Torsion in the Knot Concordance Group

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joint work with:

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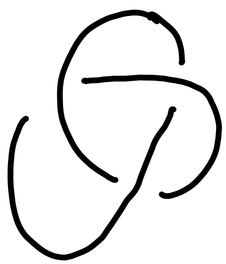
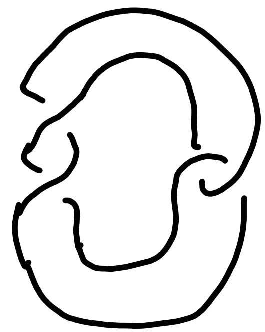
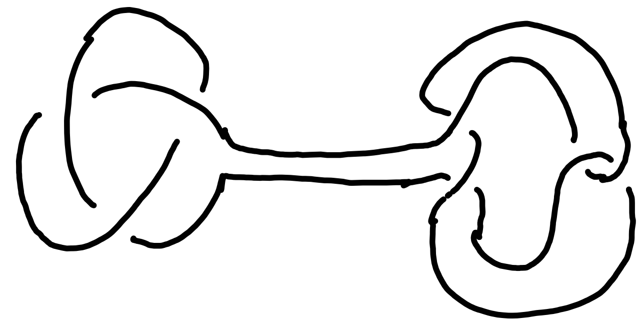
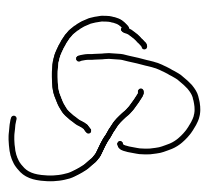
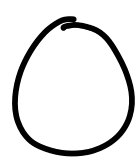
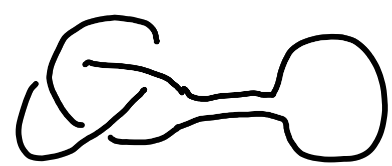
There is a binary operation on knots:

 $\#$  $:=$ 

K_1 $\qquad$ K_2 $\qquad$ $K_1 \# K_2$

connected sum of K_1 and K_2.

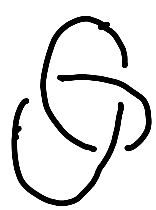
 $\#$  $=$  $=$ 

$K \# \bigcirc = K$

Thus $\mathcal{K} = (\{\text{knots}\}, \#)$ forms a monoid with unity $= \bigcirc$.

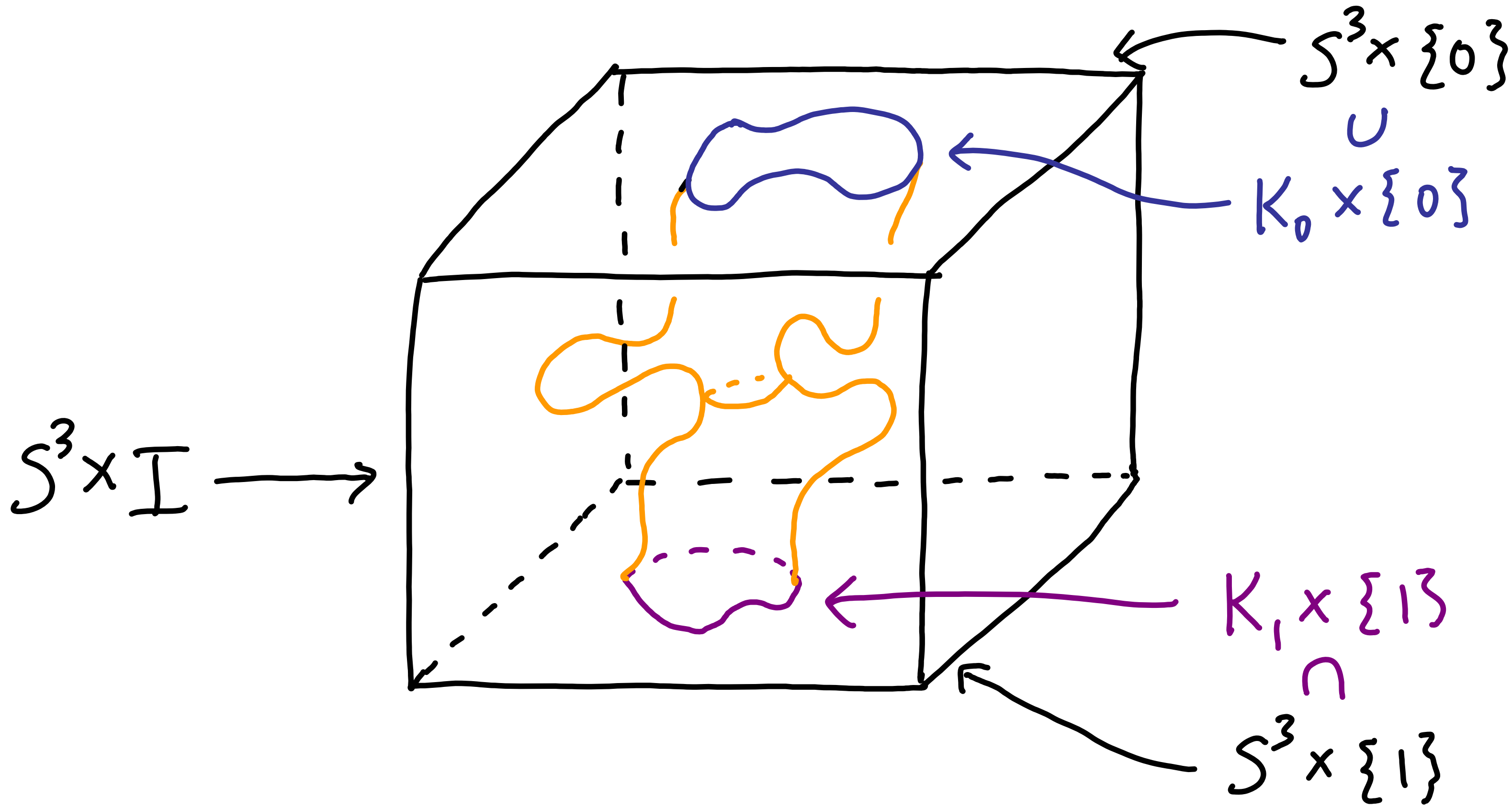
However $\mathcal{K}$ is not a group since it does not have inverses.

i.e. there is no knot K such that

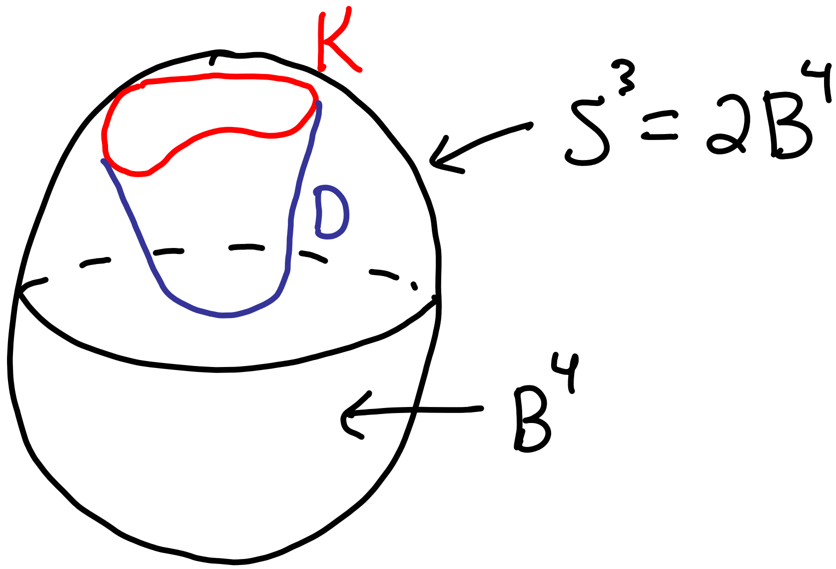
 $\# \, K = \bigcirc$.

To get a group structure, define a equivalence relation called **concordance**.

Def: Knots K_0 and K_1 are concordant if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times I$.



Def: A knot $K \subset S^3$ <u>slice</u> if $K = \partial D$ where D is a 2-dimensional disk (smoothly) embedded in $B^4 = $ 4-dim. ball.

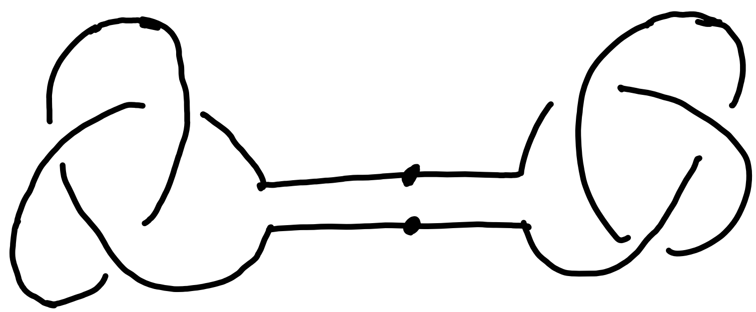


$\boxed{\text{A knot is concordant to the unknot} \Leftrightarrow \text{it is a slice knot.}}$

If K is any knot then $K \# r\bar{K}$ is slice.
(ribbon)

$r\bar{K}$ = change all crossings + change orientation
(reverse of mirror image)

Proof: "Spin" K through $\mathbb{R}^4_+$.

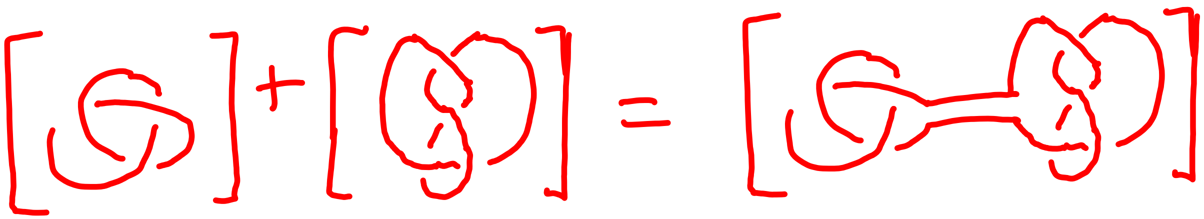


$K \quad \# \quad r\bar{K}$

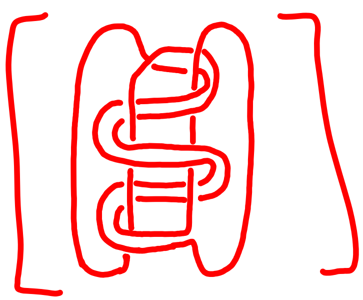
Defⁿ The knot concordance group is

$$\mathcal{C} = \{\text{knots in } S^3\}/\text{concordance}$$

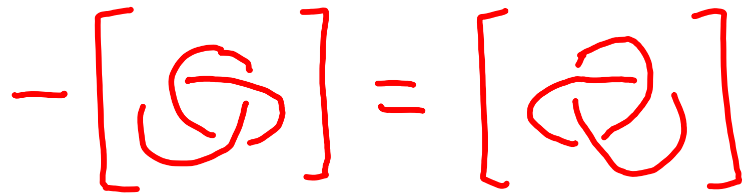
- $\mathcal{C}$ is an abelian group under the operation connected sum of knots.



- $[K] = 0 \iff K$ is slice

 $= 0$

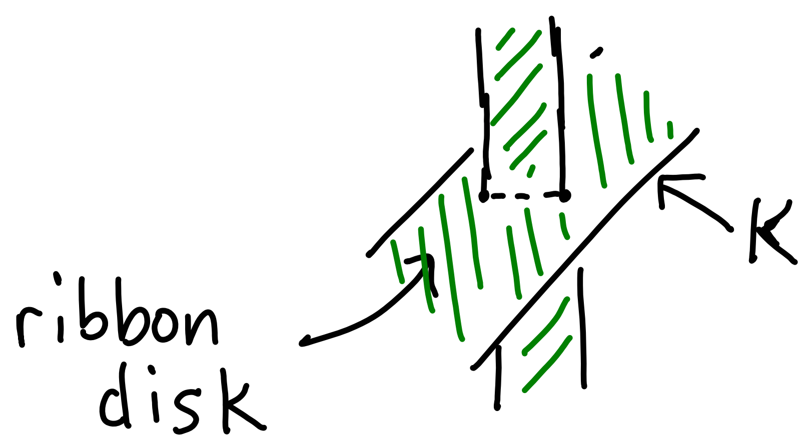
- The inverse of $[K]$ is $[r\bar{K}]$ since $K \# r\bar{K}$ is slice.



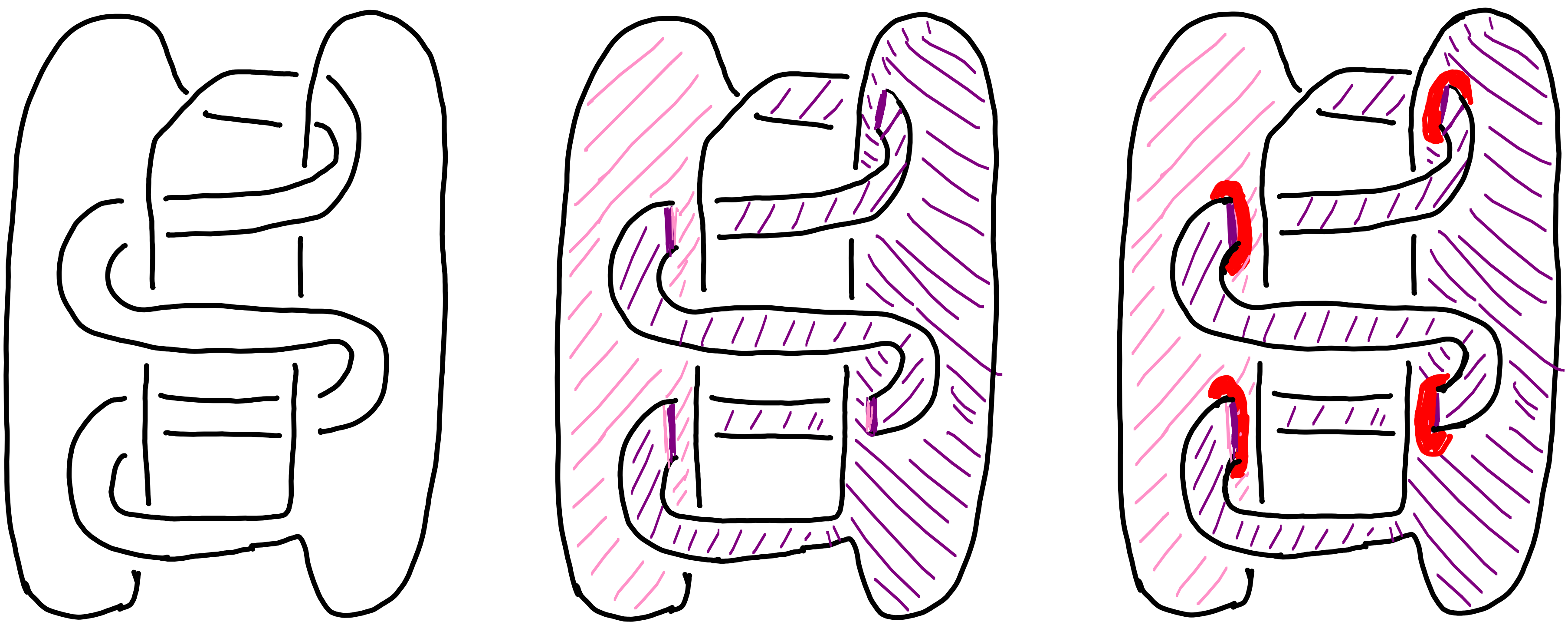
$\boxed{[K] \text{ has order } 1 \iff K \text{ is slice}}$

All known examples of (smoothly) slice knots are ribbon:

K is **ribbon** if it bounds an immersed disk in S^3 with only ribbon singularities:



Any ribbon knot is slice



To obtain embedded $D^2 \subset B^4$, push interior of red discs into interior of B^4.

Open Problem:

Ribbon-Slice Conjecture: A knot is smoothly slice $\iff$ it is ribbon.

$$\boxed{[K] \text{ has order } 2 \iff \begin{array}{l} K\#K \text{ is slice} \\ \quad\quad + \\ K \text{ is not slice} \end{array}}$$

Def: A knot K is negative amphichiral if K is isotopic to $r\overline{K}$.

If K is neg. amphichiral $\Rightarrow$

$$K\#K = K\#r\overline{K} = \text{slice}.$$

i.e. $[K]$ is of order 2 in $\mathcal{C}$ (if $K \neq$ slice).

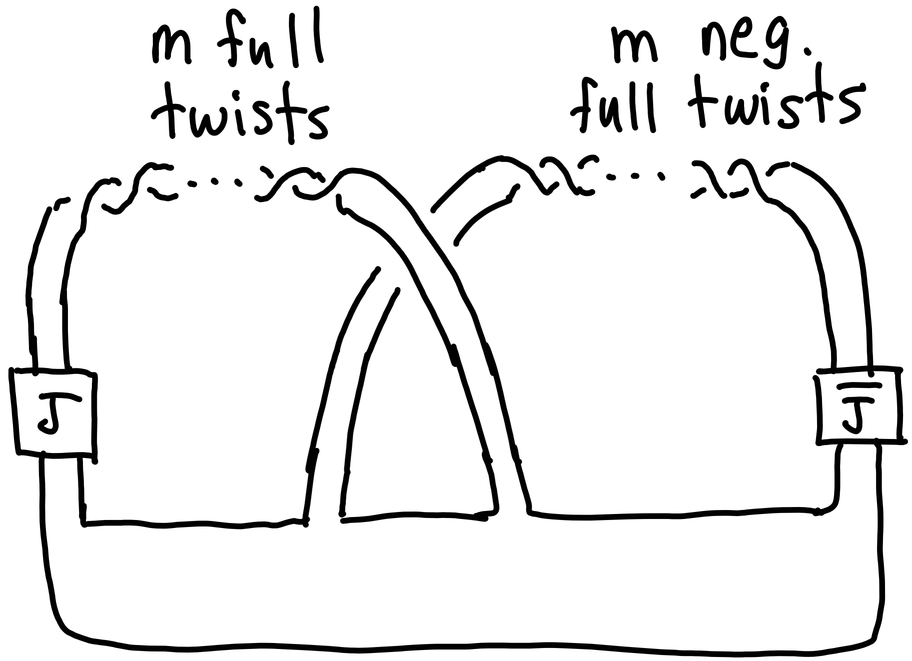
This is the only known way to create elements of finite order in $\mathcal{C}$.

Open Problems

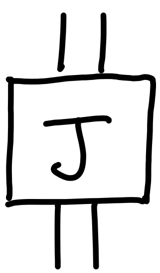
- (Gordon) If $[K]$ has order 2 in $\mathcal{C}$, then is K concordant to a negative amphichiral knot?

- Is every element of finite order in $\mathcal{C}$ of order 1 or 2?

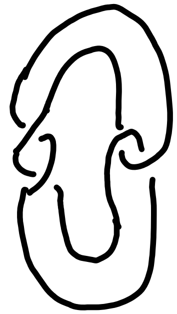
Ex: Let

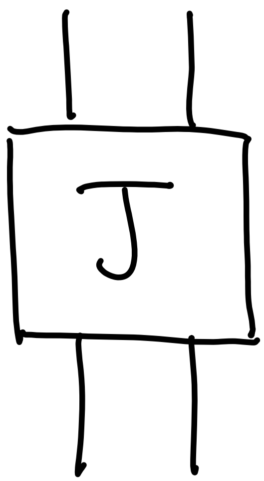
$K_m(J) =$ 

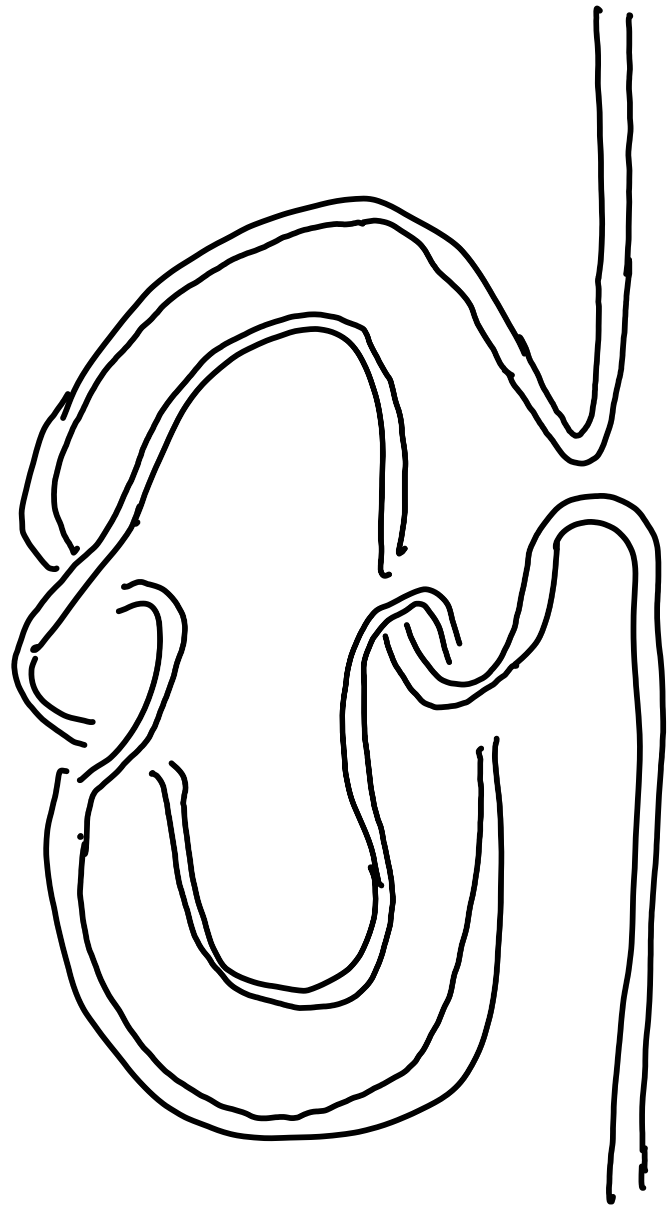
then $K_m(J)$ is neg. amphichiral.



means tie two strands into the knot J (with linking # zero).

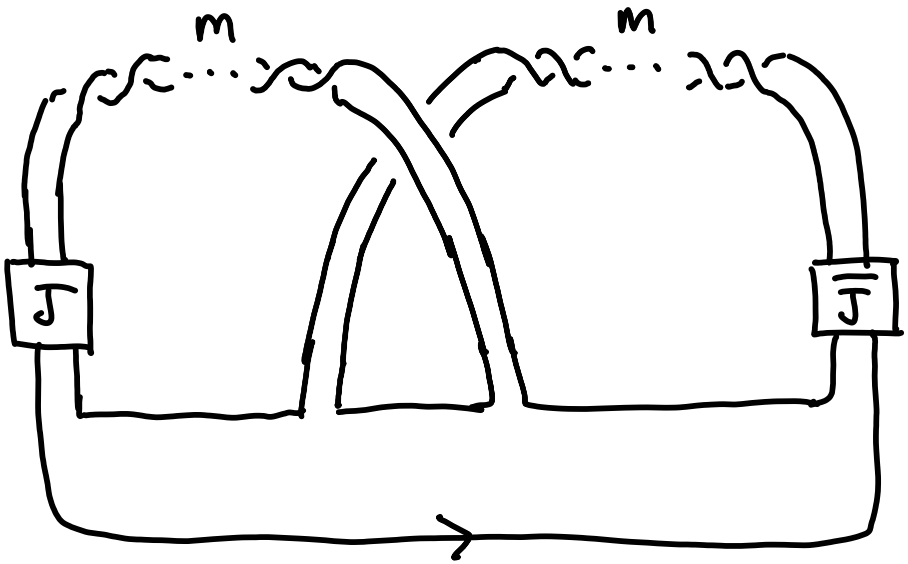
If J = 



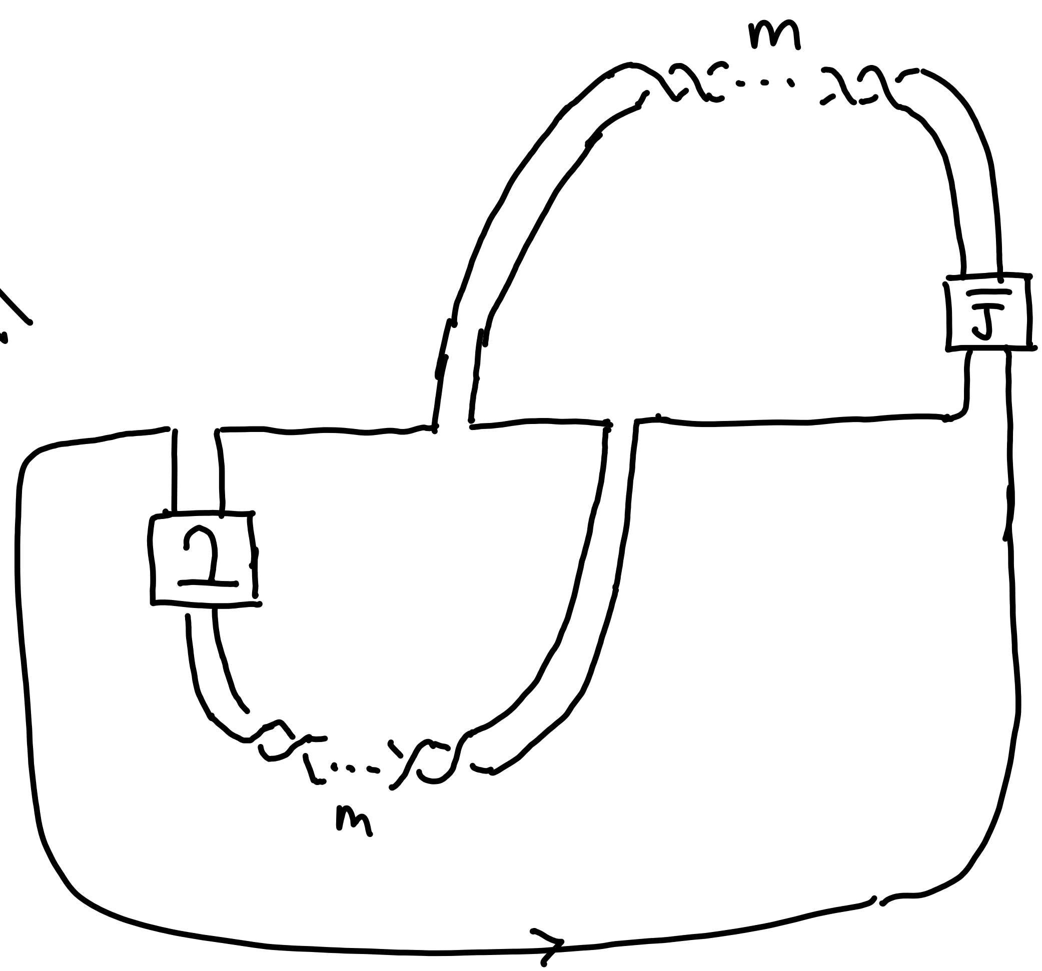
= 

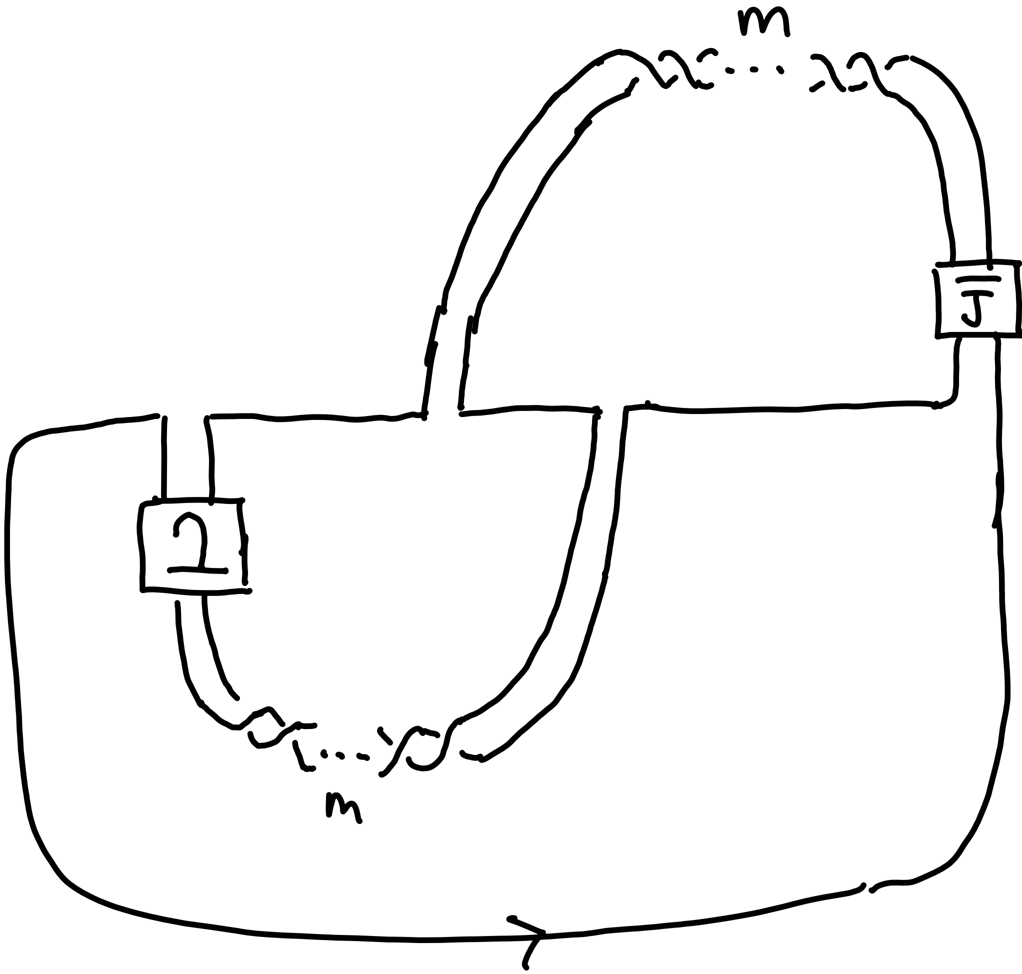
"tie strings into knot J"

Proof that $K_m(J)$ is neg. amphichiral:



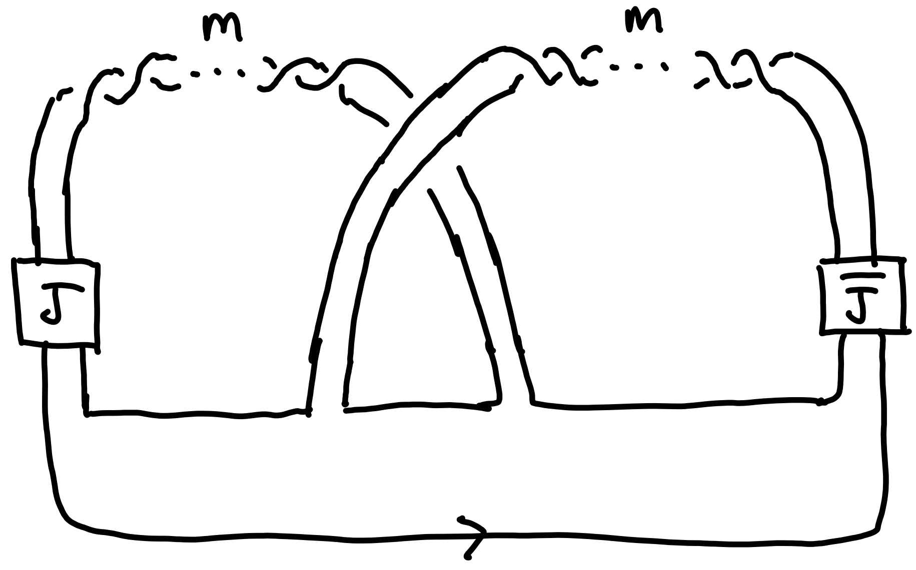
swing down

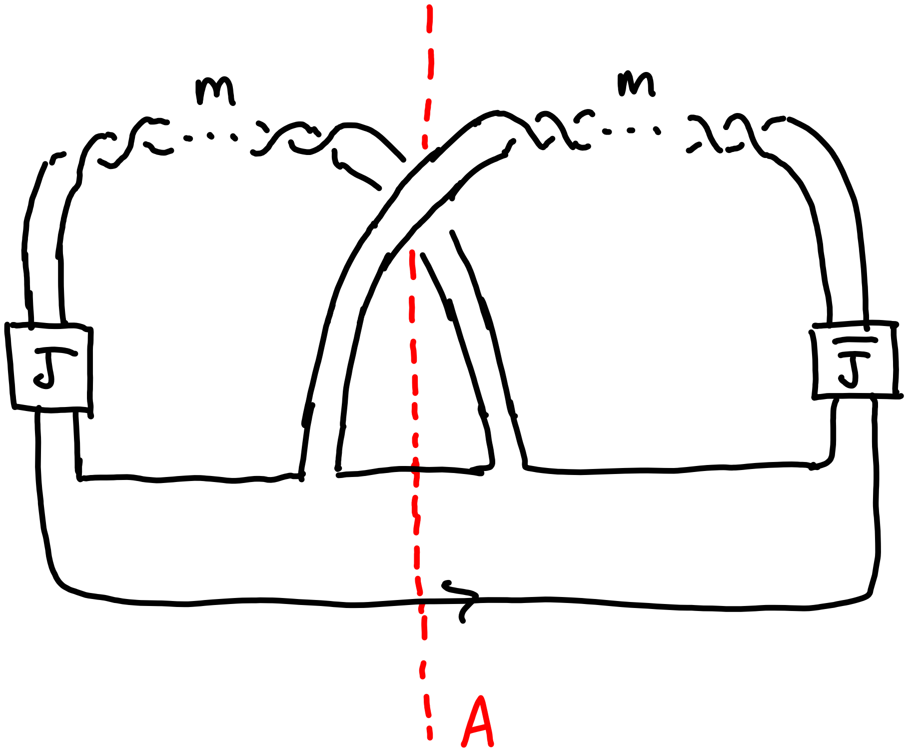




swing back up
to the "back"

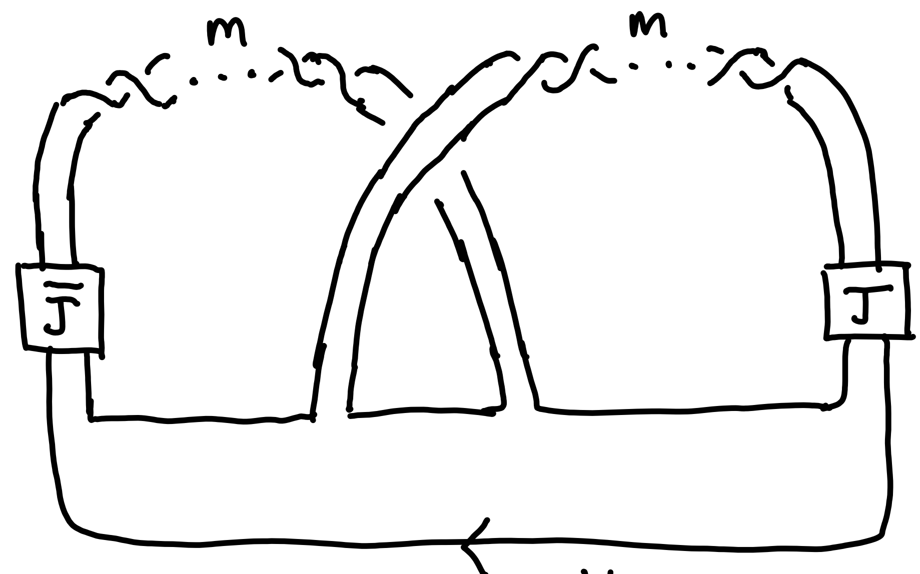
=





rotate $\mathbb{R}^3$ by π about A

$=$



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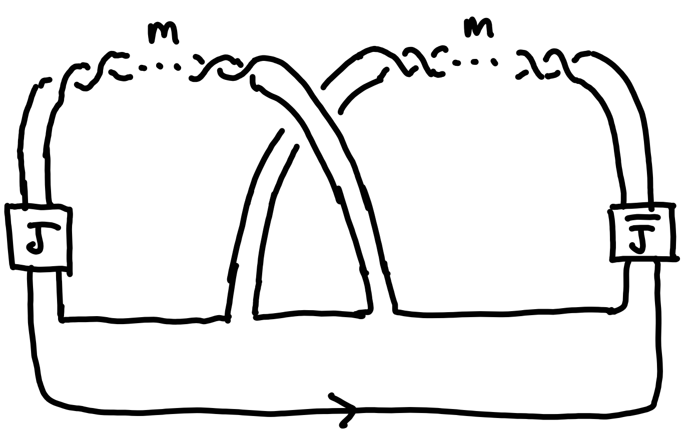
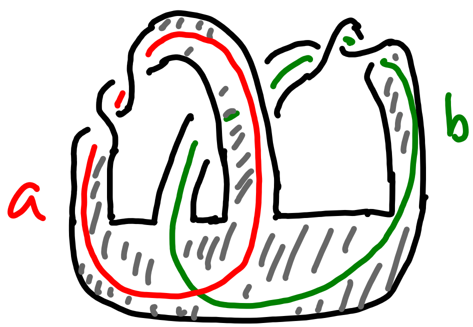
$r\,\overline{K_m(J)}$



$K_m(J)$

Levine-Tristram signatures: Sliceness obstructions



$K = \partial(\text{surface})$

$$\rightsquigarrow \quad V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad \mathrm{lk}(a, b^+)$$

"linking" matrix

For $w \in \mathbb{C}$, $|w| = 1$:

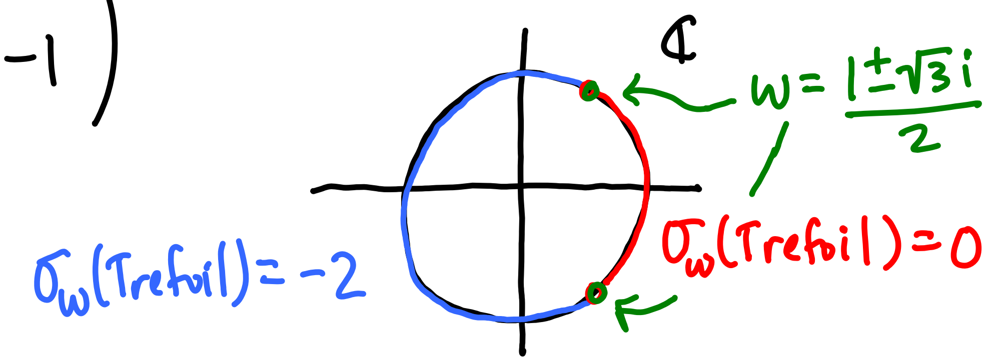
- $\sigma_w(K) := \text{signature}\left((1-w)V + (1-\bar{w})V^T\right)$
- $\rho_0(K) := \int_{S^1} \sigma_w(K)\, dw$

$$\boxed{\text{If } K \text{ is slice then } \rho_0(K) = 0.}$$

$\rho_0 : G \longrightarrow \mathbb{R}$ is a homomorphism

Thus $\rho_0(K) \neq 0 \Rightarrow K$ is of ∞-order

Ex: $V_{\text{Trefoil}} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$



$\rho_0(\text{Trefoil}) = -4/3 \neq 0$

~67, Milnor-Tristram used signatures to show $\mathcal{C}$ has infinite rank

In late 60's Levine used invariants obtained from Seifert matrix (including signatures and Arf invariant) to define epimorphism

$$\mathcal{C} \xrightarrow{\pi} \text{Algebraic concordance group} \cong \mathbb{Z}^\infty \times \mathbb{Z}_2^\infty \times \mathbb{Z}_4^\infty$$

Def: K is algebraically slice if $\pi(K) = 1$.

Moreover, Levine showed that
$\mathcal{A}$ = Algebraic concordance group can be decomposed into its $p(t)$-primary parts:

$$\mathcal{C} \xrightarrow{\pi} \mathcal{A} \cong \bigoplus_{\substack{p(t) \text{ prime} \\ p(t) \doteq p(t^{-1}) \\ p(1) = \pm 1}} \mathcal{A}(p(t))$$

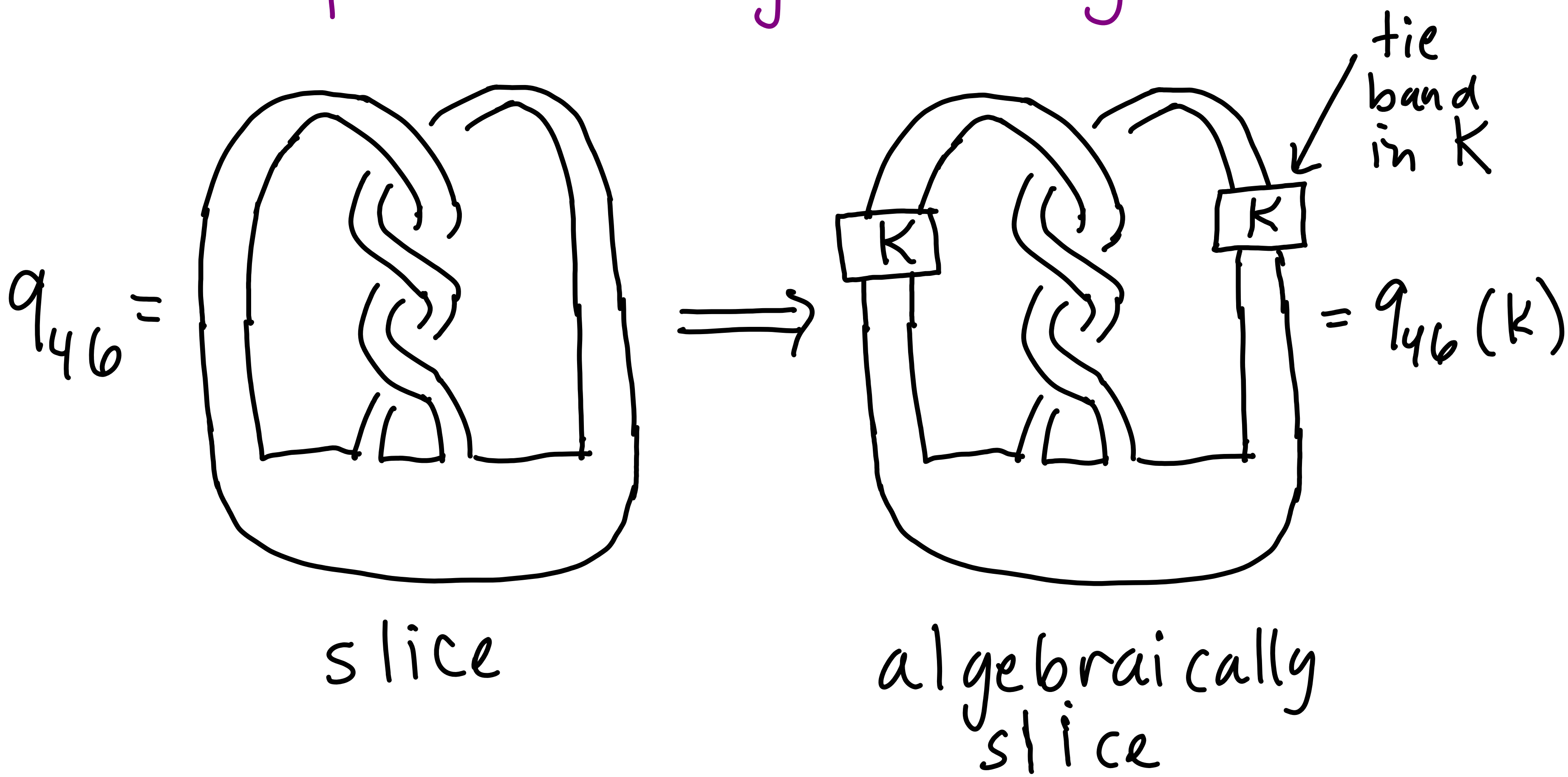
<u>Note</u>: The $\mathcal{A}(p(t))$-part of $\pi(K)$ can only be non-trivial if $p(t) \mid \Delta_K(t)$.

Open Problem:

Conjecture: If K and J have relatively prime Alexander polynomials then K is not top. concordant to J unless K and J are top. slice.

<u>Note</u>: Levine's theorem shows that for K and J not algebraically slice and with relatively prime Alexander polynomial, K is not concordant to J.

Example of an Algebraically slice knot



For K with certain non-vanishing signatures, Gilmer used Casson-Gordon invariants to show $9_{46}(K)$ is not slice.

In 1997, Cochran-Orr-Teichner defined the (n)-solvable filtration of $\mathcal{C}$ ($n \in \mathbb{N}/2$)

$$0 = \left\{ \begin{matrix} \text{slice} \\ \text{knots} \end{matrix} \right\} \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

- $\mathcal{F}_0$ = Arf invariant zero knots
- $\mathcal{F}_{0.5}$ = Algebraically slice knots
- $\mathcal{F}_{1.5} \subset$ knots with vanishing Casson-Gordon invariants.

Thm (n = 0, ~67, Milnor-Tristram; n = 1, ~81, Jiang; n = 2, ~00, Cochran-Orr-Teichner)

For n = 0, 1, 2, $\mathcal{F}_n / \mathcal{F}_{n.5}$ contains a $\mathbb{Z}^\infty$.

Thm (Livingston) $\mathcal{F}_1 / \mathcal{F}_{1.5}$ contains a $\mathbb{Z}_2^\infty$.

Thm (Cochran-Teichner, Cochran-H-Leidy): For $n \geq 3$,

$$\mathcal{F}_n / \mathcal{F}_{n.5} \supset \mathbb{Z} \oplus \mathbb{Z}^\infty.$$

CT ('02) CHL ('07)

Thm (Se-Goo Kim '05, Taehee Kim '08):

$$\bigoplus_{\substack{p(t) \text{ prime} \\ p(t) \doteq p(t^{-1}) \\ p(1) = \pm 1}} \mathbb{Z}^\infty \subset \mathcal{F}_1 / \mathcal{F}_{1.5}$$

Other work done at $\mathcal{F}_1/\mathcal{F}_{1.5}$ level by S. Friedl, T. Kim and P. Gilmer.

Let $S_n = \{(p_{i_1}(t), p_{i_2}(t), \ldots, p_{i_n}(t)) \mid i \in \mathbb{Z}\}$ be a sequence of non-unit polynomials satisfying $p_{i_j}(1) = \pm 1$, that are pairwise coordinatewise "strongly coprime".

Theorem A (Cochran-H-Leidy, '08):

For each $n \geq 2$,

$$\bigoplus_{(p_1(t), \ldots, p_n(t)) \in S_n} \mathbb{Z}^\infty \subset \mathcal{F}_n / \mathcal{F}_{n.5}.$$

Theorem B (Cochran-H-Leidy,'09):

For each $n \geq 2$,

$$\bigoplus \mathbb{Z}/2\mathbb{Z} \subset \mathcal{F}_n/\mathcal{F}_{n.5}.$$

$^*(p_1(t), p_2(t), \ldots, p_n(t)) \in \mathcal{S}_n$

- Represented by neg. amphichiral knots that are slice in a rational homology 4-ball

  $\Rightarrow \tau, s$ invts are zero.

$* \; p_1(t) \in \{ m^2t^2 - (2m^2+1)t + m^2 \mid m \neq 0 \}$

Def: We say $p(t)$ and $q(t)$ are isogenous, denoted $\widetilde{(p,q)} \neq 1$ if for some non-zero roots r_p of $p(t)$ and r_q of $q(t)$, and some $m,n \in \mathbb{Z}-\{0\}$, $r_p^n = r_q^m$. Otherwise, we say they are strongly coprime, denoted $\widetilde{(p,q)} = 1$.

Ex: $p_k(t) = (kt-(k+1))((k+1)t-k)$, $k \in \mathbb{Z}^+$

roots $R_k = \left\{\frac{k}{k+1}, \frac{k+1}{k}\right\}$.

$\widetilde{(p_k, p_\ell)} = 1$ when $k \neq \ell$.

Examples used in Theorem A:

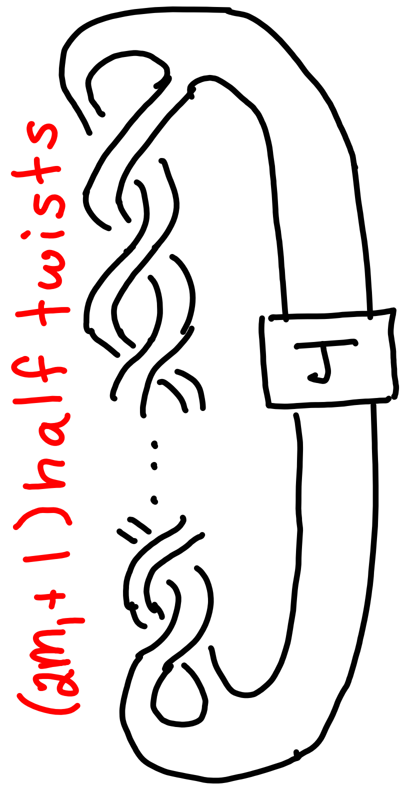
$R_m :=$ 

$(m \neq 0)$

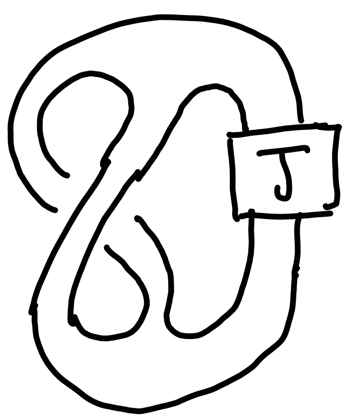
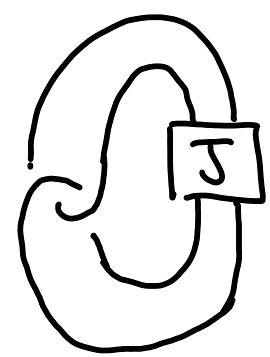
then R_m is slice and $\Delta_{R_m}(t) = p_m(t)$

where $p_m(t) = (mt-(m+1))((m+1)t-m)$ so

$(\Delta_{R_m}(t), \Delta_{R_k}(t)) = 1$ for $k \neq m$.

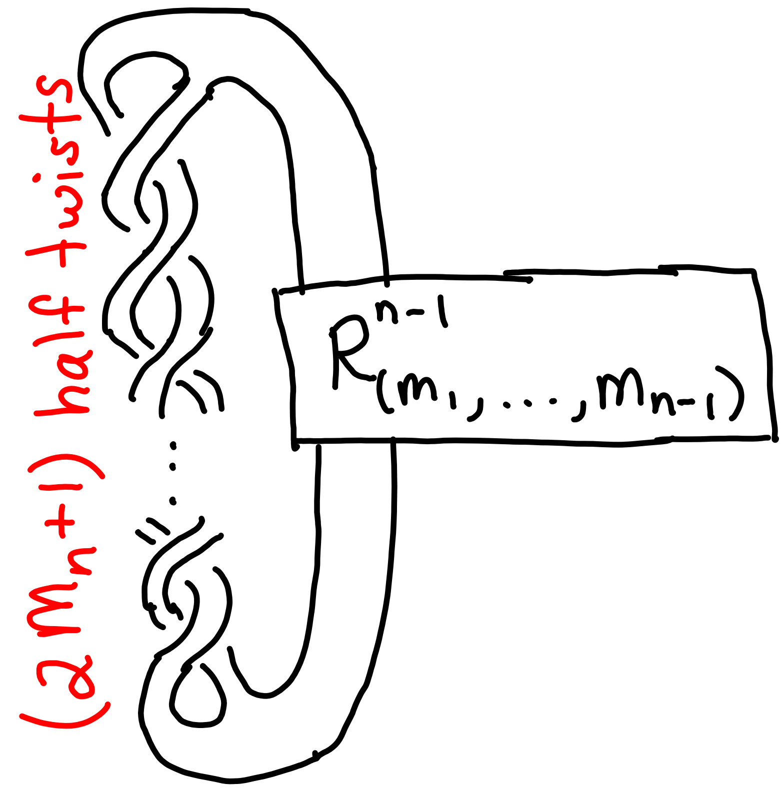
$R^1_{(m_1)}(J) =$ 

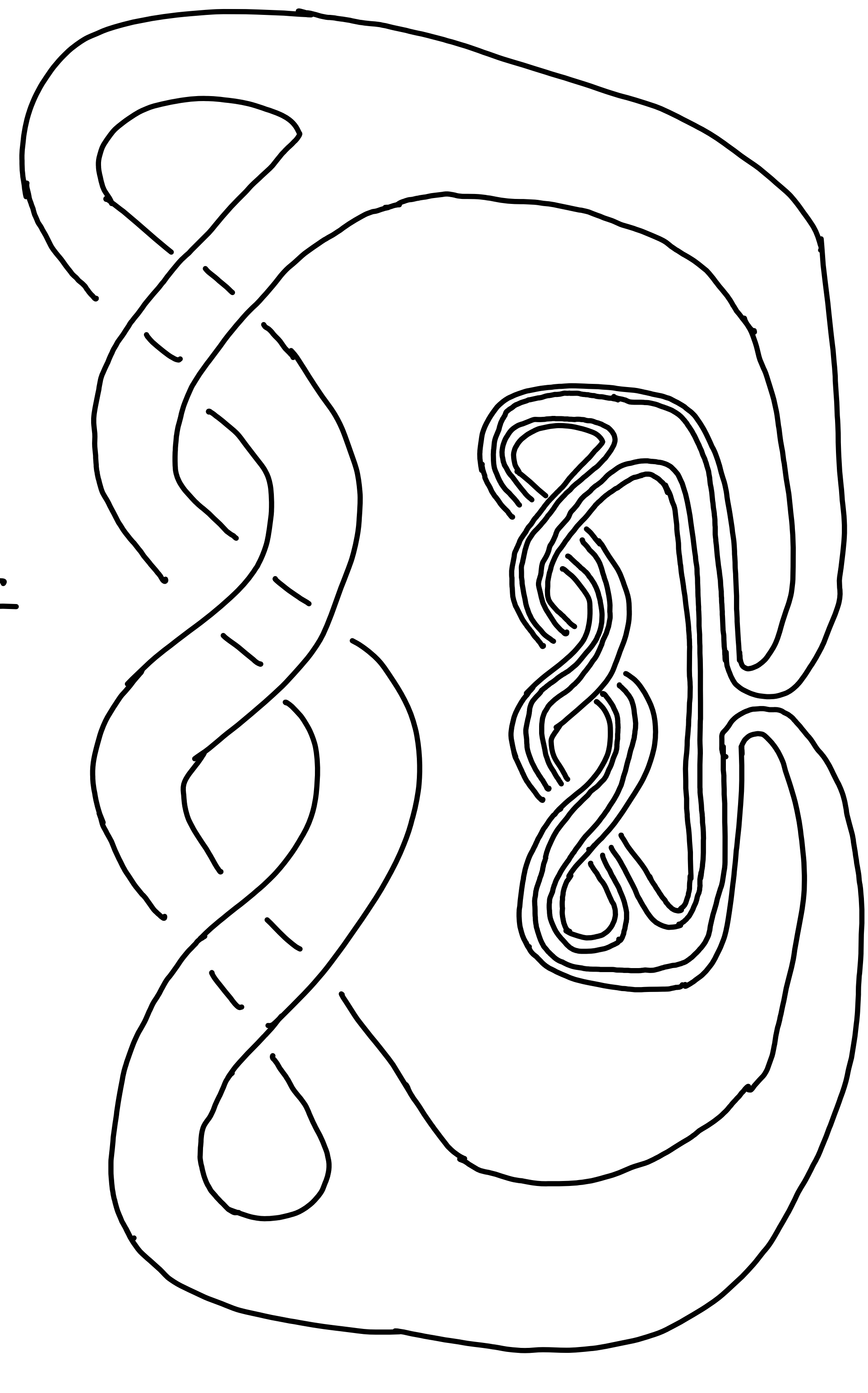
<u>Note</u>: When $m_1 = 0$,

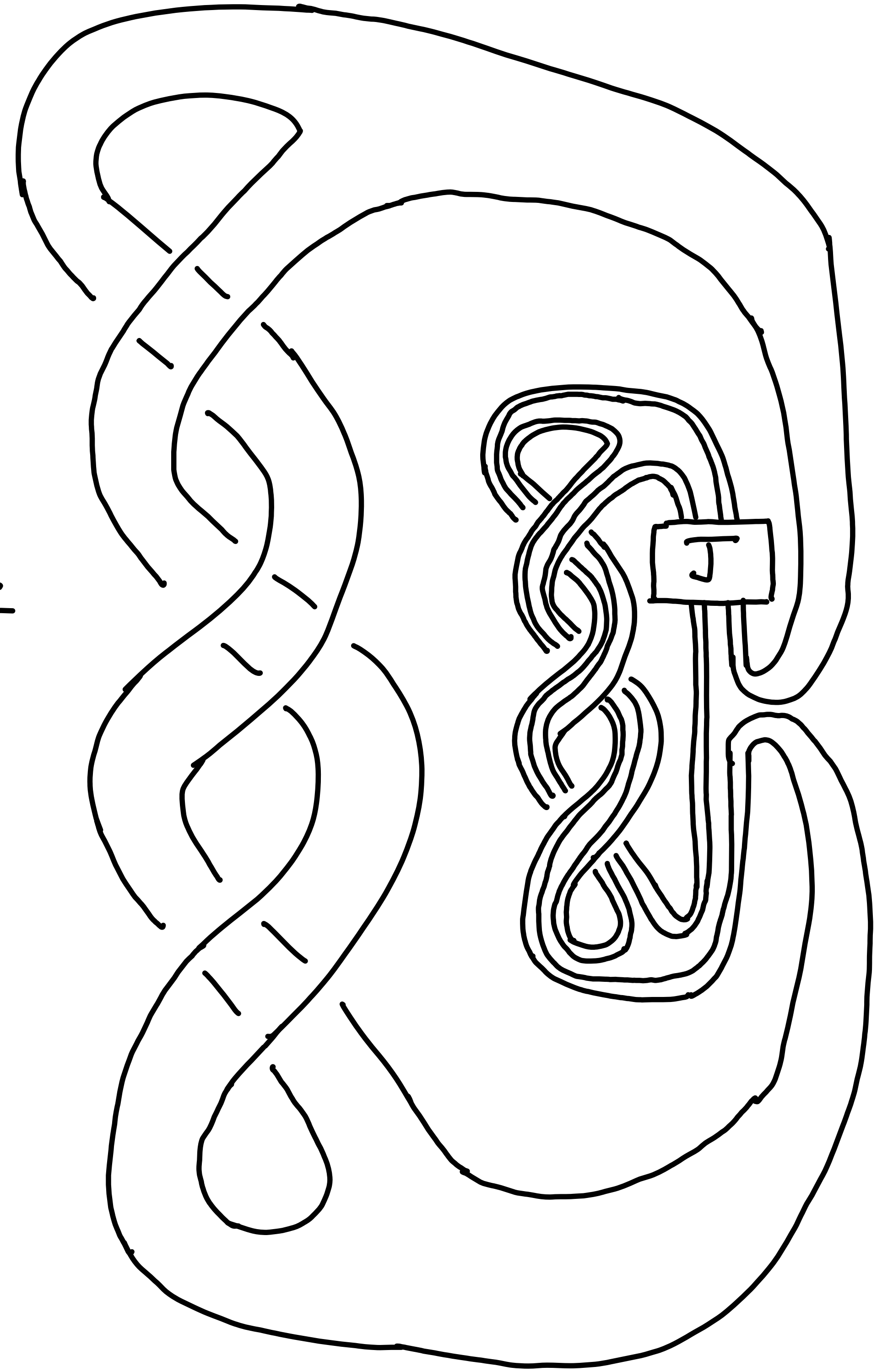
$R^1_{(0)}(J) =$  $=$ 

Whitehead double of J

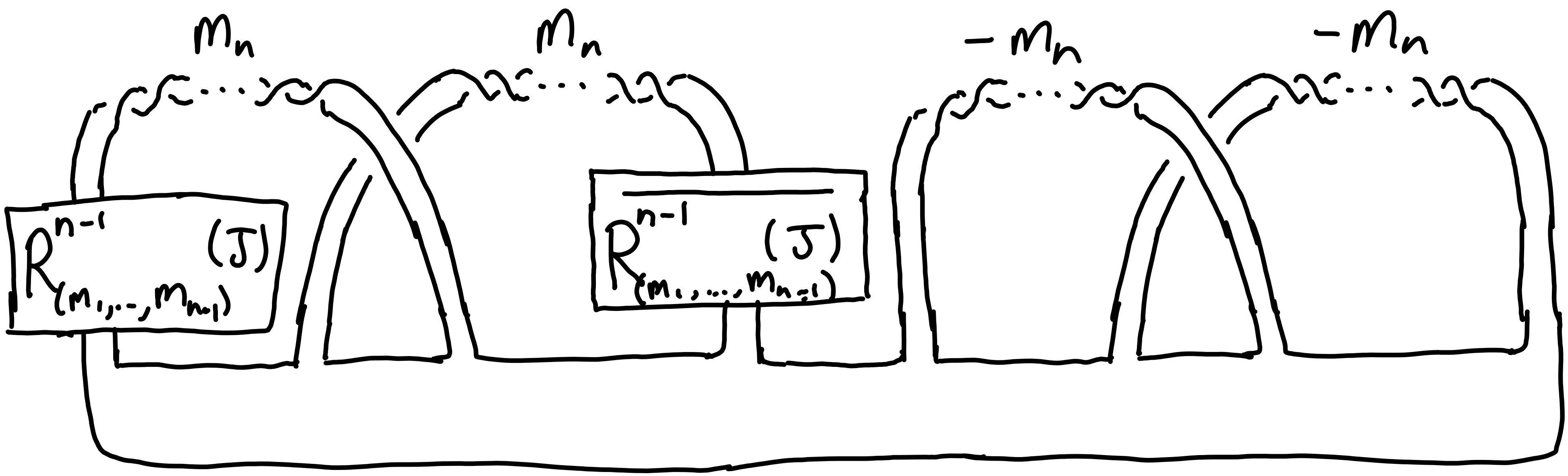
$$R^{n}_{(m_1,\ldots,m_n)}(J) =$$



$R^{2}_{(1,1)}(\text{unknot}) =$ 

$$R^2_{(1,1)}(J) =$$ 

2-torsion examples used in Theorem B



$(n \geq 2)$

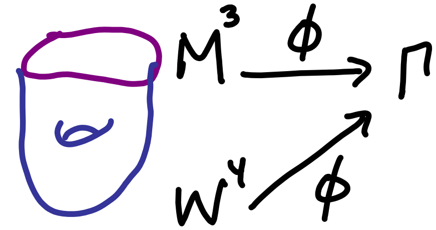
- connected sum of neg. amphichiral
- slice in a rational homology ball

These are inspired by examples of Livingston ($n=1$).

Tools used in the proof

1. von Neumann ρ-invts; $\rho(M_K, \pi_1 M_K \to \pi_1 M_K/N) \in \mathbb{R}$
2. use non-commutative localization "at a prime" to define a commutator series $G_{\mathcal{P}}^{(n)}$ for each $\mathcal{P} = (p_1(t), \ldots, p_n(t))$.
3. for each $\mathcal{P}$, define a refinement $\{\mathcal{F}_n^{\mathcal{P}}\}$ of the (n)-solvable filtration $\{\mathcal{F}_n\}$.
4. use higher-order Alexander modules + linking forms to calculate ρ-invts.

ρ-invts



Let $M^3 = \partial W^4$ and $\phi : \pi_1 W \longrightarrow \Gamma$.

Define

$$\rho(M,\Gamma) := \sigma^{(2)}(W,\Gamma) - \sigma(W)$$

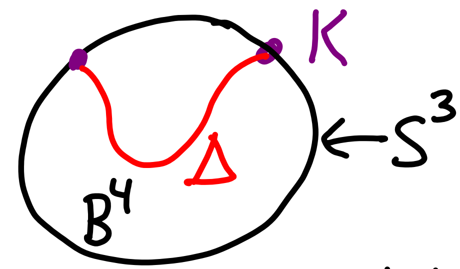
signature of Γ-equivariant intersection form on $H_2(\Gamma\text{-cover of } W)$.

For K = knot $\rightsquigarrow$ M_K = 0-surgery on K

$$\rho(M_K, \pi_1 M_K \longrightarrow \mathbb{Z} = H_1(M_K)) = \rho_0(K)$$

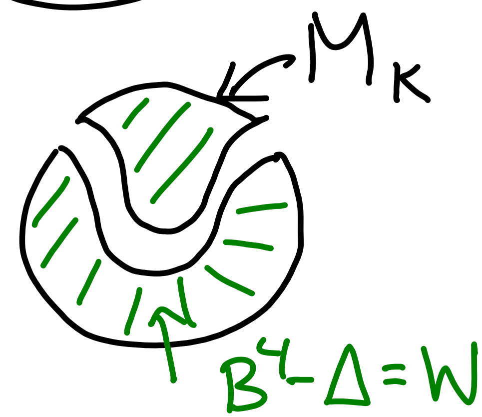
Why "$\rho = 0$" for a slice knot

Let K be a slice knot.



Then M_K = 0-surgery on K

$$= \partial(\underbrace{B^4 - \text{nbhd}(\Delta)}_{W})$$



Since $H_2(W) = 0$,

$H_2(W_\Gamma)$ = torsion (for nice Γ)

Γ-cover of W

Thus $\rho(M_K, \Gamma) = \sigma^{(2)}(W, \Gamma) - \sigma(W) = 0$

Connection between Alexander polynomials and series of groups

The derived series of a group G is:

$$G^{(0)} = G \quad \cdots \quad G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

Let K be a knot and $G = \pi_1(S^3 - K)$.

Recall, the Alexander module of K is

$$\frac{G^{(1)}}{[G^{(1)}, G^{(1)}]} = \frac{G^{(1)}}{G^{(2)}}$$

as a $\mathbb{Z}[G/G^{(1)}] = \mathbb{Z}[t, t^{-1}]$- module.

E.g. $G^{(1)}/G^{(2)} \cong \mathbb{Z}[t,t^{-1}]/\langle p(t)\rangle \rightsquigarrow \Delta_K(t) = p(t)$

If $p(t)$ is rel. prime to $q(t)$, can distinguish

$$\frac{\mathbb{Z}[t,t^{-1}]}{\langle p(t)\rangle} \quad \text{and} \quad \frac{\mathbb{Z}[t,t^{-1}]}{\langle q(t)\rangle}$$

by "killing $q(t)$-torsion."

$$\mathbb{Z}[t,t^{-1}]/\langle q(t)\rangle \longrightarrow \mathbb{Z}[t,t^{-1}]/\langle p(t),q(t)\rangle = 0$$

$$\mathbb{Z}[t,t^{-1}]/\langle p(t)\rangle \longrightarrow \mathbb{Z}[t,t^{-1}]/\langle p(t),p(t)\rangle \neq 0$$

This can be accomplished by localization:

$$G^{(1)}/G^{(2)} \longrightarrow G^{(1)}/G^{(2)} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t,t^{-1}]\, S_{p(t)}^{-1}$$

$$S_{p(t)}^{-1} = \{\text{all } q(t) \text{ relatively prime to } p(t)\}$$

Let

$$G^{(2)} \subset \underline{\underline{G^{(2)}_{p(t)}}} := \ker\left(G^{(1)} \longrightarrow G^{(1)}/G^{(2)} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t,t^{-1}]\, S_{p(t)}^{-1}\right)$$

$\rightsquigarrow$ $G^{(1)}/G^{(2)}_{p(t)} =$ $p(t)$-torsion submodule of Alexander module

Similarly, the higher-order Alexander module of K is

$$G^{(n)}/G^{(n+1)}$$

which is a module over the non-commutative <u>Ore domain</u> $\mathbb{Z}[G/G^{(n)}]$

Wish to kill certain types of torsion by enlarging $G^{(n+1)}$ to $G_{\mathcal{P}}^{(n+1)}$ where $\mathcal{P}=(p_1(t),\ldots,p_n(t))$ and considering $G^{(n)}/G_{\mathcal{P}}^{(n+1)}$.

Let $\underline{P} = (p_1(t), \dots, p_n(t))$. Define for a group G and $k \le n$ inductively:

$$G^{(1)}_{\underline{P}} = \ker\Big(G \longrightarrow \underbrace{\frac{G}{[G,G]} \otimes_{\mathbb{Z}} (\mathbb{Z}-\{0\})^{-1}}_{= H_1(G;\mathbb{Q})}\Big)$$

Let $\Gamma^{(k)}_{\underline{P}} = G/G^{(k)}_{\underline{P}}$

$$G^{(k+1)}_{\underline{P}} = \ker\Big(G^{(k)}_{\underline{P}} \longrightarrow \frac{G^{(k)}_{\underline{P}}}{[G^{(k)}_{\underline{P}}, G^{(k)}_{\underline{P}}]} \otimes_{\mathbb{Z}\Gamma^{(k)}_{\underline{P}}} \mathbb{Z}\Gamma^{(k)}_{\underline{P}} \cdot S^{-1}_{p_k(t)}\Big)$$

<u>Note</u>: $G^{(k)} \subset G^{(k)}_{\underline{P}}$

Properties of $S_{p(t)}$:

Let $\underset{\text{abelian}}{A} \trianglelefteq \Gamma = \Gamma_p^{(k)}$ and $1 \neq a \in A$.

- $\mathbb{Z}\Gamma / \langle p(a) \rangle \hookrightarrow \left(\mathbb{Z}\Gamma / \langle p(a) \rangle \right) S_{p(t)}^{-1}$

- $\left(\mathbb{Z}\Gamma / \langle q(a) \rangle \right) S_{p(t)}^{-1} = 0$

if $p(t)$ and $q(t)$ are strongly coprime.

<u>Note</u>: $MS_{p(t)}^{-1} := M \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}\Gamma \cdot S_{p(t)}^{-1}$

Def A knot is (n, P) solvable ($n \in \mathbb{N}$) if M_K bounds a smooth 4-mfld W s.t.

(1) $i_*: H_1(M_K) \xrightarrow{\cong} H_1(W)$

(2) $H_2(W)$ has a basis $\{f_i, g_i\}_{i=1}^{g}$ of embedded surfaces (w/ triv. normal bundle) all disjoint except $f_i \cdot g_i = 1$ (geometrically)

(3) $\pi_1(f_i), \pi_1(g_i) \subset \pi_1(W)^{(n)}_P$

- $K \in \mathcal{F}_n^P \Leftrightarrow K$ is (n, P)-solvable

For each $P = (p_1(t), \ldots, p_n(t))$ of polynomials (pairwise strongly coprime), our examples produce distinct $Z_P \subset \mathcal{F}_n / \mathcal{F}_{n.5}$ s.t.

$$\boxed{\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{\infty} \cong Z_P \subset \mathcal{F}_n / \mathcal{F}^{P}_{n+1}}$$

More Open Problems

1. What about $\mathcal{F}_{n.5}/\mathcal{F}_{n+1}$?

2. $\mathbb{Z}/4\mathbb{Z} \subset \mathcal{F}_n/\mathcal{F}_{n.5}$ for $n \geq 1$?

   Other types of torsion?

3. Whenever $m_i = 0$ for some i or J = top slice $\Longrightarrow$ our examples are top slice.

   $\rightsquigarrow$ lie in $\bigcap_{n\geq 1} \mathcal{F}_n$.

   Are these smoothly slice? All <u>distinct</u>?

4. $\bigcap_{n\geq 1} \mathcal{F}_n$? only known examples are top slice.