# The Geometry of Knot Concordance Spaces

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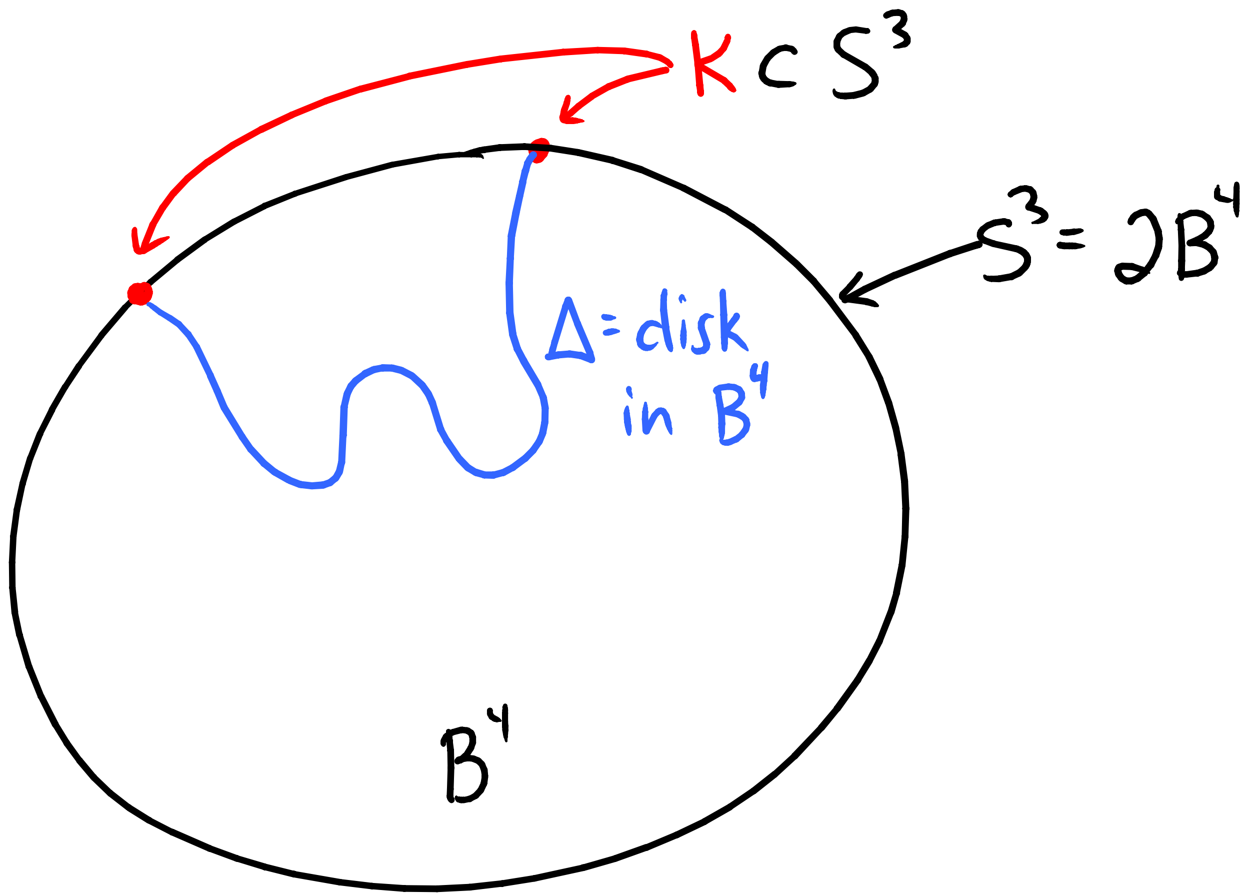
**Def**: A knot $K$ is *slice* if $K$ is the boundary of a smoothly embedded disk in $B^4$.



# (Smooth) Knot Concordance Group

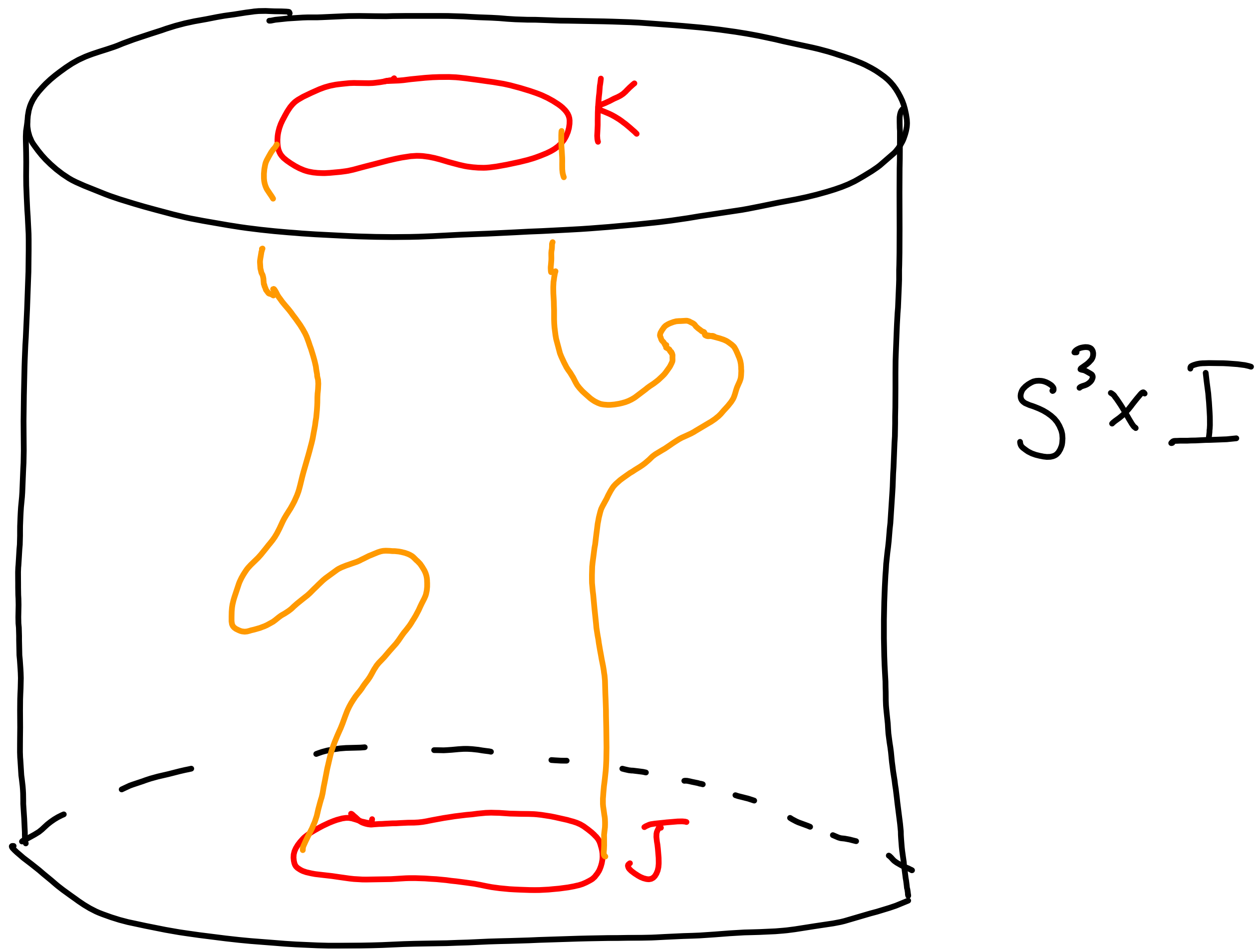
$\mathcal{C} := \{\text{oriented knots in } S^3\}/\sim$

$K \sim J$ if $K \# r\overline{J}$ is slice

reverse of mirror image.

- If $K \sim J$, we say that $K$ is **concordant** to $J$.

- $K \sim J \iff K \times \{0\}$ and $J \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times I$.



- $\mathcal{C}$ is a group under connected sum
- $[K] = 0 \iff K$ is slice
- $-[K] = [r\bar{K}]$ reverse of mirror image
- $\mathcal{C}$ is not finitely generated
- $\mathcal{C}$ has elements of $\infty$ order and order 2.

$\vdots$

We (with Connie Leidy) conjectured that $\mathcal{C}$ has the structure of a fractal set.

## Evidence

- $\exists$ filtration of $\mathcal{C}$ called the (n)-solvable filtration (defined by Cochran-Orr-Teichner).

$$\cdots \subset \mathcal{F}_n \subseteq \cdots \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_{0.5} \subsetneq \mathcal{F}_0 \subset \mathcal{C}$$

**Thm** [Cochran-H-Leidy] $\exists$ a large subgroup $S \subset \mathcal{C}$ with $S \cong \mathbb{Z}^\infty$ and an infinite family of operators $\{P_i\}_{i=1}^\infty$ s.t.

(1) $P_{i_1} \circ \cdots \circ P_{i_n} : \mathcal{C} \longrightarrow \mathcal{F}_n$

$$S \xhookrightarrow[\text{injective}]{} P_{i_1} \circ \cdots \circ P_{i_n}(S)$$

(2) $P_{i_1} \circ \cdots \circ P_{i_n}(S) \cap P_{j_1} \circ \cdots \circ P_{j_n}(S) = 0$

if $(i_1, \ldots, i_n) \neq (j_1, \ldots, j_n)$

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "P_1"] & & & \\
& \mathcal{J}_1 \arrow[r, "P_2"] & \cdots \arrow[dr, "P_n"] & \\
\mathcal{C} \arrow[ur, "P_2"'] & & & \mathcal{J}_n \\
\vdots & \vdots & & \\
\mathcal{C} \arrow[dr, "P_k"] & & & \\
& \mathcal{J}_1 \arrow[r, "P_1"] & \cdots \arrow[uuur, "P_n"'] & \\
\mathcal{C} \arrow[ur, "P_{k+1}"'] & & &
\end{tikzcd}$$

images of $S$
are getting smaller
+ all intersect
trivially

- $\exists$ a "strong winding #1" satellite operator $P: \mathcal{C} \longrightarrow \mathcal{C}$ s.t.

Thm [Cochran-Davis-Ray, 2012]

$P: \mathcal{C} \hookrightarrow \mathcal{C}$ is injective [modulo smooth 4D Poincaré conjecture]

Thm [A. Levine, 2014]

$$\cdots \subsetneq P^n(\mathcal{C}) \subsetneq \cdots\cdots \subsetneq P^2(\mathcal{C}) \subsetneq P(\mathcal{C}) \subsetneq \mathcal{C}$$

(uses $\tau$ and $\varepsilon$ from Heegaard Floer)

In order to study the fractal nature of $\mathcal{C}$, we view $\mathcal{C}$ as a metric space and study its (coarse) geometry as well as the natural operators on $\mathcal{C}$ (satellite operators)

**Def:** A **norm** on a group $G$ is a function

$$\|\cdot\| : G \to \mathbb{R}$$

s.t. $\forall\ x, y \in G$,

(1) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = e$

(2) $\|xy\| \leq \|x\| + \|y\|$

(3) $\|x^{-1}\| = \|x\|$

**Note:** If $\|x\| = 0$ for $x \neq e$, then $\|\cdot\| = 0$ is a pseudo-norm.

A group norm on $G \rightsquigarrow$ Metric on $G$. by

$$d(x,y) := \|xy^{-1}\|.$$

If $\|-\|$ is a pseudo-norm, then $d$ is a pseudo-metric.

i.e. $\exists\ x \neq y$ s.t. $d(x,y) = 0$.

There are two important metrics on $\mathcal{C}$.

(1) $\|K\|_s = g_s(K)$, slice genus.

$= \min\{g(S) \mid S$ is a smoothly embedded sfce in $B^4$ with $\partial S = K\}$

$\rightsquigarrow d_s(K,J) = \|K-J\|_s$ is the slice metric on $\mathcal{C}$.

$= \min\{g(S) \mid S$ is a smoothly embedded sfce in $S^3 \times I$ with $\partial S = K \amalg -J\}$

**Def**: $K$ is *slice in $V$* if $K$ bounds a smoothly embedded disk $\Delta$ in $V$, a smooth, oriented 4-manifold with $\partial V = S^3$ & $\pi_1(V) = \{1\}$ and s.t. $[\Delta] = 0$ in $H_2(V, \partial V)$.

(2) $\|K\|_H = \min\left\{ \frac{1}{2}\left(\beta_2(V) + |\sigma(V)|\right) \;\middle|\; K \text{ is slice in } V \right\}$

is a pseudo-norm on $\mathcal{C}$, the homology norm

$d_H(K, J) := \|K - J\|_H$, the homology metric

is a pseudo metric.

Rmk: If the 4-dim P.C. is true, then $\|\cdot\|_H$ is a norm, not just a pseudo-norm.

# Coarse geometry of $(\mathcal{C}, d_S) + (\mathcal{C}, d_H)$.

**Def:** If $(X, d_X)$ and $(Y, d_Y)$ are metric spaces, then $f : X \longrightarrow Y$ is a *quasi-isometry* if $\exists$ constants $A \geq 1$, $B \geq 0$, $C \geq 0$ s.t.

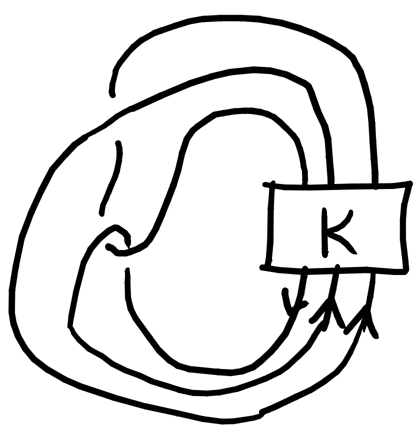
$$\boxed{\frac{1}{A} d_X(x,y) - B \leq d_Y(f(x), f(y)) \leq A\, d_X(x,y) + B}$$

$\forall\, x, y \in X$.

and $\forall\, z \in Y$, $\exists\, x \in X$ s.t. $\underbrace{d_Y(z, f(x)) \leq C.}_{\text{quasi-surjective}}$

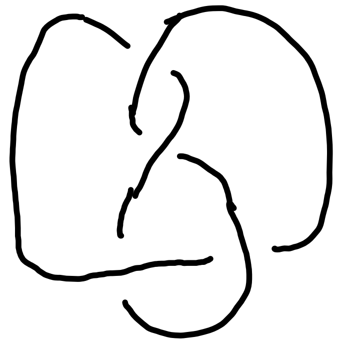
**Prop (Cochran-H):** The identity map $i : (\mathcal{C}, d_s) \longrightarrow (\mathcal{C}, d_H)$ is not a quasi-isometry.

**Pf:** Consider the operator $P : \mathcal{C} \longrightarrow \mathcal{C}$ defined by

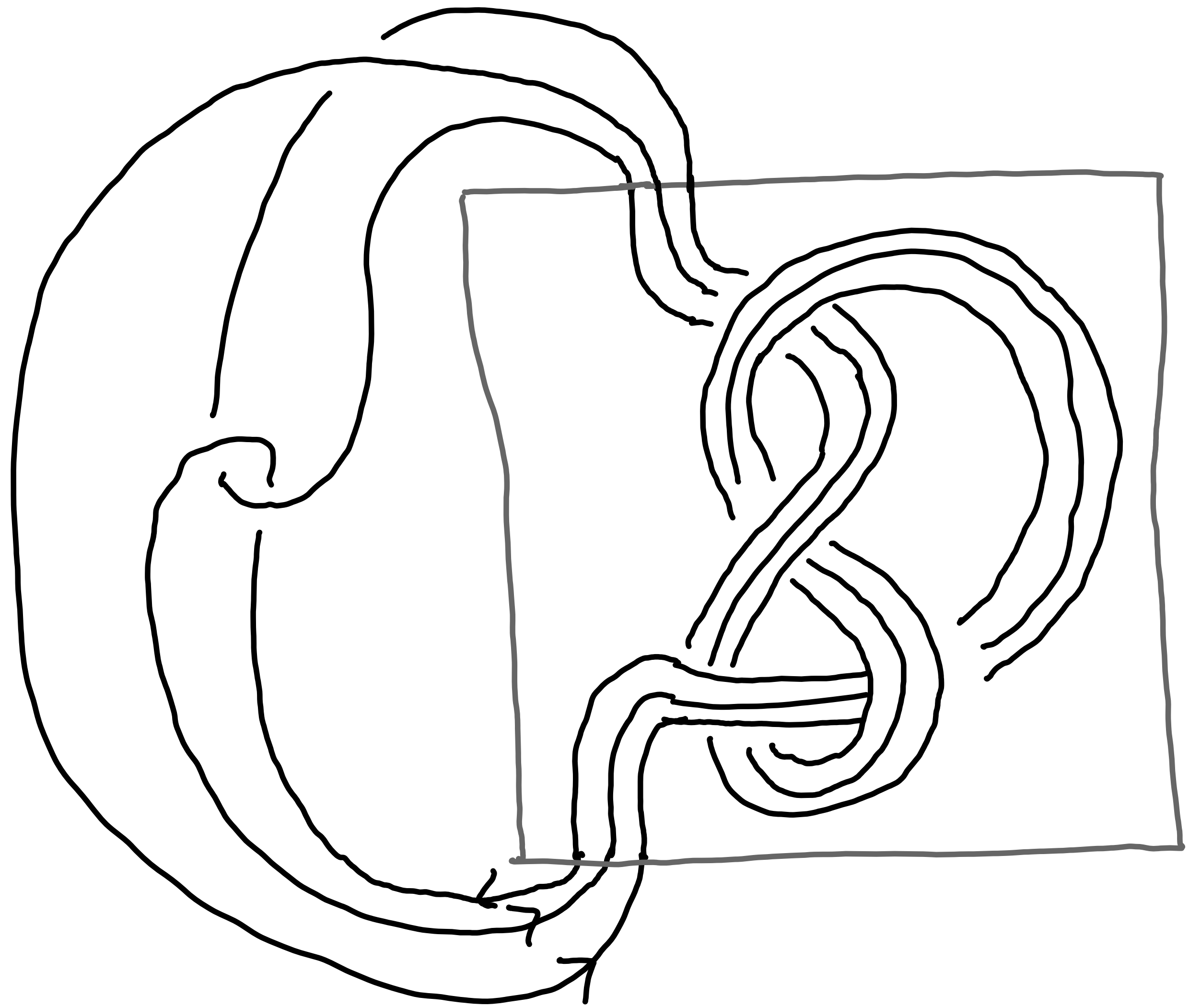
$P(K) =$ 



$\leftarrow$ tie $\downarrow\uparrow\uparrow$ into knot $K$

Ex:  $= K$ figure-8

$P(K)$



Let $M_K$ = 0-surgery on a knot $K \subset S^3$.

Thm [Cochran-Franklin-Hedden-Horn, 2011]:

For any $K$, $M_K$ is homology cobordant to $M_{P(K)}$ (+ $P(K)$ is not always concordant to $K$).

$$\overset{*}{\Longrightarrow} \quad \| \overset{n}{P}(K) \|_H = \|K\| \quad \forall\, n.$$

$\overset{n}{P}(K) = (P \circ \cdots \circ P)(K)$

* plus $\pi_1$ of the homology cobordism is normally gen by the meridian.

Thm [Ray]: $\exists$ knots $K$ (e.g. $K=$ trefoil) s.t.

$$\|P^n(K)\|_S = n+1$$

Therefore we cannot have

$$\begin{array}{ccc} \frac{1}{A} d_S(P^n(K), 0) - B & \le & d_H(P^n(K), 0) \\ \| & & \| \\ \frac{1}{A} \|P^n(K)\|_S - B & & \|P^n(K)\|_H \\ \text{unbounded} & & \text{bounded} \end{array}$$

$\blacksquare$

**Thm (Cochran-H):** $\exists$ arbitrarily large quasi-n-flats in $(\mathcal{C}, d_*)$, $* = s$ or $H$.

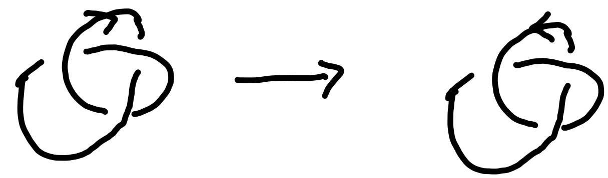
i.e. $\exists$ subspaces of $(\mathcal{C}, d_*)$ that are quasi-isometric to $(\mathbb{R}^n$, Euclidean metric$)$.

**Cor:** There is no isometric embedding of $(\mathcal{C}, d_*)$ into a finite product of (Gromov) hyperbolic spaces.

There are many natural operators on $\mathcal{C}$

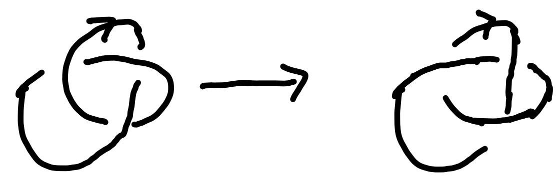
- reverse : $\mathcal{C} \longrightarrow \mathcal{C}$

  $K \longmapsto r\mathcal{C}$



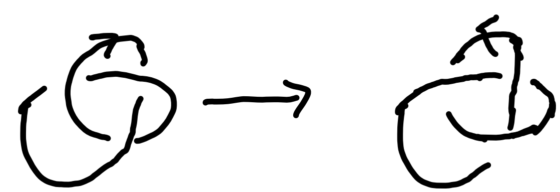
- mirror image : $\mathcal{C} \longrightarrow \mathcal{C}$

  $K \longmapsto \overline{K}$



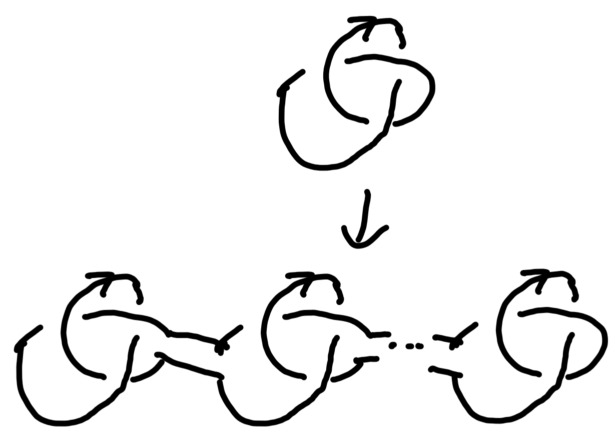
- inverse : $\mathcal{C} \longrightarrow \mathcal{C}$

  $K \longmapsto -K = r\overline{K}$



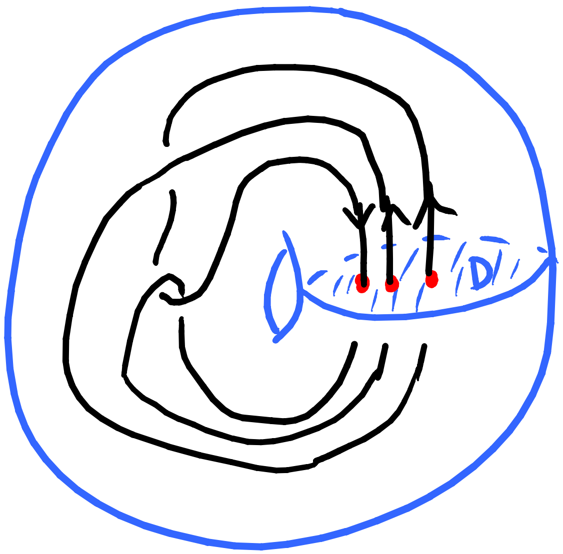
- times m : $\mathcal{C} \longrightarrow \mathcal{C}$

  $K \longmapsto mK$



★ satellite operators

# Satellite operators

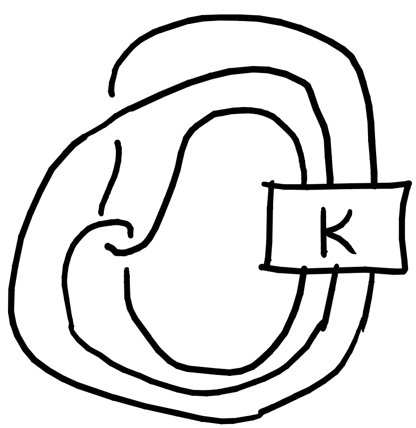
$P =$ 


pattern

$P : \mathcal{C} \to \mathcal{C}$

$K \longmapsto P(K)$

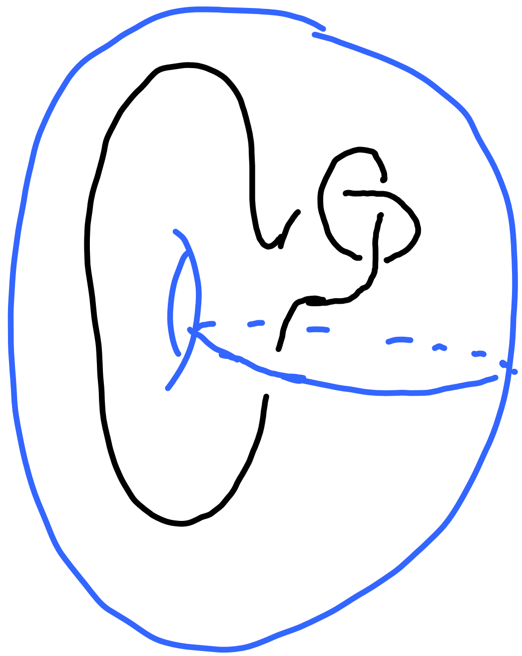
winding # of $P$
= alg intersection of knot and $D$

$P(K) =$ 

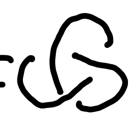

← tie bands into $K$

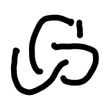
**Remark**: A satellite operator is often **not** a homomorphism.

**Ex**:

$P =$ 

$P(K) = K \#$ 

$P(K \# K) = K \# K \#$ 

$P(K) \# P(K) = K \#$  $\# \; K \#$ 

never concordant

However, it is a homomorphism in the coarse geometric sense.

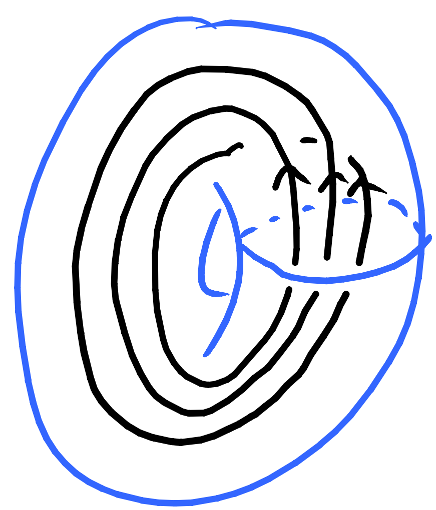
**Def:** A **quasi-homomorphism** on $(\mathcal{C}, \|\cdot\|_*)$ is a function $f: \mathcal{C} \to \mathcal{C}$ s.t. $\exists$ a constant s.t. $\forall$ $K, J \in \mathcal{C}$

$$\| f(K+J) - f(K) - f(J) \|_* \leq A_f.$$

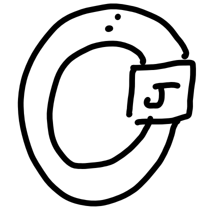
**Thm (Cochran-H):** Any satellite operator $P: \mathcal{C} \to \mathcal{C}$ is a quasi-morphism on $(\mathcal{C}, \|\cdot\|_*)$ for both $* = S$ and $H$.

The proofs has 2 steps.

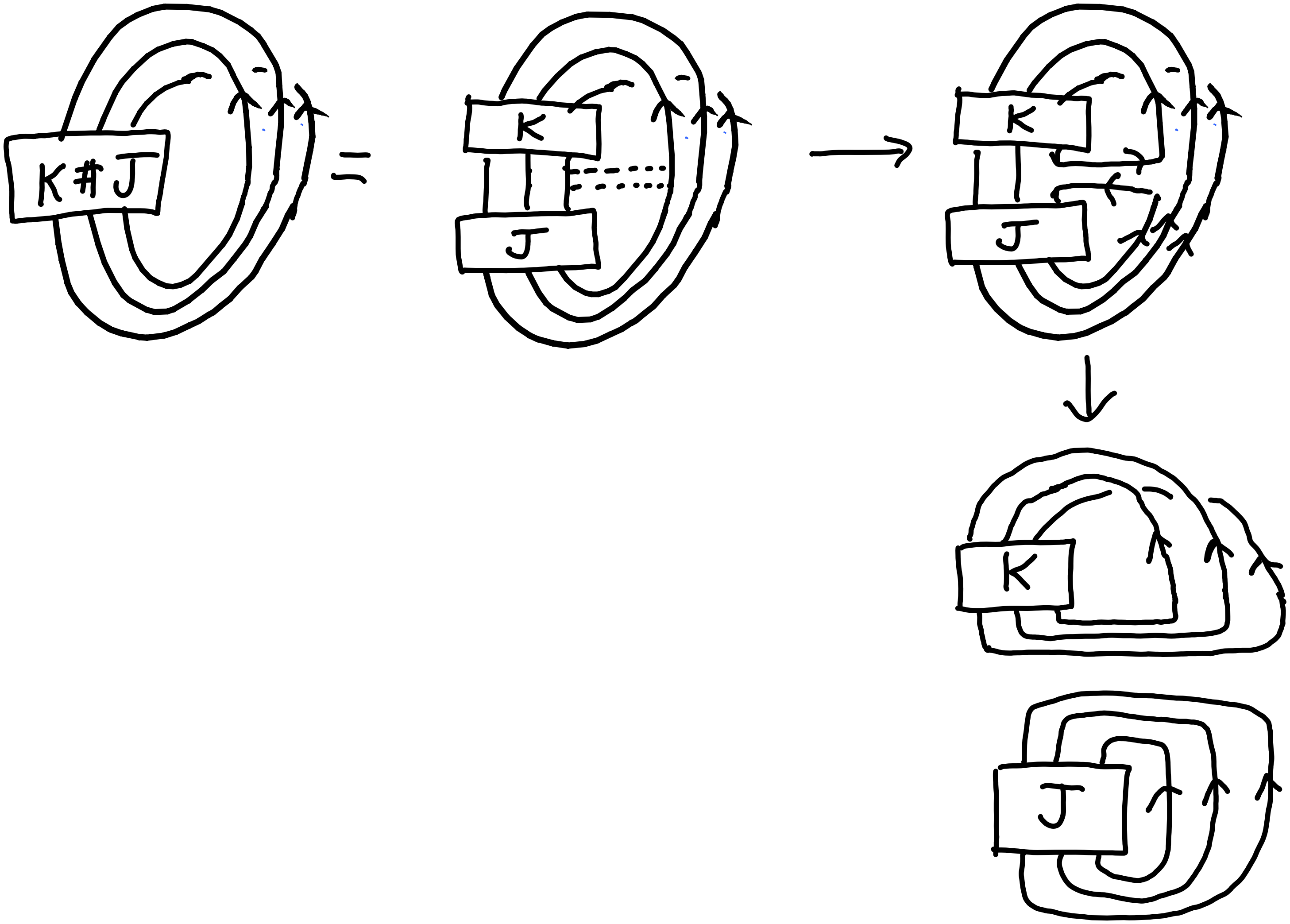
**Step 1:** $C_{n,1} = (n,1)$-cable operator

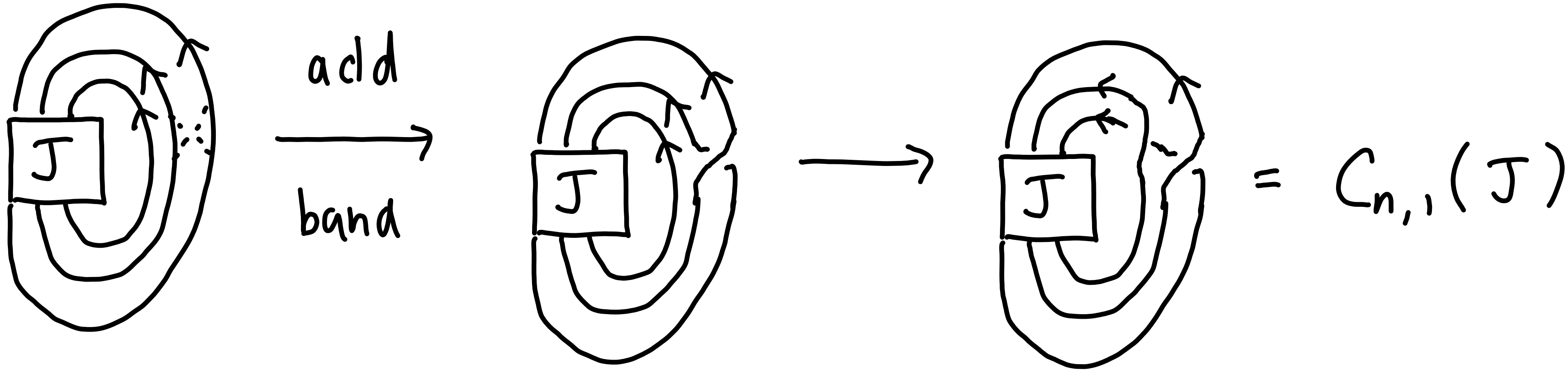
$C_{n,1} =$ 

# Claim: $C_{n,1}$ is a quasi-homomorphism

Add bands to $C_{n,1}(K\#J) \rightsquigarrow C_{n,1}(K) \sqcup$ 








$$\Rightarrow \quad C_{n,1}(K\#J) \xrightarrow[\text{2n-1 bands}]{\text{add}} C_{n,1}(K) \amalg C_{n,1}(J)$$

$$\Big\downarrow \text{add 1 bands}$$

$$C_{n,1}(K) \# C_{n,1}(J)$$

$\Rightarrow$ There is a cobordism $F$ in $S^3 \times I$ from $C_{n,1}(K \# J)$ to $C_{n,1}(K) \# C_{n,1}(J)$ + $g(F)$ only depends on $n$ (not $K$ or $J$).

$$\Rightarrow \| C_{n,1}(K\#J) - C_{n,1}(K) - C_{n,1}(J) \|_s$$
$$= d_s\big(C_{n,1}(K\#J),\, C_{n,1}(K) \# C_{n,1}(J)\big)$$
$$\leq g(F).$$

$\square$ of Claim.

## Step 2:

**Def:** Two functions $f, g : (\mathcal{C}, d) \circlearrowleft$ are within a <u>bounded distance</u> if $\forall\, x \in \mathcal{C}$, $\exists$ constant $M$ s.t.

$$d(f(x), g(x)) \leq M.$$

In this case we think of $f$ and $g$ as being equivalent.

Easy Fact: If $f$ is within a bounded distance of $g$ and $g$ is a quasi-homomorphism $\Rightarrow$ $f$ is a quasi-homomorphism.

Prop (Cochran-H): Let $P:(\mathcal{C}, d_*) \to (\mathcal{C}, d_*)$ be a satellite operator with winding # $n$ where $* = S$ or $H$. Then $P$ is within a bounded distance of $C_{n,1}$.

$\Rightarrow$ $P$ is a quasi-homomorphism

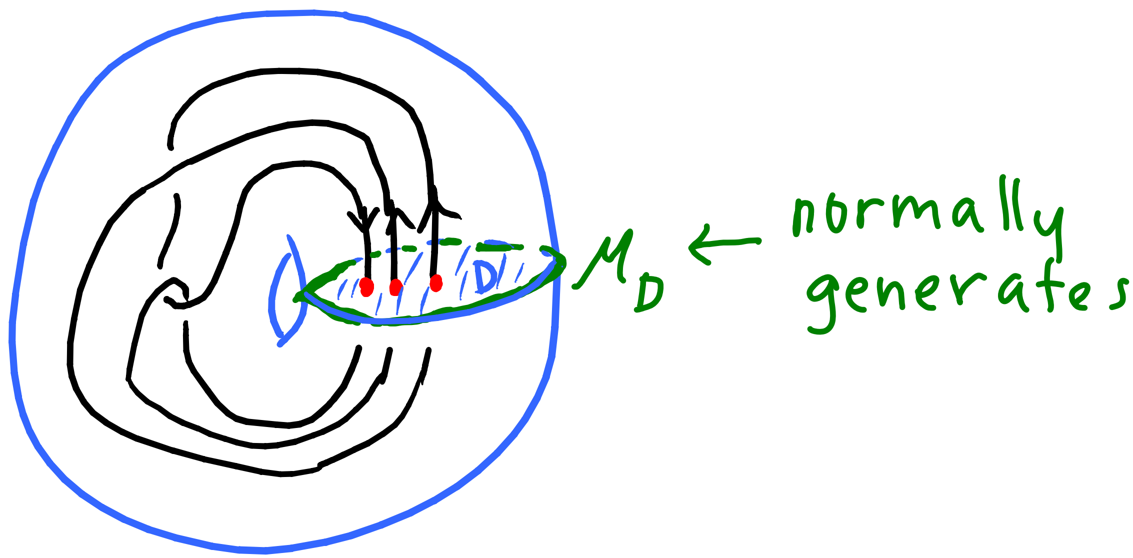
**Thm (Cochran-H):** If $P$ is a winding # $\pm 1$ operator then $P:(\mathcal{C}, d_*) \longrightarrow (\mathcal{C}, d_*)$ is a quasi-isometry for $* = S$ or $H$.

**Pf:** (1) $P$ is within a bounded distance of $C_{1,1} = id$.

(2) $id$ is an isometry.

**Cor (Cochran-H):** If $P$ is a winding # 0 operator $\Rightarrow$ $P$ is a quasi-contraction.

**Def:** $P$ is a strong winding # $\pm 1$ satellite operator if $\mu_D$ normally generates $\pi_1(S^3 \setminus P(u))$.

$P =$ 


**Note:** $P(u)$ = view knot in $P$ as in $S^3$

**Thm (Cochran-H):** Let $P$ be a strong winding # $\pm 1$ operator $P:(\mathcal{C}, d_H) \circlearrowleft$.
If the smooth 4-dimensional Poincaré conjecture is true then $P$ is an isometric embedding.

$P$ preserves the homology norm (even if 4D P.C. is not true).

**Rmk:** The proof of injectivity uses work of Cochran-Davis-Ray.