Knot ~~and Link~~ Concordance

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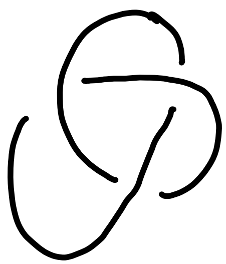
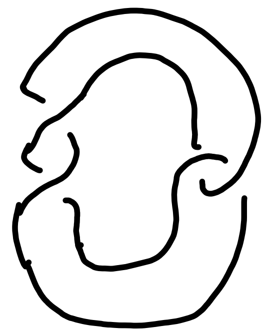
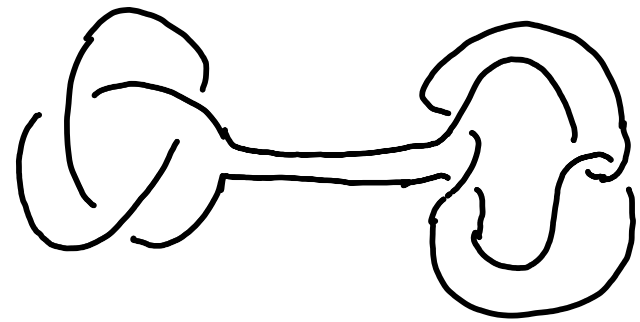
Connections for Women:
Homology Theories of Knots and Links

Spring 2010

Part I:

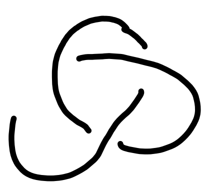
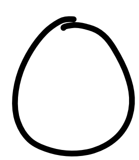
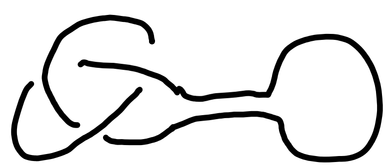
Definitions and Examples

There is a binary operation on knots:

 $\#$  $:=$ 

K_1 $\qquad$ K_2 $\qquad$ $K_1 \# K_2$

connected sum of K_1 and K_2.

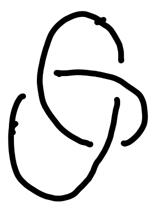
 $\#$  $=$  $=$ 

$K \# \bigcirc = K$

Thus $\mathcal{K} = (\{\text{Knots}\}, \#)$ forms a monoid with unity $= \bigcirc$.

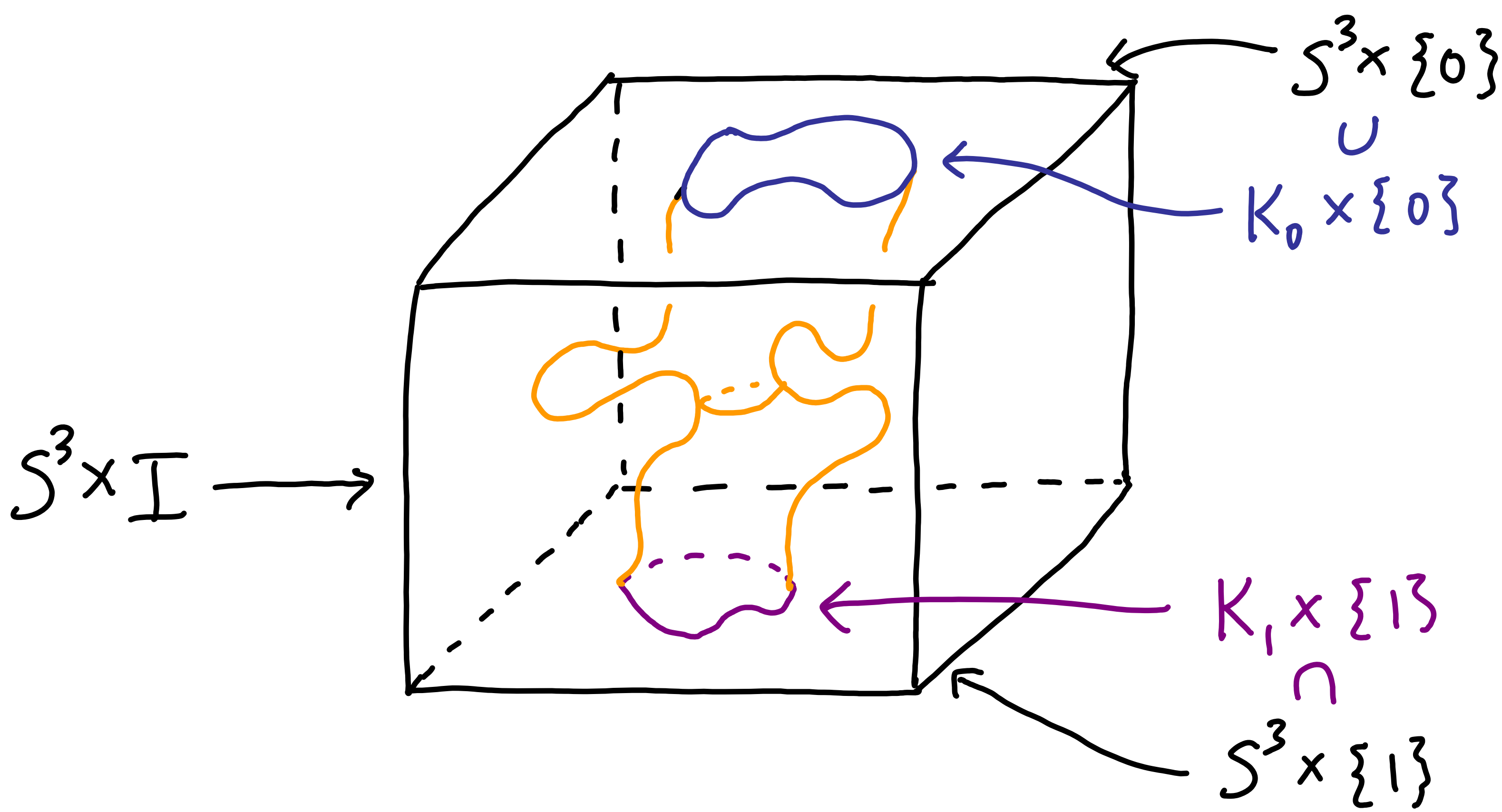
However $\mathcal{K}$ is not a group since it does not have inverses.

i.e. there is no knot K such that

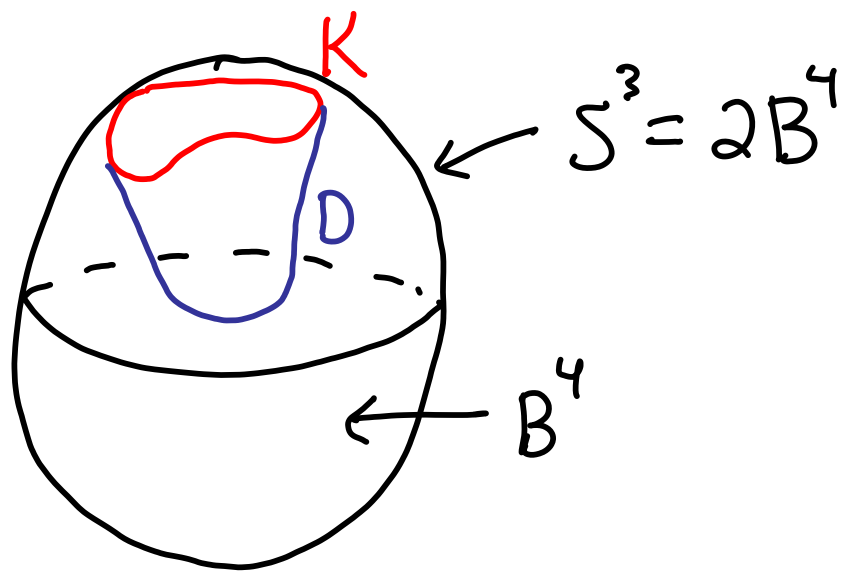
 $\# K = \bigcirc$.

To get a group structure, define a equivalence relation called concordance.

Def: Knots K_0 and K_1 are **concordant** if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times I$.



Def: A knot $K \subset S^3$ <u>slice</u> if $K = \partial D$ where D is a 2-dimensional disk (smoothly) embedded in B^4 = 4-dim. ball.

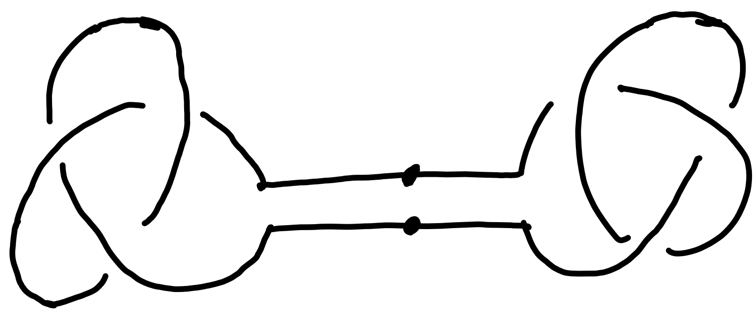


$\boxed{\text{A knot is concordant to the unknot} \Leftrightarrow \text{it is a slice knot.}}$

If K is any knot then $K \# r\bar{K}$ is slice.

$r\bar{K}$ = change all crossings + change orientation
(reverse of mirror image)

<u>Proof</u>: "Spin" K through $\mathbb{R}^4_+$.

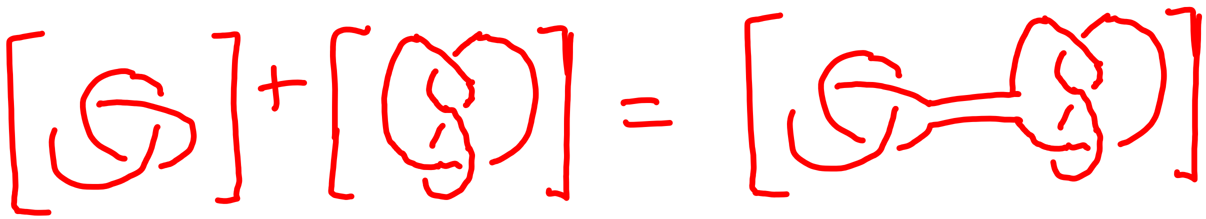


$K \quad \# \quad r\bar{K}$

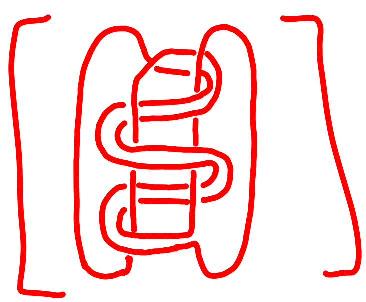
Def$^{\underline{n}}$: The knot concordance group is

$$\mathcal{C} = \{\text{knots in } S^3\} / \text{concordance}$$

- $\mathcal{C}$ is an abelian group under the operation connected sum of knots.

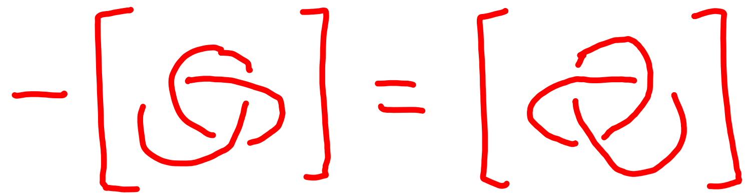


- $[K] = 0 \iff K$ is slice



$= 0$

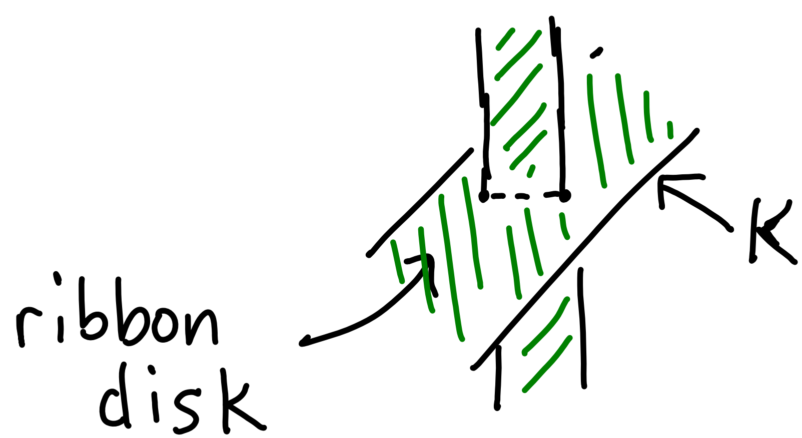
- The inverse of $[K]$ is $[r\bar{K}]$ since $K \# r\bar{K}$ is slice.



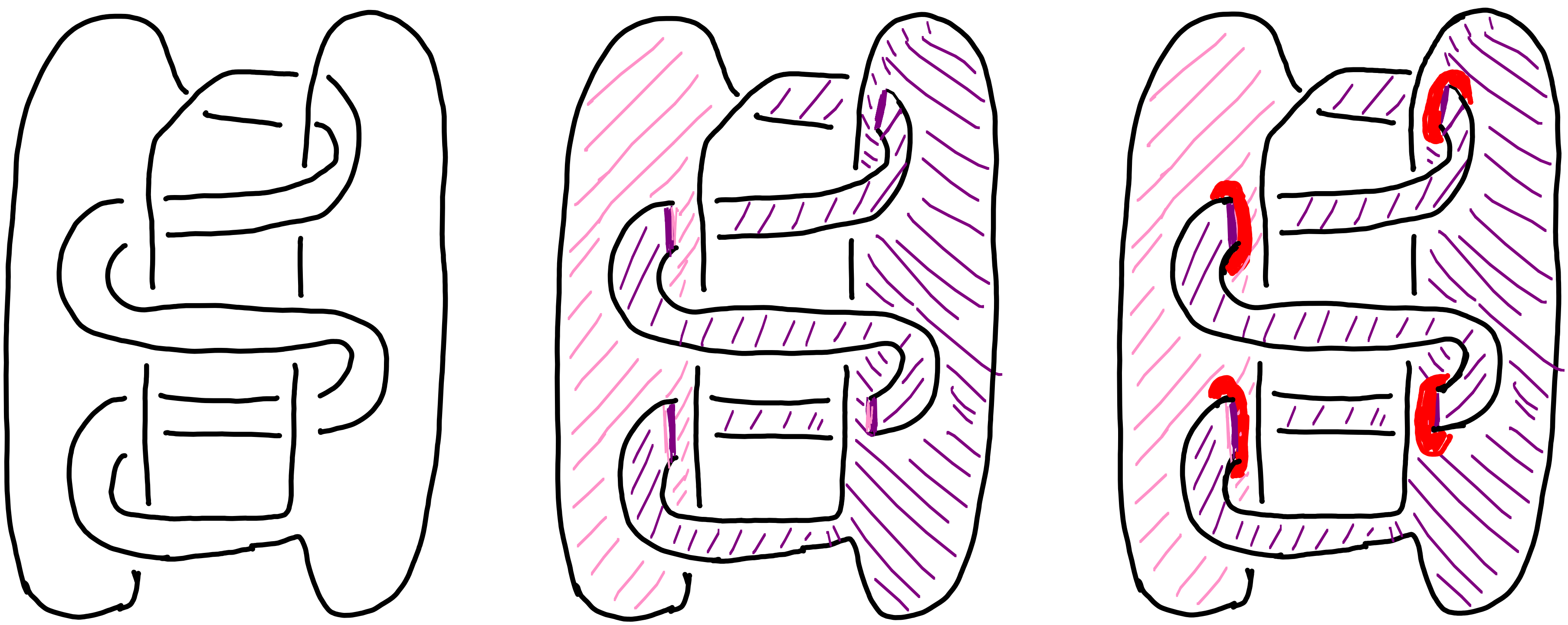
$$\boxed{[K] \text{ has order } 1 \iff K \text{ is slice}}$$

All known examples of (smoothly) slice knots are ribbon:

K is **ribbon** if it bounds an immersed disk in S^3 with only ribbon singularities:



Any ribbon knot is slice



To obtain embedded $D^2 \subset B^4$, push interior of red disco into interior of B^4.

Open Problem:

Ribbon-Slice Conjecture: A knot is smoothly slice $\iff$ it is ribbon.

$$\boxed{[K] \text{ has order } 2 \iff \begin{array}{l} K \# K \text{ is slice} \\ \quad + \\ K \text{ is not slice} \end{array}}$$

<u>Def</u>: A knot K is **negative amphichiral** if K is isotopic to $r\bar{K}$.

If K is neg. amphichiral $\Rightarrow$

$$K \# K = K \# r\bar{K} = \text{slice}.$$

i.e. $[K]$ is of order 2 in $\mathcal{C}$ (if $K \neq$ slice).

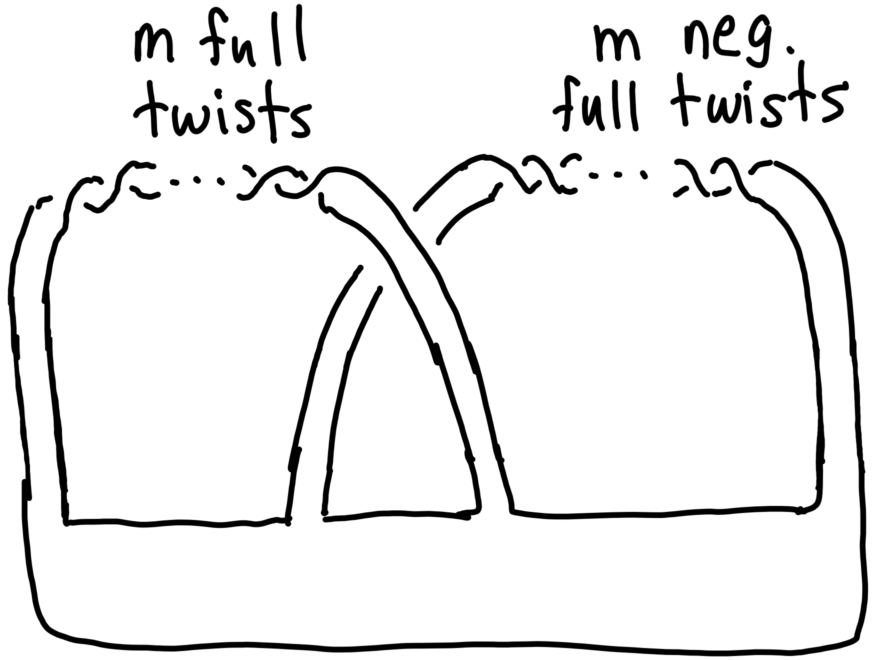
This is the only <u>known</u> way to create elements of finite order in $\mathcal{C}$.

Open Problems

- (Gordon) If $[K]$ has order 2 in $\mathcal{C}$, then is K concordant to a negative amphichiral knot?

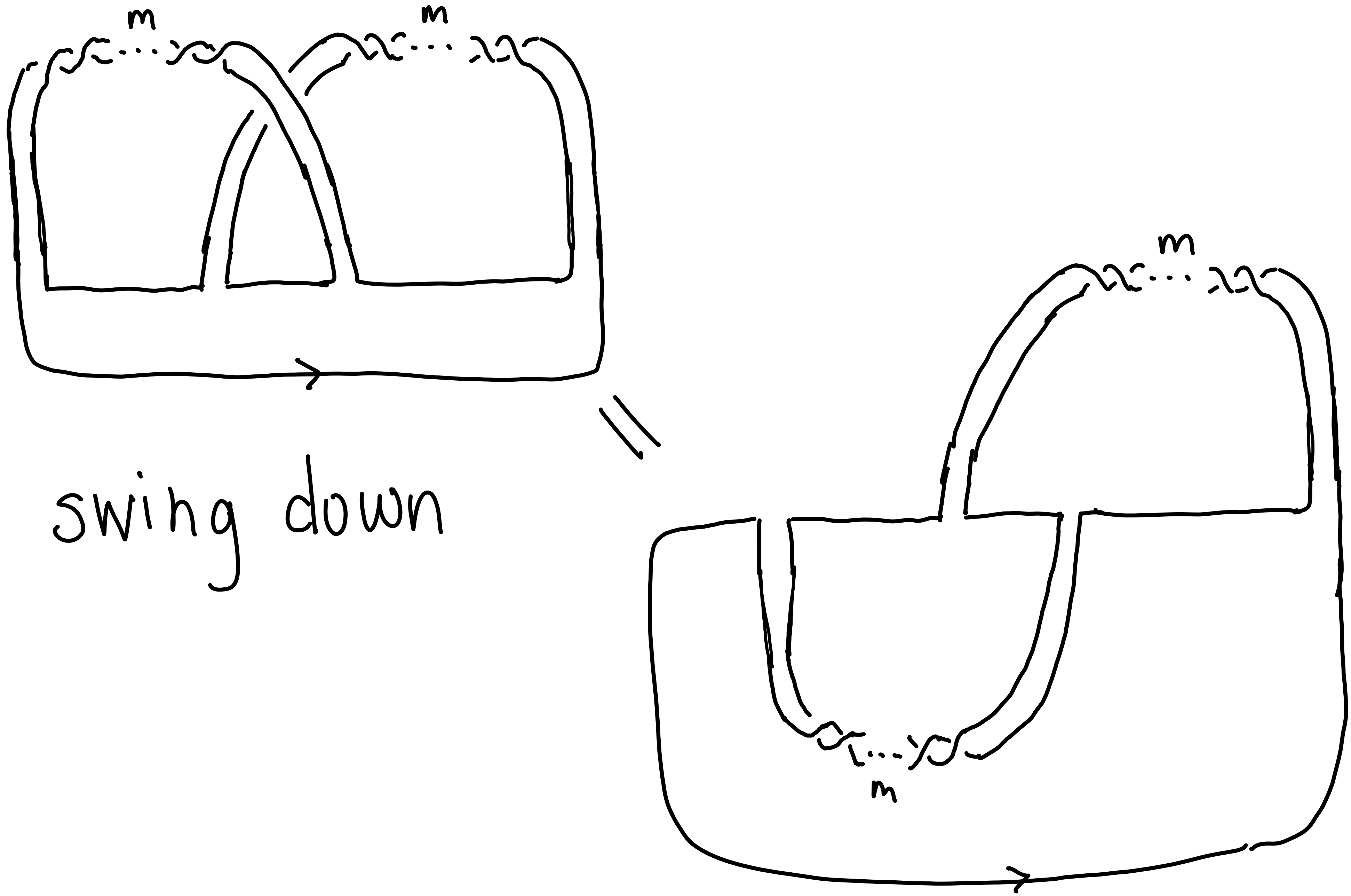
- Is every element of finite order in $\mathcal{C}$ of order 1 or 2?

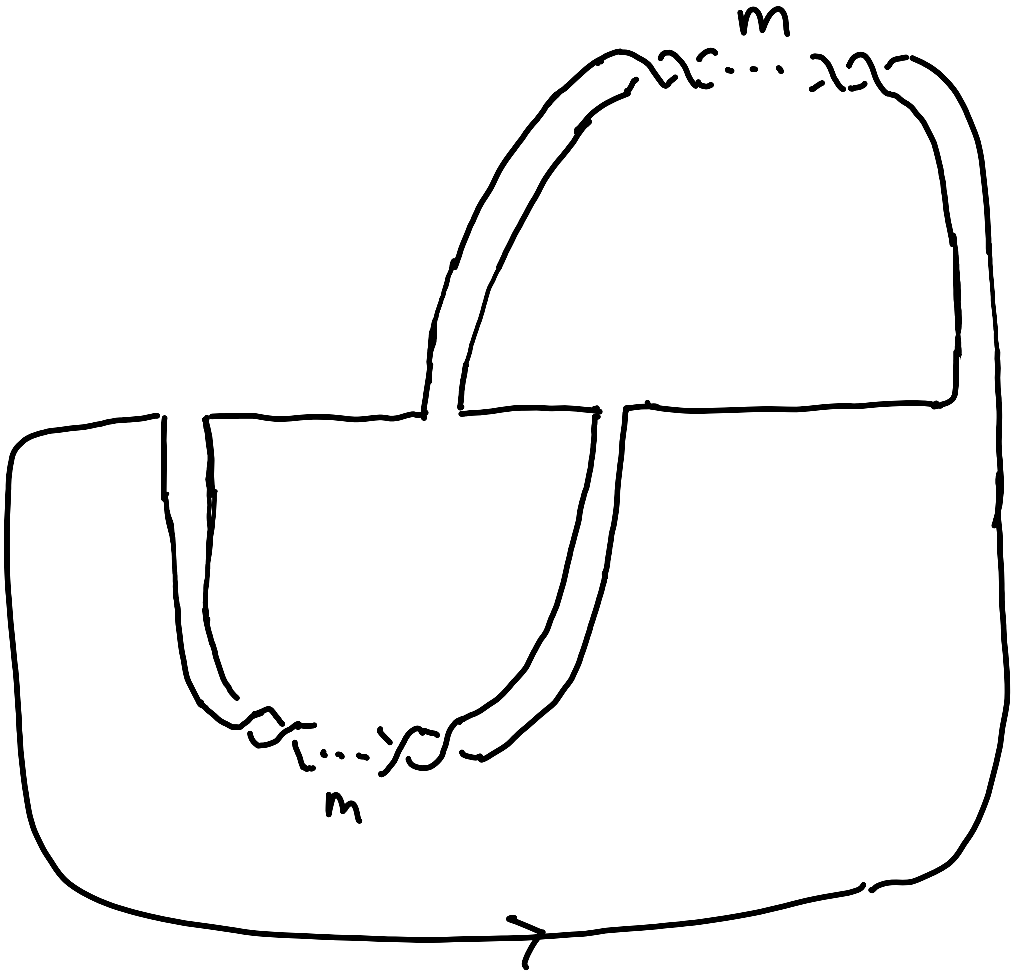
Ex: Let

$K_m =$ 

then K_m is neg. amphichiral.

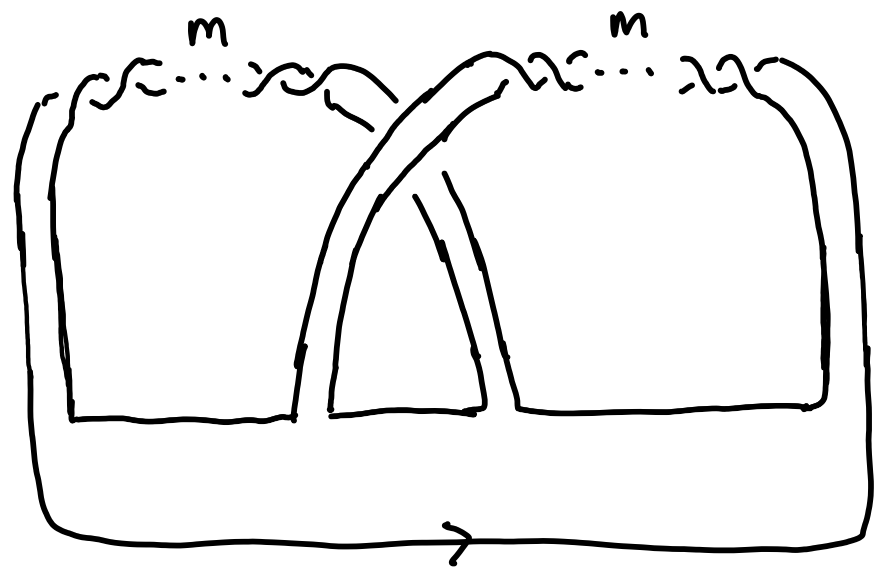
Proof that K_m is neg. amphichiral:

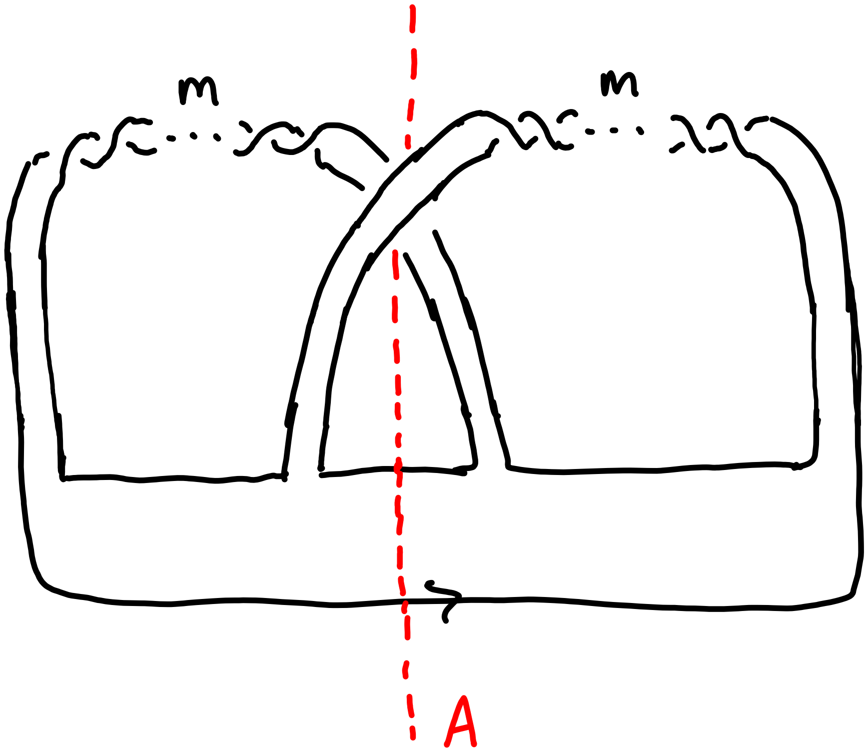




swing back up to the "back"

=





rotate $\mathbb{R}^3$ by π about A

$=$

K_m

$=$

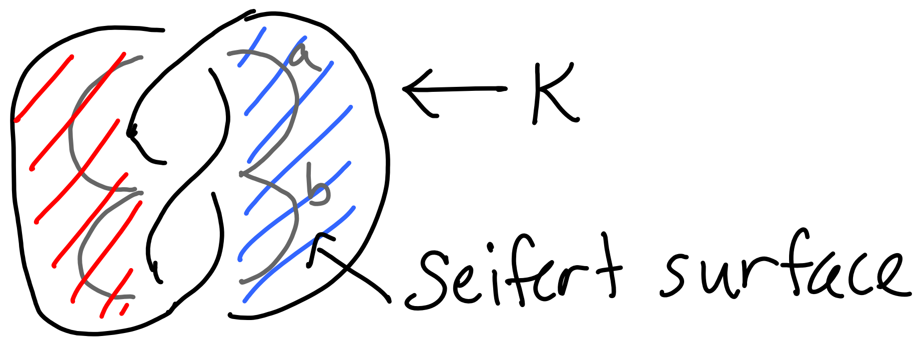
$r\overline{K_m}$

Part II:

Levine's Algebraic Concordance Group

Seifert Matrix

$K = \partial(\text{orientable surface})$, knot



V_K = Seifert ("linking" matrix)

$\ell k(a, b^+)$

$$V_K = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$\mathcal{C}_A$ = group of equivalence classes of Seifert matrices where $V \sim \emptyset$

if $V = \left(\begin{array}{c|c} * & * \\ \hline * & 0_{g\times g} \end{array}\right)_{2g\times 2g}$

- $V_1 + V_2 = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$

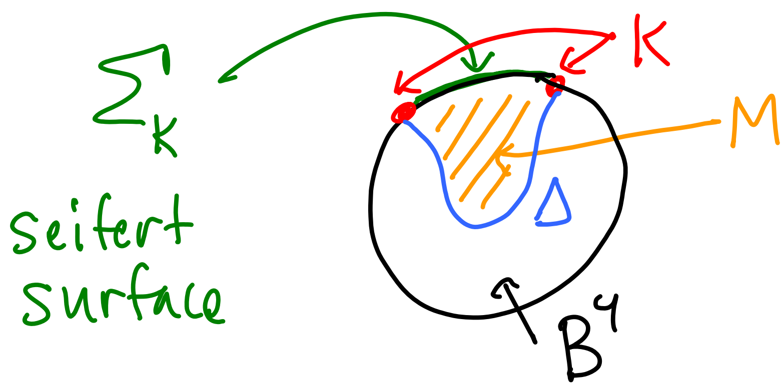
<u>Levine</u>: $\mathcal{C} \xrightarrow{\pi_A} \mathcal{C}_A \cong \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \oplus \mathbb{Z}_4^\infty$

$K \longmapsto [V_K]$

Why is $\mathcal{C} \xrightarrow{\pi_A} \mathcal{C}_A$ well-defined?

- If K is slice $\Rightarrow$ $V_K = \left(\begin{array}{c|c} * & * \\ \hline * & 0 \end{array}\right)$

Idea: If K is slice $\Rightarrow$ $K = \partial\Delta$, $\Delta \subset B^4$



Show:

(1) $\Sigma_K \cup \Delta$ bounds a 3-mfld M in B^4

(2) use a homological argument to show $\exists$ half basis of curves on Σ_K that vanish in $H_1(M;\mathbb{Q})$.

Examples of knot concordance invariants from the Seifert matrix

$$I : \mathcal{C}_A \longrightarrow \begin{matrix} \mathbb{Z} \\ \mathbb{Z}/2 \\ \mathbb{Z}/4 \end{matrix}$$

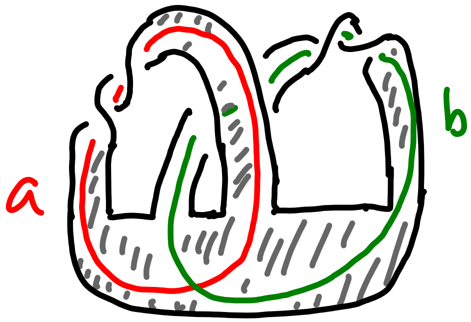
Levine-Tristram signatures

($\mathbb{Z}$-invariants)

For $\omega \in \mathbb{C}$, $|\omega| = 1$:

- $\sigma_\omega(K) := \text{signature}\left((1-\omega)V_K + (1-\bar{\omega})V_K^T\right) \in \mathbb{Z}$

- If ω is not a root of the Alex. poly and K is slice $\Rightarrow$ $\sigma_\omega(K) = 0$.

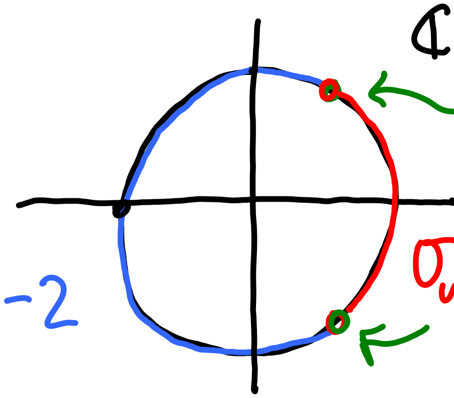


$K = \partial(\text{surface})$

$= \text{trefoil}$

$\rightsquigarrow \quad V_K = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ $\quad lk(a, b^+)$

"linking" matrix



$w = \dfrac{1 \pm \sqrt{3}\,i}{2}$ (roots of Alex. polynomial)

$\sigma_w(\text{Trefoil}) = -2$

$\sigma_w(\text{Trefoil}) = 0$

Since $\sigma_{w=-1}(\text{trefoil}) \neq 0$, trefoil $\neq$ slice.

Examples of $\mathbb{Z}/2$-invariants

<u>Def</u>$^{\underline{n}}$ Alexander polynomial of K:

$$\Delta_K(t) := \det(V_K - tV_K^T)$$

For any prime polynomial $p(t)$

$$\Delta_K(t) = p(t)^{\varepsilon(p)} \cdot g(t)$$

where $p(t) \nmid g(t)$.

Define $e_{p(t)}(K) := \varepsilon(p) \mod 2$ ($p(t)$-exponent)

If K is slice then $V_K = \left(\begin{array}{c|c} * & * \\ \hline * & 0 \end{array}\right) \Rightarrow$

$$\Delta_K(t) \doteq f(t)f(t^{-1})$$

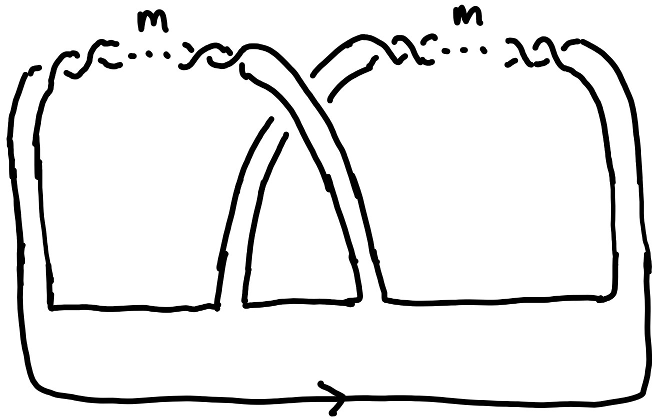
$\uparrow$ (up to units)

Thus for each prime $p(t)$ that is <u>symmetric</u> $(p(t) \doteq p(t^{-1}))$, $\Delta_K(t) = p(t)^{2\cdot m} g(t)$.

$$\Rightarrow \varepsilon_{p(t)}(K) = 0 \pmod 2.$$

- $\varepsilon_{p(t)} : \mathcal{C}_A \longrightarrow \mathbb{Z}_2$ for $p(t) \doteq p(t)$ prime.

Ex:



K_m

$$V_{K_m} = \begin{pmatrix} m & 1 \\ 0 & -m \end{pmatrix}$$

$$\Delta_{K_m} = \det(V_K - tV_K^T) = m^2t^2 - (1+2m^2)t + m^2$$

irreducible
all distinct

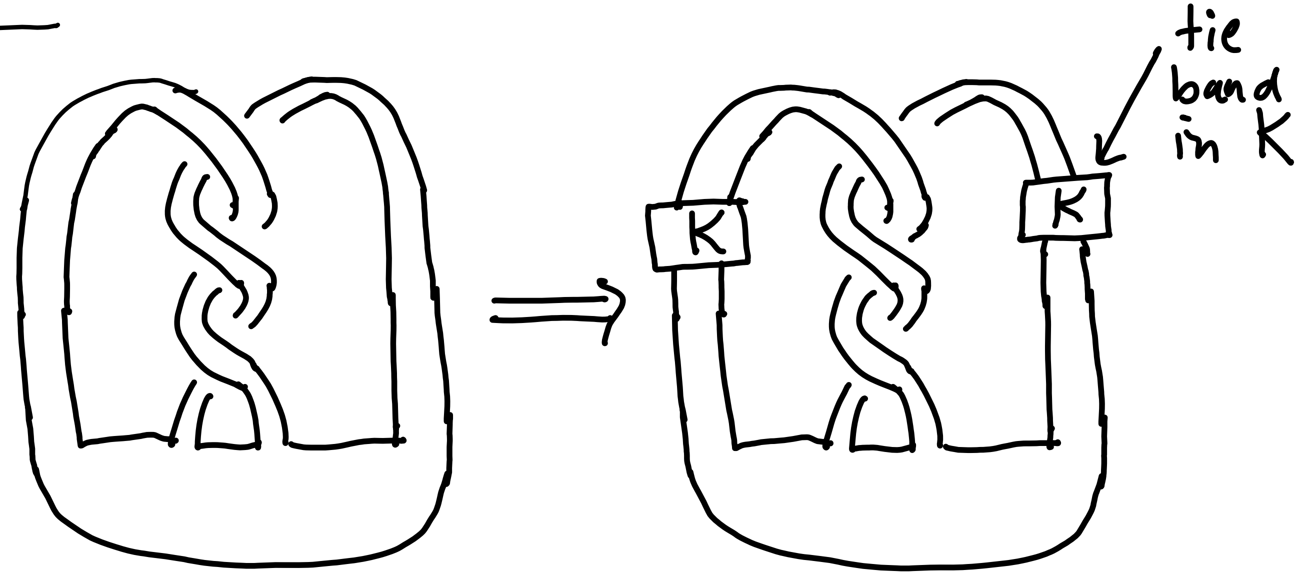
$$\Rightarrow \varepsilon_{\Delta_{K_m}}(K_m) = 1 \pmod 2$$

So K_m are independent in $\mathcal{C}$, $\mathcal{C}_A$

Def: K is __algebraically slice__ if $\pi_A(K) = 0 \in \mathcal{C}_A$.

Define $\mathcal{F}_{0.5} := \ker(\mathcal{C} \xrightarrow{\pi} \mathcal{C}_A)$, alg. slice knots.

__Example__:



slice $\Longrightarrow$ algebraically slice

Part $\underline{\text{III}}$:

(n)-solvable filtration of $\mathcal{C}$

In 1997, Cochran-Orr-Teichner define a filtration of algebraically slice knots:

$$
\begin{array}{l}
\bigcap_n \mathcal{F}_n \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{C} \\
\quad \cup \\
\mathcal{C}^{top} := \text{subgp of top. slice knots} \\
\quad \cup \\
\mathcal{C}_\Delta := \text{subgp of knots with Alex. polynomial } 1
\end{array}
$$

Def A knot is **(n)-solvable** ($n \in \mathbb{N}$) if M_K bounds a smooth 4-mfld W s.t.

(1) $i_* : H_1(M_K) \xrightarrow{\cong} H_1(W)$

(2) $H_2(W)$ has a basis $\{f_i, g_i\}_{i=1}^{g}$ of embedded surfaces (w/ triv. normal bundle) all disjoint except $f_i \cdot g_i = 1$ (geometrically)

(3) $\pi_1(f_i), \pi_1(g_i) \subset \pi_1(W)^{(n)}$

- $K \in \mathcal{F}_n \iff K$ is (n)-solvable

Thm (n=0, ~67, Milnor-Tristram; n=1, ~81, Jiang; n=2, ~00, Cochran-Orr-Teichner)

For $n=0,1,2$, $\mathcal{F}_n/\mathcal{F}_{n.5}$ contains a $\mathbb{Z}^\infty$.

Thm (Livingston) $\mathcal{F}_1/\mathcal{F}_{1.5}$ contains a $\mathbb{Z}_2^\infty$.

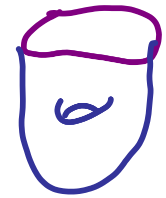
Thm (Cochran-Teichner, Cochran-H-Leidy): For $n \geq 2$,

$$\mathcal{F}_n/\mathcal{F}_{n.5} \supset \underset{\text{CT ('02)}}{\mathbb{Z}} \oplus \underset{\text{CHL ('07)}}{\mathbb{Z}^\infty} \oplus \underset{\text{CHL ('09)}}{\mathbb{Z}_2^\infty}$$

To prove these theorems one uses

- higher-order Alexander modules and their linking forms
- signature defects

ρ-invts



$$\begin{tikzcd} M^3 \arrow[r, "\phi"] & \Gamma \\ W^4 \arrow[ur, "\phi"'] & \end{tikzcd}$$

Let $M^3 = \partial W^4$ and $\phi : \pi_1 W \longrightarrow \Gamma$.

Define

$$\rho(M,\Gamma) := \sigma^{(2)}(W,\Gamma) - \sigma(W)$$

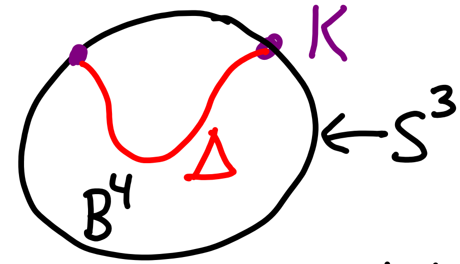
signature of Γ-equivariant intersection form on $H_2(\Gamma\text{-cover of } W)$.

<u>Ex</u>: For K = knot $\rightsquigarrow$ M_K = 0-surgery on K

$$\rho(M_K, \pi_1 M_K \longrightarrow \mathbb{Z} = H_1(M_K)) = \int_{S^1} \sigma_\omega(K)\, d\omega$$

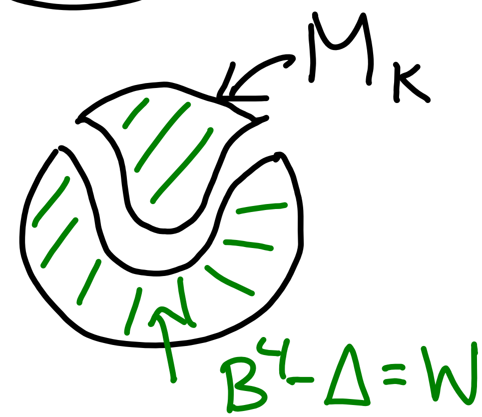
Why "$\rho = 0$" for a slice knot

Let K be a slice knot.



Then M_K = 0-surgery on K

$= \partial(\underbrace{B^4 - \text{nbhd}(\Delta)}_{W})$



Since $H_2(W) = 0$,

$H_2(W_\Gamma) =$ torsion (for nice Γ)

$\uparrow$ Γ-cover of W

Thus $\rho(M_K, \Gamma) = \sigma^{(2)}(W, \Gamma) - \sigma(W) = 0$

Open Problems

1. What about $\mathcal{J}_{n.5}/\mathcal{J}_{n+1}$?

2. $\mathbb{Z}/4\mathbb{Z} \subset \mathcal{J}_n/\mathcal{J}_{n.5}$ for $n \geq 1$?

   Other types of torsion?

Part III:

Topologically slice knots

$$\mathcal{C}_{top} \subset \mathcal{F}_\omega := \bigcap_{n\in\mathbb{Z}} \mathcal{F}_n$$

Recall $\mathcal{C}_{top}$:= knots that bound a top. locally flat disk in B^4 (top slice)

<u>Open question</u>: Is $\mathcal{F}_\omega / \mathcal{C}_{top} = 0$?

<u>Thm</u> (Freedman): If $\Delta_K(t) = 1 \Rightarrow K$ is top. slice.

$$\Rightarrow \mathcal{C}_\Delta := \left\{ \begin{matrix} \text{knot with} \\ \Delta_K = 1 \end{matrix} \right\} \subset \mathcal{C}_{top}$$

As a consequence of Donaldson's work on intersection forms on 4-mflds:

<u>Thm</u>: There are knots with $\Delta_K = 1$ that are not smoothly slice.

$$\therefore \quad 0 \neq \mathcal{C}_\Delta \subset \mathcal{C}_{top}$$

<u>Thm</u> (Endo): $\mathbb{Z}^\infty \subset \mathcal{C}_\Delta$

- uses work of Furuta, Fintushel-Stern

More recent invariants from ...
Heegaard-Floer and Khovanov homology

Heegaard Floer homology:

- $\tau : \mathcal{C} \twoheadrightarrow \mathbb{Z}$ (Ozsváth-Szabó, Rasmussen)
- $\delta : \mathcal{C} \twoheadrightarrow \mathbb{Z}$ (Manolescu-Owens, Ozsváth-Szabó)

More generally, have τ_p (Grigsby-Ruberman-Strle) and d_p-invariants associated to p-fold cyclic branched covers of S^3 over K.

$\boxed{\tau : \mathcal{C} \twoheadrightarrow \mathbb{Z}}$

A knot K in S^3 induces a bi-grading on $\widehat{CF}(S^3)$. The Alexander grading induces a $\mathbb{Z}$-filtration on $\widehat{CF}(S^3)$.

$$\{\vec{x} \mid \underset{\substack{\uparrow \\ \text{Alex. grading}}}{A(\vec{x})} \leq m\} = \mathcal{F}(K,m) \xhookrightarrow{i^m} \widehat{CF}(S^3)$$

$$\tau(K) := \min \left\{ m \in \mathbb{Z} \;\middle|\; \begin{array}{c} i^m_* : H_*(\mathcal{F}(K,m)) \longrightarrow H_*(\widehat{CF}(S^3)) \cong \mathbb{Z} \\ \text{is non-trivial} \end{array} \right\}$$

<u>Thm</u> (OS): $|\tau(K)| \leq g_4(K)$

Using τ and δ:

Thm (Manolesu-Owens, Livingston):

$$\mathcal{C}_\Delta \cong \underset{\substack{\uparrow \\ \tau}}{\mathbb{Z}} \oplus \underset{\substack{\uparrow \\ \delta}}{\mathbb{Z}} \oplus \text{other}$$

Khovanov homology

Rasmussen used Lee's spectral sequence to define a combinatorial concordance invt

$$s : \mathcal{C} \twoheadrightarrow \mathbb{Z}$$

Thm (Rasmussen) $|s(K)| \leq g_4(K)$

Uses s to give a purely combinatorial proof of Milnor's conjecture: $g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}$

Note: For alternating knots

$$\tau(K) = s(K) = \delta(K) = -\sigma(K)/2.$$

However, they are known to be independent

Recently Hedden-Livingston-Ruberman use Ozváth-Szabo's correction terms (d-invts) to define $\bar{d}(\Sigma(K), s)$ for each $s \in H_1(\Sigma(K))$. They use these to show

<u>Thm</u> (HLR): $\mathcal{C}_{top}/\mathcal{C}_\Delta \supset \mathbb{Z}^\infty$

Open Question:

- Torsion in $\mathcal{C}_{top}$ or $\mathcal{C}_{top}/\mathcal{C}_\Delta$?

Not...

THE END

... just the beginning