Fractal nature of the space of knotted curves

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Recall, the 3-dimensional sphere is
\[ S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} = \mathbb{C}^2 \]
\[ = \mathbb{R}^3 \cup \{\infty\} \]

**Def:** A **knot** is a smooth embedding
\[ f : S^1 \hookrightarrow S^3 \]
where \( S^1 = \{z \mid |z|^2 = 1\} \subseteq \mathbb{C} \) is the unit circle.
Exs:

unknot

trefoil

figure-eight
Knots can arise from singularities.

**Ex:** Let $C$ be the complex curve defined by

$$z^2 - w^3 = 0$$

It has a singularity at $(z,w) = (0,0)$.

**Def:** The link of this singularity is

$$L = C \cap \partial B(\varepsilon) = \{(z,w) \mid (z,w) \in C, |z|^2 + |w|^2 = \varepsilon^2\}$$

for small $\varepsilon$. 
Note: $L$ is the intersection of a $2$- and $3$-dimensional space in $\mathbb{C}^2 = \mathbb{R}^4$ so it is a $1$-dimensional real curve (or multicurve - called a link).
Write \( z = re^{\pi i \theta} \), \( w = Re^{\pi i \psi} \) \( \forall r, R \geq 0 \)

- \( z^2 = w^3 \Rightarrow r^2 = R^3 \Rightarrow z = R^{3/2} e^{2\pi i \theta} \)
  \[ \Rightarrow 2\theta = 3\psi \mod \pi \text{ so} \]
  \[ \psi = \frac{2}{3} \theta + k/3 \text{ for some } k \in \mathbb{Z} \]

- \( |z|^2 + |w|^2 = \varepsilon^2 \Rightarrow R^3 + R^\varepsilon = \varepsilon^2 \)

\( \exists R > 0 \text{ satisfying this.} \)
\[2\theta \equiv 3\psi \mod \mathbb{Z}\]
\[\psi = \frac{2}{3}\theta + \frac{k}{3}, \quad k \in \mathbb{Z}\]

\[L = \begin{array}{c}
\text{(2,3) torus knot}
\end{array}\]

\[S^1 \times S^1 \cong S^3\]
In knot theory, one typically studies knots up to isotopy: you can deform one knot to the other without passing the knot through itself.

![Diagram of three knots](https://via.placeholder.com/150)
Two knots can be isotopic but "look" very different!
Not allowed to "change crossings"!
Note: The unknot is the unique knot that bounds a (smooth) disk in $S^3$ (or $\mathbb{R}^3$).

Q. What are the knots in $S^3$ (or $\mathbb{R}^3$) that bound a smooth disk in $B^4 = \{ (z,w) \mid 1 \leq z^2 + w^2 \leq 1 \}$ (or $\mathbb{R}^4_+ = \{ (x,\ldots,x_4) \mid x_4 \leq 0 \}$).

Such a knot is called (smoothly) slice.
Fox and Milnor studied these in the 60's as a way to smooth singularities.

If $L$ is slice, can replace singularity with smooth disk $D \subseteq B^4$. 

$S^3 = 2B^4$
However, it turns out that the link of a singularity is never slice (except in the trivial case)!
Def: A knot $K \subseteq S^3 = \partial B^4$ is slice if $K = \partial D$ is the boundary of a smoothly embedded disk $D$ in $B^4$. 

$S^3 = \text{boundary of } B^4$
Equivalently:
A knot $K \subseteq \mathbb{R}^3 = 2B_{-}$ is slice if $K = \partial D$ with $D$ a smoothly embedded in $\mathbb{R}^4$. 
Ex: A knot is ribbon if it is the boundary of an immersed disk in $\mathbb{R}^3$ (or $S^3$) with "ribbon singularities":

Observation: Every ribbon knot is slice. Pf: Take a small disk around singularity and push it into $\mathbb{R}^4_+$ (or $B^4$)
push interior of
into interior of $\mathbb{R}^4$.

What is left in $\mathbb{R}^3$?

To get disk in $\mathbb{R}^4$, attach lower hemisphere of
$S^2 = \{(x, y, 0, v) \mid x^2 + y^2 + v^2 = 1\} \subseteq \mathbb{R}^4$
to red curve.
$K = 8q$

$8q$ is ribbon

Slice disk for $8q$

$\Rightarrow 8q$ is slice but does not bound an embedded disk in $\mathbb{R}^3$!
Biggest open problem in knot concordance:

Slice-ribbon conjecture: Every (smoothly) slice knot is ribbon.

Note: This problem is extremely difficult since every ribbon knot has a slice disk that is not even isotopic to any ribbon disk!

Thus, cannot start with a slice disk and deform it to become a ribbon disk.
[For the experts]

Let $S$ be a smoothly embedded non-trivial 2-knot, $S^2 \hookrightarrow S^4$. Let $U = \text{unknot}$, and $D = \text{standard disk}$ with $\partial D = U$. Push $U$ into $B^4$ and then take a connected sum with $S$. Then $U = 2S$ (S punctured) and $\pi_1(B^4 \setminus S^0) = \pi_1(S^4 \setminus S)$ is non-abelian since $S$ is non-trivial.
Fact: If $D$ is a ribbon disk for $K$ then
\[ \pi_1(S^3 \setminus K) \xrightarrow{i_*} \pi_1(B^4 \setminus D) \]
is surjective.

In example:
\[ \pi_1(S^3 \setminus \text{unknot}) \rightarrow \pi_1(B^4 \setminus \hat{S}) \]
is non-abelian

\[ \Rightarrow \hat{S} \text{ is a slice disk that is not isotopic to any ribbon disk}. \]
Another example (using "movie moves")
We can look at level set of a disk in $\mathbb{R}^4$.

9.46 is slice
$t = \frac{1}{4}$
$t = \frac{3}{8}$

$\mathbb{R}^3$ (t > 0)
purple disk is behind the blue

$t \geq \frac{1}{2}$

$t = \text{time}$

$\mathbb{R}^4_+ (t \geq 0)$
We can put an 4-dimensional equivalence relation on knots.

**Def:** Let $K$ and $J$ be knots in $\mathbb{R}^3$. We say that $K$ is **concordant** to $J$ if $K \times \{0\}$ and $J \times \{1\}$ co-bound a smoothly embedded annulus in $\mathbb{R}^3 \times [0,1]$. 

![Diagram of a 4-dimensional concordance](image)
Concordance group

Let $C = \{\text{knots}\}/\sim$ if they are concordant.
Then $C$ is a group under connected sum.

* need oriented knots.
Identity

Inverse of $K$ is $\bar{K}$.

For any $K$, $K \# \bar{K}$ is slice where $\bar{K} = \text{mirror image}$.
Prove that \( K \# \overline{K} \) is slice (ribbon)

Make an immersed disk by lines from \( K \) to \( \overline{K} \). The only self-intersection are ribbon singularities.
\( C \) is a non-finitely generated abelian group.

We don't know what \( C \) is.

- \( C \) contains elements that are 2-torsion.

\[
\begin{align*}
\Downarrow & = \Downarrow \\
4_1 & = \overline{4}_1
\end{align*}
\]

\[
\Rightarrow 24_1 = 0 \quad \text{and} \quad 4_1 \text{ is not slice } (4, \neq 0)
\]
- $C$ contains elements of infinite order

\[ \bigcirc \ # \ \# \ \# \ \bigcirc \ \text{is never slice.} \]

**Thm (Levine '60's) \exists \text{ surjective homomorphism}**

\[ C \xrightarrow{\pi} A \cong \mathbb{Z}^9 \oplus \mathbb{Z}_2^6 \oplus \mathbb{Z}_4^6 \]

\[ \uparrow \text{algebraic concordance group} \]

(\text{Witt group of Seifert matrices})
Q. Are all torsion elements, 2-torsion?

- $\ker(\pi)$ is non-trivial (in higher dimensions, $\pi$ is an $\equiv$)

Thm (Casson- Gordon, Gilmer): $\ker \pi \neq 0$.

$K = \text{trefoil}$
Tie strand into $K$

$K =$

$=$

$K =$
$n$-solvable filtration

Cochran-Orr-Teichner defined filtration

$E_n \subset \cdots \subset E_{1} \subset E_{0.5} \subset E_{0} \subset \mathbb{C}$

$K \in \mathcal{F}_{0} \iff \text{Arf}(K) = 0 \quad \text{Arf invariant}$

$K \in \mathcal{F}_{0.5} \iff K \in \ker(\Pi) \quad \text{Algebraically slice}$

$K \in \mathcal{F}_{1.5} \quad \Rightarrow \quad \text{Casson-Gordon invariants vanish.}$
Thm (Cochran-H-heidy): For each \( n \geq 0 \), \( J^*_n / J^*_n S \) contains \( \bigoplus \binom{Z^* \otimes Z^*}{p(t)} \)

symmetric
irreducible

\( n = 0 \) : Milnor-Tristram, Levine (60's)
\( n = 1 \) : Jiang, Livingston (80's)
\( n = 2 \) : Cochran-Teichner (02)
Operators on $C$

Def: A pattern $P$ is a slice knot $R$ and unknot $\eta$ disjoint from $R$, such that $\eta$ bounds a surface disjoint from $R$.

$P =$ 

$\leftarrow R$ slice

$\eta$
\( P : C \rightarrow C \) (not a homomorphism)

\[ P(K) = \text{satellite operator.} \]

tie strands going through \( \eta \) into \( K \)
Ex:

\[ P(K) = \]

\[ k = \mathcal{M} \]
\[ P : \mathcal{F}_n \rightarrow \mathcal{F}_{n+1}. \]

Hence \( P^n(k) = P(P(... P(k)) \in \mathcal{F}_n \)

for any \( k \) with Art invariant zero.

Exs of \( \mathcal{F}^o \) and \( \mathcal{F}_2 \in \mathcal{F}_n/\mathcal{F}_{n-1} \) are constructed this way!
Q. When is \( P \) injective?

Conjecture: \( Q \) is injective.

\[ Q(K) = \]

\( Q(K) \) is slice \( \iff \)
\( K \) is slice.

Every such \( P \) would re-embed \( C \) into itself.
Known: There is a subgroup of \( C \) on which \( P^n \) is injective for all \( n \).

Satellite operators give a way to construct elements in \( \mathfrak{fr}_n \). The difficult part is to show \( P^n(K) \) is not slice (or even in \( \mathfrak{fr}_n \)?)

- Use invariants of knots such as \( \text{L}^2 \)-signatures, d-invts and \( \mathcal{F} \) invariants from Heegaard Floer homology, etc.
Note: There is no known algorithm to determine if a knot is slice!!

Q. Is the Conway knot slice?

known to topologically slice but unknown if it is smoothly slice!
Would like some notion of distance where image \((P^n)\) is getting smaller as \(n \to \infty\).
Symmetric gropes

**Def:** A grope of height 1 is a compact oriented surface $G_i$ with $|\omega| = 1$.

Let $\{\alpha_1, \ldots, \alpha_{2g}\}$ be a standard symplectic basis of curves for $H_1(G_i)$ on $G_i$, $g = \text{genus}(G_i)$.
A grope of height \( n+1 \) is obtained by attaching gropes of height \( n \) to \( \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \).

height 2 grope
Def: A **branched symmetric grope** is defined as follows:

Let $\Sigma_{1:1}$ be a compact connected orientable surface of genus $g$, with a standard symplectic basis of curves $\{a_1, \ldots, a_{2g}\}$ with $a_{2i-1}$ dual to $a_{2i}$. Attach to each $a_i$, a grope of height $m_i$ s.t. $m_{2i-1} = m_{2i}$, no subsurface of which is a disk.

Let $n_i = m_{2i}$.
\[ n_1 = m_1 = m_2 = 0 \quad n_2 = m_3 = m_4 = 2 \quad n_3 = m_5 = m_6 = 1 \]
Let \( \Sigma \) be a branched symmetric grope.

Define

\[ g_1 = \text{genwo} (\Sigma) \]

\[ g_2 = \text{sum of genera of first stage surfaces attached to } \alpha_{2i-1}, \alpha_{2i}. \]

\[ g_{n+1} = \text{sum of genera of } n_i \text{ stage surfaces attached to } \alpha_{2i-1}, \alpha_{2i}. \]
$g_1 = 3$
No $g_2^1$ since $n_1 = m_1 = m_2$.

$g_2^2 = 1 + 2 = 3$

$g_2^3 = 2 + 2 = 4$
\[ g_3^2 = 2 + 1 + 3 + 1 + 1 + 1 = 9 \]
Note: For each $1 \leq i \leq g_i$ and $2 \leq k \leq n_i + 1$,

$$g_k \geq 2g_{k-1}$$

$$g_k^i \geq 2^{k-1}$$
Let $q \geq 1$ be a real number and $\Sigma$ a branched symmetric grope. Define

$$
\| \Sigma \|_q := \sum_{i=1}^{g_1} \frac{1}{q^{n_i}} \left( 1 - \sum_{k=2}^{n_i+1} \frac{1}{q^k} \right)
$$

**Def:** If $K, J$ are knots, define

$$
d^q(K, J) := \inf \left\{ \| \Sigma \|_q \left| \Sigma \text{ is a branched symmetric grope embedded in } S^3 \times I \text{ with boundary } K \times \{0\} \text{ and } J \times \{1\} \right. \right\}
$$

**Note:** Any two knots cobound a surface.
Ex: If $K$ has bounds a genus 1 surface $\Sigma$ and $\text{Arf}(K) \neq 0$ then $K$ cannot bound a (symmetric) height 2 grope, so

$$d(k, \text{unknot}) = g(\Sigma) = 1.$$ 

Ex: $$\frac{1}{2q} \leq d^q(\emptyset, \emptyset) \leq \frac{27}{16q}.$$
$$\sum =$$

\[
\|\Sigma\|_{q_0} = \left( \frac{1}{q_0} \cdot 1 \right) + \frac{1}{q_2} \left( 1 - \frac{1}{3} - \frac{1}{q} \right) + \frac{1}{q_i} \left( 1 - \frac{1}{4} \right) = 1 + \frac{5}{q^{2}} + \frac{3}{4q}
\]

\(i = 1\) \quad \(i = 2\) \quad \(i = 3\)
Note: The only way one could get zero is to have a (symmetric) grope of arbitrarily long height with all genus 1 surfaces at each stage, or an annulus.

\[ |E|_q = \left| \frac{1}{q_0} \left( 1 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2^{n_k}} \right) \right| \rightarrow 0 \quad \text{as } n_k \rightarrow \infty \]
Prop (Cochran-H-Powell): For $q \geq 1$, the function $d^q$ determines a pseudo-metric on $C$.

Need to show $\|E\|_q \geq 0$ for any $E$.

Prop: If $K$ does not bound a grope of height $n$ then

\[ d(K, \text{unknot}) \geq \frac{1}{(2q)^{n-2}}. \]
Thm (Cochran-Orr-Teichner): If $K$ bounds a height $n$ groove then $K \in \mathcal{F}_{n-2}$.

Prop (Cochran-H-Powell): If $P$ is a pattern then $P : C \rightarrow C$ is a contraction w.r.t. $d^q$ for $q > gw(P)$, the number of times $R$ goes through $\pi$. 
Thm (Cochran-H-Powell): For any $q > 1$ there exists uncountably many sequences of knots $\{K_i\}$ s.t.

$$d^q(K_i, \text{unknot}) > 0 \quad \forall i \quad \text{but}$$

$$d^q(K_i, \text{unknot}) \to 0 \quad \text{as} \quad i \to \infty.$$  

Hence the topology on $(C, d^q)$ is not discrete for $q > 1$. 