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Classical Knot Concordance and Blanchfield Duality

Geometric Topology Conference

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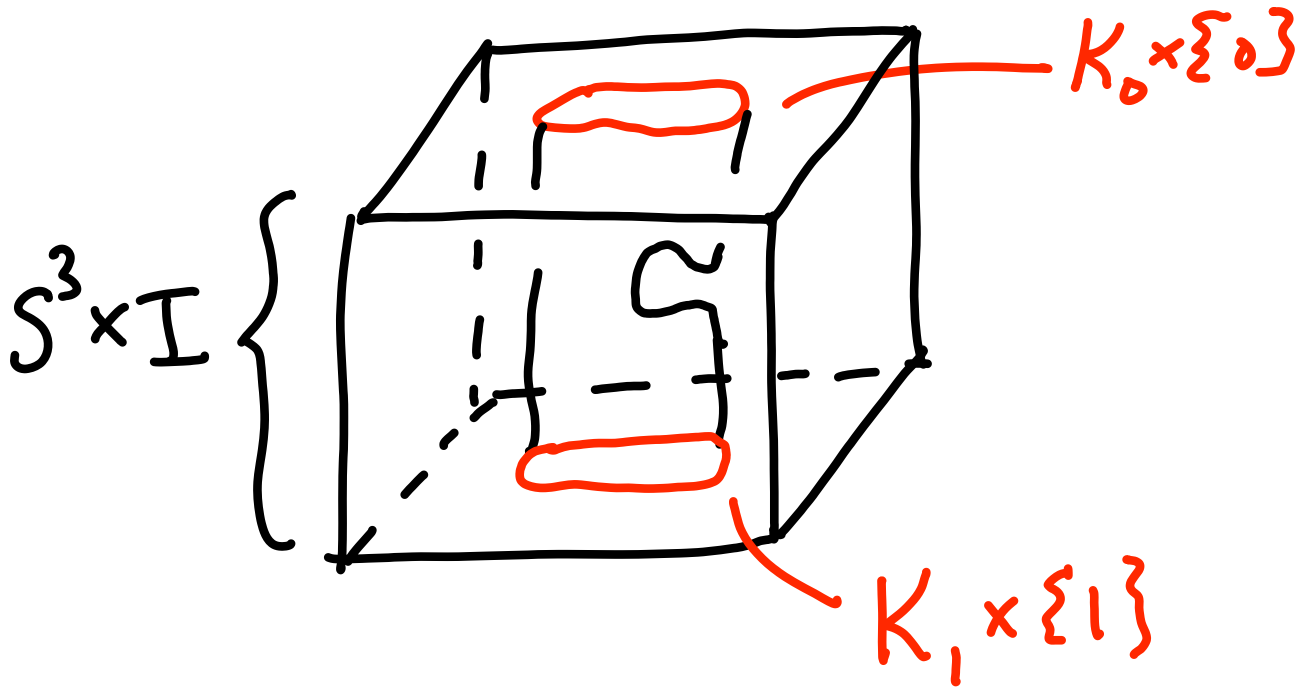
<u>Goal</u>: Show the successive quotients of the Cochran-Orr-Teichner filtration of the knot concordance group (smooth and topological) have <u>infinite</u> <u>rank</u>.

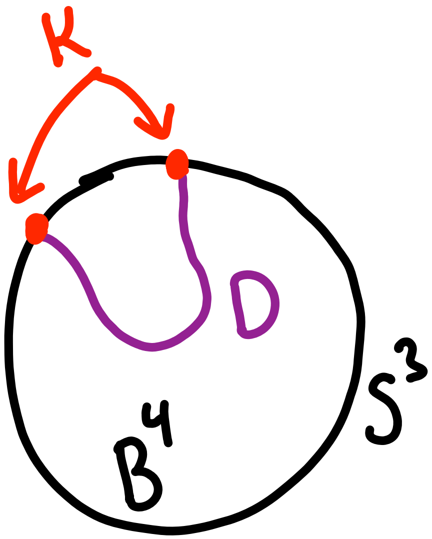
Knots: Let K_0, K_1 be knots in S^3

K_0 is (topologically/smoothly) concordant to K_1 if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a (locally flat/smooth) annulus in $S^3 \times I$.



K is slice $\iff$ K is concordant to unknot

$\iff$ K bounds 2-disc D in B^4



$\mathcal{C}$ = knot concordance group (abelian)

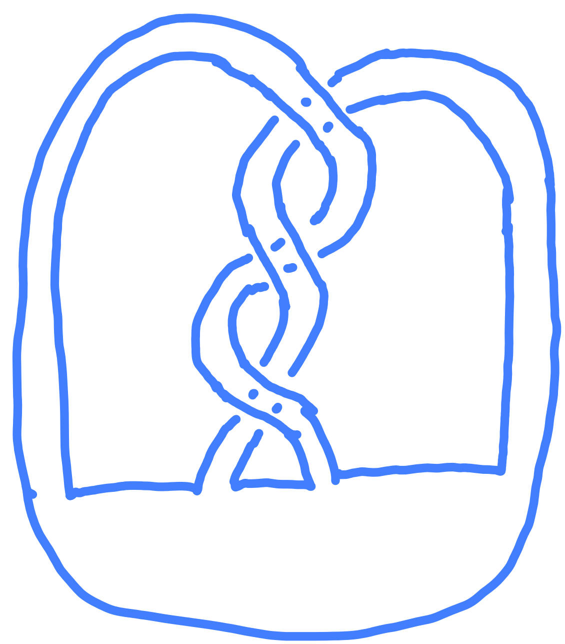
$= \{\text{knots}\}/\{\text{concordance}\}$

- Addition is connected sum: $K_1 \# K_2$
- $[K] = 0 \iff K$ is slice

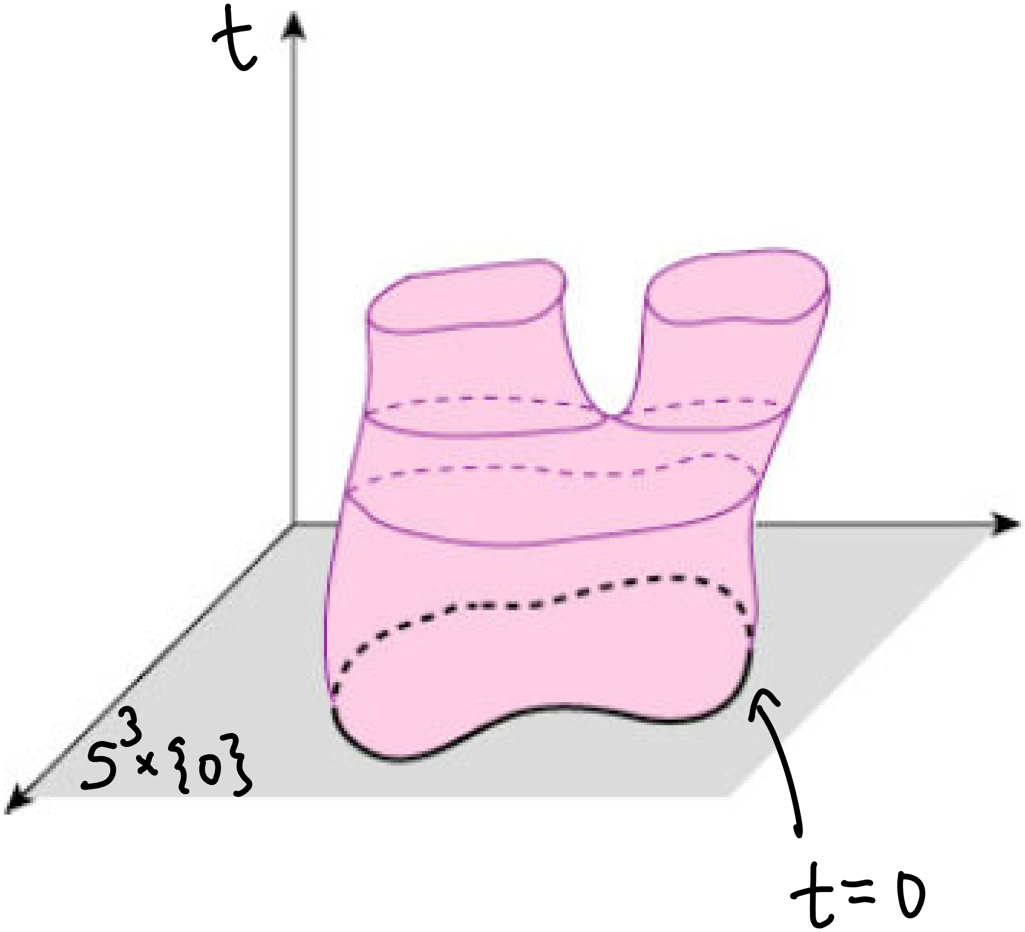
__Remark:__ In this talk, all results are valid for both smooth and topological concordance.

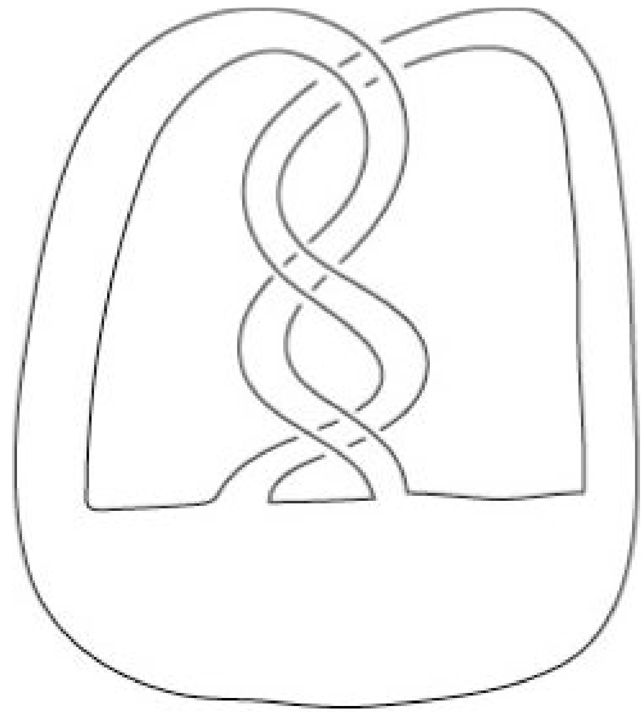
Ex: The 9_{46} knot is (smoothly and topologically) slice.

i.e. $[9_{46}] = 0$ in $\mathcal{C}^{\text{smooth}}$ and $\mathcal{C}^{\text{top}}$

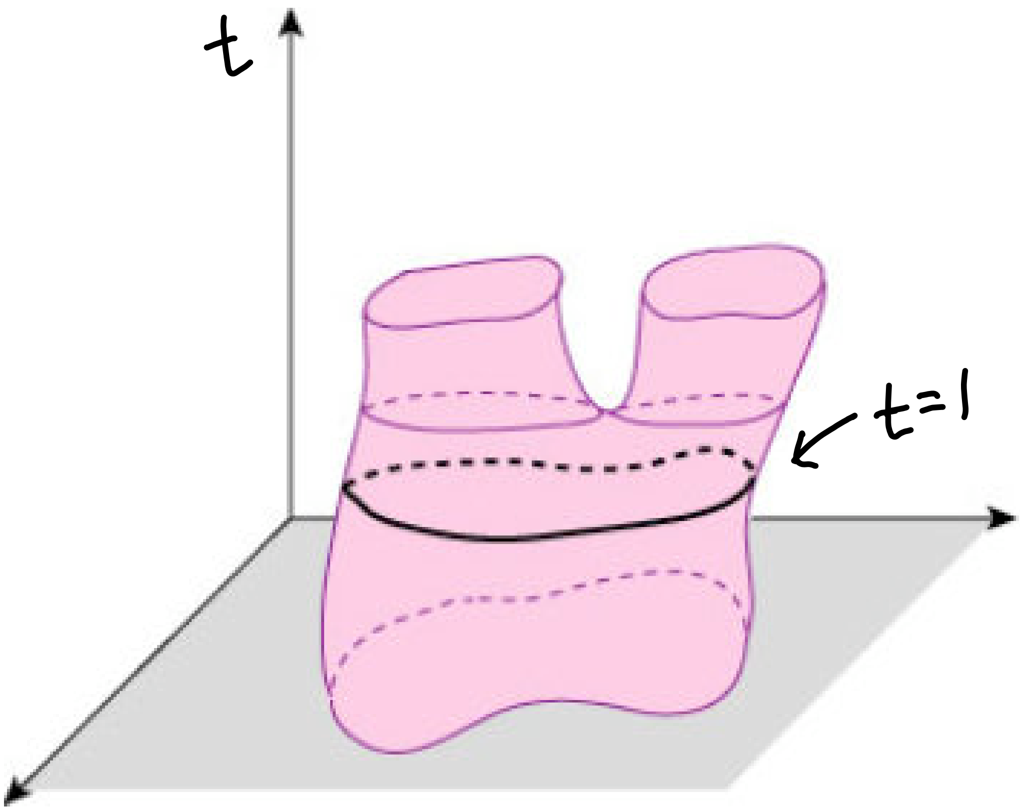
$9_{46} =$ 

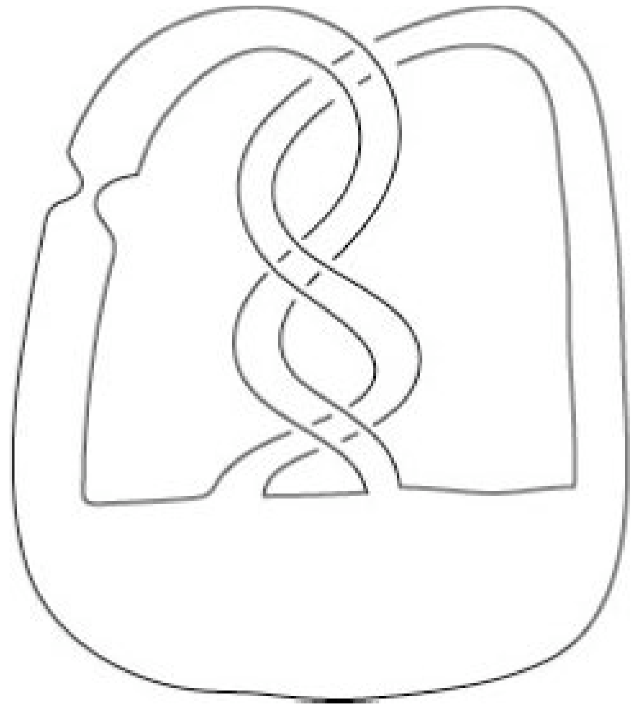
How to build a slice disc



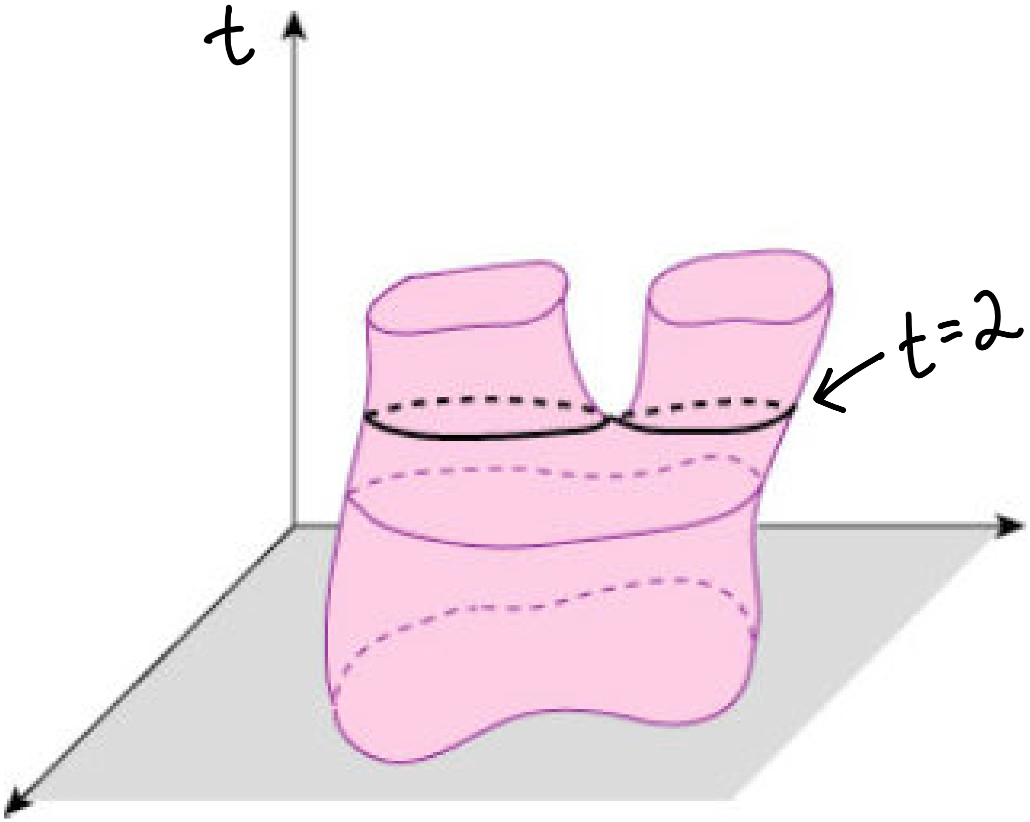


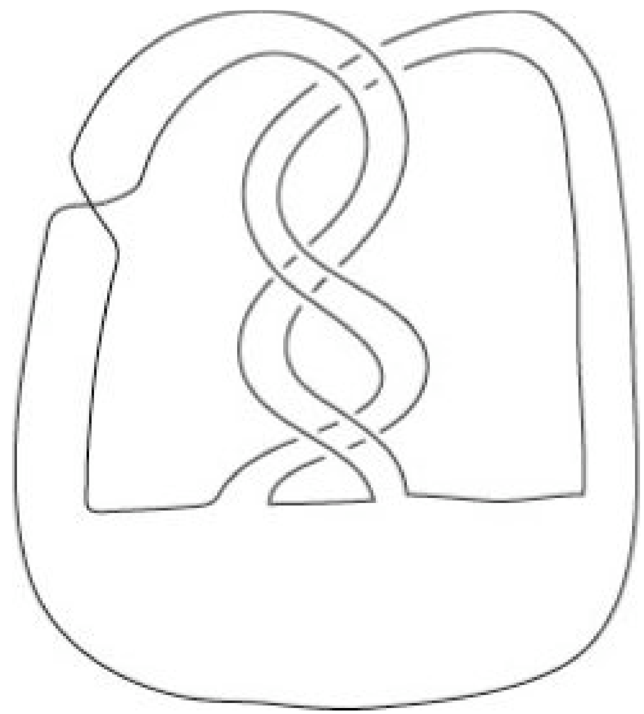
t=0



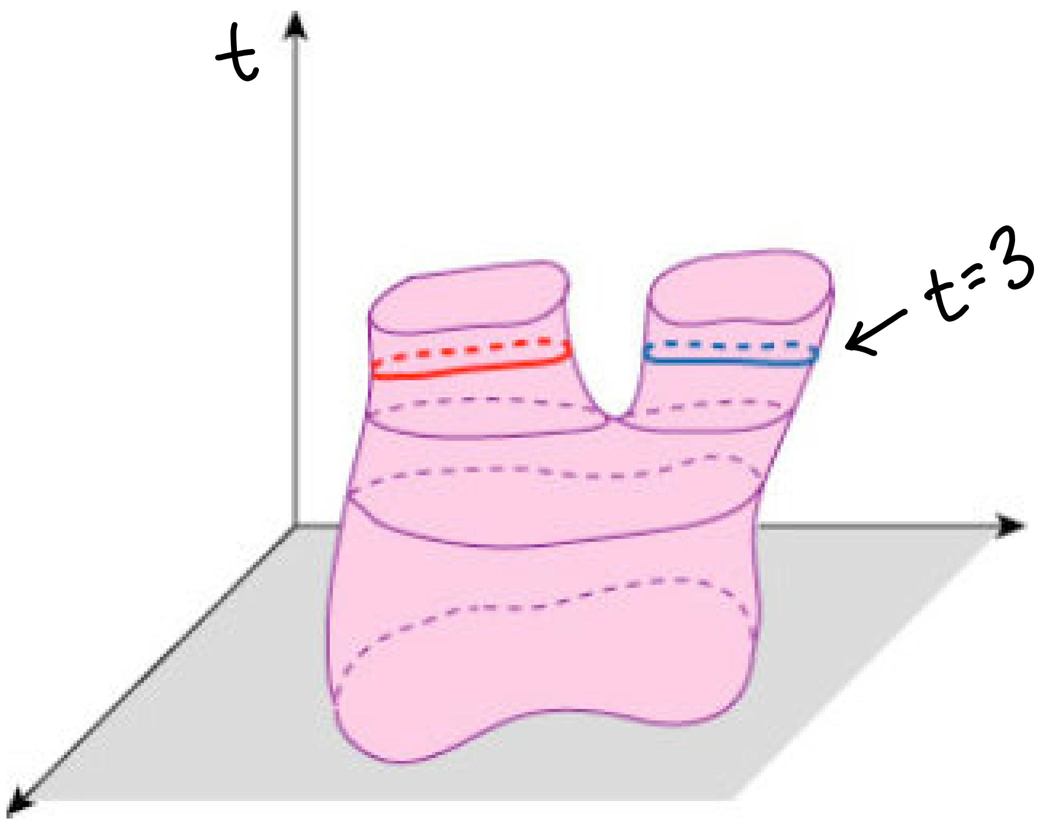


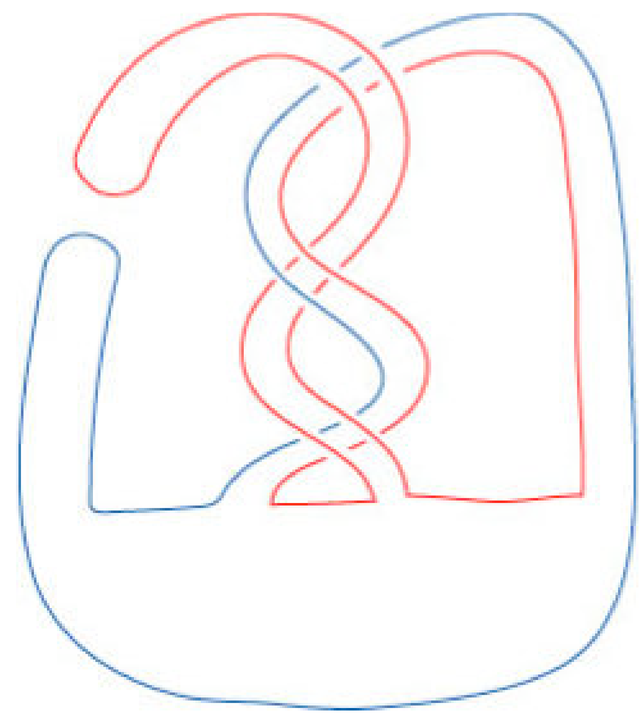
t = 1



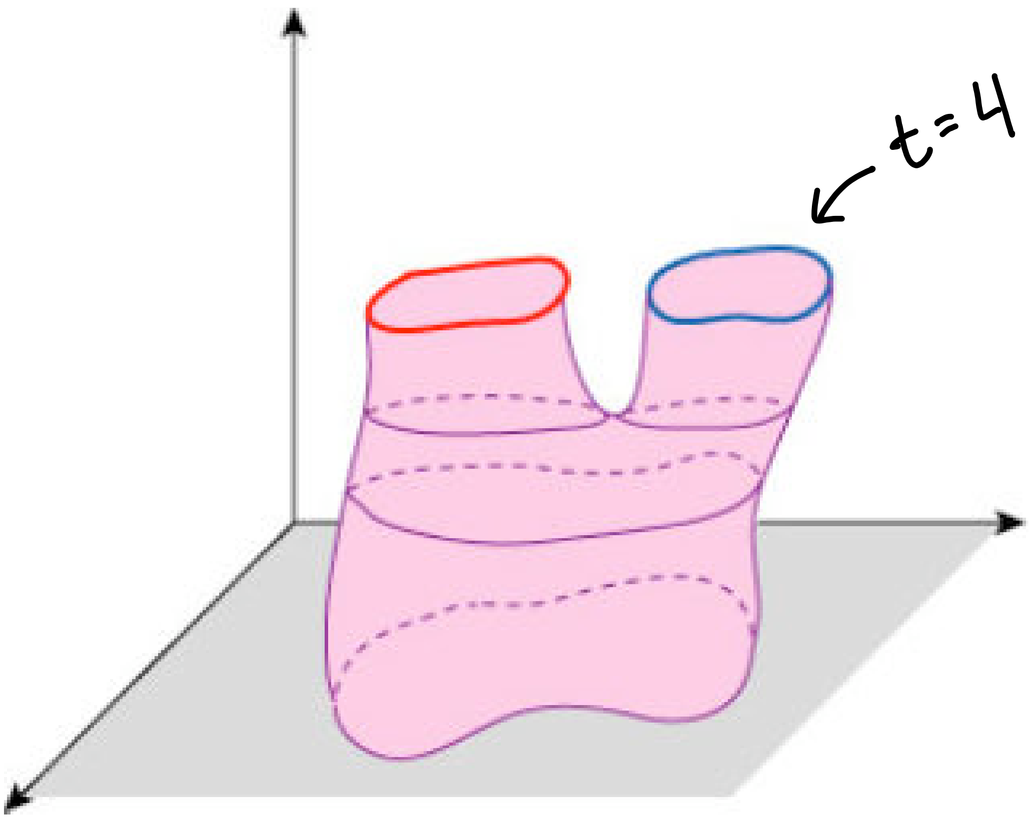


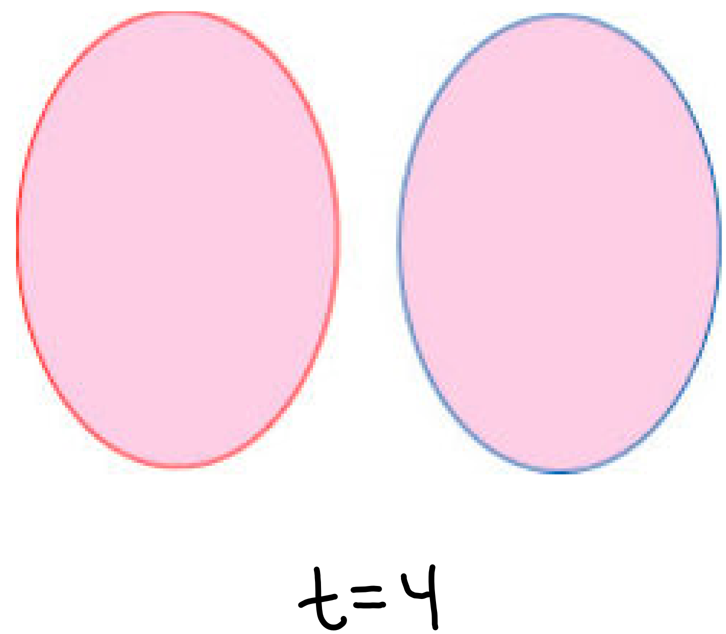
t= 2



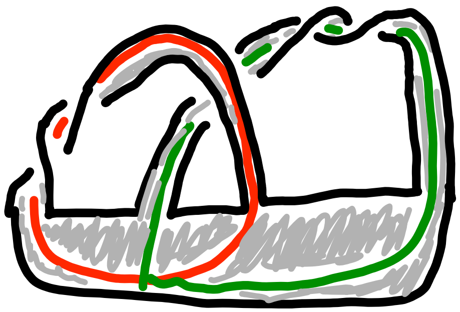


t = 3


t=4


t=4

Levine-Tristram Signatures



$\longrightarrow$ $V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

$K = \partial(\text{seifert surface})$ Seifert Matrix

For $w \in S^1 \subset \mathbb{C}$,

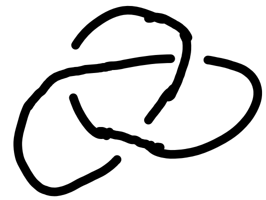
$$\sigma_w(K) := \text{signature}\left((1-w)V + (1-\bar{w})V^T\right)$$

$$\rho_0(K) := \int_{S^1} \sigma_w(K)\, dw \quad \text{(average)}$$

- If K Slice (and w not a root of $\Delta_K(t) = $ Alex. poly) then $\sigma_w(K) = 0 \Rightarrow \rho_0(K) = 0$.

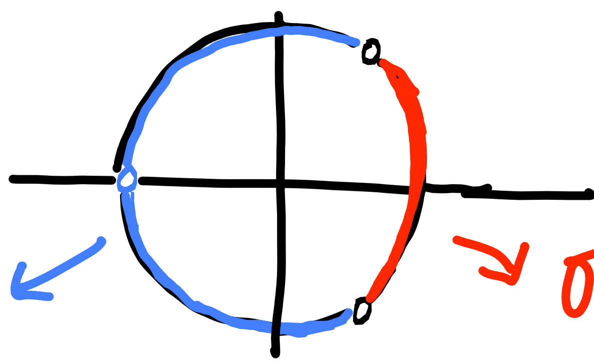
Ex: Trefoil is not Slice

= isotopic



$$V_{Trefoil} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow \sigma_{-1} = \text{sign}\left[2\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \right] = -2 \neq 0$$

In fact



$$\boxed{\Delta_{Trefoil} = t^2 - t + 1 \;\xrightarrow{\text{roots}}\; \frac{1 \pm \sqrt{3}\, i}{2}}$$

$$\rho_0(K) = \int_{S^1} \sigma_\omega(K)\, d\omega = -\frac{4}{3} \neq 0$$

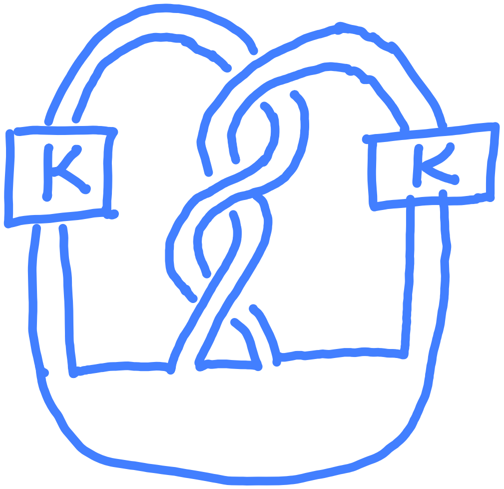
• [Milnor, Tristram ~67] $\mathcal{C}$ has infinite rank.

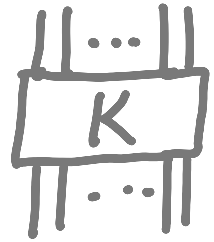
(late 60's) Levine used invariants obtained from Seifert matrix (including knot signatures and Arf invariant) to define an epimorphism

$$\mathcal{C} \xrightarrow{\pi} \mathbb{Z}^\infty \times \mathbb{Z}_2^\infty \times \mathbb{Z}_4^\infty .$$

<u>Def</u>: K is <u>algebraically slice</u> if $K \in \ker \pi$

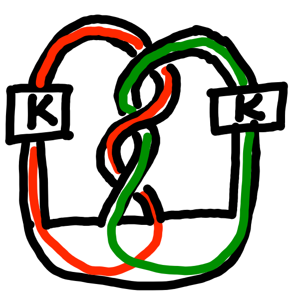
Ex:

$9_{46}(K) :=$ 



means grab strings $|| \cdots ||$ and tie them into knot K

Seifert matrix for $9_{46}(K)$ is same as a seifert matrix for 9_{46}, a slice knot.



$\longrightarrow V = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

$\Rightarrow 9_{46}(K)$ is algebraically slice.

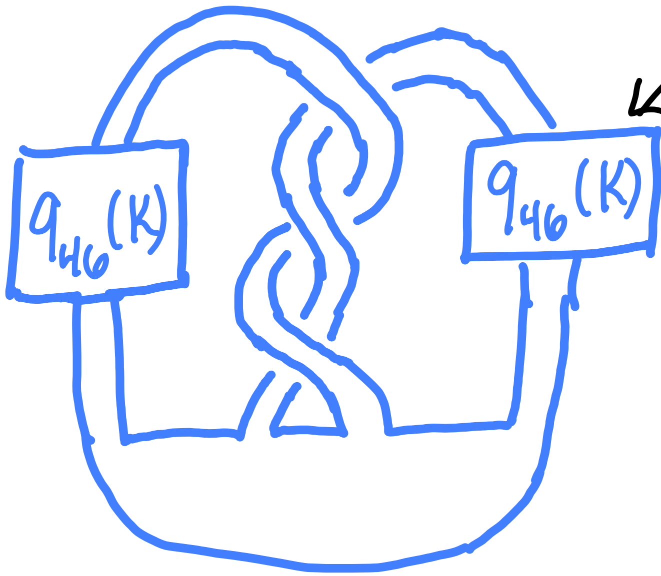
When is $9_{46}(K)$ Slice?

- If K is slice $\Rightarrow$ $9_{46}(K)$ is Slice

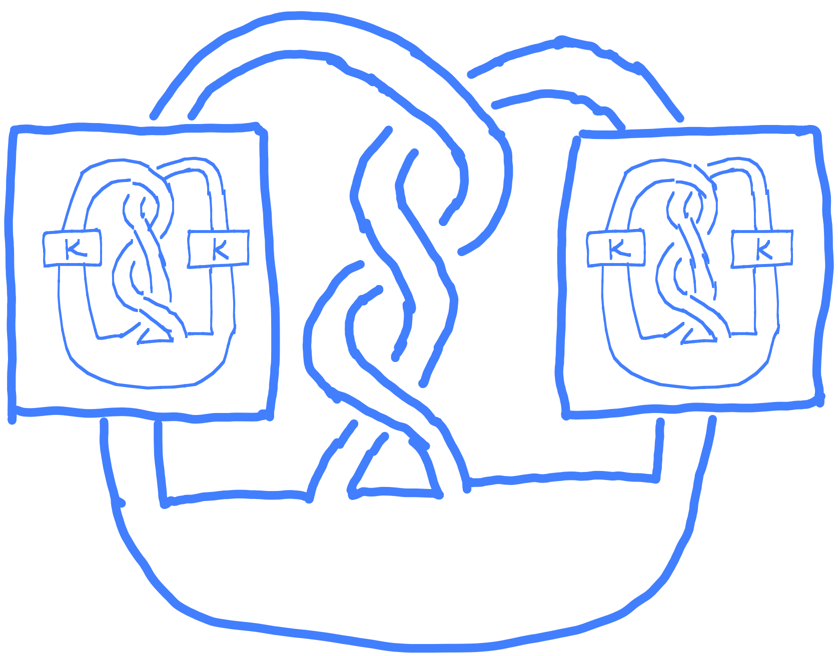
- [Casson-Gordon, Gilmer early 80's]: Casson-Gordon (70's) defined "higher-order signatures" of a knot. Gilmer used Casson-Gordon signatures to show that if $\underline{K}$ had certain ordinary signatures non-zero (K not algebraically Slice) then $9_{46}(K)$ is not Slice.

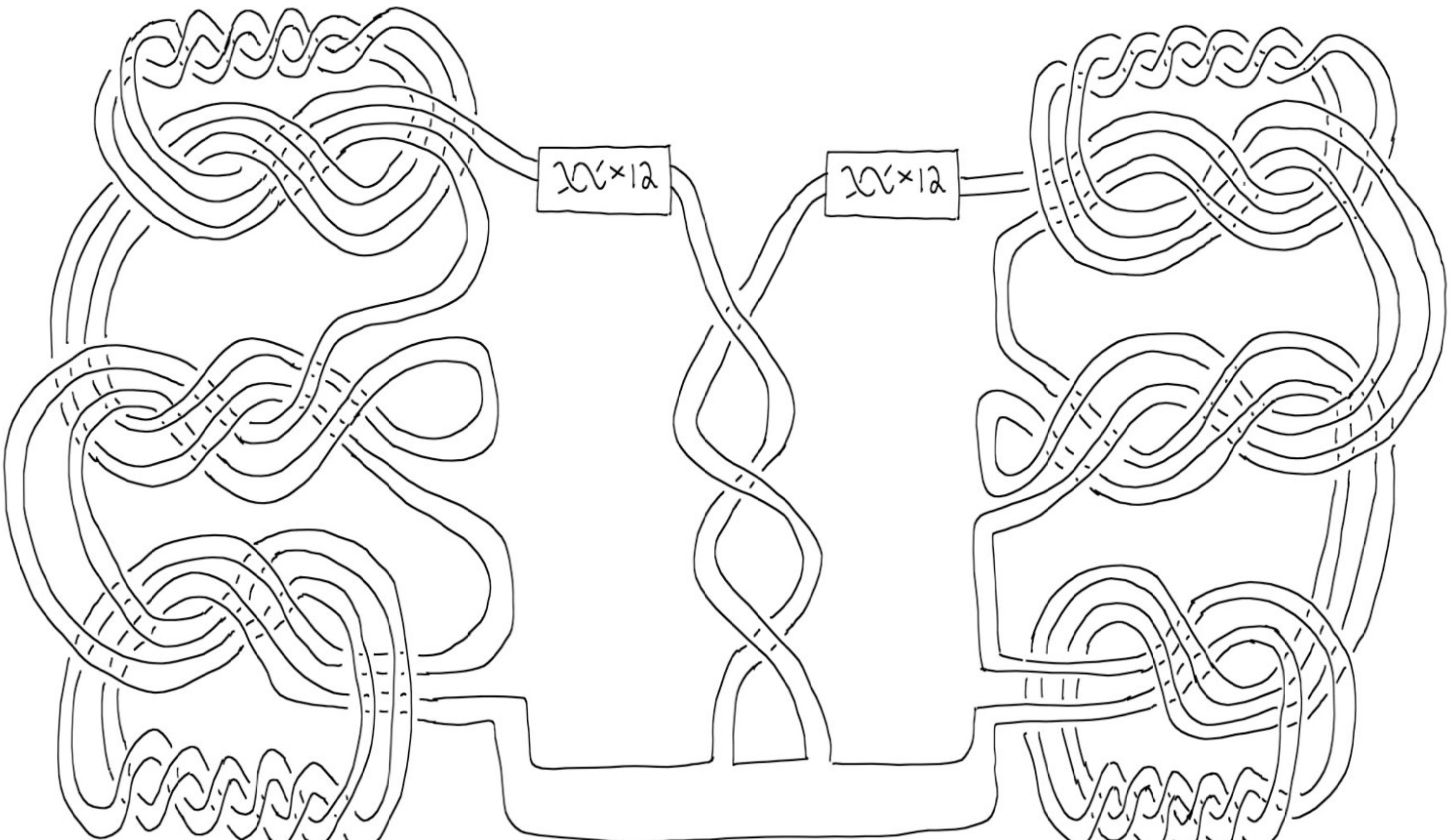
What if K is algebraically slice?

Ex:

$J_2(K) :=$ 

[tie in an algebraically slice knot, $9_{46}(K)$.]

$=$ 



$$J_2(\text{trefoil}) = 9_{46}(9_{46}(\text{trefoil}))$$

Def: For K a knot, let $G = \pi_1 M_K$ and define $\rho'(K) := \rho(M_K, G \to G/G'')$.

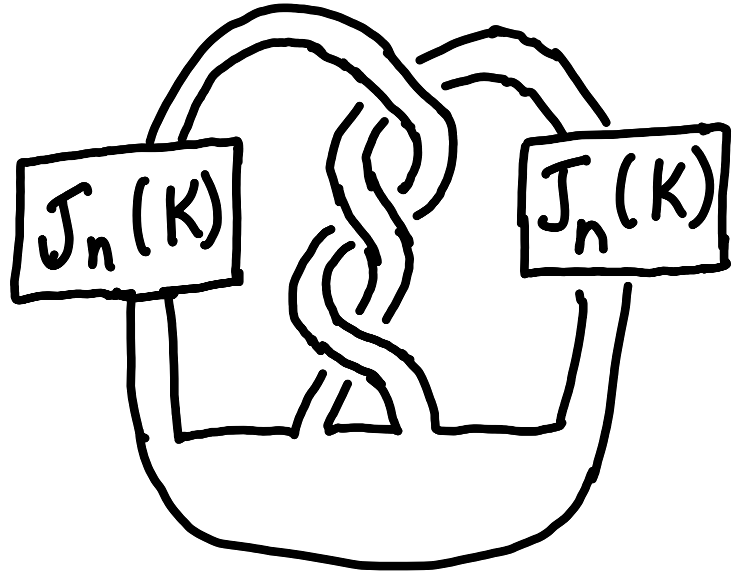
[Cheeger-Gromov ρ-invt]

$G'' = [[G,G],[G,G]]$

Thm (Cochran-H-Leidy-07) If $\rho_0(K) \notin \left\{0, -\frac{\rho'(9_{46})}{2}\right\}$ then $J_2(K)$ is not slice.

Cor: For all but a single integer m $J_2(\#_m \text{trefoils})$ is not slice $(m \neq 0)$

Define $J_0(K) := K$ and for $n \geq 0$,

$J_{n+1}(K) :=$ 

Thm (Cochran-H-Leidy-07) There is a constant C s.t. if $|\rho_0(K)| > C$ then for each n, $J_n(K)$ is not slice.

Cor: If $|m| > C$ then for each n, $J_n(\#_m \text{trefoil})$ is not slice.

In 1997, Cochran-Orr-Teichner defined the (n)-solvable filtration ($n \in \mathbb{N}/2$)

$$0 = \left\{\begin{matrix}\text{slice}\\ \text{knots}\end{matrix}\right\} \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

- $\mathcal{F}_0$ = Arf invariant zero knots
- $\mathcal{F}_{0.5}$ = Algebraically Slice knots
- $\mathcal{F}_{1.5} \subset$ knots with vanishing Casson-Gordon invariants

- $\mathcal{F}_n = \{(n)\text{-solvable knots}\}$

<u>Why is (n)-solvable filtration important?</u>

related to classification of 4-manifolds since obstructs knots bounding gropes as used in Freedman-Quinn.

Recall, if G is a group, the <u>derived series</u> of G is defined by: $G^{(0)} := G$ and

$$G^{(n+1)} := [G^{(n)}, G^{(n)}].$$

Def: K is (n)-solvable $(n \in \mathbb{N})$ if $M_K = \begin{smallmatrix}\text{0-surgery} \\ \text{on } K\end{smallmatrix}$ bounds a spin 4-mfld W [an (n)-solution]

(1) $i_* : H_1(M_K) \xrightarrow{\cong} H_1(W)$

(2) $H_2(W)$ has a basis $\{f_i, g_i\}$ of embedded surfaces (with triv. normal bundle) all disjoint except $f_i \cdot g_i = 1$ (geometrically)

(3) $\pi_1(f_i), \pi_1(g_i) \subset \pi_1(W)^{(n)}$

Def: K is (n.5)-solvable if K is (n)-solvable and (4) $\pi_1(f_i) \subset \pi_1(W)^{(n+1)}$.

Thm: $\begin{pmatrix} n=0 & \text{Milnor, Tristram} & \sim 67 \\ n=1 & \text{B. Jiang} & \sim 81 \\ n=2 & \text{Cochran-Orr Teichner} & \sim 02 \end{pmatrix}$

For $n \in \{0,1,2\}$, $\mathcal{F}_n / \mathcal{F}_{n.5}$ has infinite rank.

Thm (Cochran-Teichner ~02) For each $n \geq 3$, $\mathcal{F}_n / \mathcal{F}_{n.5}$ has rank ≥ 1.

<u>Thm</u> (Cochran-H-Leidy-07) For each $n \geq 0$, $\mathcal{F}_n / \mathcal{F}_{n.5}$ has <u>infinite</u> rank.

Moreover, our examples are linearly independent of the Cochran-Orr-Teichner examples.

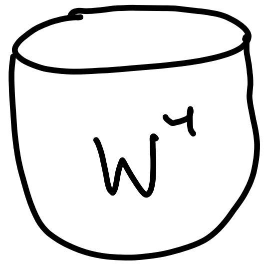
Key ideas in proof:

① For a 3-mfld M and $\phi : \pi_1 M \longrightarrow \Gamma$ (PTFA), define "higher-order Alex. module" $H_1(M;\mathbb{Z}\Gamma) = H_1(M_\Gamma)$ where M_Γ is regular Γ-cover of M wrt ϕ.

- Use (non-localized) higher-order Blanchfield form studied by C. Leidy:

$$\beta\ell_{(M,\phi)} : TH_1(M;\mathbb{Z}\Gamma) \times TH_1(M;\mathbb{Z}\Gamma) \longrightarrow \frac{\mathcal{K}(\Gamma)}{\mathbb{Z}\Gamma}$$

Suppose $M^3 = \partial W^4$ and ϕ extends to $\bar{\Phi} : \pi_1 W \longrightarrow \Gamma$.



$$M^3 \xrightarrow{\bar{\Phi}} \Gamma$$

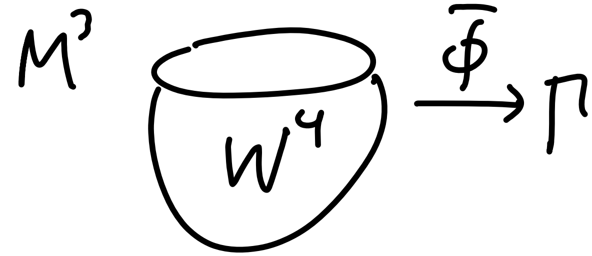
Let $P = \ker\left(H_1(M;\mathbb{Z}\Gamma) \longrightarrow H_1(W;\mathbb{Z}\Gamma)\right)$

<u>Lemma</u> (CHL) : $P \subset P^{\perp}$ w.r.t. $B\ell_{(M,\phi)}$.

$\therefore$ If $B\ell_{(M,\phi)}(x,y) \neq 0$ then either $x \notin P$ or $y \notin P$ (or both).

i.e. at least one of x or y <u>survives</u> under $H_1(M;\mathbb{Z}\Gamma) \longrightarrow H_1(W;\mathbb{Z}\Gamma)$!

② For (M,ϕ), $\exists$ Cheeger-Gromov ρ-invt
$$\rho(M,\phi) \in \mathbb{R}.$$



- If $(M,\phi) = \partial(W,\Phi)$ then

$$\boxed{\rho(M,\phi) = \sigma^{(2)}_{\Gamma}(W) - \sigma(W)}$$

(L^2-signature of Γ-equivariant intersection form on $H_2(W;\mathbb{Z}\Gamma)$)

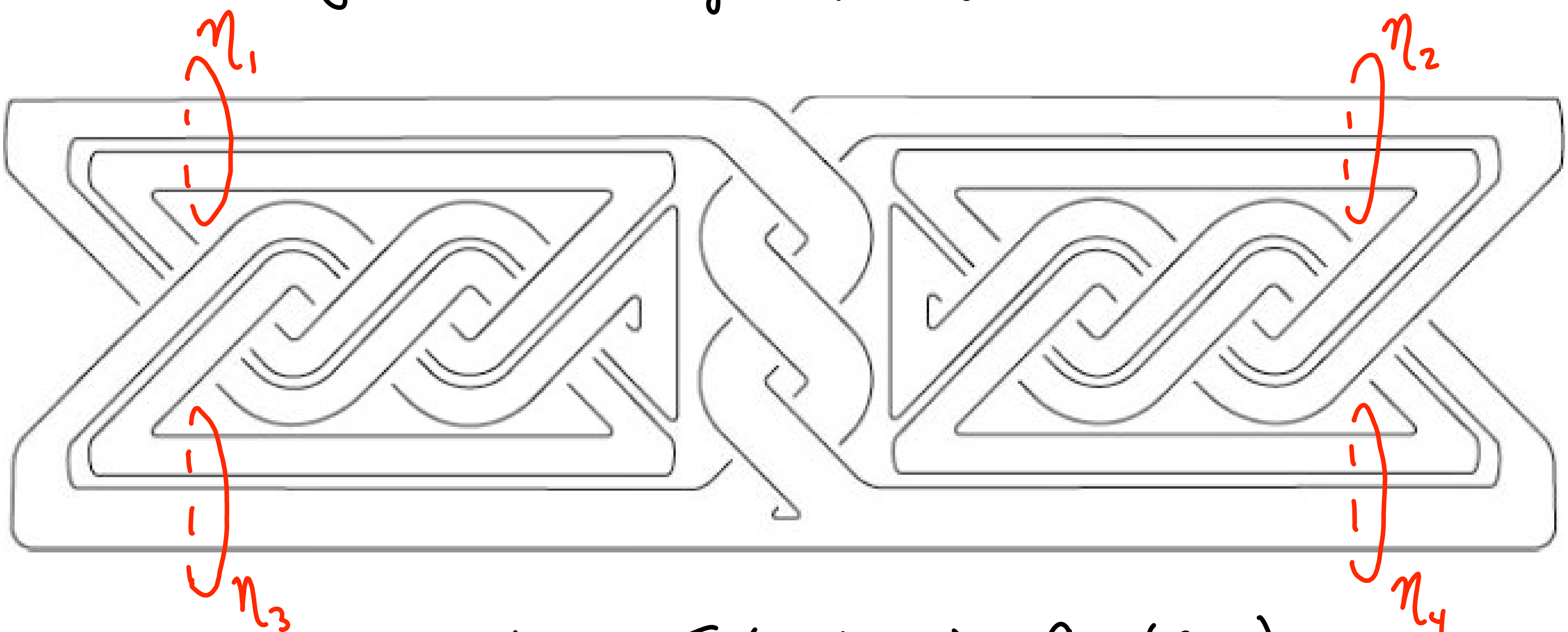
<u>Thm</u> (Cochran-Orr-Teichner): If $K \in \mathcal{F}_{n.5}$, and $\phi: \pi_1 M_K \longrightarrow \Gamma$ with $\Gamma^{(n+1)} = \{e\}$ s.t. ϕ extends over $(n.5)$-solution then $\rho(M_K,\phi) = 0$

Sketch of Proof: Assume $\rho'(9_{46}) \neq 0$.

- Choose $\{K_i\}_{i=1}^{\infty}$ s.t. $\{\rho_0(K_i)\}$ is a $\mathbb{Q}$-linearly independent set and subspace spanned by $\{\rho_0(K_i)\}$ does not contain $\rho'(9_{46})$.

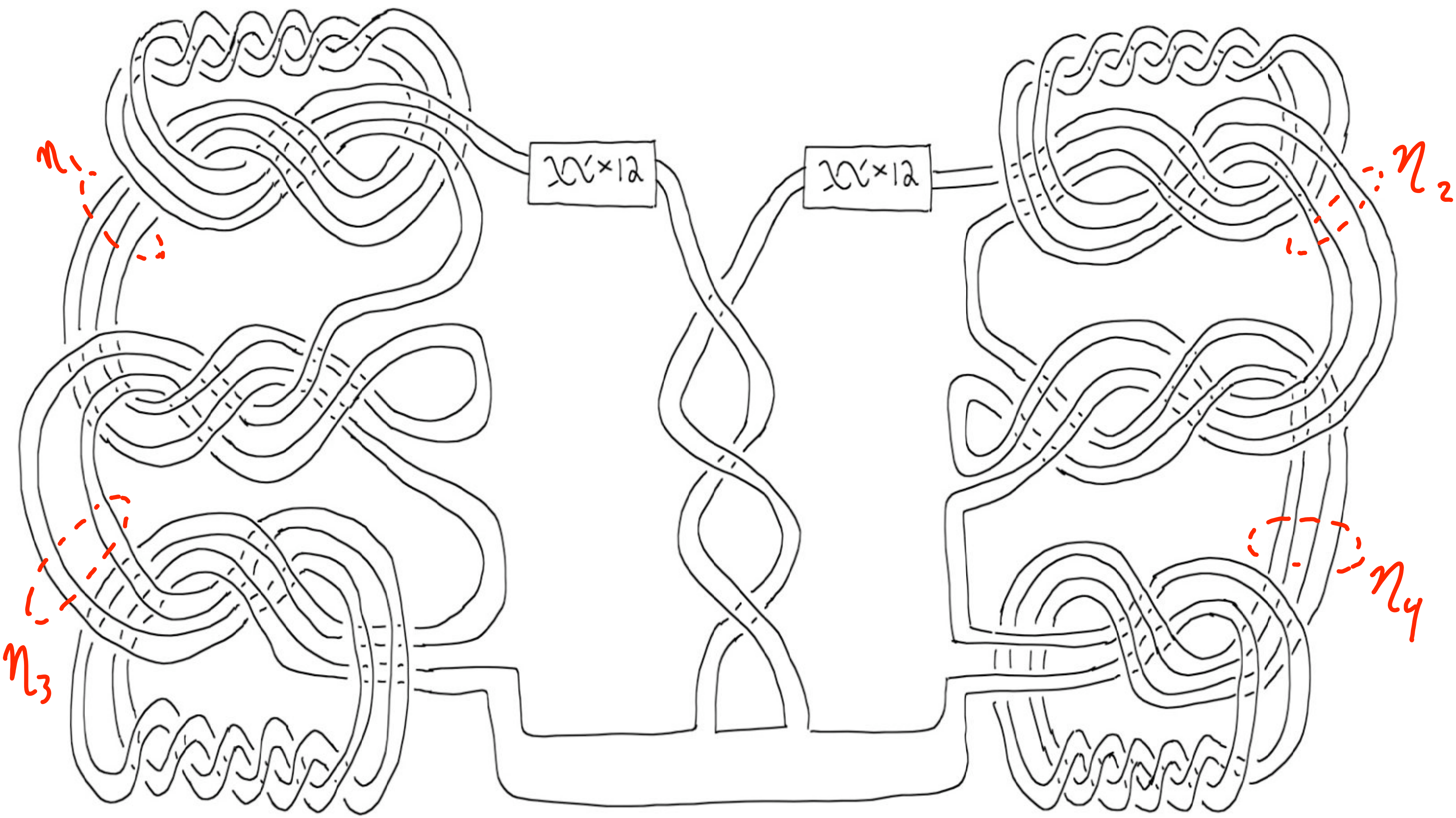
- Let $J_n^i = J_n(K_i)$, then J_n^i is in $\mathcal{F}_n$ since J_n^i can be obtained by "subtly" tying 2^n copies of K_i into the slice knot $J_n(\text{unknot})$ as follows:

To obtain $J_2(K)$, tie strings passing through curves η_j into knot K. Here $\eta_j \in G^{(2)}$ where $G = \pi_1(S^3 - J_2(\text{unknot}))$. $J_2(K)$ is obtained by "infecting" $J_2(u)$ along $\{\eta_j\}$ by knot K.



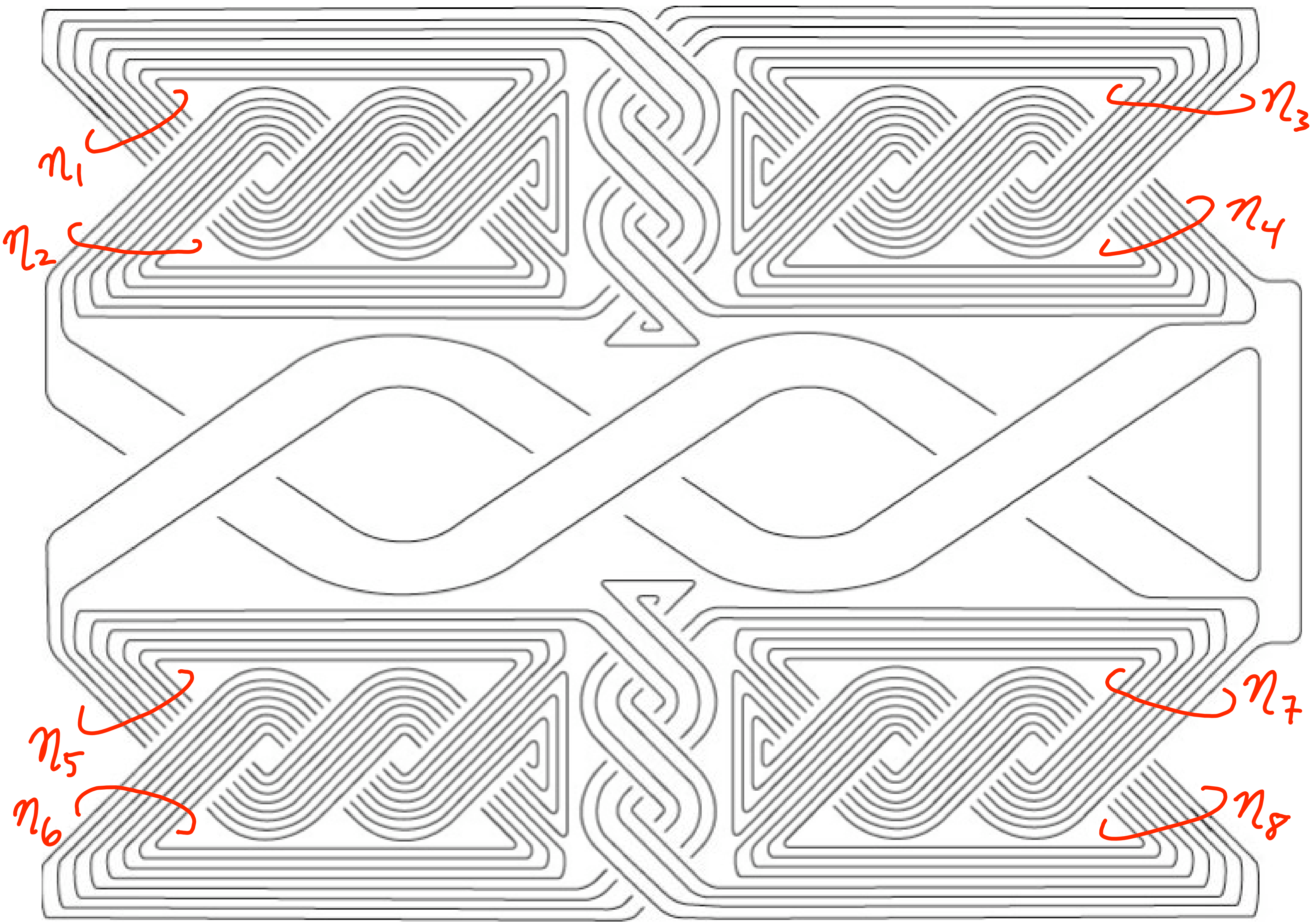
Slice knot $J_2(\text{unknot}) = 9_{46}(9_{46})$.

Example:

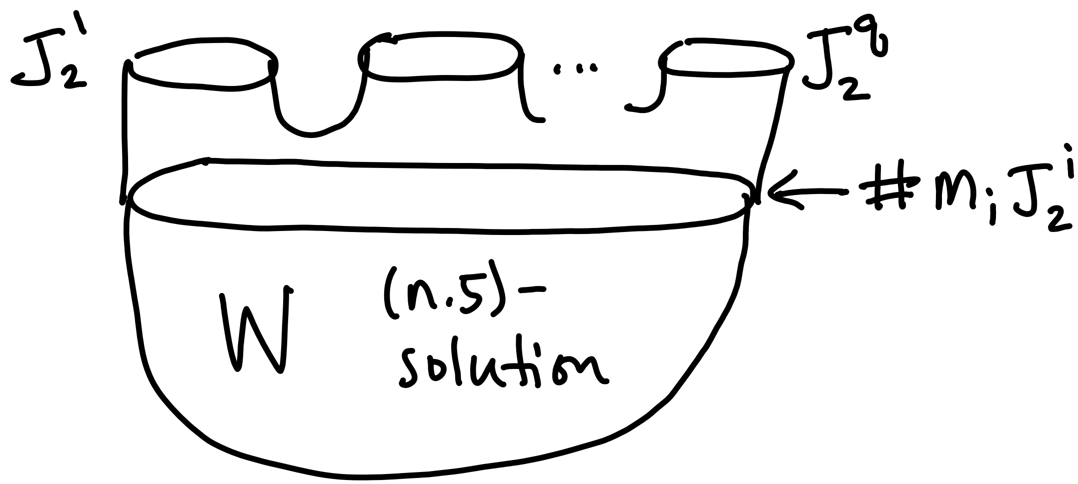


J_2(trefoil)

$J_3(K)$ = tie $\eta_1, \ldots, \eta_8$ into copies of K.

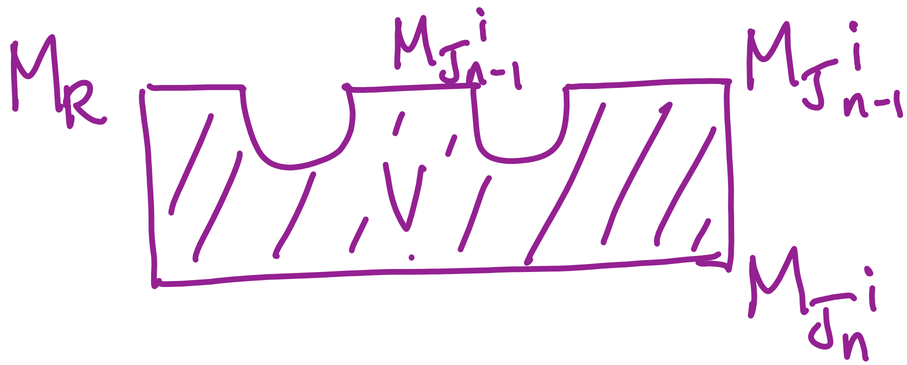


Claim: $\{J_n^i\}$ is $\mathbb{Q}$-linearly independent [only show for $n=2$]. Assume $\#m_iJ_2^i$ is (2.5)-solvable, with $m_1>0$. Get cobordism:



Since J_n^i = infect 9_{46} along curves η_1 and η_2 with knot J_{n-1}^i, $\exists$ cobordism:

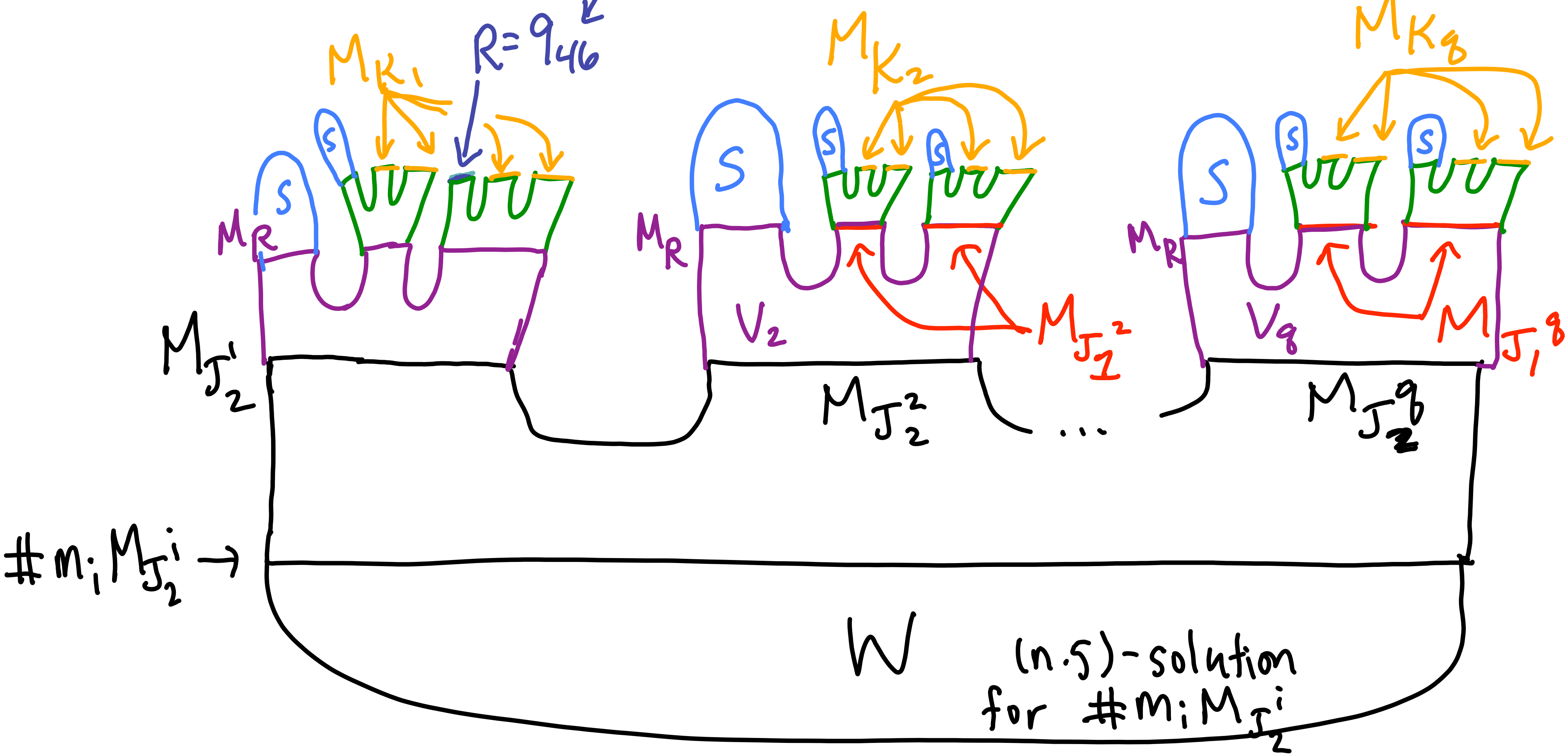
$\boxed{R = 9_{46}}$



[For any coeff system $\pi_1 V \xrightarrow{\psi} \Gamma$, $\sigma^{(2)}(V,\psi) = \sigma(V) = 0$.]

Glue on some copies of V along with S = Slice disk complement for 9_{46}.

Consider E^4:



$$\phi : \pi_1(E) \longrightarrow \pi / \pi^{(3)}$$

$$\pi_1(E) = \pi$$

S = Slice disk complement for R.

- Easy to see that for boundary components of E:

- For each copy of K_i, $\pi_1(M_{K_i}) \subset \pi^{(2)}$

$$\Rightarrow \phi(\pi_1(M_{K_i})) \subset \pi^{(2)}/\pi^{(3)} \text{ is } \underline{\underline{\mathbb{Z}}} \text{ or } \underline{\underline{0}}.$$

$$\therefore \rho(M_{K_i}, \phi) = \underline{\underline{\rho_0(K_i)}} \quad \text{or} \quad \underline{\underline{0}}$$

- $\phi(\pi_1(M_R)) \subset \pi^{(1)}/\pi^{(3)}$ so

$$\rho(M_R, \phi) \text{ is } \rho'(\underline{\underline{9_{46}}}) \quad \text{or} \quad \underline{\underline{0}}$$

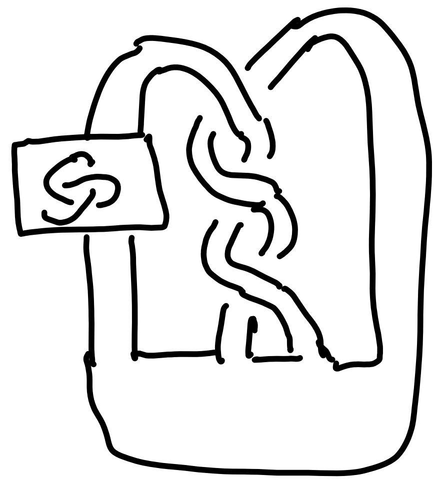
Since $\sigma^{(2)}(E,\phi) - \sigma(E) = 0$

$$\rho(M_{9_{46}}, \phi) + \sum_i \varepsilon_i \rho_0(K_i) = 0$$

where $\varepsilon_i \in \mathbb{Z}$.

- Harder to show that for at least one K_1, $\phi(\pi_1(M_{K_1})) \neq 0 \Rightarrow \varepsilon_1 \geq 1$. To do this, we use higher-order Blanchfield forms + $\rho'(9_{46}) \neq 0$ several times. This contradicts lin. independence of $\{\rho_0(K_i), \rho'(9_{46})\}$.

- If $\rho'(9_{46}) = 0$ we use "starting slice knot"

$R' =$ 

instead of 9_{46} since then $\rho'(R') \neq 0$.

$\blacksquare$