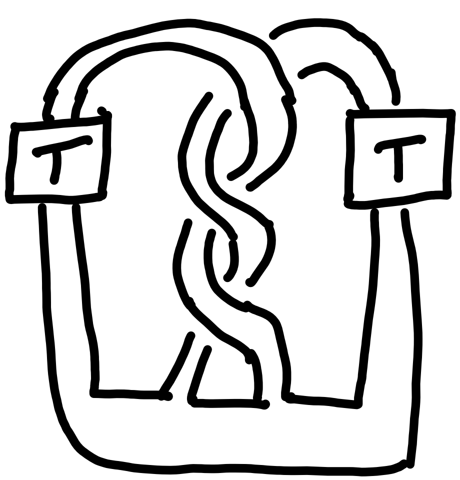
New Phenomena in Knot and Link Concordance

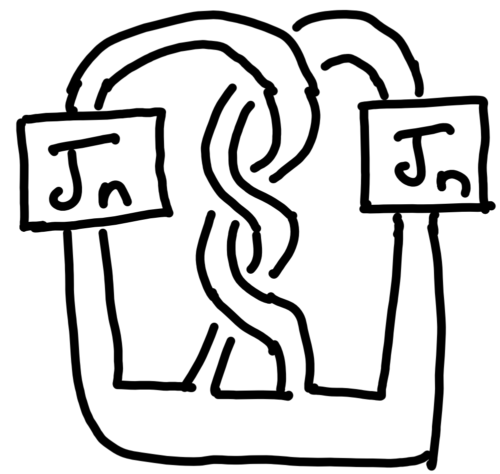
Shelly Harvey

joint work with Tim Cochran + Constance Leidy

Goal: Show certain classes of knots and links are not topologically slice.

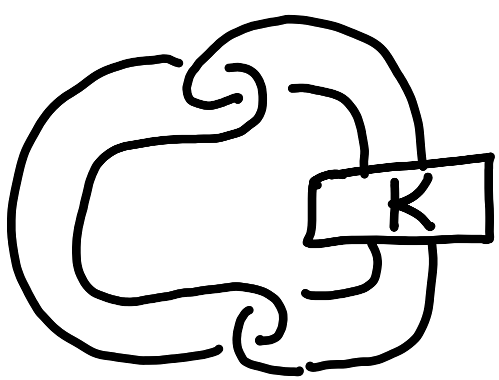
knots:

$J_1 =$ 

$\cdots$ $J_{n+1} =$ 

$(n \geq 2)$

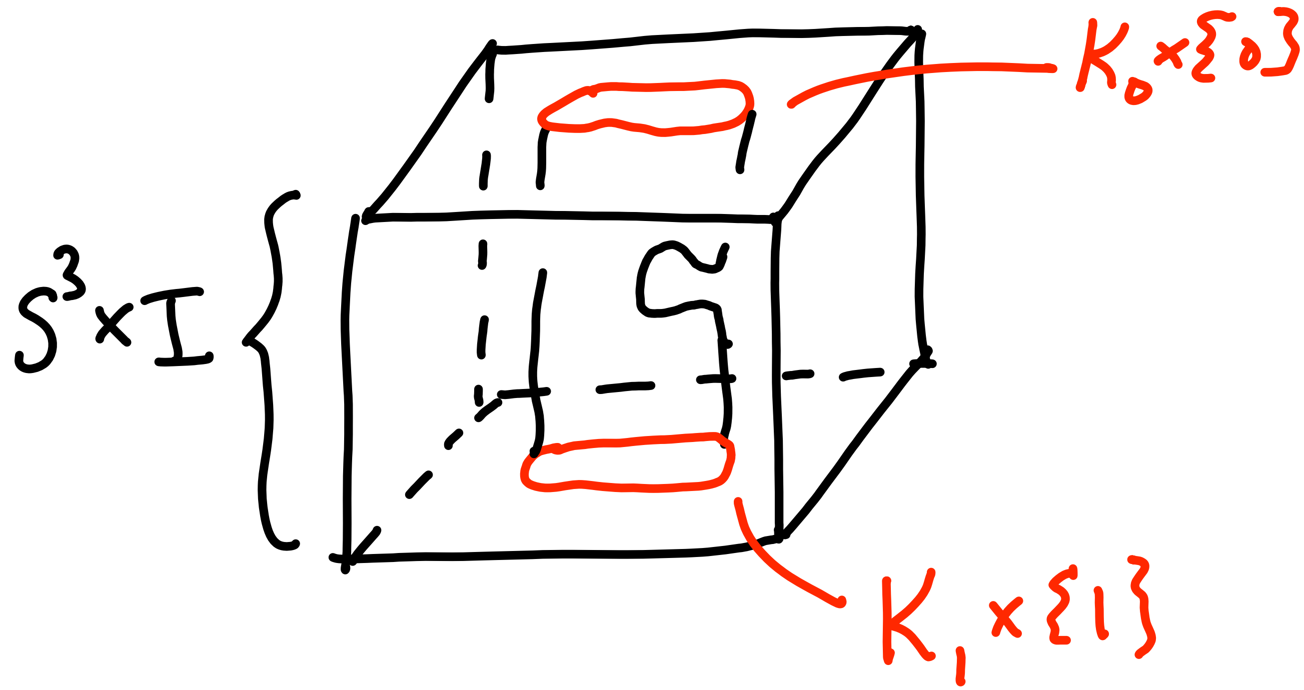
links:



(K algebraically slice)

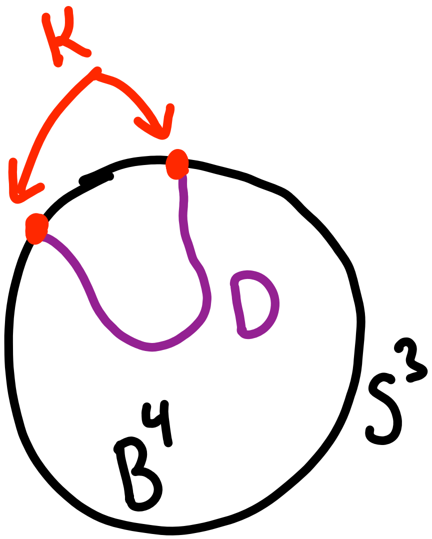
Knots: Let K_0, K_1 be knots in S^3

K_0 is (topologically/smoothly) concordant to K_1 if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a (locally flat/smooth) annulus in $S^3 \times I$.



K is <u>slice</u> $\iff$ K is concordant to unknot

$\iff$ K bounds 2-disc D in B^4

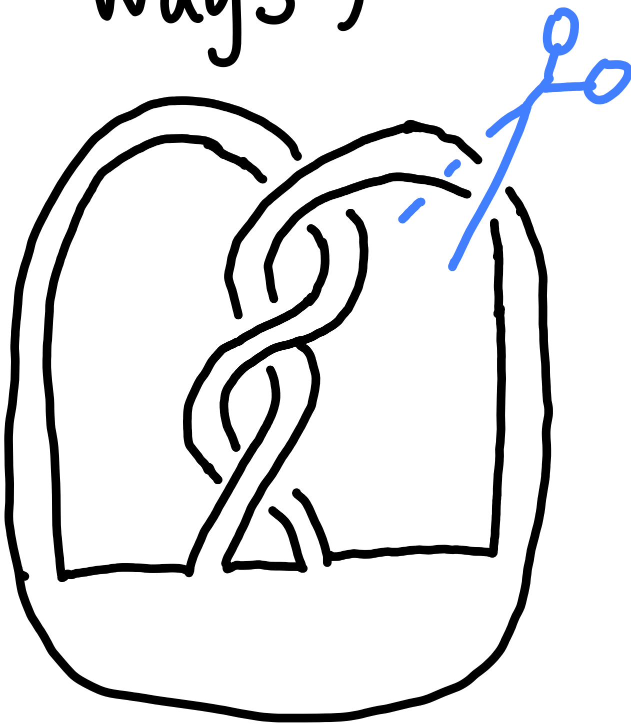


$\mathcal{C}$ = knot concordance group

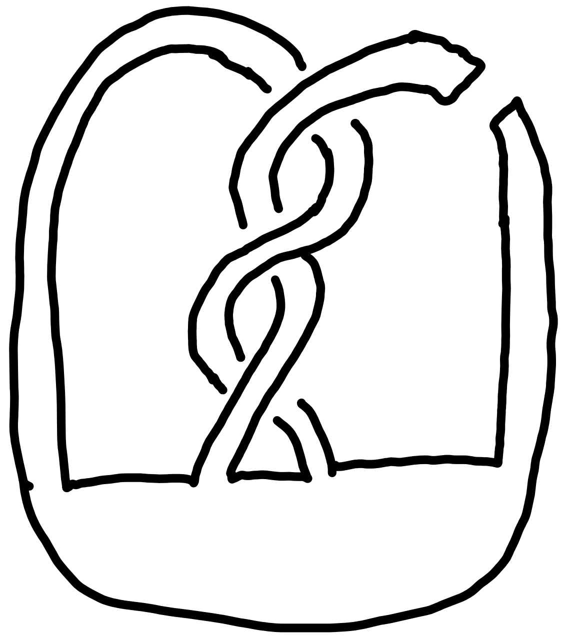
= {knots}/{concordance}

Ex 1: Stevedore's knot is Slice (in many ways)

$R_0 =$

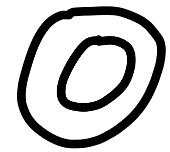


cut band $\longrightarrow$

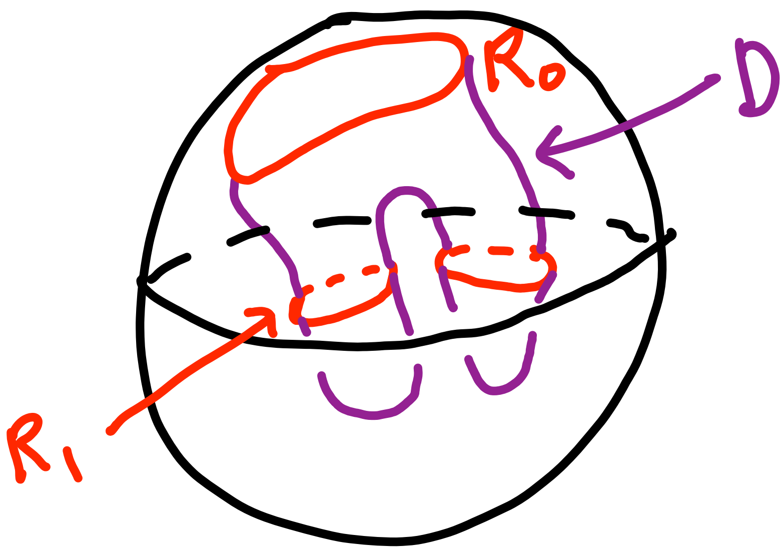


$= R_1$

$=$



$\uparrow$ cap off

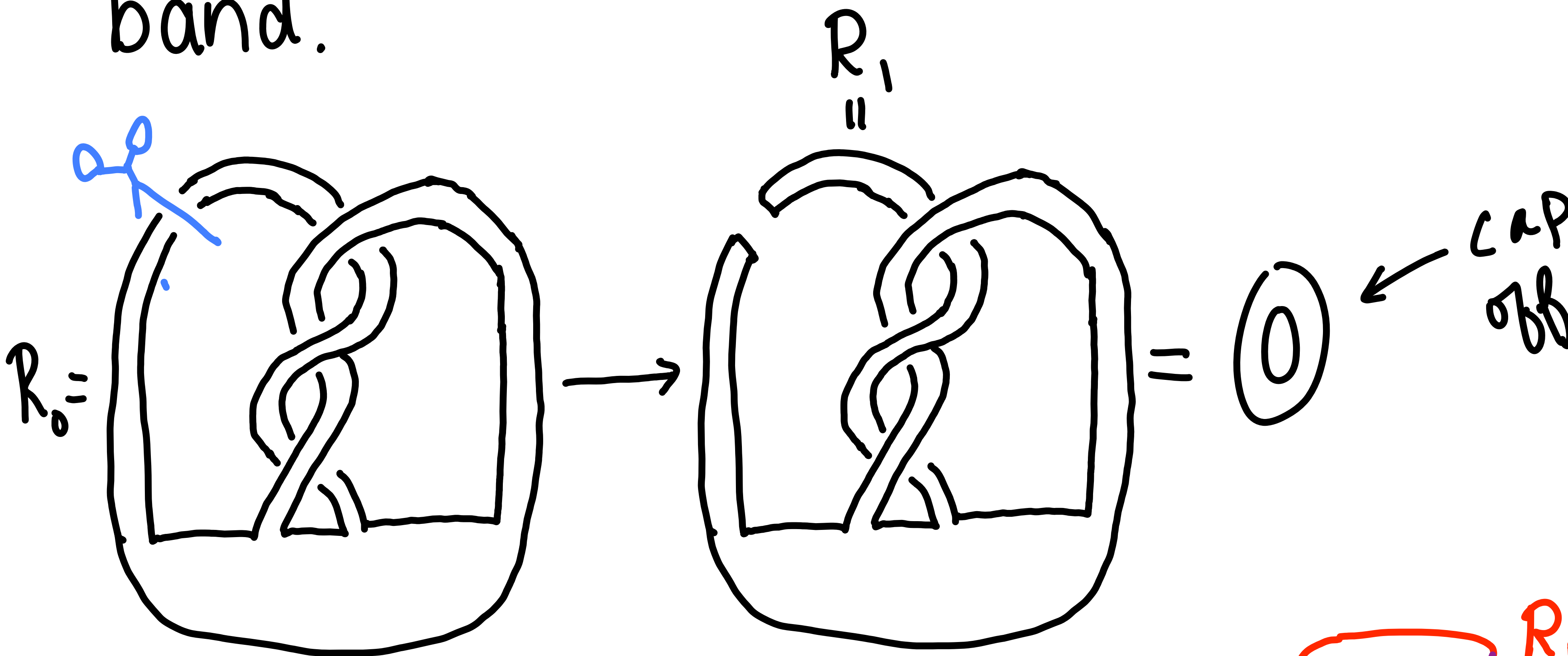


$\Longrightarrow$

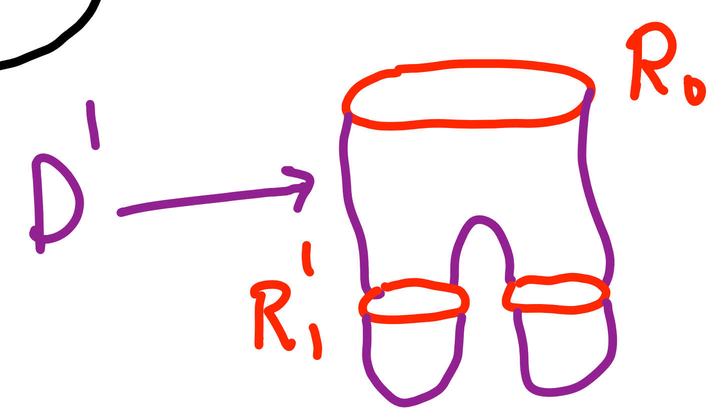
$R_0 = \partial D$

R_0 is Slice

Can also slice R_0 by cutting other band.

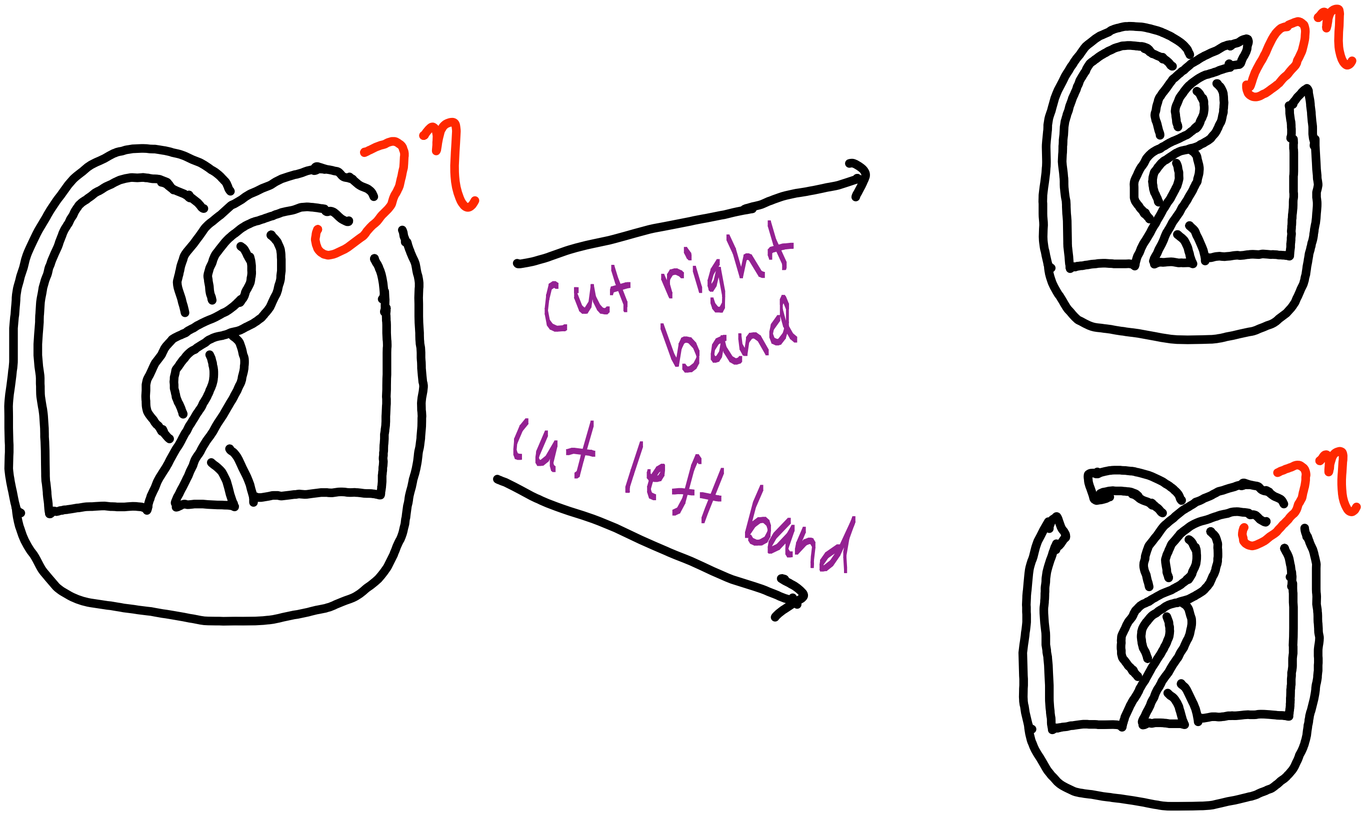


$R = \partial D'$ $\quad D' \subset B^4$



There are "different" ways to slice R_0

Let $W = B^4 - D$ and $W' = B^4 - D'$

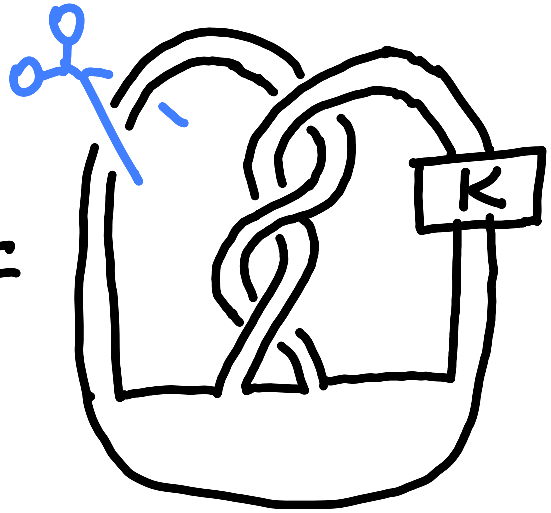


η is trivial in $\pi_1(W)$

η is non-trivial in $\pi_1(W')$

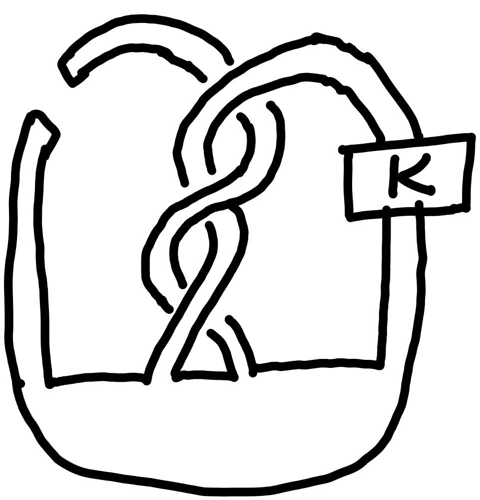
Ex 2: Tie knot K into band of $R_0 \to R_1$

Is R_1 slice?

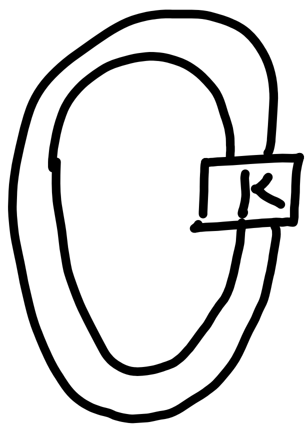
$R_1 =$ 

① Try to Slice R_1 by cutting left band

$\downarrow$



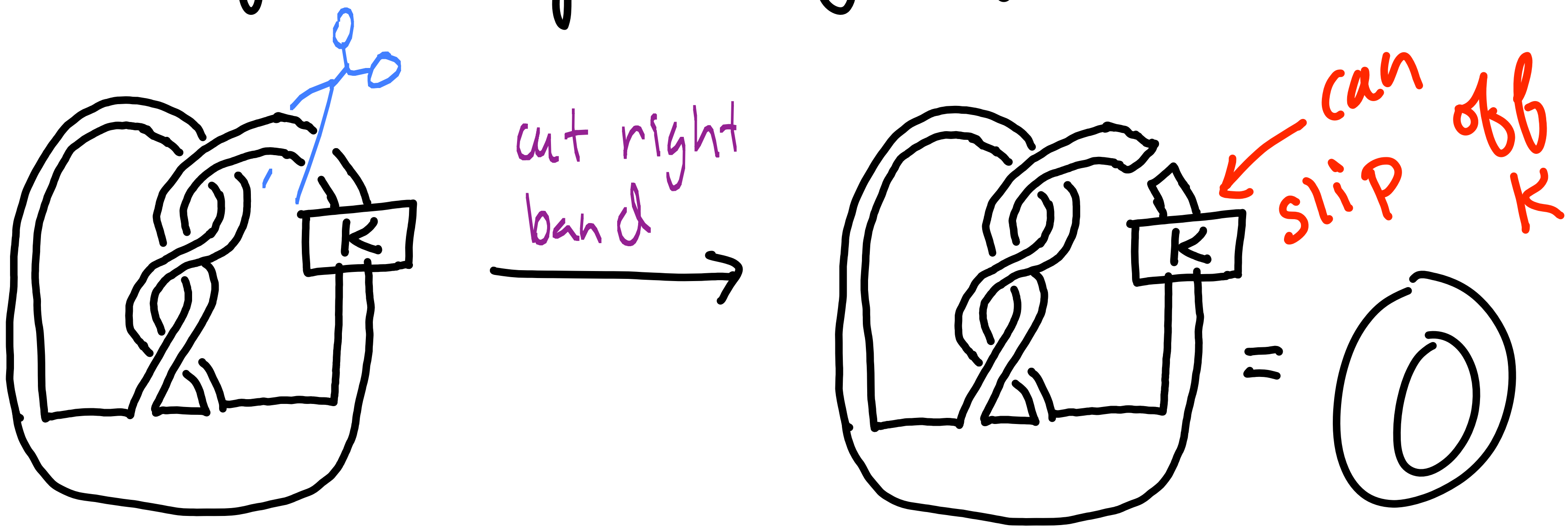
$=$



Cannot cap off

② Slice by cutting cutting right band

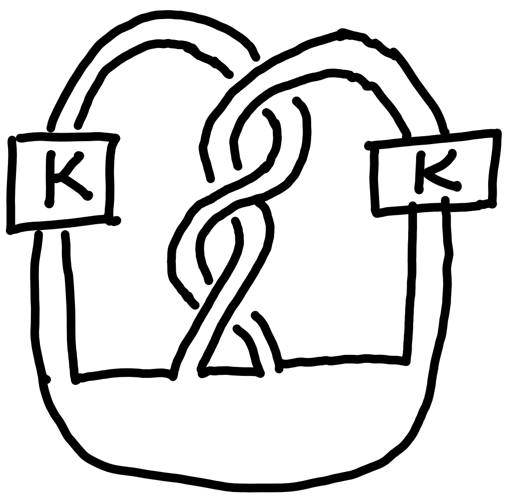
$R_1 =$



$\Longrightarrow R_1$ is still slice

$\therefore$ Necessary to tie knot in both bands to be not slice.

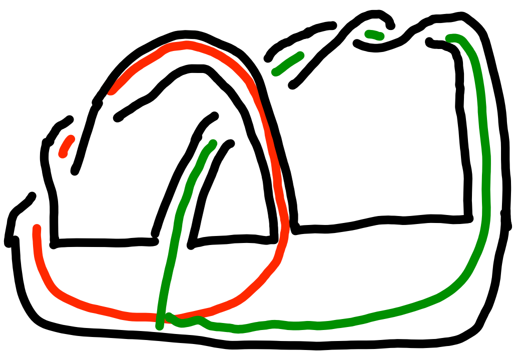
Q. For which K (nontrivial) is

$J_1(K) =$ 

Slice?

<u>Note</u>: If K Slice then $J_1(K)$ is Slice.

Levine-Tristram Signatures



$\longrightarrow$ $V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

K = knot

Seifert Matrix

For $\omega \in S^1 \subset \mathbb{C}$ define

$$\boxed{\sigma_\omega(K) = \text{signature}(\omega V + \bar{\omega} V^T)}$$

- If K is slice and ω is not a root of $\Delta_K(t)$ [Alex. poly] then $\sigma_\omega(K) = 0$.

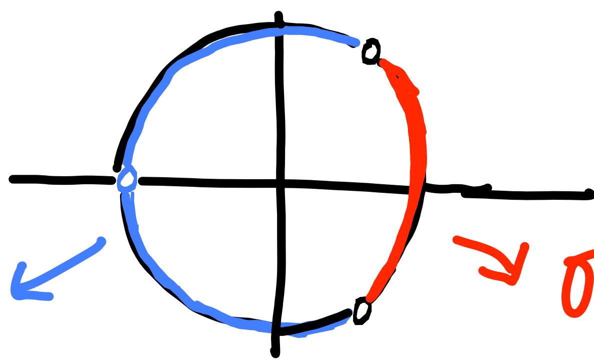
Note: If K Slice then

$$\rho_0(K) \equiv \int \sigma_\omega(K)\,d\omega = 0 \,!$$

Ex 3: Trefoil is not Slice

$$V_{\text{Trefoil}} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow \sigma_1 = \text{signature}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = -2 \neq 0$$

In fact



$\sigma_\omega = 0$

$\sigma_\omega = -2$

$$\boxed{\begin{array}{c} \Delta_{\text{Trefoil}} = t^2 - t + 1 \\ \downarrow \text{roots} \\ \dfrac{1 \pm \sqrt{3}\,i}{2} \end{array}}$$

$$\rho_0(K) = \int_{S^1} \sigma_\omega(K)\,d\omega = -\frac{2}{3} \neq 0$$

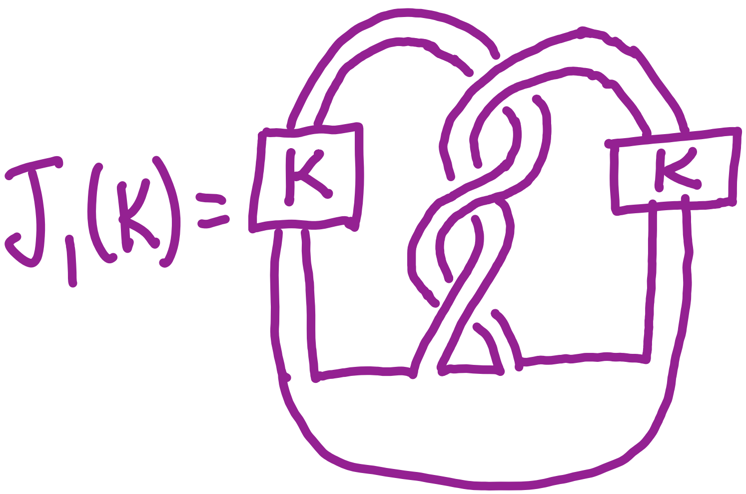
In 1960's Levine used invariants obtained from Seifert matrix (including knot signatures and Arf invariant) to define epimorphism

$$\mathcal{C} \xrightarrow{\pi} \mathbb{Z}^{\infty} \times \mathbb{Z}_2^{\infty} \times \mathbb{Z}_4^{\infty}$$

$\ker \pi$ = "Algebraically Slice knots"

hence $\mathcal{C}$ is infinitely generated.

Ex 4: Casson-Gordon-Gilmer knot



$K =$ trefoil

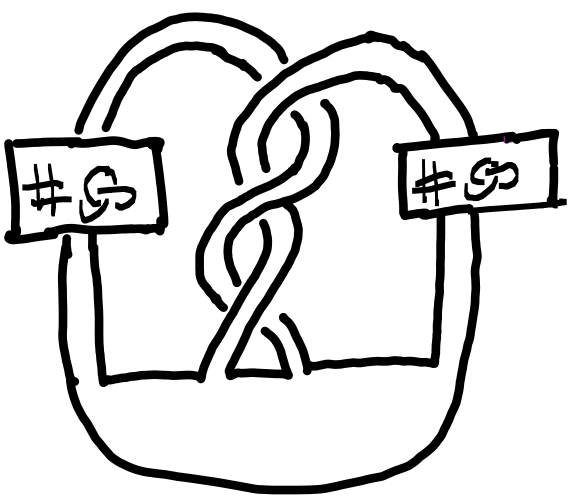
has same seifert matrix as $R =$ slice knot

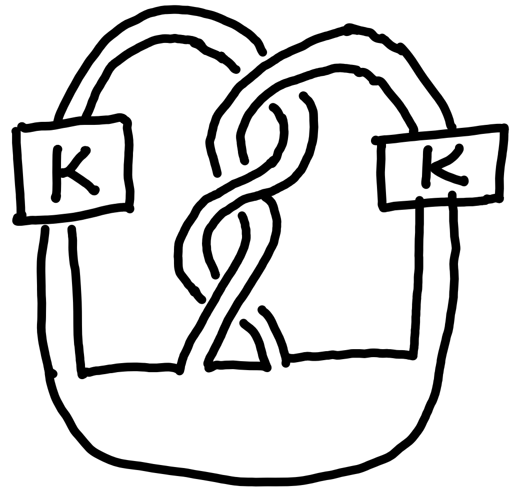
So is Algebraically slice

- Shown not to be a slice knot using Casson-Gordon invariants: signature invariants associated to metabelian covering spaces.

However if the "infecting knot" K is Algebraically Slice then all previously known methods (including Casson-Gordon + Cochran-Orr-Teichner invariants) fail to distinguish $J_1(K)$ from slice knot!

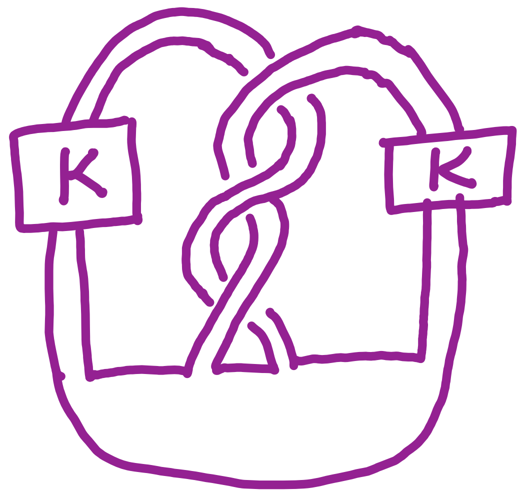
e.g. let $K = J_1(\#\text{trefoils})$

$K =$ 

$J_2 = J_1(K) =$ 

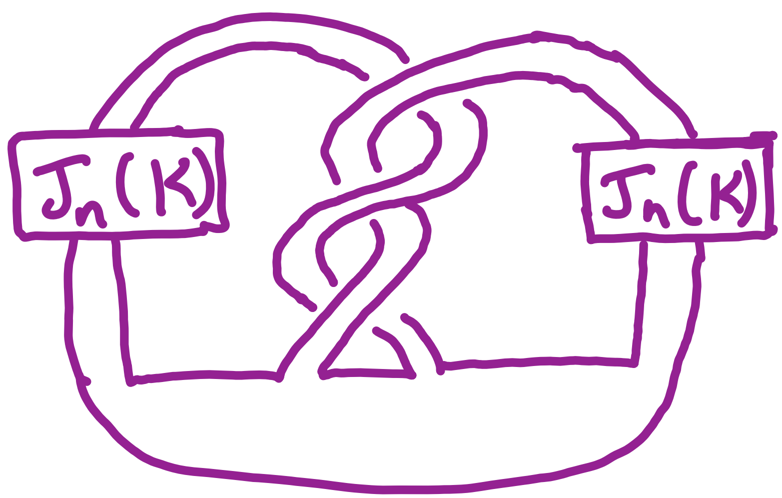
Let K be a knot. Define

$J_1(K) =$



$\vdots$

$J_{n+1}(K) =$



$(n \geq 1)$

- For $n \geq 2$, previously unknown if J_n is slice.
- ($\sim$1980) Gilmer showed that examples similar to J_2 are not ribbon [unpublished]

Theorem [Cochran-H-Leidy]:
For each $n \geq 1$, there is a constant C_n such that if $\rho_0(K) > C_n$ then $J_n(K)$ is not slice.

Remark: For m (depends on n) sufficiently large $J_n(\#_m \text{trefoils})$ is not slice.

Cochran-Orr-Teichner defined the (n)-solvable $(n \in \mathbb{N}/2)$ filtration of $\mathcal{C}$:

$$0 \subset \cdots \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} = \mathcal{C}$$

- $\mathcal{C}/\mathcal{F}_{(0.5)}$ = Algebraically Slice
- $K \in \mathcal{F}_{(1.5)} \Rightarrow$ Casson-Gordon invts vanish

<u>Theorem</u> (Cochran-Teichner): For each $n \geq 0$

$$\operatorname{rank}_{\mathbb{Z}} \mathcal{F}_{(n)}/\mathcal{F}_{(n.5)} \geq 1$$

[Levine $n=0$, CG $n=1$, COT $n=2$ $\leftarrow$ rank $= \infty$]

<u>Note</u>: If $K \in \mathcal{F}_{(n)} \Rightarrow J_1(K) \in \mathcal{F}_{(n+1)}$

hence $J_n(K) \in \mathcal{F}_{(n)}$. We show that

$J_n(K) \notin \mathcal{F}_{(n.5)}$ for some K.

Theorem [Cochran-H-Leidy]: For $n \geq 3$,

$\operatorname{rank}_{\mathbb{Z}} \mathcal{F}_{(n)} / \mathcal{F}_{(n.5)} \geq 2$.

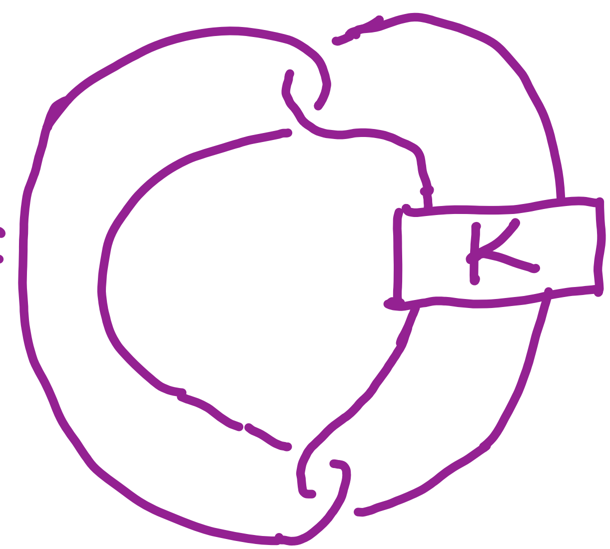
Moreover, we give an easier proof

that $\operatorname{rank}_{\mathbb{Z}} \mathcal{F}_{(n)} / \mathcal{F}_{(n.5)} \geq 1$.

Before we give an outline of proof, we consider a similar problem for links.

Knot $K \longrightarrow$ Bing Double of K

$BD(K) =$



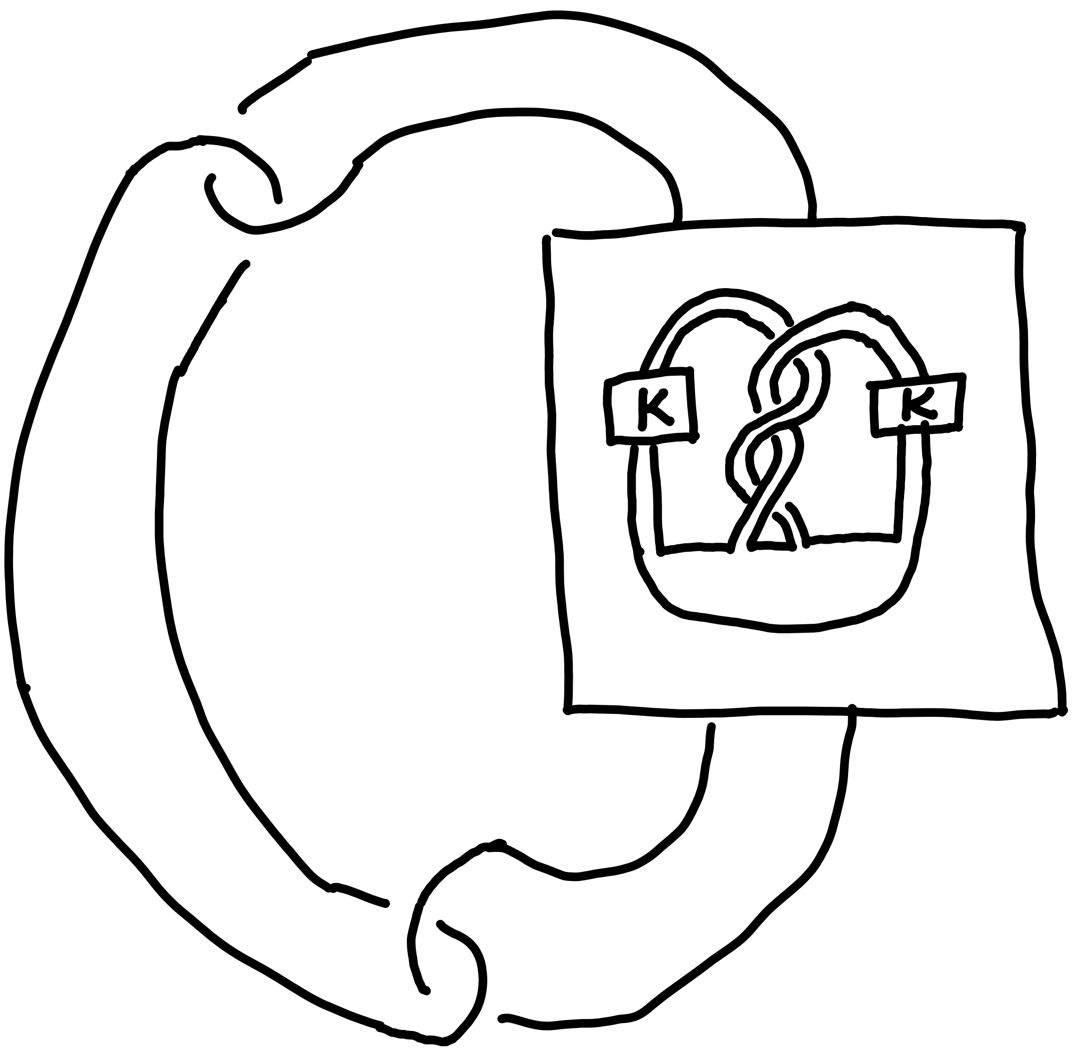
If K is slice then $BD(K)$ is slice.

Q. If $BD(K)$ is slice, is K slice?

Theorem (H): If $\rho_0(K) \neq 0$ then $BD(K)$ is not slice.

- We can show that for certain Algebraically Slice K, $BD(K)$ is not Slice

Theorem (Cochran-H-Leidy): There is a constant C such that if $|\rho_0(K)| > C$ then $BD(J_1(K))$ is not slice.



$J_1(K)$ is Algebraically Slice $\Longrightarrow$ $\rho_0(J_1(K))=0$

One can define a homomorphism
$\rho_n : B(m) \longrightarrow \mathbb{R}$ where $B(m)$ = subgroup of $C(m)$ generated by boundary links.

This descends to

[Cochran-H] $\rho_n : \dfrac{\mathcal{F}B^m_{(n)} / \mathcal{F}B^m_{(n.5)}}{\text{local knotting}} \longrightarrow \mathbb{R} \quad (n \geq 1)$

where $\mathcal{F}B^m_{(n)}$ is (n)-solvable filtration of $B(m)$.

Thm(H): For $m \geq 2$, $n \geq 0$

$$\text{image}\left(\rho_n : \frac{\mathcal{FB}^m_{(n)} / \mathcal{FB}^m_{(n.5)}}{\text{local knotting}} \longrightarrow \mathbb{R}\right)$$

is infinitely generated.

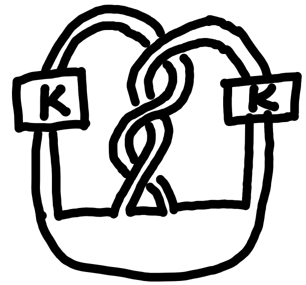
We show $BD(J,(K))$ is not (2.5)-solvable.

Since $BD(J,(K))$ is (2)-solvable and a boundary link we have

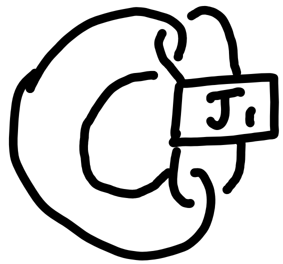
Thm(Cochran-H-Leidy): $\ker(\rho_n) \neq 0$ for $m = n = 2$.

as above

We will outline proof of $BD(J_1(K))$ is not slice when $\rho_0(K)$ is large. The proof that knots $J_n(K)$ are not slice when $\rho_0(K)$ is sufficiently large is similar (but technically more difficult).

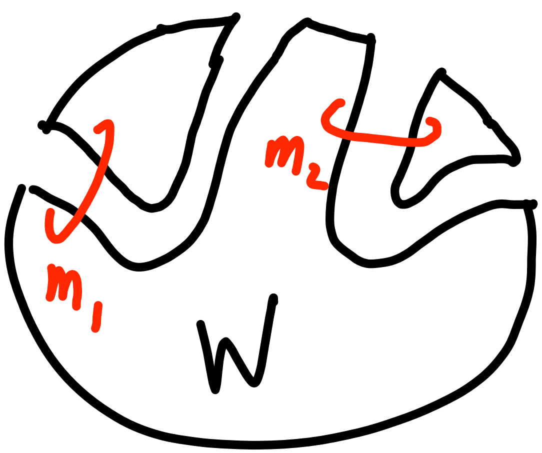
Proof: Let $J_1 =$ 

.

Suppose $BD(J_1) =$ 

is slice with slice discs $D_1 \cup D_2 \subset B^4$. Let

$W = B^4 - N(D_1 \cup D_2)$. Then $\partial W = M_{BD(J_1)}$



$= 0\text{-surgery on } BD(J_1)$,

$H_1(W) = \underset{m_1}{\mathbb{Z}} \times \underset{m_2}{\mathbb{Z}}$

and $H_2(W) = 0$

Consider $\phi : \pi_1 W \longrightarrow \Gamma$ a coefficient system with Γ = PTFA and $\pi = \pi_1(W)$.

Since $H_2(W) = 0$ by Cochran-Orr-Teichner,

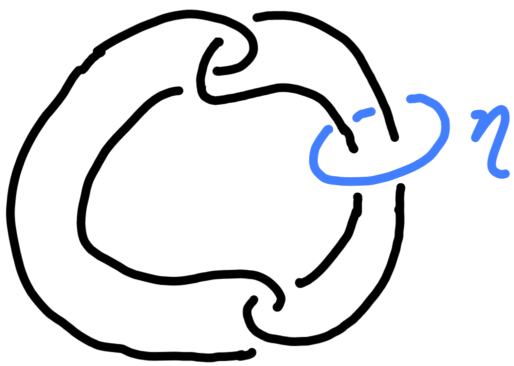
$$\boxed{\rho(M_{BD(J_1)}, \phi) \equiv \sigma^{(2)}(W, \phi) - \sigma(W) = 0.}$$

We will choose $\Gamma = \pi / \pi_r^{(3)}$ where $\pi_r^{(n)}$ = n^{th} term of rational derived series and show this cannot be the case.

Recall: $\pi_r^{(0)} = \pi$ $\quad \pi_r^{(n+1)} = \{ g \in \pi_r^{(n)} \mid g^k \in [\pi_r^{(n)}, \pi_r^{(n)}] \text{ for some } k \neq 0. \}$

• $M_{BD(J_1)} = (S^1 \times S^2) \# (S^1 \times S^2) - (\eta \times D^2) \cup_f (S^3 - J_1)$

where

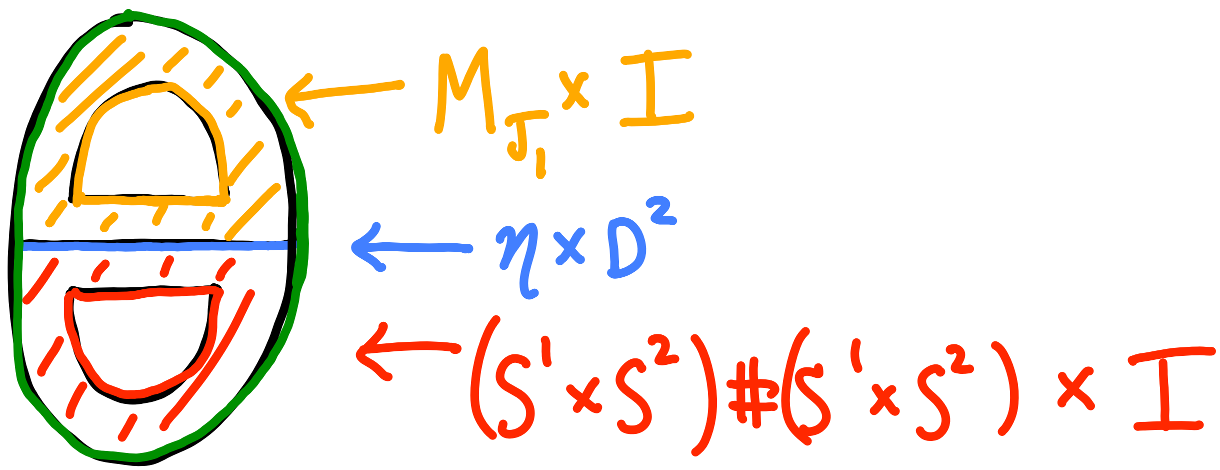


$+$ f: $\mathrm{long}_{J_1} \sim \mathrm{meridian}_\eta$

$\mathrm{long}_\eta \sim \mathrm{meridian}_{J_1}^{-1}$

Hence we have a 4 manifold V

$V =$



with $\partial V = \overline{(S^1 \times S^2) \# (S^1 \times S^2)} \cup \overline{M}_{J_1} \cup M_{BD(J_1)}$

- Let $\phi : \pi_1(M_{BD(J_1)}) \longrightarrow \pi/\pi_r^{(3)} =: \Gamma$ be defined by restricting $\pi_1(W) \xrightarrow{\phi} \pi/\pi_r^{(3)}$. Since $l_J \in \pi_1(M_{BD(J_1)})_r^{(3)}$, ϕ extends over V giving $\bar{\phi} : \pi_1(V) \longrightarrow \pi/\pi_r^{(3)}$.
- One can easily check that $H_2(V; \mathcal{K}(\pi/\pi_r^{(3)}))$ comes from the boundary

non-commutative skew field of $\mathbb{Z}\pi/\pi_r^{(3)}$

and $\sigma(V) = 0 \Rightarrow \boxed{\sigma^{(2)}(V, \bar{\phi}) - \sigma(V) = 0}$

Using $\rho(\partial V, \bar{\Phi}) = \sigma^{(2)}(V, \bar{\Phi}) - \sigma(V) = 0$

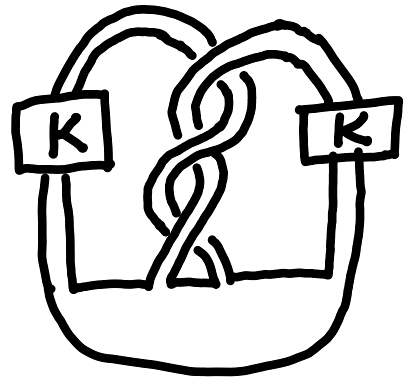
We see

$$\boxed{0 = \rho(M_{BD(J_1)}, \Phi) = \rho(S^1 \times S^1 \# S^1 \times S^1, \bar{\Phi}) + \rho(M_{J_1}, \bar{\Phi})}$$

- Since $S^1 \times S^2 \# S^1 \times S^2 = \partial(\underbrace{S^1 \times B^3 \#_b S^1 \times B^3}_{=\mathcal{U}})$ where inclusion induces $\cong$ on π_1, Φ extends over $\mathcal{U}$. Hence $\rho(S^1 \times S^2 \# S^1 \times S^2, \bar{\Phi}) = 0$.

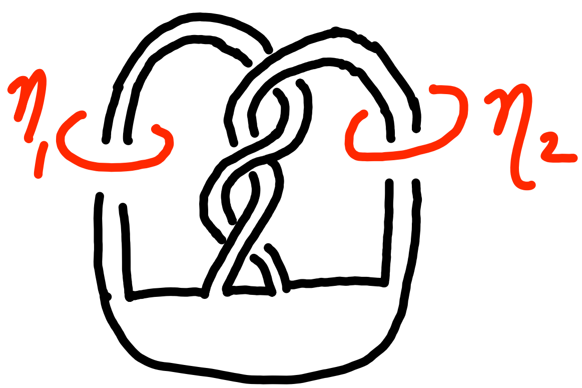
$$\therefore \quad \boxed{\rho(M_{J_1}, \bar{\Phi}) = 0 \qquad \bar{\Phi} : \pi_1(M_{J_1}) \longrightarrow \pi/\pi_r^{(3)}}$$

Analyzing $\rho(M_{J_1}, \bar{\Phi})$

Recall $J_1 =$  so as before

$$M_{J_1} = \left[M_{R_0} - (\eta_1 \times D^2) - (\eta_2 \times D^2) \right] \cup (S^3 - K_1) \cup (S^3 - K_2)$$

$(K_1 = K_2 = K)$

where  . We have a

cobordism X with $\partial X = M_{K_1} \cup M_{K_2} \cup M_{R_0} \cup M_{J_1}$

and can extend $\bar{\Phi} : \pi_1(M_{J_1}) \longrightarrow \pi/\pi_r^{(3)}$ over X

to $\bar{\bar{\phi}} : \pi_1(X) \longrightarrow \pi/\pi_r^{(3)}$. Moreover,

$$\sigma^{(2)}(X, \bar{\bar{\phi}}) - \sigma(X) = 0$$

hence

$$\underbrace{\rho(M_{J_1}, \bar{\phi})}_{\overset{\shortparallel}{0}} = \rho(M_{R_0}, \bar{\bar{\phi}}) + \rho(M_{K_1}, \bar{\bar{\phi}}) + \rho(M_{K_2}, \bar{\bar{\phi}})$$

By Cheeger-Gromov, $|\rho(M_{R_0}, \bar{\bar{\phi}})| \le C$

a constant that depends only on M_{R_0}

not on coefficient system.

<u>Goal</u>: Choose $K_1 = K_2 = K$ with $\rho_0(K) > C$.

Will Show that the image of

$$\bar{\bar{\phi}} : \pi_1(M_{K_i}) \longrightarrow \pi/\pi_r^{(3)}$$

is $\mathbb{Z}$ for $i = 1$ or 2 hence

$$|\rho(M_{K_i}, \bar{\bar{\phi}})| = |\rho_0(K)| > C \text{ for } i = 1 \text{ or } 2$$

which gives a contradiction and hence $BD(J_1)$ is not Slice.

Note:

$$\begin{tikzcd} \pi_1(S^3-K_i) \arrow[r, "{\text{since } \eta_i \in \pi_1(S^3-R_0)^{(1)}}"] \arrow[d] & \pi_1(S^3-J_1)^{(1)}_r \arrow[r] & \pi_1(BD(J_1))^{(2)}_r \arrow[d] \\ \pi_1(M_{K_i}) \arrow[rr, "\bar{\Phi}"] & & \pi/\pi_r^{(3)} \end{tikzcd}$$

hence $\quad \mathrm{image}(\bar{\Phi}) \subset \pi_r^{(2)}/\pi_r^{(3)}$ [torsion-free abelian]

so suffices to show $j_*(\eta_i) \notin \pi_r^{(3)}$ where

$j: S^3-K_i \hookrightarrow W$ is inclusion.

- We will translate this into a question about higher order Alexander modules

Let $\Lambda = \pi/\pi_r^{(2)}$ and

$$i_* : TH_1(M_{BD(J_1)}; \mathbb{Q}\Lambda) \longrightarrow TH_1(W; \mathbb{Q}\Lambda) \subset \frac{\pi_r^{(2)}}{\pi_r^{(3)}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

$\shortparallel$

$[\mathbb{Q}\Lambda\text{-torsion sub-module of } H_1(M_{BD(J_1)}; \mathbb{Q}\Lambda)]$

$\shortparallel$

$$\mathcal{A}_0^{\mathbb{Q}}(J_1) \otimes_{\mathbb{Q}[t^{\pm 1}]} \mathbb{Q}\Lambda \quad \text{since } i_*(\eta) \in \pi_r^{(1)} - \pi_r^{(2)}$$

$\shortparallel$

Classical Alex. module of J_1

- Consider $\eta_i \in \mathcal{A}_0(J_1)$, we wish to show $i_*(\eta_i \otimes 1) \neq 0$ in $TH_1(W; \mathbb{Q}\Lambda)$ for $i = 1$ or 2

Using work of Constance Leidy on higher-order (non-localized) Blanchfield forms, there are commuting maps (with $\mathbb{Q}\Lambda$-coeffs)

$$\begin{tikzcd} TH_2(W,M) \arrow[r, "\pi"] \arrow[d, "\psi"] & TH_1(M_{BD(J_1)}) \arrow[r, "i_*"] \arrow[d, "B\ell"] & TH_1(W) \\ TH_1(W)^{\#} \arrow[r, "\tilde{i}"] & TH_1(M_{BD(J_1)})^{\#} & \end{tikzcd}$$

(top exact)

Let $P = \ker_*$ and $P^\perp$ be defined w.r.t. $B\ell$.

<u>Lemma</u>: $P \subset P^\perp$ <u>Proof</u>: $x \in P \Rightarrow \exists\, y\ \ \pi(y) = x$

if $z \in P \Rightarrow B\ell(x)(z) = \tilde{i}(\psi(y))(z) = \psi(y)(i_*(z)) = \psi(y)(0) = 0$

$\Rightarrow$ If $P = TH_1(M_{BD(J_1)}; \mathbb{Q}\Lambda)$ then $P^{\perp} = TH_1(M; \mathbb{Q}\Lambda)$

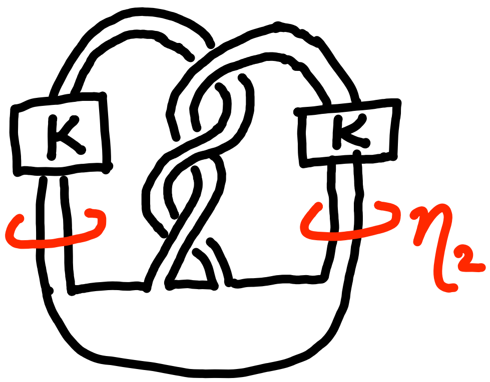
$$\Rightarrow \quad \beta\ell = 0$$

We will show that $\beta\ell \neq 0$ which will imply that $P = \ker i_* \neq TH_1(M; \mathbb{Q}\Lambda)$.

We will see that $\eta_1 \otimes 1$ and $\eta_2 \otimes 1$ generate $TH_1(M_{BD(J_1)}; \mathbb{Q}\Lambda)$ hence either $i_*(\eta_i \otimes 1) \neq 0$ or $i_*(\eta_2 \otimes 1) \neq 0$.

This will complete the proof.

Recall, $\mathcal{A}_0^{\mathbb{Q}}(J_1) \cong H_1(M_{R_0}; \mathbb{Q}[t^{\pm 1}])$

$J_1 =$ 

and η_1 and η_2 generate $\mathcal{A}_0^{\mathbb{Q}}(J_1)$.

[Hence $\eta_1 \otimes 1$ and $\eta_2 \otimes 1$ generate
$H_1(M_{\partial D(J_1)}; \mathbb{Q}\Lambda) = \mathcal{A}_0^{\mathbb{Q}}(J_1) \otimes \mathbb{Q}\Lambda$]

C. Leidy shows that

$$Bl_{\mathcal{A}_0^{\mathbb{Q}}(J_1) \otimes \mathbb{Q}\Lambda} = Bl_{\mathcal{A}_0^{\mathbb{Q}}(J_1)} \text{ "tensored up"}$$

hence $$Bl_{\mathcal{A}_0^{\mathbb{Q}}(J_1) \otimes \mathbb{Q}\Lambda}(x \otimes 1, y \otimes 1) = Bl_{\mathcal{A}_0(J_1)}(x, y)$$

In particular, since $\beta\ell_{A_0^{\mathbb{Q}}(J_1)}$ is nonsingular,

$$\beta\ell_{A_0^{\mathbb{Q}}(J_1)\otimes\mathbb{Q}\Lambda}(x\otimes 1, y\otimes 1) \neq 0 \quad \forall\, x, y \in A_0^{\mathbb{Q}}(J_1)$$

$$\Rightarrow \beta\ell_{A_0^{\mathbb{Q}}(J_1)\otimes\mathbb{Q}\Lambda} \neq 0$$

$\blacksquare$