Subgroups of the mapping class group and higher-order signature cocycles

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Goal: For certain subgroups $J(H)$ of the mapping class group of a surface, define

• quasi-homomorphisms
  \[ \rho^\psi : J(H) \to \mathbb{R} \]

• signature 2-cocycles
  \[ \sigma_\psi : J(H) \times J(H) \to \mathbb{R} \].
In some cases, we will show that these are non-trivial.

In fact, when $J(H) = \text{Torelli subgroup}$ and $\Psi : H_1(\Sigma) \rightarrow \mathbb{U}(1)$, we will show that $\{\rho^\Psi\}$ has infinite rank.
Let $\Sigma = \Sigma_{g,1}$ be a compact, oriented surface with 1 boundary component.

\[ \Sigma = \begin{array}{c}
\xymatrix{
& x_1 & \ar[r] & y_1 & \ar[r] & x_g & \ar[r] & y_g \\
& \ar[ru] & \ar[l] & \ar[u] & \ar[l] & \ar[u] & \ar[l] & \ar[u] \\
& x_0 & \ar[u] & \ar[r] & \ar[l] & \ar[u] & \ar[r] & \ar[l] \\
} \end{array} \]

$F := \text{free group generated by } \{x_1, \ldots, x_g, y_1, \ldots, y_g\}$

\[ \cong \pi_1(\Sigma, x_0), \]
Def: The mapping class group of $\Sigma$, $\text{Mod}(\Sigma)$, is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma$ that fix the boundary of $\Sigma$ pointwise.

Ex: (for a closed surface)

Hyperelliptic involution

rotate by $\frac{2\pi}{5}$

rotate by $\pi$
Mod(\Sigma) is related to 3-manifolds (Assume \Sigma \neq \emptyset). Let \( f: \Sigma \rightarrow \Sigma \).

1. Let \( U_i \) be a thickened graph.

Glue \( U_1 \) to \( U_2 \) using \( f: \partial U_1 \rightarrow \partial U_2 \).

Get a Heegaard decomposition of \( M = U_1 \cup_f U_2 \).
2. $M_f = \Sigma \times I \big/ (x, 1) \sim (f(x), 0)$

\[ \text{glue top to bottom by } f \]

$M_f$ is a fiber bundle over $S^1$ with fiber $\Sigma$. 
Dehn twists

Let $\alpha$ be a s.c.c. on $\Sigma$. Let $A = \text{nbhd of } \gamma$.

Define $D_\alpha : \Sigma \to \Sigma$ by $D_\alpha |_{\Sigma - A} = \text{id}$.
Theorem (Dehn, Ξ) \( \text{Mod}(\Sigma) \) is generated by \underline{finitely many} Dehn twists.

\[ \text{Diagram of Dehn twists} \]

Theorem (McCoul, Hatcher-Thurston, Deligne-Mumford):
For any surface \( \text{Mod}(\Sigma) \) is finitely presented.

- Hatcher-Thurston found an algorithm to construct a presentation. Harer produced the first explicit presentation.
Let $f \in \text{Mod}(\Sigma_{g,1}) \Rightarrow f|_{\partial\Sigma} = \text{id}$ so by choosing a base point $x_0 \in \partial\Sigma$, $f_* \in \text{Aut}(\pi_1(\Sigma, x_0))$

**Theorem (Dehn-Nielsen-Baer):**

$$\Theta: \text{Mod}(\Sigma_{g,1}) \leftrightarrow \text{Aut}(\pi_1(\Sigma, x_0)) = \text{Aut}(F_{2g})$$

$$f \leftrightarrow f_*$$

Can approximate $\Theta$ by send $f \mapsto f_*^k$

where $f_*^k: F/F_k \to F/F_k$ where $\{F_k\}$ is a characteristic series that approximates $F$. 
Example: \[ F_2 = [F, F] = \text{commutator subgp} \]

**Def:** The Torelli group of \( \Sigma \), \( \Gamma(\Sigma) \), is the subgroup of \( \text{Mod}(\Sigma) \) consisting of diffeomorphisms that induce the identity on \( H_1(\Sigma) = F/F_2 \).

\[ \Gamma(\Sigma) = \ker(\tau_2: \text{Mod}(\Sigma) \to \text{Aut}(H_1(\Sigma))) \]
Thm (Powell, Birman, Johnson): \( I(\Sigma) \) generated by Dehn twists on bounding pairs of curves, \( D_\alpha \cdot D_{\beta}^{-1} \).

\[ \alpha, \beta \text{ disjoint homologous non-separating curves} \]

Thm (Johnson): \( I(\Sigma) \) is finitely generated for \( g \geq 3 \).

Thm (Mess): \( I(\Sigma) \) is an infinitely generated free group when \( g = 2 \).
We wish to investigate the homology and cohomology of certain special subgroups of $\mathcal{I}(\Sigma)$ using $p$-invariants of 3-manifolds.
Def: Let $G$ be a group. Then the lower-central series of $G$ is defined as

- $G_2 := [G, G] = \text{ commutator subgroup of } G$.
- for $k \geq 3$, $G_{k+1} := [G, G_k]$.

Def: The $k^{th}$ generalized Johnson subgroup of $\Sigma$, $J_k = J_k(\Sigma)$ is the subgroup of $\text{Mod}(\Sigma)$ that induce the identity on $F/F_k$. 
• Note that $J_2(\Sigma) = I(\Sigma)$.

**Thm (Johnson):** $J_3(\Sigma) =$ subgroup generated by Dehn twists on separating simple closed curves on $\Sigma$.

$[ J_3(\Sigma),$ denoted $K_g,$ is often called the $]$

$[\text{Johnson subgroup}.]$
**Question (Morita):** Is $H_1(K_g; \mathbb{Z})$ finitely generated for $g \geq 3$? [Note: $\chi_2 = I_2 \text{ finitely gen}$.]

**Note:** Still unknown if $K_g$ (or more generally $J_k(\Sigma), k \geq 3$) is infinitely generated for $g \geq 3$!
Construction of Invariants

Let $H \leq F=\pi_1(\Sigma)$, a characteristic subgroup, and $J(H)$ = subgroup of $\text{Mod}(\Sigma)$ consisting of diffeos of $\Sigma$ that induce the identity on $F/H$.

e.g. $H \leq F_k \Rightarrow J(H) \leq k^{th}$ Johnson subgroup.
Given $f \in J(H) \rightarrow$ closed 3-manifold $N_f$ by

1. $M_f = \text{mapping torus of } f$
   
   $$= \Sigma \times I \Big/ (x, 1) \sim (f(x), 1) \quad \forall x \in \Sigma$$

2. $N_f = M_f \Big/ (y, t) \sim (y, s)$
   for $t, s \in I$ and $y \in \Sigma$

$\alpha$ identified to a point in $N_f$. 
\[ \Sigma = \] 

Then \[ \pi_1(M_f) = \langle x_1, \ldots, x_{2g}, t \mid tx_i t^{-1} = f(x_i) \rangle \]

\[ \downarrow \text{kill } [\alpha] = t \]

\[ \pi_1(N_f) = \langle x_1, \ldots, x_{2g} \mid x_i = f(x_i) \rangle \]

but since \( f(x_i) \equiv x_i \mod H \) \implies \[ \pi_1(N_f)_H \cong F/H \]

Thus, we have \[ \Psi_H : \pi_1(N_f) \to F/H. \]

\( \uparrow \) independent of \( f \)
Def: $\rho_H(f) = \rho^{(2)}(N_f, \Psi_H: \pi_1(N_H) \to F/H) \in \mathbb{R}$, the Cheeger Gromov $L^{(2)}$-\(\rho\) invariant associated to \((N_f, \Psi_H)\).

Note: $\rho^{(2)}(M, \Psi)$ is an invariant associated to any 3-mfld $M$ and $\Psi: \pi_1(M) \to \Gamma$ (group)

Note: $L^{(2)}$-\(\rho\) invariants have been useful recently in the study of 3-manifolds and knot and link concordance (Cochran, H, Leidy, Kim, Cha, Orr, Teichner, Horn, Heck)
Lemma. For each \((M^3, \psi)\) there exists \((W^4, \phi)\) s.t.

\[
\begin{align*}
\pi_1 \partial W & = \pi_1 M & \xrightarrow{\psi} & \Gamma \\
\downarrow & & \downarrow & \\
\pi_1 W & \xrightarrow{\phi} \Lambda
\end{align*}
\]

Thm (Ramachandran)

\[\rho^{(2)}(M, \Gamma) = \left( L^2\text{-signature of } \Gamma\text{-equivariant} \right) \]

\[
\left( \text{intersection form on } H_2(W, \mathbb{Z} \Lambda) \right) - \left( \text{signature of } W \right), \epsilon \in \mathbb{R}
\]
• If we have finite unitary repr. \( \psi : F/H \rightarrow U(n) \) then can define

\[
\rho^\psi_H(f) := \rho(N_f, \pi_1(N_f) \rightarrow F/H \xrightarrow{\psi} U(n)) \in \mathbb{R}
\]

the Atiyah–Patodi–Singer (APS) \( \rho \)-invt.

• Thm(APS): If \( \psi : \pi_1 N_f \rightarrow U(n) \) extends to

\( \phi : \pi_1 W \rightarrow U(n) \) then

\[
\rho^\psi_H(f) = \text{sign}^\psi(W) - n \text{sign}(W) \in \mathbb{Z}
\]

\[\uparrow\]

twisted signature (signature of finite Hermitian matrix over \( \mathbb{C} \)).
Ex: $H = [F,F] + \pi_1(N_f) \xrightarrow{\psi_H} F/F,F) = \mathbb{R}^2 \xrightarrow{\phi} U(1) \\
\xrightarrow{x_i \mapsto w_i}$

where $w_i \in \mathbb{C}$ with $|w_i| = 1$.

When $w_i = w$, define $\rho_w(f) = \rho(N_f, \pi_1 N_f \rightarrow U(1))$.

We will show that $\rho_w(f)$ is often non-trivial.
Ex: For each $k \geq 2$, consider $H = [F_k, F_k]$. Since $[F_k, F_k] \leq F_{k+1} = [F, F_k]$, $J([F_k, F_k]) \leq J_{k+1}$, $g$ ($k+1$st Johnson subgroup).

Def: For $f \in J([F_k, F_k])$, define $\rho_{k+1}(f) = \rho^{(2)}(N_f, \prod N_f \to F/[F_k, F_k]) \in IR$.

Thm (Cochran-H-Horn): For $k \geq 2$, $\rho_{k+1} : H_1(J([F_k, F_k])); \mathbb{Z} \to IR$ is a homomorphism.
Question: What is the image of \( \overline{P_{k+1}} \)? If infinitely generated, then \( H_1(J([F_k, F_k])_j \mathbb{Q}) \) is infinitely generated.

For an arbitrary characteristic subgroup \( H \leq F_j \),

\[ \rho_H : J(H) \longrightarrow R \]

is a function but not necessarily a homomorphism!
Bounded Cohomology

Let $\rho_H^\psi : J(H) \to \mathbb{R}$ be one of the $\rho$-invt\^{s} as above (associated to a finite or infinite unitary representation).

- $\rho_H^\psi : J(H) \to \mathbb{R}$ is a 1-cochain in group cohomology. Its coboundary

$$\delta_H^\psi := \delta \rho_H^\psi : J(H) \times J(H) \to \mathbb{R}$$

is the 2-cocycle:

$$\delta_H^\psi (f,g) := \delta \rho_H^\psi (f, g) = \rho_H^\psi (fg) - \rho_H^\psi (f) - \rho_H^\psi (g).$$
Def: A function $\Psi: G \to \mathbb{R}$ is called a quasi-homomorphism if there exists a constant $D(\Psi)$ such that for all $x, y \in G$, \[ |\Psi(xy) - \Psi(x) - \Psi(y)| \leq D(\Psi) \]

$QH(G) := \{\text{quasi-homomorphisms on } G\}$ \(\text{bounded functions}\)

For each $G$, there is an exact sequence:

$$0 \to H^1(G;\mathbb{R}) \to QH(G) \xrightarrow{\delta} H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R})$$
Proposition (Cochran-H-Horn): The higher $\rho$-in vs $\rho^t_H$ are quasi-homomorphisms and hence define elements $\Phi^t_H = S\rho^t_H \in \mathcal{H}_b^2(\mathcal{T}(H); \mathbb{R})$.

Note: $\Phi^t_H$ is not necessarily $0$ in $\mathcal{H}_b^2(\mathcal{T}(H); \mathbb{R})$ since $\rho^t_H$ may be an unbounded function.
Idea of Proof ($\rho^i_H$ is a quasi-homomorphism):

For simplicity, assume $\Sigma$ is closed.

Then for $f, h \in J(H)$, can construct a 4-mfld $W_{fh}$ by: glue $M_f \times I$ to $M_h \times I$ along $(\Sigma_g \times (0, \frac{1}{2})) \times \{1\} \subset M_f \times I + M_h \times I$.

\[ W_{fh} = \begin{array}{c}
M_f \\
\uparrow \Sigma_g \times (0, \frac{1}{2}) \times \{1\}
\end{array} \quad M_{fh} \]  

\[ \Psi_1 : \pi_1 M_f \rightarrow F/H \]

\[ \Psi_2 : \pi_1 M_{fg} \rightarrow \]

\[ \Psi_3 : \pi_1 M_g \rightarrow \]

Can show $\Pi \Psi_i$ extends $\pi_1 (W_{gh}) \rightarrow F/H$. 
One can show that

$$\Rightarrow \quad \left| \rho_H^\varphi(fh) - \rho_H^\varphi(f) - \rho_H^\varphi(h) \right| = \left| \sigma^{(2)}(W, F/H) - \sigma_0(W) \right|$$

$$\leq 2 \text{ genus } (\Sigma) = D(g)$$

[D(g) is independent of f and h.]
For $k \geq 1$, consider

$$
\rho_{w_k}(\mathfrak{f}) := \rho \left( N_f, \pi_1(N_f) \to U(1) \right) \xrightarrow{X_i} w_k = e^{2\pi i / 4^k}.
$$

**Theorem (Cochran-H-Horn)** \(\{\rho_{w_k}\}_{k=1}^{\infty}\) is a linearly independent subset of \(\Omega H(^{9}K_9)\) for \(g \geq 2\).
To prove this, we produce a formula for
\[ \rho_{w_k}(\left(D_{\alpha_m} \cdot D_{\beta_m}\right)^n) \]
for \( k \geq m \) and \( n \geq 0 \).
There is a long exact sequence:

\[ 0 \to H^1(\mathcal{O}_{K_g}; \mathbb{R}) \to \mathbb{Q}H(\mathcal{O}_{K_g}) \xrightarrow{\delta} H^2_b(\mathcal{O}_{K_g}; \mathbb{R}) \to \mathbb{R}^\infty \cong \{ \rho_{w_K} \} \]

What is \( \text{im}(\delta|_\{\rho_{w_K}\}) \), \( \text{ker}(\delta|_\{\rho_{w_K}\}) \)?

**Note:** If \( \text{ker}(\delta) = \mathbb{R}^\infty \Rightarrow \exists K^\infty \subset H_1(\mathcal{O}_{K_g}; \mathbb{R}) \) which answers Morita's question!
Other Computations of \( \rho^4 \)

Let \( D \hookrightarrow \Sigma \) be an embedding of

\[ D = D^2 - (n \text{-disks}) \]

\( \text{e.g.} \)

\[ D \times \{0\} \subset \Sigma = \mathcal{L}(D \times I) \]

\( P_n = \) pure braid group (framed) is the mapping class group of \( D \).
This embedding gives an embedding

\[ \Theta: P_n \hookrightarrow \text{Mod}(\Sigma). \]

**Proposition (Cochran-H-Horn):** Let \( \beta \in P_k \). The higher order \( p \)-invariants of \( \Theta(\beta) \) can be calculated in terms of the higher-order \( p \)-invariants of the zero framed surgery on \( \widehat{\beta} \).

\[ \beta = \begin{array}{c}
\text{Diagram}
\end{array} \quad \xrightarrow{\text{Closure}} \quad \begin{array}{c}
\text{Closure}
\end{array} = \widehat{\beta} \]
Idea of proof:

\[ M_f = \left( (\Sigma \times S^1) - \left( \begin{array}{c} \text{cylinder} \\ \text{cylinder} \end{array} \right) \right) \cup \left( \begin{array}{c} \text{cylinder} \\ \text{cylinder} \end{array} \right) \]

This gives a cobordism \( W \) between \( \Sigma \times S^1, M_f \), and \( \hat{M}_\hat{\beta} = 0 \)-surgery on \( \hat{\beta} \).

Signature defect of \( E \) gives the sum of \( \rho \)-invariants on the boundary of \( E \).