Subgroups of the mapping class group and higher-order signature cocycles

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Goal: For certain subgroups $\mathcal{J}(H)$ of the mapping class group of a surface, define

- quasi-homomorphisms

$$\rho^{\psi} : \mathcal{J}(H) \longrightarrow \mathbb{R}$$

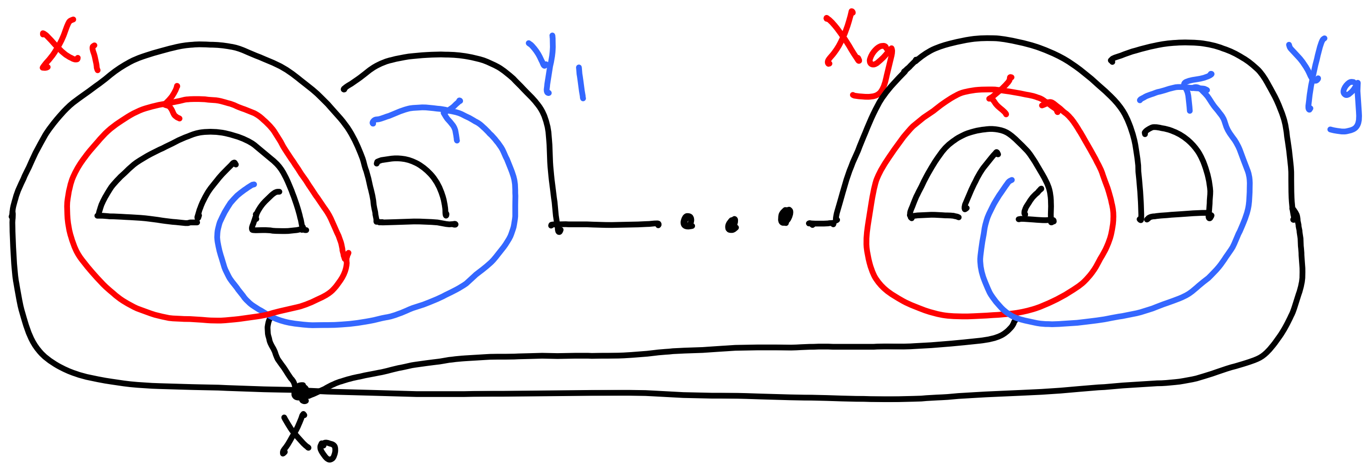
- signature 2-cocycles

$$\sigma_{\psi} : \mathcal{J}(H) \times \mathcal{J}(H) \longrightarrow \mathbb{R}.$$

In some cases, we will show that these are non-trivial.

In fact, when $J(H)$ = Torelli subgroup and $\psi : H_1(\Sigma) \longrightarrow U(1)$, we will show that $\{\rho^\psi\}$ has infinite rank.

Let $\Sigma = \Sigma_{g,1}$ be a compact, oriented surface with 1 boundary component.

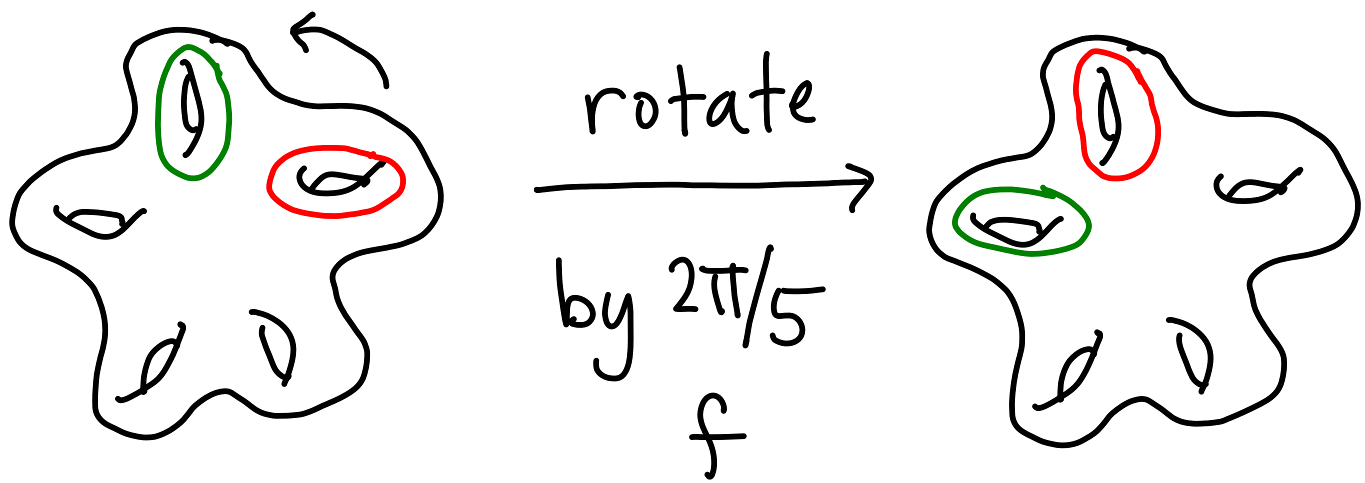
$\Sigma =$ 

$F :=$ free group generated by $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$
$\cong \pi_1(\Sigma, x_0)$,

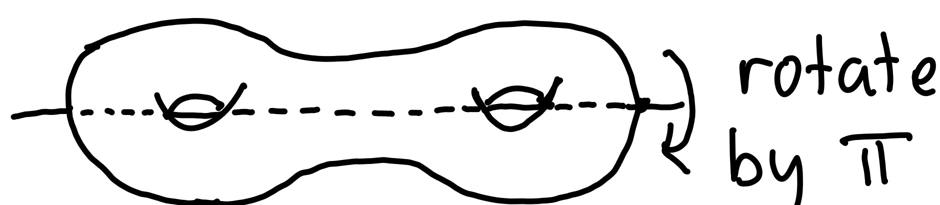
Def: The mapping class group of Σ, $\boxed{\mathrm{Mod}(\Sigma)}$ is the group of isotopy classes of orientation preserving diffeomorphisms of Σ that fix the boundary of Σ pointwise.

Ex:

(for a closed surface)

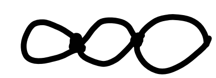


Hyperelliptic involution



$\mathrm{Mod}(\Sigma)$ is related to 3-manifolds

(Assume $\Sigma \neq \emptyset$). Let $f: \Sigma \to \Sigma$.

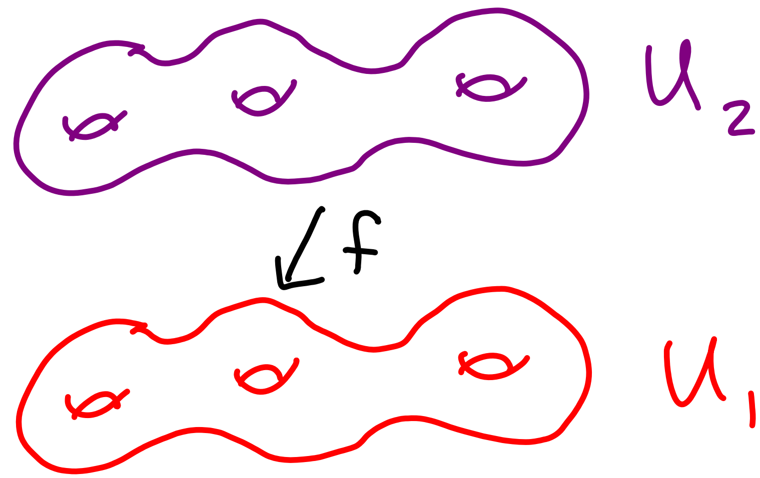
1. Let U_i be a thickened graph 

Glue U_1 to U_2

using $f: \partial U_1 \to \partial U_2$

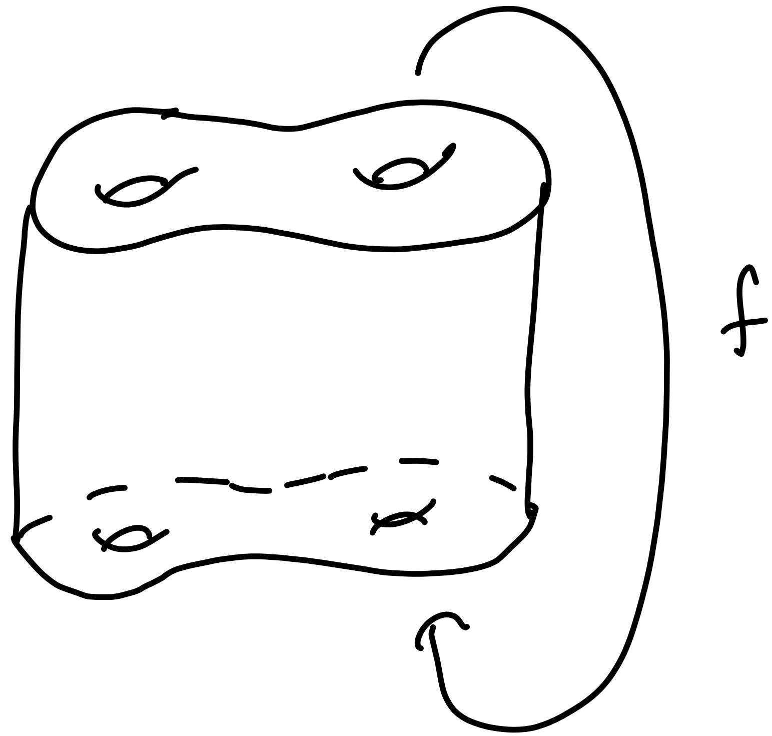
$\qquad\qquad\qquad \parallel \qquad\quad \parallel$

$\qquad\qquad\qquad \Sigma \qquad\quad \Sigma$



Get a Heegaard decomposition of $M = U_1 \cup_f U_2$

2. $M_f = \Sigma \times I \Big/ (x,1) \sim (f(x),0)$

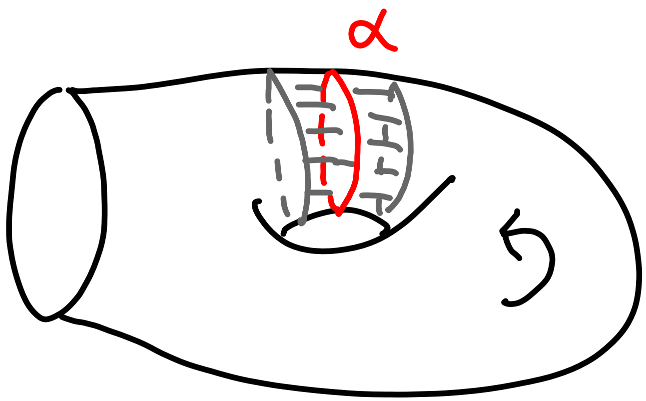


glue top to bottom by f

M_f is a fiber bundle over S^1 with fiber Σ.

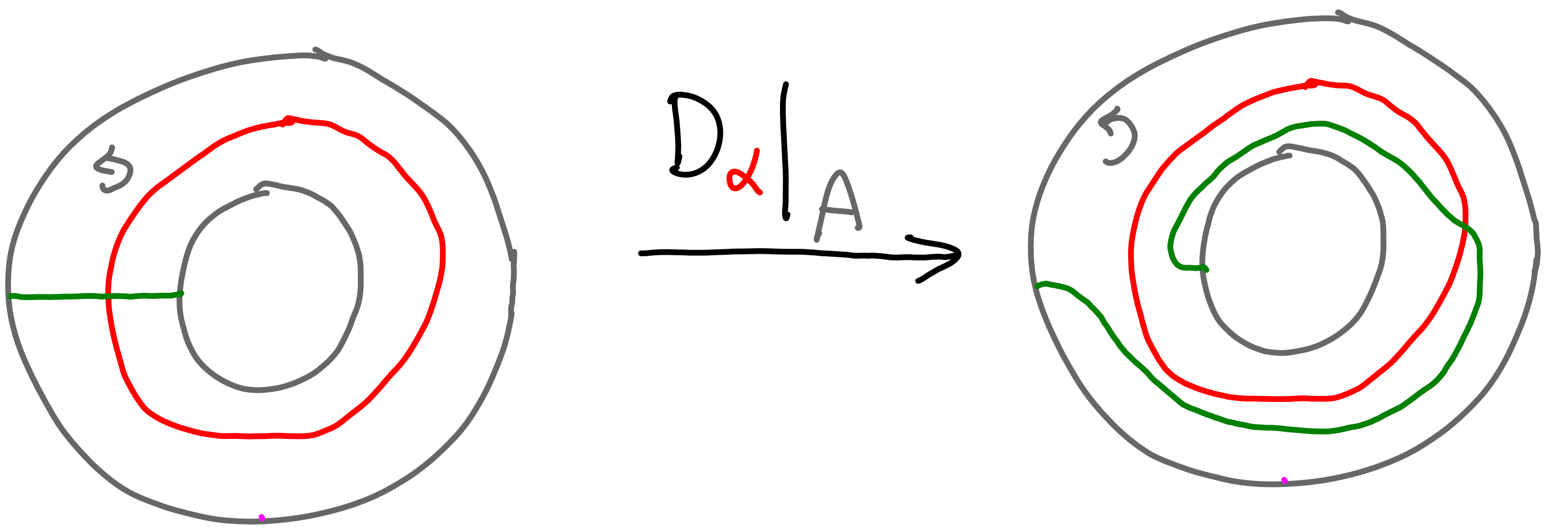
Dehn twists

Let α be a s.c.c. on Σ. Let A = nbhd of γ.

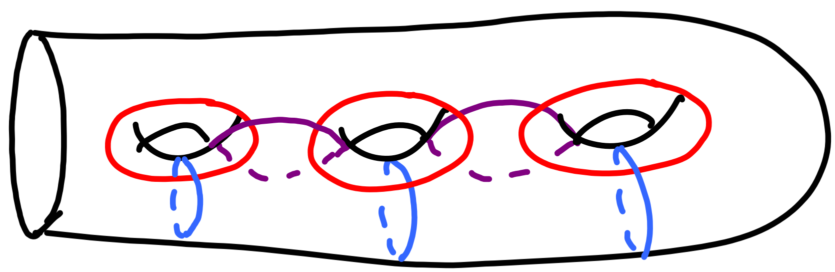


Define $D_\alpha : \Sigma \to \Sigma$ by • $D_\alpha|_{\Sigma - A} = \mathrm{id}$

•



Theorem (Dehn, $\partial\partial$) $\mathrm{Mod}(\Sigma)$ is generated by finitely many Dehn twists.



Theorem (McCool, Hatcher-Thurston, Deligne-Mumford): For any surface $\mathrm{Mod}(\Sigma)$ is finitely presented.

- Hatcher-Thurston found an algorithm to construct a presentation. Harer produced the first explicit presentation.

Let $f \in \mathrm{Mod}(\Sigma_{g,1}) \Rightarrow f|_{\partial\Sigma} = id$ so by choosing a base point $x_0 \in \partial\Sigma$, $f_* \in \mathrm{Aut}(\pi_1(\Sigma, x_0))$

Theorem (Dehn-Nielsen-Baer):

$$\begin{aligned} \mathcal{I} : \mathrm{Mod}(\Sigma_{g,1}) &\hookrightarrow \mathrm{Aut}(\pi_1(\Sigma, x_0)) = \mathrm{Aut}(F_{2g}) \\ f &\longmapsto f_* \end{aligned}$$

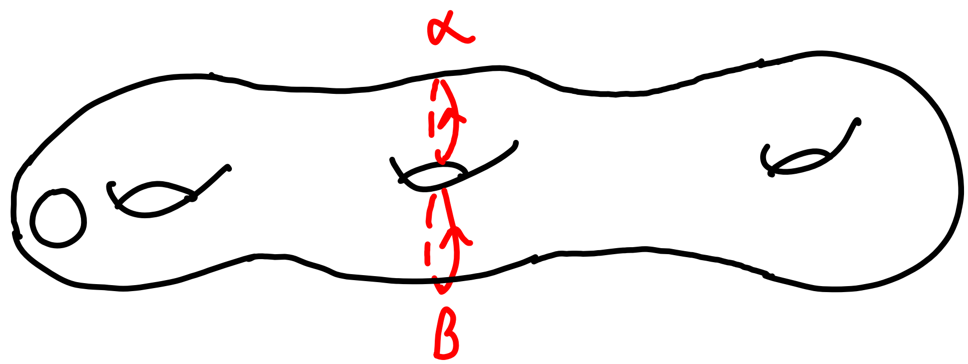
Can approximate $\mathcal{I}$ by send $f \longmapsto f_*^k$ where $f_*^k : F/F_k \longrightarrow F/F_k$ where $\{F_k\}$ is a characterstic series that approximates F.

Example: $F_2 = [F, F]$ = commutator subgp

Def: The <u>Torelli group</u> of Σ, $\mathcal{I}(\Sigma)$ is the subgroup of $\mathrm{Mod}(\Sigma)$ consisting of diffeos that induces the identity on $H_1(\Sigma) = F/F_2$.

$$\mathcal{I}(\Sigma) = \ker\left(\tau_2 : \mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Aut}(H_1(\Sigma))\right)$$

Thm (Powell, Birman, Johnson): $\mathcal{I}(\Sigma)$ generated by Dehn twists on bounding pairs of curves; $D_\alpha \circ D_\beta^{-1}$.



[α, β disjoint homologous non-separating curves]

Thm (Johnson): $\mathcal{I}(\Sigma)$ is finitely generated for $g \geq 3$.

Thm (Mess): $\mathcal{I}(\Sigma)$ is an infinitely generated free group when $g = 2$.

We wish to investigate the homology and cohomology of certain special subgroups of $\mathcal{I}(\Sigma)$ using ρ-invariants of 3-manifolds.

Def: Let G be a group. Then the lower-central series of G is defined as

- $G_2 := [G, G] =$ commutator subgp of G.
- for $k \geq 3$, $G_{k+1} := [G, G_k]$.

Def: The $k^{\underline{th}}$ generalized Johnson subgroup of Σ, $J_k = J_k(\Sigma)$ is the subgroup of $\mathrm{Mod}(\Sigma)$ that induce the identity on F/F_k.

- Note that $\mathcal{J}_2(\Sigma) = \mathcal{I}(\Sigma)$.

<u>Thm</u> (Johnson): $\mathcal{J}_3(\Sigma)$ = subgroup generated by Dehn twists on separating simple closed curves on Σ.

[$\mathcal{J}_3(\Sigma)$, denoted $\underline{\mathcal{K}_g}$, is often called the <u>Johnson subgroup</u>.]

Question (Morita): Is $H_1(\mathcal{K}_g;\mathbb{Z})$ finitely generated for $g \geq 3$? [Note: $\mathcal{K}_2 = \mathcal{I}_2$ infinitely gen]

Note: Still unknown if $\mathcal{K}_g$ (or more generally $\mathcal{J}_k(\Sigma), k \geq 3$) is infinitely generated for $g \geq 3$!

Construction of Invariants

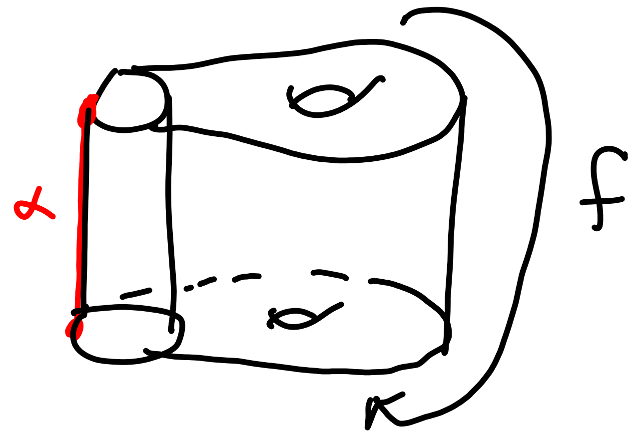
Let $H \trianglelefteq F = \pi_1(\Sigma)$, a characteristic subgroup, and $J(H)$ = subgroup of $\mathrm{Mod}(\Sigma)$ consisting of diffeos of Σ that induce the identity on F/H.

e.g. $H \subseteq F_k \Rightarrow J(H) \subseteq k^{\text{th}}$ Johnson subgroup.

Given $f \in \mathcal{J}(H) \rightsquigarrow$ closed 3-mfld N_f by

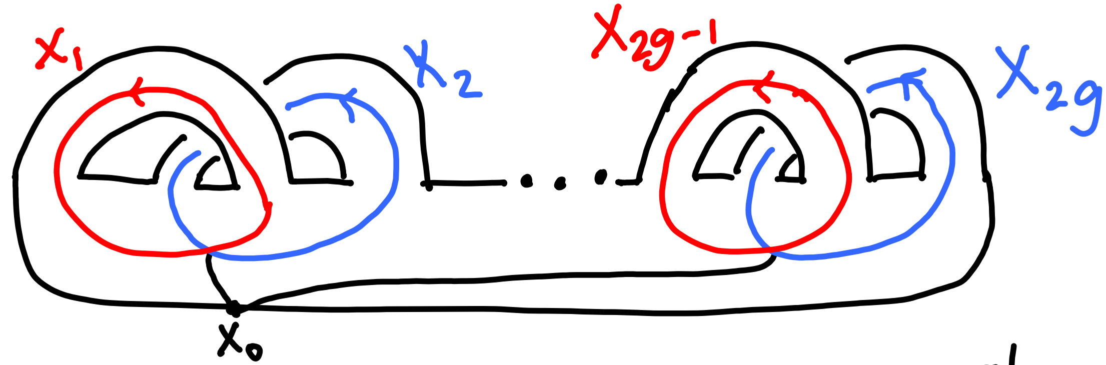
- M_f = mapping torus of f

$$= \Sigma \times I \Big/ \begin{array}{l}(x,1) \sim (f(x),1) \\ \forall\, x \in \Sigma\end{array}$$



- $N_f = M_f \Big/ \begin{array}{l}(y,t) \sim (y,s) \\ \text{for } t,s \in I \\ \text{and } y \in \partial\Sigma\end{array}$

α identified to a point in N_f.

$\Sigma =$ 

Then $\pi_1(M_f) = \langle x_1, \dots, x_{2g}, t \mid t x_i t^{-1} = f(x_i) \rangle$

$$\Big\downarrow \text{ kill } [\alpha] = t$$

$$\pi_1(N_f) = \langle x_1, \dots, x_{2g} \mid x_i = f(x_i) \rangle$$

but since $f(x_i) \equiv x_i \mod H \Rightarrow$

$$\pi_1(N_f)/H \cong F/H$$

Thus, we have $\varphi_H : \pi_1(N_f) \longrightarrow F/H$.

$\uparrow$ independent of f

Def: $\rho_H(f) = \rho^{(2)}(N_f, \psi_H : \pi_1(N_H) \twoheadrightarrow F/H) \in \mathbb{R}$, the Cheeger Gromov $L^{(2)}$-ρ invariant associated to (N_f, ψ_H).

Note: $\rho^{(2)}(M, \psi)$ is a invariant associated to any 3-mfld M^3 and $\psi : \pi_1(M) \longrightarrow \Gamma$ (group)

Note: $L^{(2)}$-ρ invts have been useful recently in the study of 3-manifolds and knot and link concordance (Cochran, H, Leidy, Kim, Cha, Orr, Teichner, Horn, Heck)

Lemma For each (M^3, ψ) there exists (W^4, ϕ)

s.t.

$$\begin{tikzcd} \pi_1 \partial W = \pi_1 M \arrow[r, "\psi"] \arrow[d] & \Gamma \arrow[d, hook] \\ \pi_1 W \arrow[r, "\phi"] & \Lambda \end{tikzcd}$$

Thm (Ramachandran)

$$\rho^{(2)}(M, \Gamma) = \begin{pmatrix} L^2\text{-signature of } \Gamma\text{-equivariant} \\ \text{intersection form on } H_2(W, \mathbb{Z}\Lambda) \end{pmatrix} - (\text{signature of } W). \in \mathbb{R}$$

- If we have finite unitary repr. $\psi : F/H \longrightarrow U(n)$ then can define

$$\rho_H^{\psi}(f) := \rho(N_f, \pi_1(N_f) \twoheadrightarrow F/H \xrightarrow{\psi} U(n)) \in \mathbb{R}$$

the Atiyah-Patodi-Singer (APS) ρ-invt.

- <u>Thm</u> (APS): <u>If</u> $\psi : \pi_1 N_f \longrightarrow U(n)$ extends to $\phi : \pi_1 W \longrightarrow U(n)$ then

$$\rho_H^{\psi}(f) = \operatorname{sign}^{\psi}(W) - n \operatorname{sign}(W) \in \mathbb{Z}$$

twisted signature (signature of finite Hermitian matrix over $\mathbb{C}$)

Ex: $H = [F,F]$ & $\pi_1(N_f) \xrightarrow{\psi_H} F/[F,F] = \mathbb{Z}^{2g} \xrightarrow{\phi} U(1)$

$$x_i \longmapsto w_i$$

where $w_i \in \mathbb{C}$ with $|w_i| = 1$.

When $w_i = w$, define $\rho_w(f) = \rho(N_f, \pi_1 N_f \longrightarrow U(1))$

We will show that $\rho_w(f)$ is often non-trivial.

Ex: For each $k \geq 2$, consider $H = [F_k, F_k]$.

Since $[F_k, F_k] \leq F_{k+1} = [F, F_k]$,

$$\mathcal{J}([F_k, F_k]) \leq \mathcal{J}_{k+1,g} \quad (k+1^{st} \text{ Johnson subgroup}).$$

Def: For $f \in \mathcal{J}([F_k, F_k])$, define

$$\bar{\rho}_{k+1}(f) = \rho^{(2)}(N_f, \pi_1 N_f \longrightarrow F/[F_k, F_k]) \in \mathbb{R}.$$

Thm (Cochran-H-Horn): For $k \geq 2$,

$$\bar{\rho}_{k+1} : H_1(\mathcal{J}([F_k, F_k]); \mathbb{Z}) \longrightarrow \mathbb{R}$$

is a homomorphism.

Question: What is the image of $\bar{\rho}_{k+1}$? If infinitely generated, then $H_1(\mathcal{J}([F_k,F_k]);\mathbb{Z})$ is infinitely generated.

For an arbitrary characteristic subgroup $H \trianglelefteq F$,

$$\rho_H : \mathcal{J}(H) \longrightarrow \mathbb{R}$$

is a function but not necessarily a homomorphism!

Bounded Cohomology

Let $\rho_H^\psi : J(H) \longrightarrow \mathbb{R}$ be one of the ρ-invts as above (associated to a finite or infinite unitary representation).

- $\rho_H^\psi : J(H) \longrightarrow \mathbb{R}$ is a 1-cochain in group cohomology. Its coboundary $\sigma_H^\psi := \delta \rho_H^\psi : J(H) \times J(H) \longrightarrow \mathbb{R}$ is the 2-cocycle:

$$\boxed{\sigma_H^\psi(f,g) := \delta\rho_H^\psi(f,g) = \rho_H^\psi(fg) - \rho_H^\psi(f) - \rho_H^\psi(g).}$$

Def: A function $\varphi : G \to \mathbb{R}$ is called a <u>quasi-homomorphism</u> if $\exists$ constant $D(\varphi)$ such that $\forall\, x, y \in G$,

$$|\varphi(xy) - \varphi(x) - \varphi(y)| \leq D(\varphi).$$

$$QH(G) := \{\text{quasi-homomorphisms on } G\} \big/ \text{bounded functions}$$

For each G, there is an exact sequence:

$$0 \to H^1(G;\mathbb{R}) \longrightarrow QH(G) \xrightarrow{\delta} H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R})$$

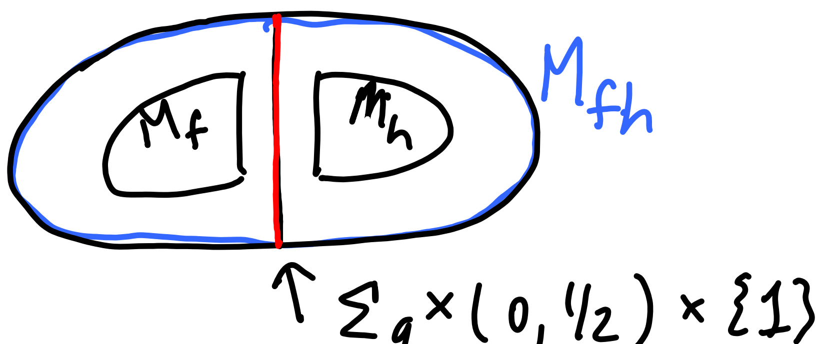
Proposition (Cochran-H-Horn): The higher ρ-invts ρ^4_H are quasi-homomorphisms and hence define elements $\sigma^4_H := \delta\rho^4_H \in H^2_b(\mathcal{J}(H);\mathbb{R})$.

Note: σ^4_H is not necessarily 0 in $H^2_b(\mathcal{J}(H);\mathbb{R})$ since ρ^4_H may be an unbounded function.

<u>Idea of Proof</u> (ρ_H^φ is a quasi-homomorphism):

For simplicity, assume Σ is closed.

Then for $f, h \in \mathcal{J}(H)$, can construct a 4-mfld W_{fh} by: glue $M_f \times I$ to $M_h \times I$ along $(\Sigma_g \times (0, 1/2)) \times \{1\} \subset M_f \times I$ & $M_h \times I$.

$W_{fh} =$ 

$$\coprod \begin{cases} \varphi_1 : \pi_1 M_f \to F/H \\ \varphi_2 : \pi_1 M_{fg} \to F/H \\ \varphi_3 : \pi_1 M_g \to F/H \end{cases}$$

Can show $\coprod \varphi_i$ extends $\pi_1(W_{gh}) \longrightarrow F/H$.

One can show that

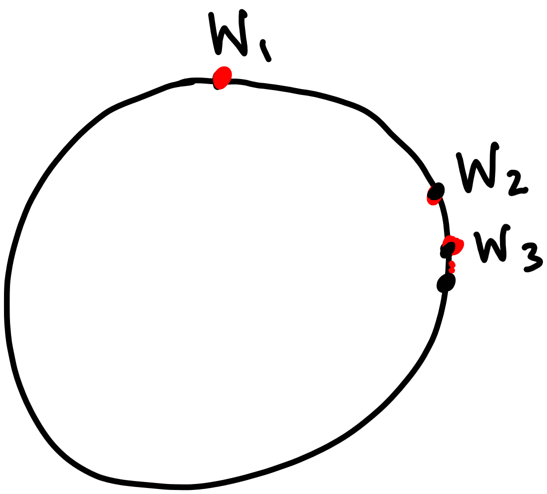
$$\Rightarrow \left| \rho_H^{\varphi}(fh) - \rho_H^{\varphi}(f) - \rho_H^{\varphi}(h) \right| = \left| \sigma^{(2)}(W, F/H) - \sigma_0(W) \right|$$

$$\leq 2\,\text{genus}(\Sigma) = D(g)$$

[$D(g)$ is independent of f and h.]

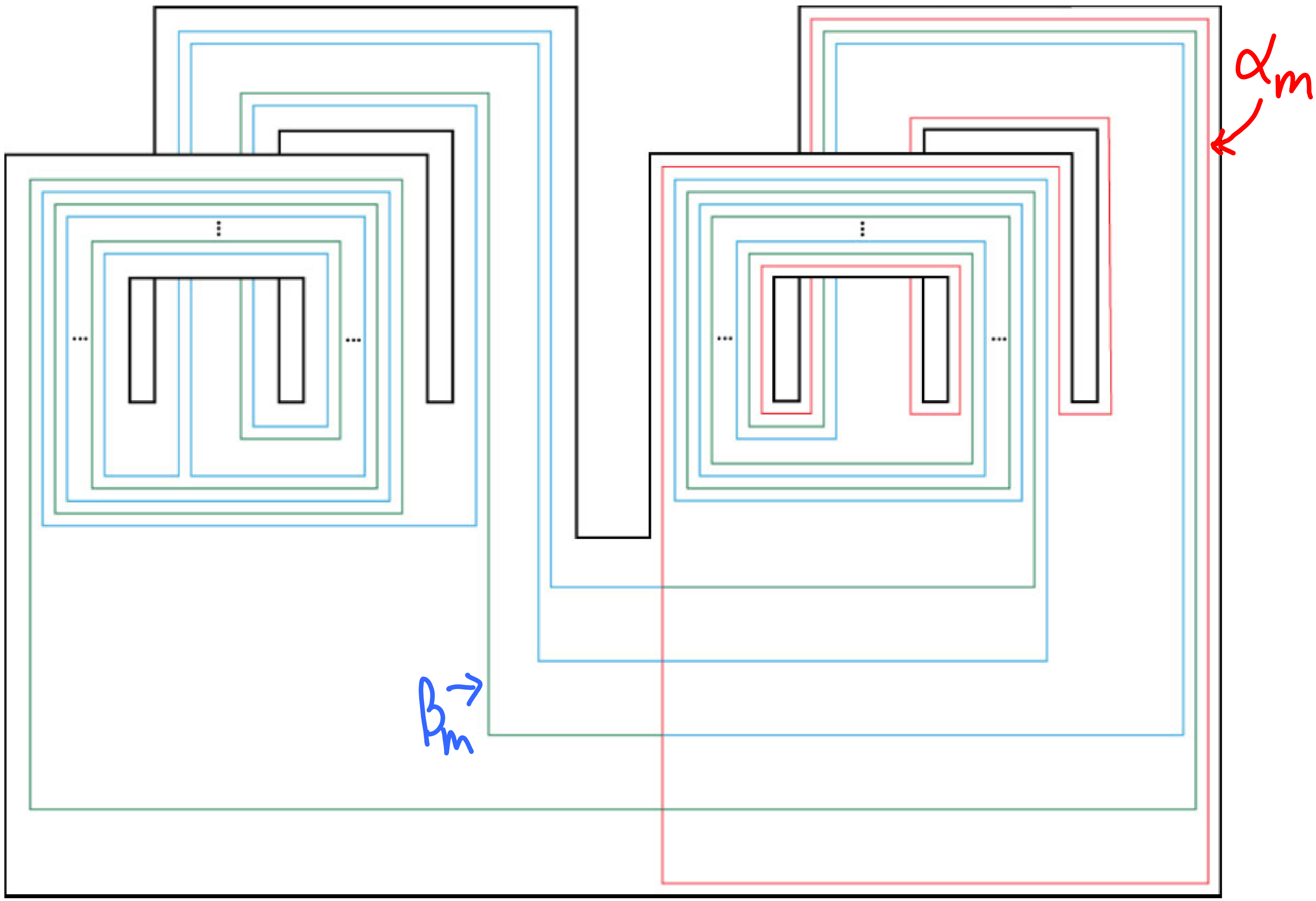
For $k \geq 1$, consider

$$\rho_{\omega_k}(f) := \rho\left(N_f,\ \pi_1(N_f) \longrightarrow U(1)\right)$$
$$x_i \longmapsto \omega_k = e^{2\pi i/4^k}.$$



Theorem (Cochran-H-Horn) $\{\rho_{\omega_k}\}_{k=1}^{\infty}$ is a linearly independent subset of $QH(\mathcal{K}_g)$ for $g \geq 2$.

To prove this, we produce a formula for $\rho_{w_k}\left(\left(D_{\alpha_m}\cdot D_{\beta_m}\right)^n\right)$ for $k\geq m$ and $n\geq 0$.



$\blacksquare$

There is a long exact sequence:

$$0 \to H^1(\mathcal{K}_g;\mathbb{R}) \longrightarrow QH(\mathcal{K}_g) \xrightarrow{\delta} H^2_b(\mathcal{K}_g;\mathbb{R}) \longrightarrow H^2(\mathcal{K}_g;\mathbb{R})$$

$$\mathbb{R}^\infty \cong \{\rho_{w_k}\} \subset QH(\mathcal{K}_g)$$

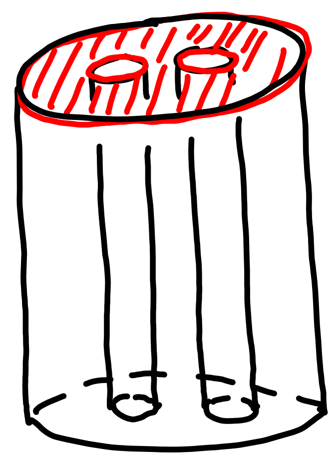
What is $\mathrm{im}(\delta|_{\{\rho_{w_k}\}})$, $\ker(\delta|_{\{\rho_{w_k}\}})$?

Note: If $\ker(\delta) = \mathbb{R}^\infty \Rightarrow \exists\, \mathbb{Z}^\infty \subset H_1(\mathcal{K}_g;\mathbb{Z})$ which answer's Morita's question!

Other Computations of ρ^4

Let $D \hookrightarrow \Sigma$ be an embedding of

$D = D^2 - (n\text{-disks})$

e.g. 

$D \times \{0\} \subset$

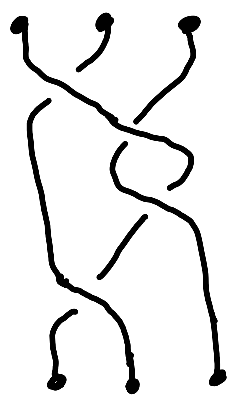
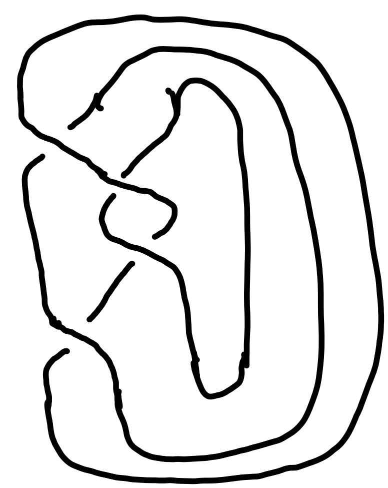
$\Sigma = \partial(D \times I)$

P_n = pure braid group (framed) is the mapping class group of D.

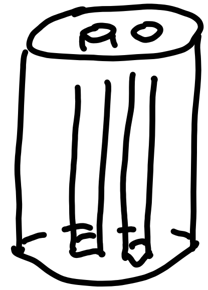
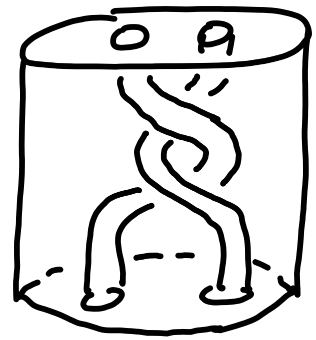
This embedding gives an embedding

$$\theta : P_n \hookrightarrow \mathrm{Mod}(\Sigma).$$

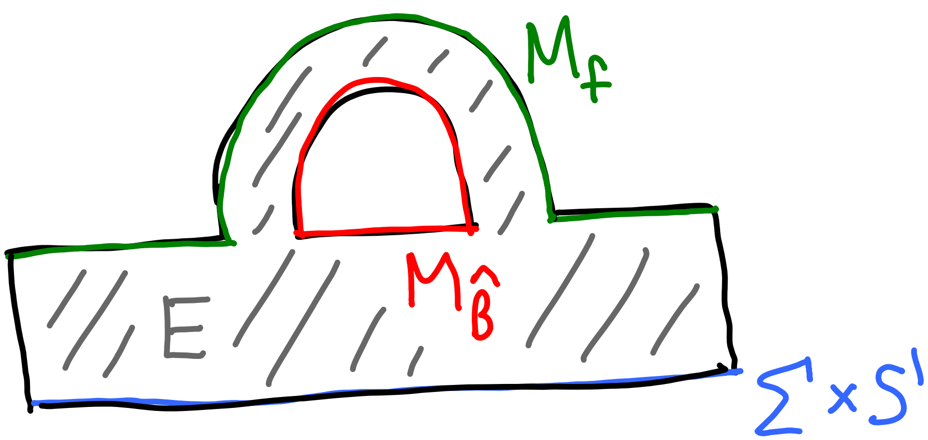
Proposition (Cochran-H-Horn): Let $\beta \in P_k$. The higher order ρ-invariants of $\theta(\beta)$ can be calculated in terms of the higher-order ρ-invts of the zero framed surgery on $\hat{\beta}$.

$\beta =$  $\xrightarrow{\text{Closure}}$  $= \hat{\beta}$

Idea of proof:

$M_f = \Big[(\Sigma' \times S^1) \setminus \Big($  $\Big) \Big] \cup$ 

This gives a cobordism W between



$\Sigma' \times S^1$, M_f, and

$M_{\hat{\beta}} = 0$-surgery on $\hat{\beta}$.

Signature defect of E gives the sum of ρ-invariants on the boundary of E. $\square$