4-Dimensional Equivalence Relations on Knots

Fall Southeastern Sectional Meeting 2011

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Def: A knot is a smooth embedding

\[ f: S^1 \rightarrow \mathbb{R}^3 \]

i.e. take a rope, tie it up and attach the ends

unknot  trefoil  figure-eight
In topology, we are interested in knots up to isotopy.

Def: Two knots are equivalent if one can be deformed into the other through embeddings.
Note: A knot is trivial (equivalent to the unknot) if and only if it bounds an embedded disk in $\mathbb{R}^3$. 
However, if we allow the disk to move into $\mathbb{R}^4$, we can get more knots bounding disks.

**Def**: A knot $K \subset \mathbb{R}^3$ is **slice** if the boundary of a smoothly embedded disk in $\mathbb{R}^4$.

2-dim disk = $D$ (slice disk)

$\mathbb{R}^3 = 2\mathbb{R}_+^4$
Examples of slice knots:

**Def:** $K$ is ribbon if it bounds an immersed disk in $\mathbb{R}^3$ with only ribbon singularities.
A ribbon knot is slice

**Pf:** To obtain a disk embedded in $R_x^y$, push the interior of red disks into interior of $R_x^y$. 
So is slice.

Conjecture: A knot is (smoothly) slice if it is ribbon.
There is a binary operation on (oriented) knots:

\[ K_1 \# K_2 := \text{connected sum} \]
If $K$ is a knot then we change change all crossings and get a new knot $\overline{K}$, its mirror image.
In general, \( K \neq \overline{K} \).

\[ \text{Ex:} \quad \includegraphics[width=0.3\textwidth]{example1} \neq \includegraphics[width=0.3\textwidth]{example2} \]

\[ \text{Ex:} \quad \includegraphics[width=0.3\textwidth]{example3} = \includegraphics[width=0.3\textwidth]{example4} \]
For any knot $K$, $K \# r\bar{K}$ is slice

* need to orient knots and $r\bar{K}$ is \bar{K} with the reversed orientation.

We will show that $K \# r\bar{K}$ is ribbon.
$K \# r\bar{K}$
Continue this "vertical drape" around the arc.
Intersections look like

\[ \Rightarrow k \# r \bar{R} \text{ is ribbon!} \]
Another example:

The $9_{46}$ knot is slice.
How to build a slice disk:

- look at slices of the disk in time viewing $R^4_t = R^3 \times R \times \text{time}$
It will be convenient to view knots in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

**Def:** Knots $K_0$ and $K_1$ are **concordant** if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times [0,1]$. 
The relation, is concordance to, is an equivalence relation. We denote it by $\sim_c$.

**Def:** $C := \{\text{equivalence classes of knots}\} = \{\text{knots}\}/\sim_c$

**Claim:** $C$ is an abelian group under the operation of connected sum.
• Identity = [0] = \{slice knots\}

• Inverse of [K] = [rR]:
  \[
  K \# rR = \text{slice} \implies [K \# rR] = [0].
  \]

\[
\Rightarrow -[K] = [rR]
\]

What is this group?
Is it ... trivial?
  finitely generated?
torsion-free?
Ex: \( \mathcal{G} \) is not slice
\[ \Rightarrow \mathcal{C} \text{ is non-trivial} \]
In fact, \([\mathcal{G}]\) has infinite order in \( \mathcal{C} \).

Ex: \( K = \mathcal{G} \) is not slice but
\[ K = r\overline{K} \Rightarrow [K] \neq [0] \text{ but } 2[K] = [K \# K] = [K \# r\overline{K}] = [0] \]
\[ C = \mathbb{Z} \oplus \mathbb{Z}/2. \]

**Theorem (Levine, 60's):** There is a surjective map \( \Pi : C \rightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \oplus \mathbb{Z}/4^\infty. \)

To define this map, Levine defined an infinite number of invariants from the Seifert matrix of a knot.
**Def:** A Seifert surface $\Sigma$ for $k$ is a 2-sided surface embedded in $S^3$ with $\partial \Sigma = k$. 

**Seifert Surface $\Sigma$**
From a Seifert surface $\rightarrow$ Seifert matrix

$$V = \begin{pmatrix} \text{lk}(a,a^+) & \text{lk}(a,b^+) \\ \text{lk}(b,a^+) & \text{lk}(b,b^+) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$b^+ = \text{push } b \text{ off } \Sigma \text{ into } + \text{ direction}$

$\text{lk}(a,b^+) = \text{linking number of } a \text{ and } b^+$.
For \( w \in \mathbb{C} \),
\[(1-w)V + (1-\overline{w})V^T \] is a Hermitian matrix.

**Def:** \( \sigma_w(K) := \text{signature of } (\lambda - \overline{w})V + (1-\overline{w})V^T \) for \( \lambda \in \mathbb{C} \).

If \( K \) is slice and \( w = (p^k)^{th} \) root of unity \( \Rightarrow \sigma_w(K) = 0 \).
\[
\Rightarrow \quad \oplus \sigma_w : \mathbb{C} \rightarrow \mathbb{Z}
\]
Ex:

\[ V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \]

\[ w = -1 \implies \delta_w = -2 \]

\[ \implies \text{is not slice} \]
The knots in $\ker (\pi)$ are those with a Seifert matrix of the form

$$
\begin{pmatrix}
0_g & * \\
* & *_g
\end{pmatrix}
$$

called algebraically slice.
Lemma: If $K$ is slice then its Seifert matrix has the form $(\ast \ast)$.

Proof: Let $\Sigma$ be a Seifert surface for $K$ and $\Delta = $ slice disk for $K$. Consider $\hat{\Sigma} = \Sigma \cup \Delta$.

$\hat{\Sigma} = 2M$, $M = 2$-sided orientable 3-manifold
Consider the map $h : H_1(\Sigma; \mathbb{Q}) \to H_1(M; \mathbb{Q})$ induced by inclusion. Can show that
\[ \text{rank } \ker(h) = \frac{1}{2} \text{rank } H_1(\Sigma; \mathbb{Q}) \]
Using that $H_1(\Sigma) \cong H_1(\Sigma) = \mathbb{Z}^{2g}$, there is a basis $a_1, \ldots, a_{2g}$ for $H_1(\Sigma)$ s.t.
$a_1, \ldots, a_g$ bound disjoint 2-chains in $D^g$.
Thus $\text{lk}(a_i, a_j^*) = 0$ for $1 \leq i \leq g$.

\[ \Rightarrow \quad V = \begin{pmatrix} O_{g \times g} & * \\ * & * \end{pmatrix}_{2g \times 2g}. \]
Note: If \( K \) is slice the \( k = 2\Delta \)
where \( \Delta = \text{disk in } B^4 \). Let \( W = \text{closed smooth } 4\text{-dimensional manifold}. \) Then \( K \) is "slice in \( W\)."
Let \( W \) be a smooth 4-dimensional manifold with \( \partial W = S^3 \). If \( K \) is any knot, we can ask if \( K \) is slice in \( W \), i.e. does \( K \) bound a smoothly embedded disk in \( W \).

Then we could try to filter \( C \) by considering various 4-manifolds \( W_i \).
Prop: Every knot is slice in $(\#_k \mathbb{C}P^2)\#(\#_n \overline{\mathbb{C}P^2})\# \mathbb{B}^4$.

Recall $\mathbb{C}P^n = \{\text{complex lines in } \mathbb{C}^{n+1}\}$. $\overline{\mathbb{C}P^n} = \mathbb{C}P^n$ with opposite orientation.

Pf: Every knot can be changed into the unknot by changing crossings.

\[ \begin{array}{c}
\kappa \\
\xrightarrow{\text{changing crossings}}
\kappa'
\end{array} = \bigcirc \]
A crossing change gives a homotopy $S^1 \times I \to S^3 \times I$ starting at $K \times \{0\}$ and ending at $K' \times \{1\}$ which is an embedding except at one point $p$.

\[ \begin{align*}
& \text{(a)} \quad \text{t=0} \quad \text{p} \quad \text{t=1} \\
& \text{(b)} \quad \text{t=1/2} \\
\end{align*} \]

Take a 4-ball $B \subset S^3 \times I$ around $p$. Let $A = \text{image of } S^1 \times I$. Then $A \cap B$
is two complex disks in $\mathbb{B}^4 \subset \mathbb{C}^2$ and $\partial(\mathbb{A} \cap \mathbb{B}) = \mathbb{C}$, Hopf link. We can replace $\mathbb{B}$ with $\mathbb{C}\mathbb{P}^2 \setminus \mathbb{B}^4$ or $(\mathbb{C}\mathbb{P}^2 - \mathbb{B}^4)$ and then $\mathbb{C}$ will bound disjointly embedded disks (blowing up at $p$).
Note: Whether you use $\mathbb{CP}^1$ or $\overline{\mathbb{CP}}^1$ depends on if the crossing was a + or - crossing. We have shown that $K$ and $K'$ cobound an embedded annulus in $(S^3 \times I) \# \mathbb{CP}^2$. Doing this for every crossing change and capping of $S^3$ with a $B^4$, we see that $K$ is slice in $\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \# \cdots \# \overline{\mathbb{CP}}^2 \setminus B^4$. 

\[ \text{ unknot } \]
Remark: If $K$ bounds a disk $\Delta$ in $W=(\# \mathbb{CP}^2)\#(\# \mathbb{CP}^2)\setminus B^4$ and $\Delta$ misses the copies of $\mathbb{CP}^1$ (the generators of $H_2(W)$) then one can blowdown the $\mathbb{CP}^2$s to get $\Delta \subset B^4$.

We filter this condition.

- For a group $G$, $G^{(0)} = G$, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. 
**Def:** A knot $K$ is $n$-positive if there exists a smooth 4-manifold $W$ with $\partial W = S^3$ and a disk $\Delta$ smoothly embedded in $W$ with $\partial \Delta = K$ such that:

1. $H_1(W) = 0$
2. There exist disjunctly embedded surfaces $S_1, \ldots, S_s$ freely generating $H_2(W)$ with $S_i \cap S_i^+ = \emptyset \neq 3$, a positive intersection $\forall i$.
3. $S_i \cap \Delta = \emptyset \quad \forall i$
4. $\pi_1(S_i) < \pi_1(W - \Delta)^{(n)} \quad \forall i$
Similarly for $n$-negative except

$S_i \cap S_i^\perp = \text{negative intersection point}$

(Euler class of normal bundle is $-1$).

$P_n = \{ n \text{-positive knots} \} \subset C$

$N_n = \{ n \text{-negative knots} \} \subset C$.

$NP_n = P_n \cap N_n$ is a filtration by subgroups.
... c \text{NP}_2 \subset \text{NP}_1 \subset \text{NP}_0 \subset \mathcal{C}

\textbf{Prop (Cochran-H-Horn):} If \( K \) can be changed to the unknot by changing only positive crossings \( \Rightarrow K \in \text{P}_0 \).

\textbf{Idea:} \( K \) is slice in \( \#_k \mathbb{C}P^2 \). Can find representative of \( H_2(\#_k \mathbb{C}P^2) \) in exterior of \( \Delta \).
**Ex: Twist knots**

\[ T_{tw} = \begin{array}{c}
\text{n full twists}
\end{array} \]

\[ T_{tw} \in P_0 \quad \text{since can change half the + crossings to a - crossing to unknot} \]

\[ T_{tw} \in N_0 \quad \text{since can change one - crossing to + crossing to unknot} \]

\[ \Rightarrow T_{tw} \in NP_0 \]
\begin{itemize}
  \item If \( K \in N_0 \) \( \Rightarrow \sigma_w(K) = \tau(K) = 0 \)
    \[ \uparrow \]
    Heegaard-Floer \( \tau \)-invts.
  \item If \( K \in P_1 \) \( \Rightarrow \) certain \( d \)-invariants
      associated to \( p \)-fold branched covers are zero.
      (similar for \( K \in N_1 \)).
\end{itemize}
Theorem (Cochran-H-Horn):

\[ \Theta_p(\mathbb{Z}^\infty \otimes \mathbb{Z}/2) \subset NP_n/NP_{n+1}, \quad \forall n. \]

We are interested in

\[ T = \text{topologically slice knots} \subset \mathcal{C}. \]

Define \[ T_n = T \cap NP_n. \]
Theorem (Cochran-H-Horn):
\[ \mathbb{Z} \leq T_1 / T_2. \]
In particular \( T_1 / T_2 \neq 0 \).

Ex:
untwisted Whitehead double of \( \mathbb{Z} \)

Twist knot w/ 5 twists → \( T_{W5} \)

Wh(R)
Proof uses $d$-invariants from Heegaard-Floer homology and Casson-Gordon signature invariants.