A non discrete metric on the group of topologically slice knots

Topology in dimensions 3, 3.5, and 4

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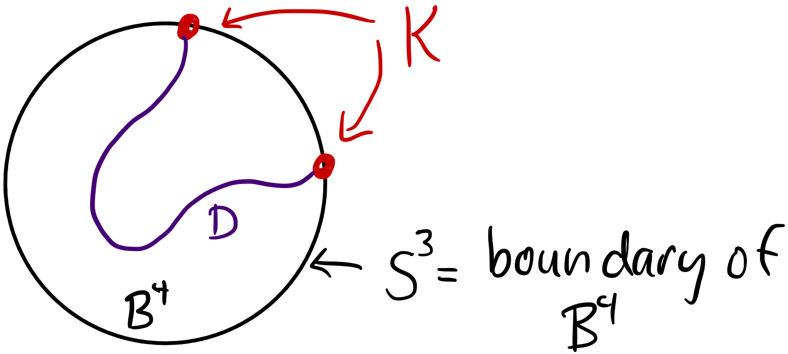
Def: A **Knot** is a smooth embedding

$$f : S^1 \hookrightarrow S^3.$$

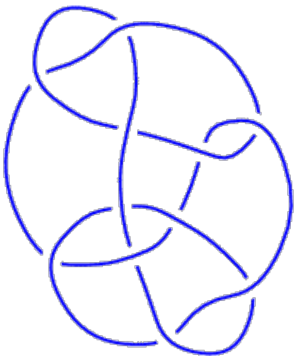
Remark: A knot K is the unknot $\Longleftrightarrow$ K bounds a disk in S^3.

Def: A knot $K \subseteq S^3 = \partial B^4$ is <u>slice</u> if $K = \partial D$ is the boundary of a smoothly embedded disk D in B^4.



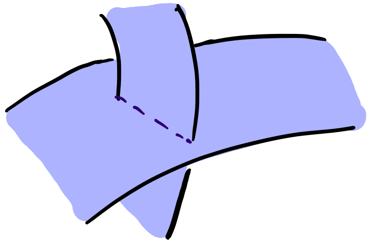
<u>Note</u>: There is no known algorithm to determine if a knot is slice!!!
(or ribbon)

Q. Is the Conway knot slice?



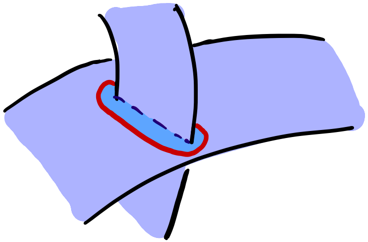
known to topologically slice but unknown if it is smoothly slice!

Ex: A knot is ribbon if it is the boundary of an immersed disk in S^3 with "ribbon singularities":

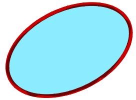


Observation: Every ribbon knot is slice.

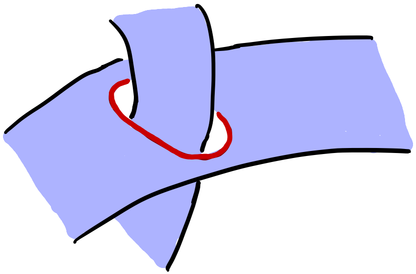
Pf: Take a small disk around singularity and push it into B^4.



push interior of

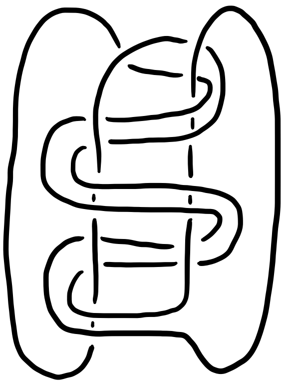


into interior of B^4

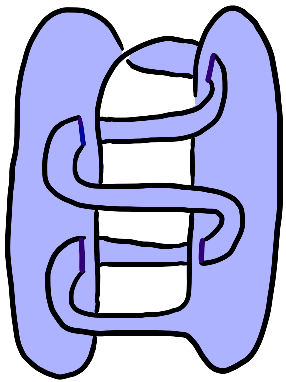


$\leftarrow$ (what is left in S^3)

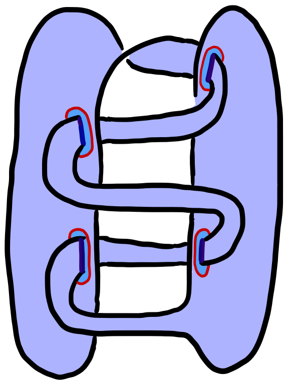
$K = 8_9$



8_9 is ribbon



Slice disk for 8_9



$\Rightarrow$ 8_9 is slice but does not bound an embedded disk in $\mathbb{R}^3$!

Slice-ribbon conjecture: Every (smoothly) slice knot is ribbon.

Note: This problem is extremely difficult since every ribbon knot has a slice disk that is not even is isotopic to any ribbon disk!

Ex: Let S be a smoothly embedded non-trivial 2-knot, $S^2 \hookrightarrow S^4$. Let U = unknot, and D = standard disk with $\partial D = U$. Push U into B^4 and then take a connected sum with S. Then $U = \partial \mathring{S}$ (S punctured) and $\pi_1(B^4 \setminus S^\circ) = \pi_1(S^4 \setminus S)$ is non-abelian since S is non-trivial.

Fact: If D is a ribbon disk for K then

$$\pi_1(S^3 \setminus K) \xrightarrow{i_*} \mathrel{\mkern-14mu}\rightarrow \pi_1(B^4 \setminus \underset{\substack{\uparrow \\ \text{pushed in}}}{D})$$

is surjective.

In example:

$$\underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}}}{\pi_1(S^3 \setminus \text{unknot})} \longrightarrow \underset{\substack{\uparrow \\ \text{non-abelian}}}{\pi_1(B^4 \setminus \mathring{S})}$$

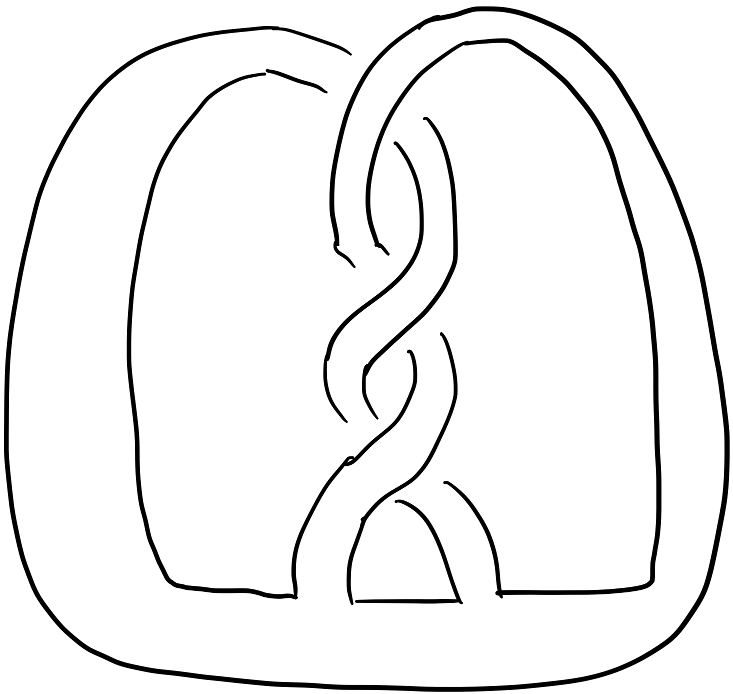
$\Rightarrow \mathring{S}$ is a slice disk that is not isotopic to <u>any</u> ribbon disk.

Another example (using "movie moves")

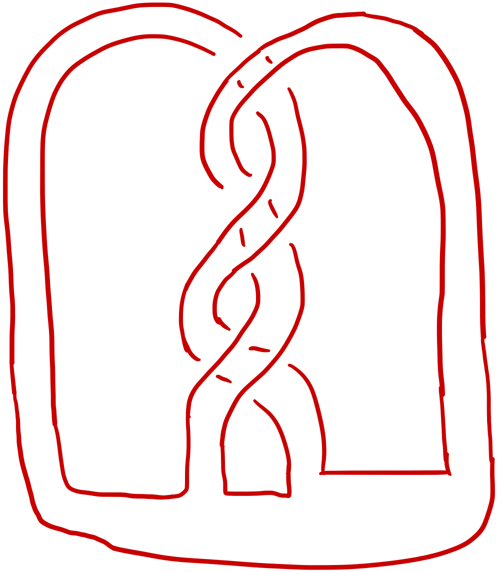
We can look at level set of a disk in $\mathbb{R}^4_+$.

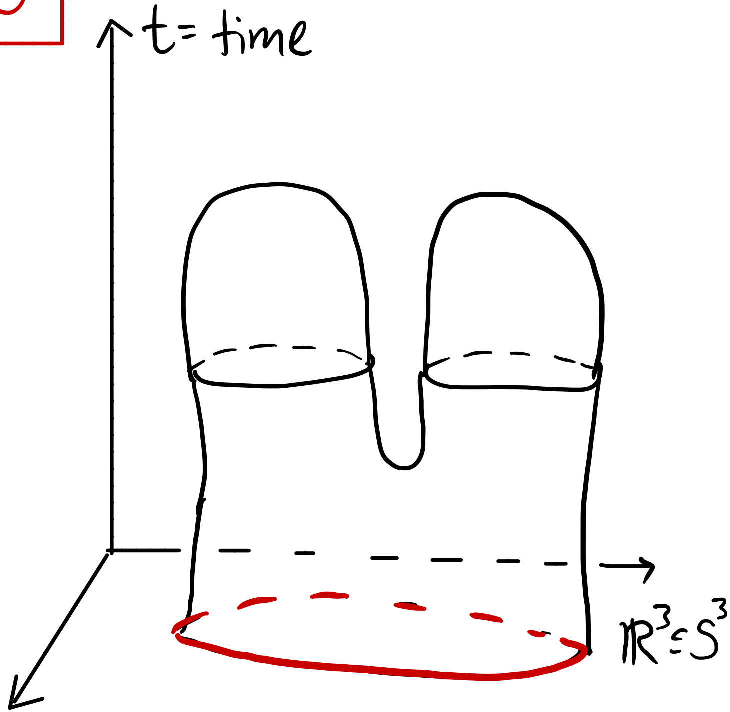
$\overset{\cong}{B^4}$

9_{46} is slice



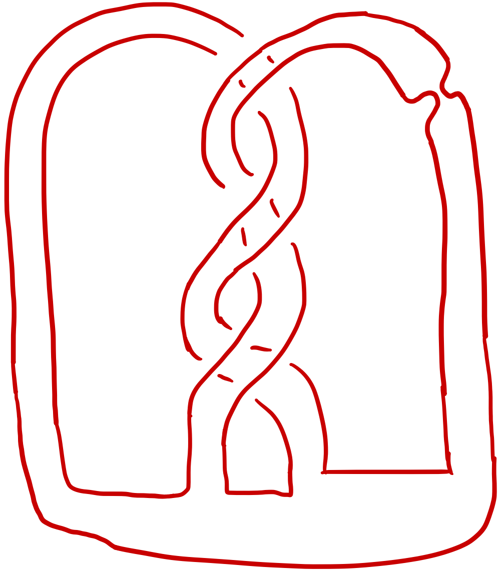
$\boxed{t = 0}$

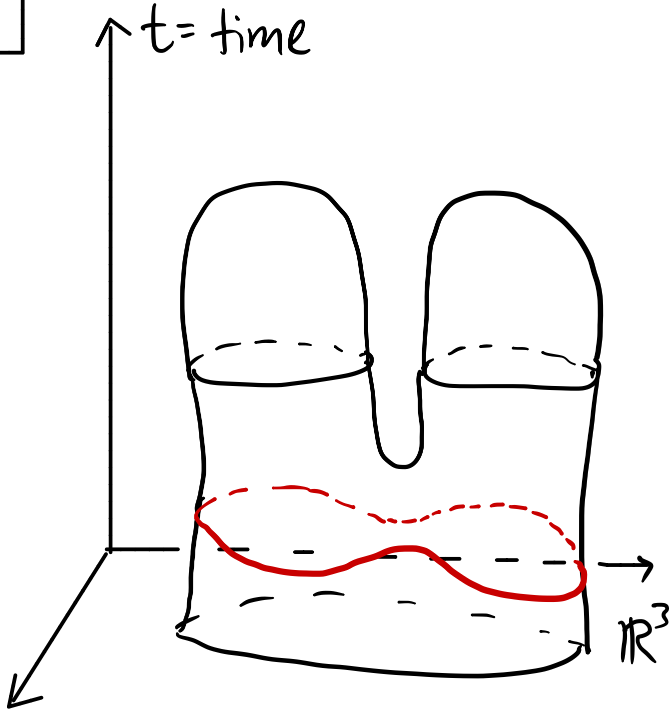




$$B^4 \supseteq \mathbb{R}^4_+ \quad (t \geq 0)$$

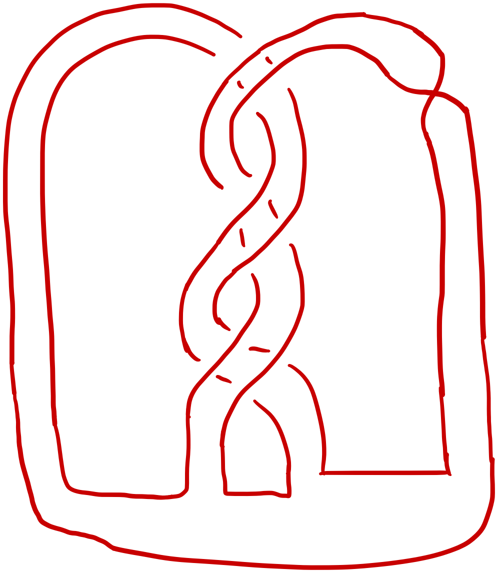
$t = 1/8$

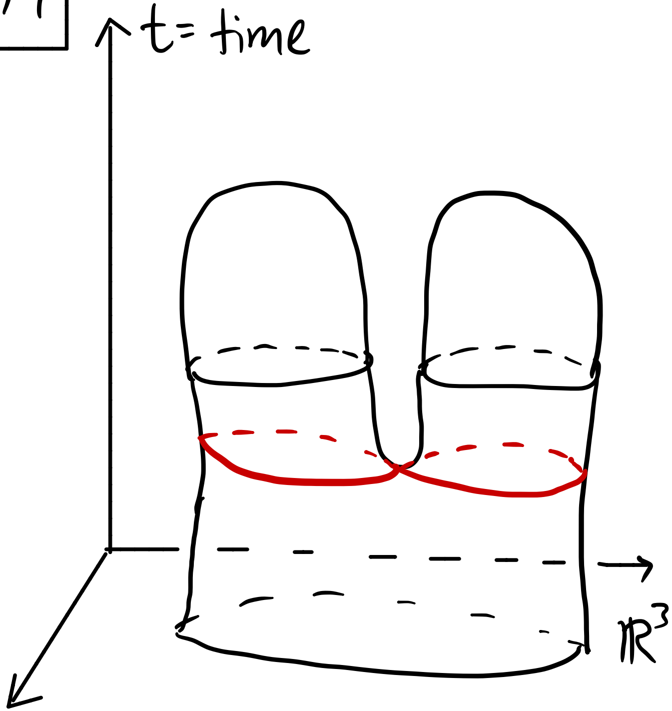




$\mathbb{R}^4_+ \quad (t \geq 0)$

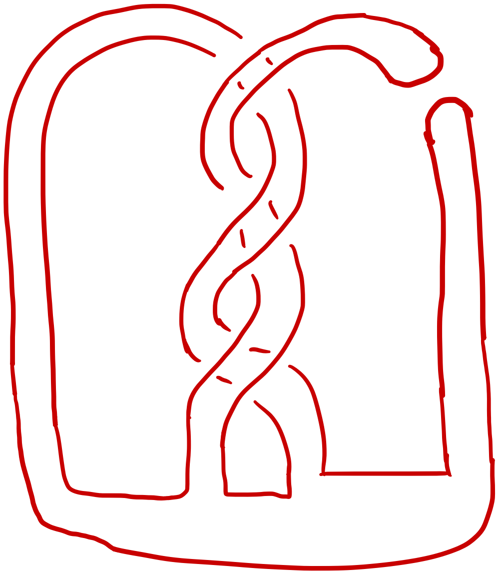
$\boxed{t = 1/4}$

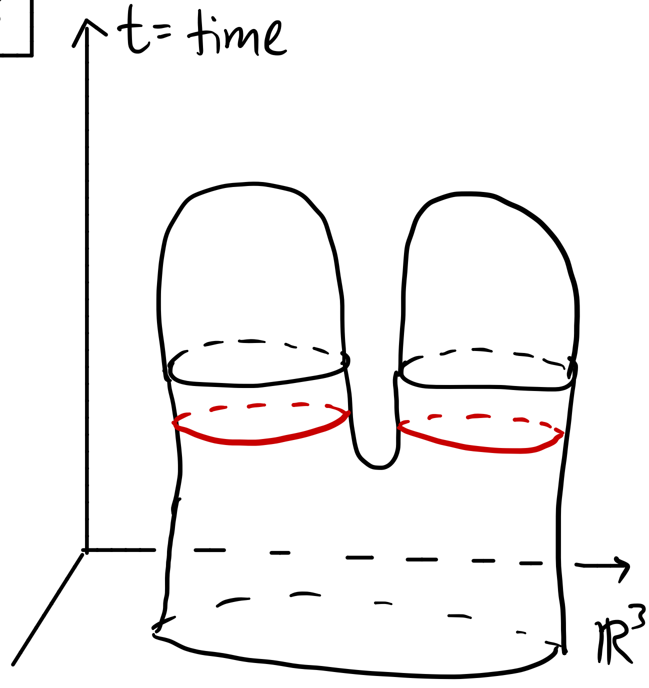




$\mathbb{R}^4_+ \quad (t \geq 0)$

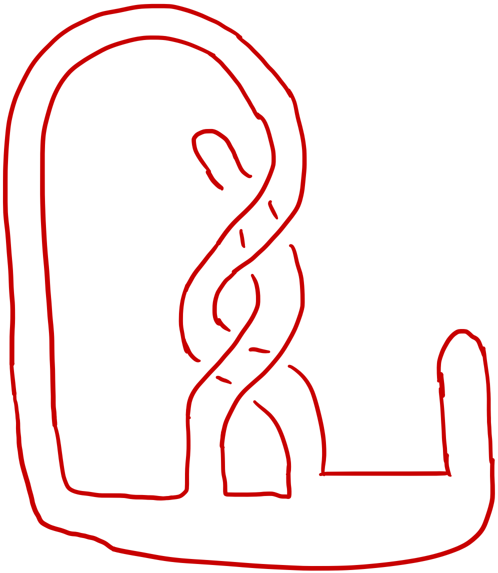
$\boxed{t = 3/8}$

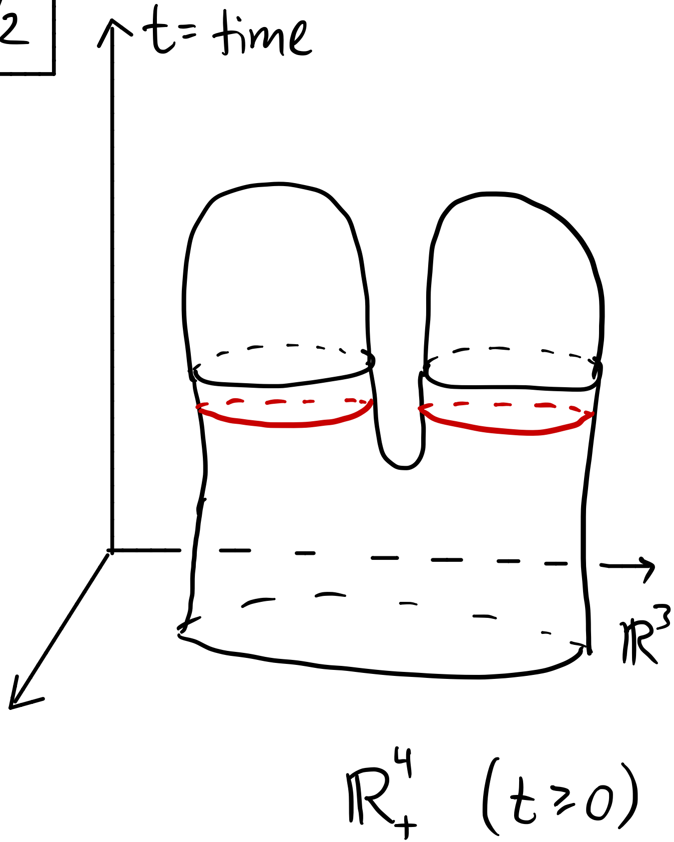




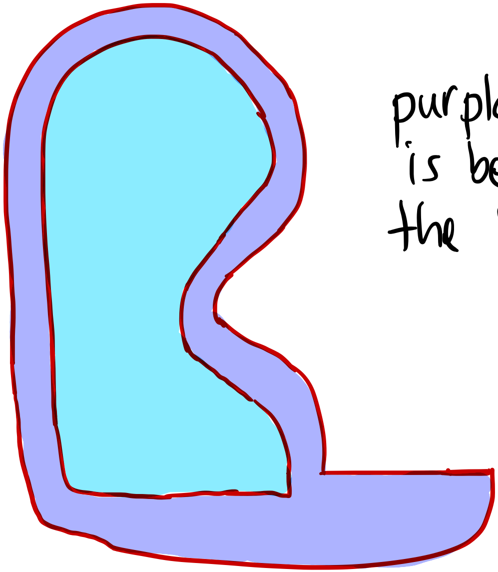
$\mathbb{R}^4_+ \quad (t \geq 0)$

$\boxed{t = 1/2}$

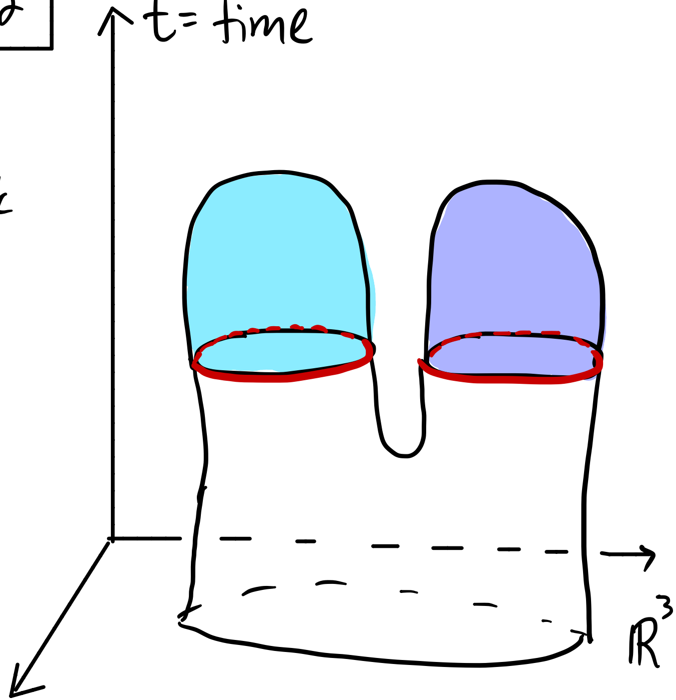




$\boxed{t \geq 1/2}$



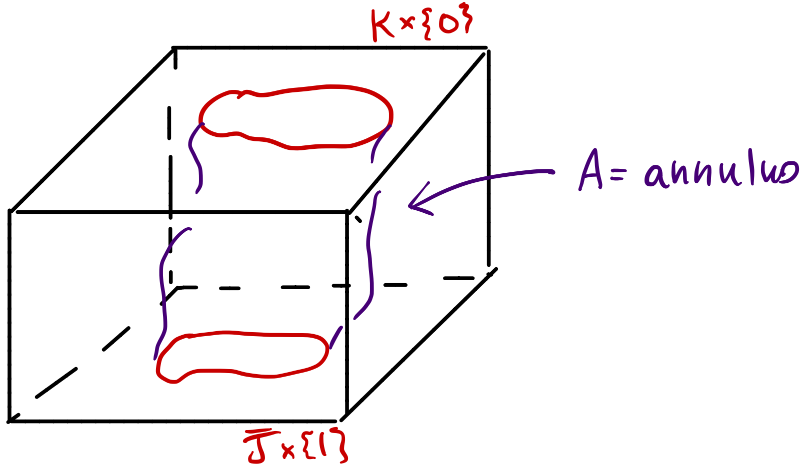
purple disk is behind the blue



$\mathbb{R}^4_+ \ (t \geq 0)$

We can put an 4-dimensional equivalence relation on knots.

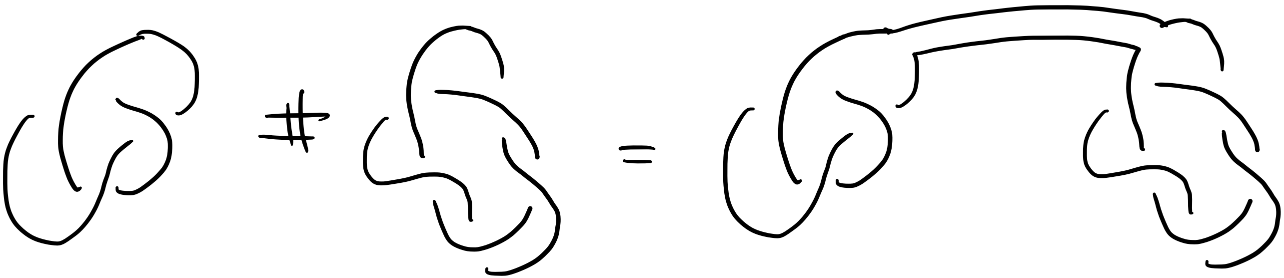
Def: Let K and J be knots in S^3. We say that K is concordant to J if $K \times \{0\}$ and $\bar{J} \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times [0,1]$.



Concordance group

Let $\mathcal{C} = \{\text{knots}\}/\sim$ $\quad K \sim J$ if they are concordant.

Then $\mathcal{C}$ is a group under connected sum.

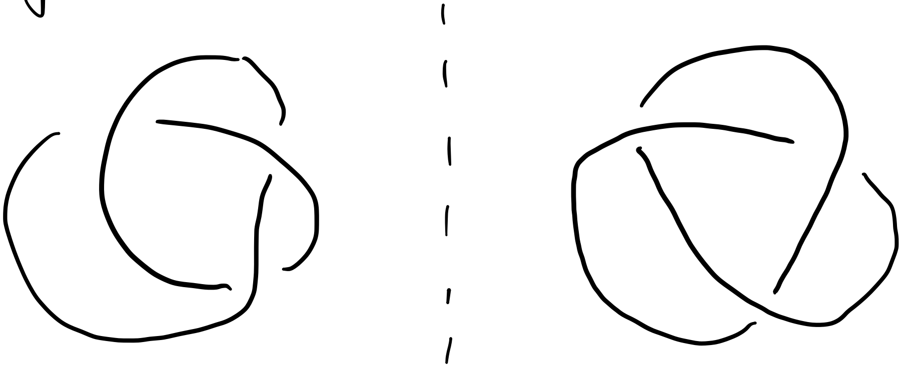


* need oriented knots.

$O = \{\text{Slice knots}\}$

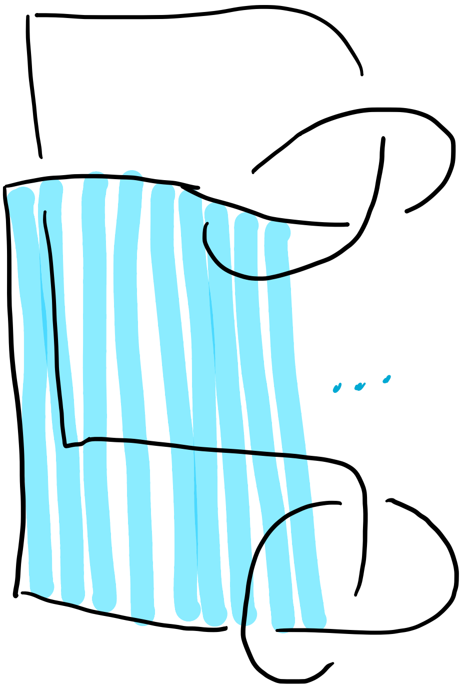
Inverse of K is $\bar{K}$.

For any K, $K \# \bar{K}$ is slice where

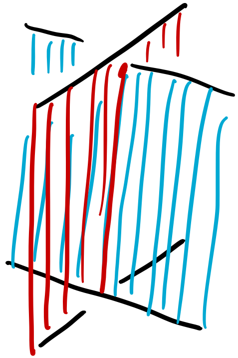


K $\quad$ $\bar{K}$ = mirror image

Pf that $K \# \overline{K}$ is slice (ribbon)



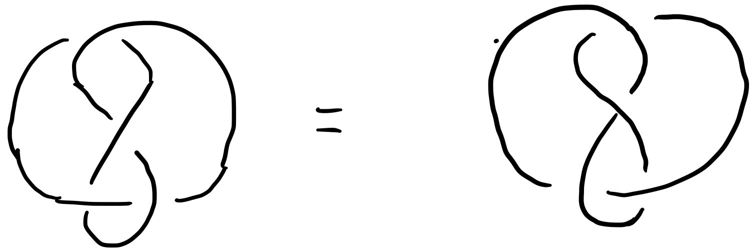
Make immersed disk by lines from K to $\overline{K}$.
The only self-intersection are ribbon singularities



$\mathcal{C}$ is a non finitely generated abelian group.

We don't know what $\mathcal{C}$ is.

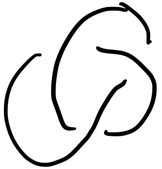
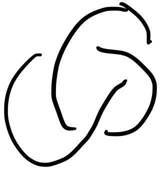
- $\mathcal{C}$ contains elements that are 2-torsion.



4_1 $\qquad\qquad$ $\overline{4_1}$

$\Rightarrow 24_1 = 0$ and 4_1 is not slice $(4_1 \neq 0)$

• $\mathcal{C}$ contains elements of infinite order

 # . . . . #  is never slice.

<u>Thm</u> (Levine '60's) $\exists$ surjective homomorphism

$$\mathcal{C} \overset{\pi}{\twoheadrightarrow} \mathcal{A} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$$

$\uparrow$

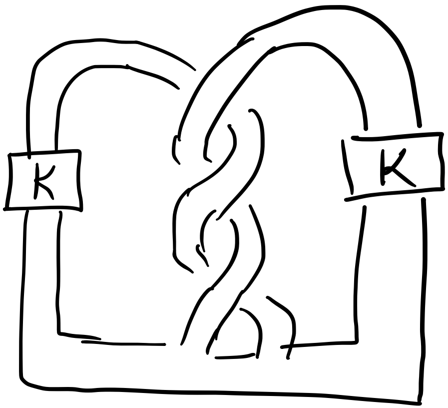
algebraic concordance group

(Witt group of Seifert matrices)

Q. Are all torsion elements, 2-torsion?

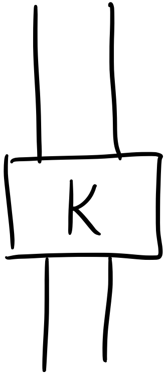
- $\ker(\pi)$ is non-trivial (in higher dimensions π is an $\cong$)

<u>Thm</u> (Casson-Gordon, Gilmer): $\ker \pi \neq 0$.



K = trefoil

Tie strand into K



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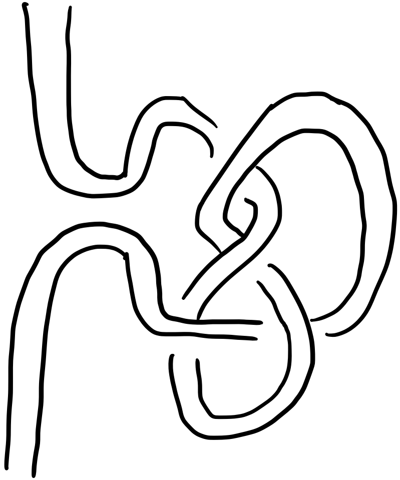
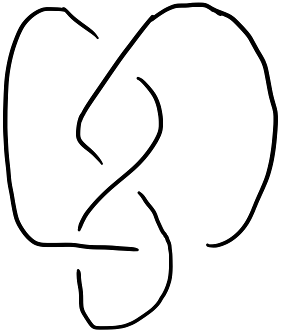


K =



n-solvable filtration

Cochran-Orr-Teichner defined filtration

$$\cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$

$K \in \mathcal{F}_0 \iff \mathrm{Arf}(K) = 0$ Arf invariant

$K \in \mathcal{F}_{0.5} \iff K \in \ker(\pi)$ Algebraically slice

$K \in \mathcal{F}_{1.5} \implies$ Casson-Gordon invariants vanish.

Def: If G is a group, define

$$G^{(0)} := G \quad \text{and}$$

$$G^{(n+1)} = [G^{(n)}, G^{(n)}],$$

$\{G^{(n)}\}$ is the derived series of G.

Def: A knot K is (n)-solvable (in $\mathcal{F}_n$) if there is a smooth 4-mfld W with $\partial W = S^3$ and smoothly embedded disk $\Delta \subseteq W$ with $\partial \Delta = K$ s.t.

(1) $H_1(S^3 \setminus K) \xrightarrow{\cong} H_1(W \setminus \Delta)$

(2) $[\Delta] = 0$ in $H_2(W, S^3)$

(3) $H_2(W) \cong \mathbb{Z}^{2g}$ has a basis represented by surfaces $\Sigma_i, d_i \subseteq W \setminus \Delta$ s.t. $\Sigma_i \cdot d_j = \delta_{ij}$, $d_i \cdot d_j = 0 = \Sigma_i \cdot \Sigma_j$.

(4) $\pi_1(\Sigma_i), \pi_1(d_i) \subseteq \pi_1(W \setminus \Delta)^{(n)}$

Thm (Cochran-H-Leidy): For each $n \geq 0$,

$\mathcal{F}_n / \mathcal{F}_{n.5}$ contains $\bigoplus_{\substack{p(t) \\ \text{symmetric} \\ \text{irreducible}}} \left(\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \right)$

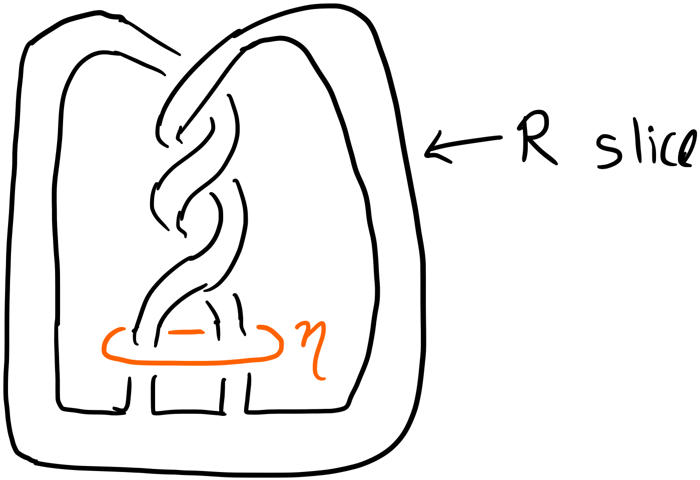
$n=0$: Milnor-Tristram, Levine (60's)

$n=1$: Jiang, Livingston (80's)

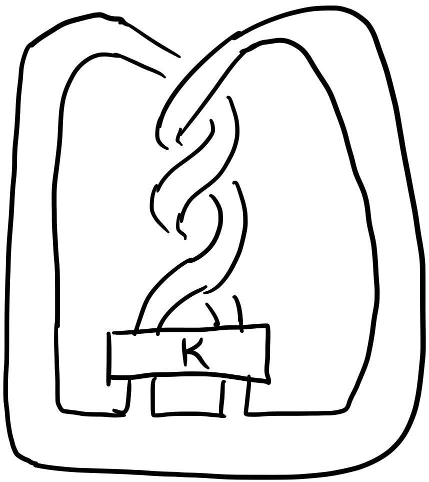
$n=2$: Cochran-Teichner ('02)

Operators on $\mathcal{C}$

Def: A pattern P is a slice knot R and unknot η disjoint from R, such that η bounds a surface disjoint from R.

$P =$ 

$P : \mathcal{C} \longrightarrow \mathcal{C}$ (not a homomorphism)

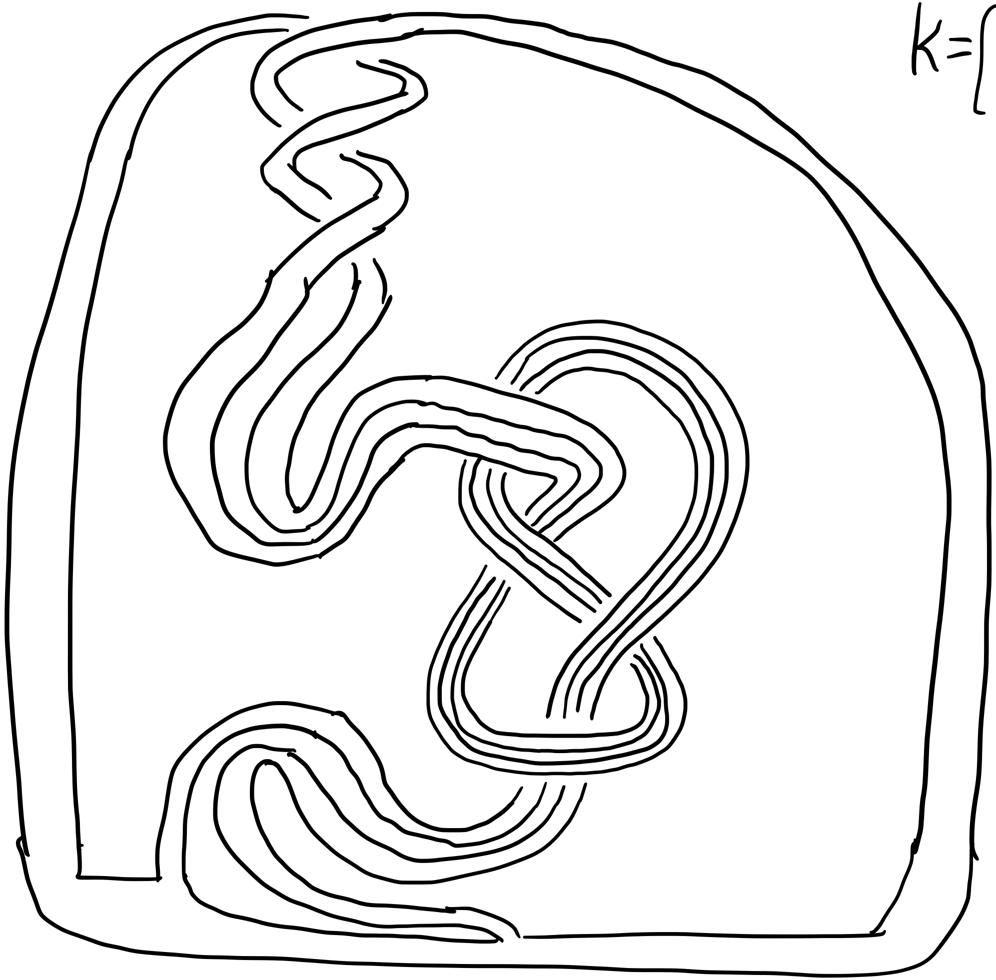
$P(K) =$ 

tie strands going through η into K

satellite operator.

Ex:

K = 

P(K) = 

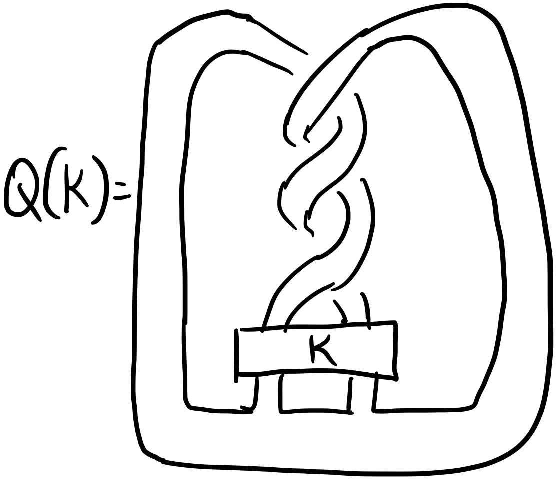
$$P : \mathcal{F}_n \longrightarrow \mathcal{F}_{n+1}.$$

Hence $\underbrace{P^n(K) = P(P(\cdots P(K))}_{n \text{ times}} \in \mathcal{F}_n$ for any K with Arf invariant zero.

Exs of $\mathbb{Z}^\infty$ and $\mathbb{Z}_2^\infty \in \mathcal{F}_n / \mathcal{F}_{n.5}$ are constructed this way!

Q. When is P injective?

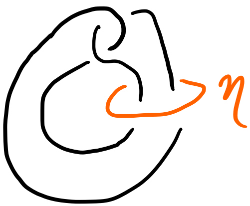
<u>Conjecture:</u> Q is injective

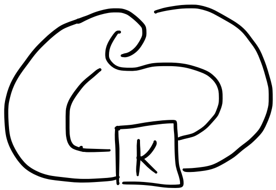


$Q(K)$ is slice $\Leftrightarrow$ K is slice

<u>Known:</u> There is a subgroup of $\mathcal{C}$ on which Q^n is injective for $\forall n$.

Ex: Whitehead double

$Wh =$ 

$Wh : K \longmapsto$ 

$Wh(K)$

Conjecture: $Wh(K)$ is smoothly slice $\Longleftrightarrow$ K is smoothly slice (i.e. Wh is weakly injective).

Remark: For any K, the Alexander polynomial of $Wh(K)$ is 1 $\Longrightarrow$ by Freedman, $Wh(K)$ is always topologically slice (bounds a topologically locally flat disk in B^4).

Satellite operators give a way to construct elements in $\mathcal{F}_n$. The difficult part is to show $P^n(K)$ is not slice (or even in $\mathcal{F}_{n.5}$)!!!

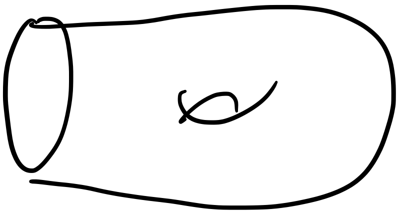
- Use invariants of knots such as L^2-signatures, d-invts and τ invariants from Heegaard Floer homology, etc.

We conjecture that $\mathcal{C}$ has the structure of a "fractal set".

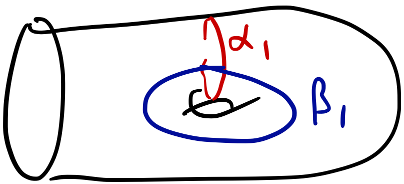
Would like some notion of distance where image(P^n) is getting smaller as $n \to \infty$.

Symmetric gropes

Def: A grope of height 1 is a compact oriented surface G_1 with $|\partial\Sigma| = 1$.

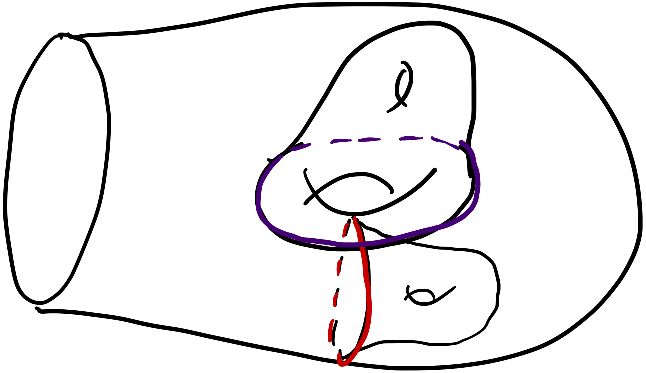


Let $\{\alpha_1, \dots, \alpha_{2g}\}$ be a standard symplectic basis of curves for $H_1(G_1)$ on G_1, $g = \text{genus}(G_1)$



A grope of height n+1 is obtained by attaching gropes of height n to $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$.

height 2 grope

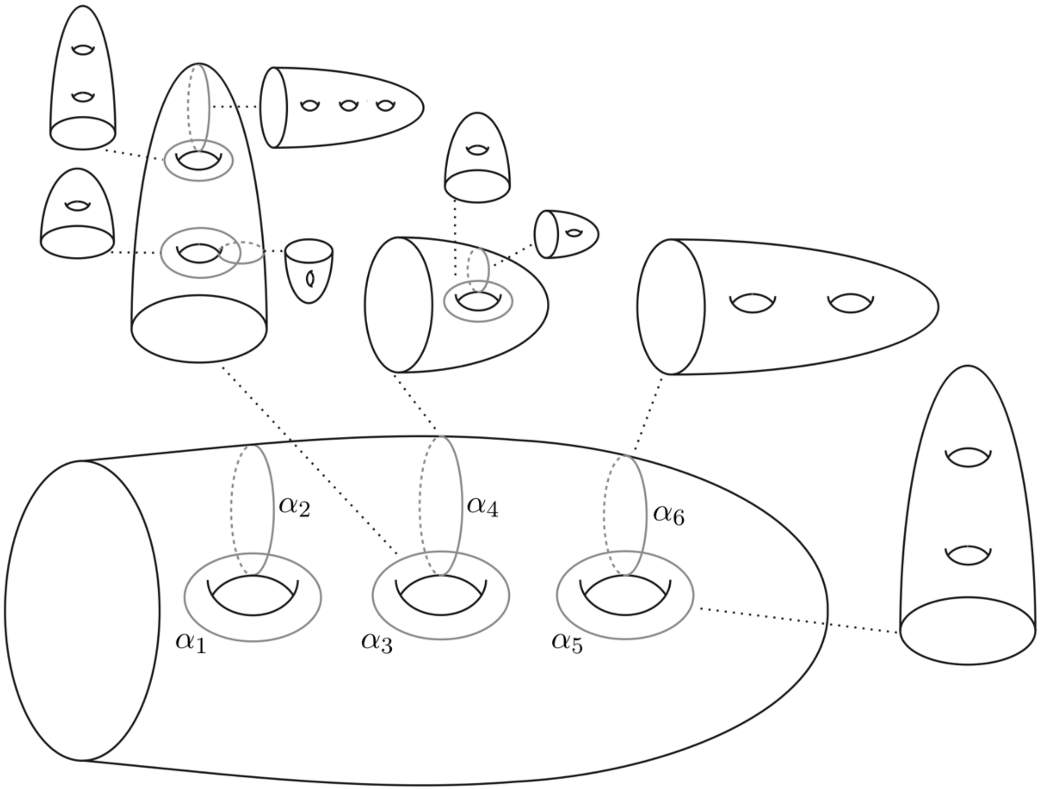


Def: A **branched symmetric grope** is defined as follows:

Let Σ_1 be a compact connected orientable surface of genus g, with a standard sympl. basis of curves $\{\alpha_1, \dots, \alpha_{2g}\}$ with α_{2i-1} dual to α_{2i}. Attach to each α_i, a grope of height m_i s.t. $m_{2i-1} = m_{2i}$, no subsurface of which is a disk.

Let $n_i = m_{2i}$.

$n_1 = m_1 = m_2 = 0$ $n_2 = m_3 = m_4 = 2$ $n_3 = m_5 = m_6 = 1$



Let Σ be a branched symmetric grope.

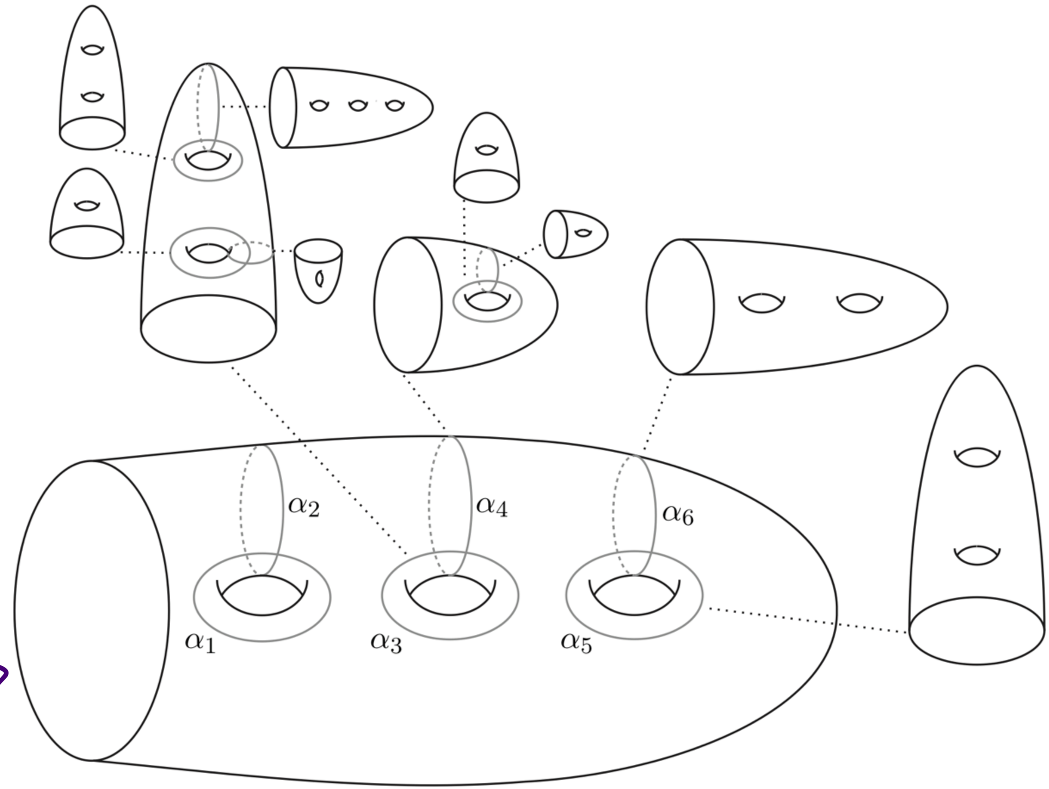
Define $g_1 = \text{genus}(\Sigma_1)$

g_2^i = sum of genera of first stage surfaces attached to $\alpha_{2i-1}, \alpha_{2i}$.

$\vdots$

$g_{n_i+1}^i$ = sum of genera of n_i stage surfaces attached to $\alpha_{2i-1}, \alpha_{2i}$.

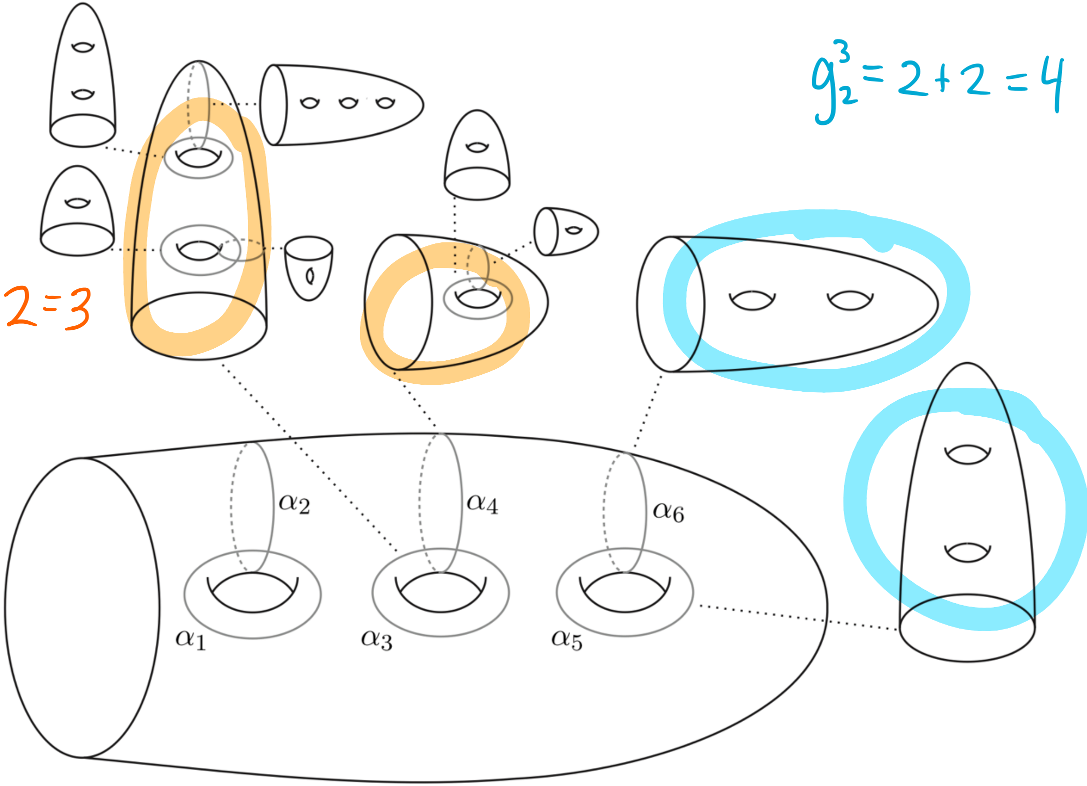
$g_1 = 3$



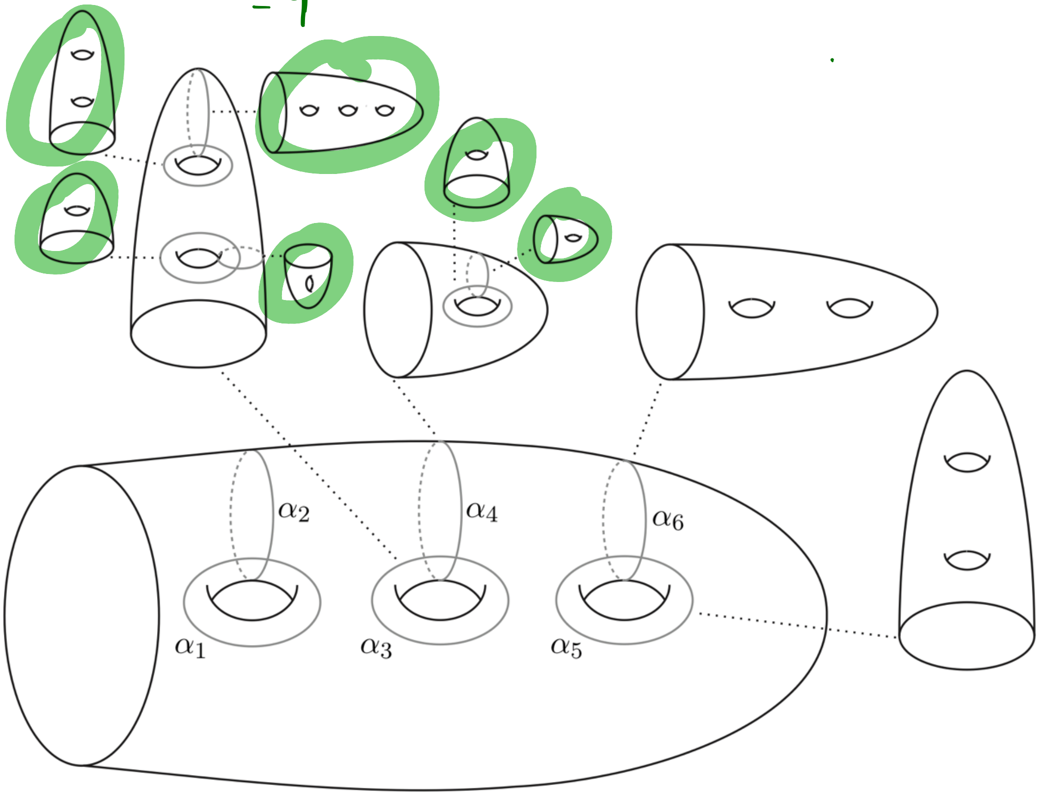
No g_2^1 since $n_1 = m_1 = m_2$.

$g_2^2 = 1 + 2 = 3$

$g_2^3 = 2 + 2 = 4$



$g_3^2 = 2+1+3+1+1+1$
$= 9$



Note: For each $1 \le i \le g$, and $2 \le k \le n_i + 1$,

$$g_k^i \ge 2 g_{k-1}^i$$

$$\Longrightarrow \quad \boxed{g_k^i \ge 2^{k-1}}$$

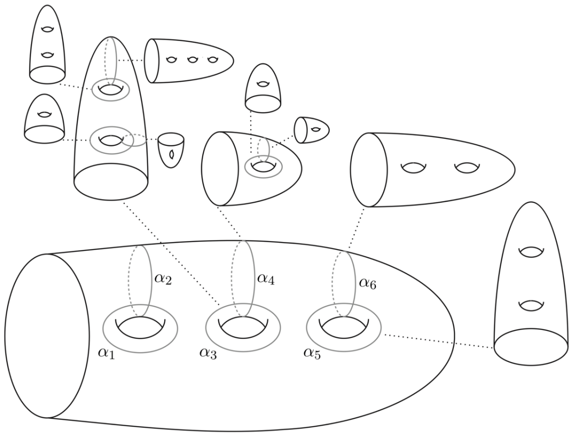
Let $q \geq 1$ be a real number and Σ a branched symmetric grope. Define

$$\|\Sigma\|_q := \sum_{i=1}^{g_1} \frac{1}{q^{n_i}} \left(1 - \sum_{k=2}^{n_i+1} \frac{1}{g_k^i}\right)$$

<u>Def</u>: If K, J are knots, define

$$d^q(K,J) := \inf \left\{ \|\Sigma\|_q \;\middle|\; \begin{array}{c} \Sigma \text{ is a branched symmetric grope} \\ \text{embedded in } S^3 \times I \text{ with boundary} \\ K \times \{0\} \text{ and } \bar{J} \times \{1\} \end{array} \right\}$$

<u>Note</u>: Any two knot cobound a surface.

$\Sigma =$ 

$$\|\Sigma\|_q = \left(\frac{1}{q^0}\cdot 1\right) + \frac{1}{q^2}\left(1 - \frac{1}{3} - \frac{1}{9}\right) + \frac{1}{q^1}\left(1 - \frac{1}{4}\right) = 1 + \frac{5}{9q^2} + \frac{3}{4q}$$

$i=1$ $\qquad$ $i=2$ $\qquad$ $i=3$

Ex: If K has bounds a genus 1 surface Σ and $\mathrm{Arf}(K) \neq 0$ then K cannot bound a (symmetric) height 2 grope. So

$$d^q(K, \text{unknot}) = g(\Sigma) = 1.$$

Ex: $\frac{1}{2q} \leq d^q($  ,  $) \leq \frac{27}{16q}$

Prop (Cochran-H-Powell): For $q \geq 1$, the function d^q determines a pseudo-metric on $\mathcal{C}$.

- Need to show $\|\Sigma\|_q \geq 0$ for any Σ.

Prop: If K does not bound a grope of height n then

$$d(K, \text{unknot}) \geq \frac{1}{(2q)^{n-2}}.$$

Thm (Cochran-Orr-Teichner): If K bounds a height n grope then $K \in \mathcal{F}_{n-2}$.

Prop (Cochran-H-Powell): If P is a pattern then $P: \mathcal{C} \longrightarrow \mathcal{C}$ is a contraction w.r.t. d^q for $q > gw(P)$:= # of times R goes through η.

<u>Thm</u> (Cochran-H-Powell): For any $q>1$ there exists uncountably many sequences of knots $\{K_i\}$ s.t.

$$d^q(K_i, \text{unknot}) > 0 \quad \forall i \quad \text{but}$$
$$d^q(K_i, \text{unknot}) \longrightarrow 0 \text{ as } i \to \infty.$$

Hence the topology on $(\mathcal{C}, d^q)$ is not discrete for $q>1$.

<u>Topologically slice knots</u>

Let $\mathcal{T} = \{\text{topologically slice knots}\} \leq \mathcal{C}$.

This is an interesting and subtle subgp of $\mathcal{C}$.

<u>Thm</u> (Hom): $\mathcal{T}$ has a $\mathbb{Z}^\infty$ summand.

(Endo showed that $\mathbb{Z}^\infty \leq \mathcal{C}$).

Remark 1: If $K \in T$, then $K \in \mathcal{F}_n$ $\forall n$.

Remark 2: If $K \in T$, then K bounds an arbitrarily long symmetric grope all of whose first stage genus is fixed.

Hence for $q > 1$,

$$d^q(K, \text{unknot}) = 0.$$

Remark 3: For $g=1$, the only way to get $d'(K, \text{unknot}) = 0$ would be for K to bound a arbitrarily long grope with each stage having genus 1.

$$\|\Sigma\|_1 = 1 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2^{n_1}} \longrightarrow 0 \quad \text{as } n_1 \to \infty$$

It is unknown if there is a non-slice knot that bounds such a rope!

Conj: $\exists\, K \in T$ s.t. $d'(K, \text{unknot}) > 0$.

More generally

<u>Conjecture</u>: $d'(K,J) > 0 \quad \forall \quad K \neq J.$

Bipolar Filtration

Cochran, Horn and I defined a filtration

$$\cdots \subseteq B_1 \subseteq B_0 \subseteq \mathcal{C}$$

that is a refinement of $\{\mathcal{F}_n\}$ and Gompf and Cochran's notion of positivity of knots.

<u>Thm</u> (Cochran-H-Horn): $\mathbb{Z}^\infty \subseteq B_n / B_{n+1}$ $\forall n$.

Unlike $\{\mathcal{F}_n\}$ this is an interesting filtration for topologically slice knots.

Def: A knot $K \in P_n$ (is n-positive) if $K = \partial\Delta$, Δ is a smoothly embedded disk in a smooth mfld W s.t.

- $\partial W = S^3$, $\pi_1(W) = 1$
- $[\Delta] = 0$ in $H_2(W, S^3)$.
- The int. form on $H_2(W)$ is pos. def.
- $H_2(W)$ has a basis repr. by surfaces $\{S_i\}$, disjointly embedded in $W \setminus \Delta$ s.t.

$$\pi_1(S_i) \leq \pi_1(W - \Delta)^{(n)}.$$

* Can also define when $K \in N_n$ (n-negative).

$B_n := N_n \cap P_n \quad \forall\, n.$ (n bipolar knots)

<u>Prop</u> (CHH): $B_n \subseteq \mathcal{Y}_n$.

<u>Prop</u> (CHH): If $K \in B_0 \implies$

$$\tau(K) = s(K) = \delta(K) = d(+1 \text{ surj. on } K) = \varepsilon(K) = 0$$

<u>Prop</u> (CHH): If $K \in B_1$ and $Y = p^r$-fold branched cyclic cover of K, $s_0 \in \mathrm{Spin}^c(Y)$ corresponding to a spin structure on $Y \implies$

$$d(Y, s_0) = 0.$$

Def: $T_n := T \cap B_n$.

Thm (Cochran-H-Horn): $T_0/T_1 \neq 0$

Thm (Cochran-Horn): $T_1/T_2 \neq 0$.

Thm (Cha-Kim) $T_n/T_{n+1} \neq 0 \quad \forall n$.

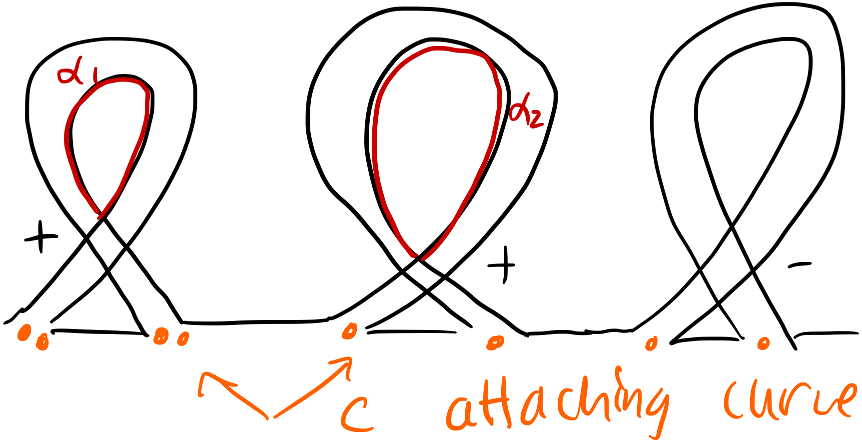
Pf uses L^2-ρ invts and d-invts of p-fold branched covers for an infinite # of p.

Tower metrics

(Cochran, H, Powell, Ray)

For $q \geq 1$, can define a metric $d_q^B : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ based on kinky disks and gropes.

"Def" A generalized positive plumbed handle (GPH$^+$)

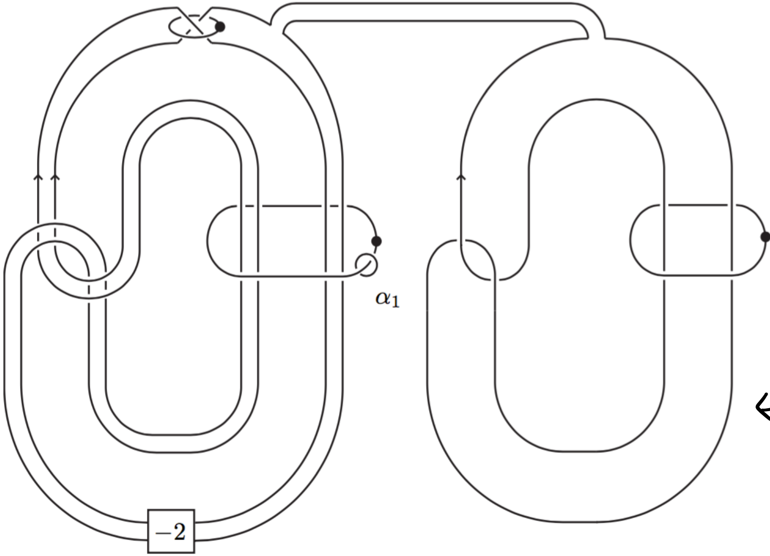


attach gropes of height n_i to α_i

$g_i^+ = \# +$

$g_i^- = \# -$

Ex:



attach grope to α $\quad g_1^+ = 1 \quad g_1^- = 1$

$am_1 = gm_1 = 2 \qquad am_2 = gm_2 = 1$

Def: A positive tower for K is an embedding of a GPH^+ into B^4 with $C \to K$. (C has 0-framing)

If K, J bound a positive tower, Σ,

$$d^+_{\Sigma,q}(K,J) = \sum_{i=1}^{g_1} \frac{m_i}{g^{n_i}} \left(1 - \frac{1}{2} \sum_{k=2}^{n_i+1} \frac{1}{g^i_k}\right)$$

grope

$$g_1 = g_1^+ + g_1^-$$

$$m_i = \frac{|\text{alg mult}_i| + \text{geom. mult}_i}{2}$$

$$d_q^+(K,J) = \min\{d^+_{\Sigma,q}(K,J) \mid \Sigma\}.$$

can define d_q^- sim.

$$d_q^B(K,J) = \max\{d_q^+(K,J), d_q^-(K,J)\}.$$

<u>Prop</u> (CHPR): If $\|K\|_q^+ < \frac{1}{(2q)^n} \Rightarrow K \in P_n$

<u>Cor</u>: $\exists$ topologically slices knots K_i with $d_q^B(K_i, \text{unknot}) > 0$.

<u>Conjecture</u>: (1) There are topologically slice knots K_i s.t. $d_a^B(K_i, \text{unknot}) \xrightarrow[i\to\infty]{} 0$ and $d_q^B(K_i, \text{unknot}) \neq 0$ for all $q \geq 1$.

(2) d_q^B is a metric (not just pseudo-metric) $\forall q \geq 1$.