

Torsion in the Knot Concordance Group

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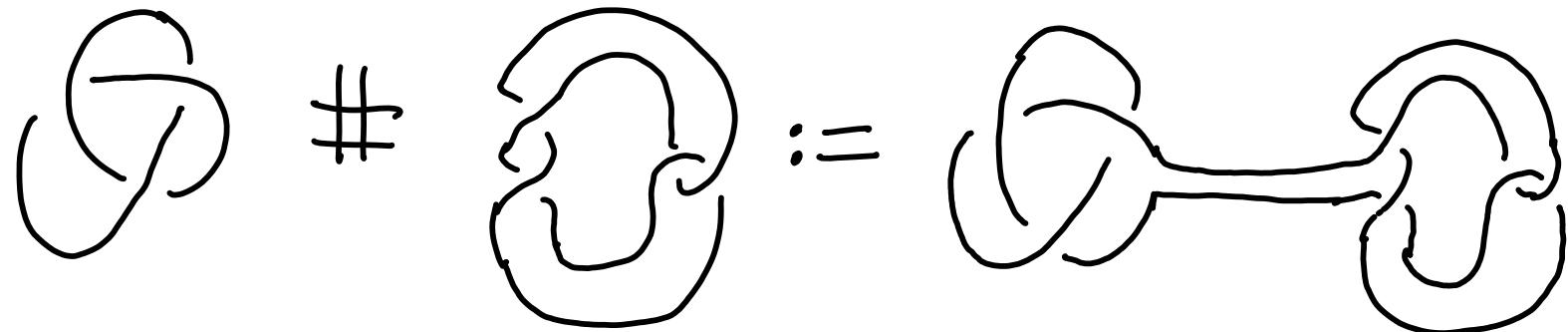
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joint work with:

Tim Cochran (Rice University)

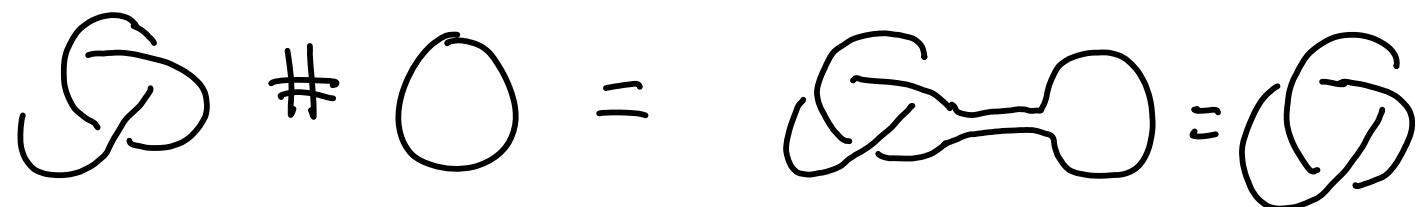
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There is a binary operation on knots:



$$K_1 \quad K_2 \quad K_1 \# K_2$$

connected sum of
 K_1 and K_2 .



$$K \# O = K$$

Thus $\mathcal{K} = (\{\text{knots}\}, \#)$ forms a monoid with unity = O .

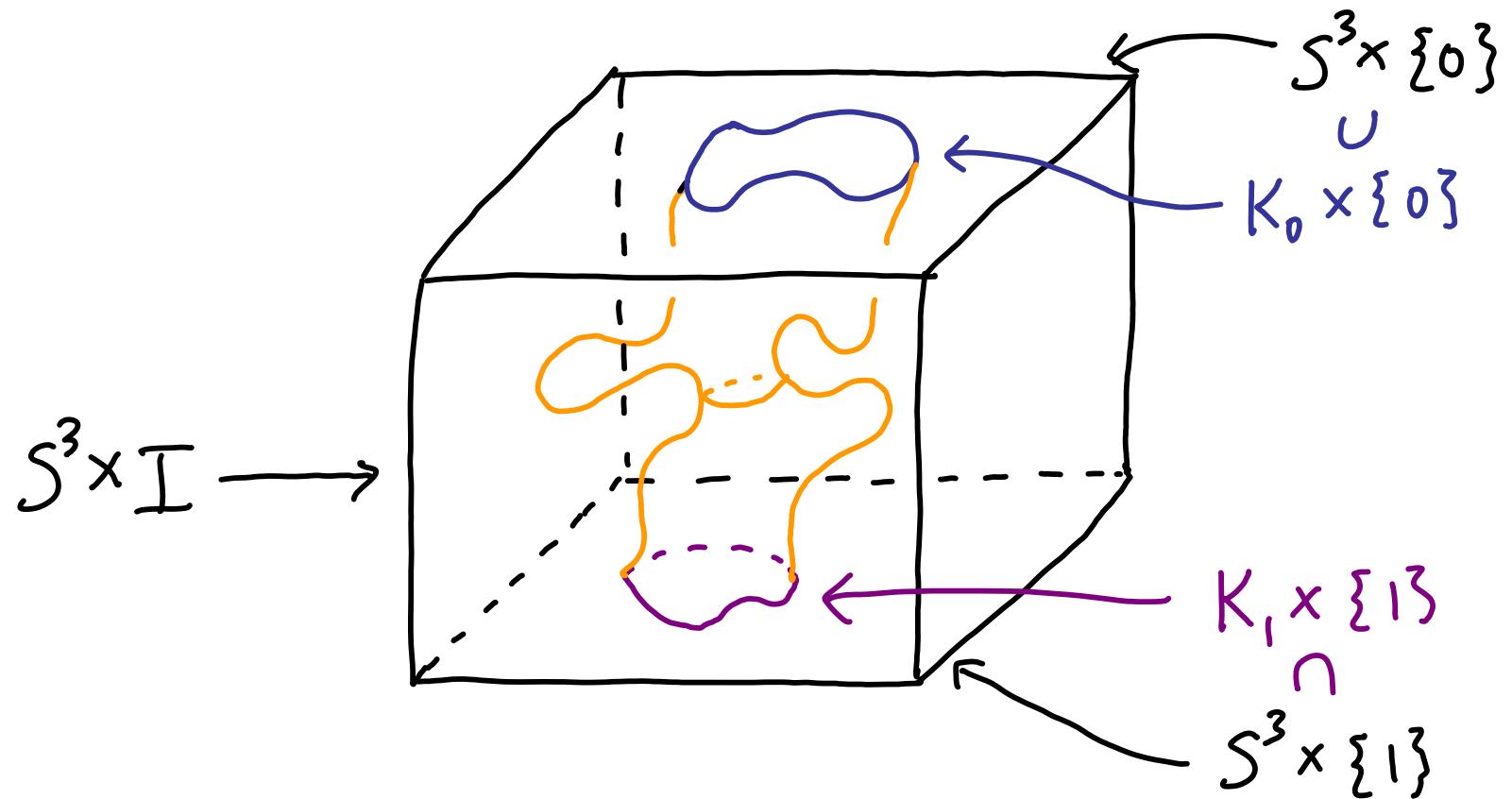
However \mathcal{K} is not a group since it does not have inverses.

i.e. there is no knot K such that

$$\text{G} \# K = \text{O}.$$

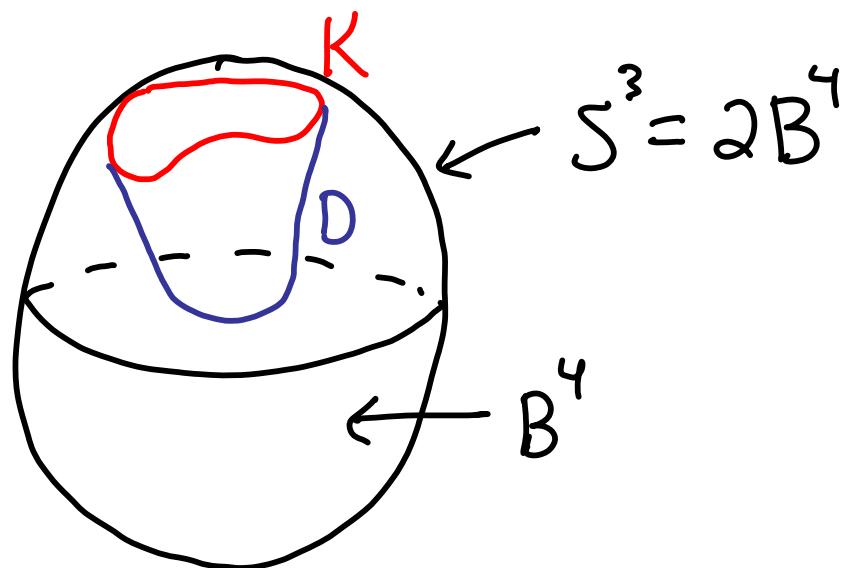
To get a group structure, define an equivalence relation called **concordance**.

Def: Knots K_0 and K_1 are concordant
 if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a
 smoothly embedded **annulus** in $S^3 \times I$.



A knot is concordant to the unknot
 \Leftrightarrow it is a **slice** knot.

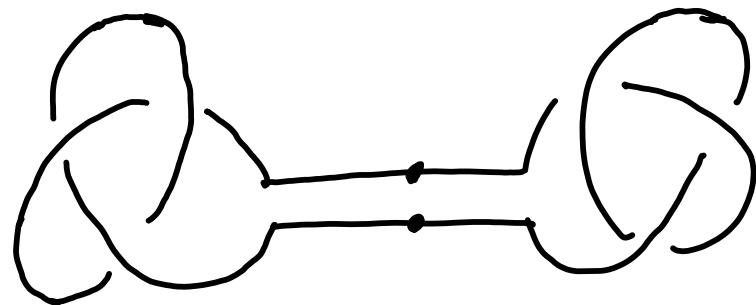
Def: A knot $K \subset S^3$ slice if $K = \partial D$ where
D is a 2-dimensional disk (smoothly) embedded
in $B^4 = 4\text{-dim. ball}$.



If K is any knot then $K \# r\bar{K}$ is slice.
(ribbon)

$r\bar{K}$ = change all crossings + change orientation
(reverse of mirror image)

Proof: "Spin" K through \mathbb{R}^4_+ .



$K \ # \ r\bar{K}$

Defn The knot concordance group is

$$\mathcal{C} = \{\text{knots in } S^3\} / \text{concordance}$$

- \mathcal{C} is an abelian group under the operation connected sum of knots.

$$[S] + [S] = [S \# S]$$

- $[K] = 0 \iff K \text{ is slice}$

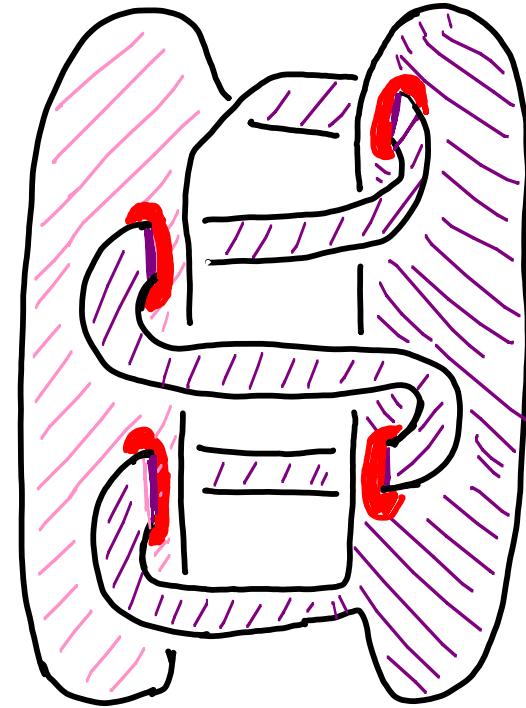
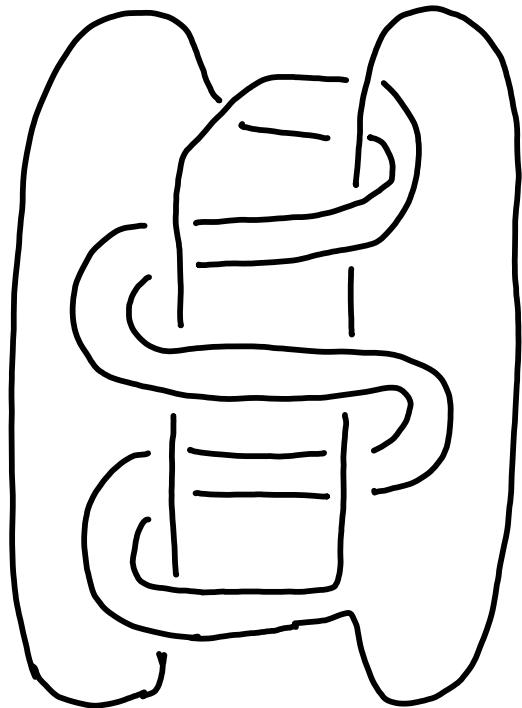
$$[K] = 0$$

- The inverse of $[K]$ is $[r\bar{K}]$ since $K \# r\bar{K}$ is slice.

$$-[\text{G}] = [\text{O}]$$

$[K]$ has order 1 $\Leftrightarrow K$ is slice

Ex: Any ribbon knot is slice.

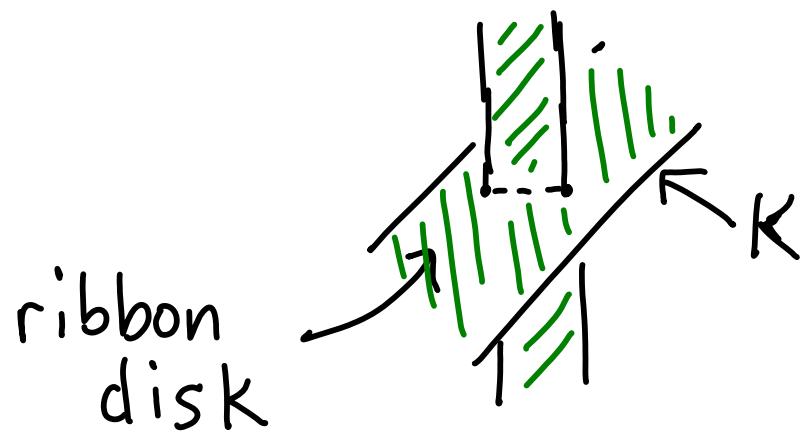


To obtain embedded $D^2 \times B^4$, push interior of red discs into interior of B^4 .

Open Problem :

Ribbon-Slice Conjecture : A knot is ribbon \Leftrightarrow it is smoothly slice.

K is **ribbon** if it bounds an immersed disk in S^3 with only ribbon singularities:



$[K]$ has order 2 \iff $K \# K$ is slice
 $+$
 K is not slice

Def: A knot K is **negative amphichiral** if
 K is isotopic to $r\bar{K}$.

If K is neg. amphichiral and not slice

$$K \# K = K \# r\bar{K} = \text{slice}.$$

i.e. $[K]$ is of order 2 in \mathcal{C} .

This is the only known way to create elements of finite order in \mathcal{C} .

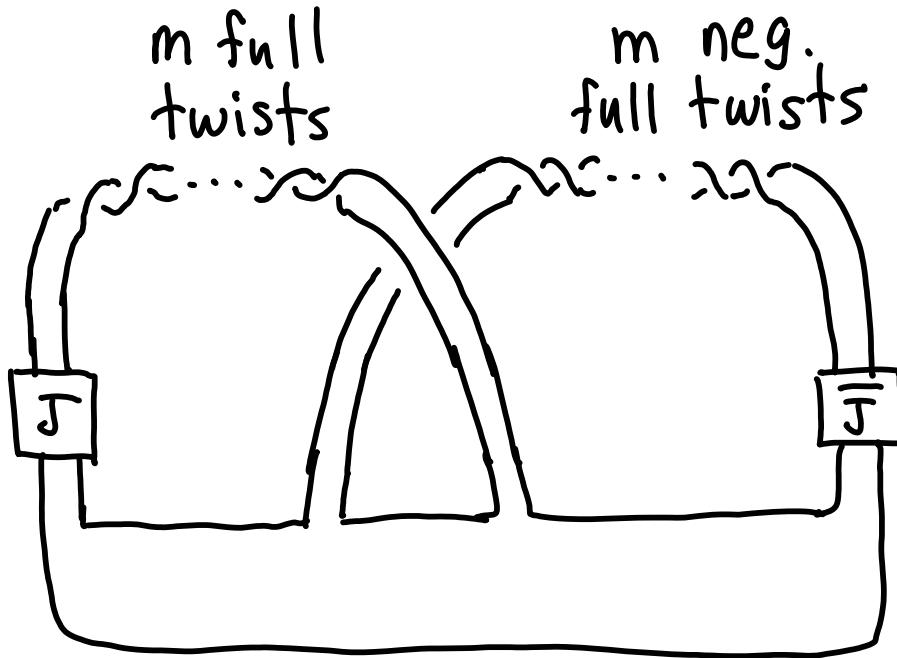
Open Problems

(Gordon) If $[K]$ has order 2 in \mathcal{C} , then is K concordant to a negative amphichiral knot?

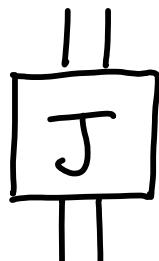
Is every element of finite order in \mathcal{C} of order 1 or 2?

Ex: Let

$$K_m(J) =$$

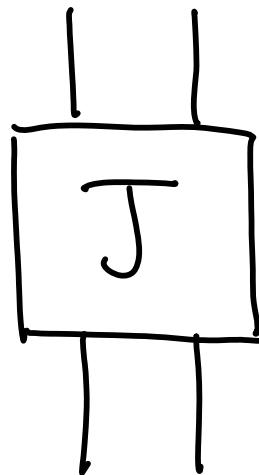
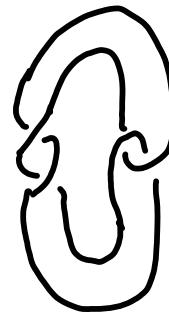


then $K_m(J)$ is neg. amphichiral.

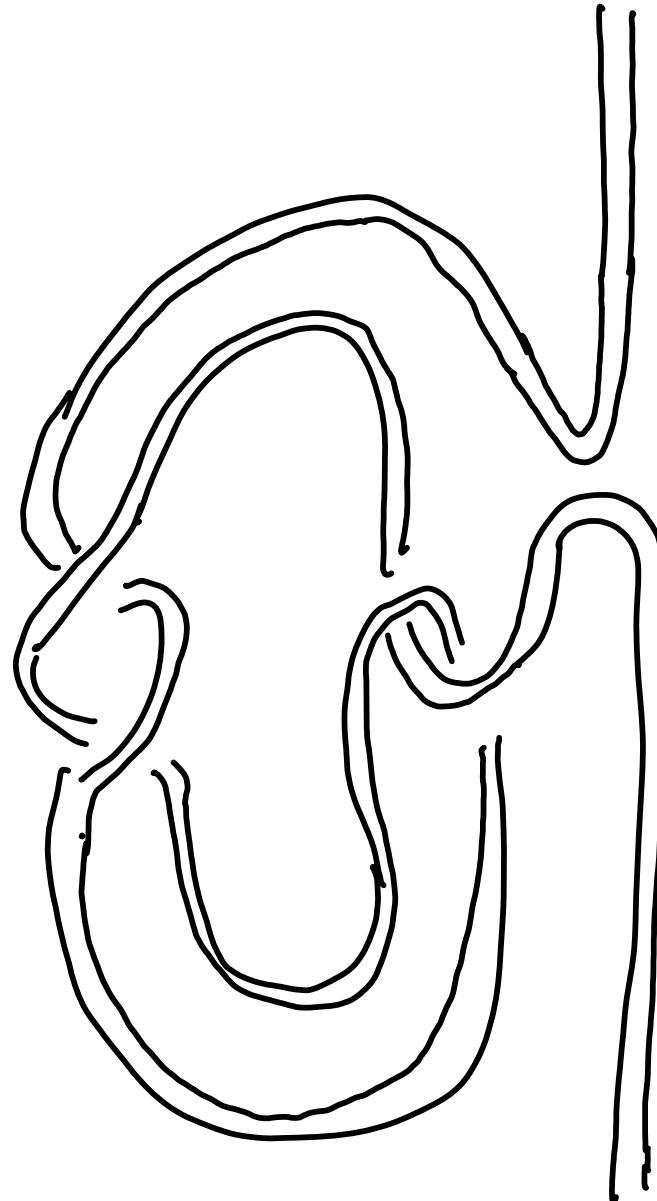


means tie two strands into the knot J (with linking # zero).

If $J =$

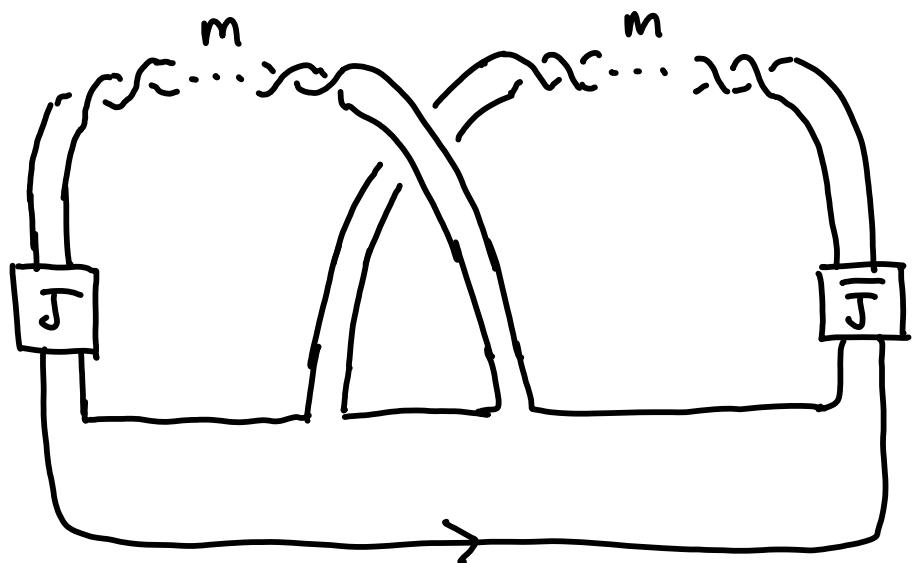


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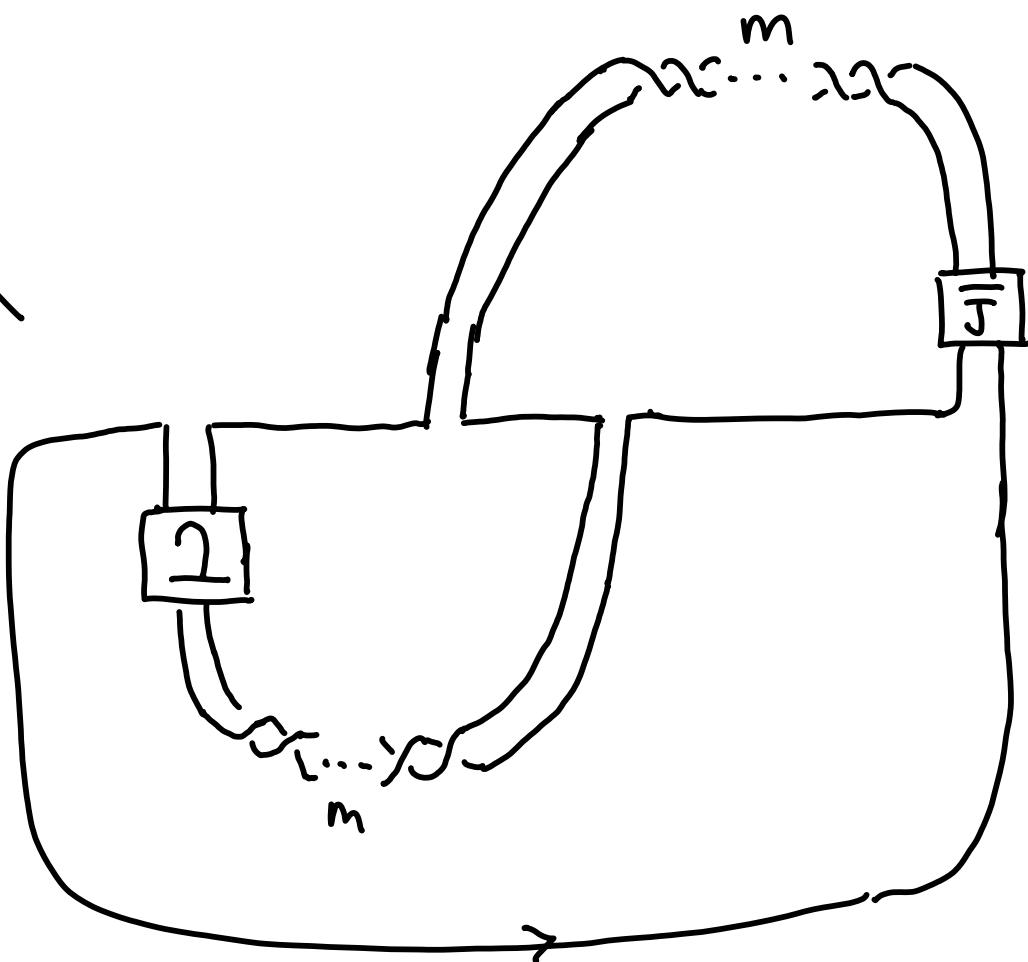
"tie strings
into knot
 J "

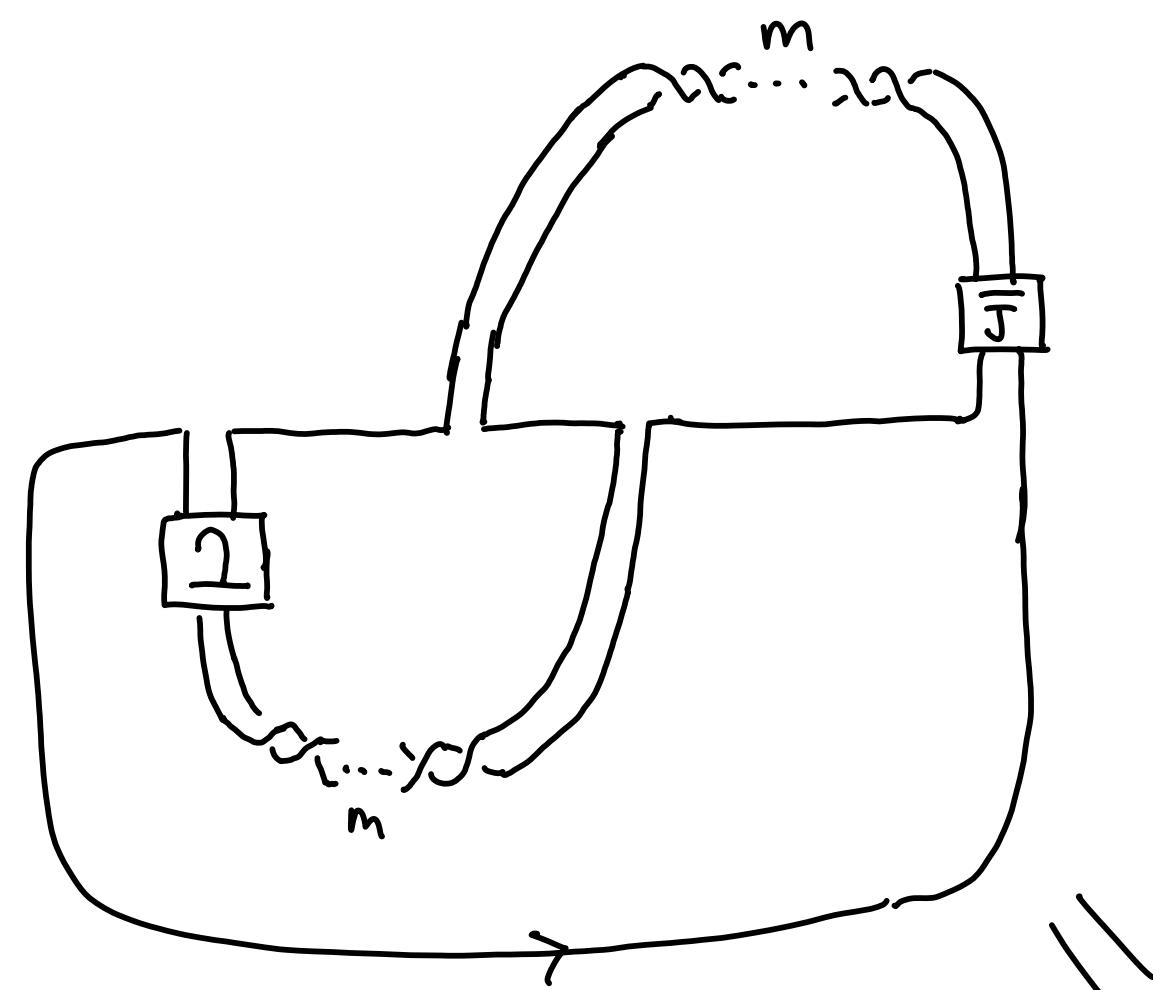
Proof that $K_m(J)$ is neg. amphichiral:



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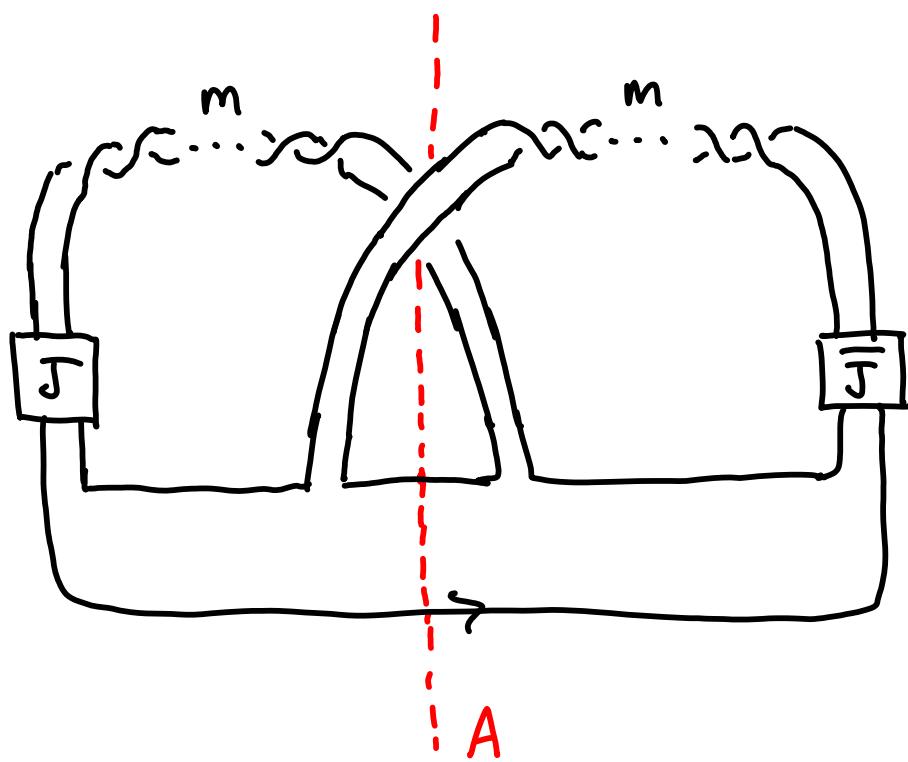
swing down





swing back up
to the "back"



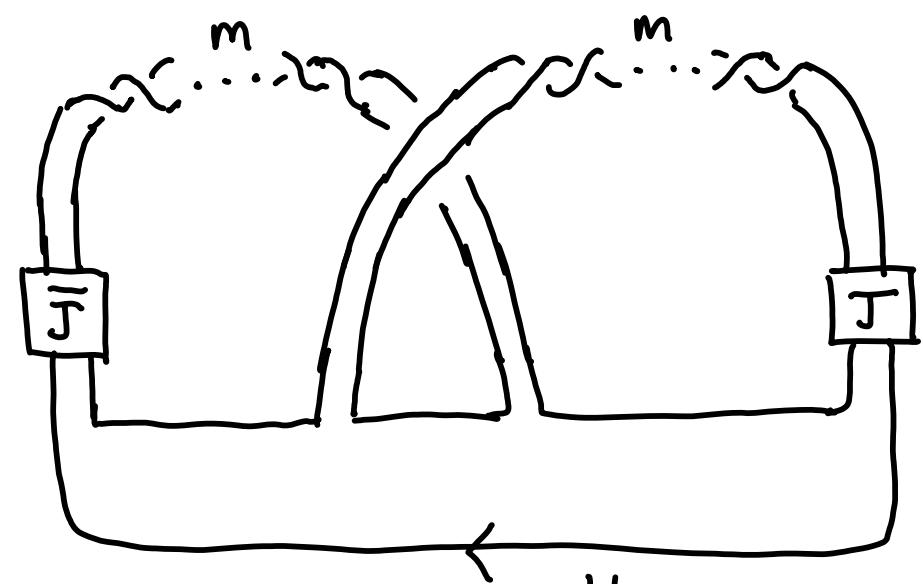


rotate R^3 by π
about A

\Rightarrow

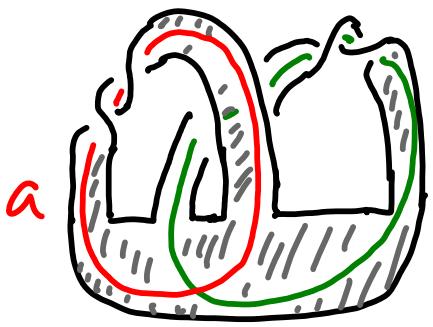


$K_m(J)$



$r \overline{K_m(J)}$

Levihe-Tristram signatures: Sliceness Obstructions


$$K = \partial(\text{surface})$$

$\text{lk}(a, b^+)$

$$V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

"linking" matrix

For $w \in \mathbb{C}$, $|w| = 1$:

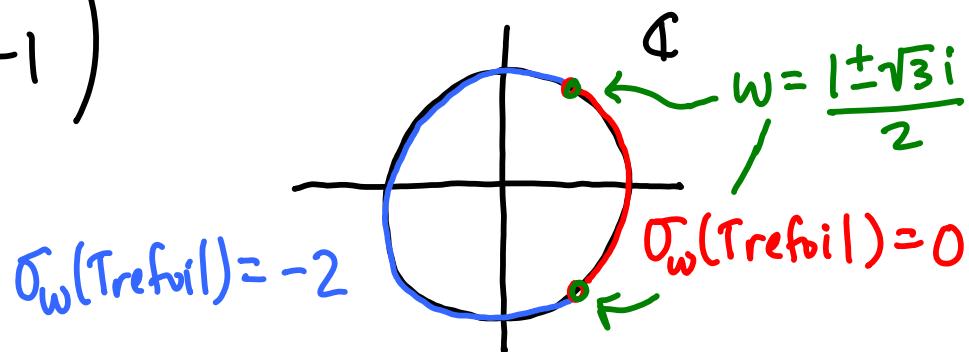
- $\sigma_w(K) := \text{signature}((1-w)V + ((-w)\bar{V})^T)$
- $\rho_0(K) := \int_{S^1} \sigma_w(K) dw$

If K is slice then $\rho_0(K) = 0$.

$\rho_0 : \mathbb{C} \longrightarrow \mathbb{R}$ is a homomorphism

Thus $\rho_0(K) \neq 0 \Rightarrow K$ is of ∞ -order

Ex: $V_{\text{Trefoil}} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$



$$\rho_0(\text{Trefoil}) = -4/3 \neq 0$$

~ 67 , Milnor-Tristram used signatures
to show C has infinite rank

In late 60's Levine used invariants
obtained from Seifert matrix
(including signatures and Arf invariant)
to define epimorphism

$$C \xrightarrow{\pi} \text{Algebraic concordance} \cong \mathbb{Z}^\infty \times \mathbb{Z}_2^\infty \times \mathbb{Z}_4^\infty$$

group

Def: K is algebraically slice if $\pi(K) = 1$.

Moreover, Levine showed that
 $A = \text{Algebraic concordance group}$ can be
decomposed into its $p(t)$ -primary parts:

$$C \xrightarrow{\pi} A \cong \bigoplus_{p(t) \text{ prime}} A(p(t))$$

$$p(t) \doteq p(t')$$

$$p(1) = \pm 1$$

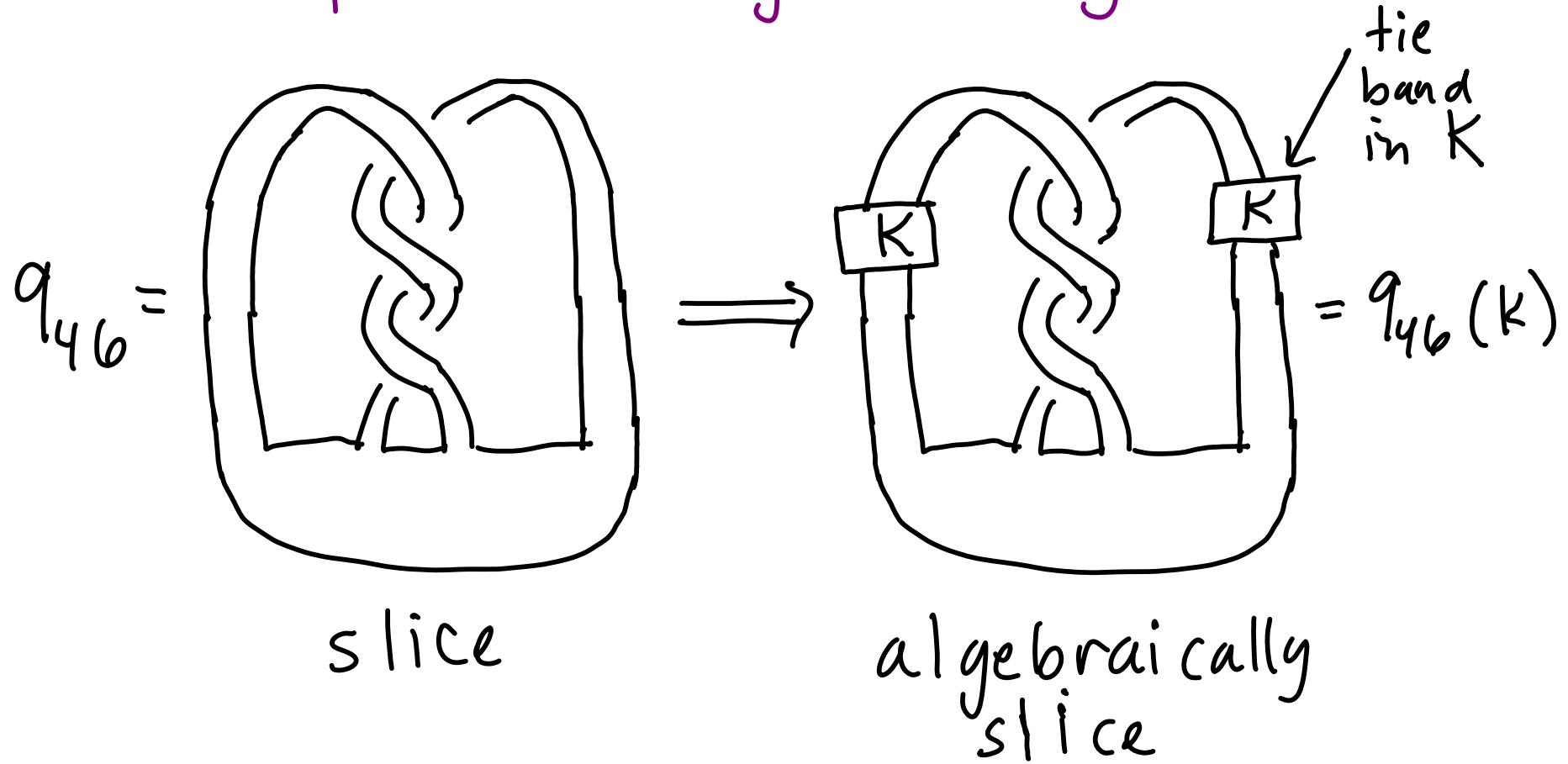
Note: The $A(p(t))$ -part of $\pi(K)$ can
only be non-trivial if $p(t) \mid \Delta_K(t)$.

Open Problem:

Conjecture: If K and J have relatively prime Alexander polynomials then K is not top. concordant to J unless K and J are top. slice.

Note: Levine's theorem shows that for K and J not algebraically slice and with relatively prime Alexander polynomial, K is not concordant to J .

Example of an Algebraically slice knot



For K with certain non-vanishing signatures, Gilmer used Casson-Gordon invariants to show $q_{46}(K)$ is not slice.

In 1997, Gchran-Orr-Teichner defined the (n) -solvable filtration of G ($n \in \mathbb{N}/2$)

$$0 = \left\{ \begin{matrix} \text{slice} \\ \text{knots} \end{matrix} \right\}_0 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset G$$

- \mathcal{F}_0 = Arf invariant zero knots
- $\mathcal{F}_{0.5}$ = Algebraically slice knots
- $\mathcal{F}_{1.5} \subset$ knots with vanishing Casson-Gordon invariants.

Thm $\left(\begin{array}{l} n=0, \sim 67, \text{Milnor-Tristram}; n=1, \sim 81, \text{Jiang}; \\ n=2, \sim 00, \text{Cochran-Orr-Teichner} \end{array} \right)$

For $n=0,1,2$, $\mathcal{Y}_n/\mathcal{Y}_{n,5}$ contains a \mathbb{Z}^∞ .

Thm (Livingston) $\mathcal{Y}_1/\mathcal{Y}_{1,5}$ contains a \mathbb{Z}_2^∞ .

Thm (Cochran-Teichner, Cochran-H-Leidy): For $n \geq 3$,

$$\mathcal{Y}_n/\mathcal{Y}_{n,5} \supset \mathbb{Z} \oplus \mathbb{Z}^\infty.$$

\uparrow \uparrow
 CT ('02) CHL ('07)

Thm (Se-Goo Kim '05, Taehee Kim '08):

$$\bigoplus_{\substack{p(t) \text{ prime} \\ p(t) = p(t') \\ p(1) = \pm 1}} \mathbb{Z}^\infty \subset \mathfrak{f}_1 / \mathfrak{f}_{1,5}$$

Other work done at $\mathfrak{f}_1 / \mathfrak{f}_{1,5}$ level by S. Friedl, T. Kim and P. Gilmer.

Much work done by Jae Choon Cha for links.

Let $S_n = \{(p_{i_1}(t), p_{i_2}(t), \dots, p_{i_n}(t)) \mid i \in \mathbb{Z}\}$
 be a sequence of non-unit polynomials
 satisfying $p_{ij}(1) = \pm 1$, that are pairwise
 coordinatewise "strongly coprime".

Theorem A (Cochran-H-Leidy, '08):

For each $n \geq 2$,

$$\bigoplus \mathbb{Z}^\omega \subset \frac{\alpha f_n}{\alpha f_{n.5}}.$$

$$(p_1(t), \dots, p_n(t)) \in S_n$$

Theorem B (Cochran - H - Leidy, '09):

For each $n \geq 2$,

$$\bigoplus \mathbb{Z}/2\mathbb{Z} \subset \frac{\alpha}{f_n} / \frac{\alpha}{f_{n,5}}.$$

$$^*(p_1(t), p_2(t), \dots, p_n(t)) \in S_n$$

- Represented by neg. amphichiral knots that are slice in a rational homology 4-ball
 $\Rightarrow \tau, s$ invts are zero.

$$| * p_i(t) \in \left\{ m^2 t^2 - (2m^2 + 1)t + m^2 \mid m \neq 0 \right\}$$

Def: We say $p(t)$ and $q(t)$ are isogenous, denoted $(\tilde{p}, \tilde{q}) \neq 1$ if for some non-zero roots r_p of $p(t)$ and r_q of $q(t)$, and some $m, n \in \mathbb{Z} - \{0\}$, $r_p^n = r_q^m$. Otherwise, we say they are Strongly coprime, denoted $(\tilde{p}, \tilde{q}) = 1$.

Ex: $p_k(t) = (kt - (k+1))((k+1)t - k)$, $k \in \mathbb{Z}^+$

roots $R_k = \left\{ \frac{k}{k+1}, \frac{k+1}{k} \right\}$.

$$(\tilde{p}_k, \tilde{p}_l) = 1 \text{ when } k \neq l.$$

Examples used in Theorem A:

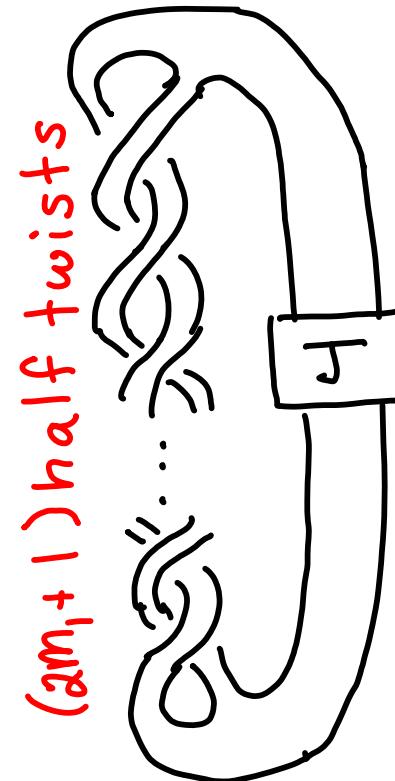
$$R_m := \\ (m \neq 0)$$



then R_m is slice and $\Delta_{R_m}(t) = P_m(t)$

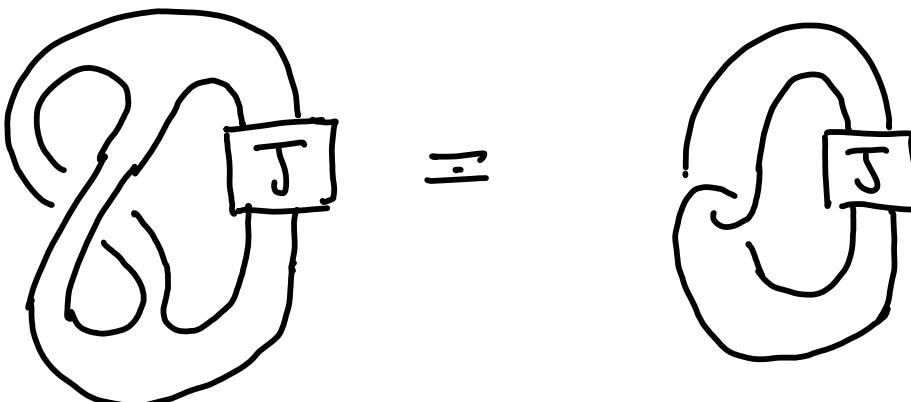
where $P_m(t) = (mt - (m+1))((m+1)t - m)$ so
 $(\Delta_{R_m}(t), \Delta_{R_k}(t)) = 1$ for $k \neq m$.

$$R_{(m_1)}^1(J) =$$



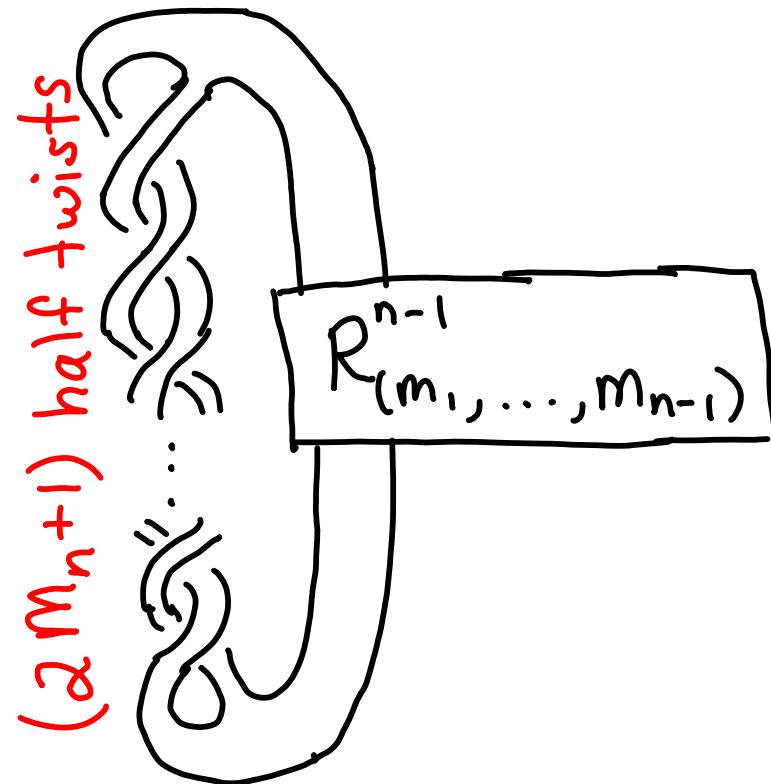
Note: When $m_1 = 0$,

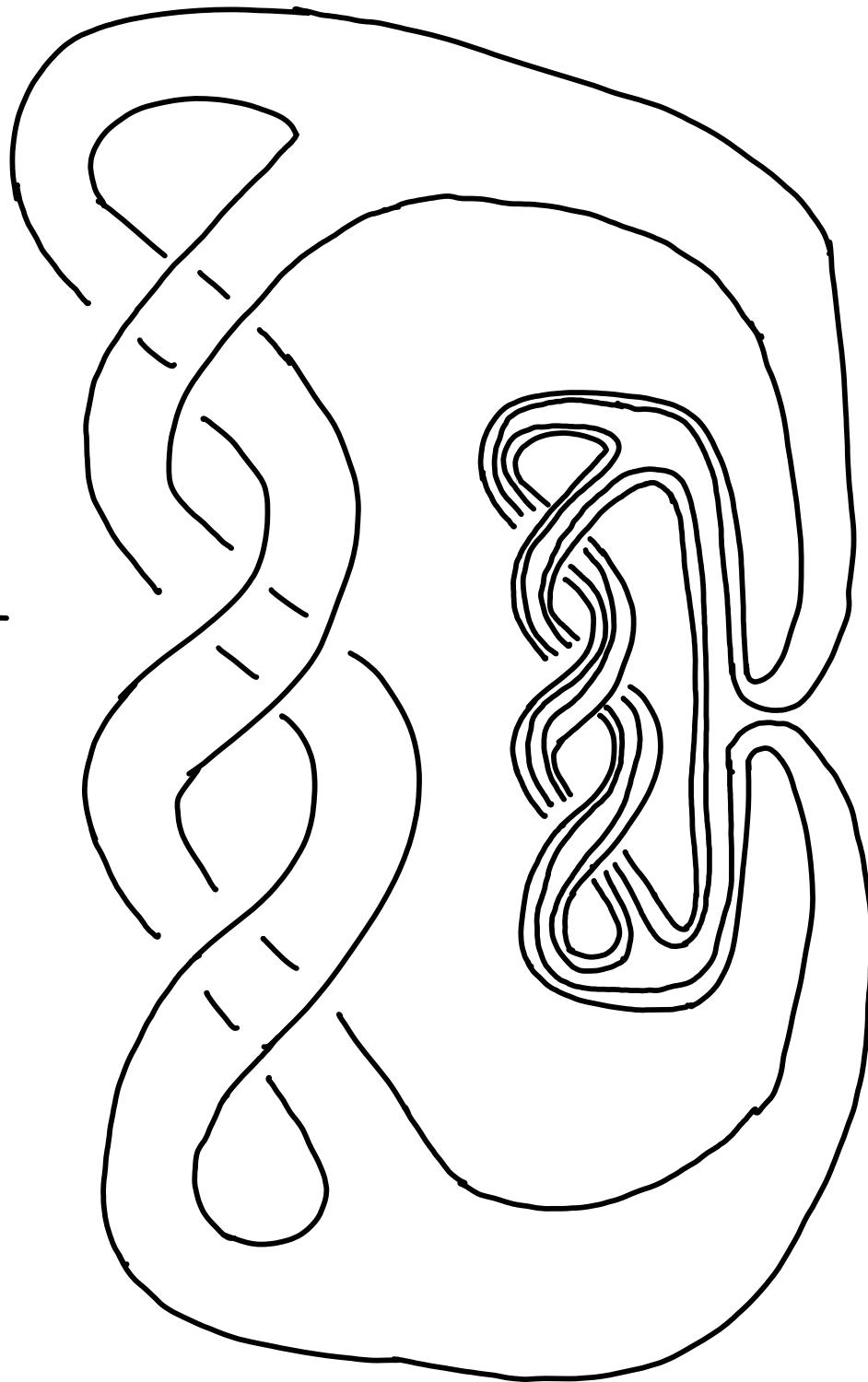
$$R_{(0)}^1(J) =$$



Whitehead double of J

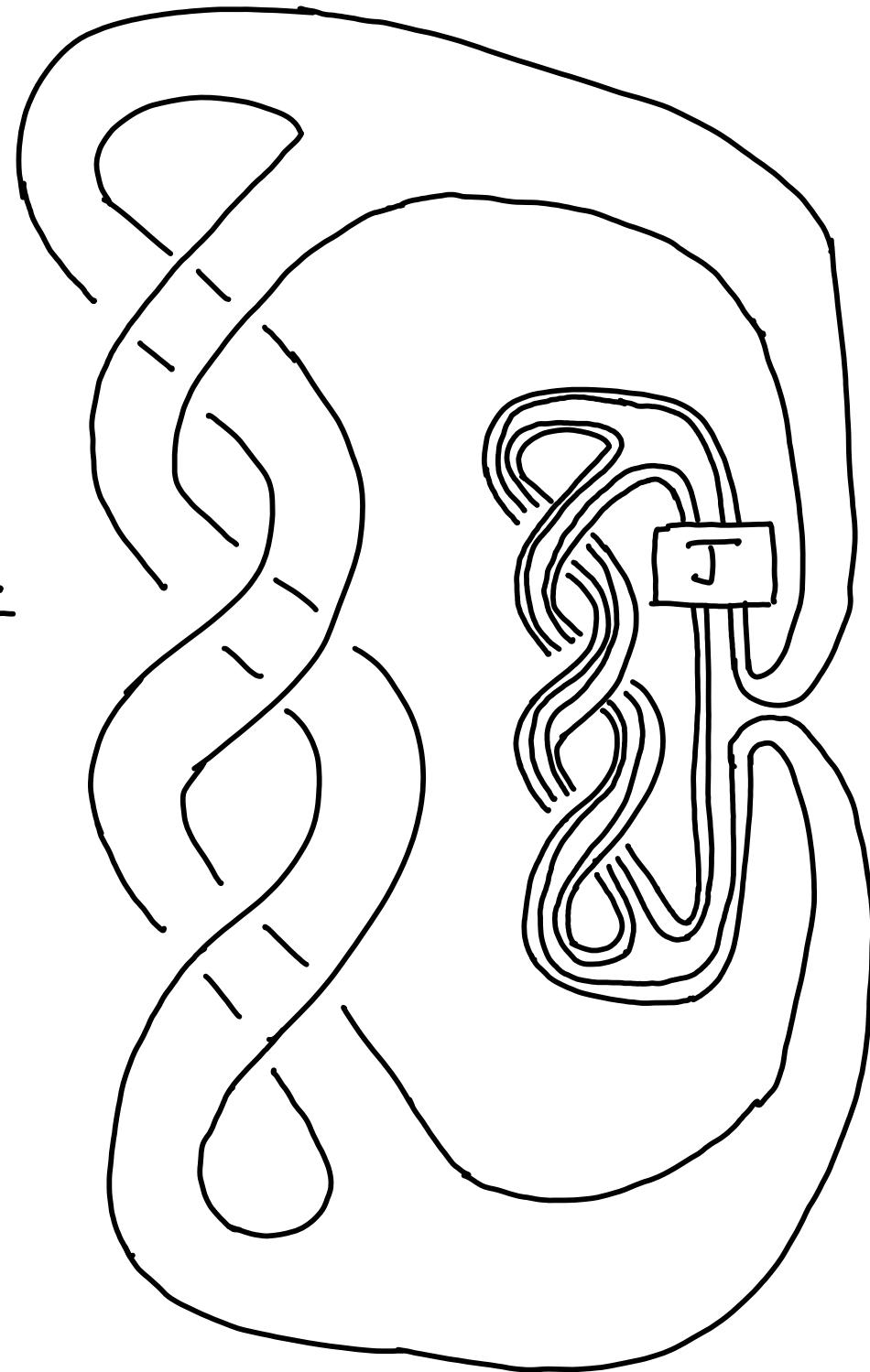
$$R_{(m_1, \dots, m_n)}^n(J) =$$



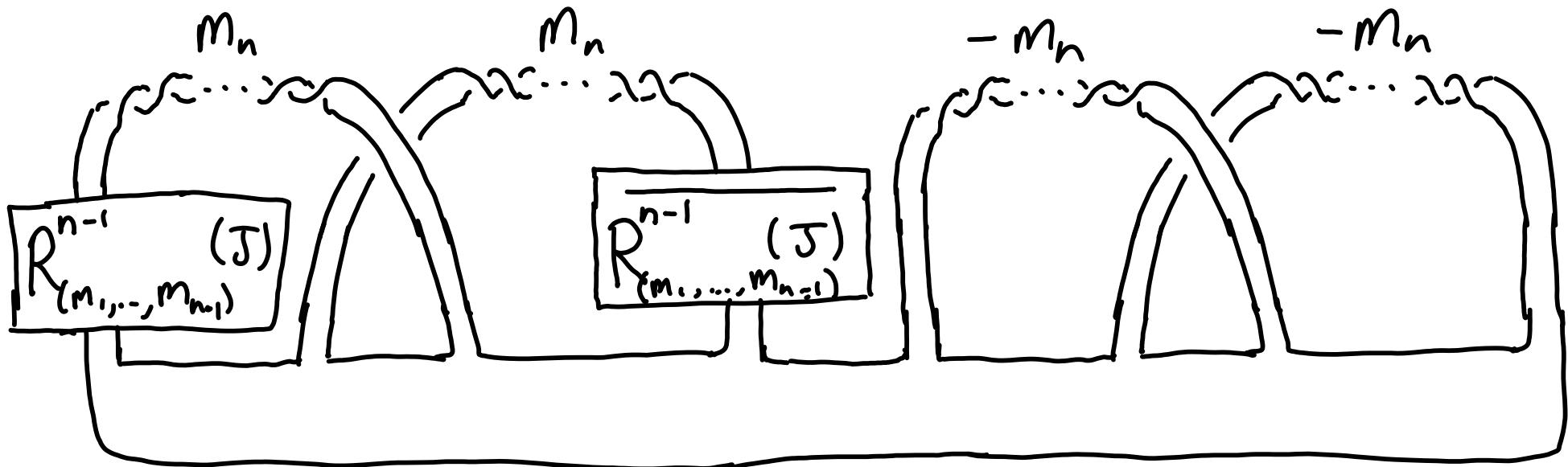
$$R_{(1,1)}^2(\text{unknot}) =$$


$$R^2_{(1,1)}(J)$$

=



2 -torsion examples used in Theorem B



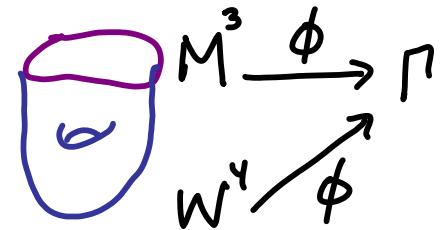
- connected sum of neg. amphichiral ($n \geq 2$)
- slice in a rational homology ball
(follows from a proof of J.C. Cha)

These are inspired by examples of Livingston ($n=1$).

Tools used in the proof

1. von Neumann p -invs; $p(M_k, \pi, M_k \rightarrow \pi_1 M_k / N) \in \mathbb{R}$
2. use non-commutative localization "at a prime"
to define a commutator series $G_P^{(n)}$ for each
 $P = (P_1(t), \dots, P_n(t))$.
3. for each P , define a refinement $\{\alpha f_n^P\}$ of
the (n) -solvable filtration $\{f_n\}$.
4. use higher-order Alexander modules +
linking forms to calculate p -invs.

ρ -invs



Let $M^3 = 2W^4$ and $\phi: \pi_1 W \rightarrow \Gamma$.

Define

$$\rho(M, \Gamma) := \sigma^{(2)}(W, \Gamma) - \sigma(W)$$

↑
signature of Γ -equivariant
intersection form on

$H_2(\Gamma\text{-cover of } W)$.

For $K = \text{knot} \rightsquigarrow M_K = 0\text{-surgery on } K$

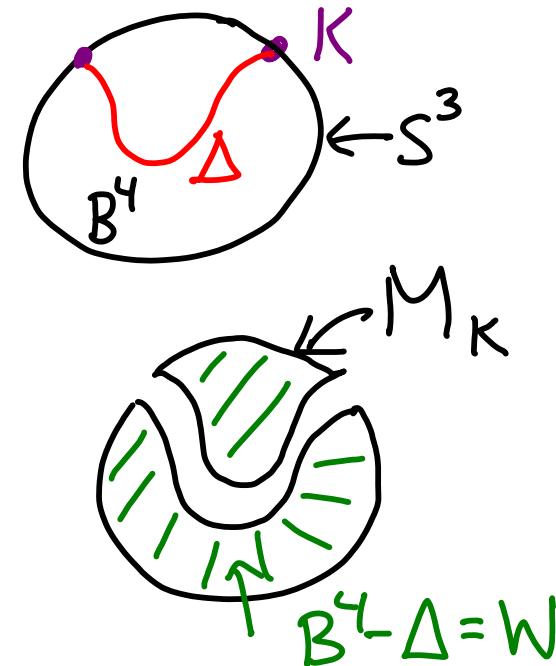
$$\rho(M_K, \pi_1 M_K \rightarrow \mathbb{Z} = H_1(M_K)) = \rho_0(K)$$

Why " $\rho = 0$ " for a slice knot

Let K be a slice knot.

Then $M_K = 0\text{-surgery on } K$

$$= 2 \left(\underbrace{B^4 - \text{nbhd}(\Delta)}_{W} \right)$$



Since $H_2(W) = 0$,

$H_2(W_\Gamma) = \text{torsion}$ (for nice Γ)
 $\nwarrow \Gamma\text{-cover of } W$

Thus $\rho(M_K, \Gamma) = \sigma^{(2)}(W, \Gamma) - \sigma(W) = 0$

Connection between Alexander polynomials and series of groups

The derived series of a group G is:

$$G^{(0)} = G \quad \dots \quad G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

Let K be a knot and $G = \pi_1(S^3 - K)$.

Recall, the Alexander module of K is

$$\frac{G^{(1)}}{[G^{(1)}, G^{(1)}]} = \frac{G^{(1)}}{G^{(2)}}$$

as a $\mathbb{Z}[G/G^{(1)}] = \mathbb{Z}[t, t^{-1}]$ -module.

$$\text{E.g. } G^{(1)}/G^{(2)} \cong \mathbb{Z}[t, t^{-1}]/\langle p(t) \rangle \implies \Delta_K(t) = p(t)$$

If $p(t)$ is rel. prime to $q(t)$, can distinguish

$$\frac{\mathbb{Z}[t, t^{-1}]}{\langle p(t) \rangle} \quad \text{and} \quad \frac{\mathbb{Z}[t, t^{-1}]}{\langle q(t) \rangle}$$

by "killing $q(t)$ -torsion."

$$\mathbb{Z}[t, t^{-1}]/\langle q(t) \rangle \longrightarrow \mathbb{Z}[t, t^{-1}]/\langle p(t), q(t) \rangle = 0$$

$$\mathbb{Z}[t, t^{-1}]/\langle p(t) \rangle \longrightarrow \mathbb{Z}[t, t^{-1}]/\langle p(t), p(t) \rangle \neq 0$$

This can be accomplished by localization:

$$G^{(1)} / G^{(2)} \longrightarrow G^{(1)} / G^{(2)} \otimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Z}[t, t^{-1}] S_{p(t)}^{-1}$$

$$S_{p(t)}^{-1} = \left\{ \text{all } q(t) \text{ relatively prime to } p(t) \right\}$$

Let

$$\underline{G^{(2)} \subset G_{p(t)}^{(2)}} := \ker \left(G^{(1)} \longrightarrow G^{(1)} / G^{(2)} \otimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Z}[t, t^{-1}] S_{p(t)}^{-1} \right)$$

$\rightsquigarrow G^{(1)} / G_{p(t)}^{(2)}$ = $p(t)$ -torsion submodule
of Alexander module

Similarly, the higher-order Alexander module of K is

$$G^{(n)} / G^{(n+1)}$$

which is a module over the non-commutative ring $\mathbb{Z}[G/G^{(n)}]$.

Wish to kill certain types of torsion by enlarging $G^{(n+1)}$ to $G_P^{(n+1)}$ where

$P = (p_1(t), \dots, p_n(t))$ and considering $G^{(n)} / G_P^{(n+1)}$.

Let $P = (p_1(t), \dots, p_n(t))$. Define for a group G and $k \leq n$ inductively:

$$G_P^{(1)} = \ker\left(G \rightarrow \frac{G}{[G, G]} \otimes_{\mathbb{Z}} (\mathbb{Z} - \{0\})^{-1}\right)$$

"

$$H_1(G; \mathbb{Q})$$

$$\text{Let } \Gamma_P^{(k)} = G/G_P^{(k)}$$

$$G_P^{(k+1)} = \ker\left(\Gamma_P^{(k)} \rightarrow \frac{G_P^{(k)}}{[G_P^{(k)}, G_P^{(k)}]} \otimes_{\mathbb{Z}^{\Gamma_P^{(k)}}} \mathbb{Z}^{\Gamma_P^{(k)} \cdot S_{P_k(t)}^{-1}}\right)$$

~~~~~

Note:  $G^{(k)} \subset G_P^{(k)}$

## Properties of $S_{p(t)}$ :

Let  $A \triangleleft \Gamma = \Gamma_p^{(k)}$  and  $1 \neq a \in A$ .  
abelian

- $(\mathbb{Z}\Gamma/\langle p(a) \rangle) \hookrightarrow (\mathbb{Z}\Gamma/\langle p(a) \rangle) S_{p(t)}^{-1}$
- $(\mathbb{Z}\Gamma/\langle q(a) \rangle) S_{p(t)}^{-1} = 0$

if  $p(t)$  and  $q(t)$  are strongly coprime.

-----  
Note:  $MS_{p(t)}^{-1} := M \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}\Gamma \cdot S_{p(t)}^{-1}$

Def A knot is  $(n, P)$  solvable ( $n \in \mathbb{N}$ ) if  $M_K$  bounds a smooth 4-mfld  $W$  s.t.

(1)  $i_{*f} : H_1(M_K) \xrightarrow{\cong} H_1(W)$

(2)  $H_2(W)$  has a basis  $\{f_i, g_i\}_{i=1}^g$  of embedded surfaces (wl triv. normal bundle) all disjoint except  $f_i \cdot g_i = 1$  (geometrically)

(3)  $\pi_1(f_i), \pi_1(g_i) \subset \pi_1(W)_P^{(n)}$

•  $K \in \mathcal{G}_n^P \Leftrightarrow K$  is  $(n, P)$ -solvable

For each  $P = (p_1(t), \dots, p_n(t))$  of polynomials (pairwise strongly coprime), our examples produce distinct  $Z_P \subset \mathcal{F}_n / \mathcal{F}_{n,5}$  s.t.

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^\infty \cong Z_P \subset \mathcal{F}_n / \mathcal{F}_{n+1}^P$$

## More Open Problems

1. What about  $\mathfrak{f}_{n,5}/\mathfrak{f}_{n+1}$ ?
2.  $\mathbb{Z}/4\mathbb{Z} \subset \mathfrak{f}_n/\mathfrak{f}_{n,5}$  for  $n \geq 1$ ?
- Other types of torsion?
- Whenever  $m_i = 0$  for some  $i$  or  $J = \text{top slice} \Rightarrow$  our examples are top slice.  
 $\rightsquigarrow$  lie in  $\bigcap_{n \geq 1} \mathfrak{f}_n$ .  
Are these smoothly slice? All distinct?
- $\bigcap_{n \geq 1} \mathfrak{f}_n$ ? only known examples are top slice.