4-Dimensional Equivalence Relations on Knots

Fall Southeastern Sectional Meeting 2011

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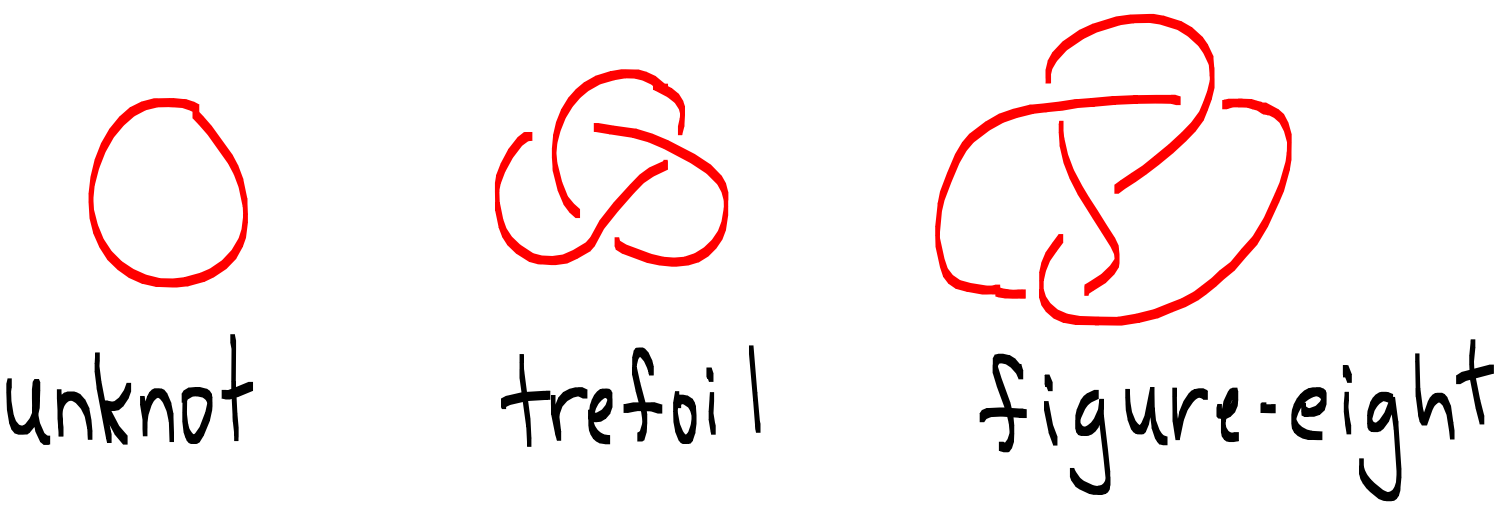
Def: A knot is a smooth embedding

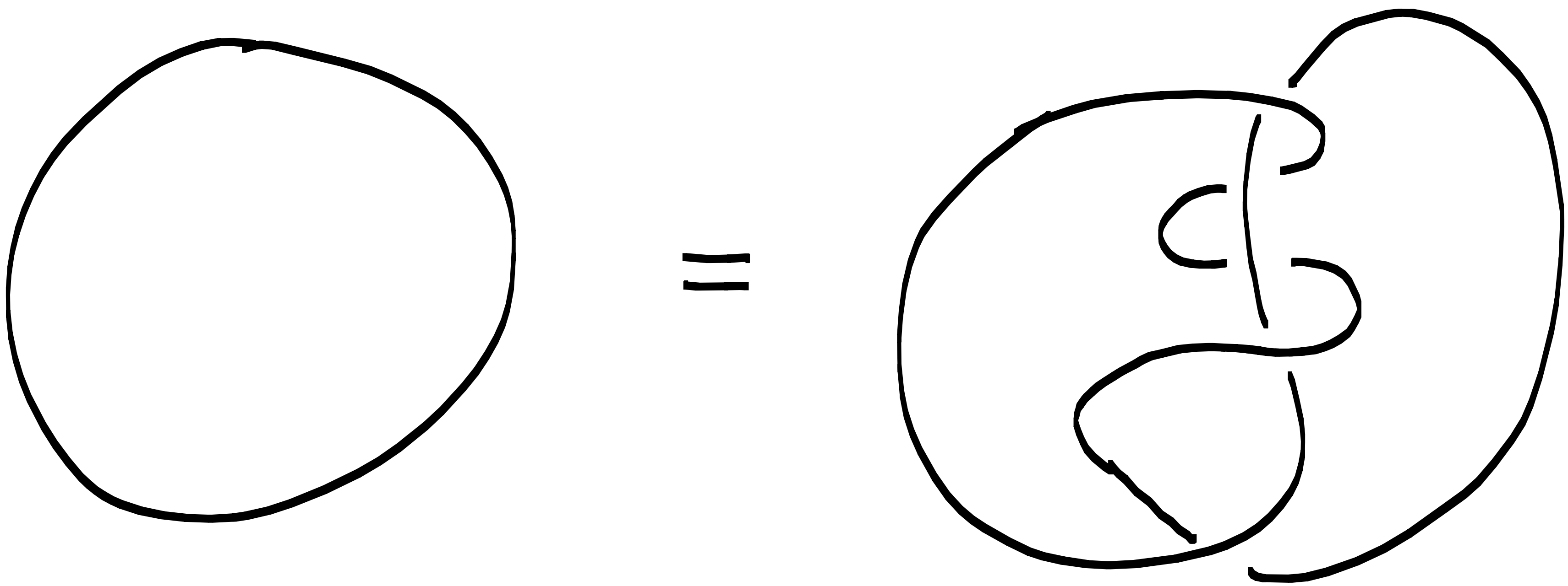
$$f : S^1 \longrightarrow \mathbb{R}^3$$

i.e. take a rope, tie it up and attach the ends



In topology, we are interested in knots up to isotopy.

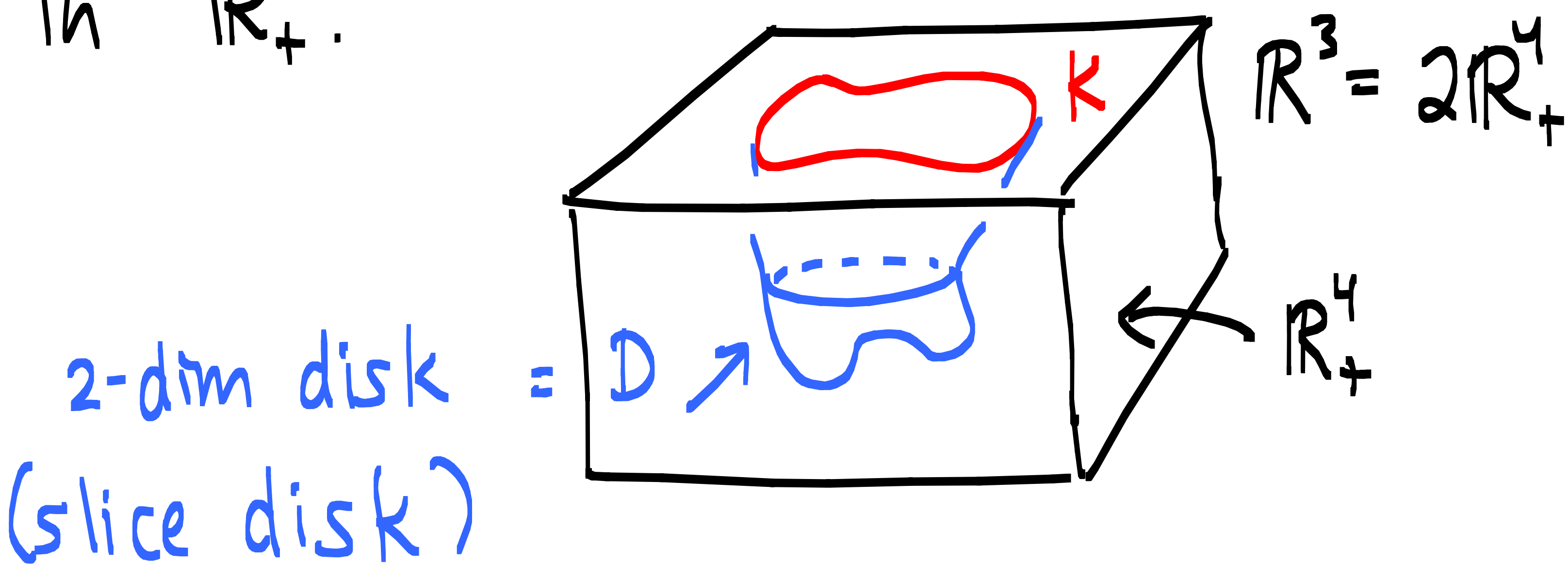
<u>Def</u>: Two knots are equivalent if one can be deformed into the other through embeddings.



__Note__: A knot a trivial (equivalent to the unknot) if and only if it bounds an embedded disk in $\mathbb{R}^3$.

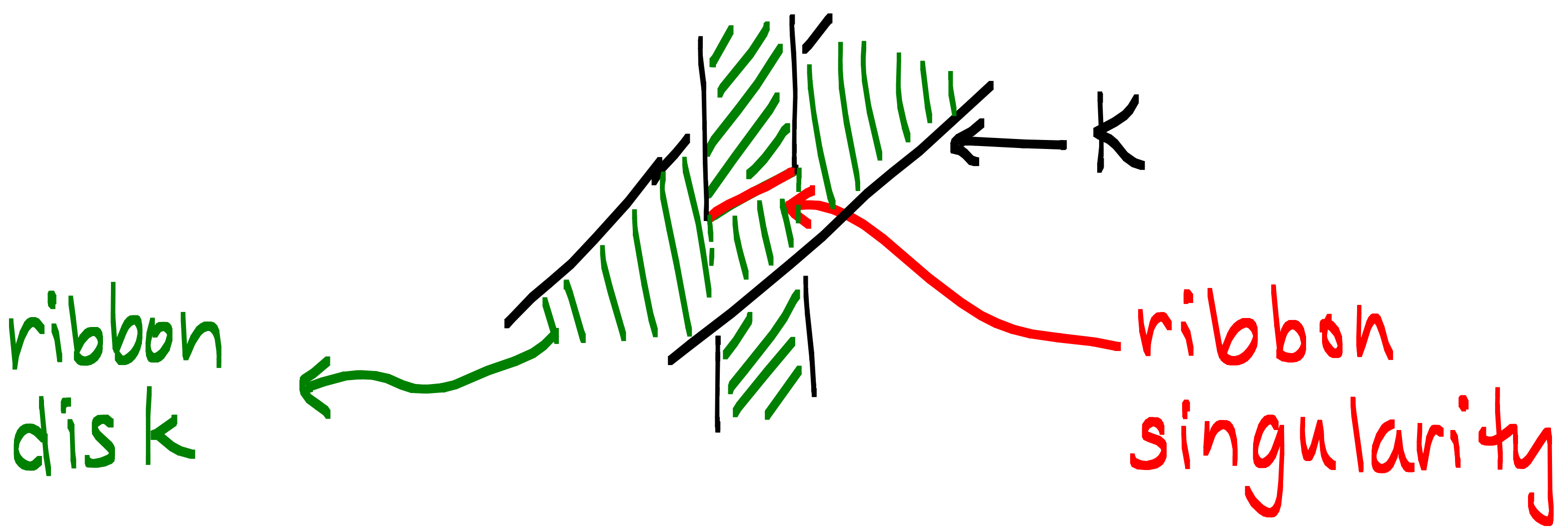
However, if we allow the disk to move into $\mathbb{R}^4$, we can get more knots bounding disks.

<u>Def</u>: A knot $K \subset \mathbb{R}^3$ is slice if the boundary of a smoothly embedded disk in $\mathbb{R}^4_+$.

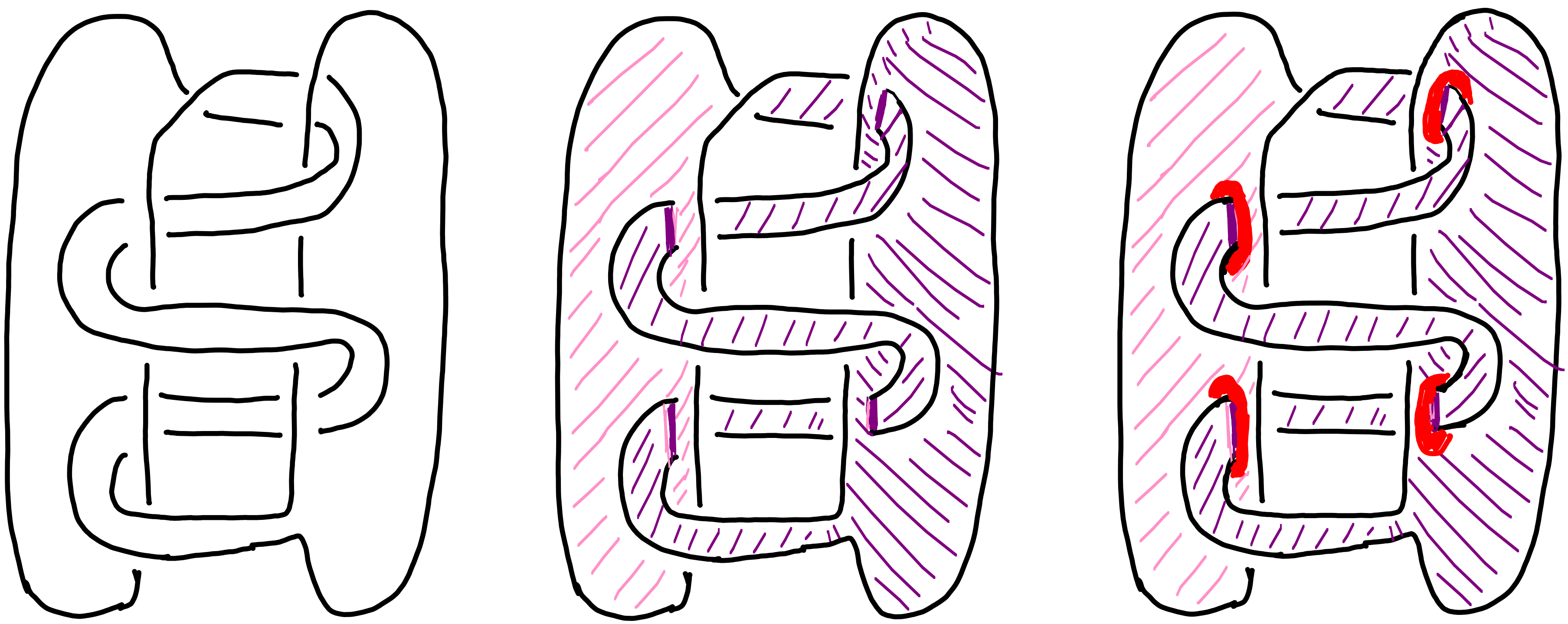


Examples of slice knots:

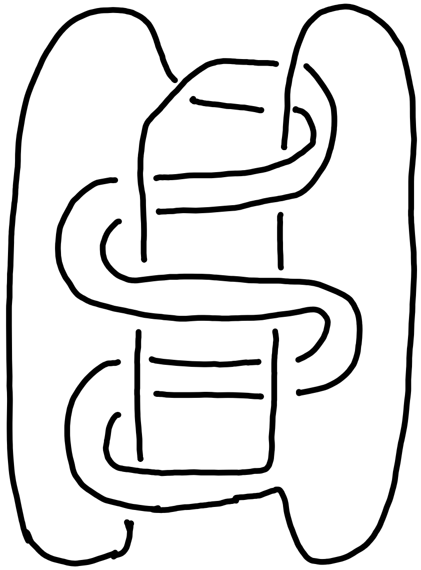
Def: K is ribbon if it bounds an immersed disk in $\mathbb{R}^3$ with only ribbon singularities.



A ribbon knot is slice

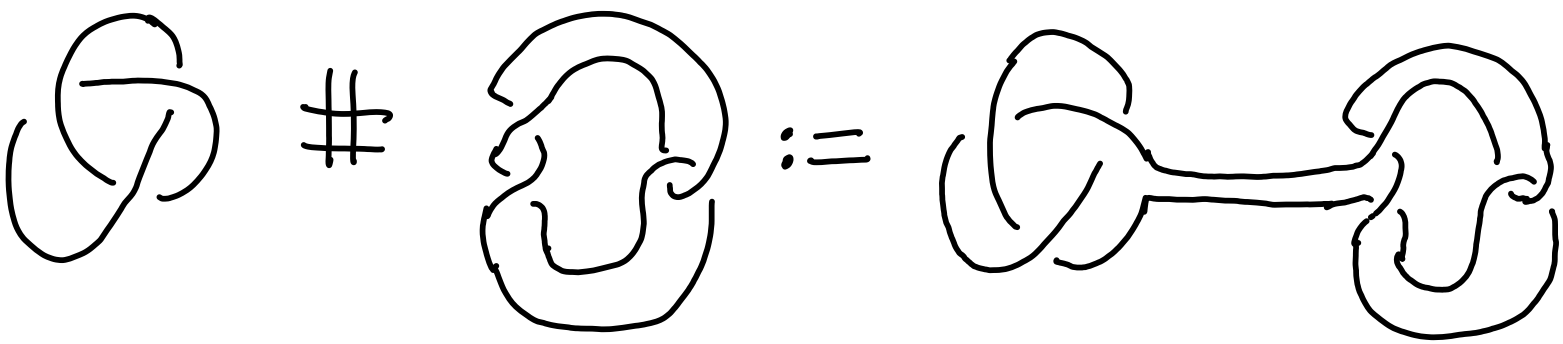


Pf: To obtain a disk <u>embedded</u> in $\mathbb{R}^4_+$, push the interior of red disks into interior of $\mathbb{R}^4_+$.

So  is slice.

<u>Conjecture</u>: A knot is (smoothly) slice $\Leftrightarrow$ it is ribbon.

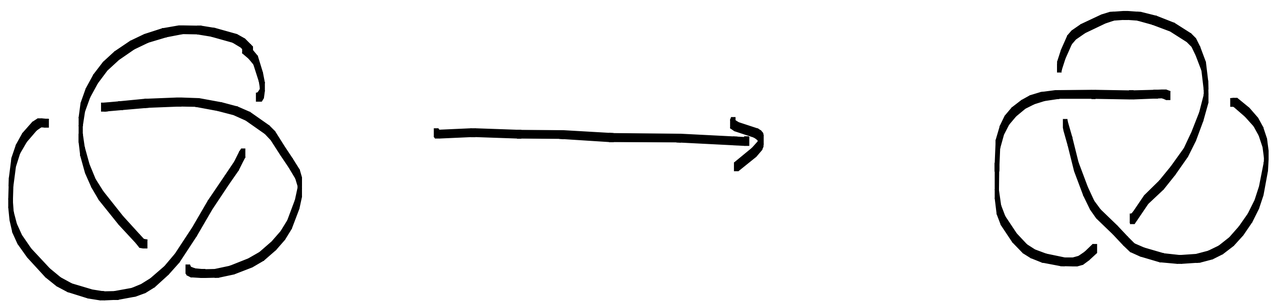
There is a binary operation on (oriented) knots:



$K_1 \quad\quad\quad\quad K_2 \quad\quad\quad\quad K_1 \# K_2$

connected sum

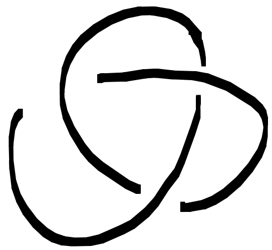
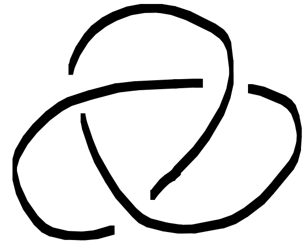
If K is a knot then we change change all crossings and get a new knot $\bar{K}$, its mirror image.



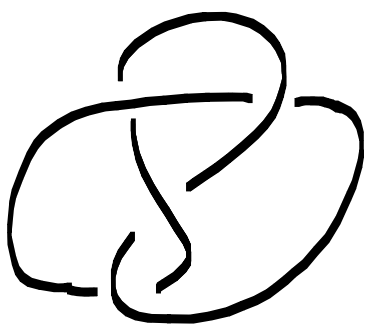
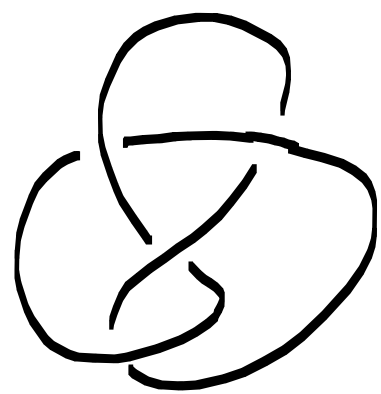
K $\qquad\qquad\qquad$ $\bar{K}$

In general, $K \neq \bar{K}$.

Ex:

 $\neq$ 

Ex:

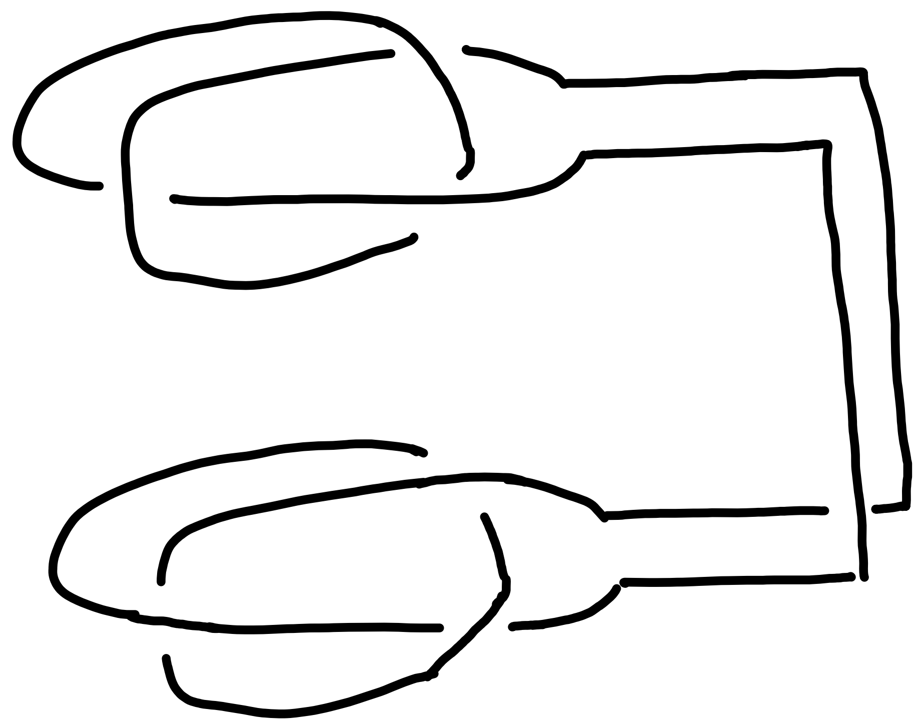
 $=$ 

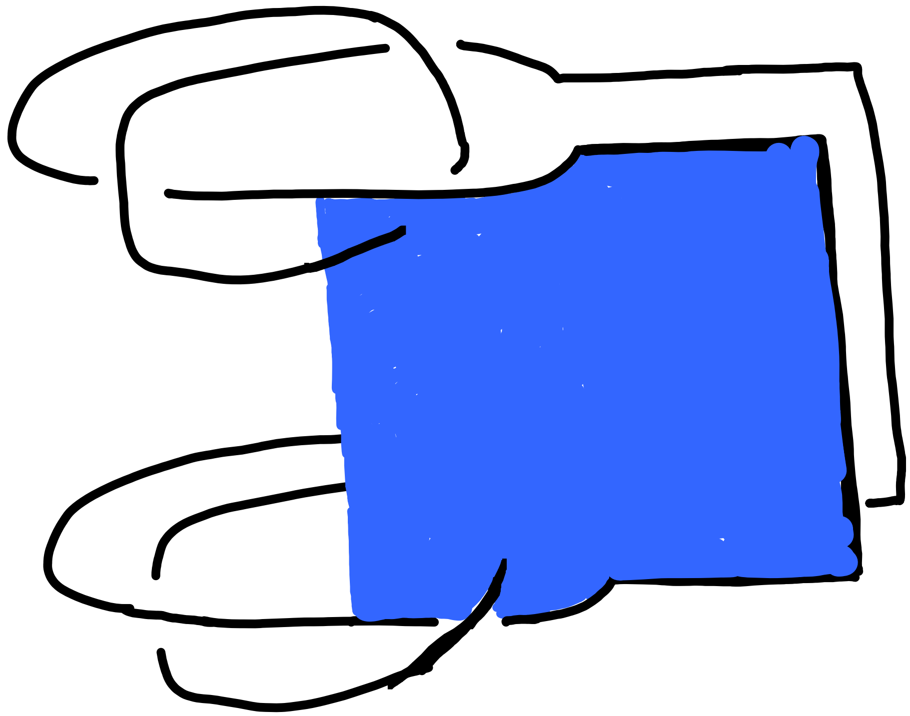
$\boxed{\text{For any knot } K,\ K \# r\bar{K} \text{ is slice}}^{*}$

* need to orient knots and $r\bar{K}$ is $\bar{K}$ with the reversed orientation.

We will show that $K \# r\bar{K}$ is ribbon.

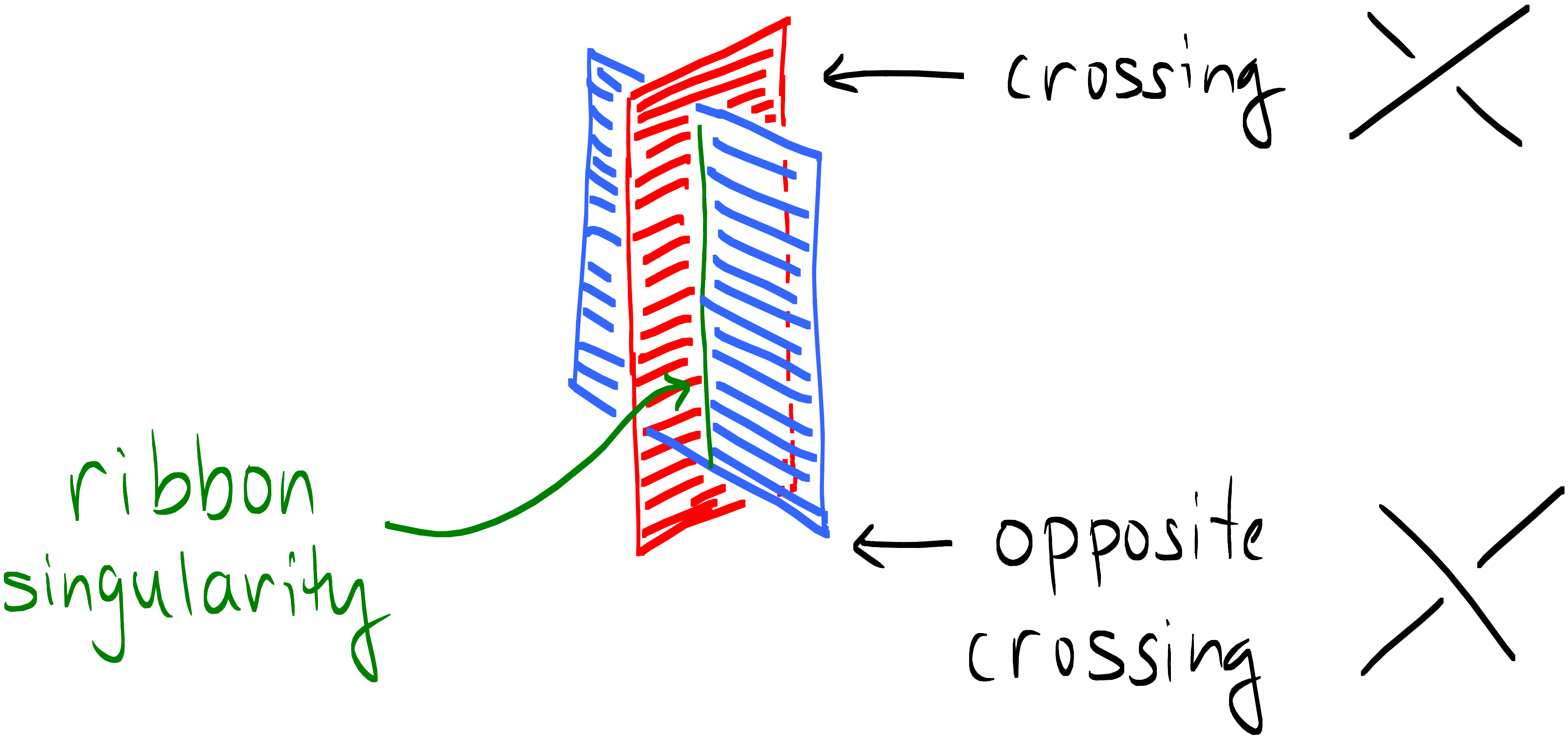
$K \# r\bar{K}$





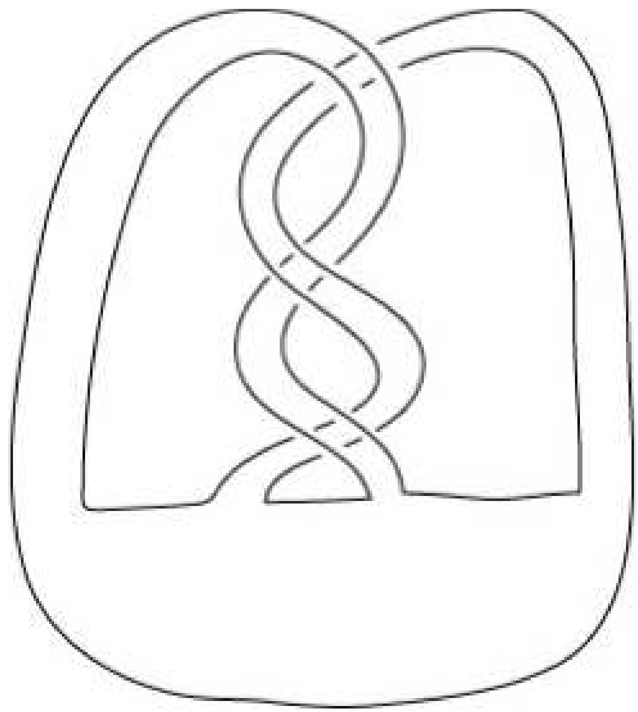
Continue this "vertical drape" around the arc.

Intersections look like



$\Rightarrow K \# r\bar{K}$ is ribbon!

<u>Another example</u> :

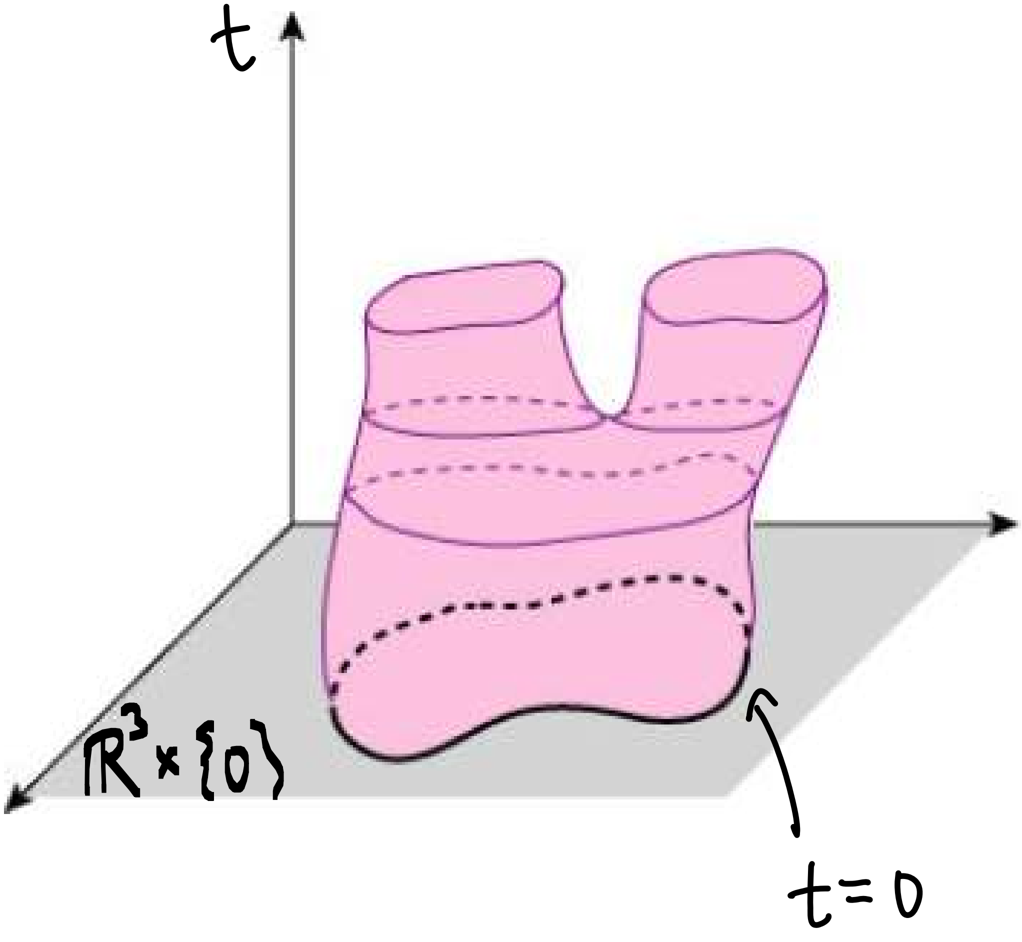


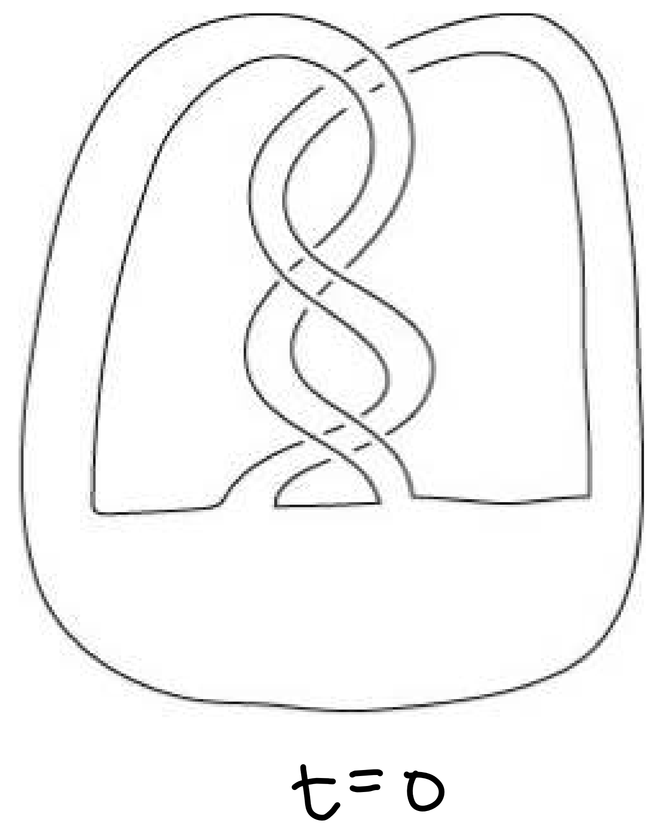
$=: 9_{46}$

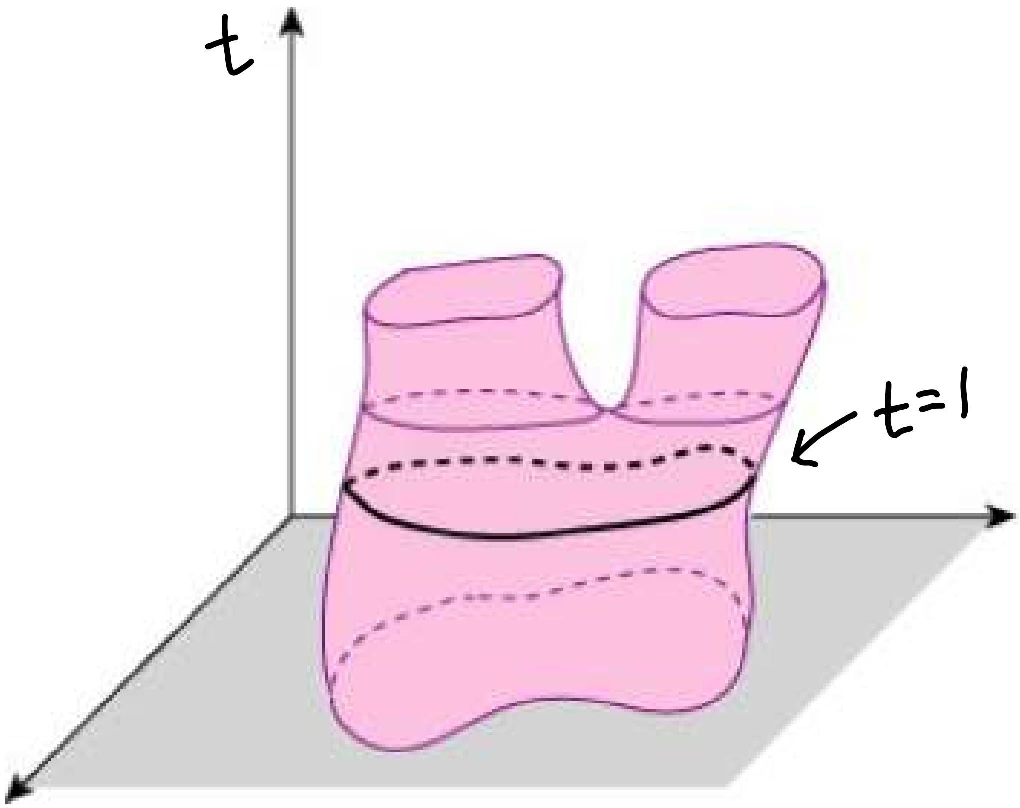
The 9_{46} knot is slice .

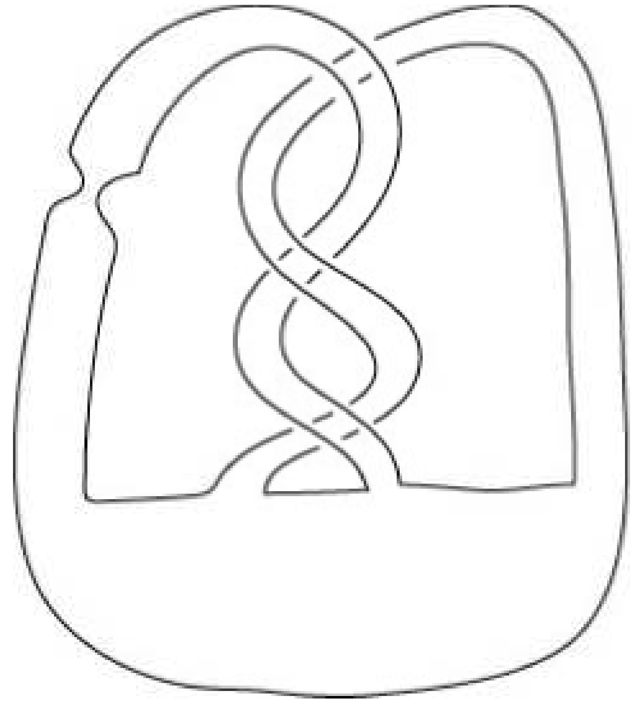
How to build a slice disk:

- look at slices of the disk in time viewing $\mathbb{R}^4_+ = \mathbb{R}^3 \times \mathbb{R}$ ← time

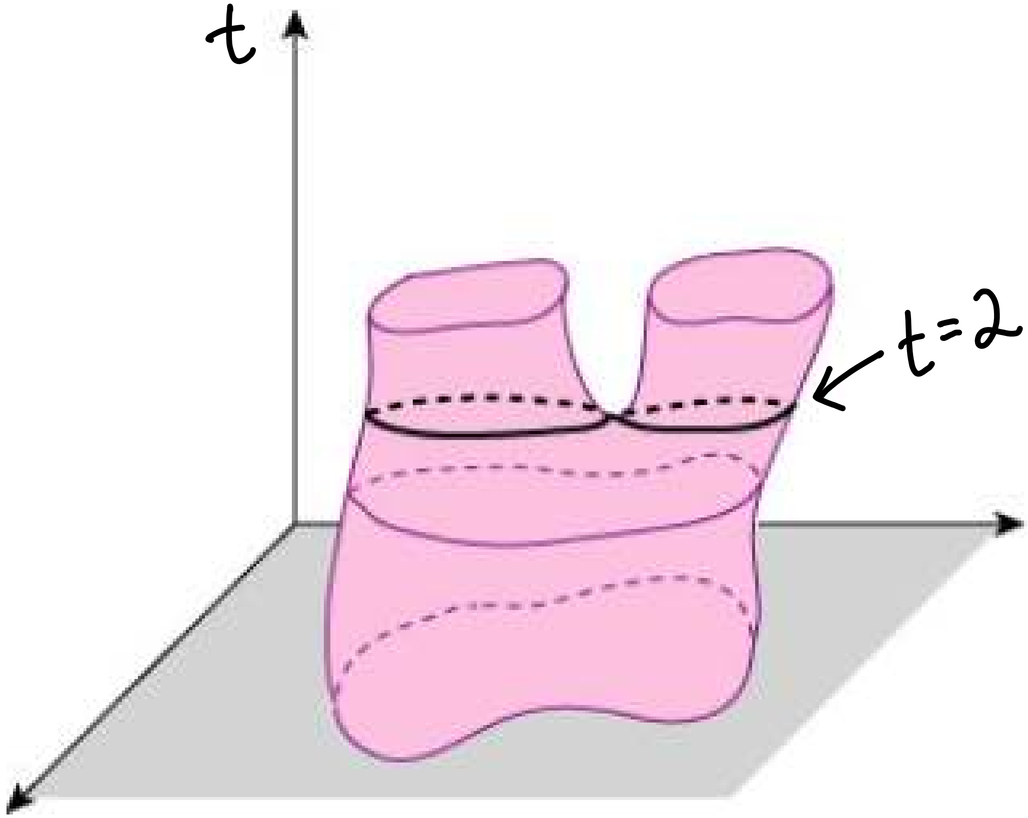


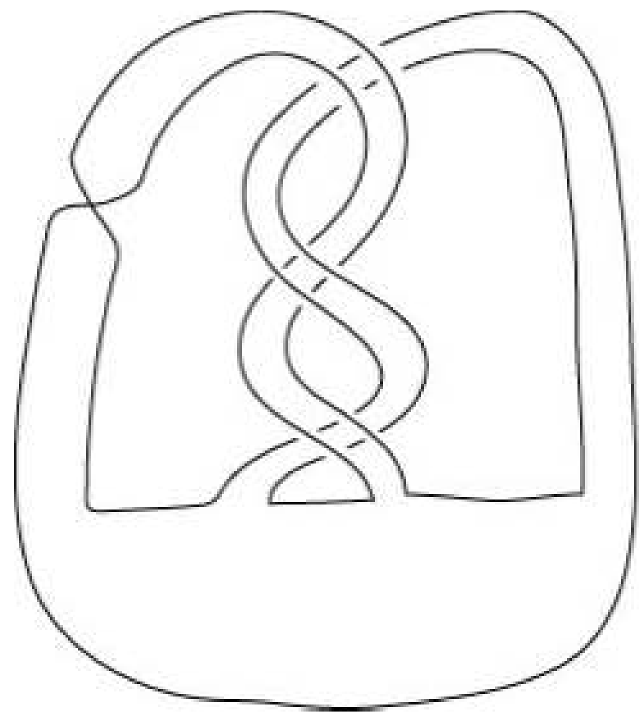




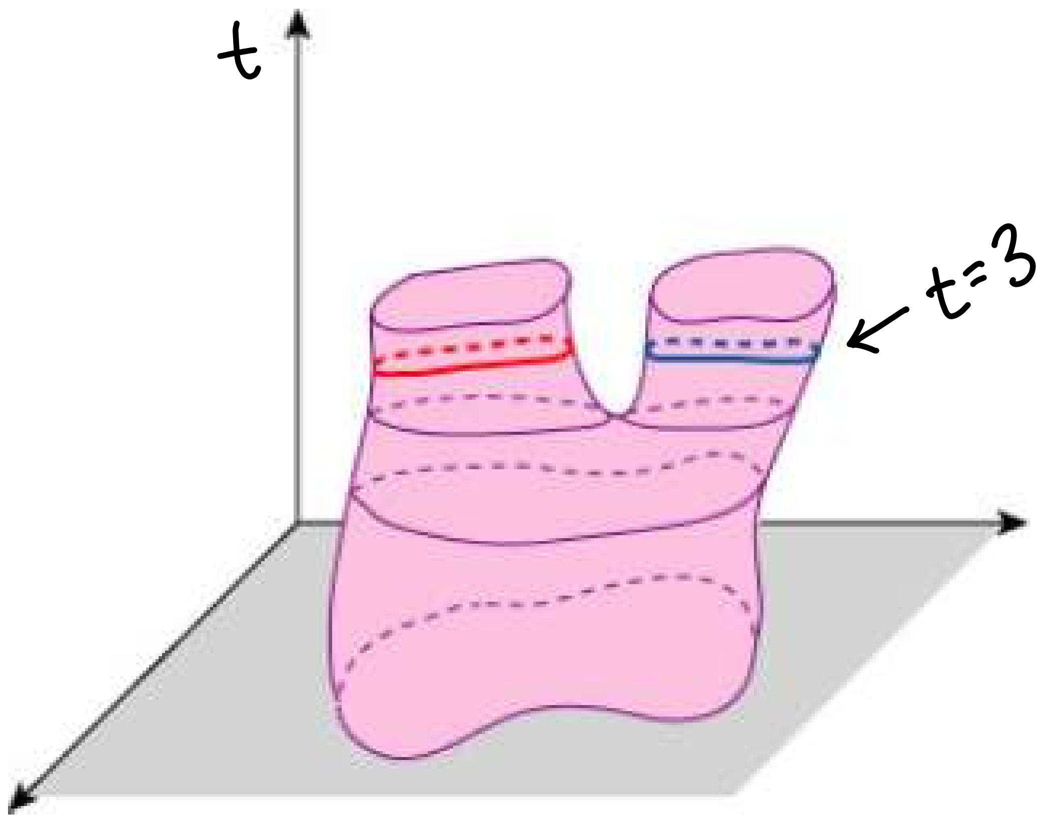


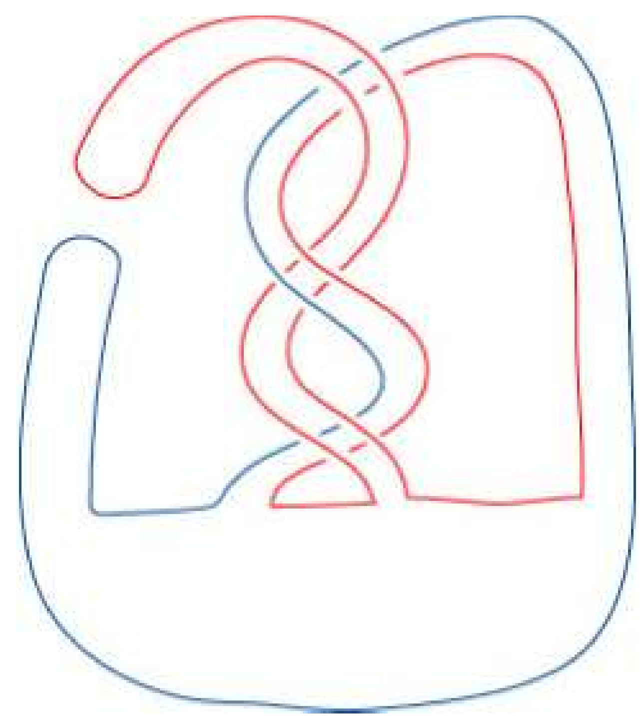
t = 1



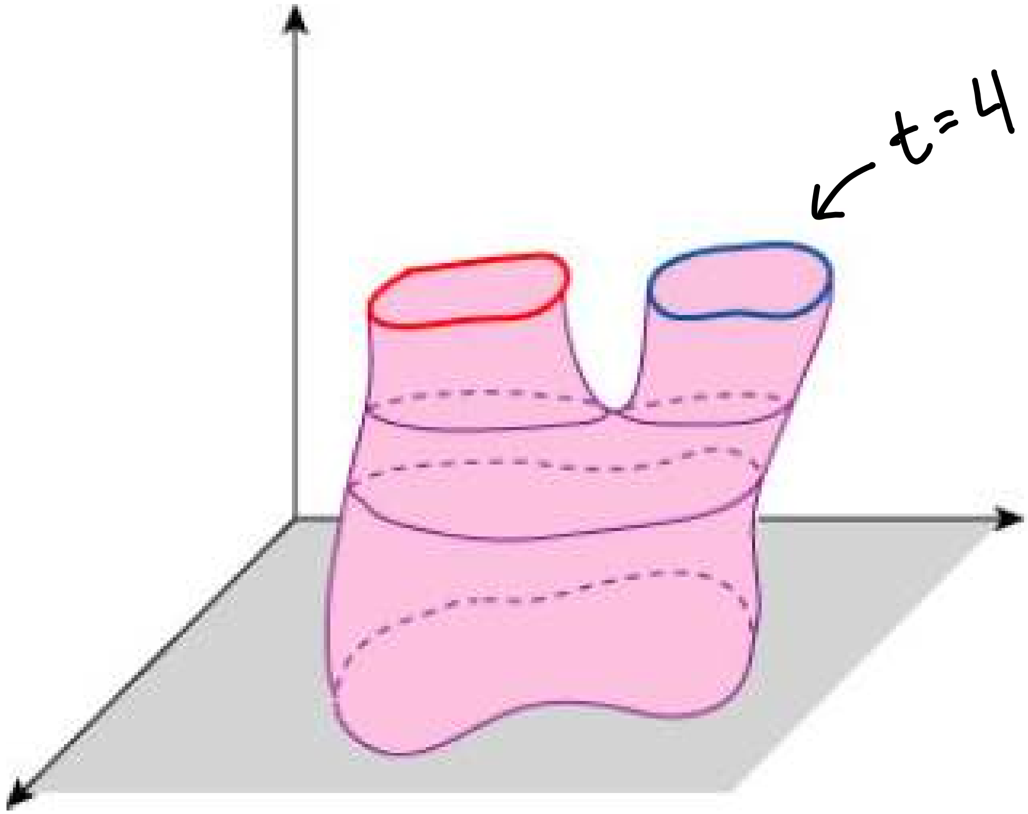


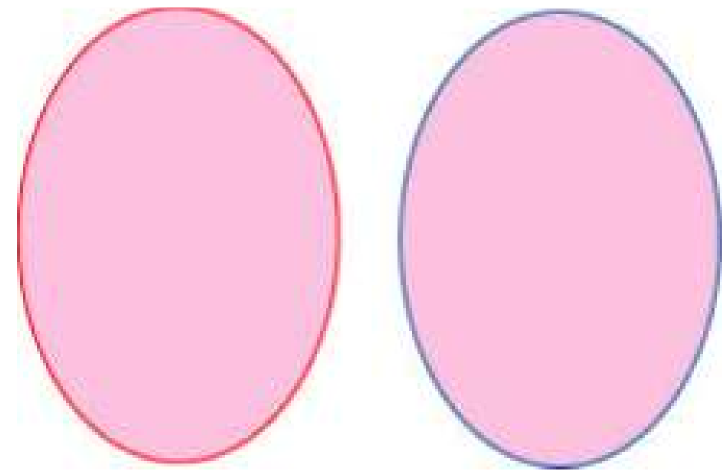
t = 2





t = 3

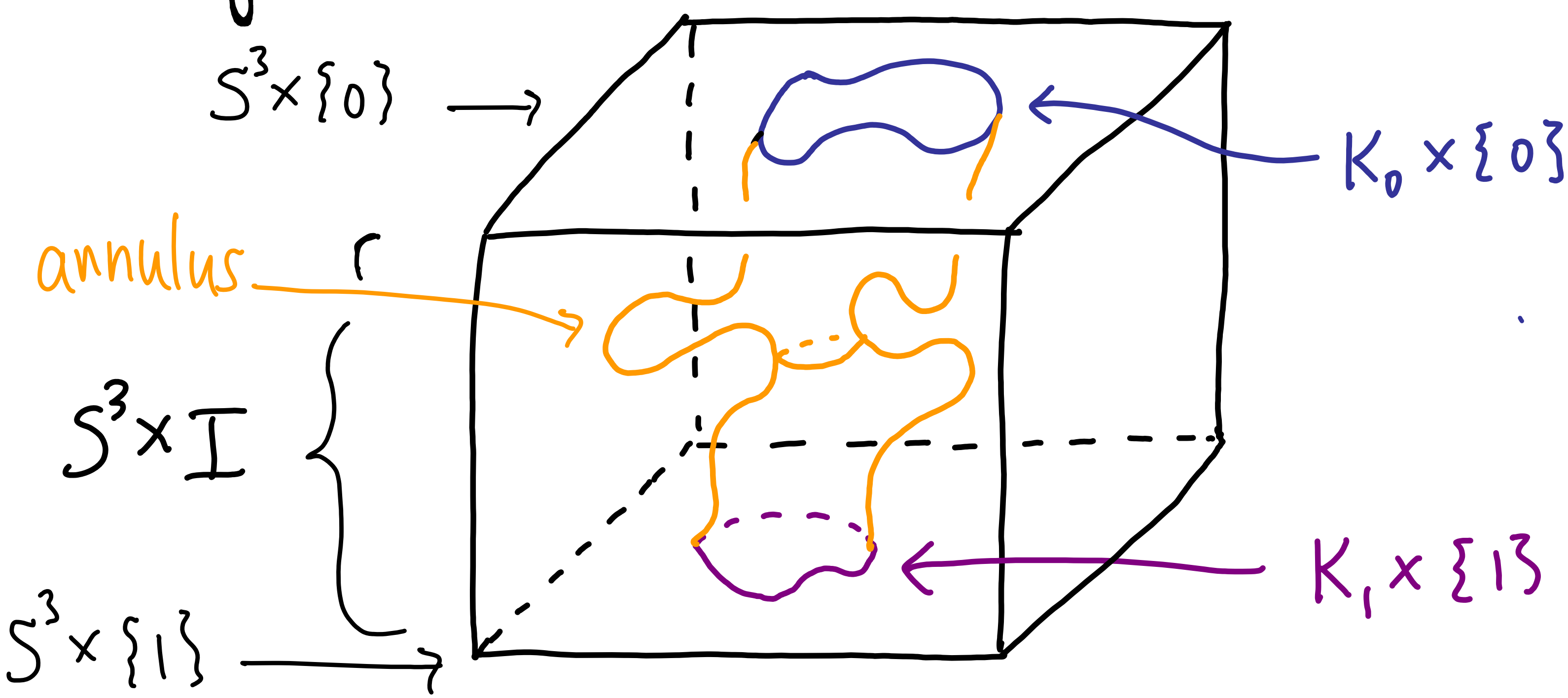




$t=4$

It will be convenient to view knots in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

Def: Knots K_0 and K_1 are concordant if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times [0,1]$



The relation, is concordance to, is an equivalence relation. We denote it by $\sim_c$.

Def: $\mathcal{C} := \{\text{equivalence classes of knots}\}$
$$= \{\text{knots}\}/\sim_c$$

Claim: $\mathcal{C}$ is an abelian group under the operation of connected sum.

- Identity $= [0] = \{\text{slice knots}\}$
- Inverse of $[K]$ is $[r\overline{K}]$:

$$K \# r\overline{K} = \text{slice} \Rightarrow [K \# r\overline{K}] = [0].$$

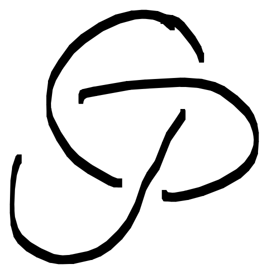
$$\Rightarrow \quad -[K] = [r\overline{K}]$$

What is this group?

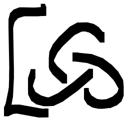
Is it ... trivial?

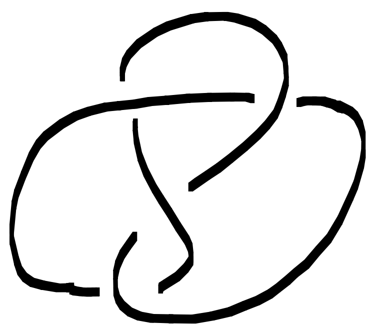
finitely generated?

torsion-free?

Ex:  is not slice

$\Rightarrow$ $\mathcal{C}$ is non-trivial

In fact, [] has infinite order in $\mathcal{C}$.

Ex: $K =$  is not slice but

$K = r\bar{K} \Rightarrow [K] \neq [0]$ but

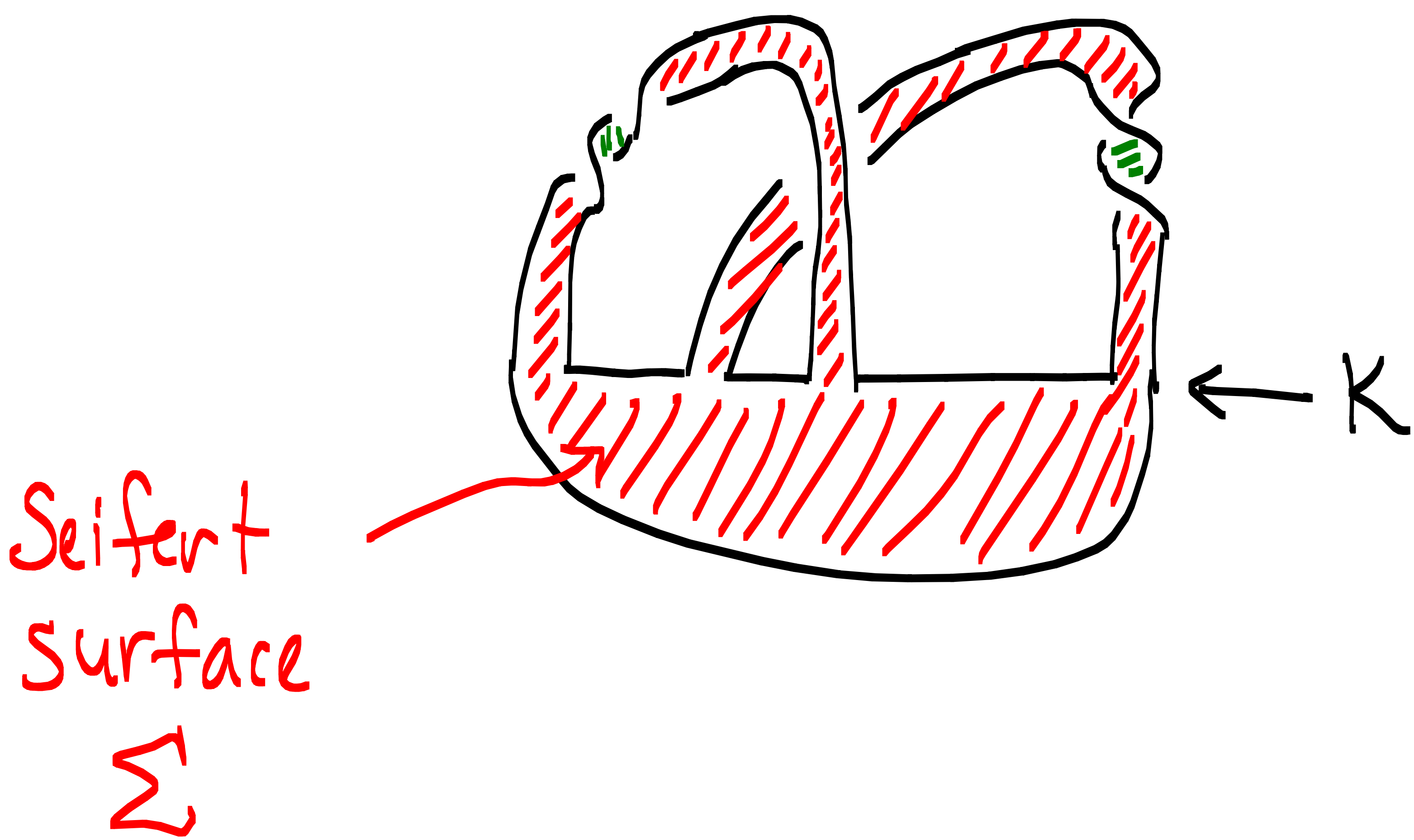
$$2[K] = [K \# K] = [K \# r\bar{K}] = [0]$$

$\Rightarrow \quad \mathcal{C} \supset \mathbb{Z} \oplus \mathbb{Z}/2.$

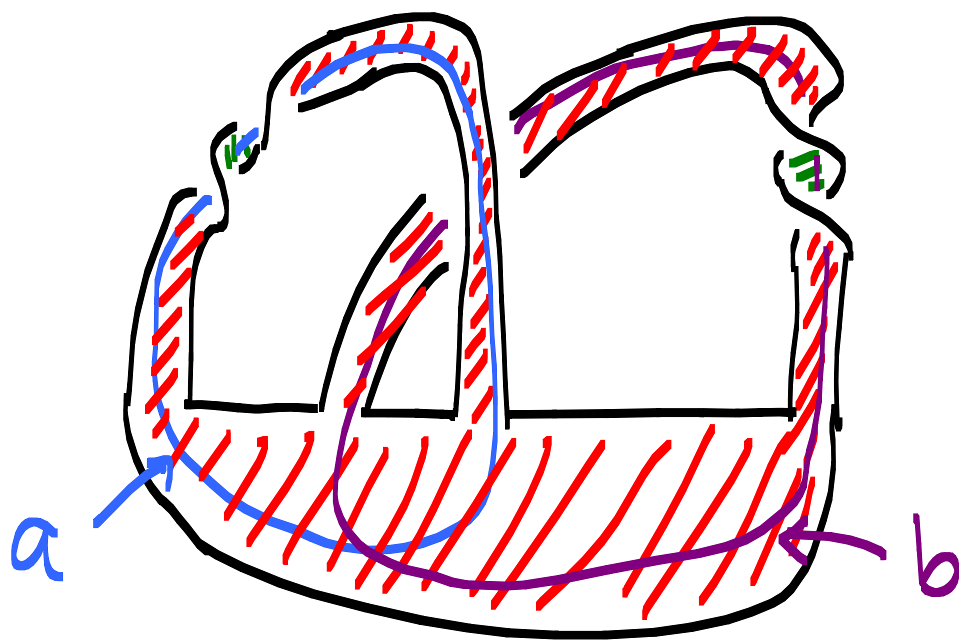
<u>Theorem</u> (Levine, 60's): There is a surjective map $\pi : \mathcal{C} \twoheadrightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \oplus \mathbb{Z}/4^\infty$.

To define this map, Levine defined an infinite number of invariants from the Seifert matrix of a knot.

Def: A Seifert surface Σ for K is a 2-sided surface embedded in S^3 with $\partial\Sigma = K$.



From a Seifert surface $\rightsquigarrow$ Seifert matrix



$$V = \begin{pmatrix} lk(a,a^+) & lk(a,b^+) \\ lk(b,a^+) & lk(b,b^+) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

b^+ = push b off Σ into + direction

$lk(a,b^+)$ = linking number of a and b^+.

For $\omega \in \mathbb{C}$,

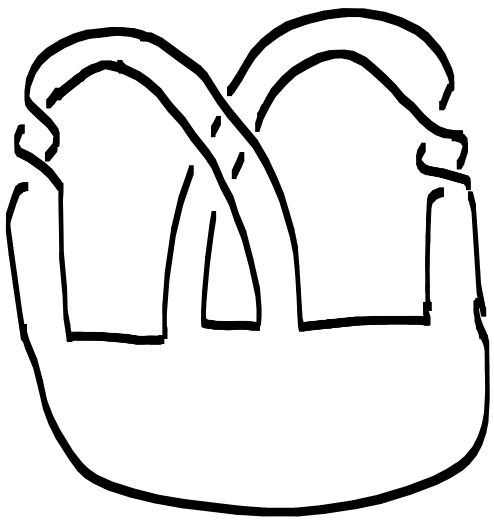
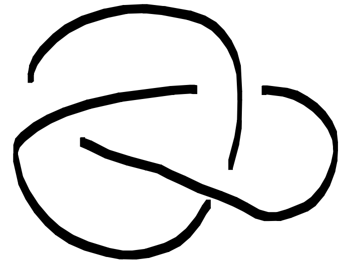
$(1-\omega)V + (1-\bar{\omega})V^T$ is a Hermitian matrix

Def: $\sigma_\omega(K) :=$ signature of $\left((1-\omega)V + (1-\bar{\omega})V^T\right)$
$\in \mathbb{Z}$.

If K is slice and $\omega = (p^k)^{th}$ root of unity $\Rightarrow \sigma_\omega(K) = 0$.

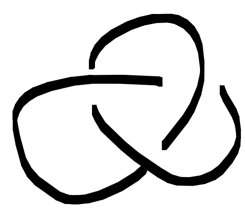
$$\rightsquigarrow \quad \bigoplus \sigma_\omega : \mathcal{C} \longrightarrow \mathbb{Z}^\infty$$

Ex:

 = 

$$V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\omega = -1 \Rightarrow \sigma_\omega = -2$$

$\Rightarrow$  is not slice

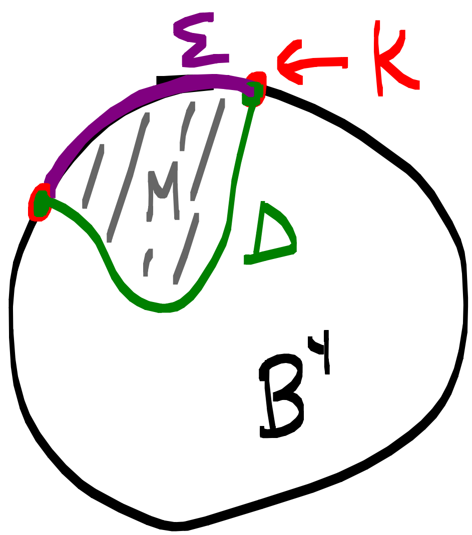
The knots in $\ker(\pi)$ are those with a Seifert matrix of the form

$$\left(\begin{array}{c|c} 0_g & * \\ \hline * & *_g \end{array}\right)$$

called algebrically slice.

Lemma: If K is slice then its Seifert matrix has the form $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$.

Pf: Let Σ be a Seifert surface for K and Δ = slice disk for K. Consider $\hat{\Sigma} = \Sigma \cup \Delta$.



$\hat{\Sigma} = \partial M$, M = 2-sided orientable 3-manifold

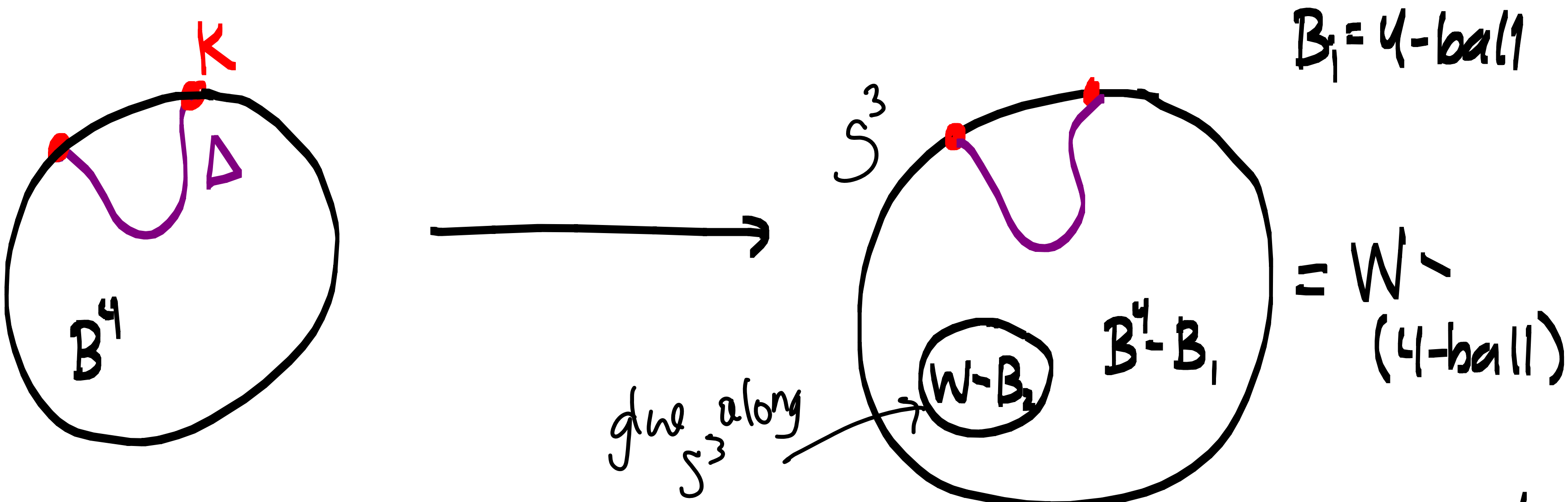
Consider the map $h: H_1(\hat{\Sigma};\mathbb{Q}) \longrightarrow H_1(M;\mathbb{Q})$ induced by inclusion. Can show that

$$\operatorname{rank} \ker(h) = \frac{1}{2} \operatorname{rank} H_1(\hat{\Sigma};\mathbb{Q})$$

Using that $H_1(\Sigma) \cong H_1(\hat{\Sigma}) \cong \mathbb{Z}^{2g}$, there is a basis $a_1, \dots, a_{2g}$ for $H_1(\Sigma)$ s.t. $a_1, \dots, a_g$ bound disjoint 2-chains in D^4. Thus $\mathrm{lk}(a_i, a_j^+) = 0$ for $1 \leq i \leq g$.

$$\Rightarrow \quad V = \begin{pmatrix} 0_{g\times g} & * \\ * & * \end{pmatrix}_{2g\times 2g}. \qquad \blacksquare$$

Note: If K is slice the $K = \partial\Delta$ where Δ = disk in B^4. Let W = closed smooth 4-dimensional manifold. Then K is "slice in W".



connected sum of W and B^4

Let W be a smooth 4-dimensional manifold with $\partial W = S^3$. If K is any knot, we can ask if K is slice in W, i.e. does K bound a smoothly embedded disk in W.

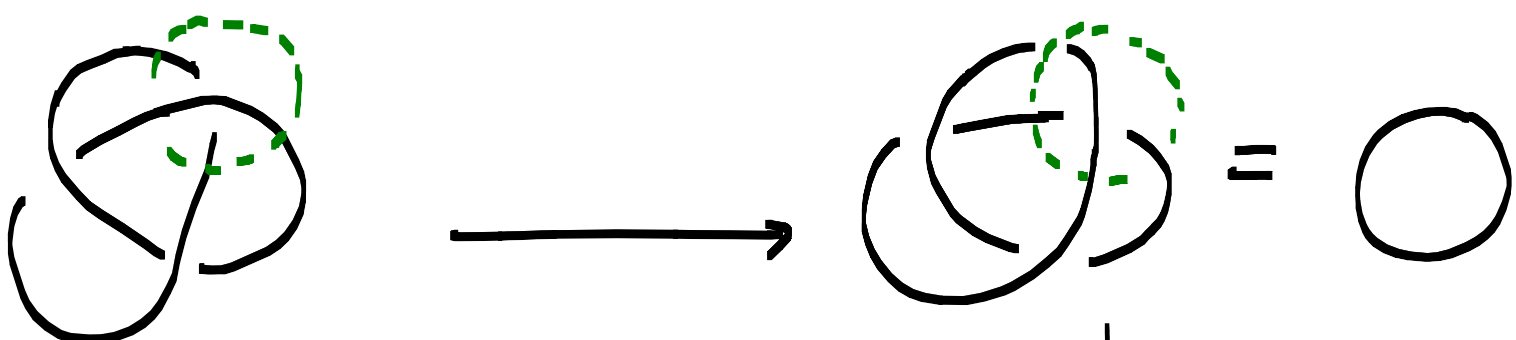
Then we could try to filter $\mathcal{C}$ by considering various 4-manifolds W_i.

Prop: Every knot is slice in $\left(\#_k \mathbb{CP}^2\right) \# \left(\#_\ell \overline{\mathbb{CP}^2}\right) \setminus B^4$.

Recall $\mathbb{CP}^n = \{\text{complex lines in } \mathbb{C}^{n+1}\}$.

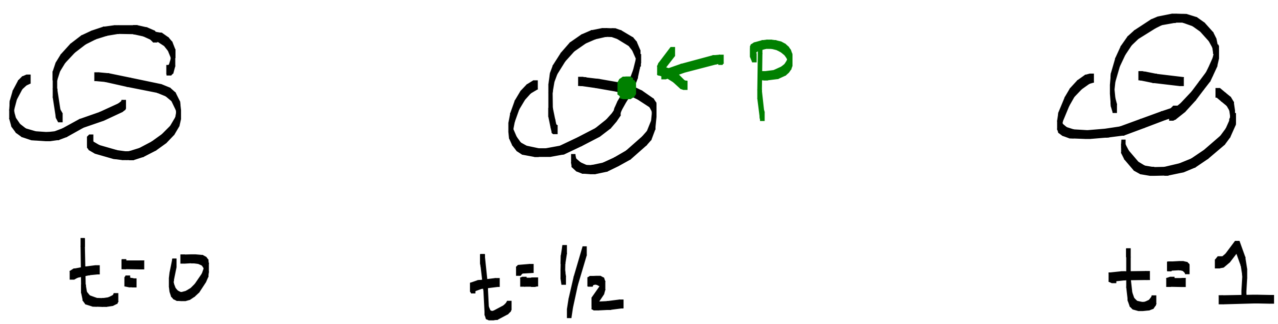
$\overline{\mathbb{CP}^n} = \mathbb{CP}^n$ w/ opposite orientation

Pf: Every knot can be changed into the unknot by changing crossings.



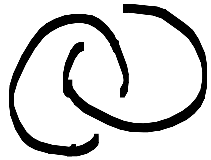
K $\qquad\qquad$ K'

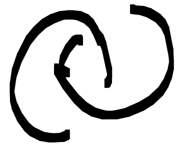
A crossing change gives a homotopy $S^1 \times I \longrightarrow S^3 \times I$ starting at $K \times \{0\}$ and ending at $K' \times \{1\}$ which is an embedding except at one point p.

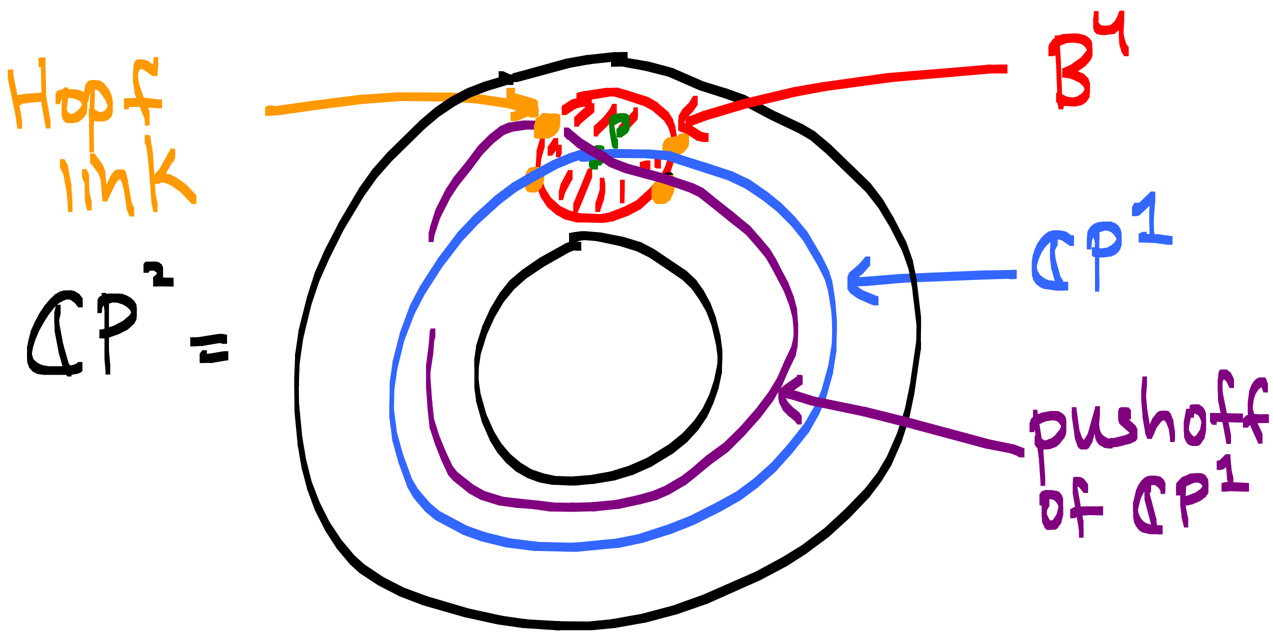


Take a 4-ball $B \subset S^3 \times I$ around p.
Let A = image of $S^1 \times I$. Then $A \cap B$

is two complex disks in $B^4 \subset \mathbb{C}^2$ and

$\partial(A \cap B) =$  , Hopf link.

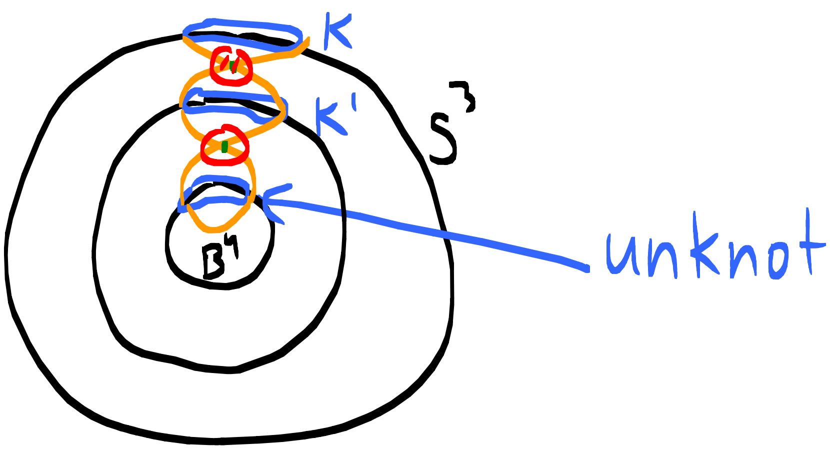
We can replace B with $\mathbb{CP}^2 \setminus B^4$ or $(\overline{\mathbb{CP}}^2 \setminus B^4)$ and then  will bound disjointly embedded disks (blowing up at p).

$\mathbb{CP}^2 =$ 

replace B^4 with $\mathbb{CP}^2 \setminus B^4$

Note: Whether you use $\mathbb{C}P^2$ or $\overline{\mathbb{C}P}^2$ depends on if the crossing was a + or − crossing. We have shown that K and K' cobound an embedded annulus in $(S^3 \times I) \# \pm \mathbb{C}P^2$.

Doing this for every crossing change and capping of S^3 w/ a B^4, we see that K is slice in $\mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \cdots \# \overline{\mathbb{C}P}^2 \setminus B^4$.



$\boxtimes$

Remark: If K bounds a disk Δ in $W = (\#_k \mathbb{C}P^2) \# (\#_\ell \overline{\mathbb{C}P}^2) \setminus B^4$ and Δ misses the copies of $\mathbb{C}P^1$ (the generators of $H_2(W)$) then one can blowdown the $\mathbb{C}P^1$'s to get $\Delta \subset B^4$.

We filter this condition.

- For a group G, $G^{(0)} = G$,
  $G^{(n+1)} = [G^{(n)}, G^{(n)}]$.

Def: A knot K is n-positive if $\exists$ a smooth 4-manifold W with $\partial W = S^3$ and a disk Δ smoothly embedded in W with $\partial\Delta = K$ s.t.

(1) $H_1(W) = 0$

(2) $\exists$ disjointly embedded surfaces $S_1, \ldots, S_j$ freely generating $H_2(W)$ with $S_i \cap S_i^+ = \{pt\}$, a positive intersection $\forall\ i$.

(3) $S_i \cap \Delta = \phi \quad \forall\ i$

(4) $\pi_1(S_i) \subset \pi_1(W \setminus \Delta)^{(n)} \quad \forall\ i$

Similarly for n-negative except

$$S_i \cap S_i^+ = \text{negative intersection point}$$

(Euler class of normal bundle is -1).

$P_n = \{n\text{-positive knots}\} \subset \mathcal{C}$

$N_n = \{n\text{-negative knots}\} \subset \mathcal{C}$.

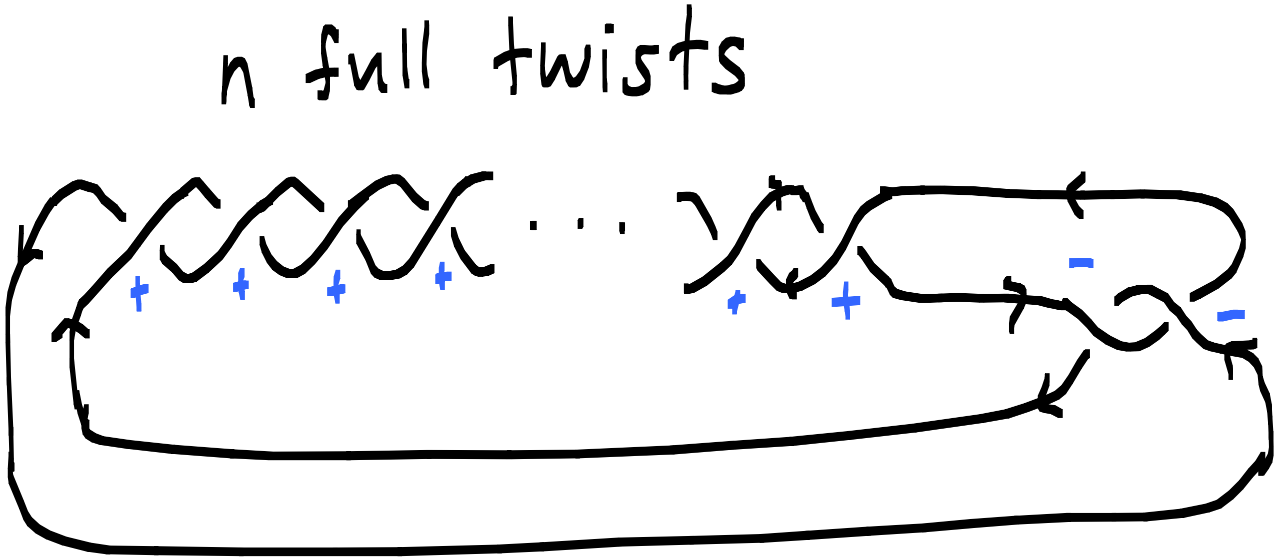
$NP_n = P_n \cap N_n$ is a filtration by subgroups.

$$\cdots \subset NP_2 \subset NP_1 \subset NP_0 \subset \mathcal{C}$$

Prop (Cochran-H-Horn): If K can be changed to the unknot by changing only positive crossings $\Rightarrow K \in P_0$.

Idea: K is slice in $\#_k \mathbb{C}P^2$. Can find representative of $H_2(\#_k \mathbb{C}P^2)$ in exterior of Δ.

Ex: Twist knots

$Tw_n =$ 

$Tw_n \in P_0$ since can change half the + crossings to a − crossing to unknot

$Tw_n \in N_0$ since can change one − crossing to + crossing to unknot

$\Rightarrow Tw_n \in NP_0$

- If $K \in NP_0 \Rightarrow \sigma_\omega(K) = \tau(K) = 0$

  Heegaard-Floer τ-invt.

- If $K \in P_1 \Rightarrow$ certain d-invariants associated to p^r-fold branched covers are zero.

  (similar for $K \in N_1$).

Theorem (Cochran-H-Horn):

$$\bigoplus_{p(t)} \left(\mathbb{Z}^{\infty} \oplus \mathbb{Z}/2\right) \subset NP_n / NP_{n+1} \quad \forall\, n.$$

We are interested in

$$T = \text{topologically slice knots} \subset \mathcal{C}.$$

Define $T_n = T \cap NP_n$.

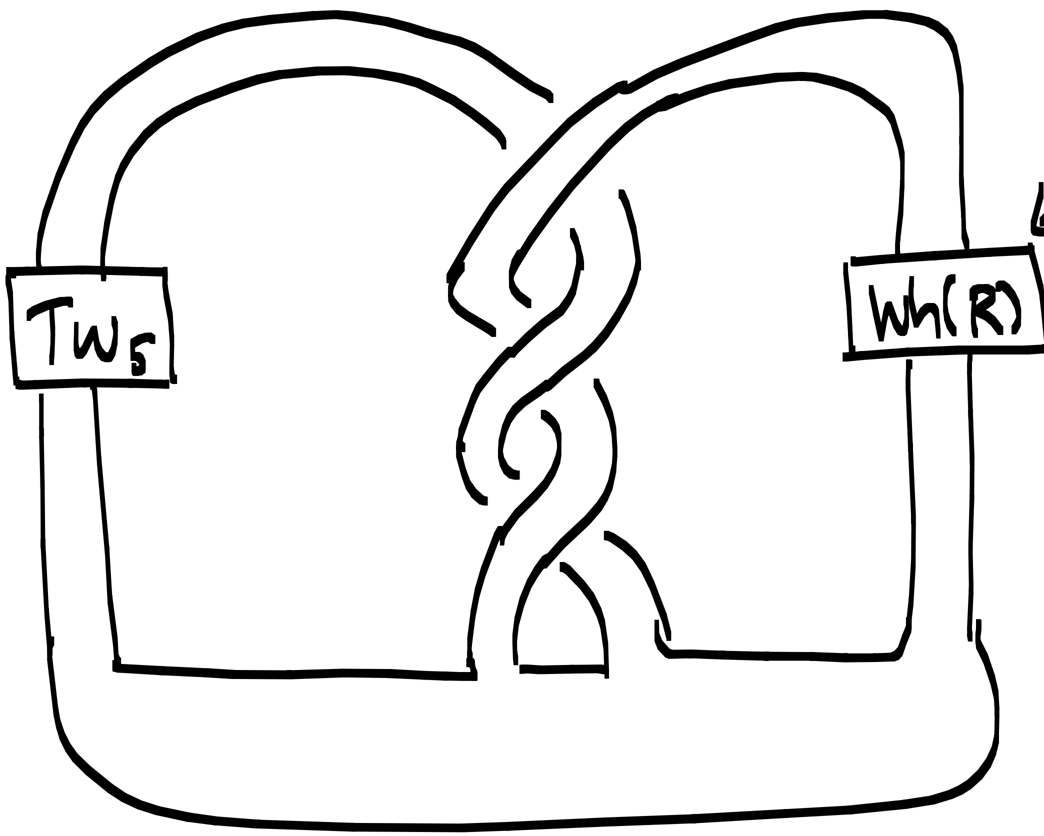
Theorem (Cochran-H-Horn):

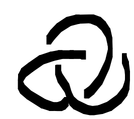
$$\boxed{\mathbb{Z} \subset T_1/T_2.}$$

In particular $T_1/T_2 \neq 0$.

Ex:

twist knot w/ 5 twists →



← untwisted Whitehead double of 

Proof uses d-invariants from Heegaard-Floer homology and Casson-Gordon signature invariants.