \rangle$ generated by t . We wish to define $\bar{\delta}_\Gamma(\psi)$ as a non-negative integer. However, in order to do this, we need some compatibility conditions on Γ and ψ .

Definition 1.4. Let G be a group, $\phi_\Gamma: G \rightarrow \Gamma$, and $\psi: G \rightarrow \mathbb{Z}$ where Γ is a PTFA group. We say that (ϕ_Γ, ψ) is an **admissible pair** for G if there exists a surjection $\alpha_{\Gamma, \mathbb{Z}}: \Gamma \rightarrow \mathbb{Z}$ such that $\psi = \alpha_{\Gamma, \mathbb{Z}} \circ \phi_\Gamma$. If $\alpha_{\Gamma, \mathbb{Z}}$ is an isomorphism then we say that (ϕ_Γ, ψ) is **initial**.

Let (ϕ_Γ, ψ) be an admissible pair for G . We define $\Gamma' := \ker(\alpha_{\Gamma, \mathbb{Z}})$. It is clear that (ϕ_Γ, ψ) is initial if and only if $\Gamma' = 1$. Since Γ is PTFA by Remark 1.2, Γ' is PTFA. Hence Γ' embeds in its right ring of quotients which we call $\mathbb{K}_{\Gamma'}$. Moreover, $\mathbb{Z}\Gamma' - \{0\}$ is known to be a right divisor set of $\mathbb{Z}\Gamma'$ [Pa, p. 609] hence we can define the right quotient ring $R_{\Gamma'} := \mathbb{Z}\Gamma'(\mathbb{Z}\Gamma' - \{0\})^{-1}$. After choosing a splitting $\xi: \mathbb{Z} \rightarrow \Gamma'$, we see that any element of $R_{\Gamma'}$ can be written uniquely as $\sum t^{n_i} k_i$ where $t = \xi(1)$ and $k_i \in \mathbb{K}_{\Gamma'}$. In this way, one sees that $R_{\Gamma'}$ is isomorphic to the skew polynomial ring $\mathbb{K}_{\Gamma'}[t^{\pm 1}]$ (see the proof of proposition 4.5 of [Ha1] for more details). Moreover, the embedding $g_\psi: \mathbb{Z}\Gamma' \rightarrow \mathbb{K}_{\Gamma'}$ extends to this isomorphism $R_{\Gamma'} \rightarrow \mathbb{K}_{\Gamma'}[t^{\pm 1}]$ (here we are identifying $\mathbb{K}_{\Gamma'}$ and $t^0 \mathbb{K}_{\Gamma'}$).

The abelian group $(G_\Gamma)_{ab} := \ker \phi_\Gamma / [\ker \phi_\Gamma, \ker \phi_\Gamma]$ is a right $\mathbb{Z}\Gamma$ -module via conjugation,

$$[g]\gamma = [\gamma^{-1}g\gamma]$$

for $\gamma \in \Gamma$ and $g \in \ker \phi_\Gamma$. Moreover, $(G_\Gamma)_{ab}$ is a $\mathbb{Z}\Gamma'$ -module via the inclusion $\mathbb{Z}\Gamma' \hookrightarrow \mathbb{Z}\Gamma$. Thus, $(G_\Gamma)_{ab} \otimes_{\mathbb{Z}\Gamma} \mathcal{K}_\Gamma$ and $(G_\Gamma)_{ab} \otimes_{\mathbb{Z}\Gamma'} \mathbb{K}_{\Gamma'}$ are right \mathcal{K}_Γ and $\mathbb{K}_{\Gamma'}$ -modules respectively.

Definition 1.5. Let G be a group and $\phi_r : G \rightarrow \Gamma$ a coefficient system with Γ a PTFA group. We define the Γ -rank of G to be

$$r_r(G) = \text{rank}_{\mathcal{K}_\Gamma} \left(\frac{\ker \phi_r}{[\ker \phi_r, \ker \phi_r]} \otimes_{\mathbb{Z}\Gamma} \mathcal{K}_\Gamma \right).$$

For a general group G and coefficient system ϕ_r , this rank may be infinite. However, if G is finitely generated and ϕ_r is non-zero then by Proposition 2.11 of [COT], $r_r(G) \leq \beta_1(G) - 1$ and hence is finite. In the case that ϕ_r is the zero map, $r_r(G) = \beta_1(G)$.

Definition 1.6. Let G be a finitely generated group and (ϕ_r, ψ) an admissible pair for G . We define the Γ -degree of ψ to be

$$\bar{\delta}_\Gamma(\psi) = \text{rank}_{\mathbb{K}_\Gamma} \left(\frac{\ker \phi_r}{[\ker \phi_r, \ker \phi_r]} \otimes_{\mathbb{Z}\Gamma'} \mathbb{K}_\Gamma \right)$$

if $r_r(G) = 0$ and $\bar{\delta}_\Gamma(\psi) = 0$ otherwise.

We remark that $(G_\Gamma)_{ab} \otimes_{\mathbb{Z}\Gamma'} \mathbb{K}_\Gamma$ is merely $(G_\Gamma)_{ab} \otimes_{\mathbb{Z}\Gamma} \mathbb{K}_\Gamma[t^{\pm 1}]$ viewed as a \mathbb{K}_Γ -module. Since G is a finitely generated group, $(G_\Gamma)_{ab} \otimes_{\mathbb{Z}\Gamma} \mathbb{K}_\Gamma[t^{\pm 1}]$ is a finitely generated $\mathbb{K}_\Gamma[t^{\pm 1}]$ -module. Moreover, since $\mathbb{K}_\Gamma[t^{\pm 1}]$ is a (noncommutative left and right) principal ideal domain, [Co2, 2.1.1, p. 49], the latter is isomorphic to

$$\bigoplus_{i=1}^l \mathbb{K}_\Gamma[t^{\pm 1}] / \langle p_i(t) \rangle \oplus (\mathbb{K}_\Gamma[t^{\pm 1}])_{r_r(G)}$$

[Ja, theorem 16, p. 43]. Thus, $(G_\Gamma)_{ab} \otimes_{\mathbb{Z}\Gamma'} \mathbb{K}_\Gamma$ is a finitely generated \mathbb{K}_Γ -module if and only if $r_r(G) = 0$. In particular, if $r_r(G) = 0$ then $\bar{\delta}_\Gamma(\psi)$ is the sum of the degrees of the $p_i(t)$. Therefore, $\bar{\delta}_\Gamma(\psi)$ as defined above is always finite.

Let us consider the case when $\Gamma = \mathbb{Z}^m$. Let X be a CW-complex with $\pi_1(X) = G$ and X_{ϕ_r} be the regular \mathbb{Z}^m -cover of X corresponding to ϕ_r . Consider an admissible pair $(\phi_{\mathbb{Z}^m}, \psi)$ for G . This is one such that $\psi = \psi' \circ \phi_r$ where $\psi' : \mathbb{Z}^m \rightarrow \mathbb{Z}$. In this case, $H_1(X_{\phi_r}; \mathbb{Z}) = \ker \phi_r / [\ker \phi_r, \ker \phi_r]$ is a module over the Laurent polynomial ring with m variables, $\mathbb{Z}[\mathbb{Z}^m]$. Moreover, $H_1(X_{\phi_r}; \mathbb{Z})$ can be considered as a module over the Laurent polynomial ring with $m - 1$ variables $\mathbb{Z}\Gamma' = \mathbb{Z}[\mathbb{Z}^{m-1}]$. Note that the $m - 1$ variables in $\mathbb{Z}[\mathbb{Z}^{m-1}]$ correspond to a choice of basis elements of $\Gamma' = \ker(\alpha_{\mathbb{Z}^m, \mathbb{Z}} : \mathbb{Z}^m \rightarrow \mathbb{Z})$. Therefore, as long as the rank of $H_1(X_{\phi_r}; \mathbb{Z})$ as a $\mathbb{Z}[\mathbb{Z}^m]$ -module is 0, $\bar{\delta}_{\mathbb{Z}^m}(\psi)$ is equal to the rank of $H_1(X_{\phi_r}; \mathbb{Z})$ as a $\mathbb{Z}[\mathbb{Z}^{m-1}]$ -module. In particular, when $m = 1$, $\bar{\delta}_{\mathbb{Z}}(\psi)$ is equal to the rank of $H_1(X_\psi; \mathbb{Z})$ as an abelian group where X_ψ is the infinite cover corresponding to ψ as long as this rank is finite (otherwise $\bar{\delta}_{\mathbb{Z}}(\psi) = 0$). When \mathbb{Z}^m is the abelianization of G , $\bar{\delta}_{\mathbb{Z}^m}(\psi) = \bar{\delta}_0(\psi)$ (see below for the definition of $\bar{\delta}_0$) is equal to the Alexander norm (see [McM] for the definition of the Alexander norm) of ψ by [Ha1, proposition 5.12].

We now define the higher-order degrees and ranks associated to a group G . For each $n \geq 0$, let $\Gamma_n = G / G_r^{(n+1)}$ where $G_r^{(n+1)}$ is the $(n + 1)$ st-term of the rational derived series of G as defined in Definition 1.3. We define the n th-order rank of X to be

$$r_n(X) = r_{\Gamma_n}(X).$$

Next, we remark that if $\psi \in H^1(G; \mathbb{Z}) \cong \text{Hom}(G; \mathbb{Z})$, then $\psi(G_r^{(n+1)}) = 1$. Hence for each primitive $\psi \in H^1(G; \mathbb{Z})$ the pair (ϕ_{Γ_n}, ψ) is an admissible pair for G . For primitive ψ , we define the n th-order degree of ψ to be

$$\bar{\delta}_n(\psi) = \bar{\delta}_{\Gamma_n}(\psi).$$

For non-primitive ψ , there is a primitive cohomology class $\psi' \in H^1(X; \mathbb{Z})$ such that $\psi = m\psi'$. Define $\bar{\delta}_n(\psi) = m\bar{\delta}_n(\psi')$.

Thus, for each group G and $n \geq 0$ we have defined a function $\bar{\delta}_n : H^1(G; \mathbb{Z}) \rightarrow \mathbb{Z}$ which is “linear on rays through the origin”. We put a partial ordering on these functions by $\bar{\delta}_i \leq \bar{\delta}_j$ if $\bar{\delta}_i(\psi) \leq \bar{\delta}_j(\psi)$ for all $\psi \in H^1(G; \mathbb{Z})$. Also, we say that $\bar{\delta}_i = 0$ provided $\bar{\delta}_i(\psi) = 0$ for all $\psi \in H^1(G; \mathbb{Z})$.

Suppose $f : E \rightarrow G$ is a surjective homomorphism and (ϕ_r, ψ) is an admissible pair for G . Then there is an induced admissible pair $(\phi_r \circ f, \psi \circ f)$ for E . In particular, we can speak $\bar{\delta}_r^Y E(\psi \circ f)$. When we have this situation, unless otherwise noted, we will use this admissible pair induced by G . When there is no confusion, we will suppress the f and just write (ϕ_r, ψ) when we mean $(\phi_r \circ f, \psi \circ f)$ or ψ when we mean $\psi \circ f$.

In this paper, we will often use the notation $r_r(X)$ and $\bar{\delta}_r^X(\psi)$ for X a CW-complex and ψ an element of $H^1(X; \mathbb{Z}) \cong H^1(\pi_1(X); \mathbb{Z})$. By this, we mean $r_r(\pi_1(X))$ and $\bar{\delta}_r^{\pi_1(X)}(\psi)$ for an admissible pair (ϕ_r, ψ) for $\pi_1(X)$. These are equivalent to the homological definitions given in [Ha1]. That is, if (ϕ_r, ψ) is an admissible pair for $\pi_1(X)$ then $H_1(X; \mathbb{K}_\Gamma[t^{\pm 1}])$ and $H_1(X; \mathcal{K}_\Gamma)$ are right \mathbb{K}_Γ and \mathcal{K}_Γ -modules respectively and since \mathcal{K}_Γ and $\mathbb{K}_\Gamma[t^{\pm 1}]$ are flat left $\mathbb{Z}\Gamma$ -modules [Ste, proposition II.3.5], we see that

$$r_r(X) = \text{rank}_{\mathcal{K}_\Gamma} H_1(X; \mathcal{K}_\Gamma)$$

and

$$\bar{\delta}_r(\psi) = \text{rank}_{\mathbb{K}_\Gamma} H_1(X; \mathbb{K}_\Gamma[t^{\pm 1}])$$

if $r_r(X) = 0$ and $\bar{\delta}_r(\psi) = 0$ otherwise.

2. Main results

We seek to study the behavior of $\bar{\delta}_n(\psi)$ as n increases. More generally, we would like to compare $\bar{\delta}_\Gamma$ as we vary the group Γ . We show that the $\bar{\delta}_\Gamma$ satisfy a monotonicity condition provided the groups satisfy a compatibility condition. We describe this condition below.

Definition 2.1. Let G be a group, $\phi_\Lambda : G \twoheadrightarrow \Lambda$, $\phi_\Gamma : G \twoheadrightarrow \Gamma$, and $\psi : G \twoheadrightarrow \mathbb{Z}$ where Λ and Γ are PTFA groups. We say that $(\phi_\Lambda, \phi_\Gamma, \psi)$ is an **admissible triple** for G if there exist surjections $\alpha_{\Lambda, \Gamma} : \Lambda \twoheadrightarrow \Gamma$ and $\alpha_{\Gamma, \mathbb{Z}} : \Gamma \twoheadrightarrow \mathbb{Z}$ such that $\phi_\Gamma = \alpha_{\Lambda, \Gamma} \circ \phi_\Lambda$, $\psi = \alpha_{\Gamma, \mathbb{Z}} \circ \phi_\Gamma$, and $\alpha_{\Lambda, \Gamma}$ is not an isomorphism. If $\alpha_{\Gamma, \mathbb{Z}}$ is an isomorphism then we say that $(\phi_\Lambda, \phi_\Gamma, \psi)$ is **initial**.

Note that if $(\phi_\Lambda, \phi_\Gamma, \psi)$ an admissible triple then (ϕ_Λ, ψ) and (ϕ_Γ, ψ) are both admissible pairs. Hence, in this case, we can define both $\bar{\delta}_\Lambda(\psi)$ and $\bar{\delta}_\Gamma(\psi)$. We also remark that $(\phi_\Lambda, \phi_\Gamma, \psi)$ is initial if and only if (ϕ_Γ, ψ) is initial. Moreover, (ϕ_Λ, ψ) is never initial since $\Lambda \twoheadrightarrow \Gamma$ is not an isomorphism. We will show that $\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi)$ as long as the triple is not initial. We point out that even if $\alpha_{\Lambda, \Gamma}$ is an isomorphism, we can define both the Λ - and Γ -degrees and in this case $\delta_\Gamma(\psi) = \delta_\Lambda(\psi)$!

We now proceed to state and prove the main theorems.

THEOREM 2.2. Let G be a finitely presented group with $\text{def}(G) \geq 1$ and $(\phi_\Lambda, \phi_\Gamma, \psi)$ be an admissible triple for G . If $(\phi_\Lambda, \phi_\Gamma, \psi)$ is not initial then

$$\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi) \tag{3}$$

otherwise

$$\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi) - 1. \tag{4}$$

Before proving Theorem 2.2, we will state a Corollary of the theorem and make some remarks about the deficiency hypothesis in the theorem. First, let Γ_n be the quotient of G by the $(n + 1)$ st term of the rational derived series as in Definition 1.3. Recall that for any $\psi \in H^1(G; \mathbb{Z})$, (ϕ_{Γ_n}, ψ) is an admissible pair. Moreover, $(\phi_{\Gamma_{n+1}}, \phi_{\Gamma_n}, \psi)$ is an admissible triple unless $G_r^{(n+1)} = G_r^{(n+2)}$ which is initial if and only if $\beta_1(G) = 1$ and $n = 0$. Hence by Theorem 2.2 we see that the $\bar{\delta}_n$ are a nondecreasing function of n (for $n \geq 1$). This behavior was first established for the fundamental groups of knot complements in S^3 by T. Cochran in [Co, theorem 5.4]. Recall that $\bar{\delta}_{n+1} \geq \bar{\delta}_n$ (respectively $\bar{\delta}_n = 0$) means that $\bar{\delta}_{n+1}(\psi) \geq \bar{\delta}_n(\psi)$ (respectively $\bar{\delta}_n(\psi) = 0$) for all $\psi \in H^1(G; \mathbb{Z})$.

COROLLARY 2.3. *Let G be a finitely presented group with $\text{def}(G) \geq 1$. If $\beta_1(G) \geq 2$ then*

$$\bar{\delta}_0 \leq \bar{\delta}_1 \leq \dots \leq \bar{\delta}_n \leq \dots .$$

If $\beta_1(G) = 1$ and ψ is a generator of $H^1(G; \mathbb{Z})$ then $\bar{\delta}_0(\psi) - 1 \leq \bar{\delta}_1(\psi) \leq \dots \leq \bar{\delta}_n(\psi) \leq \dots$.

Proof. Let ψ be a primitive class in $H^1(G; \mathbb{Z})$. We can assume that $G_r^{(n+1)} \neq G_r^{(n+2)}$ since if $G_r^{(n+1)} = G_r^{(n+2)}$ then $\bar{\delta}_{n+1}(\psi) = \bar{\delta}_n(\psi)$ (note that in the case $\beta_1(G) = 1$ and $n = 0$, $\bar{\delta}_1(\psi) = \bar{\delta}_0(\psi) \geq \bar{\delta}_0(\psi) - 1$ is also satisfied). Therefore $T = (\phi_{\Gamma_{n+1}}, \phi_{\Gamma_n}, \psi)$ is an admissible triple. As mentioned above, T is initial if and only if $\beta_1(G) = 1$ and $n = 0$. Hence if $\beta_1(G) = 1$ and $n = 0$ then by Theorem 2.2, $\bar{\delta}_1(\psi) \geq \bar{\delta}_0(\psi) - 1$. Otherwise, $\bar{\delta}_{n+1}(\psi) \geq \bar{\delta}_n(\psi)$.

If $\beta_1(G) \geq 2$ and ψ is not primitive then $\psi = m\psi'$ for some primitive ψ' and $m \geq 2$. Hence, $\bar{\delta}_{n+1}(\psi) = m\bar{\delta}_{n+1}(\psi') \geq m\bar{\delta}_n(\psi') = \bar{\delta}_n(\psi)$.

We now make some remarks about the condition $\text{def}(G) \geq 1$. First, if G has deficiency at least 2 then the results of Theorem 2.2 and Corollary 2.3 hold simply because all of the degrees are zero.

Remark 2.4. If G is a finitely presented group with $\text{def}(G) \geq 2$ and (ϕ_Γ, ψ) is an admissible pair for G then $r_\Gamma(G) \geq 1$ and hence $\bar{\delta}_\Gamma(\psi) = 0$.

To see this, let X_G be a finite, connected 2-complex with one 0-cell x_0 , m 1-cells, r 2-cells where $m - r \geq 2$ and $G = \pi_1(X_G, x_0)$. Then $H_1(X_G, x_0; \mathcal{K}_\Gamma)$ has a presentation with m generators and r relations so $\text{rank}_{\mathcal{K}_\Gamma} H_1(X_G, x_0; \mathcal{K}_\Gamma) \geq 2$ and hence $r_\Gamma(G) = r_\Gamma(X_G) = \text{rank}_{\mathcal{K}_\Gamma} H_1(X_G, x_0; \mathcal{K}_\Gamma) - 1 \geq 1$ [Ha1, sections 4 and 5]. Therefore, $\bar{\delta}_\Gamma(\psi) = 0$ for all $\psi \in H^1(G; \mathbb{Z})$.

However, if the deficiency of G is not positive, we can create an infinite number of examples where the theorem is false! We construct finitely presented groups for which the degrees are “large” up to (but not including) the n th stage but the degree at the n th stage is zero! For simplicity, we only describe examples when $\beta_1(G) = 1$. However, the reader should notice that the same type of behavior can be seen for groups with $\beta_1(G) \geq 2$ using the same techniques.

PROPOSITION 2.5. *For each $g \geq 1$ and $n \geq 1$ there exist examples of finitely presented groups $G_{n,g}$ with $\text{def}(G_{n,g}) \leq 0$ and $\beta_1(G_{n,g}) = 1$ such that $\bar{\delta}_0(\psi) = 2g$, $\bar{\delta}_i(\psi) = 2g - 1$ for $1 \leq i \leq n - 1$ and $\bar{\delta}_n(\psi) = 0$ whenever ψ is a generator of $H^1(G_{n,g}; \mathbb{Z})$.*

Proof. We will construct these examples by adding relations to the fundamental group of a fibered knot complement G that kill the generators the $G^{(n+1)}/G^{(n+2)} \otimes \mathbb{K}_n$. Let G be the fundamental group of a fibered knot K in S^3 of genus $g \geq 1$ and $n \geq 1$. Since K is fibered, $G^{(1)}$ is free, so $G_r^{(n+1)}/G_r^{(n+2)} = G^{(n+1)}/G^{(n+2)}$ and $\mathcal{A}_n = G^{(n+1)}/G^{(n+2)} \otimes_{\mathbb{Z}\Gamma_n} \mathbb{K}_n$ is a finitely generated free right \mathbb{K}_n -module of rank $2g - 1$. [Co, proposition 9.1]. Let a_1, \dots, a_{2g-1} be the generators of \mathcal{A}_n . Since \mathbb{K}_n is an Ore domain, we can find $k_j \in \mathbb{K}_n$ such that $a_j k_j \in G^{(n+1)}/G^{(n+2)} \otimes 1$. Pick $\gamma_1, \dots, \gamma_{2g-1} \in G^{(n+1)}$ such that $[\gamma_j] = a_j k_j$ and let $H = G/\langle \gamma_1, \dots, \gamma_{2g-1} \rangle$ and $\eta: G \rightarrow H$. Note that since any knot group has deficiency 1, H has a presentation with m generators and $m + 2g - 2$ relations. Since $\gamma_1, \dots, \gamma_{2g-1} \in G^{(n+1)}$, we have an isomorphism $G/G^{(n+1)} \cong H/H^{(n+1)} \cong H/H_r^{(n+1)}$. Therefore, $\delta_0^H(\psi) = \bar{\delta}_0^G(\psi) = 2g$ and $\delta_i^H(\psi) = \bar{\delta}_i^G(\psi) = 2g - 1$ for $1 \leq i \leq n - 1$.

Since $G' \rightarrow H'$, we have $H'/H^{(n+1)} \cong G'/G^{(n+1)}$ for $0 \leq i \leq n$ hence $\mathbb{K}_n = \mathbb{K}_n^G \cong \mathbb{K}_n^{H'}$. Moreover, since $G^{(n+1)} \rightarrow H^{(n+1)}$, the map $G^{(n+1)}/G^{(n+2)} \otimes \mathbb{K}_n \rightarrow H^{(n+1)}/H^{(n+2)} \otimes \mathbb{K}_n$ is surjective. But the generators of \mathcal{A}_n are sent to zero under this map, so $H^{(n+1)}/H^{(n+2)} \otimes \mathbb{K}_n = 0$. Finally, $H_r^{(n+1)} = H^{(n+1)}$ so

$$\frac{H_r^{(n+1)}}{H_r^{(n+2)}} \otimes \mathbb{K}_n \cong \frac{H^{(n+1)}}{H_r^{(n+2)}} \otimes \mathbb{K}_n \cong \left(\frac{H^{(n+1)}}{H^{(n+2)}} \Big/ \{ \mathbb{Z}\text{-torsion} \} \right) \otimes \mathbb{K}_n = 0$$

(see lemma 3.5 of [Ha1] for the second isomorphism) hence $\bar{\delta}_n(\psi) = 0$.

We will now prove Theorem 2.2.

Proof of Theorem 2.2. If the deficiency of G is at least 2 then by Remark 2.4, all of the degrees are zero hence the conclusions of the theorem are true. Now we prove the case when $\text{def}(G) = 1$. We can assume that $r_\Gamma(G) = 0$, otherwise $\bar{\delta}_\Gamma(\psi) = 0$ and hence the statement of the theorem is true since $\bar{\delta}_\Lambda(\psi)$ is always non-negative. Since G is finitely presented, there is a finite 2-complex X such that $G = \pi_1(X)$ and $\chi(X) = 1 - \text{def}(G) = 0$. Recall that X is obtained from the presentation of G with deficiency 1 by starting with one 0-cell, attaching a 1-cell for each generator and a 2-cell for each relation in the presentation of G . Since $\Gamma \rightarrow \mathbb{Z}$ and ϕ_Γ is surjective, $H_i(X; \mathcal{K}_\Gamma) = 0$ for $i \neq 1, 2$ [COT, proposition 2.9]. Moreover, $\chi(X) = 0$ implies that $\text{rank}_{\mathcal{K}_\Gamma} H_2(X; \mathcal{K}_\Gamma) = \text{rank}_{\mathcal{K}_\Gamma} H_1(X; \mathcal{K}_\Gamma) = r_\Gamma(G) = 0$ since the Euler characteristic can be computed using \mathcal{K}_Γ -coefficients as mentioned in Section 1. Since $r_\Gamma(X) = 0$, it follows that $r_\Lambda(X) = 0$ [Ha2]. Replacing Γ by Λ in the above argument, it follows that $\text{rank}_{\mathcal{K}_\Lambda} H_2(X; \mathcal{K}_\Lambda) = 0$.

Let X_ψ be the infinite cyclic cover of X corresponding to ψ . There is a coefficient system for X_ψ , $\phi'_\Gamma: \pi_1(X_\psi) \rightarrow \Gamma'$, given by restricting ϕ_Γ to $\pi_1(X_\psi)$. Moreover, as \mathbb{K}_Γ -modules $H_1(X; \mathcal{K}_\Gamma) \cong H_1(X_\psi; \mathbb{K}_\Gamma)$ so $H_1(X_\psi; \mathbb{K}_\Gamma)$ is a finitely generated free \mathbb{K}_Γ -module of rank $\bar{\delta}_\Gamma(\psi)$ (similarly for Λ). Since Γ' is PTFA (and hence $\mathbb{Z}\Gamma'$ is an Ore domain), there exists a wedge of e circles W and a map $f: W \rightarrow X_\psi$ such that

$$f_*: H_1(W; \mathbb{K}_\Gamma) \rightarrow H_1(X_\psi; \mathbb{K}_\Gamma)$$

is an isomorphism. Here, the coefficient system on W is given by $\phi'_\Gamma \circ f_*$. By the proof of lemma 2.1 in [COT], $\ker \phi_\Gamma \neq \ker \psi$ if and only if $\phi'_\Gamma \circ f_*$ is non-trivial. Moreover, since W is a finite connected 2-complex with $H_2(W) = 0$, if $\ker \phi_\Gamma \neq \ker \psi$ then $H_1(W; \mathbb{K}_\Gamma) \cong \mathbb{K}_\Gamma^{e-1}$ [COT, lemma 2.12]; otherwise $H_1(W; \mathbb{K}_\Gamma) \cong \mathbb{K}_\Gamma^e$.

Up to homotopy we can assume that W is a subcomplex of X_ψ by replacing X_ψ with the mapping cylinder of f . Consider the long exact sequence of the pair (X_ψ, W)

with coefficients in \mathbb{K}_Γ :

$$H_2(X_\psi; \mathbb{K}_\Gamma) \longrightarrow H_2(X_\psi, W; \mathbb{K}_\Gamma) \longrightarrow H_1(W; \mathbb{K}_\Gamma) \longrightarrow H_1(X_\psi; \mathbb{K}_\Gamma).$$

Since X has no 3-cells, there is a cell complex, $C_i(X; \mathbb{Z}\Gamma)$, which has no 3-cells. Therefore, $TH_2(X; \mathbb{Z}\Gamma)$, the $\mathbb{Z}\Gamma$ -torsion submodule of $H_2(X; \mathbb{Z}\Gamma)$, is zero. Now, the kernel of the map $H_2(X; \mathbb{Z}\Gamma) \rightarrow H_2(X; \mathbb{K}_\Gamma)$ is $TH_2(X; \mathbb{Z}\Gamma)$. Moreover, we have shown that $H_2(X; \mathbb{K}_\Gamma) = 0$ hence $H_2(X; \mathbb{Z}\Gamma) = 0$. Thus, $H_2(X_\psi; \mathbb{K}_\Gamma) \cong H_2(X; \mathbb{K}_\Gamma[t^{\pm 1}]) \cong H_2(X; \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathbb{K}[t^{\pm 1}] = 0$. Since the last arrow in the sequence is an isomorphism, $H_2(X_\psi, W; \mathbb{K}_\Gamma) = 0$. Our goal is to show that $H_2(X_\psi, W; \mathbb{K}_\Lambda) = 0$. Then by analyzing the long exact sequence of the pair (X_ψ, W) with coefficients in \mathbb{K}_Λ , it will follow that $H_1(W; \mathbb{K}_\Lambda) \rightarrow H_1(X_\psi; \mathbb{K}_\Lambda)$ is a monomorphism. We note that $\ker \phi_\Lambda \neq \ker \phi_\Gamma$ implies that $\text{rank}_{\mathbb{K}_\Lambda} H_1(W; \mathbb{K}_\Lambda) = e - 1$ as above. Thus, if $\ker \phi_\Gamma \neq \ker \psi$ then (assuming the monomorphism above) $\bar{\delta}_\Lambda(\psi) \geq e - 1 = \bar{\delta}_\Gamma(\psi)$; otherwise $\bar{\delta}_\Lambda(\psi) \geq e - 1 = \bar{\delta}_\Gamma(\psi) - 1$.

Consider the relative chain complex of (X_ψ, W) with coefficients in $\mathbb{Z}\Gamma'$:

$$0 \longrightarrow C_2(X_\psi, W; \mathbb{Z}\Gamma') \xrightarrow{\partial_2^{\Gamma'}} C_1(X_\psi, W; \mathbb{Z}\Gamma') \longrightarrow \dots$$

Since W has no 2-cells, X_ψ has no 3-cells. Therefore $H_2(X_\psi, W; \mathbb{Z}\Gamma')$ is $\mathbb{Z}\Gamma'$ -torsion free, so $H_2(X_\psi, W; \mathbb{K}_\Gamma) = 0$ implies that $H_2(X_\psi, W; \mathbb{Z}\Gamma') = 0$ and hence $\partial_2^{\Gamma'}$ is injective.

Let $A = \ker(\alpha_{\Lambda, \Gamma|_{\Lambda'}} : \Lambda' \rightarrow \Gamma')$. Since A is a subgroup of a PTFA group, A is PTFA by Remark 1.2. If M is any right $\mathbb{Z}\Lambda'$ -module then $M \otimes_{\mathbb{Z}A} \mathbb{Z}$ has the structure of a right $\mathbb{Z}\Gamma'$ -module given by

$$\left(\sum \sigma \otimes n \right) \gamma = \sum \sigma \gamma \otimes n$$

for any $\gamma \in \Gamma'$. Moreover, one can check that $C_*(X_\psi, W; \mathbb{Z}\Lambda') \otimes_{\mathbb{Z}A} \mathbb{Z}$ is isomorphic to $C_*(X_\psi, W; \mathbb{Z}\Gamma')$ as right $\mathbb{Z}\Gamma'$ -modules. Thus, after making this identification, $\partial_2^{\Lambda'} : C_2(X_\psi, W; \mathbb{Z}\Lambda') \rightarrow C_1(X_\psi, W; \mathbb{Z}\Lambda')$ is injective by the following result of Strebel.

PROPOSITION 2.6. ([Str, p. 305]). *Suppose Γ is a PTFA group and R is a commutative ring. Any map between projective right $R\Gamma$ -modules whose image under the functor $-\otimes_{R\Gamma}$ is injective, is itself injective.*

Since \mathbb{K}_Λ is flat as a $\mathbb{Z}\Lambda'$ -module, $H_2(X_\psi, W; \mathbb{K}_\Lambda) = 0$ as desired.

Suppose Λ and Γ are abelian groups and G is the fundamental group of a compact orientable manifold with toroidal (or empty) boundary. In this case, it can easily be shown, using the results in [McM] and [Ha1], that the inequalities in Theorem 2.2 (and Theorem 2.9 below) are in fact *equalities* for all ψ which lie in the cone of an open face of the Alexander norm ball. We show below that even in this case, there are ψ for which the inequality in Theorem 2.2 is necessary.

Example 2.7. Let X be the exterior of the Borromean rings in S^3 and let G be the fundamental group of the X . A Wirtinger presentation of G is given by $\langle x, y, z \mid [z, [x, y^{-1}]], [y, [z, x^{-1}]] \rangle$ (see [F, p.10] for a similar presentation). Thus, there is an epimorphism $f: G \rightarrow \langle y, z \rangle$ by sending x to 1. Let $\psi_{(0,m,n)}: G \rightarrow \mathbb{Z}$ be the homomorphism defined by $\psi(x) = 1, \psi(y) = t^m, \psi(z) = t^n$ where $\text{gcd}(m, n) = 1$. Since f factors through $\psi_{(0,m,n)}$, the rank of H_1 of the infinite cyclic cover of X corresponding to $\psi_{(0,m,n)}$ is non-zero (see, for example, [Ha, proposition 2.2]). It follows that $\bar{\delta}_\mathbb{Z}(\psi_{(0,m,n)}) = 0$. However, one can compute the Alexander polynomial

of X (from the presentation of G) to be $\Delta_X = (x - 1)(y - 1)(z - 1)$. Therefore, $\bar{\delta}_{\mathbb{Z}^3}(\psi_{(0,m,n)}) = |m| + |n| > 0$.

Now we consider the case when G is the fundamental group of a closed 3-manifold. In this case, the deficiency of G is 0 so Theorem 2.2 does not suffice to prove a monotonicity result for G . The proof that the degrees satisfy a monotonicity relation will use Theorem 2.2 for 2-complexes but will also use some additional topology of the 3-manifold. Before stating the corresponding theorem for closed 3-manifolds, we introduce an important lemma which will be used in the proof of Theorem 2.9.

LEMMA 2.8. *Let K be a nullhomologous knot in a 3-manifold X , M_K be the 0-surgery on K , $\psi : \pi_1(M_K) \rightarrow \mathbb{Z}$ be a homomorphism that maps the meridian of K to a nonzero element of \mathbb{Z} , and (ϕ_r, ψ) be an admissible pair for $\pi_1(M_K)$. If $r_\Gamma(M_K) = 0$ and (ϕ_r, ψ) is not initial then the longitude of K is not 0 in $H_1(X \setminus K; \mathbb{K}_\Gamma[t^{\pm 1}])$.*

Proof. Let $l \subset N(K)$ be the longitude of K . Here, $N(K)$ is an open neighbourhood of K in X . Note that $M_K = (X \setminus N(K)) \cup e^2 \cup e^3$ where the attaching circle of e^2 is l . Since $X \setminus N(K)$ is homotopy equivalent to $X \setminus K$ we use the latter. Consider the diagram below.

$$\begin{array}{c}
 \mathbb{K}_\Gamma[t^{\pm 1}] \\
 \downarrow \partial_3 \\
 H_2(X \setminus K \cup e^2; \mathbb{K}_\Gamma[t^{\pm 1}]) \xrightarrow{\pi} \mathbb{K}_\Gamma[t^{\pm 1}] \xrightarrow{\partial_3} H_1(X \setminus K; \mathbb{K}_\Gamma[t^{\pm 1}]) \\
 \downarrow i_* \\
 H_2(M_K; \mathbb{K}_\Gamma[t^{\pm 1}]).
 \end{array}$$

The horizontal (respectively vertical) sequence is the long exact sequence of the pair $(X \setminus K \cup e^2, X \setminus K)$ (respectively $(M_K, X \setminus K \cup e^2)$) and the $\mathbb{K}_\Gamma[t^{\pm 1}]$ term in the sequence is generated by the relative class coming from e^2 (respectively e^3). We note that the boundary of the class represented by e^2 is the class represented by the longitude of K in $H_1(X \setminus K; \mathbb{K}_\Gamma[t^{\pm 1}])$. By analyzing the attaching map of ∂e^3 , we see that $\pi \circ \partial_3$ is the map which sends 1 to $t^r - 1$ where t^r is the image of the meridian of K under ϕ . Since $r \neq 0$ we see that this map is never surjective since $t^r - 1$ is not a unit in $\mathbb{K}_\Gamma[t^{\pm 1}]$.

Since $r_\Gamma(M_K) = 0$, by remark 2.8 of [COT] we have $H_2(M_K; \mathbb{K}_\Gamma[t^{\pm 1}]) \cong H^1(M_K; \mathbb{K}_\Gamma[t^{\pm 1}]) \cong \text{Ext}_{\mathbb{K}_\Gamma[t^{\pm 1}]}^1(H_0(M_K; \mathbb{K}_\Gamma[t^{\pm 1}]), \mathbb{K}_\Gamma[t^{\pm 1}])$. By the proof of proposition 2.9 in [COT], $H_0(M_K; \mathbb{K}_\Gamma[t^{\pm 1}]) = \mathbb{K}_\Gamma[t^{\pm 1}] / (\mathbb{K}_\Gamma[t^{\pm 1}] \cdot I)$ where I is the augmentation ideal of $\mathbb{Z}\pi_1(M_K)$ acting via $\mathbb{Z}\pi_1(M_K) \rightarrow \mathbb{Z}\Gamma \rightarrow \mathbb{K}_\Gamma[t^{\pm 1}]$. Thus, $H_0(M_K; \mathbb{K}_\Gamma[t^{\pm 1}]) \neq 0$ if and only if (ϕ_r, ψ) is initial. Thus, if (ϕ_r, ψ) is not initial, ∂_3 is surjective. Suppose $[l] = 0$ in $H_1(X \setminus K; \mathbb{K}_\Gamma[t^{\pm 1}])$, then π would be surjective, making $\pi \circ \partial_3$ surjective which is a contradiction.

Consider the situation when $X = S^3 \setminus K$, $G = \pi_1(S^3 \setminus K)$, ψ is the abelianization map of G , and $\phi_r : G \twoheadrightarrow \Gamma = G/G^{(2)}$ be the quotient map where $G^{(n)}$ is the n th term

of the derived series of G . It is known that Γ is a PTFA group [Str]. Let l be the longitude of K . Since $l \in G^{(2)}$, ϕ_Γ extends to a map $\pi_1(M_K) \rightarrow G/G^{(2)}$. We note that in this case, the pair (ϕ_Γ, ψ) is initial if and only if the Alexander polynomial is 1. The longitude being nonzero in $H_1(S^3 \setminus K; \mathbb{K}_\Gamma[t^{\pm 1}])$ implies that l is nonzero in $H_1(S^3 \setminus K; \mathbb{Z}\Gamma) = G^{(2)}/G^{(3)}$. Hence, if the Alexander polynomial of K is not 1 then $l \notin G^{(3)}$. This was first proved by Cochran in proposition 12·5 of [Co].

We now state our main monotonicity theorem for closed 3-manifolds.

THEOREM 2·9. *Let G be the fundamental group of a closed, orientable, connected 3-manifold and $(\phi_\Lambda, \phi_\Gamma, \psi)$ be an admissible triple for G . If $(\phi_\Lambda, \phi_\Gamma, \psi)$ is not initial then*

$$\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi) \tag{5}$$

otherwise

$$\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi) - 2. \tag{6}$$

As we saw for finitely presented groups with deficiency 1 (Corollary 2·3), for groups of closed 3-manifolds, when $n \geq 1$, the $\bar{\delta}_n$ are a nondecreasing function of n .

COROLLARY 2·10. *Let G be the fundamental group of a closed, orientable, connected 3-manifold. If $\beta_1(G) \geq 2$ then*

$$\bar{\delta}_0 \leq \bar{\delta}_1 \leq \dots \leq \bar{\delta}_n \leq \dots$$

If $\beta_1(G) = 1$ and ψ is a generator of $H^1(G; \mathbb{Z})$ then $\bar{\delta}_0(\psi) - 2 \leq \bar{\delta}_1(\psi) \leq \dots \leq \bar{\delta}_n(\psi) \leq \dots$.

Proof of Theorem 2·9. Let X be a closed, orientable, connected 3-manifold with $G = \pi_1(X)$. We will need the following lemma which is an extension of a lemma of Lescop [L].

LEMMA 2·11. *Let X be a closed, connected, orientable 3-manifold and $\psi: \pi_1(X) \rightarrow \mathbb{Z} = \langle t \rangle$ be a surjective map. X can be presented as surgery on an framed link, $L = \amalg_{i=1}^{\beta_1(X)} L_i$, in a rational homology sphere R such that:*

- (i) *the components of L are null-homologous in R ;*
- (ii) *the surgery coefficients on L_i are all 0;*
- (iii) *$\text{lk}(L_i, L_j) = 0$ for $i \neq j$ and;*
- (iv) *$\psi(\mu_i) = t^{\delta_{ij}}$ when μ_i is a meridian of L_i and δ_{ij} is the Kronecker delta.*

Proof. By lemma 5·1·1 in [L], X can be obtained by surgery on a framed link L with $\beta_1(X)$ components such that (ii), (iii), and (i) are satisfied. Now we note that any automorphism of $H_1(X)/\{\mathbb{Z}\text{-torsion}\} \cong \mathbb{Z}^{\beta_1(X)}$ corresponds to a sequence of handleslides and reordering or reorienting of the components of L . Moreover, since ψ is a surjective map to \mathbb{Z} , there exists an automorphism of $H_1(X)/\{\mathbb{Z}\text{-torsion}\}$ that sends the first basis element to t (a generator of \mathbb{Z}) and the other basis elements to $t^0 = 1$. That is, we can do a sequence of handleslides (along with possible reorienting or reordering) to get a new link L' for which the meridian of the first component maps to t and the other meridians map to 1. Since the original surgery coefficients and linking numbers of L were 0, the same is true for L' . We also note that the components of L' are null-homologous in R .

By Lemma 2·11 above, X can be presented as surgery on a framed link $L = \amalg_{i=1}^{\beta_1(X)} L_i$, in a rational homology sphere R such that the first component, L_1 , has surgery coefficient 0, $\text{lk}(L_1, L_i) = 0$ for $i \neq 1$ and $\psi(\mu_1) = t$ when μ_1 is a meridian of L_1 . Let l

be the longitude of L_1 and Y' be the space obtained by performing 0-surgery in R on the components L_2, \dots, L_k . Let $Y = Y' - N(L_1)$ where $N(L_1)$ is an open neighborhood of L_1 in Y' . Finally, $X' = Y \cup_l D^2$ be the space obtained by adding a 2-disk to Y which identifies ∂D^2 with l .

After picking a basepoint in Y (hence in X and X'), we note that the inclusion map induces an isomorphism $i_*: \pi_1(X') \xrightarrow{\cong} \pi_1(X)$. Thus any coefficient system ϕ_Γ for X induces a coefficient system for X' . Moreover, if M is any $\mathbb{Z}\Gamma$ -module then $H_1(X; M) \cong H_1(X'; M)$. In particular, $r_r(X) = r_r(X')$ and $\bar{\delta}_\Gamma^X(\psi) = \bar{\delta}_\Gamma^{X'}(\psi)$ for all $\psi \in H^1(X) \cong H^1(X')$. Since l is null-homologous in Y , we can identify $H^1(X')$ and $H^1(Y)$. We define the coefficient systems and admissible pairs for $\pi_1(Y)$ by pre-composing the coefficient systems and admissible pairs for $\pi_1(X)$ with $\pi_1(Y) \rightarrow \pi_1(X)$ induced by the inclusion $Y \subset X$.

We pick the splitting $s: \mathbb{Z} \rightarrow \Gamma$ which sends t to $\phi_\Gamma(\mu_1)$. Now we consider the long exact sequence of the pair (X', Y) :

$$\longrightarrow H_2(X', Y; \mathbb{K}_\Gamma[t^{\pm 1}]) \xrightarrow{\partial_2} H_1(Y; \mathbb{K}_\Gamma[t^{\pm 1}]) \longrightarrow H_1(X'; \mathbb{K}_\Gamma[t^{\pm 1}]) \longrightarrow 0. \tag{7}$$

As a $\mathbb{K}_\Gamma[t^{\pm 1}]$ -module $H_2(X', Y; \mathbb{K}_\Gamma[t^{\pm 1}]) \cong \mathbb{K}_\Gamma[t^{\pm 1}]$ generated by the relative 2-cell α . Hence as a \mathbb{K}_Γ -module, $H_2(X', Y; \mathbb{K}_\Gamma[t^{\pm 1}])$ is an infinitely generated free module, generated by αt^k for $k \in \mathbb{Z}$. Since the 2-cell is attached along l , we have $\partial\alpha = [l]$. We note that l and μ_1 live on $\partial N(L_1)$, hence $[l, \mu_1] = 1 \in \pi_1(Y)$. Thus, $[l](t - 1) = 0$ in $H_1(Y; \mathbb{K}_\Gamma[t^{\pm 1}])$. Equivalently, $[l] = [l]t^k$ for all k hence the image of ∂_2 as a \mathbb{K}_Γ -module has at most one dimension and is generated by $[l]$.

Using the same argument as in the first paragraph of Theorem 2.2, we can assume that $r_r(X) = 0$ and $\text{rank}_\Gamma H_2(X; \mathcal{K}_\Gamma) = 0$. Since $[l]$ is $t - 1$ torsion, the ∂_2 map in the long exact sequence of the pair (X', Y) with coefficients in \mathcal{K}_Γ is 0. Since $r_r(X') = r_r(X) = 0$, we see that $r_r(Y) = 0$. By the theorem in [Ha2], $r_\Lambda(Y) = 0$. Thus, $H_1(Y; \mathbb{K}_\Gamma[t^{\pm 1}])$ and $H_1(X'; \mathbb{K}_\Gamma[t^{\pm 1}])$ are finitely generated right \mathbb{K}_Γ -modules of dimensions $\bar{\delta}_\Gamma^Y(\psi)$ and $\bar{\delta}_\Gamma^{X'}(\psi) = \bar{\delta}_\Gamma^X(\psi)$ respectively.

Since $r_r(X) = 0$, if (ϕ_Γ, ψ) is not initial then $[l] \neq 0$ in $H_1(Y; \mathbb{K}_\Gamma[t^{\pm 1}])$ by Lemma 2.8. Also, we note that if (ϕ_Γ, ψ) is initial, then $\Gamma = \mathbb{Z}$ so all of the meridians except ψ_1 lift to the Γ -cover. Moreover, since l is nullhomologous in Y , it bounds a surface F in Y . F will lift to the Γ -cover which implies that $l = 0$ in $H_1(Y; \mathbb{K}_\Gamma[t^{\pm 1}])$. Thus, (ϕ_Γ, ψ) is initial if and only if $[l] = 0$ in $H_1(Y; \mathbb{K}_\Gamma[t^{\pm 1}])$. Recall that (ϕ_Λ, ψ) is never initial.

Suppose $(\phi_\Lambda, \phi_\Gamma, \psi)$ is not initial. Then (ϕ_Γ, ψ) is not initial hence $\bar{\delta}_\Gamma^Y(\psi) = \bar{\delta}_\Gamma^{X'}(\psi) + 1$ (similarly for Γ). Since Y is homotopy equivalent to a 2-complex with $\chi(Y) = 0$, $\bar{\delta}_\Lambda^X(\psi) \geq \bar{\delta}_\Gamma^Y(\psi)$ by Theorem 2.2. Therefore

$$\bar{\delta}_\Lambda^X(\psi) = \bar{\delta}_\Lambda^{X'}(\psi) = \bar{\delta}_\Lambda^Y(\psi) - 1 \geq \bar{\delta}_\Gamma^Y(\psi) - 1 = \bar{\delta}_\Gamma^{X'}(\psi) = \bar{\delta}_\Gamma^X(\psi).$$

Now suppose $(\phi_\Lambda, \phi_\Gamma, \psi)$ is initial. Then (ϕ_Γ, ψ) is initial so $\bar{\delta}_\Gamma^Y(\psi) = \bar{\delta}_\Gamma^{X'}(\psi)$ but $\bar{\delta}_\Lambda^X(\psi) = \bar{\delta}_\Lambda^{X'}(\psi) + 1$. Since Y is homotopy equivalent to a 2-complex with $\chi(Y) = 0$, $\bar{\delta}_\Lambda^X(\psi) \geq \bar{\delta}_\Gamma^Y(\psi) - 1$ by Theorem 2.2. Therefore

$$\bar{\delta}_\Lambda^X(\psi) = \bar{\delta}_\Lambda^{X'}(\psi) = \bar{\delta}_\Lambda^Y(\psi) - 1 \geq (\bar{\delta}_\Gamma^Y(\psi) - 1) - 1 = \bar{\delta}_\Gamma^{X'}(\psi) - 2 = \bar{\delta}_\Gamma^X(\psi) - 2.$$

We point out that there are other higher-order degrees, $\delta_n(\psi)$, for a CW-complex X defined in terms of the $\mathbb{K}_n[t^{\pm 1}]$ -torsion submodule of $H_1(X; \mathbb{K}_n[t^{\pm 1}])$ (see [Ha1]). These are equal to $\bar{\delta}_n(\psi)$ when $r_n(X) = 0$. It would be very interesting to understand

the monotonicity behavior of these $\delta_n(\psi)$. In particular, for $n \geq 1$ are the $\delta_n(\psi)$ a nondecreasing function of n ?

3. Applications

3.1. Deficiency of a group and obstructions to a group being the fundamental group of a 3-manifold

One consequence of the theorems in the previous section is that the higher-order degrees give obstructions to a finitely presented group having positive deficiency or being the fundamental group of a 3-manifold.

PROPOSITION 3.1. *Let G be a finitely presented group and $(\phi_\Lambda, \phi_\Gamma, \psi)$ be an admissible triple for G .*

- (1) *Suppose $(\phi_\Lambda, \phi_\Gamma, \psi)$ is not initial. If $\bar{\delta}_\Lambda(\psi) < \bar{\delta}_\Gamma(\psi)$ then $\text{def}(G) \leq 0$ and G cannot be the fundamental group of a compact, orientable 3-manifold (with or without boundary).*
- (2) *Suppose $(\phi_\Lambda, \phi_\Gamma, \psi)$ is initial. If $\bar{\delta}_\Lambda(\psi) < \bar{\delta}_\Gamma(\psi) - 1$ then $\text{def}(G) \leq 0$ and G cannot be the fundamental group of a compact, orientable 3-manifold with at least one boundary component which is not a 2-sphere. In addition, if $\bar{\delta}_\Lambda(\psi) < \bar{\delta}_\Gamma(\psi) - 2$ then G cannot be the fundamental group of a compact, orientable 3-manifold (with or without boundary).*

Proof. First, suppose that $\text{def}(G) \geq 1$. Then, by Theorem 2.2, $\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi)$ when $(\phi_\Lambda, \phi_\Gamma, \psi)$ is not initial and $\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi) - 1$ when $(\phi_\Lambda, \phi_\Gamma, \psi)$ is initial.

Now, suppose that G is the fundamental group of a closed, orientable, 3-manifold X . Then, by Theorem 2.9, $\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi)$ when $(\phi_\Lambda, \phi_\Gamma, \psi)$ is not initial and $\bar{\delta}_\Lambda(\psi) \geq \bar{\delta}_\Gamma(\psi) - 2$ when $(\phi_\Lambda, \phi_\Gamma, \psi)$ is initial. Finally, suppose G is the fundamental group of a connected, orientable 3-manifold with boundary. If at least 1 boundary component is not a 2-sphere then $\text{def}(G) \geq 1$ in which case the paragraph above applies. Moreover, if all the boundary components X are 2-spheres then G is the fundamental group of a closed 3-manifold.

We point out that Proposition 3.1 is sometimes very easy to use computationally since the groups Λ and Γ can be taken to be finitely generated free abelian groups. Using Proposition 3.1, one can easily prove the well-known fact that \mathbb{Z}^m cannot be the group of a compact 3-manifold when $n \geq 4$.

Example 3.2. Consider the initial triple $(\text{id}_{\mathbb{Z}^m}, \psi, \psi)$ for \mathbb{Z}^m where $\psi: \mathbb{Z}^m \rightarrow \mathbb{Z}$ is any surjective map. Since $\ker(\psi) \cong \mathbb{Z}^{m-1}$, we see that $\bar{\delta}_{\mathbb{Z}}(\psi) = m - 1$. Moreover, since $\ker(\text{id}_{\mathbb{Z}^m}) = 0$, we see that $\bar{\delta}_{\mathbb{Z}^m}(\psi) = 0$. Therefore, if $m \geq 4$, $0 = \bar{\delta}_{\mathbb{Z}^m}(\psi) < \bar{\delta}_{\mathbb{Z}}(\psi) - 2 = m - 3$. Thus, by Proposition 3.1, for $m \geq 4$, $\text{def}(\mathbb{Z}^m) \leq 0$ and \mathbb{Z}^m cannot be the fundamental group of any compact, connected, orientable 3-manifold. If \mathbb{Z}^m were the fundamental group of a non-orientable 3-manifold X then the orientable double cover of X would also have fundamental group \mathbb{Z}^m . Hence \mathbb{Z}^m cannot be the fundamental group of a compact, connected 3-manifold. Note that when $m = 3$ we see that $\bar{\delta}_{\mathbb{Z}^3}(\psi) < \bar{\delta}_{\mathbb{Z}}(\psi) - 1$ so \mathbb{Z}^3 cannot be the fundamental group of a compact 3-manifold with at least one boundary component which is not a 2-sphere.

If we consider the case when the groups Γ and Λ are quotients of G by the terms of its rational derived series we have the following immediate corollary to Proposition 3.1.

COROLLARY 3.3. *Let G be a finitely presented group.*

- (1) *Suppose $\beta_1(G) \geq 2$. If there exists a $\psi \in H^1(G; \mathbb{Z})$, and $m, n \in \mathbb{Z}$ such that $n > m \geq 0$ and $\bar{\delta}_n(\psi) < \bar{\delta}_m(\psi)$ then $\text{def}(G) \leq 0$ and G cannot be the fundamental group of a compact, orientable 3-manifold.*
- (2) *Suppose $\beta_1(G) = 1$ and ψ in a generator of $H^1(G; \mathbb{Z})$.*
 - (a) *If there exists $m, n \in \mathbb{Z}$ such that $n > m \geq 1$ and $\bar{\delta}_n(\psi) < \bar{\delta}_m(\psi)$ then $\text{def}(G) \leq 0$ and G cannot be the fundamental group of a compact, orientable 3-manifold.*
 - (b) *If there exists an $n \in \mathbb{Z}$ such that $n \geq 1$ and $\bar{\delta}_n(\psi) < \bar{\delta}_0(\psi) - 1$ then $\text{def}(G) \leq 0$ and G cannot be the fundamental group of a compact, orientable 3-manifold with at least one boundary component that is not a 2-sphere. In addition, if $\bar{\delta}_n < \bar{\delta}_0 - 2$ then G cannot be the fundamental group of a compact, orientable 3-manifold.*

Example 3.4. We saw that the examples $G_{n,g}$ in Proposition 2.5 satisfy $\bar{\delta}_1(\psi) < \bar{\delta}_0(\psi) - 1$ when $n = 1$ and $g = 1$, $\bar{\delta}_1(\psi) < \bar{\delta}_0(\psi) - 2$ when $n = 1$ and $g \geq 2$, and $\bar{\delta}_n(\psi) < \bar{\delta}_{n-1}(\psi)$ when $n \geq 2$. Thus, by Corollary 3.3, for each $n \geq 1$ and $g \geq 1$ the groups $G_{n,g}$ in Proposition 2.5 have $\text{def}(G_{n,g}) \leq 0$. Moreover, except in the case that $g = 1$ and $n = 1$, for each $n \geq 1$ and $g \geq 1$, the group $G_{n,g}$ cannot be the fundamental group of a compact, orientable 3-manifold (with or without boundary). The group $G_{1,1}$ cannot be the fundamental group of a compact, orientable 3-manifold with at least one boundary component that is not a 2-sphere.

3.2. Obstructions to $X \times S^1$ admitting a symplectic structure

We will show that a consequence of Corollaries 2.3 and 2.10 is that the $\bar{\delta}_n(\psi)$ give obstructions to a 4-manifold of the form $X \times S^1$ admitting a symplectic structure. It is well known that if X is a closed 3-manifold that fibers over S^1 then $X \times S^1$ admits a symplectic structure. Taubes asks whether the converse is true.

QUESTION 3.5 (Taubes). *Let X be a 3-manifold such that $X \times S^1$ admits a symplectic structure. Does X admit a fibration over S^1 ?*

In [Ha1], we showed that if X is a 3-manifold that fibers over S^1 with $\beta_1(X) \geq 2$ and ψ representing the fibration then $\bar{\delta}_n(\psi)$ is equal to Thurston norm $\|\psi\|_T$ of ψ . This generalized the work of McMullen who showed that the Alexander norm gives a lower bound for the Thurston norm which is an equality when ψ represents a fibration.

THEOREM 3.6 ([Ha1]). *Let X be a compact, orientable 3-manifold (possibly with boundary). For all $\psi \in H^1(X; \mathbb{Z})$ and $n \geq 0$*

$$\bar{\delta}_n(\psi) \leq \|\psi\|_T$$

except for the case when $\beta_1(X) = 1$, $n = 0$, $X \cong S^1 \times S^2$, and $X \cong S^1 \times D^2$. In this case, $\bar{\delta}_0(\psi) \leq \|\psi\|_T + 1 + \beta_3(X)$ whenever ψ is a generator of $H^1(X; \mathbb{Z}) \cong \mathbb{Z}$. Moreover, equality holds in all cases when $\psi: \pi_1(X) \rightarrow \mathbb{Z}$ can be represented by a fibration $X \rightarrow S^1$.

Using the work of Meng–Taubes and Kronheimer–Mrowka, Vidussi [Vi] has recently given a proof of McMullen’s inequality (that the Alexander norm gives a lower bound for the Thurston norm of a 3-manifold) using Seiberg–Witten theory. This generalizes the work of Kronheimer [K2] who dealt with the case that X is the

0-surgery on a knot. Moreover, Vidussi shows that if $X \times S^1$ admits a symplectic structure (and $\beta_1(X) \geq 2$) then the Alexander and Thurston norms of X coincide on a cone over a face of the Thurston norm ball of X , supporting a positive answer to question 3.5 asked by Taubes.

THEOREM 3.7 (Kronheimer, Vidussi [K2, V, Vi]). *Let X be a closed, irreducible 3-manifold such that $X \times S^1$ admits a symplectic structure. If $\beta_1(X) \geq 2$ there exists a $\psi \in H^1(X; \mathbb{Z})$ such that $\|\psi\|_A = \|\psi\|_T$. If $\beta_1(X) = 1$ then for any generator ψ of $H^1(X; \mathbb{Z})$, $\|\psi\|_A = \|\psi\|_T + 2$.*

In [Ha1, theorem 12.5], we used Vidussi’s result and our result that the $\bar{\delta}_n$ give lower bounds for the Thurston norm [Ha1, theorem 10.1] to show that the higher-order degrees of a 3-manifold X give algebraic obstructions to a 4-manifold of the form $X \times S^1$ admitting a symplectic structure. As a result, we were able to show that the closed, irreducible 3-manifolds (with $\beta_1(X) \geq 2$) in theorem 11.1 of [Ha1] have $\bar{\delta}_0 < \bar{\delta}_1 < \dots < \bar{\delta}_n$ hence cannot admit a symplectic structure. However, it was still unknown at this time whether Vidussi’s Theorem holds if one replaces the Alexander norm with $\bar{\delta}_n$. In [Ha1, conjecture 12.7], we conjectured this to be true. Since the Alexander norm is equal to $\bar{\delta}_0$, Vidussi’s theorem gives us the case when $n = 0$. We will show that conjecture 12.7 of [Ha1] is true when $n \geq 1$. This is theoretically important since it gives more evidence that the only symplectic 4-manifolds of the form $X \times S^1$ are such that X fibers over S^1 , supporting a positive answer to the question of Taubes.

THEOREM 3.8. *Let X be a closed, orientable, irreducible 3-manifold such that $X \times S^1$ admits a symplectic structure. If $\beta_1(X) \geq 2$ there exists a $\psi \in H^1(X; \mathbb{Z})$ such that*

$$\bar{\delta}_0(\psi) = \bar{\delta}_1(\psi) = \dots = \bar{\delta}_n(\psi) = \dots = \|\psi\|_T.$$

If $\beta_1(X) = 1$ then for any generator ψ of $H^1(X; \mathbb{Z})$,

$$\bar{\delta}_0(\psi) - 2 = \bar{\delta}_1(\psi) = \dots = \bar{\delta}_n(\psi) = \dots = \|\psi\|_T.$$

Proof. If X is a closed, orientable, irreducible, 3-manifold with $\beta_1(X) \geq 2$ such that $X \times S^1$ admits a symplectic structure then by Theorem 3.7 there exists a $\psi \in H^1(X; \mathbb{Z})$ such that $\bar{\delta}_0(\psi) = \|\psi\|_A = \|\psi\|_T$. By Corollary 2.10 and Theorem 3.6, $\bar{\delta}_0(\psi) \leq \bar{\delta}_n(\psi) \leq \|\psi\|_T$ hence for all $n \geq 0$, $\bar{\delta}_0(\psi) = \bar{\delta}_n(\psi) = \|\psi\|_T$. Similarly, if $\beta_1(X) = 1$ then for ψ a generator of $H^1(X; \mathbb{Z})$, $\bar{\delta}_0(\psi) - 2 = \|\psi\|_T$. Since $S^1 \times S^2$ is not irreducible, for $n \geq 1$ we have $\bar{\delta}_0(\psi) - 2 \leq \bar{\delta}_n(\psi) \leq \|\psi\|_T$ hence $\bar{\delta}_0(\psi) - 2 = \bar{\delta}_n(\psi) = \|\psi\|_T$.

3.3. Behavior of the Thurston norm under a continuous map that is surjective on π_1

An important problem in 3-manifold topology is determine the behavior of the Thurston norm under continuous maps $f: X \rightarrow Y$ between 3-manifolds. It was shown by Gabai in [Ga] that if f is a p -fold covering map then $\|f^*(\psi)\|_T = p \|\psi\|_T$. Moreover, Gabai showed that if f is a degree d map then $\|f^*(\psi)\|_T \geq |d| \|\psi\|_T$. These statements were first conjectured by Thurston his original paper on the Thurston norm in [Th, conjecture 2(b)]. We sketch a proof of the latter, since it does not seem to explicitly appear in [Ga].

THEOREM 3·9 (Gabai). *Let $f: X \rightarrow Y$ be a degree d map between closed, orientable, 3-manifolds. Then for each $\psi \in H^1(Y; \mathbb{Z})$, $\|f^*(\psi)\|_T \geq |d| \|\psi\|_T$.*

Proof. Let $\psi \in H^1(Y; \mathbb{Z})$ and F be an embedded (possibly disconnected) surface in X such that $[F]$ is dual to $f^*(\psi)$ and $\chi_-(F) = \|f^*(\psi)\|_T$. Since the following diagram commutes [Mu, theorem 67·2], $[f(F)] = f_*([F]) = d(\psi \cap \Gamma_Y)$

$$\begin{array}{ccc}
 H^1(X; \mathbb{Z}) & \xrightarrow{\cap \Gamma_X} & H_2(X; \mathbb{Z}) \\
 \uparrow f^* & \cong & \downarrow f_* \\
 H^1(Y; \mathbb{Z}) & \xrightarrow{\cap d\Gamma_Y} & H_2(Y; \mathbb{Z}).
 \end{array}$$

By corollary 6·18 of [Ga], $\| - \|_T = x_s(-)$ where x_s is the singular norm. Hence $|d| \|\psi\|_T \leq \chi_-(f(F)) \leq \chi_-(F) = \|f^*(\psi)\|_T$.

Recall that a degree one map is surjective on π_1 . Hence one could ask if the existence of a map $f: X \rightarrow Y$ between compact, orientable, 3-manifolds, that is surjective on π_1 suffices to guarantee that $\|f^*(\psi)\|_T \geq \|\psi\|_T$ for all $\psi \in H^1(Y; \mathbb{Z})$. We will give some (algebraic) conditions on X and Y (i.e. that do not depend on the map f) that will guarantee $\|f^*(\psi)\|_T \geq \|\psi\|_T$.

This question was first asked by Simon (see Kirby’s Problem List [Ki, question 1·12(b)]) for knot complements. Recall that if K is a nontrivial knot in S^3 then $H^1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$ generated by ψ and $\|\psi\|_T = 2g(K) - 1$ where $g(K)$ is the genus of K .

Question 1·12(b) of [Ki]. If J and K are knots in S^3 and $f: S^3 \setminus L \rightarrow S^3 \setminus K$ is surjective on π_1 , is $g(L) \geq g(K)$?

The answer to the above question is known to be yes when $\bar{\delta}_0(K) = 2g(K)$. We strengthen this result to the case when $\bar{\delta}_n(K) = 2g(K) - 1$ in Corollary 3·13. By $\bar{\delta}_n(K)$ we mean $\bar{\delta}_n(\psi)$ for a generator ψ of $H^1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$. Note that by Theorems 5·4 and 7·1 of [Co],

$$\bar{\delta}_0(K) - 1 \leq \bar{\delta}_1(K) \leq \dots \leq \bar{\delta}_n(K) \leq \dots \leq 2g(K) - 1.$$

Moreover, by corollary 7·4 of [Co], there exist knots K for which $\bar{\delta}_0(K) - 1 < \bar{\delta}_1(K) < \dots < \bar{\delta}_n(K)$. Therefore, the result in Corollary 3·13 is strict generalization of the previously known result.

Before we state the results concerning the behavior of the Thurston norm under a surjective map on π_1 , we state and prove the following theorem which describes the behavior of $\bar{\delta}_n$ under a surjective map on π_1 . We only consider the case that $\text{def}(G) = 1$ since if $\text{def}(G) \geq 2$ then by Remark 2·4, $r_0(G) \geq 1$.

THEOREM 3·10. *Let G be either (1) a finitely presented group with $\text{def}(G) = 1$ or (2) the fundamental group of closed, connected, orientable 3-manifold. If P is a group with $\beta_1(P) = \beta_1(G)$, $r_0(G) = 0$, and $\rho: G \twoheadrightarrow P$ is a surjective map then for each $n \geq 0$ and $\psi \in H^1(P; \mathbb{Z})$,*

$$\bar{\delta}_n(\rho^*(\psi)) \geq \bar{\delta}_n(\psi).$$

Proof. We will first show that the theorem holds for primitive elements of $H^1(P; \mathbb{Z})$. It will then follow for arbitrary elements of $H^1(P; \mathbb{Z})$ since for any $k \in \mathbb{Z}$, $\rho^*(k\psi) = k\rho^*(\psi)$, $\bar{\delta}_n(k\rho^*(\psi)) = |k|\bar{\delta}_n(\rho^*(\psi))$ and $\bar{\delta}_n(k\psi) = |k|\bar{\delta}_n(\psi)$. Let ψ be a primitive element of $H^1(P; \mathbb{Z})$, $G_n = G/G_r^{(n+1)}$, and $P_n = P/P_r^{(n+1)}$. For each $n \geq 0$, we have two coefficient systems for G : $\phi_n^1: G \rightarrow G_n$ and $\phi_n^2: G \rightarrow P_n$, defined by $\phi_n^1(g) = [g]$ and $\phi_n^2(g) = [\rho(g)]$. Note that ρ induces a surjection $\bar{\rho}: G_n \rightarrow P_n$. Moreover, $\bar{\rho}$ has non-trivial kernel if and only if $(\phi_n^1, \phi_n^2, \rho^*(\psi))$ is an admissible triple.

If $\bar{\rho}$ is an isomorphism, then $\bar{\delta}_{G_n}(\rho^*(\psi)) = \bar{\delta}_{P_n}(\rho^*(\psi))$. Suppose $\bar{\rho}$ is not an isomorphism. We remark that $(\phi_n^1, \phi_n^2, \rho^*(\psi))$ is initial if and only if $\beta_1(P) = 1$ and $n = 0$. However, since ρ is surjective and $\beta_1(G) = \beta_1(P)$, we have $\bar{\rho}: G_0 \xrightarrow{\cong} P_0$. Thus $(\phi_n^1, \phi_n^2, \rho^*(\psi))$ is never initial and hence, by Theorems 2.2 and 2.9, $\bar{\delta}_{G_n}(\rho^*(\psi)) \geq \bar{\delta}_{P_n}(\rho^*(\psi))$.

To finish the proof, we will show that $\bar{\delta}_{P_n}(\rho^*(\psi)) \geq \bar{\delta}_{P_n}(\psi)$. First, since ρ is surjective, we have a surjective map

$$\rho_* : H_1(G; \mathbb{Z}P_n) = \frac{\ker(\phi_n^2)}{[\ker(\phi_n^1), \ker(\phi_n^2)]} \twoheadrightarrow \frac{P_r^{(n+1)}}{[P_r^{(n+1)}, P_r^{(n+1)}]} = H_1(P; \mathbb{Z}P_n).$$

Since $\mathbb{K}_n^P[t^{\pm 1}]$ is a flat (right) $\mathbb{Z}P_n$ -module, the homomorphism $\rho_* : H_1(G; \mathbb{K}_n^P[t^{\pm 1}]) \rightarrow H_1(P; \mathbb{K}_n^P[t^{\pm 1}])$ is surjective. The condition $r_0(G) = 0$ implies that both of these modules are torsion [Ha2] hence $\text{rank}_{\mathbb{K}_n^P} H_1(G; \mathbb{K}_n^P[t^{\pm 1}]) \geq \text{rank}_{\mathbb{K}_n^P} H_1(P; \mathbb{K}_n^P[t^{\pm 1}])$ which completes the proof.

COROLLARY 3.11. *Suppose there exists an epimorphism $\rho: \pi_1(X) \rightarrow \pi_1(Y)$, where X and Y are compact, connected, orientable 3-manifolds, with toroidal or empty boundaries, such that $\beta_1(X) = \beta_1(Y)$ and $r_0(X) = 0$. Let $\psi \in H^1(\pi_1(Y); \mathbb{Z})$. If any of the following conditions is satisfied:*

- (a) $\beta_1(Y) \geq 2$ and $\bar{\delta}_n(\psi) = \|\psi\|_T$ for some $n \geq 0$;
- (b) $\beta_1(Y) = 1$ and $\bar{\delta}_n(\psi) = \|\psi\|_T$ for some $n \geq 1$;
- (c) $\beta_1(Y) = 1$, $\beta_3(X) \leq \beta_3(Y)$, ψ is primitive and $\bar{\delta}_0(\psi) = \|\psi\|_T + 1 + \beta_3(Y)$;

then

$$\|\rho^*(\psi)\|_T \geq \|\psi\|_T.$$

Proof. Let $G = \pi_1(X)$ and $P = \pi_1(Y)$. If X is $S^1 \times D^2$ or $S^1 \times S^2$ then $\pi_1(X) \cong \mathbb{Z}$ and $\pi_1(Y) \cong \mathbb{Z}$ hence $\bar{\delta}_n(\psi) = 0$ for all n . Thus, we are in case (b) and have $\|\psi\|_T = 0$ which trivially satisfies the conclusion of the corollary. Therefore, we can assume that X is neither $S^1 \times D^2$ nor $S^1 \times S^2$. We also remark that since $r_0(X) = 0$, $\text{def}(\pi_1(X)) \leq 1$ by Remark 2.4. Thus, if (a) or (b) is satisfied then by theorem 10.1 of [Ha1], $\bar{\delta}_n(\rho^*(\psi)) \leq \|\rho^*(\psi)\|_T$. Hence by Theorem 3.10 we have $\|\rho^*(\psi)\|_T \geq \bar{\delta}_n(\rho^*(\psi)) \geq \bar{\delta}_n(\psi) = \|\psi\|_T$. If c is satisfied then by theorem 10.1 of [Ha1] we have $\bar{\delta}_0(\rho^*(\psi)) \leq \|\rho^*(\psi)\|_T + 1 + \beta_3(X)$. Therefore, $\|\rho^*(\psi)\|_T \geq \bar{\delta}_0(\rho^*(\psi)) - 1 - \beta_3(X) \geq \bar{\delta}_0(\psi) - 1 - \beta_3(Y) = \|\psi\|_T$.

We will now discuss the case when G is the fundamental group of a knot complement.

COROLLARY 3.12. *If J and K are knots in S^3 such that there exists a surjective homomorphism $\rho: \pi_1(S^3 \setminus L) \rightarrow \pi_1(S^3 \setminus K)$ then for each $n \geq 0$, $\bar{\delta}_n(L) \geq \bar{\delta}_n(K)$.*

Proof. Let $G = \pi_1(S^3 \setminus L)$, $P = \pi_1(S^3 \setminus K)$, $\psi_P: P \rightarrow P/P^{(1)} \cong \mathbb{Z}$ be the abelianization map, and $\psi_G = \psi_P \circ \rho$. Since ρ is surjective and $\beta_1(S^3 - L) = 1$, ψ_G is a generator

of $H^1(S^3 - L; \mathbb{Z})$. By [COT, proposition 2.11], $r_0(G) = 0$ hence by Theorem 3.10, $\bar{\delta}_n(L) = \bar{\delta}_n(\psi_G) \geq \bar{\delta}_n(\psi_P) = \bar{\delta}_n(K)$.

COROLLARY 3.13. *Suppose J and K are knots in S^3 such that there exists a surjective homomorphism $\rho: \pi_1(S^3 \setminus L) \rightarrow \pi_1(S^3 \setminus K)$. If $\bar{\delta}_0(K) = 2g(K)$ or $\bar{\delta}_n(K) = 2g(K) - 1$ for some $n \geq 1$ then $g(L) \geq g(K)$.*

This corollary follows immediately from Corollary 3.11. Instead of omitting any proof, we will supply a proof which is a simplified version of the proof of Corollary 3.11.

Proof. We can assume that L is not the unknot since ϕ is surjective. If $n \geq 1$ we have $\bar{\delta}_n(L) \leq 2g(L) - 1$ by theorem 7.1 of [Co] or theorem 10.1 of [Ha1]. Hence, by Corollary 3.12, $2g(K) - 1 = \bar{\delta}_n(K) \leq \bar{\delta}_n(L) \leq 2g(L) - 1$. In the other case, we have $\bar{\delta}_0(L) \leq 2g(L)$ so $2g(K) = \bar{\delta}_0(K) \leq \bar{\delta}_0(L) \leq 2g(L)$.

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