Knot Theory and its Combinatorial Invariants

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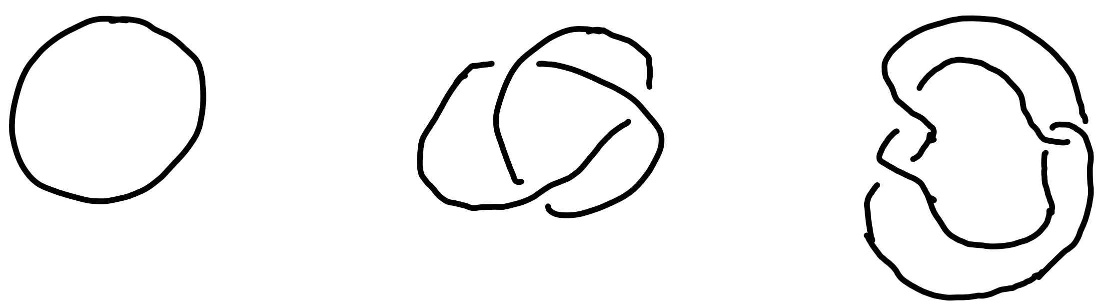
Def: A knot is a smooth embedding

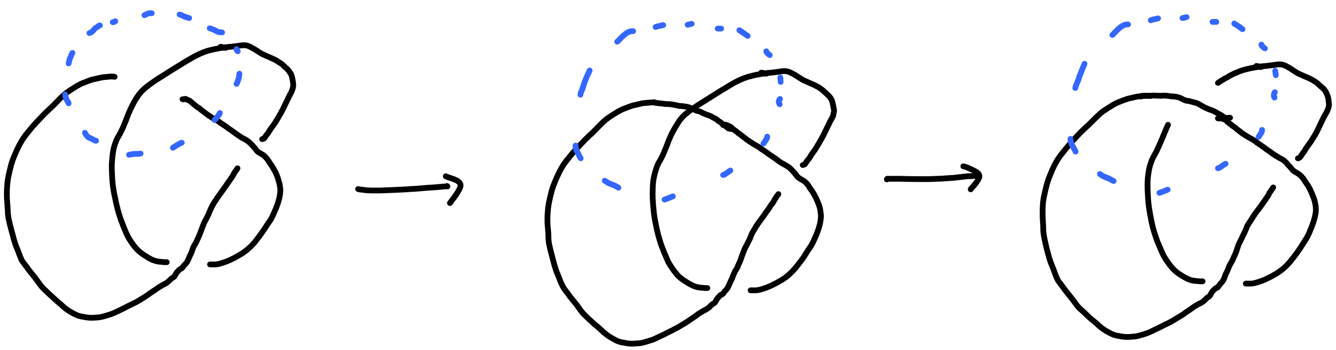
$$f: S^1 \longrightarrow \mathbb{R}^3.$$

i.e. take a rope, tie it up and attach the ends.



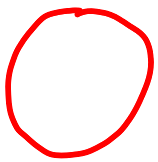
We say that two knots are <u>equivalent</u> if you can deform one into the other without allowing the curve to pass through itself.

all equivalent knots

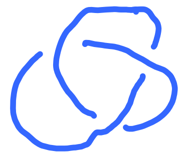


NOT ALLOWED

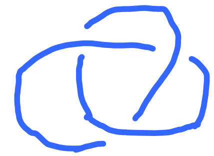
Examples of Distinct Knots:



unknot



left-handed trefoil



right-handed trefoil

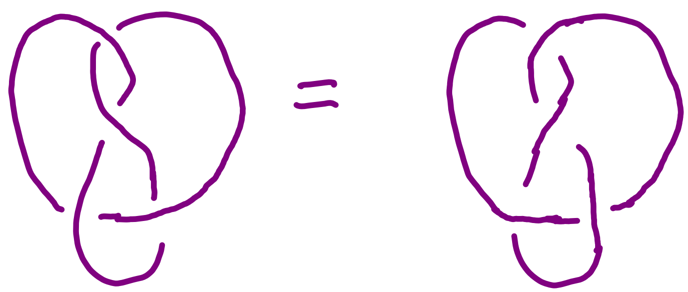
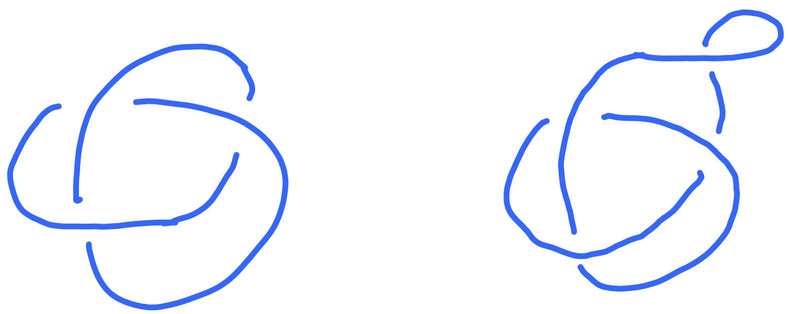


figure-eight

Instead of thinkings of knots as subsets of $\mathbb{R}^3$, we project them to plane and remember the crossing information. This is called a diagram of the knot.



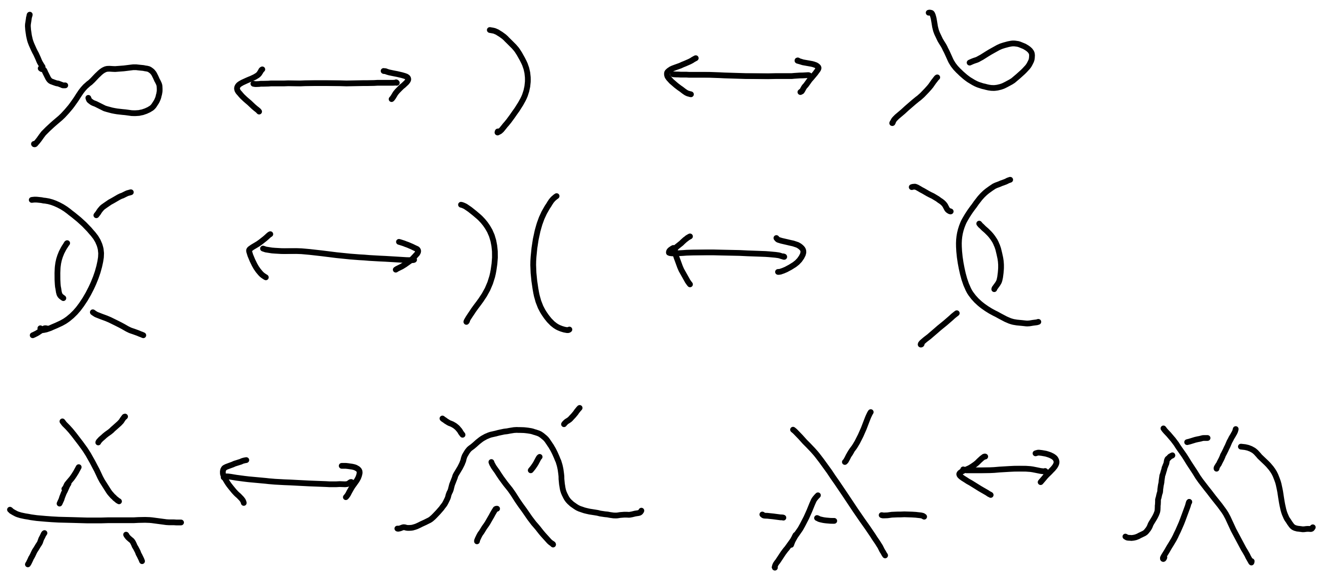
Different diagram for the same knot

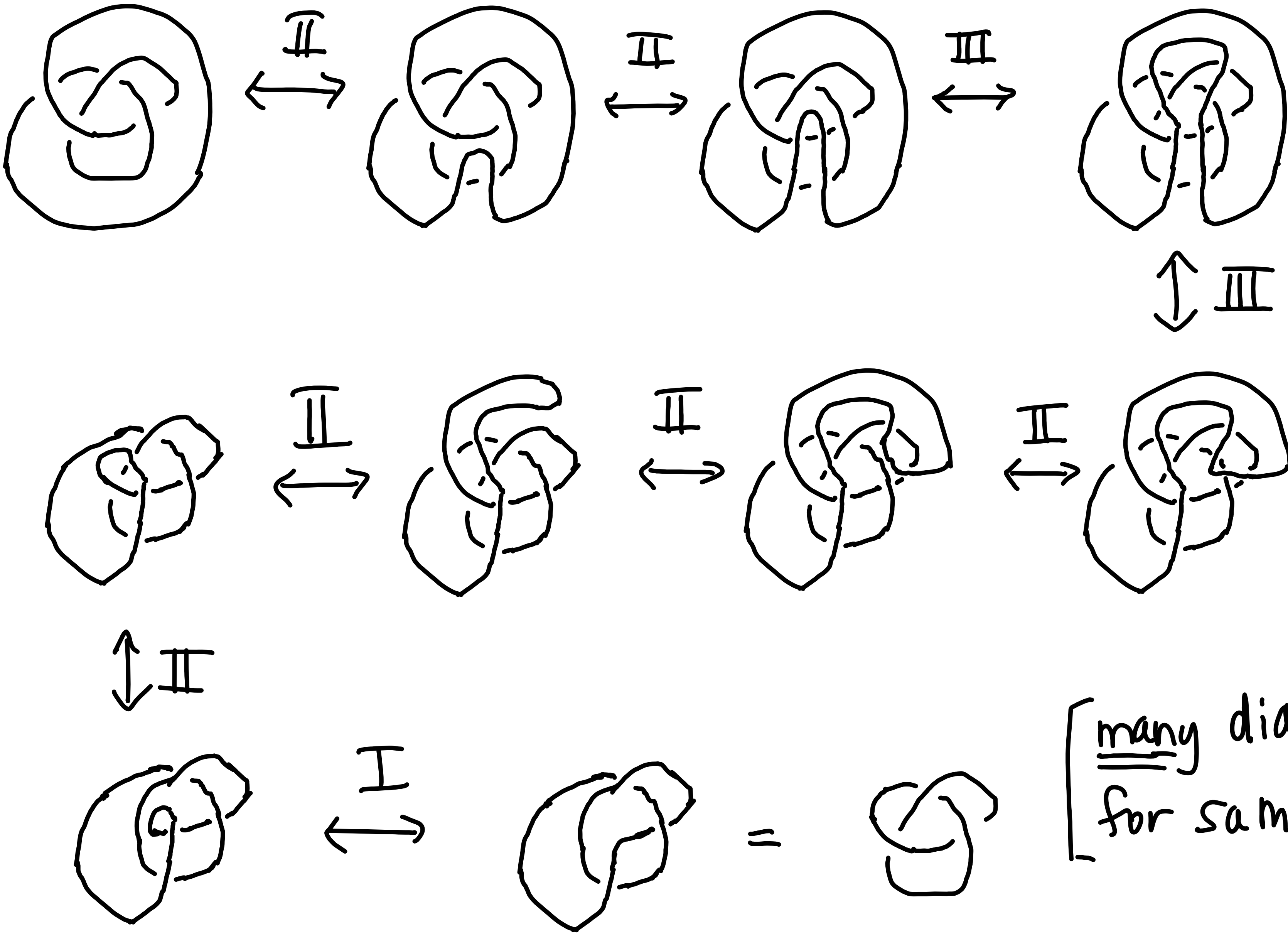
Theorem (Alexander, Briggs): Two diagrams are associated to the same knot $\Longleftrightarrow$ they are related by a sequence of the Reidemeister moves (plus planar isotopies).

I.

II.

III.





[many diagrams for same knot!]

Thus to define a knot invariant, it suffices to consider invariants of diagrams that are preserved under Reidemeister moves:

$$I : \{\text{diagrams}\} \longrightarrow A = \left\{ \begin{array}{c} \text{rings,} \\ \text{algebra, etc.} \end{array} \right\}$$

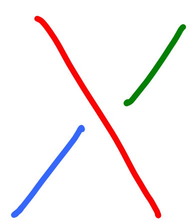
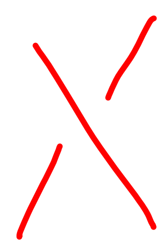
$I(D_1) = I(D_2)$ if D_1 is related to D_2 by Reidemeister moves I, II, III.

The simplest invariant is 3-colorability:

$$C : \{\text{Diagrams}\} \longrightarrow \{0,1\}.$$

$C(D) = 1$ if D is <u>3-colorable</u>, i.e.

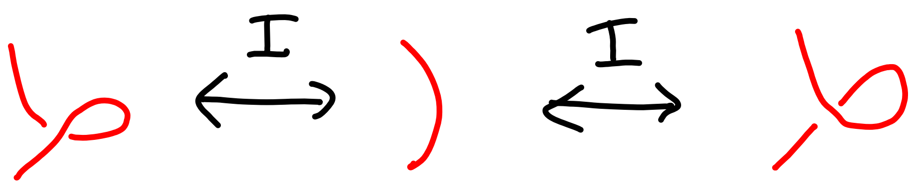
the arcs in the projection can be colored with $\{R, G, B\}$ s.t. every colored is used and the crossings look like:

(all 3 colors appear)  or  (all one color)

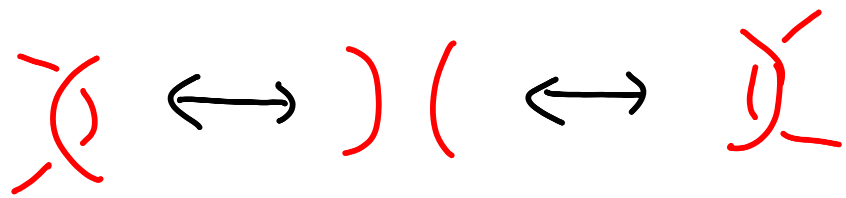
Otherwise $c(D) = 0$.

Can check that if D_1 and D_2 differ by Reidemeister moves, then $c(D_1) = c(D_2)$.

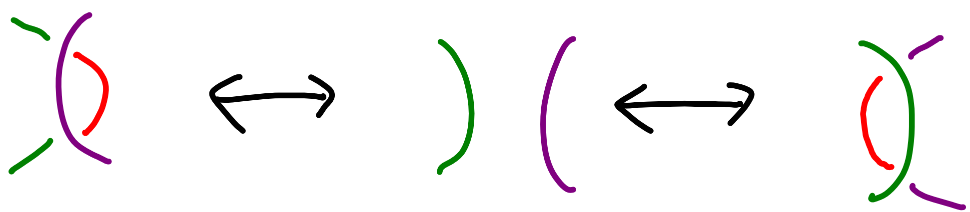
RI:



RII: (Case 1)



(Case 2)



RIII is similar

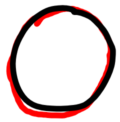
Therefore 3-colorability is a property of the knot, not just its diagram.

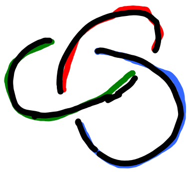
Def $\quad C(K) := c(D)$ for some diagram D of K.

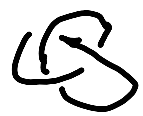
knot

$K: S^1 \longrightarrow \mathbb{R}^3$

D = a diagram or "picture" of K

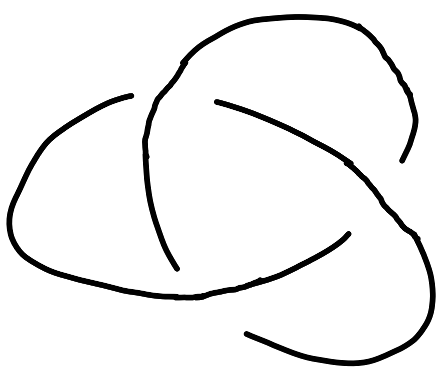
Ex:  is not 3-colorable $\Rightarrow c(O) = \emptyset$

 is 3-colorable $\Rightarrow$ c() = 1

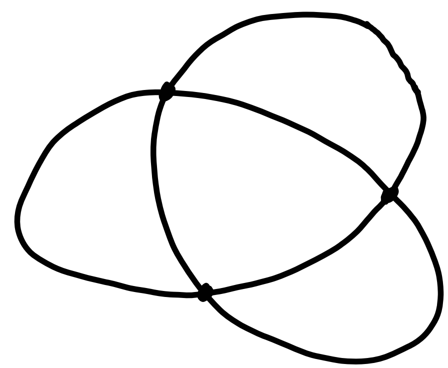
$\therefore$ O $\neq$  (as knots)

Let K be a knot and D be a diagram for K. If we forget about whether crossings are over/under, we get a planar graph:

crossings $\longleftrightarrow$ vertices

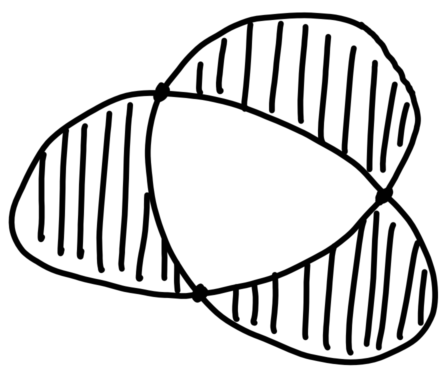


D = diagram

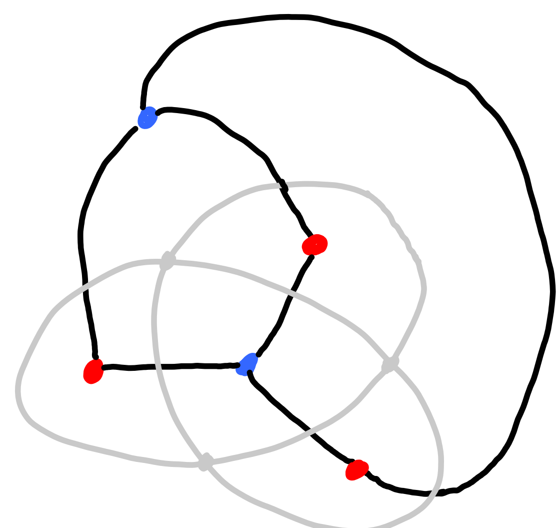


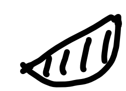
P_D = projection, 4-regular graph

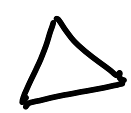
The dual of P_D is a planar bipartite graph, each of whose regions has 4 edges.



Checkerboard coloring of P_D

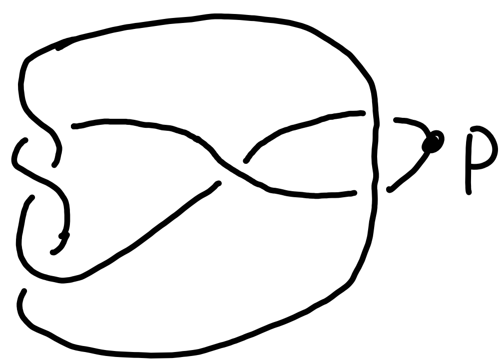


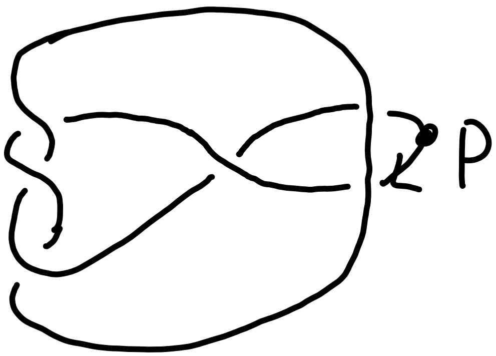
Red vertices $\leftrightarrow$ 

Blue vertices $\leftrightarrow$ 

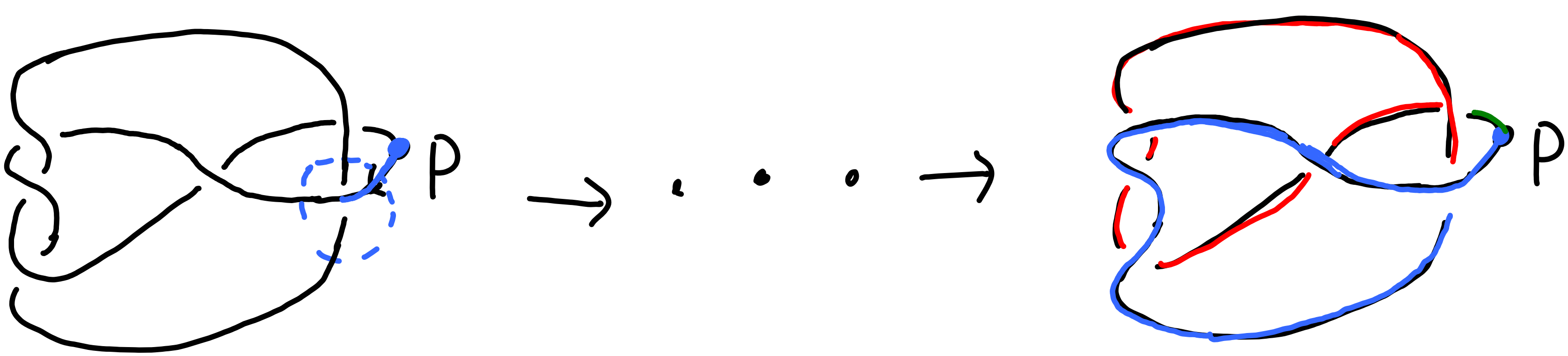
Fact: every knot can be made into the unknot by changing a subset of crossings in a diagram

Proof: Let D be a diagram for K.



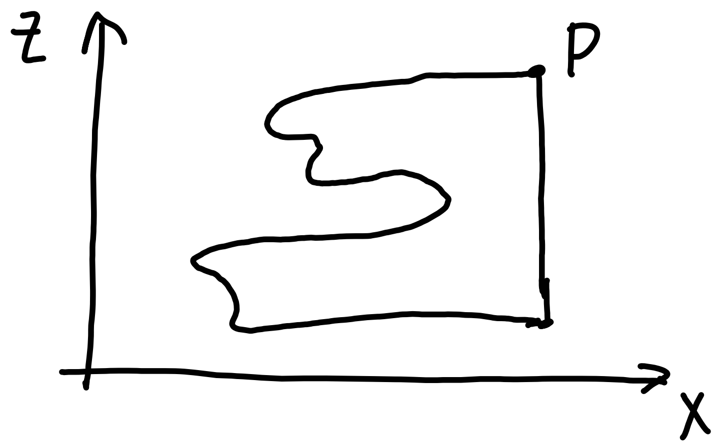


Start at P, traverse the knot. Each time you hit a crossing, if it is the first time you hit a crossing, make sure you are crossing over the other arc (if not change crossing).



Then the z-value of the points on knot decreases as you traverse the knot except near p where there is a nearly vertical arc.

From side:



This is always a diagram for the unknot (O).

∎

Unknotting Number

Def: $u(D) =$ min # of crossing of D needed to change D to a diagram of unknot.

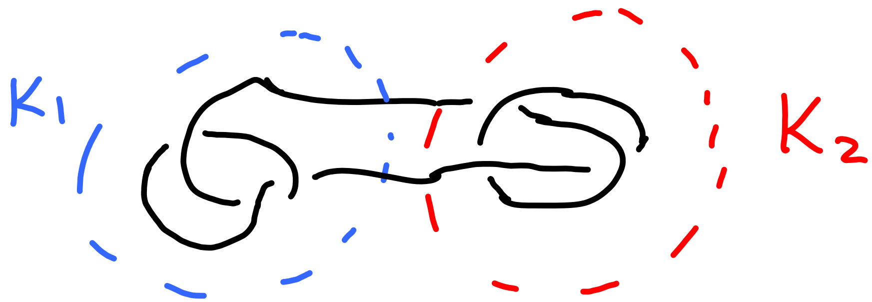
$$u(K) = \min\{u(D) \mid D \text{ is a diagram for } K\}$$

$u(K)$ is called the unknotting number of K.

Unless $u(K)=1$ or $u(K)=0$, it is very difficult to determine $u(K)$!

Open

<u>Conjecture</u>: $u(K_1 \# K_2) = u(K_1) + u(K_2)$



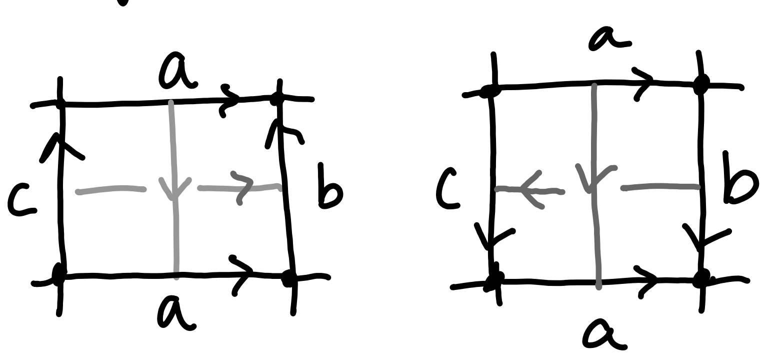
$K_1 \# K_2$

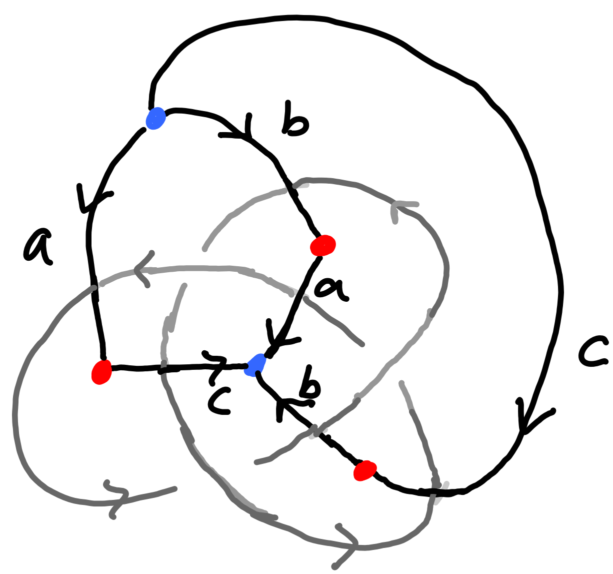
Easy to see that $u(K_1 \# K_2) \leq u(K_1) + u(K_2)$.

Since any knot can be changed to the unknot by changing crossings, to get knotting information, we need to remember crossing information.

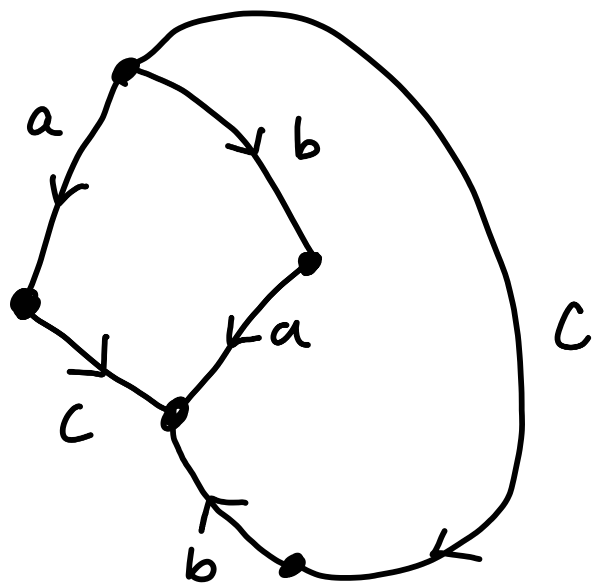
To do this, we consider (oriented) colored graphs:

- orient knot
- orient and color edges w/ a, b, ... s.t.





$dg(D) :=$

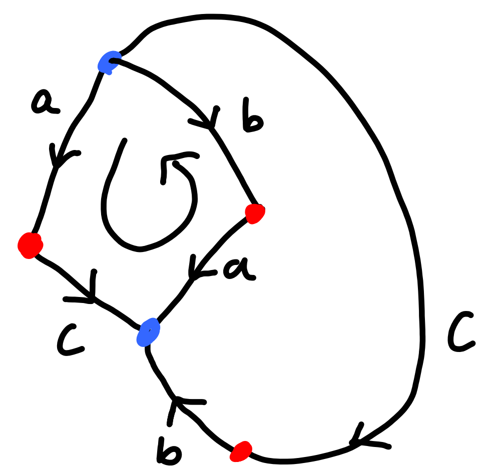


Note: Can recover the knot from the colored graph (orientations on edges gives an orientation on knot).

Define $G(D) := \langle a, b, c, \ldots \mid$ relations coming from compact regions in $dg(D)$ $\rangle$

(finitely presented group)

Ex:

$dg(D) :=$ 

$$G(D) = \langle a, b, c \mid aca^{-1}b^{-1},\ ab^{-1}c^{-1}b \rangle$$

It is easy to show that if D_1 and D_2 are related by Reidemeister moves then $G(D_1) \underset{\text{as groups}}{\cong} G(D_2)$.

Hence $G(K) := G(D)$, for any diagram D, is a knot invariant, called the <u>knot group</u>.

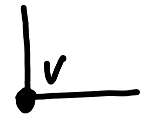
In fact, $G(K) \cong \pi_1(\underset{\substack{\uparrow \\ \text{independent of diagram.}}}{\mathbb{R}^3 \setminus K})$

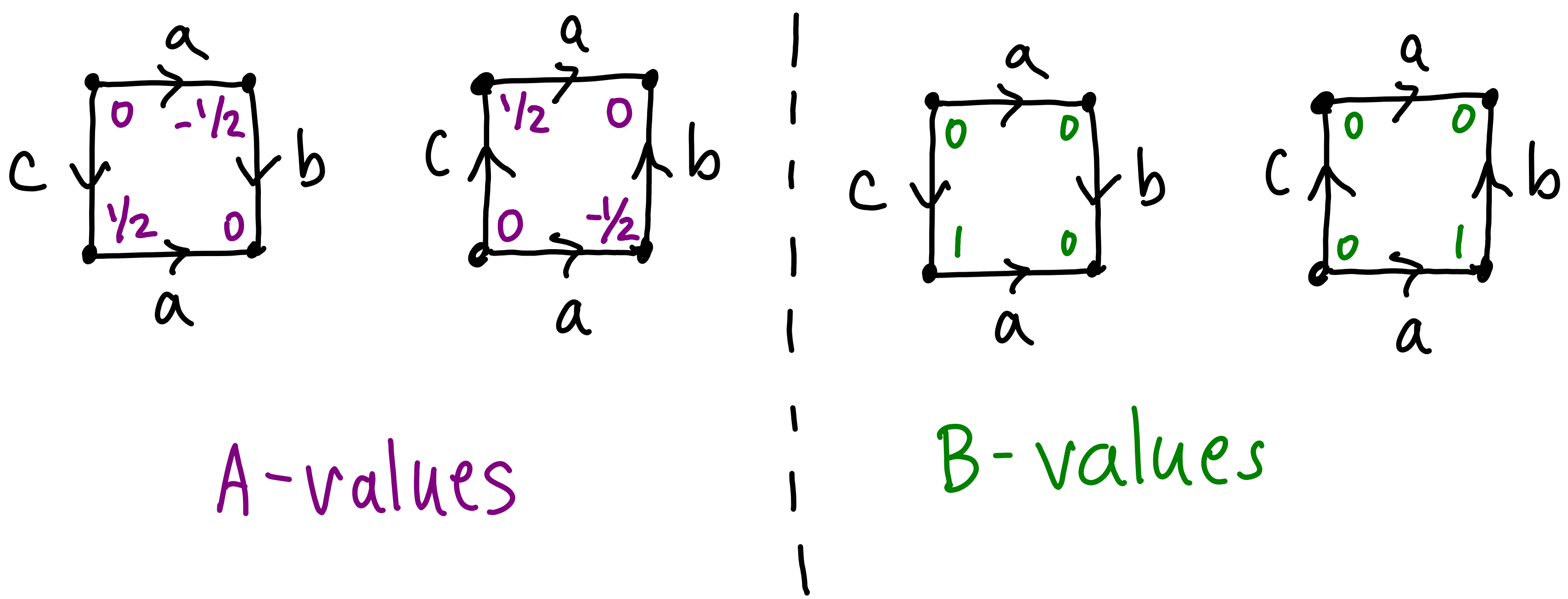
Theorem (Waldhausen, Gordon-Luecke): If there is an isomorphism $\varphi: G(K_1) \longrightarrow G(K_2)$ that preserves the "peripheral structure" of K_1 and K_2 then $K_1 = K_2$ (up to mirror image).

Thus the group of a knot is a complete invariant of knot (up to mirrors).

However, it is difficult to distinguish finitely presented groups!

Alexander Polynomial

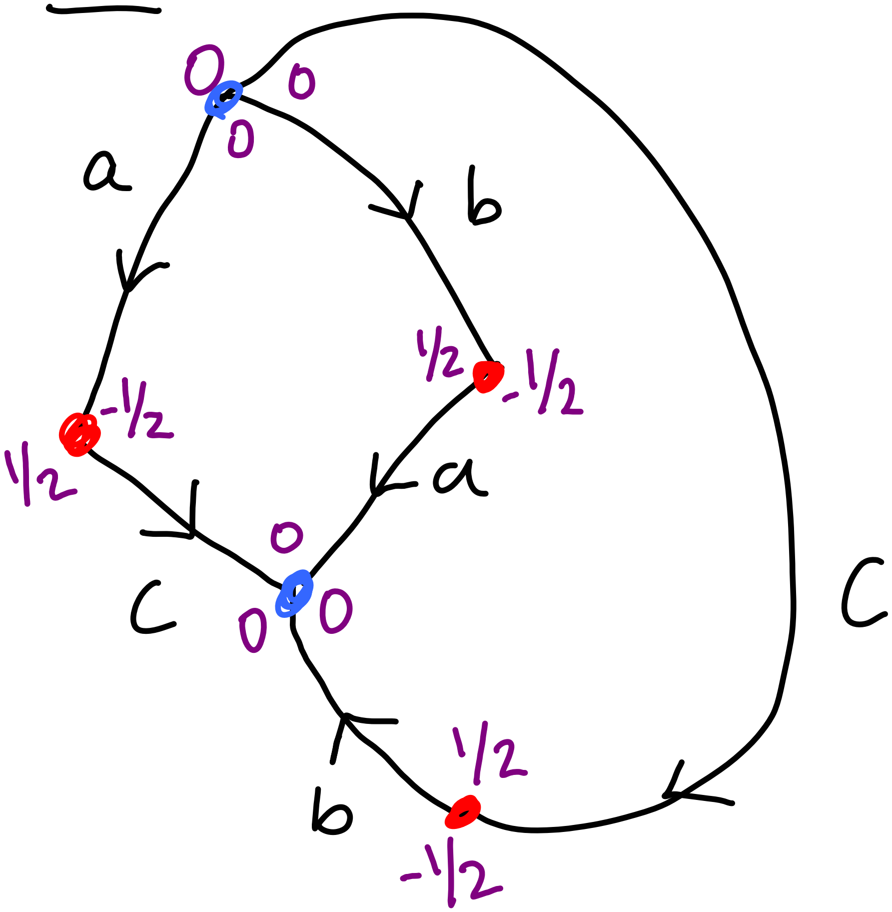
To each corner  in a region of dg(D) assign values $A(v)$ and $B(v)$ as follows:



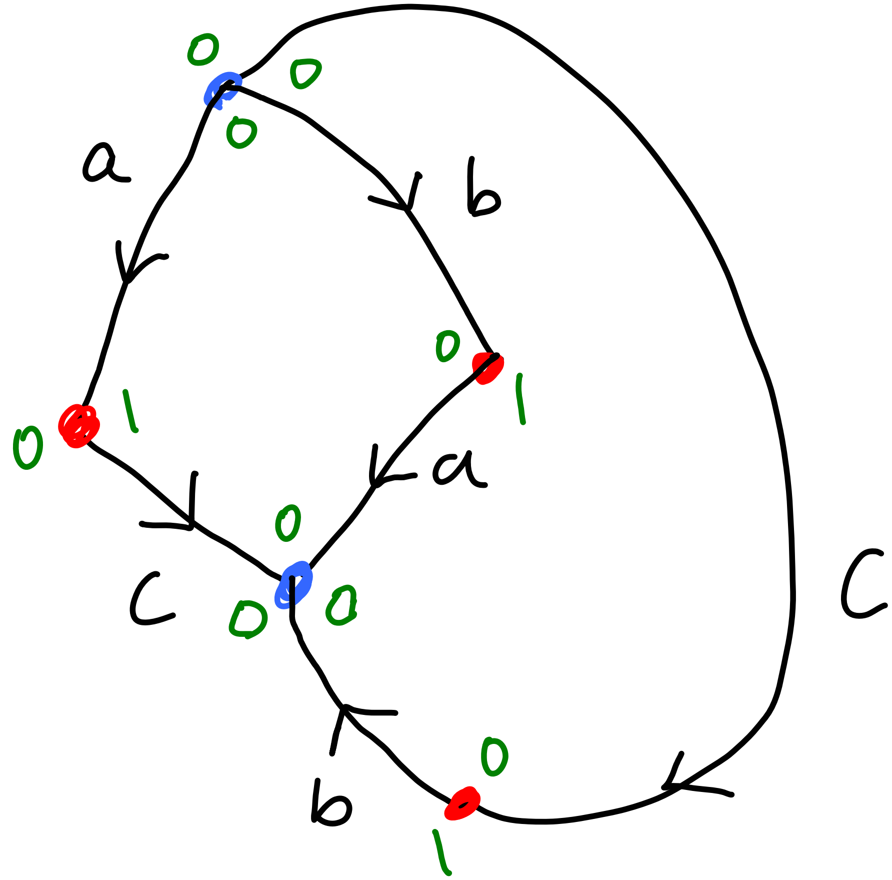
A-values

B-values

Ex:

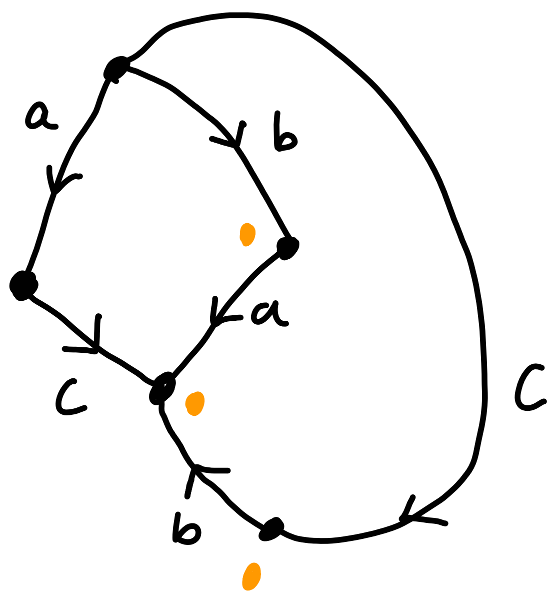


A-values



B-values

A (Kauffman) state of dg(D) is a choice of one vertex per region s.t. no vertex is used twice.

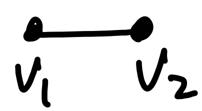


the orange dots constitute a single state.

Let s be a state. Then

$$A(s) := \sum_{v \in \{\text{corners in } s\}} A(v) \qquad B(s) := \sum_{v \in \{\text{corners in } s\}} B(v)$$

Let v_1 and v_2 be vertices of $dg(G)$ that share an edge



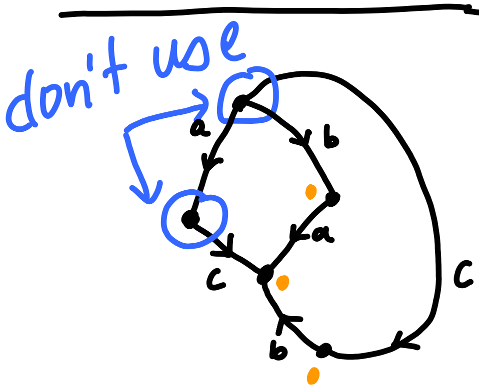
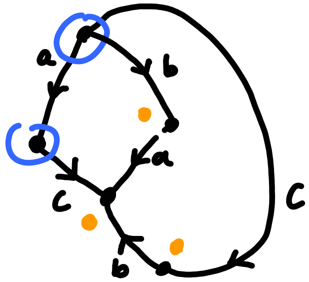
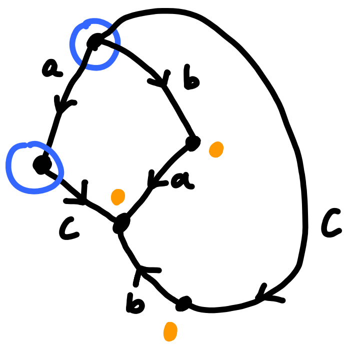
.

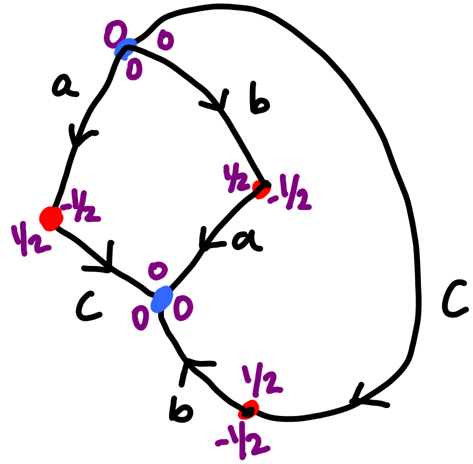
<u>Def:</u> $$\Delta_D(t) = \sum_{\substack{\text{states that do} \\ \text{not contain } v_1 \text{ or } v_2}} (-1)^{B(s)}\, t^{A(s)}$$

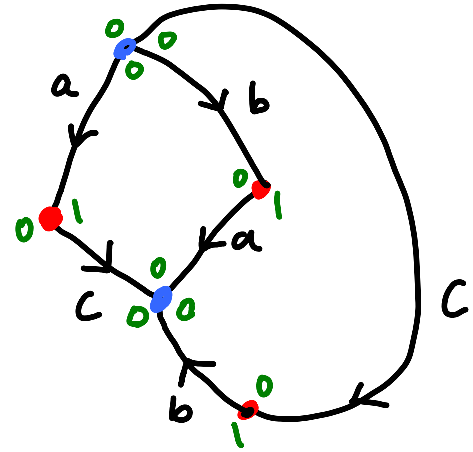
If D_1 and D_2 are related by Reidmeister moves then $\Delta_{D_1}(t) = \Delta_{D_2}(t)$. Define $\Delta_K(t) := \Delta_D(t)$ for any diagram D for K.

$\Delta_K(t)$ is called the Alexander polynomial of K and was originally defined* by Alexander in 1928.

* Alexander gave an equivalent but different definition.

states	A	B
	0	1
	1	0
	-1	2





$$\Delta_K(t) = \sum_{\substack{\text{allowable} \\ \text{states}}} (-1)^{B(s)} t^{A(s)}$$

$$= t^{-1} - 1 + t$$

Categorifications

In 2002, P. Ozsváth - Z. Szabó and J. Rasmussen defined a bi-graded homology theory that is a categorification of the Alexander polynomial.

(i.e. the Euler characteristic is $\Delta_K(t)$).

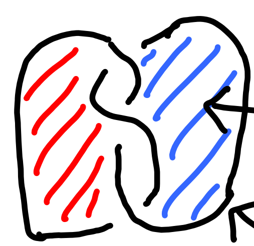
knot $K \longrightarrow \widehat{HFK}_j(K,i)$ abelian group $i,j \in \mathbb{Z}$

(Knot Floer homology of K)

$$\chi(\widehat{HFK}_j(K,i)) := \sum_{i,j} (-1)^j \operatorname{rank} \widehat{HFK}_j(K,i) = \Delta_K(t)$$

Let K be a knot. A seifert surface for K is an oriented surface embedded in $\mathbb{R}^3 - K$ whose boundary is K.

$$g(K) := \min \left\{ \frac{1-\chi(\Sigma)}{2} \;\middle|\; \Sigma \text{ is a seifert surface for } K \right\}$$



$$g(K) = 0 \iff K = \text{unknot}$$

<u>Thm</u> (Ozsváth-Szabó):

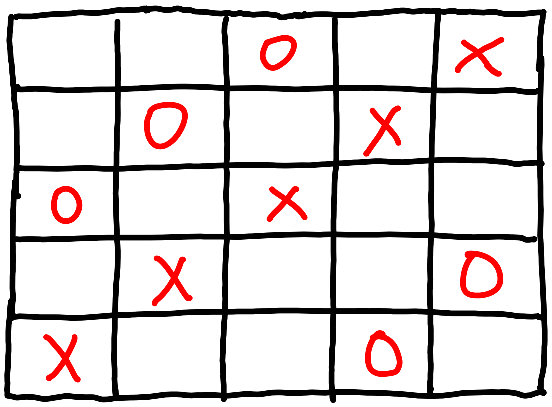
$$g(K) = \max i \text{ s.t. } \widehat{HFK}_*(K,i) \neq 0.$$

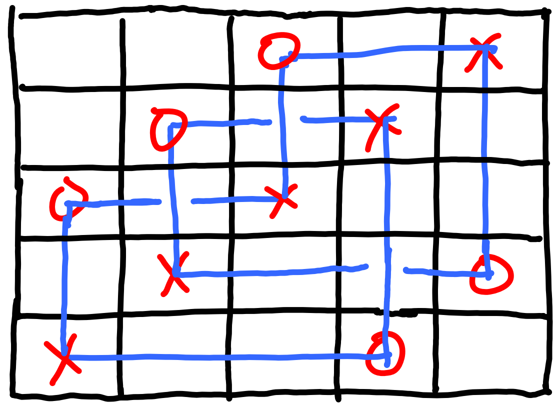
<u>Thm</u> (Ozsváth-Szabó, Ghinni-Ni 06):

$$K \text{ is fibered} \iff \text{for } \max i \text{ s.t. } \widehat{HFK}_*(K,i) \neq 0 \quad \operatorname{rank}(\widehat{HFK}_*(K,i)) = 1$$

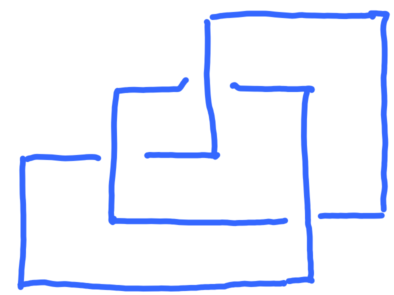
The chain groups are generated by Kauffman states. However, the boundary map is defined by counting pseudoholomorphic curves so we cannot determine the groups for an arbitrary knot.

Grid Diagrams





trefoil



Grid diagram $\Longrightarrow$ knot

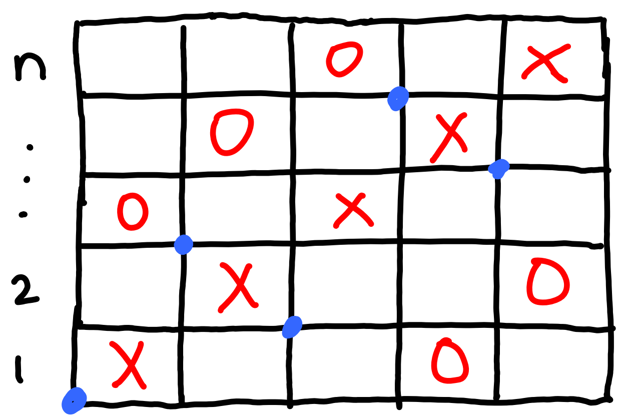
One X and O per row/column

Connect X to O in columns
O to X in rows

vertical strands cross over horizontal strands

Define chain complex $\widetilde{CK}$ over $\mathbb{Z}/2$.

$n!$ generators: Matchings between horizontal and vertical gridlines



one • on each horizontal and vertical gridline (except top and right).

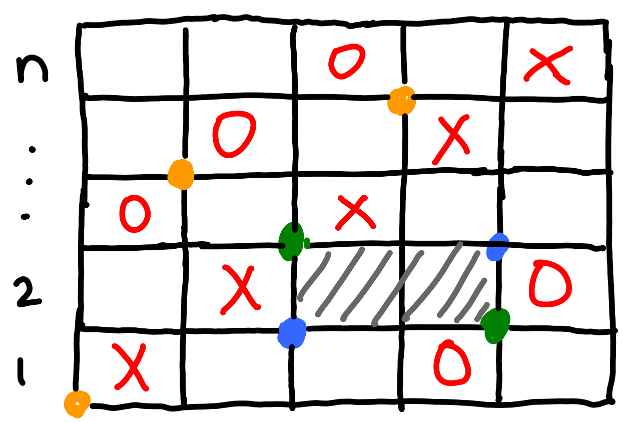
We view grid as living on a torus by identifying left/right and top bottom.

Generators $\longleftrightarrow S_n$

$$\partial : \widetilde{CK} \longrightarrow \widetilde{CK}$$

$$\partial x = y + \cdots$$

counts all ways to switch SW-NE corners of an empty rectangle to a NW-SE corner.



$x = \bullet$

$y = \bullet$

• all points where x, y agree

M (j-grading):

$M(x) - M(y) = (1 - \#$ of O's) in rectangle with x and y corners

A (i-grading): ($\#$ of X's $- \#$ of O's) in a rectangle w/ x and y corners.

∂ preserves i-grading and lowers j-grading by 1.

Theorem (Manolescu-Ozsváth-Sarkar):

If G is a grid diagram for K,

$$H_*(\widetilde{CK}(G)) \cong \widehat{HFK}(K) \otimes V^{\otimes n-1}$$

where $V = \mathbb{Z}/2_{(0,0)} \oplus \mathbb{Z}/2_{(-1,-1)}$

Hence $\chi(H_*(\widetilde{CK}(G))) = \Delta_K(t)(1-t)^{n-1}$.

This gives the simplest algorithm to compute knot genus!

If $p_K(t)$ is any knot invariant, a categorification of $p_K(t)$ is a bi-graded chain complex $(C_{i,j}(K), \partial)$ s.t.

$$\chi(C_{i,j}(K), \partial) = \sum_{i,j} (-1)^j \operatorname{rank} C_{i,j}(K) \cdot t^i$$

$$= \sum_{i,j} (-1)^j \operatorname{rank} \underbrace{H_{i,j}(K)}_{\text{homology of } (C_{i,j}(K), \partial)} \cdot t^i$$

$$= p_K(t).$$

Other Categorifications

Knot Polynomial	Categorification
Alexander poly	knot Floer homology (Heegaard Floer) ~2002
Jones poly.	Khovanov homology (Khovanov) ~1999
HOMFLY poly.	Khovanov-Rosanky hom. ~2004

Categorification of graph polynomials

Polynomial	Categorification
chromatic polynomial	Helme-Guizon + Rong ~2005
Tutte polynomial	Jasso-Hernandex + Rong ~2006
Yamada polynomial	Vershinin + Vesnin ~2006

Thank you!