Notes on web: http://math.rice.edu/~shelly

Group Theoretic Invariants of Links and 3-manifolds

Shelly Harvey
Rice University

Some work joint w/ T. Cochran or S. Friedl

Let M^3 = compact, orientable 3-manifold, $G = \pi_1 M$.

We will investigate invariants of M associated to $\phi: G \longrightarrow \Gamma$.
when Γ is a "nice" group (i.e. $\mathbb{Z}\Gamma$ is an Ore domain) and Γ is canonically associated to G.

3 invariants associated to pair $(M, \phi: \pi_1 M \longrightarrow \Gamma)$:

1. Γ-ranks of M:

$$r_\Gamma^i(M) = \operatorname{rank}_{\mathbb{Z}\Gamma} H_i(M_\Gamma)$$

2. Γ-degrees of M:

$$\delta_\Gamma : H^1(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

3. Γ ρ-invariants of M:

$$\rho_\Gamma(M) \in \mathbb{R}$$

Remark: C. Leidy has studied the higher-order Blanchfield forms $\mathcal{B}\ell_{\Gamma}(M)$ associated to (M, Γ)

Examples of $\Gamma = G/H$, fix an $n \geq 0$:

1. $H = G_n^r$ = $n^{\underline{th}}$ term of (rational) lower central series of G.

*2. $H = G_r^{(n)}$ = $n^{\underline{th}}$ term of (rational) derived series of G.

*3. $H = G_H^{(n)}$ = $n^{\underline{th}}$ term of torsion-free derived series of G.

4. $H = G_*^{(n)} = G_H^{(n)} \cap G_{2^n}^r$ (refined torsion-free...)

$\rightsquigarrow \quad G_r^{(n)} \subset G_*^{(n)} \subset G_{2^n}^r$

<u>Note</u>: Each of the groups Γ on last page are solvable with torsion-free quotients (PTFA) hence $\mathbb{Z}\Gamma$ is an Ore domain

$$\rightsquigarrow \mathbb{Z}\Gamma \hookrightarrow K(\Gamma) = \begin{bmatrix} \text{quotient field} \\ \text{of } \mathbb{Z}\Gamma \end{bmatrix}$$

e.g. $\mathbb{Z}[\mathbb{Z}^m] \hookrightarrow K(\mathbb{Z}^m) = \left\{ \dfrac{p(x_1,\dots,x_m)}{q(x_1,\dots,x_m)} \right\}$

p, q multivariable polynomials

In particular,

- if A is a right Γ-module then A has a well-defined rank; $\text{rank}_\Gamma A$.
- $0 \to A \to B \to C \to 0$

  $\leadsto \text{rank}_\Gamma B = \text{rank}_\Gamma A + \text{rank}_\Gamma C$

All of the invariants defined above are homeomorphism invariants, we would like to understand which give invariants of homology cobordism (or concordance for knots and links)!

Homeomorphism Invts of 3-Manifolds

[Isotopy Invts of Knots and Links]

1) Higher-order ranks

Let $\Gamma_n = G/G_r^{(n)}$ where $G_r^{(0)} = G$ and

$$G_r^{(n)} = \left\{ g \in G_r^{(n-1)} \;\middle|\; g^k \in [G_r^{(n-1)}, G_r^{(n-1)}] \text{ for some } k \neq 0 \right\}$$

$$\boxed{r_n(M) := \operatorname{rank}_{\Gamma_n} H_1(M_n)}$$

Where M_n = regular Γ_n-cover of M corresponding to

$$G = \pi_1 M \longrightarrow \Gamma_n = G/G_r^{(n)}$$

Properties of r_n:

(i) r_n only depends on $\pi_1(M)$

- can be defined for any group G

(ii) r_n is a decreasing in n:

Thm (H): For any M^3:

$$0 \le \cdots \le r_n(M) \le r_{n-1}(M) \le \cdots \le r_0(M) \le b_1(M) - 1$$

(iii) If M fibers over S^1 then $r_n(M) = 0$

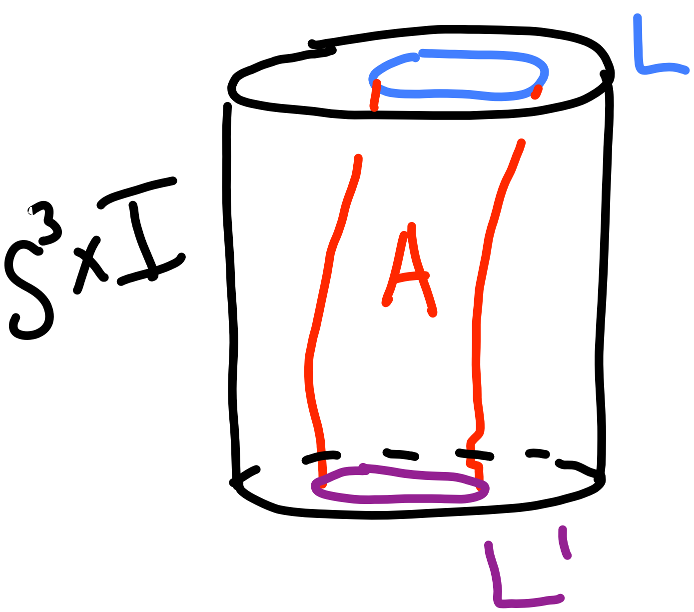
(iv) $r_n(M)$ can be interpreted as the $\ell^{(2)}$-first betti number $b_1^{(2)}(M_{\Gamma_n})$ corresponding to cover M_n

(v) r_n generalizes Alexander nullity $r_0(S^3 - L) = \alpha_0(L)$

<u>Recall</u>: $\alpha_0(L) = \text{rank} \quad H_1((S^3 - L)_{ab})$

where M_{ab} = t.f. abelian cover of M.
and L = m-component link in S^3.

- $\alpha_0(L)$ is a <u>concordance</u> invariant



If L and L' cobound a topologically flat annulus in $S^3 \times I$ then $\alpha_0(L) = \alpha_0(L')$

$r_n(S^3-L)$ <u>not</u> concordance Invariant

Ex: $L =$ $(2,0)$ cable of $K \# -K$

$=$ boundary link

$K =$ trefoil 

$K \# -K$ = slice (concordant to $\bigcirc$)

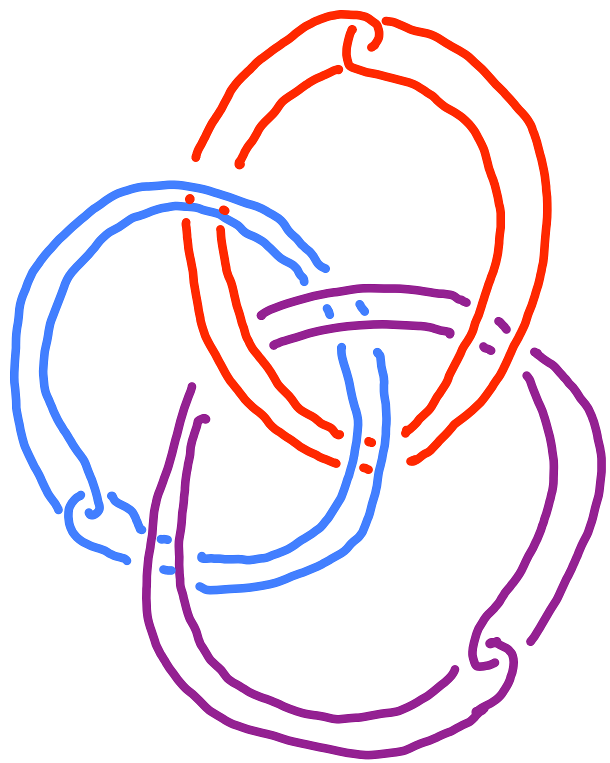
$\Rightarrow L$ = slice ($\sim$ to $\bigcirc \bigcirc$)

$$r_n(S^3-L) = 0 \neq 1 = r_n(S^3 - \bigcirc\bigcirc)$$

for $n \geq 1$.

(vi) Prop (H): If L is an m-comp good boundary link (first homology of free cover is trivial) then $r_n(S^3 - L) = m-1$ (maximal).

Ex:



L = Whitehead double of Borromean rings

$$\boxed{r_n(S^3 - L) = 2}$$

$\forall\, n \geq 0$.

2. Higher-order degrees of M

Given $\psi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{Z})$
get $\bar{\psi} : G/G_r^{(n+1)} \longrightarrow \mathbb{Z}$ for each $n \geq 0$.
Then $\Gamma_n' = \ker(\bar{\psi})$ is PTFA.
Since $H_1(M_n)$ is a right $\mathbb{Z}\Gamma_n$-module, $H_1(M_n)$ is a $\mathbb{Z}\Gamma_n'$-module via $\mathbb{Z}\Gamma' \subset \mathbb{Z}\Gamma$.

$$\boxed{\delta_n(\psi) = \operatorname{rank}_{\mathbb{Z}\Gamma'} H_1(M_n)}$$

Properties of $\delta_n : H^1(M) \longrightarrow \mathbb{Z}$

(i) Thm (Friedl-H): δ_n can be extended to a (semi-)norm on $H^1(M;\mathbb{R})$. In particular,

$$\boxed{\delta_n(\psi_1+\psi_2) \leq \delta_n(\psi_1) + \delta_n(\psi_2)}$$

for each $n \geq 0$ and $\psi_i \in H^1(M;\mathbb{Z})$.

(ii) Thm (H): If $b_1(M) \geq 2$,

$$\boxed{\delta_0(-) \leq \cdots \leq \delta_{n-1}(-) \leq \delta_n(-) \leq \cdots \leq \| - \|_T}$$

(iii) δ_0 can be interpretted as the Alexander norm (defined by C. McMullen)

(iv) Thm (Friedl-H): There is a multivariable skew Laurent polynomial $f_n = \sum a_\gamma x^\gamma$ where $\gamma = (\gamma_1, \dots, \gamma_m)$ and $a_\gamma \in \mathbb{K}_n =$ quotient field of $\mathbb{Z}\Gamma'$ (generalizing the multivariable Alexander polynomial Δ_M) s.t.

$$\delta_n(\psi) = \sup\{\psi(x^\gamma) - \psi(x^\beta) \mid a_\gamma a_\beta \neq 0\}$$

(v) If ψ represents a fibration of M over S^1 ($b_1 M \geq 2$) then

$$\|\psi\|_T = \delta_n(\psi)$$

(vi) If $M \times S^1$ (M closed irreducible) admits a symplectic structure then there is a $\psi \in H^1(M;\mathbb{Z})$ s.t. $\|\psi\|_T = \delta_n(\psi)$ for all $n \geq 0$.

(vii) Thm (H): There exist examples w/

$$\delta_0(-) < \delta_1(-) < \cdots < \delta_m(-) \quad \left(\begin{matrix} m \\ \text{arbitrary} \end{matrix}\right)$$

$\rightsquigarrow$ If X is one of previous examples then $X \times S^1$ does not admit a symplectic structure (nor does X fiber over S^1).

(viii) Prop (H): If $f: \pi_1 M \twoheadrightarrow \pi_1 N$ $b_1(M) = b_1(N) \geq 2$, $r_0(M) = 0$ then for all $\psi \in H^1(N; \mathbb{Z})$,

$$\delta_n(f^* \psi) \geq \delta_n(\psi)$$

<u>Corollary</u>: If J and K knots,
$f: \pi_1(S^3 - J) \twoheadrightarrow \pi_1(S^3 - K)$ surjective
and $\delta_n(K) = 2g(K) - 1$ $(n \geq 1)$ then

$$g(J) \geq g(K) \qquad (g = \text{genus})$$

- Gives partial answer to question of J. Simon: "If J, K knots, $\varphi: S^3 - J \to S^3 - K$ surjective on π_1, is $g(J) \geq g(K)$?"

  Known when $\delta_0(K) = \deg \Delta_K = 2g(K)$

- similar statement for $\| \ \|_T$.

(iv) Can define δ_n for any G and $\phi: G \longrightarrow \Gamma$ where Γ PTFA, $\delta_n: H^1(\Gamma; \mathbb{Z}) \to \mathbb{Z}$

<u>Thm(H)</u>: If $G \longrightarrow \Lambda \longrightarrow \Gamma$ (not "initial"), $\operatorname{def}(G) \geq 1$ or $G = \pi_1 M^3$ then

$$\delta_\Lambda(\psi) \geq \delta_\Gamma(\psi) \quad \forall \ \psi \in H^1(\Gamma; \mathbb{Z}).$$

$\rightsquigarrow$ δ_n gives obstructions to a group being the fundamental group of a 3-manifold (or having positive deficiency)

Homology Cobordism Invariants of 3-manifolds

[link concordance Invts]

Recall: M is homology cobordant to N if there is a 4-manifold W s.t. $\partial W = M_1 \cup \overline{M}_2$ and $i: M \to W$ $(j: N \to W)$ induces inclusions on $H_*(-;\mathbb{Z})$.

Ex: $L_1, L_2 \hookrightarrow S^3$ links

If L_1 concordant to L_2 then M_{L_1} = 0-surgery on L_1 is homology cobordant to M_{L_2}.

Hence $i : M \longrightarrow W$ is a <u>homology equivalence</u>. What is preserved under $i_* : \pi_1 M \longrightarrow \pi_1 W$?

Ex: $i_* : \dfrac{\pi_1(M)}{[\pi_1 M, \pi_1 M]} \xrightarrow{\cong} \dfrac{\pi_1 W}{[\pi_1 W, \pi_1 W]}$

<u>Thm</u> (Stallings): Let $\phi : A \longrightarrow B$ be hom. of groups s.t. ϕ induces $\cong$ on H_1 and epimorphism on H_2. Then for all n,

$$\phi_* : \frac{A}{A_n} \xrightarrow{\cong} \frac{B}{B_n}$$

[A_n = lower central series of A]

$\rightsquigarrow$ Can define various concordance invariants of links (like Milnor's invariants, etc).

What about derived series?

<u>Ex:</u> K = knot in S^3 with $\Delta_K \neq 1$

$G = \pi_1(S^3 - K)$, $\phi : G \longrightarrow \mathbb{Z}$ abelianization

- $\phi_* \cong$ on H_1, H_2 ($S^3 - K$ aspherical)
- $\mathbb{Z}^{(n)} = 0$, $G^{(1)}/G^{(2)} \neq 0 \rightsquigarrow G/G^{(2)}$ "big"

$$\phi_* : G/G^{(2)} \twoheadrightarrow \mathbb{Z}/\mathbb{Z}^{(2)} = \mathbb{Z} \quad \left(\overset{\text{not}}{\cong}\right)$$

Thm (Cochran-H): If $\phi: F \longrightarrow B$ induces mono on $H_1(-;\mathbb{Q})$ and an epimorphism on $H_2(-;\mathbb{Q})$ [F free gp, B fin. related] then $\forall\ n \geq 1$,

$$\phi_*: \frac{F}{F^{(n)}} \hookrightarrow \frac{B}{B^{(n)}}.$$

Note: For applications, only need a monomorphism (as above)!!!

As a corollary, define higher-order Alexander nullity $\alpha_n(L)$ for link L:

$$x \longrightarrow \mu(x)$$

1. Consider $F(m) \xrightarrow{\mu} \pi_1(S^3 - L) =: G$ meridinal map.

$H_1((S^3-L)_n)$ is a module over $\mathbb{Z}[F/F^{(n)}]$ via $F/F^{(n)} \longrightarrow G/G^{(n)}$.

<u>Define</u>: $\alpha_n(L) = \operatorname{rank}_{F/F^{(n+1)}} H_1((S^3-L)_n)$

$(S^3-L)_n$ = cover corresponding to $G^{(n)}$

Properties

- generalizes Alexander nullity
- Thm (H): If L is slice then $\alpha_n(L) = m-1 = \alpha_n(\text{trivial})$ for all n!

Conjecture: α_n is a concordance invariant.

To get invt of 3-mfld need new series: <u>Torsion-free derived series</u>

- $G_H^{(0)} := G.$
- $G_H^{(n+1)} := \left\{ g \in G_H^{(n)} \;\middle|\; \begin{array}{l} \exists\ 0 \neq \sum k_i \gamma_i \in \mathbb{Z}[G/G_H^{(n)}] \\ \text{s.t} \\ \prod \gamma_i^{-1} g^{k_i} \gamma_i \in [G_H^{(n)}, G_H^{(n)}] \end{array} \right.$

<u>Note:</u> $G_H^{(n+1)} = \operatorname{Ker}\left(G_H^{(n)} \to G_H^{(n)} / [G_H^{(n)}, G_H^{(n)}] \otimes_{G/G_H^{(n)}} K(G/G_H^{(n)}) \right)$

- $G^{(n)} \subset G_r^{(n)} \subset G_H^{(n)}$

Ex: F = free group

Since $F^{(n)}/F^{(n+1)}$ is torsion-free as a $\mathbb{Z}[F/F^{(n)}]$-module $\rightsquigarrow$

$$\boxed{F_H^{(n)} = F^{(n)}} \quad \forall\ n \geq 0$$

Ex: K = knot in S^3, $G = \pi_1(S^3 - K)$

Since $G^{(1)}/G^{(2)}$ is a torsion module,

$$\boxed{G_H^{(n)} = [G,G]} \quad \forall\ n \geq 1.$$

Thm (Cochran-H): If $\phi: A \to B$ is mono on $H_1(-;\mathbb{Q})$ and epi on $H_2(-;\mathbb{Q})$ [A f.g., B f.related] then for each $n \geq 1$,

$$\phi_*: \frac{A}{A_H^{(n)}} \hookrightarrow \frac{B}{B_H^{(n)}}$$

is a monomorphism.

If ϕ is $\cong$ on $H_1(-;\mathbb{Q})$ then $A_H^{(n)}/A_H^{(n+1)}$ and $B_H^{(n)}/B_H^{(n+1)}$ have same rank (over respective rings).

2. Higher-order ranks of M^3 $(G = \pi_1 M)$

$$R_n(M) = \operatorname{rank}_{G/G_H^{(n)}} H_1(M_n^{tf})$$

where M_n^{tf} = covering space of M corresponding to $G_H^{(n)}$

Corollary: If M and N are homology cobordant then $R_n(M) = R_n(N)$.

Q. Is $R_n(S^3 - L) = \alpha_n(L)$ for all links and $n \geq 1$?

3. Higher-order ρ-invts : $\rho_n(M) \in \mathbb{R}$.

Let $\phi_n : G \longrightarrow G/G_H^{(n+1)}$ then (M, ϕ_n) is stably nullbordant, $\exists$ 4-mfld W and $\pi_1 W \xrightarrow{\psi} \Lambda$ s.t. $\partial W = M$ and

$$\begin{tikzcd} G = \pi_1 M \arrow[r, "\phi_n"] \arrow[d, "i_*"] & G/G_H^{(n)} \arrow[d, hook] \\ \pi_1 W \arrow[r, "\psi"] & \Lambda \end{tikzcd}$$

(W, ψ) is called a s-nullbordism for (M, ϕ_n)

Lemma: If (W_i, Ψ_i) are s-null bordisms then $\sigma^{(2)}(W_1, \Psi_1) - \sigma(W) = \sigma^{(2)}(W_2, \Psi_2) - \sigma(W)$

Define $\boxed{\rho_n(M) = \sigma^{(2)}(W, \Psi) - \sigma(W)}$ for any s-null bordism (W, Ψ) for (M, ϕ_n).

Properties

(i) Same as Cheeger-Gromov ρ-invariant

(ii) K = knot in S^3, σ_ω = Levine Tristram sign.

$$\rho_0(M_K) = \int_{S^1} \sigma_\omega(K)\, d\omega \in \mathbb{R}$$

0-surgery on K

Thm (H): ρ_n is an invariant of homology cobordism

Thm (H): For each $n \geq 0$, the image of $\rho_n : \{3\text{-manifolds}\} \longrightarrow \mathbb{R}$ is dense and infinitely generated in $\mathbb{R}$.

Idea of Proof: Use Bing double of knot $K \rightsquigarrow \{L_K\}$

L_K

Show $\rho_n(M_{L_K}) = \rho_0(M_K)$

Consider the Cochran-Orr-Teichner filtration of (string) link concordance group:

$$\mathcal{F}_{(n)} \subset \mathcal{F}_{(n-1)} \subset \cdots \subset \mathcal{F}_{(1)} \subset \mathcal{F}_{(0)} \subset \mathcal{C}(m)$$

Thm (H): If $L \in \mathcal{F}_{(n+1)}$ then $\rho_n(L) = 0$.

Thm (H): For each $n \geq 0$ $(m \geq 2)$, $\mathcal{F}_{(n)} / \mathcal{F}_{(n+1)}$ contains an infinitely generated subgroup (unknown for knots $(m=1)$ when $n \geq 3$).

For applications, it is useful to weaken H_2 condition in Stallings' theorem.

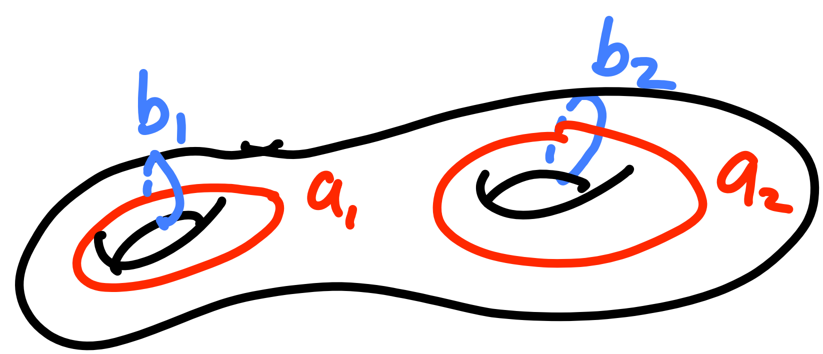
Thm (W. Dwyer): Let $\phi: A \longrightarrow B$ be s.t. ϕ induces $\cong$ on H_1. Then for any n, the following are equivalent:

- ϕ induces $A/A_{n+1} \cong B/B_{n+1}$
- ϕ induces epimorphism
$$H_2(A)/\Phi_n(A) \longrightarrow H_2(B)/\Phi_n(B)$$

where $\Phi_n(A) = \ker(H_2(A) \longrightarrow H_2(A/A_n))$

For a group A, let $X = K(A,1)$.

Known that $\Phi_n(A) =$ subgroup of $x \in H_2(X)$ s.t. x can be represented by an oriented surface $f: \Sigma_g \longrightarrow X$ s.t. for some symplectic basis of curves $\{a_i, b_i \mid i \leq g\}$ of Σ_g $f_*([a_i]) \subset \pi_1(X)_n$.

$\Sigma_2 =$ 

Let A = group, $X = K(A,1)$.

Define $\Phi_H^{(n)} \subset H_2(X) = H_2(A)$ by

$\Phi_H^{(n)}$ = subgroup of $x \in H_2(X)$ that can be represented by $f: \Sigma_g \longrightarrow X$ s.t. for some symplectic basis $\{a_i, b_i \mid i \le g\}$ of curves in Σ_g,

$f_*([a_i]) \subset \pi_1(X)_H^{(n)}$ & $f_*([b_i]) \subset \pi_H^{(n)}$.

Thm (Cochran - H): Let $\phi : A \to B$ (A fin. gen, B fin rel) s.t. ϕ induces a mono on $H_1(-;\mathbb{Q})$. If

$$\phi_* : H_2(A) \longrightarrow H_2(B)/\Phi_H^{(n)}(B)$$

is surjective then ϕ induces a monomorphism

$$\frac{A}{A_H^{(k+1)}} \hookrightarrow \frac{B}{B_H^{(k+1)}}$$

for $k \leq n$.

Applications

1. Given set $\{g_1, \ldots, g_m\}$ of G, that are linearly independent in $H_1(G)$.
If $H_2(G)$ is represented by "n-gropes" then $F/F^{(n+1)} \subset G/G^{(n+1)}$.
In particular, G is not nilpotent.

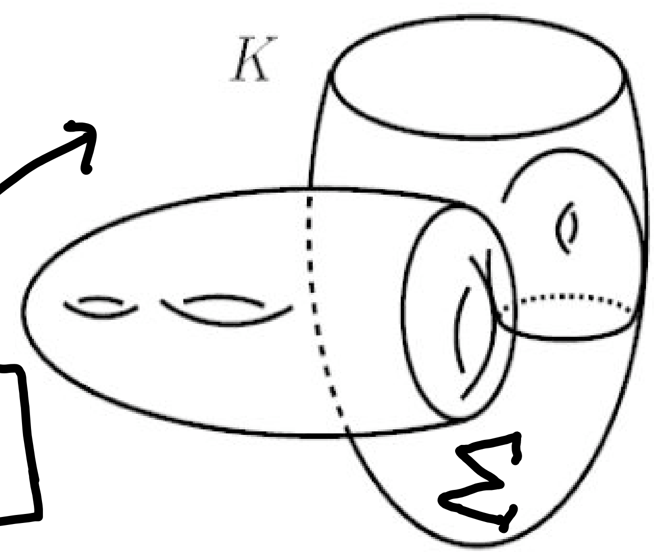
2. <u>Thm</u> (Cochran-H): If link L bounds disjoint embedded gropes of height $(n+2)$ then

$$\frac{\pi_1(S^3-L)}{\pi_1(S^3-L)^{(n+1)}_H} \hookrightarrow \frac{\pi_1(B^4-\Sigma)}{\pi_1(B^4-\Sigma)^{(n+1)}_H}$$

Σ = bottom stage of grope

$\rightsquigarrow$ $\boxed{\rho_{n-1}(L) = 0}$

[K bounds grope, height 2]



Questions

1. Massey products are higher-order cohomology operation related to lower central series. Are there cohomology operations related to derived series?

2. Mixed Hodge Structures associated to $\pi_1(V)/\pi_1(V)_n$ (V = algebraic variety) exist, have been useful in Algebraic Geometry. Can one do this for $\pi_1(V)/\pi_1(V)^{(n)}$?

3. C. Leidy and L. Maxim have studied δ_n for plane curves in $\mathbb{C}^2$. What more can we say about curves using these types of invariants?

4. Generalizations of Stallings' and Dwyer's $\mathbb{Z}_p$-theorems for derived series?

5. "Rational Homotopy Theory for solvable groups"?