Torsion in the Knot Concordance Group

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There is a binary operation on knots:

\[ K_1 \# K_2 := \]

Connected sum of \( K_1 \) and \( K_2 \).

\[ S \# O = S - O = O \]

\[ K \# O = K \]
Thus $\mathcal{H} = (\{\text{knots}\}, \#)$ forms a monoid with unity $= O$.

However $\mathcal{H}$ is not a group since it does not have inverses.

i.e. there is no knot $K$ such that

$\mathcal{O} \# K = O$.

To get a group structure, define a equivalence relation called concordance.
Def: Knots $K_0$ and $K_1$ are **concordant** if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a smoothly embedded **annulus** in $S^3 \times I$. 
**Def:** A knot $K \subset S^3$ **slice** if $K = 2D$ where $D$ is a 2-dimensional disk (smoothly) embedded in $B^4 = 4$-dim. ball.

A knot is concordant to the unknot $\iff$ it is a slice knot.
If $K$ is any knot then $K \# r\overline{K}$ is slice. (ribbon)

$r\overline{K} =$ change all crossings + change orientation
(reverse of mirror image)

Proof: "Spin" $K$ through $\mathbb{R}_+^4$.

\[ K \# r\overline{K} \]
**Def** The knot concordance group is

\[ C = \{ \text{knots in } S^3 \}/\text{concordance} \]

- \( C \) is an abelian group under the operation connected sum of knots.

\[ [\text{\includegraphics[scale=0.5]{knot1}}] + [\text{\includegraphics[scale=0.5]{knot2}}] = [\text{\includegraphics[scale=0.5]{knot3}}] \]

- \([K] = 0 \iff K \text{ is Slice}\)

\[ [\text{\includegraphics[scale=0.5]{knot4}}] = 0 \]
• The inverse of $[K]$ is $[r\bar{K}]$ since $K \# r\bar{K}$ is slice.

\[-[\mathcal{O}] = [\mathcal{O}]\]
[K] has order 1 $\iff$ K is slice

All known examples of (smoothly) slice knots are ribbon:

K is **ribbon** if it bounds an immersed disk in $S^3$ with only ribbon singularities:
Any ribbon knot is slice.

To obtain embedded $D^2 \subset B^4$, push interior of red disc \emph{into} interior of $B^4$. 
Open Problem:

Ribbon-Slice Conjecture: A knot is smoothly slice if and only if it is ribbon.
\[ [K] \text{ has order 2 } \iff K \# K \text{ is slice } K \text{ is not slice} \]

**Def:** A knot \( K \) is **negative amphichiral** if \( K \) is isotopic to \( rK \).

If \( K \) is neg. amphichiral \( \Rightarrow \)

\[ K \# K = K \# rK = \text{ slice} . \]

i.e. \([K]\) is of order 2 in \( \mathcal{C} \) (if \( K \neq \text{slice} \)).
This is the only known way to create elements of finite order in C.

Open Problems

• (Gordon) If \([K]\) has order 2 in \(C\), then is \(K\) concordant to a negative amphichiral knot?

• Is every element of finite order in \(C\) of order 1 or 2?
Ex: Let

\[ K_m(J) = \]

then \( K_m(J) \) is neg. amphichiral.

\( \square \) means tie two strands into the knot \( J \) (with linking ≠ zero).
If $J = 0$

"tie strings into knot $J$"
Proof that $K_m(J)$ is neg. amphichiral:

$J \rightsquigarrow \text{swing down} \Rightarrow \overline{J}$
swing back up to the "back"
rotate $\mathbb{R}^3$ by $\pi$
about $A$

$K_\mathcal{m}(J)$

$\overline{rK_\mathcal{m}(J)}$
Levine-Tristram signatures: Sliceness Obstructions

\[ V = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ \ell K(a, b^+) \]

\[ K = \partial (\text{surface}) \]

For \( w \in \mathbb{C}, \ |w| = 1 \):

- \( \sigma_w(K) := \text{signature} \left( (1-w)V + (1-\bar{w})V^* \right) \)
- \( \rho_o(K) := \int_{S'} \sigma_w(K) \, dw \)

If \( K \) is slice then \( \rho_o(K) = 0 \).
\( \rho_0 : G \rightarrow \mathbb{R} \) is a homomorphism

Thus \( \rho_0(K) \neq 0 \) \( \Rightarrow \) \( K \) is of \( \infty \)-order

**Ex:** \( V_{Trefoil} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \)

\( \sigma_w(Trefoil) = -2 \)

\( \sigma_w(Trefoil) = 0 \)

\( w = \frac{1 + \sqrt{3}i}{2} \)

\( \rho_0(Trefoil) = -4/3 \neq 0 \)
~67, Milnor-Tristram used signatures to show $C$ has infinite rank

In late 60's Levine used invariants obtained from Seifert matrix (including signatures and Arf invariant) to define epimorphism

\[ C \xrightarrow{\Pi} \text{Algebraic concordance} \cong \mathbb{Z}_2^\infty \times \mathbb{Z}_2^\infty \times \mathbb{Z}_4^\infty \]

group

*Def*; $K$ is algebraically slice if $\pi(K) = 1$. 
Moreover, Levine showed that 
\( A = \) Algebraic concordance group can be decomposed into its \( p(t) \)-primary parts:

\[
C \xrightarrow{\pi} A \cong \bigoplus A(p(t))
\]

\[
p(t) \text{ prime} \\
p(t) = p(t^{-1}) \\
p(1) = \pm 1
\]

\textbf{Note:} The \( A(p(t)) \)-part of \( \pi(k) \) can only be non-trivial if \( p(t) \mid \Delta_k(t) \).
Open Problem:

Conjecture: If $K$ and $J$ have relatively prime Alexander polynomials then $K$ is not top. concordant to $J$ unless $K$ and $J$ are top. slice.

Note: Levine's theorem shows that for $K$ and $J$ not algebraically slice and with relatively prime Alexander polynomial, $K$ is not concordant to $J$. 
Example of an Algebraically slice knot

For $K$ with certain non-vanishing signatures, Gilmer used Casson-Gordon invariants to show $9_{46}(K)$ is not slice.
In 1997, Cochran-Orr-Teichner defined the \((n)\)-solvable filtration of \(G\) \((n \in \mathbb{N}/2)\)

\[
0 = \{\text{slice knots}\} \subset \cdots \subset F_n \subset \cdots \subset F_1 \subset F_{0.5} \subset F_0 \subset G
\]

- \(\mathcal{F}_0 = \text{Arf invariant zero knots}\)
- \(\mathcal{F}_{0.5} = \text{Algebraically slice knots}\)
- \(\mathcal{F}_{1.5} \subset \text{knots with vanishing Casson-Gordon invariants}\)
Thm (n = 0, 67, Milnor-Tristram; n = 1, 81, Jiang; n = 2, 00, Cochran-Orr-Teichner)

For n = 0, 1, 2, \( \mathcal{F}_n / \mathcal{F}_{n,5} \) contains a \( \mathbb{Z}^\infty \).

Thm (Livingston) \( \mathcal{F}_1 / \mathcal{F}_{1,5} \) contains a \( \mathbb{Z}^\infty \).

Thm (Cochran-Teichner, Cochran-H-L-Keidy): For n ≥ 3, \( \mathcal{F}_n / \mathcal{F}_{n,5} \subset \mathbb{Z} \oplus \mathbb{Z}^\infty \).

CT (02) \quad CHL (07)
Thm (Se-Goo Kim ‘05, Taehee Kim ‘08):

\[ \bigoplus \mathbb{Z}^\infty \subset \mathbb{F}_{1,5} \]

- \( p(t) \) prime
- \( p(t) = p(t') \)
- \( p(1) = \pm 1 \)

Other work done at \( \mathbb{F}_{1,5} \) level by S. Friedl, T. Kim and P. Gilmer.
Let $S_n = \{(p_{i_1}(t), p_{i_2}(t), \ldots, p_{i_n}(t)) \mid i \in \mathbb{Z}\}$ be a sequence of non-unit polynomials satisfying $p_{ij}(1) = \pm 1$, that are pairwise coordinatewise “strongly coprime”.

**Theorem A (Cochran-H-Leidy, '08):**

For each $n \geq 2$,

$$\bigoplus_{(p_1(t), \ldots, p_n(t)) \in S_n} \mathbb{Z}^\infty \subset \mathfrak{f}_n / \mathfrak{f}_{n.5}.$$
**Theorem B** (Cochran-H-Leidy, '09):

For each \( n \geq 2 \),

\[
\bigoplus \mathbb{Z} / 2\mathbb{Z} \subseteq \gamma_n / \alpha_{4n-5}.
\]

\[ (p_1(t), p_2(t), \ldots, p_n(t)) \in S_n \]

- Represented by neg. amphichiral knots that are slice in a rational homology 4-ball

\[ \Rightarrow \tau, s \text{ invariants are zero.} \]
Def: We say \( p(t) \) and \( q(t) \) are **isogenous**, denoted \((\tilde{p}, \tilde{q}) \neq 1\) if for some non-zero roots \( r_p \) of \( p(t) \) and \( r_q \) of \( q(t) \), and some \( m, n \in \mathbb{Z} - \{0\} \), \( r_p^m = r_q^n \). Otherwise, we say they are **strongly coprime**, denoted \((\tilde{p}, \tilde{q}) = 1\).

Ex: \( p_k(t) = (kt-(k+1))(k+1)t-k \), \( k \in \mathbb{Z}^+ \)

roots \( R_k = \{ \frac{k}{k+1}, \frac{k+1}{k} \} \).

\( (\tilde{p}_k, \tilde{p}_\ell) = 1 \) when \( k \neq \ell \).
Examples used in Theorem A:

\[ R_m := \begin{cases} \text{(m ≠ 0)} \end{cases} \]

then \( R_m \) is slice and \( \Delta_{R_m}(t) = p_m(t) \)

where \( p_m(t) = (mt - (m+1))(m+1)t - m \) so

\( (\Delta_{R_m}(t), \Delta_{R_k}(t)) = 1 \) for \( k \neq m \).
\[ R^1_{(m_i)}(J) = \text{ (am+1) half twists } \]

Note: When \( m_i = 0 \),

\[ R^1_{(0)}(J) = \text{ whitehead double of } J \]
$$R^n_{(m_1, \ldots, m_n)}(J) = (2^{M_{n+1}}) \text{ half twists}$$
$R_{(1,1)}(\text{unknot}) =$
$R^{2}_{(1,1)}(J) = \text{Diagram}$
$\alpha$-torsion examples used in Theorem B

- connected sum of neg. amphichiral $R_{(m_1, \ldots, m_{n-1})}$
- Slice in a rational homology ball

These are inspired by examples of Livingston ($n=1$).
Tools used in the proof

1. von Neumann $p$-invariants, $p(M_k, \pi_1 M_k \to \pi_1 M_k \times \mathbb{Z}) \in \mathbb{R}$

2. use non-commutative localization "at a prime" to define a commutator series $G_p^{(n)}$ for each $P = (p_1(t), \ldots, p_n(t))$.

3. for each $P$, define a refinement $\{ \alpha P \}$ of the $(n)$-solvable filtration $\{ \mathcal{F}_n \}$.

4. use higher-order Alexander modules + linking forms to calculate $p$-invariants.
Let $M^3 = 2W^4$ and $\phi: \pi_1 W \to \Gamma$.

Define

$$\rho(M, \Gamma) := \sigma^{(2)}(W, \Gamma) - \sigma(W)$$

signature of $\Gamma$-equivariant intersection form on $H_2(\Gamma$-cover of $W$).

For $K = \text{knot} \Rightarrow M_K = 0$-surgery on $K$

$$\rho(M_K, \pi_1 M_K \to \mathbb{Z} = H_1(M_K)) = \rho_0(K)$$
Why \( \rho = 0 \) for a slice knot

Let \( K \) be a slice knot.

Then \( M_K = 0 \)-surgery on \( K \)
\[
= 2(B^4 - \text{nbhd}(\Delta)) \quad \underline{W}
\]

Since \( H_2(W) = 0 \),
\[ H_2(W_\pi) = \text{torsion} \ (\text{for nice } \pi) \]
\[ \pi \text{-cover of } W \]

Thus \( \rho(M_K, \pi) = \sigma^{(2)}(W, \pi) - \sigma(W) = 0 \)
Connection between Alexander polynomials and series of groups

The derived series of a group $G$ is:

$$G^{(0)} = G \quad \ldots \quad G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

Let $K$ be a knot and $G = \pi_1(S^3 - K)$. Recall, the Alexander module of $K$ is

$$\frac{G^{(1)}}{[G^{(1)}, G^{(1)}]} = \frac{G^{(1)}}{G^{(2)}}$$

as a $\mathbb{Z}[G/G^{(1)}] = \mathbb{Z}[t, t^{-1}]$-module.
E.g. $G^{(2)}/G^{(1)} \cong \mathbb{Z}[t, t^{-1}]/\langle p(t) \rangle \rightarrow \Delta_k(t) = p(t)$

If $p(t)$ is rel. prime to $q(t)$, can distinguish

\[
\frac{\mathbb{Z}[t, t^{-1}]}{\langle p(t) \rangle} \quad \text{and} \quad \frac{\mathbb{Z}[t, t^{-1}]}{\langle q(t) \rangle}
\]

by "killing $q(t)$-torsion."

\[
\frac{\mathbb{Z}[t, t^{-1}]}{\langle q(t) \rangle} \rightarrow \frac{\mathbb{Z}[t, t^{-1}]}{\langle p(t), q(t) \rangle} = 0
\]

\[
\frac{\mathbb{Z}[t, t^{-1}]}{\langle p(t) \rangle} \rightarrow \frac{\mathbb{Z}[t, t^{-1}]}{\langle p(t), p(t) \rangle} \neq 0
\]
This can be accomplished by localization:

\[
\frac{G^{(1)}}{G^{(2)}} \rightarrow \frac{G^{(1)}}{G^{(2)}} \otimes \mathbb{Z}[t, t^{-1}] S_{p(t)}^{-1}
\]

\[S_{p(t)}^{-1} = \{ \text{all } q(t) \text{ relatively prime to } p(t) \}\]

Let

\[
G^{(2)}_{p(t)} := \ker \left( G^{(1)} \rightarrow \frac{G^{(1)}}{G^{(2)}} \otimes \mathbb{Z}[t, t^{-1}] S_{p(t)}^{-1} \right)
\]

\[
\sim \rightarrow \frac{G^{(1)}}{G^{(2)}} = \text{p(t)-torsion submodule of Alexander module}
\]
Similarly, the higher-order Alexander module of $K$ is

$$G^{(n)}/G^{(n+1)}$$

which is a module over the non-commutative Ore domain $\mathbb{Z}[G/G^{(n)}]$.

Wish to kill certain types of torsion by enlarging $G^{(n+1)}$ to $G^{(n+1)}_P$ where $P = (p_1(t), ..., p_n(t))$ and considering $G^{(n)}/G^{(n+1)}_P$. 
Let \( P = (p_1(t), \ldots, p_n(t)) \). Define for a group \( G \) and \( k \leq n \) inductively:

\[
G^{(k)}_P = \ker \left( G \longrightarrow \frac{G}{[G, G]} \otimes_{\mathbb{Z}} (\mathbb{Z}-\{0\})^{-1} \right)
\]

\[
H_1(G; \mathbb{Q})
\]

Let \( \Gamma_P^{(k)} = G/G^{(k)}_P \)

\[
G^{(k+1)}_P = \ker \left( G^{(k)}_P \longrightarrow \frac{G^{(k)}_P}{[G^{(k)}_P, G^{(k)}_P]} \otimes_{\mathbb{Z}} \mathbb{Z} \Gamma_P^{(k)} \cdot S_{p_k(t)}^{-1} \right)
\]

\[\underline{Note:} \quad G^{(k)} \subset G^{(k)}_P\]
Properties of $S_{p(t)}$:

Let $A < \Gamma = \Gamma_p^{(k)}$ and $1 \neq a \in A$ abelian.

- $\mathbb{Z}\Gamma \langle p(a) \rangle \rightarrow (\mathbb{Z}\Gamma \langle p(a) \rangle) S_{p(t)}^{-1}$

- $(\mathbb{Z}\Gamma \langle q(a) \rangle) S_{p(t)}^{-1} = 0$

if $p(t)$ and $q(t)$ are strongly coprime.

Note: $MS_{p(t)}^{-1} = M \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}\Gamma S_{p(t)}^{-1}$
A knot is \((n,P)\) solvable \((n \in \mathbb{N})\) if \(M_k\) bounds a smooth 4-mfld \(W\) s.t.

1. \(i_k : H_1(M_k) \xrightarrow{\sim} H_1(W)\)
2. \(H_2(W)\) has a basis \(\{f_i, g_i\}_{i=1}^{9}\) of embedded surfaces (with trivial normal bundle) all disjoint except \(f_i \cdot g_i = 1\) (geometrically)
3. \(\pi_1(f_i), \pi_1(g_i) < \pi_1(W)^{(n)}\)

\[ K \in \mathfrak{f}_n^P \iff K \text{ is } (n,P)\text{-solvable} \]
For each $P = (p_1(t), \ldots, p_n(t))$ of polynomials (pairwise strongly coprime), our examples produce distinct $\mathbb{Z}_p \subset \mathbb{F}_n / \mathbb{F}_{n+1}$ s.t.

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^\infty \cong \mathbb{Z}_p \subset \mathbb{F}_n / \mathbb{F}_{n+1}$$
More Open Problems

1. What about $G_{n,5}/G_{n+1}$?

2. $\mathbb{Z}/4\mathbb{Z} \subset G_n/G_{n,5}$ for $n \geq 1$?

Other types of torsion?

3. Whenever $m_i = 0$ for some $i$ or $J = \text{top slice}$

$\Rightarrow$ our examples are top slice.

$\Rightarrow$ lie in \bigcap_{n \geq 1} G_n.

Are these smoothly slice? All distinct?

4. $\bigcap_{n \geq 1} G_n$? only known examples are top slice.