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Group Theoretic
Invariants of
Links and 3-manifolds

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Some work joint w/ T. Cochran or S. Friedl
Let $M^3$ be a compact, orientable 3-manifold, $G = \pi_1 M$.

We will investigate invariants of $M$ associated to $\phi: G \to \Gamma$ when $\Gamma$ is a "nice" group (i.e. $\mathbb{Z}\Gamma$ is an Ore domain) and $\Gamma$ is canonically associated to $G$. 
3 invariants associated to pair $({M, \phi: \pi, M \to \Gamma})$:

1. $\Gamma$-ranks of $M$:
   $$r_{\Gamma}^i(M) = \text{rank}_{\mathbb{Z}_{\Gamma}} H_i(M_{\Gamma})$$

2. $\Gamma$-degrees of $M$:
   $$\delta_{\Gamma} : H^1(M; \mathbb{Z}) \to \mathbb{Z}$$

3. $\Gamma$ $\rho$-invariants of $M$:
   $$\rho_{\Gamma}(M) \in \mathbb{R}$$
Remark: C. Leidy has studied the higher-order Blanchfield forms $BL_p(M)$ associated to $(M, \mathfrak{p})$.
Examples of $\Gamma = G/H$, fix an $n \geq 0$:

1. $H = G_n^r = n^{th}$ term of (rational) lower central series of $G$.

2. $H = G_r^{(n)} = n^{th}$ term of (rational) derived series of $G$.

3. $H = G_{H}^{(n)} = n^{th}$ term of torsion-free derived series of $G$.

4. $H = G_\ast^{(n)} = G_{H}^{(n)} \cap G_{2n}^r$ (refined torsion-free ...)

$G_r^r \subset G_\ast^{(n)} \subset G_{2n}^r$
Note: Each of the groups $\Gamma$ on last page are solvable with torsion-free quotients (PTFA) hence $\mathbb{Z}\Gamma$ is an Ore domain

$$\frac{\mathbb{Z}\Gamma}{\sim} \hookrightarrow \mathbb{K}(\Gamma) = \left[\text{quotient field of } \mathbb{Z}\Gamma\right]$$

e.g. $\mathbb{Z}[\mathbb{Z}^m] \hookrightarrow \mathbb{K}(\mathbb{Z}^m) = \left\{ \frac{p(x_1, \ldots, x_m)}{q(x_1, \ldots, x_m)} \right\}$

$p, q$ multivariable polynomials
In particular,

- if $A$ is a right $\Gamma$-module then $A$ has a well-defined rank; $\text{rank}_p A$.

- $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$
  \[ \rightarrow \text{rank}_p B = \text{rank}_p A + \text{rank}_p C \]
All of the invariants defined above are homeomorphism invariants, we would like to understand which give invariants of homology cobordism (or concordance for knots and links)!
Homeomorphism Invts of 3-Manifolds

[Isotopy Invts of Knots and Links]
1) Higher-order ranks

Let $\Gamma_n = G/G_r^{(n)}$ where $G_r^{(0)} = G$ and $G_r^{(n)} = \{ g \in G_r^{(n-1)} \mid g^k \in [G_r^{(n-1)}, G_r^{(n-1)}] \text{ for some } k \neq 0 \}$

$$r_n(M) := \text{rank}_{\Gamma_n} H_1(M_n)$$

Where $M_n$ = regular $\Gamma_n$-cover of $M$ corresponding to

$$G = \prod_i M \longrightarrow \Gamma_n = G/G_r^{(n)}$$
Properties of $r_n$:

(i) $r_n$ only depends on $\Pi_1(M)$
- can be defined for any group $G$

(ii) $r_n$ is a decreasing in $n$:

Thm(H): For any $M^3$:

\[ 0 \leq \cdots \leq r_n(M) \leq r_{n-1}(M) \leq \cdots \leq r_0(M) \leq b_1(M) - 1 \]
(iii) If $M$ fibers over $S^1$ then $r_n(M) = 0$

(iv) $r_n(M)$ can be interpreted as the $l^{(2)}$-first betti number $b_1^{(2)}(M_{r_n})$ corresponding to cover over $M_n$

(v) $r_n$ generalizes Alexander nullity $r_0(S^3-L) = \alpha_0(L)$
Recall: \( \alpha_0(L) = \text{rank } H_1((S^3-L)_{ab}) \)

where \( M_{ab} = \text{t.f. abelian cover of } M \) and \( L = m\)-component link in \( S^3 \).

- \( \alpha_0(L) \) is a \textit{concordance invariant}

If \( L \) and \( L' \) cobound a topologically flat annulus in \( S^3 \times I \) then \( \alpha_0(L) = \alpha_0(L') \).
$r_n(S^3 - L)$ not concordance Invariant

Ex: $L = (2, 0)$ cable of $K \# - K$

= boundary link

$K = treffoil$

$K \# - K = slice \ (\text{concordant to } O)$

$\Rightarrow L = \text{slice} \ (\sim \text{ to } O \ O \ O)$

$r_n(S^3 - L) = 0 \neq 1 = r_n(S^3 - \emptyset \emptyset)$

for $n \geq 1$. 
(vi) Prop (H): If $L$ is an $m$-comp good boundary link (first homology of free cover is trivial) then $r_n(S^3-L) = m-1$ (maximal).

Ex: $L =$ Whitehead double of Borromean rings

\[ r_n(S^3-L) = 2 \]
\[ \forall \ n \geq 0. \]
2. Higher-order degrees of $M$ 
Given $\Psi \in H^1(M;\mathbb{Z}) = \text{Hom}(G,\mathbb{Z})$
get $\overline{\Psi} : G/G_r^{(n+1)} \to \mathbb{Z}$ for each $n \geq 0$.
Then $\Gamma_n' = \ker(\overline{\Psi})$ is PTFA.
Since $H_1(M_n)$ is a right $\mathbb{Z}\Gamma_n$-module, $H_1(M_n)$ is a $\mathbb{Z}\Gamma_n'$-module
via $\mathbb{Z}\Gamma' \subset \mathbb{Z}\Gamma$.

$\delta_n(\Psi) = \text{rank}_{\mathbb{Z}\Gamma_n'} H_1(M_n)$
Properties of $\delta_n : H^1(M) \to \mathbb{Z}$

(i) **Thm (Friedl-H):** $\delta_n$ can be extended to a (semi-)norm on $H^1(M; \mathbb{R})$. In particular,

$$\delta_n(\psi_1 + \psi_2) \leq \delta_n(\psi_1) + \delta_n(\psi_2)$$

for each $n \geq 0$ and $\psi_i \in H^1(M; \mathbb{Z})$.

(ii) **Thm(H):** If $b_1(M) \geq 2$,

$$\delta_0(-) \leq \cdots \leq \delta_{n-1}(-) \leq \delta_n(-) \leq \cdots \leq \| - \|_T$$
(iii) $\delta_0$ can be interpreted as the Alexander norm (defined by C. McMullen).

(iv) Thm (Friedl-H): There is a multivariable skew Laurent polynomial $f_n = \sum a_\gamma x^\gamma$ where $\gamma = (\gamma_1, \ldots, \gamma_m)$ and $a_\gamma \in \mathbb{K}_n = \text{quotient field of } \mathbb{Z}[\Gamma]$ (generalizing the multivariable Alexander polynomial $\Delta_\mathbb{M}$) s.t.

$$\delta_n(\mathcal{F}) = \sup \{ \psi(x^\gamma) - \psi(x^\beta) | a_\gamma a_\beta \neq 0 \}$$
(v) If $\psi$ represents a fibration of $M$ over $S^1$ ($b_1 M \geq 2$) then
\[ \| \psi \|_{L^r} = \delta_n(\psi) \]

(vi) If $M \times S^1$ (M irreducible) admits a symplectic structure then there is a $\psi \in \mathcal{H}(M; \mathbb{R})$ s.t. $\| \psi \|_{L^r} = \delta_n(\psi)$ for all $n \geq 0$.

(vii) Thm (H): There exist examples w/ $\delta_0(-) < \delta_1(-) < \cdots < \delta_m(-)$ ($m$ arbitrary)
If $X$ is one of previous examples then $X \times S^1$ does not admit a symplectic structure (nor does $X$ fiber over $S^1$).

(viii) **Prop (H):** If $f: \pi_1 M \to \pi_1 N, b_1(M) = b_1(N) \geq 2, r_0(M) = 0$ then for all $\psi \in H^1(N; \mathbb{Z})$,

$$\delta_n(f^*\psi) \geq \delta_n(\psi)$$
Corollary: If $J$ and $K$ knots, $f : \pi_1(S^3 - J) \to \pi_1(S^3 - K)$ surjective and $\delta_n(K) = 2g(K) - 1$ ($n \geq 1$) then $g(J) \geq g(K)$ \hspace{1cm} (g = genus)

- Gives partial answer to question of J. Simon: "If $J, K$ knots, $\psi : S^3 - J \to S^3 - K$ surjective on $\pi_1$, is $g(J) \geq g(K)$?" known when $\delta_0(K) = \text{deg } \Delta_K = 2g(K)$
- Similar statement for $\ll$ II$^\perp$.
(iv) Can define $\tilde{\delta}_n$ for any $G$ and
$\phi: G \to \Gamma$ where $\Gamma$ PTFA, $\delta_n: H^1(\Gamma; \mathbb{Z}) \to \mathbb{Z}$

**Thm(4):** If $\Lambda \to \Lambda \to \Gamma$ (not "initial"),
$\text{def}(\Theta) \geq 1$ or $G = \pi_1 M^3$ then
$\tilde{\delta}_n(\psi) \geq \delta_n(\psi)$ \quad \forall \ \psi \in H^1(\Gamma; \mathbb{Z})$.

$\tilde{\delta}_n$ gives obstructions to a group being the fundamental group of a 3-manifold (or having positive deficiency).
Homology Cobordism
Invariants of 3-manifolds
[link concordance Invs]
Recall: \( M \) is homology cobordant to \( N \) if there is a 4-manifold \( W \) s.t. 
\[ 2W = M_1 \cup \overline{M}_2 \] and \( i : M \rightarrow W \) (\( j : N \rightarrow W \)) induces inclusions on \( H_x(-j\mathbb{Z}) \).

Ex: \( L_1, L_2 \hookrightarrow S^3 \) links

If \( L_1 \) concordant to \( L_2 \) then \( M_{L_1} = 0\)-surgery on \( L_1 \) is homology cobordant to \( M_{L_2} \).
Hence $i: M \rightarrow W$ is a homology equivalence. What is preserved under $i_* : \pi_1 M \rightarrow \pi_1 W$?

**Ex:** $i_* : \frac{\pi_1(M)}{[\pi_1 M, \pi_1 M]} \cong \frac{\pi_1 W}{[\pi_1 W, \pi_1 W]}$

**Thm (Stallings):** Let $\phi: A \rightarrow B$ be hom. of groups s.t. $\phi$ induces $\cong$ on $H_1$ and epimorphism on $H_2$. Then for all $n$,

$$\phi_* : \frac{A}{A_n} \cong \frac{B}{B_n} \left[ A_n = \text{lower central series of } A \right]$$
Can define various concordance invariants of links (like Milnor's invariants, etc).

What about derived series?

\[ \text{Ex: } K = \text{knot in } S^3 \text{ with } \Delta_K \neq 1 \]

\[ G = \pi_1(S^3 - K), \phi: G \longrightarrow \mathbb{Z} \text{ abelianization} \]

\[ \cdot \phi_* \cong \text{on } H_1, H_2 \text{ (} S^3 - K \text{ aspherical)} \]

\[ \mathbb{Z}^{(n)} = 0, \ G^{(n)}/G^{(2)} \neq 0 \Rightarrow G/G^{(2)} \text{ "big"} \]

\[ \phi_*: G/G^{(2)} \longrightarrow \mathbb{Z}/\mathbb{Z}^{(2)} = \mathbb{Z} \text{ (not)} \]
Thm (Cochran-H): If \( \phi: F \rightarrow B \) induces mono on \( H_1(-; \mathbb{Q}) \) and an epimorphism on \( H_2(-; \mathbb{Q}) \) [\( F \) free gp, \( B \) fin. related] then \( \forall \, n \geq 1, \)

\[ \phi_*: \frac{F}{F^{(n)}} \rightarrow \frac{B}{B^{(n)}}. \]

Note: For applications, only need a monomorphism (as above) !!!
As a corollary, define higher-order Alexander nullity $\alpha_n(L)$ for link $L$:

1. Consider $F(m) \xrightarrow{\pi} \pi_1(S^3-L) =: G$ meridional map.

$H_1((S^3-L)_n)$ is a module over $\mathbb{Z}[F/F^{(m)}]$ via $F/F^{(m)} \longrightarrow G/G^{(m)}$.

Define: $\alpha_n(L) = \text{rank}_{F/F^{(n+1)}} H_1((S^3-L)_n) \quad (S^3-L)_n = \text{cover corresponding to } G^{(m)}$
Properties

• generalizes Alexander nullity

Thm(H): If $L$ is slice then

$$\alpha_n(L) = m-1 = \alpha_n(\text{trivial})$$ for all $n$.

Conjecture: $\alpha_n$ is a concordance invariant.
To get $inv$ of 3-mfd need new series: \textit{Torsion-free derived series}

- $G^{(0)}_H := G$.
- $G^{(n+1)}_H := \left\{ g \in G^{(n)}_H \mid \exists \ 0 \neq \sum k_i \xi_i \in \mathbb{Z}[G/G^{(n)}_H] \quad \text{s.t.} \quad \prod \xi_i^{-1} g^{k_i} \xi_i \in [G^{(n)}_H, G^{(n)}_H] \right\}$

\textbf{Note:} $G^{(n+1)}_H = \ker\left( G^{(n)}_H \to G^{(n)}_H / [G^{(n)}_H, G^{(n)}_H] \otimes K(G/G^{(n)}_H) \right)$

- $G^{(n)} \subset G^{(n)}_r \subset G^{(n)}_H$
Example: $F = \text{free group}$

Since $F^{(n)}/F^{(n+1)}$ is torsion-free as a $\mathbb{Z}[F/F^{(n)}]$-module

$$F^{(n)}_H = F^{(n)} \quad \forall \ n \geq 0$$

Example: $K = \text{knot in } S^3$, $G = \pi_1(S^3-K)$

Since $G^{(n)}/G^{(2)}$ is a torsion module,

$$G^{(n)}_H = [G,G] \quad \forall \ n \geq 1.$$
Thm (Cochran-H): If $\phi : A \to B$ is mono on $H_1(-; @)$ and epi on $H_2(-; @)$ [A f.g., B f.r. related] then for each $n \geq 1$, 
$\phi^*_n : \frac{A}{A^{(n)}_H} \to \frac{B}{B^{(n)}_H}$ 

is a monomorphism.

If $\phi$ is $\approx$ on $H_1(-; @)$ then 
$\frac{A^{(n)}_H}{A^{(n+1)}_H}$ and $\frac{B^{(n)}_H}{B^{(n+1)}_H}$ have same rank (over respective rings).
2. Higher-order ranks of $M^3$ ($G=\pi_1 M$)

$$R_n(M) = \text{rank}_{G/G_H^{(n)}} H_1(M_n^{tf})$$

where $M_n^{tf} =$ covering space of $M$

Corollary: If $M$ and $N$ are homology cobordant then $R_n(M) = R_n(N)$. 

Q. Is $R_n(S^3 - L) = \alpha_n(L)$ for all links and $n \geq 1$?
3. Higher-order p-invts: \( p_n(M) \in \mathbb{R} \).

Let \( \phi_n: G \to G/G_H^{(n+1)} \) then \((M, \phi_n)\) is stably nullbordant, \( \exists 4\)-mfd \( W \) and \( \Pi_1 W \xrightarrow{\psi} \Lambda \) s.t. \( \partial W = M \) and

\[
G = \Pi_1 M \xrightarrow{\phi_n} G/G_H^{(n)}
\]

\[\xymatrix{\Pi_1 W \ar[r]^{\psi} \ar[d]^{i_*} & \Lambda \ar[d]}
\]

\((W, \psi)\) is called a s-nullbordism for \((M, \phi_n)\)
Lemma: If \((W_i, \psi_i)\) are s-nullbordism then \(\sigma^{(2)}(W, \psi) - \sigma(W) = \sigma^{(2)}(W_2, \psi_2) - \sigma(W)\).

Define \(\rho_n(M) = \sigma^{(2)}(W, \psi) - \sigma(W)\) for any s-nullbordism \((W, \psi)\) for \((M, \phi_n)\).

**Properties**

(i) Same as Cheeger-Gromov \(\rho\)-invariant

(ii) \(K\) = knot in \(S^3\), \(\sigma_w\) = Levine Tristram sign.

\(\rho_n(M_K) = \int_{S^1} \sigma_w(K) \, dw \in \mathbb{R}\)

0-surgery on \(K\)
Thm(H): $p_n$ is an invariant of homology cobordism.

Thm(H): For each $n \geq 0$, the image of $p_n: \{3\text{-manifold}\} \rightarrow \mathbb{R}$ is dense and infinitely generated in $\mathbb{R}$.

Idea of Proof: Use Bing double of knot $K \rightsquigarrow \{L_K\}$

Show $p_n(M_{L_K}) = p_0(M_K)$
Consider the Cochran-Orr-Teichner filtration of (string) link concordance group:
\[ F_{(n)} < F_{(n-1)} < \ldots < F_{(1)} < F_{(0)} < C(m) \]

**Thm (H):** If \( L \in F_{(n+1)} \), then \( \rho_n(L) = 0 \).

**Thm (H):** For each \( n \geq 0 \) \( (m \geq 2) \),
\[ F_{(n)} / F_{(n+1)} \] contains an infinitely generated subgroup (unknown for knots \( (m=2) \) when \( n \geq 3 \)).
For applications, it is useful to weaken $H_2$ condition in Stallings' theorem.

**Thm (W. Dwyer):** Let $\phi: A \to B$ be s.t. $\phi$ induces $\cong$ on $H_1$. Then for any $n$, the following are equivalent:

- $\phi$ induces $A/\text{Ann}_n(A) \cong B/\text{Ann}_n(B)$
- $\phi$ induces epimorphism $H_2(A)/\text{Ann}_n(A) \to H_2(B)/\text{Ann}_n(B)$

Where $\text{Ann}_n(A) = \ker(H_2(A) \to H_2(A/\text{Ann}_n(A)))$
For a group $A$, let $X = K(A,1)$. Known that $\Phi_n(A)$ is a subgroup of $x \in H_2(\Sigma)$ s.t. $x$ can be represented by an oriented surface $f: \Sigma \rightarrow X$ s.t. for some symplectic basis of curves $\{a_i, b_i\}_{i \leq g}$ of $\Sigma_g$ \( f_*([a_i]) \subset \pi_1(X)_n \).
Let $A = \text{group}$, $\mathcal{X} = K(A, 1)$.

Define $\Phi_H^{(n)} \subset H_2(\mathcal{X}) = H_2(A)$ by $\Phi_H^{(n)} = \text{subgroup of } x \in H_2(\mathcal{X})$ that can be represented by $f: \Sigma_g \to \mathcal{X}$ s.t. for some symplectic basis $\{a_i, b_i | i \leq g\}$ of curves in $\Sigma_g$,

$f_*([a_i]) \in \Pi_1(X)_H^{(n)}$ \& $f_*([b_i]) \in \Pi_H^{(n)}$.  

Thm (Cochran - H): Let $\phi: A \to B$ (A fin. gen, B fin rel) s.t. $\phi$ induces a mono on $H_1(-; \mathbb{Q})$. If

$$\phi_*: H_2(A) \to H_2(B)/\text{Im}_{H}^{(n)}(B)$$

is surjective then $\phi$ induces a monomorphism

$$\begin{array}{c}
\frac{A}{A_{H}^{(k+1)}} \\
\downarrow
\end{array} \quad \quad \quad \quad \quad \quad \begin{array}{c}
\frac{B}{B_{H}^{(k+1)}}
\end{array}$$

for $k \leq n$. 
Applications

1. Given set \{g_1, ..., g_m\} of \(G\), that are linearly independent in \(H_1(G)\). If \(H_2(G)\) is represented by "\(n\)-gropes" then \(F/F^{(n+1)} \subset G/G^{(n+1)}\).

In particular, \(G\) is not nilpotent.
2. Thm (Cochran-H): If link \( L \) bounds disjoint embedded gropes of height \((n+2)\) then

\[
\frac{\Pi_1(S^3 - L)}{\Pi_1(S^3 - L)^{(n+1)}_H} \xleftarrow{\sim} \frac{\Pi_1(B^4 - \Sigma)}{\Pi_1(B^4 - \Sigma)^{(n+1)}_H}
\]

\( \Sigma = \text{bottom stage of groove} \)

\[
\rho_{n-1}(L) = 0
\]

[K bounds grope, height 2]
Questions

1. Massey products are higher-order cohomology operation related to lower central series. Are there cohomology operations related to derived series?

2. Mixed Hodge Structures associated to $\Pi_1(V)/\Pi_1(V)_n$ ($V$ = algebraic variety) exist, have been useful in Algebraic Geometry. Can one do this for $\Pi_1(V)/\Pi_1(V)^{(m)}$?
3. C. Leidy and L. Maxim have studied $S_n$ for plane curves in $\mathbb{C}^2$. What more can we say about curves using these types of invariants?

4. Generalizations of Stallings' and Dwyer's $\mathbb{Z}/p$-theorems for derived series?

5. "Rational Homotopy Theory for Solvable groups"?