A non discrete metric on the group of topologically slice knots.

Topology in dimensions 3, 3.5, and 4

Shelley Harvey
Rice University

w/ Tim Cochran, Mark Powell, & Arunima Ray.
**Def:** A **knot** is a smooth embedding
\[ f : S^1 \hookrightarrow S^3. \]

**Remark:** A knot \( K \) is the unknot \( \iff \)
\( K \) bounds a disk in \( S^3 \).
Definition: A knot $K \subseteq S^3 = \partial B^4$ is slice if $K = \partial D$ is the boundary of a smoothly embedded disk $D$ in $B^4$. 

$S^3 = \text{boundary of } B^4$
Note: There is no known algorithm to determine if a knot is slice!!

Q. Is the Conway knot slice?

known to topologically slice but unknown if it is smoothly slice!
Ex: A knot is ribbon if it is the boundary of an immersed disk in $S^3$ with “ribbon singularities”:

---

Observation: Every ribbon knot is slice.
Pf: Take a small disk around singularity and push it into $B^y$. 
push interior of $B^4$

← (what is left in $S^3$)
$K = 8q$

$8q$ is ribbon

Slice disk for $8q$

$\Rightarrow 8q$ is slice but does not bound an embedded disk in $\mathbb{R}^3$!
Slice-ribbon conjecture: Every (smoothly) slice knot is ribbon.

Note: This problem is extremely difficult since every ribbon knot has a slice disk that is not even isotopic to any ribbon disk!
Ex: Let $S$ be a smoothly embedded non-trivial 2-knot, $S^2 \hookrightarrow S^4$. Let $U=\text{unknot}$, and $D=\text{standard disk with } \partial D=U$. Push $U$ into $B^4$ and then take a connected sum with $S$. Then $U=\Sigma S$ ($S$ punctured) and $\pi_1(B^4\setminus S^0)=\pi_1(S^4\setminus S)$ is non-abelian since $S$ is non-trivial.
Fact: If $D$ is a ribbon disk for $K$ then

$$
\pi_1(S^3 \setminus K) \xrightarrow{i_*} \pi_1(B^n \setminus D)
$$

is surjective.

In example:

$$
\pi_1(S^3 \setminus \text{unknot}) \xrightarrow{=} \pi_1(B^n \setminus \hat{S})
$$

non-abelian

$\Rightarrow \hat{S}$ is a slice disk that is not isotopic to any ribbon disk.
Another example (using "movie moves")

We can look at level set of a disk in $\mathbb{R}_+^n$.

9.46 is slice
$t = 0$

$B^y = \mathbb{R}^y_+ (t \geq 0)$
$t = \frac{1}{8}$

Graph showing a time evolution process in a 4-dimensional spacetime ($\mathbb{R}^4$) for $t \geq 0$. The diagram illustrates a complex structure evolving over time, with different regions highlighted to indicate changes or states at various times.
\[ t = \frac{3}{8} \]

- Diagram showing a time \( t = 0 \) and \( t = \text{time} \) in a 3-dimensional space \( \mathbb{R}^3 \).
\[ t = \frac{1}{2} \]

Diagram of a 4-dimensional space-time with a 3-dimensional slice at \( t = 0 \) and a time axis labeled \( t = \) time.
purple disk is behind the blue
We can put an 4-dimensional equivalence relation on knots.

**Def:** Let $K$ and $J$ be knots in $S^3$. We say that $K$ is concordant to $J$ if $K \times \{0\}$ and $J \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times [0,1]$. 

![Diagram of annulus](#)
Concordance group

Let $C = \{ \text{knots} \}/\sim$ if they are concordant.

Then $C$ is a group under connected sum.

* need oriented knots.
\( O = \{ \text{slice knots}\} \)

Inverse of \( K \) is \( \overline{K} \).

For any \( K \), \( K \# \overline{K} \) is slice where

\[ K \]

\[ \overline{K} = \text{mirror image} \]
Pf that $K \# \overline{K}$ is slice (ribbon)

Make immersed disk by lines from $K$ to $\overline{K}$. The only self-intersection are ribbon singularities.
C is a non-finitely generated abelian group. We don't know what C is.

- C contains elements that are 2-torsion.

\[
\begin{align*}
\mathcal{A}_1 & \cong \mathcal{B}_1 \\
\Rightarrow 2\mathcal{A}_1 & = 0 \quad \text{and} \quad \mathcal{A}_1 \text{ is not slice } (\mathcal{A}, \neq 0)
\end{align*}
\]
\( C \) contains elements of infinite order

\[ \therefore \# \ldots \# \therefore \]

is never slice.

Thm (Levine '60's) $\exists$ surjective homomorphism

\[ C \xrightarrow{\pi} A \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \]

algebraic concordance group

(Witt group of Seifert matrices)
Q. Are all torsion elements, 2-torsion?

- \( \ker(\pi) \) is non-trivial (in higher-dimension, \( \pi \) is an \( \cong \))

**Thm (Casson-Gordon, Gilmer)**: \( \ker \pi \neq 0 \).

\[ K = \text{trefoil} \]
Tie strand into $K$

$K = \quad \quad = \quad \quad$
$n$-solvable filtration

Cochran-Orr-Teichner defined filtration

$s_0^* \subseteq \ldots \subseteq s_1^* \subseteq s_{0.5}^* \subseteq s_0 \subseteq \mathbf{C}$

$K \in s_0^* \iff \text{Arf}(K) = 0$ \hspace{1cm} \text{Arf invariant}

$K \in s_{0.5}^* \iff K \in \ker(\Pi)$ \hspace{1cm} \text{Algebraically slice}

$K \in s_{1.5}^* \Rightarrow \text{Casson-Gordon invariants vanish}$
**Def.** If \( G \) is a group, define

\[
G^{(\ell)} := G \quad \text{and} \\
G^{(n)} := [G^{(n)}, G^{(n)}],
\]

\( \{G^{(n)}\} \) is the derived series of \( G \).
**Def:** A knot $K$ is *(n)-solvable* (in $\mathfrak{y}_n$) if there is a smooth 4-mfld $W$ with $\partial W = S^3$ and smoothly embedded disk $\Delta \subseteq W$ with $\partial \Delta = K$ s.t.

1. $H_1(S^3, K) \cong \tilde{H}_1(W, \Delta)$
2. $[\Delta] = 0$ in $H_2(W, S^3)$
3. $H_2(W) \cong \mathbb{Z}^{2g}$ has a basis represented by surfaces $\Sigma_i$, $d_i \subseteq W \setminus \Delta$ s.t. $\Sigma_i \cdot c_j = \delta_{ij}$, $d_i \cdot d_j = 0 = \Sigma_i \cdot \Sigma_j$.
4. $\pi_1(\Sigma_i)$, $\pi_1(d_i) \subseteq \pi_1(W \setminus \Delta)^{(n)}$
Thm (Cochran-H-Heidy): For each \( n \geq 0 \), \( J_n / J_{n,s} \) contains \( \bigoplus_{p(t)} (\mathbb{Z}^t \oplus \mathbb{Z}_2^s) \) symmetric irreducible.

- \( n = 0 \): Milnor-Tristram, Levine (60's)
- \( n = 1 \): Jiang, Livingston (80's)
- \( n = 2 \): Cochran-Teichner ('02)
Operators on C

Def: A pattern $P$ is a slice knot $R$ and unknot $\eta$ disjoint from $R$, such that $\eta$ bounds a surface disjoint from $R$.
\[ p : C \rightarrow C \quad \text{(not a homomorphism)} \]

\[ p(K) = \]

satellite operator.

tie strands going through \( \eta \) into \( K \)
\[ P : \mathcal{F}_n \rightarrow \mathcal{F}_{n+1}. \]

Hence \( P^n(k) = P(P(\ldots P(k))) \in \mathcal{F}_n \) for any \( k \) with Art invariant zero. Exs of \( \mathbb{F}_\infty \) and \( \mathbb{Z}_\infty \in \mathcal{F}_n/\mathcal{F}_{n+1} \) are constructed this way!
Q. When is $P$ injective?

**Conjecture:** $Q$ is injective

$Q(K) = \text{slice} \iff K$ is slice

**Known:** There is a subgroup of $C$ on which $Q^n$ is injective for $\forall n$. 
Ex: Whitehead double

\[ \text{Wh} = \includegraphics{diagram.png} \]

\[ \text{Wh: } K \rightarrow \text{Wh}(K) \]
**Conjecture:** \( \text{Wh}(K) \) is smoothly slice \( \iff K \) is smoothly slice (i.e. \( \text{Wh} \) is weakly injective).

**Remark:** For any \( K \), the Alexander polynomial of \( \text{Wh}(K) \) is \( 1 \) \( \implies \) by Freedman, \( \text{Wh}(k) \) is always topologically slice (bounds a topologically locally flat disk in \( B^4 \)).
Satellite operators give a way to construct elements in $\mathfrak{g}_n$. The difficult part is to show $P^n(K)$ is not slice (or even in $\mathfrak{g}_n$s)!

- Use invariants of knots such as $L^2$-signatures, $d$-invariants and $\mathcal{I}$ invariants from Heegaard Floer homology, etc.
We conjecture that \( C \) has the structure of a "fractal set".

Would like some notion of distance where image \((P^n)\) is getting smaller as \( n \to \infty \).
Symmetric gropes

**Def:** A *grope* of height 1 is a compact oriented surface $G_1$ with $|\omega| = 1$.

Let $\{\alpha_1, ..., \alpha_{2g}\}$ be a standard symplectic basis of curves for $H_1(G_1)$ on $G_1$, $g = \text{genus}(G_1)$. 
A grope of height $n+1$ is obtained by attaching gropes of height $n$ to $\alpha_1, \ldots, \alpha_9, \beta_1, \ldots, \beta_9$. 

height 2 grope
Def: A branched symmetric groove is defined as follows:

Let $\Sigma_1$ be a compact connected orientable surface of genus $g$, with a standard symplectic basis of curves $\{\alpha_1, \ldots, \alpha_{2g}\}$ with $\alpha_{2i-1}$ dual to $\alpha_{2i}$. Attach to each $\alpha_i$, a groove of height $m_i$ s.t. $m_{2i-1} = m_{2i}$, no subsurface of which is a disk.

Let $n_i = m_{2i}$. 
$n_1 = m_1 = m_2 = 0 \quad n_2 = m_3 = m_4 = 2 \quad n_3 = m_5 = m_6 = 1$
Let $\Sigma$ be a branched symmetric grope.

Define 

$$g_1 = \text{genus}(\Sigma)$$

$$g_2 = \text{sum of genera of first stage surfaces attached to } \alpha_{2i-1}, \alpha_{2i}.$$ 

$$\vdots$$ 

$$g_{n+1} = \text{sum of genera of } n_{i+1} \text{ stage surfaces attached to } \alpha_{2i-1}, \alpha_{2i}.$$
No $g_2^1$ since $n_1 = m_1 = m_2$.

$g_2^2 = 1 + 2 = 3$

$g_2^3 = 2 + 2 = 4$
\[ g_3^2 = 2 + 1 + 3 + 1 + 1 + 1 = 9 \]
Note: For each $1 \leq i \leq g_i$ and $2 \leq k \leq n_i + 1$, \( g_k \geq 2 g_{k-1} \)
Let $q \geq 1$ be a real number and $\Sigma$ a branched symmetric grope. Define

$$
\| \Sigma \|_{q} = \sum_{i=1}^{g} \frac{1}{q^{n_{i}}} \left( 1 - \sum_{k=2}^{n_{i}+1} \frac{1}{q^{k}} \right)
$$

**Def:** If $K, J$ are knots, define

$$
d^{q}(K, J) := \inf \left\{ \| \Sigma \|_{q} \mid \Sigma \text{ is a branched symmetric grope embedded in } S^{3} \times I \text{ with boundary } K \times \{0\} \text{ and } J \times \{1\} \right\}
$$

**Note:** Any two knot cobound a surface.
\[ \sum = \]

\[ \| \Sigma \|_{q_i} = \left( \frac{1}{q_0^i} \right) + \frac{1}{q_2^i} \left( 1 - \frac{1}{3} - \frac{1}{q} \right) + \frac{1}{q_4^i} \left( 1 - \frac{1}{4} \right) = 1 + \frac{5}{q^2} + \frac{3}{4q} \]

for \( i = 1, 2, 3 \).
Ex: If $K$ has bounds a genus 1 surface $\Sigma$ and $\text{Arf}(K) \neq 0$ then $K$ cannot bound a (symmetric) height 2 grope, so

$$d^9(k, \text{unknot}) = g(\Sigma) = 1.$$ 

Ex: $\frac{1}{2q} \leq d^9(\xi, \xi) \leq \frac{27}{16q}$.
Prop (Cochran-H-Powell): For \( q \geq 1 \), the function \( d^q \) determines a pseudo-metric on \( C \).

Need to show \( \| \Xi \|_q \geq 0 \) for any \( \Xi \).

Prop: If \( K \) does not bound a grope of height \( n \) then

\[
d(K, \text{unknot}) \geq \frac{1}{(2q)^{n-2}}.
\]
Thm (Cochran-Orr-Teichner): If \( K \) bounds a height \( h \) grope then \( KE \in \mathcal{F}_{n-2} \).

Prop (Cochran-H-Powell): If \( P \) is a pattern then \( P : C \to C \) is a contraction w.r.t. \( d^q \) for \( q > gw(P) \), i.e. # of times \( R \) goes through \( P \).
Thm (Cochran-H-Powell): For any $q > 1$ there exists uncountably many sequences of knots $\{K_i\}$ s.t.

\[ d^q(K_i, \text{unknot}) > 0 \quad \forall \ i \quad \text{but} \quad d^q(K_i, \text{unknot}) \to 0 \quad \text{as} \quad i \to \infty. \]

Hence the topology on $(\mathbb{C}, d^q)$ is not discrete for $q > 1$. 
Topologically slice knots

Let \( T = \{ \text{topologically slice knots} \} \subseteq C \).

This is an interesting and subtle subgp of \( C \).

Thm (Hom): \( T \) has a \( \mathbb{Z}^\infty \) summand.

(Endo showed that \( \mathbb{Z}^\infty \subseteq C \).)
Remark 1: If \( K \in T \), then \( K \notin \mathcal{E}_n \) for \( n \).

Remark 2: If \( K \in T \), then \( K \) bounds an arbitrarily long symmetric grope all of whose first stage genus is fixed.

Hence for \( q > 1 \),

\[
cl_q^g(k, \text{unknot}) = 0.
\]
Remark 3: For \( q = 1 \), the only way to get \( d'(k, \text{unknot}) = 0 \) would be for \( K \) to bound a arbitrarily long grope with each stage having genus 1.

\[
\|\Sigma\|_1 = 1 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2^n} \rightarrow 0 \quad \text{as} \ n \rightarrow \infty
\]

It is unknown if there is a non-slice knot that bounds such a rose!

Conj: \( \exists \ K \in \mathcal{T} \text{ s.t. } d'(k, \text{unknot}) > 0. \)
More generally

Conjecture: $d'(k, J) > 0 \ \forall \ k \neq J$. 
Bipolar Filtration

Cochran, Horn and I defined a filtration

\[ \ldots \leq B_1 \leq B_0 \leq \mathcal{C} \]

that is a refinement of \( \{ \mathcal{F}_{\mathfrak{t}} \} \) and Gompf and Cochran’s notion of positivity of knots.

Thm (Cochran-H-A-Horns): \( \mathbb{Z}^\omega \leq B_n / B_{n+1} \) \( \forall n \).

Unlike \( \{ \mathcal{F}_{\mathfrak{t}} \} \) this is an interesting filtration for topologically slice knots.
Def: A knot $K \in P_n$ (is $n$-positive) if $K = \partial \Delta$, $\Delta$ is a smoothly embedded disk in a smooth mfld $W$ s.t.
- $\partial W = S^3$, $\Pi_1(W) = 1$
- $[\Delta] = 0$ in $H_2(W, S^3)$.
- The int. form on $H_2(W)$ is pos. def.
- $H_2(W)$ has a basis repr. by surfaces $\{S_i\}$, disjointly embedded in $W \setminus \Delta$ s.t.
  $\Pi_1(S_i) \leq \Pi_1(W - \Delta)^{(n)}$.
*Can also define when $K \in N_n$ (n-negative).

$B_n := N_n \cap P_n \forall n$. (n bipolar knots)

**Prop(CHH):** $B_n \subseteq \alpha_n$.

**Prop (CHH):** If $K \in B_0 \Rightarrow$

$\tau(k) = \pi(k) = \delta(k) = d(+1 \text{ surj. on } K) = 3(k) = 0$

**Prop (CHH):** If $K \in B_1$ and $Y = p^s$-fold branched cyclic cover of $K$, $s_0 \in \text{spin}^c(Y)$ corresponding to a spin structure on $Y \Rightarrow$

$d(Y, s_0) = 0$. 

**Def:** $T_n := T_n B_n$.

**Thm (Cochran-Horn):** $T_0 / T_1 \neq 0$

**Thm (Cochran-Horn):** $T_1 / T_2 \neq 0$.

**Thm (Cha-Kim):** $T_n / T_{n+1} \neq 0 \quad \forall \ n$.

Pf uses $L^2$-p invs. and d-invts. of p-fold branched covers for an infinite # of p.
Tower metrics (Cochran, H, Powell, Ray)

For $q \geq 1$, can define a metric $d_q^B : \mathcal{C} \times \mathcal{E} \to \mathbb{R}$ based on kinky disks and gropes.

"Def" A generalized positive plumbed handle (GPH):

- Attach gropes of height $N_i$ to $\alpha_i$:
  - $g_i^+ = \# +$
  - $g_i^- = \# -$

\[ \alpha_1 \quad \alpha_2 \quad \text{at} \quad \begin{cases} \text{attaching curve} \end{cases} \]
Ex: 

Attach grope to α  \( g_1^+ = 1 \)  \( g_1^- = 1 \)  \( am_1 = gm_1 = 2 \)  \( am_2 = gm_2 = 1 \)
Def: A positive tower for $K$ is an embedding of a GPHT into $B^q$ with $C \to K$. ($C$ has 0-framing)

If $K,J$ bound a positive tower, $\Sigma$,

$$d^+_\Sigma,q(K,J) = \sum_{i=1}^{g_i} \frac{m_i}{q_i^6} \left( 1 - \frac{1}{2} \sum_{k=2}^{n_i+1} \frac{1}{q_k^i} \right)$$

$$g_i = g_i^+ + g_i^-$$

$$n_i = \frac{\text{alg mult}_i + \text{geom mult}_i}{2}$$
\[ d_q^+(k, J) = \min \{ d_{\Sigma, q}(k, J) \mid \Sigma \} \]

can define \(d_q^-\) sim.

\[ d_q^B(k, J) = \max \{ d_q^+(k, J), d_q^-(k, J) \} \]

\textbf{Prop(CHPR):} If \( \|K\|_q^+ < \left( \frac{1}{2q} \right)^n \Rightarrow K \in \mathcal{P}_n \)

\textbf{Cor:} \exists \text{ topologically slices knots } K_i \text{ with } d_q^B(K_i, \text{ unknot}) > 0.
Conjecture 1. (1) There are topologically slice knots $K_i$ s.t. $d^B_q(K_i, \text{unknot}) \xrightarrow{i \to \infty} 0$ and $d^0_q(K_i, \text{unknot}) \neq 0$ for all $q \geq 1$.

2) $d^B_q$ is a metric (not just pseudo-metric) for all $q \geq 1$. 

\[ (1) \] There are topologically slice knots $K_i$ s.t. $d^B_q(K_i, \text{unknot}) \xrightarrow{i \to \infty} 0$ and $d^0_q(K_i, \text{unknot}) \neq 0$ for all $q \geq 1$.

(2) $d^B_q$ is a metric (not just pseudo-metric) for all $q \geq 1$. 

\[ (1) \] There are topologically slice knots $K_i$ s.t. $d^B_q(K_i, \text{unknot}) \xrightarrow{i \to \infty} 0$ and $d^0_q(K_i, \text{unknot}) \neq 0$ for all $q \geq 1$.

(2) $d^B_q$ is a metric (not just pseudo-metric) for all $q \geq 1$. 

\[ (1) \] There are topologically slice knots $K_i$ s.t. $d^B_q(K_i, \text{unknot}) \xrightarrow{i \to \infty} 0$ and $d^0_q(K_i, \text{unknot}) \neq 0$ for all $q \geq 1$.

(2) $d^B_q$ is a metric (not just pseudo-metric) for all $q \geq 1$. 

\[ (1) \] There are topologically slice knots $K_i$ s.t. $d^B_q(K_i, \text{unknot}) \xrightarrow{i \to \infty} 0$ and $d^0_q(K_i, \text{unknot}) \neq 0$ for all $q \geq 1$.

(2) $d^B_q$ is a metric (not just pseudo-metric) for all $q \geq 1$. 

\[ (1) \] There are topologically slice knots $K_i$ s.t. $d^B_q(K_i, \text{unknot}) \xrightarrow{i \to \infty} 0$ and $d^0_q(K_i, \text{unknot}) \neq 0$ for all $q \geq 1$.

(2) $d^B_q$ is a metric (not just pseudo-metric) for all $q \geq 1$. 

\[ (1) \] There are topologically slice knots $K_i$ s.t. $d^B_q(K_i, \text{unknot}) \xrightarrow{i \to \infty} 0$ and $d^0_q(K_i, \text{unknot}) \neq 0$ for all $q \geq 1$.

(2) $d^B_q$ is a metric (not just pseudo-metric) for all $q \geq 1$.