Filtrations of the Knot Concordance Group

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Colloquium

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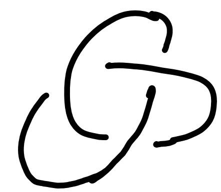
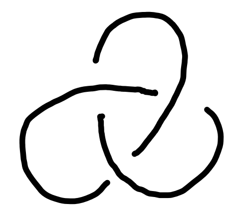
A **knot** is a smooth embedding

$$f: S^1 \longrightarrow S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} = \mathbb{R}^3 \cup \{\infty\}$$

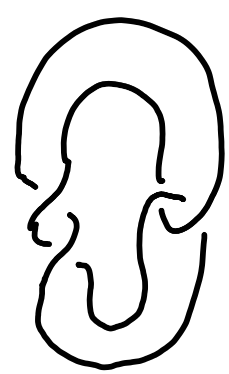
Ex: Left handed trefoil



Ex: Right handed trefoil



Ex: Figure-8 knot



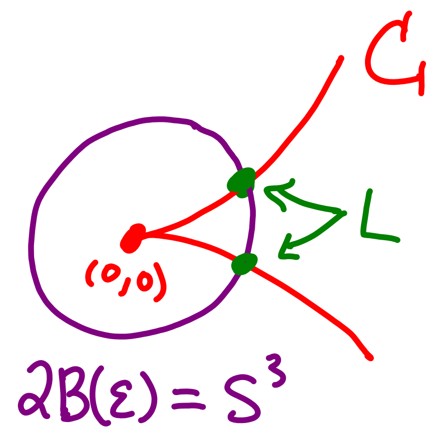
Ex: Consider the complex curve C defined by

$$z^2 - w^3 = 0.$$

It has a singularity at $(z,w) = (0,0)$.

The link of the singularity is

$L = C \cap \partial B(\varepsilon) = C \cap \{ |z|^2 + |w|^2 = \varepsilon \}.$



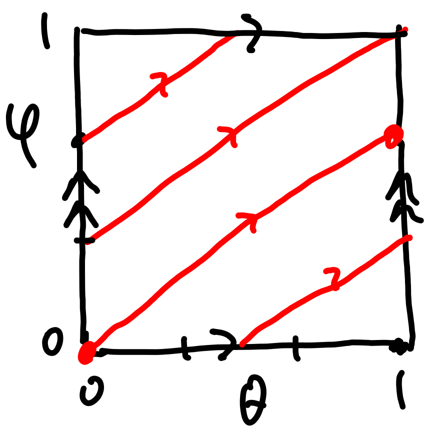
This is a 1-dimension (real) curve in S^3 (i.e. a knot or link)

several components

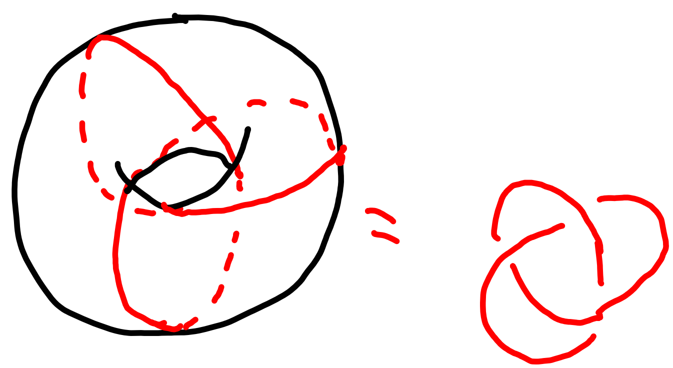
$$(z,w) \in L \iff |z|^2+|w|^2=\varepsilon \quad \text{and} \quad z^2=w^3$$

Writing $z = re^{2\pi i\theta}$ $\quad w = Re^{2\pi i \varphi}$ $\quad r, R \geq 0$

- $z^2=w^3 \Rightarrow r^2 = R^3 \Rightarrow z = R^{3/2}e^{2\pi i\theta}$
- $|z|^2+|w|^2=\varepsilon \Rightarrow R^3+R^2=\varepsilon \quad (\exists!\ R>0)$
- $z^2=w^3 \Longrightarrow 2\theta = 3\varphi \mod \mathbb{Z} \quad (\varphi = 2/3\,\theta + k/3 \text{ for some } k)$

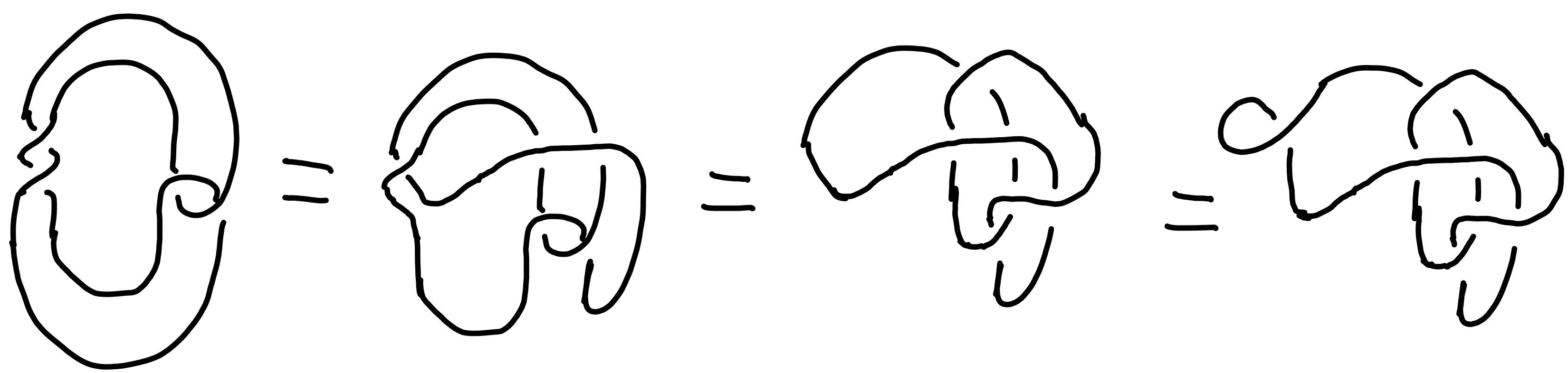


$$L = \{(R^{3/2}e^{4\pi i t}, Re^{6\pi i t})\} \subset \{(R^{3/2}e^{2\pi i\theta}, Re^{2\pi i\varphi})\}$$
$$\cong S^1 \times S^1 \subset S^3$$

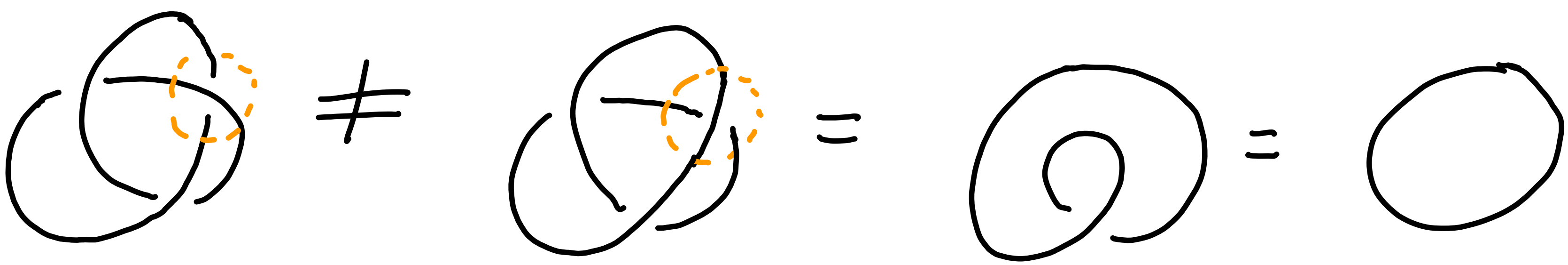


(2,3) torus knot

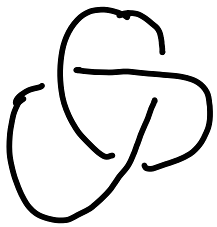
In knot theory, one usually studies knots up to <u>isotopy</u>: two knots are equivalent if you can deform one into the other in S^3 without it passing through itself.



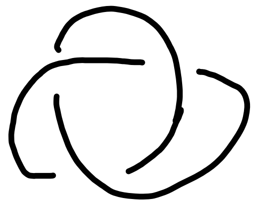
Cannot "change crossings"



Examples of Distinct knots up to isotopy



Left handed trefoil



Right handed trefoil

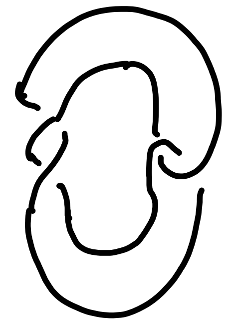
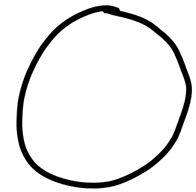
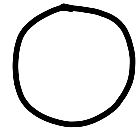
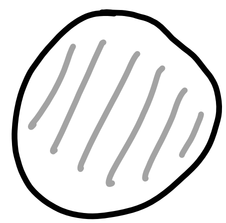


Figure 8

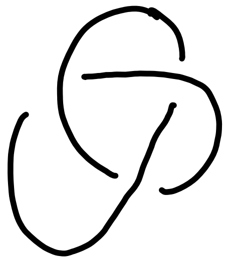
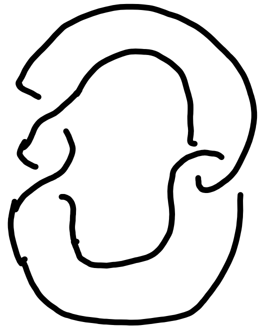
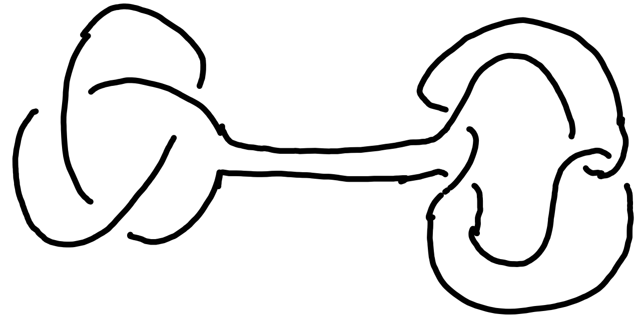


Trivial knot

The trivial knot  is the only knot that bounds an embedded disk in S^3:

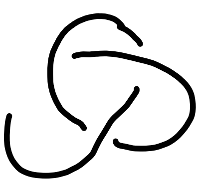
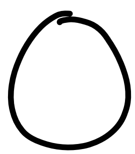
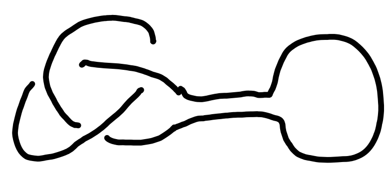
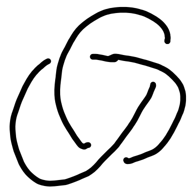


There is a binary operation on knots:

 $\#$  $:=$ 

K_1 K_2 $K_1 \# K_2$

connected sum of K_1 and K_2.

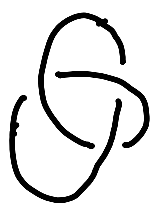
 $\#$  $=$  $=$ 

$$K \# \bigcirc = K$$

Thus $\mathcal{K} = (\{\text{knots}\}, \#)$ forms a monoid with unity $= \bigcirc$.

However $\mathcal{K}$ is <u>not</u> a group since it does not have inverses.

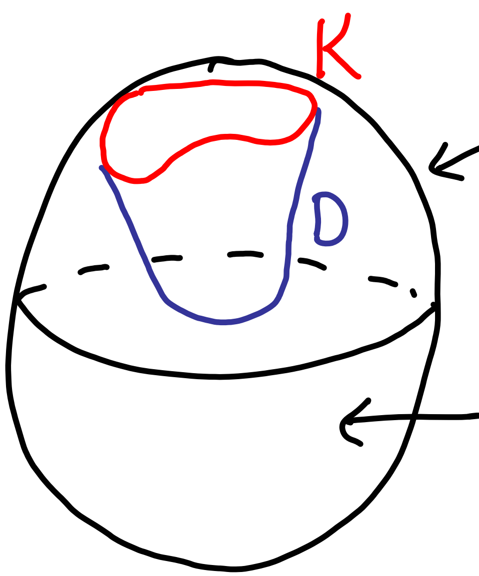
<u>Exercise</u>: There is no knot K such that

 $\# \, K = \bigcirc$.

To get a group structure, define a new equivalence relation called concordance.

We will think of "slice" knots as "trivial".

<u>Def</u>: A knot $K \subset S^3$ <u>slice</u> if $K = \partial D$ where D is a 2-dimensional disk (smoothly) embedded in B^4 = 4-dim. ball.

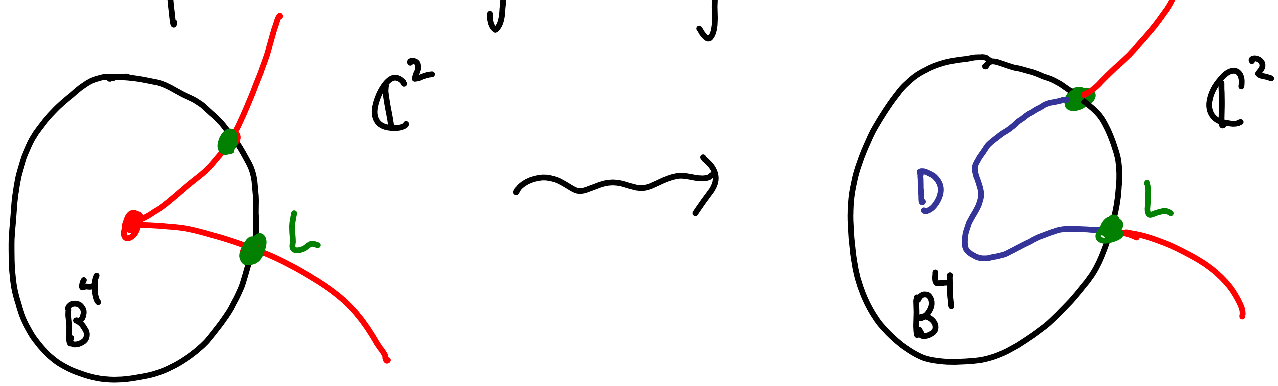


$$S^3 = \partial B^4 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

$$B^4 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 \leq 1\}$$

Fox-Milnor first studied the notion of a knot being slice to understand when one could "remove" a plane curve singularity.

- If the link of a singularity is slice, we can replace singularity with a smooth disk

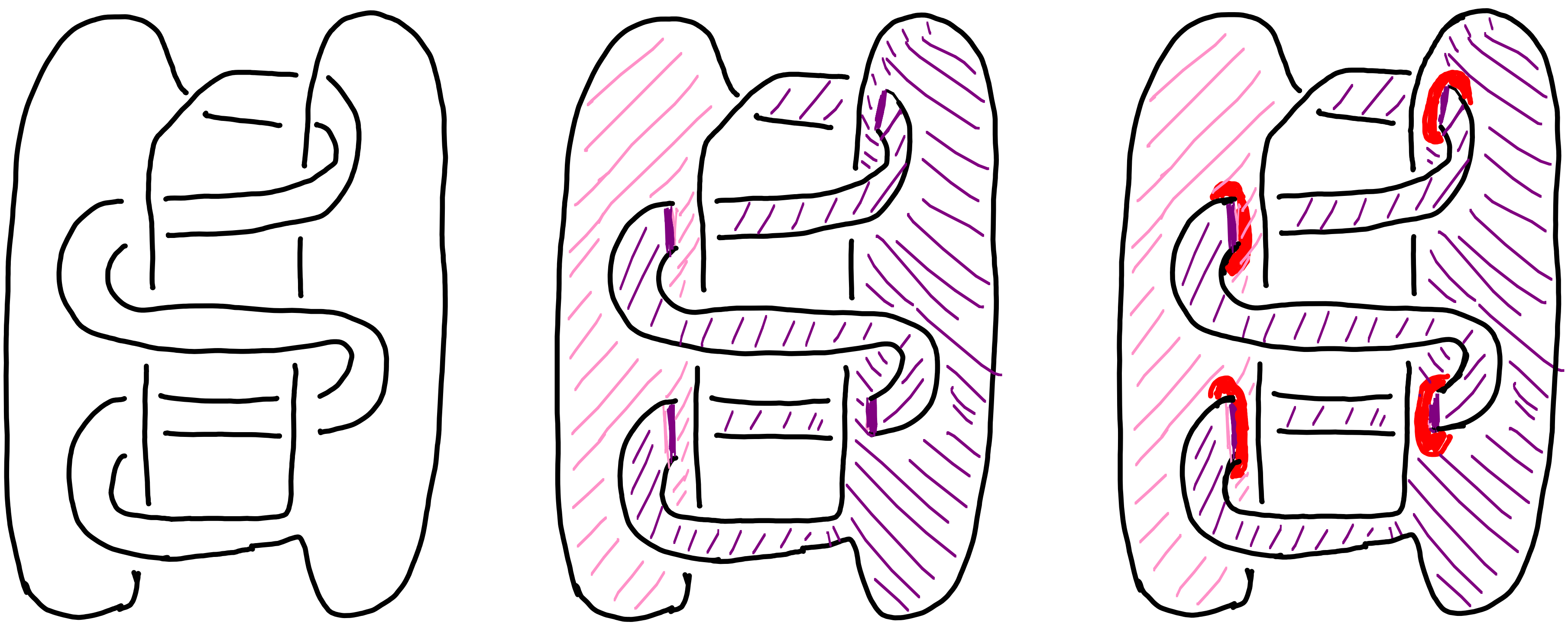


However, it turns out that the link of a singularity is never slice!

Ex: Any ribbon knot is slice.

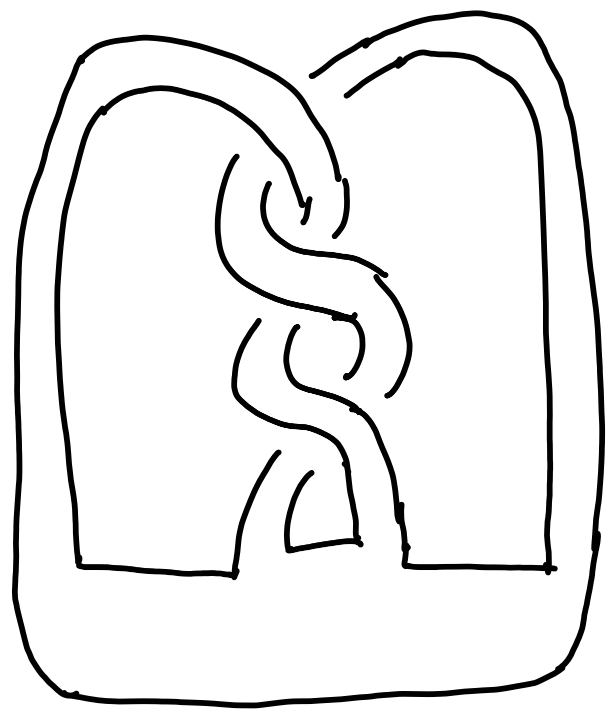
8_9

$8_9 = 2\left(\begin{matrix}\text{immersed}\\ \text{disc in } S^3\end{matrix}\right)$

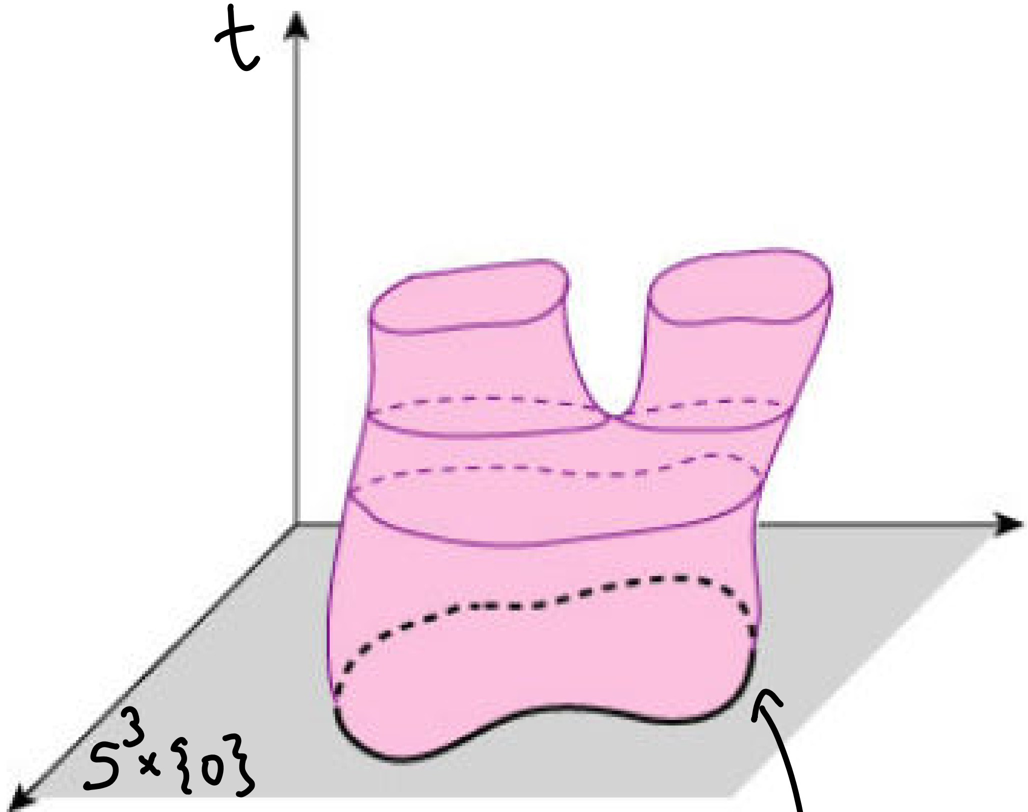


To obtain embedded in B^4, push interior of red disco into interior of B^4.

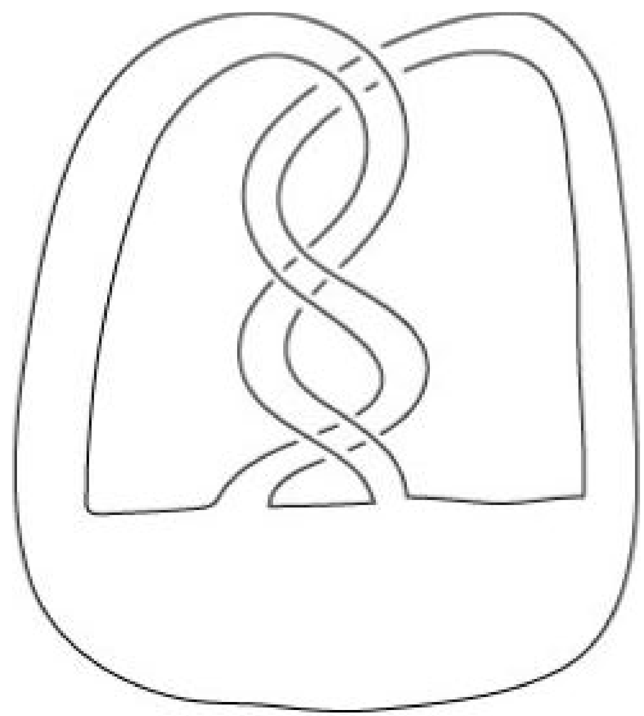
The 9_{46} knot is slice



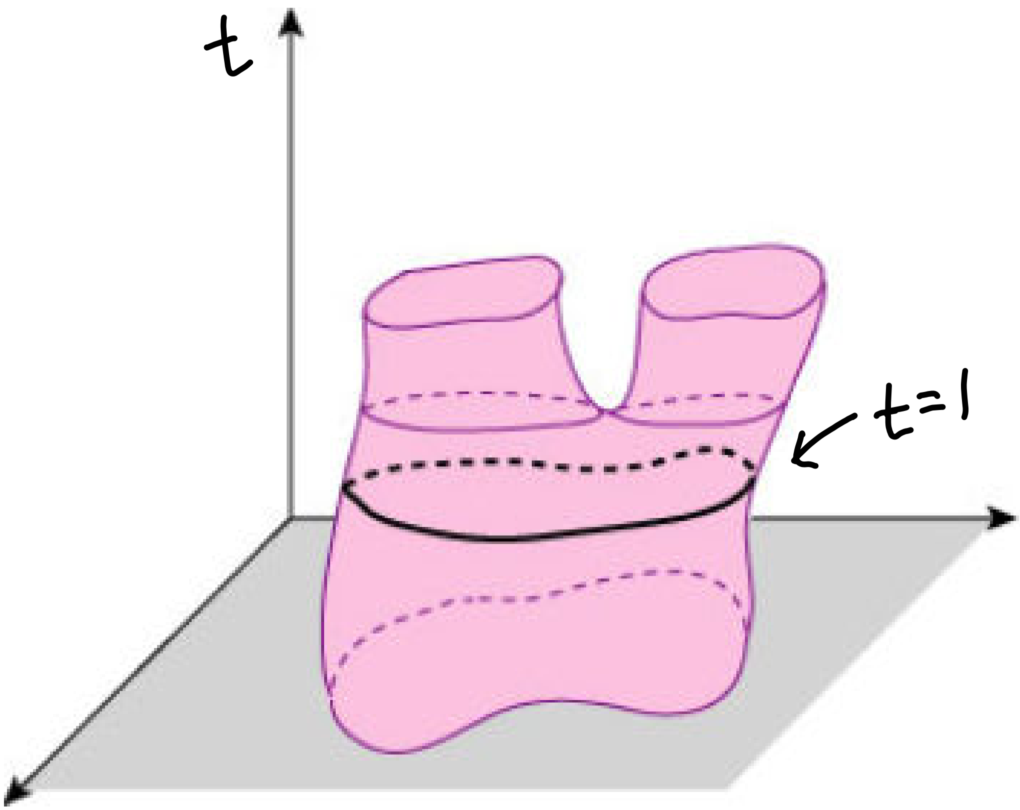
How to build a slice disc

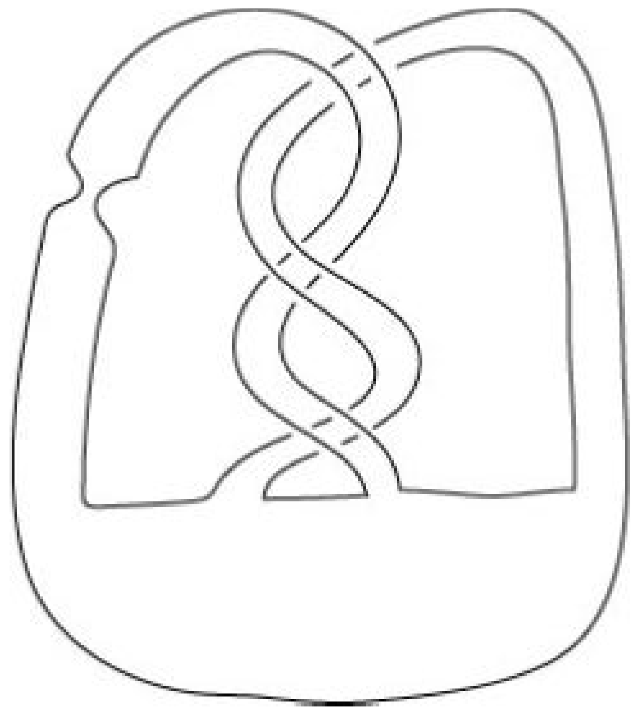


t=0

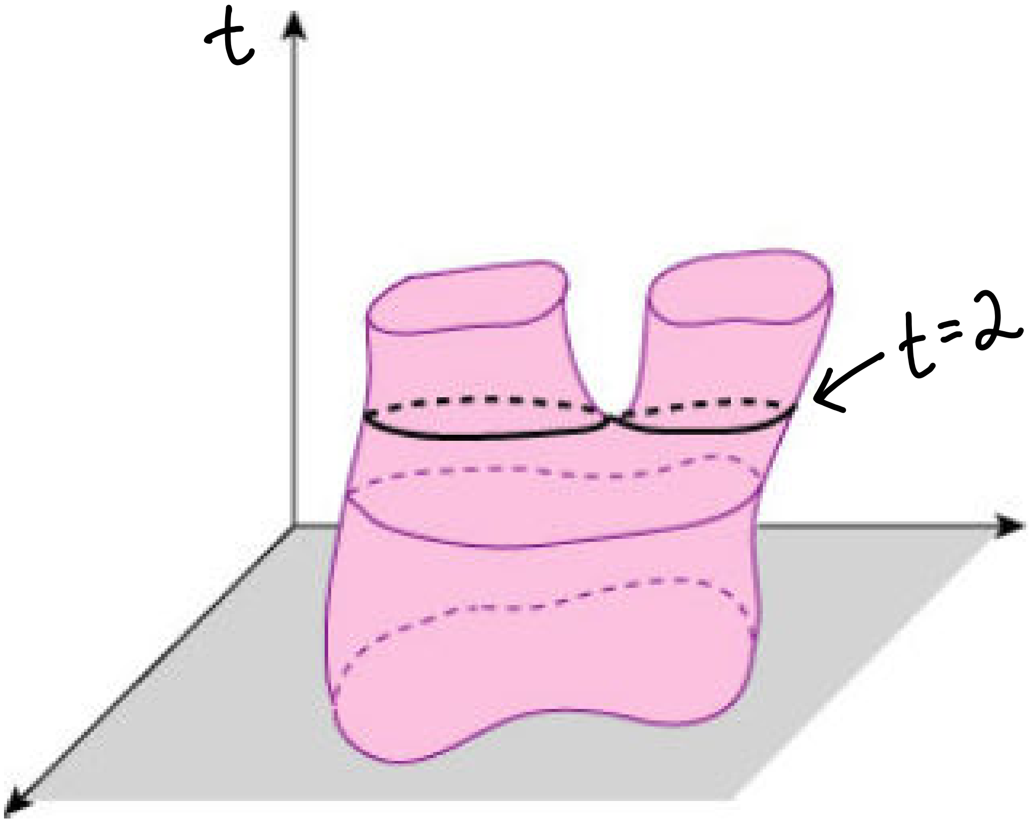


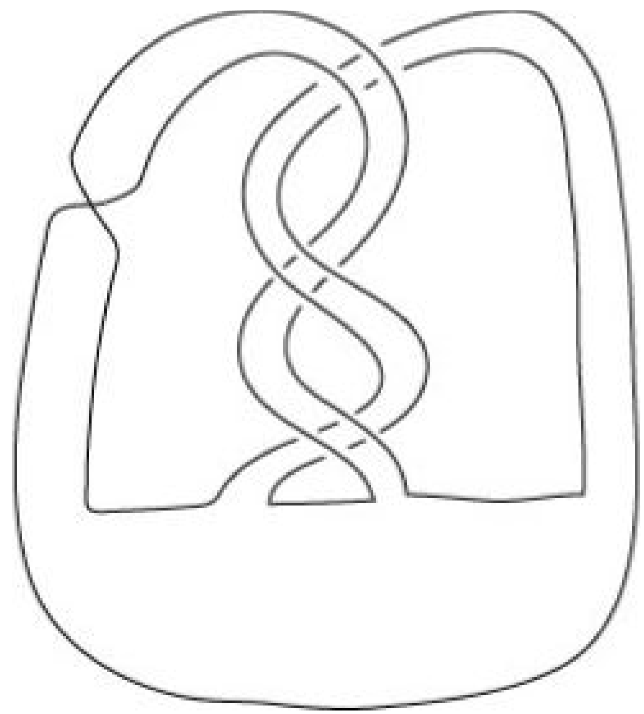
t=0



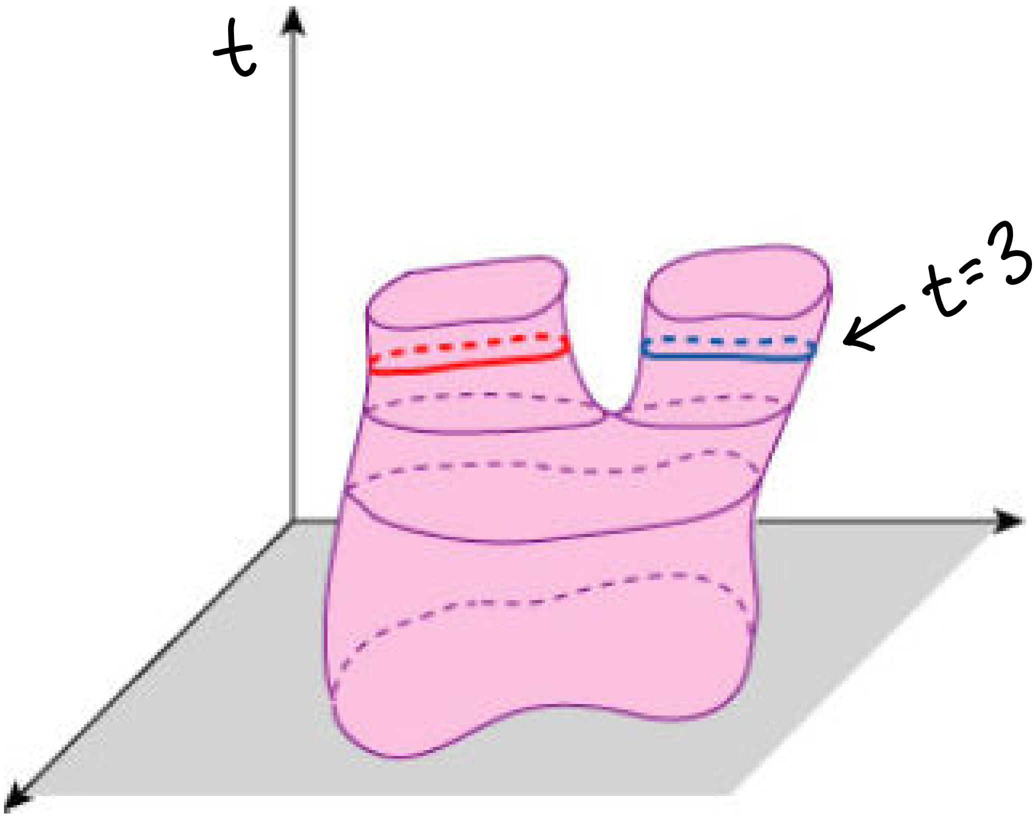


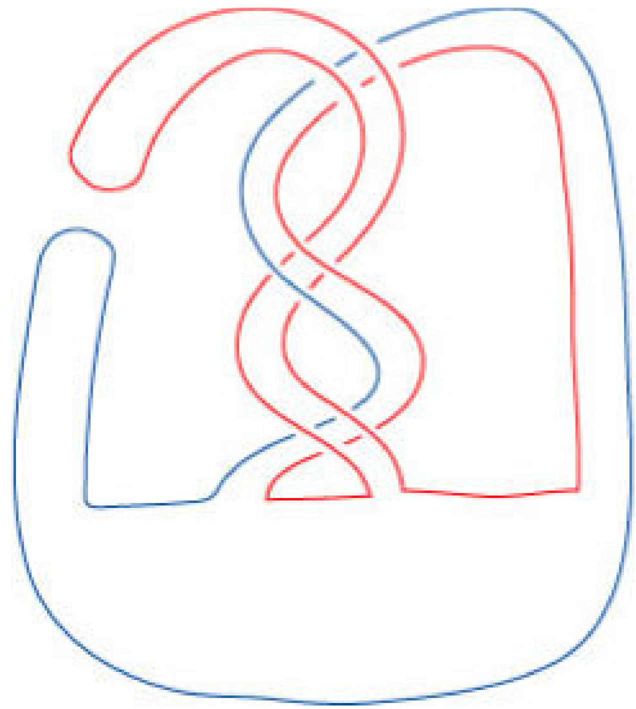
t = 1



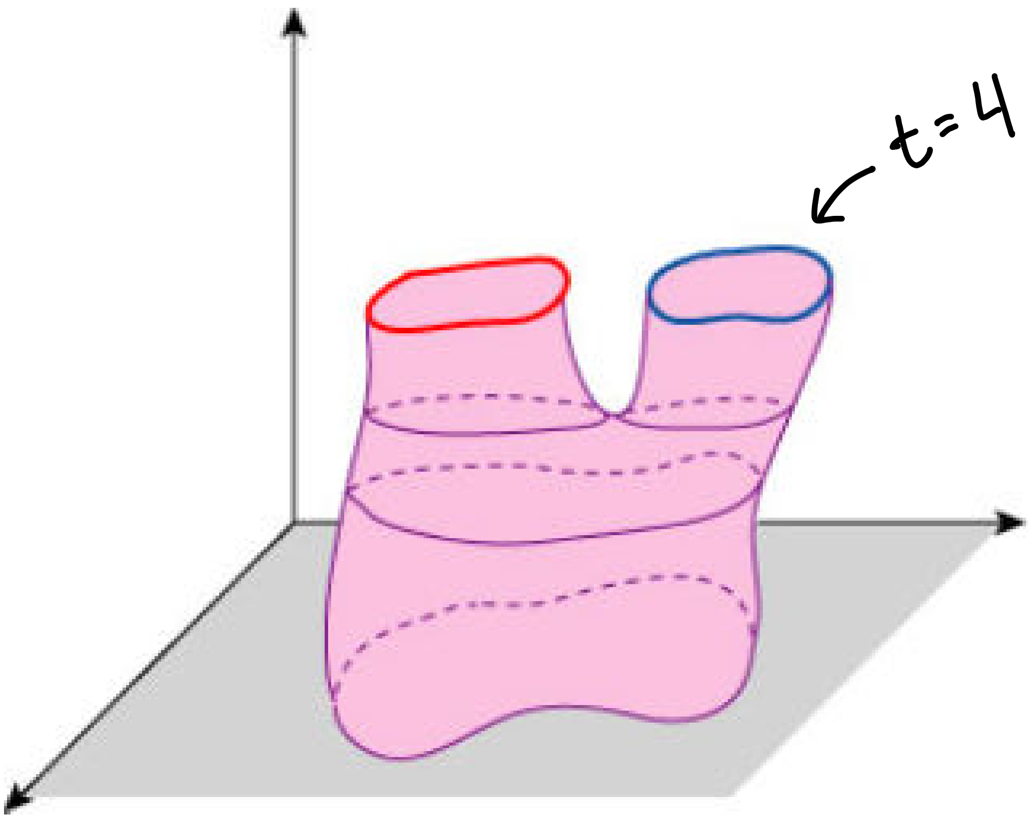


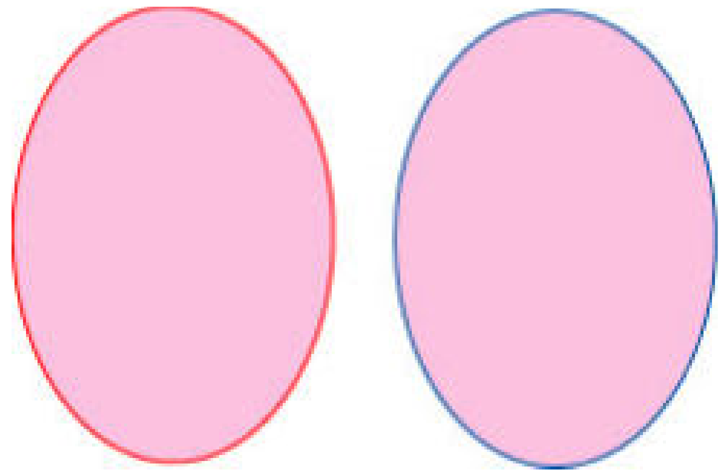
t= 2





t = 3



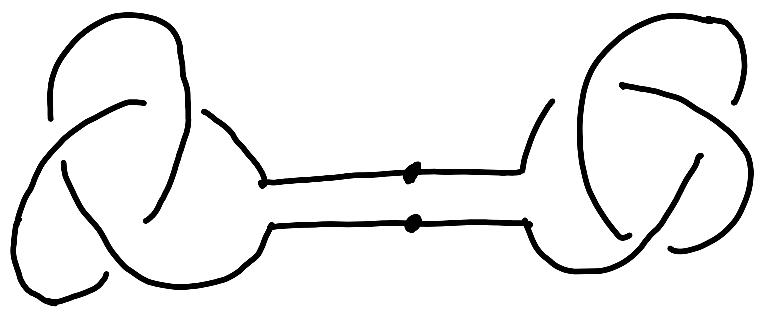


t=4

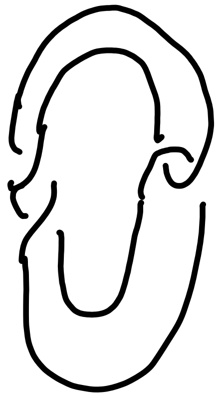
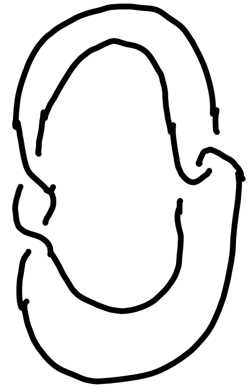
If K is any knot then $K \# \bar{K}$ is slice.
(ribbon)

[$\bar{K}$ = mirror image of K = reverse all crossings]

<u>Proof</u>: "Spin" K through $\mathbb{R}^4_+$.



$K \quad \# \quad \bar{K}$

Ex: $4_1 :=$  $\underset{\text{Exercise}}{=}$  $=: \overline{4}_1$

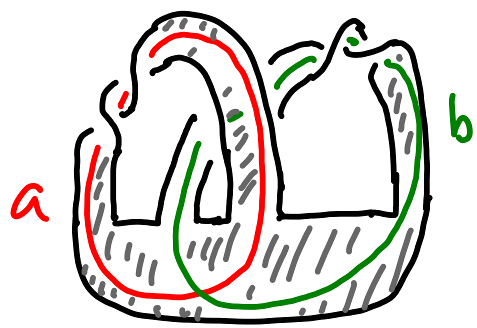
(figure-8)

Claim: 4_1 is not slice* but $4_1 = \overline{4}_1$ so

$$4_1 \# 4_1 = 4_1 \# \overline{4}_1 \text{ is slice.}$$

* Since $\mathrm{Arf}(4_1) \neq 0$

Levine-Tristram signature: Sliceness obstructions



$K = \partial(\text{surface})$

$\rightsquigarrow \quad V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ $lk(a, b^+)$

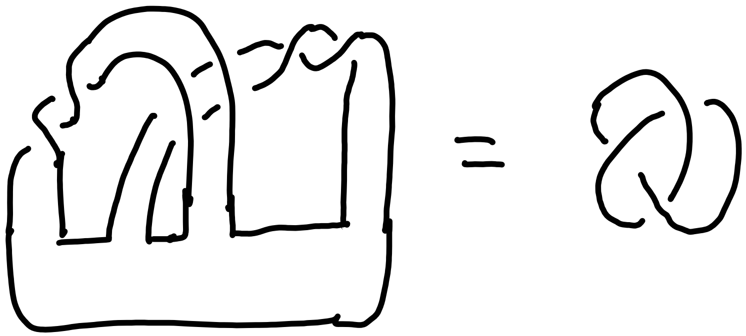
"linking" matrix

For $w \in \mathbb{C}$, define

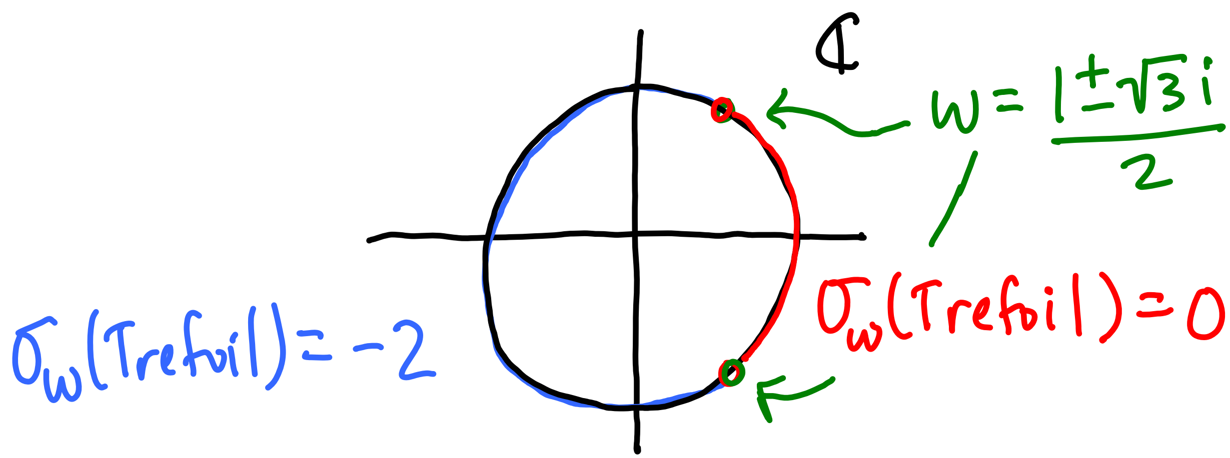
- $\sigma_w(K) := \text{signature}\left((1-w)V + (1-\bar{w})V^T\right)$
- $\rho_0(K) : \int_{S^1} \sigma_w(K)\, dw$

$$\boxed{\text{If } K \text{ is slice then } \rho_0(K) = 0.}$$

Ex: Trefoil is not Slice



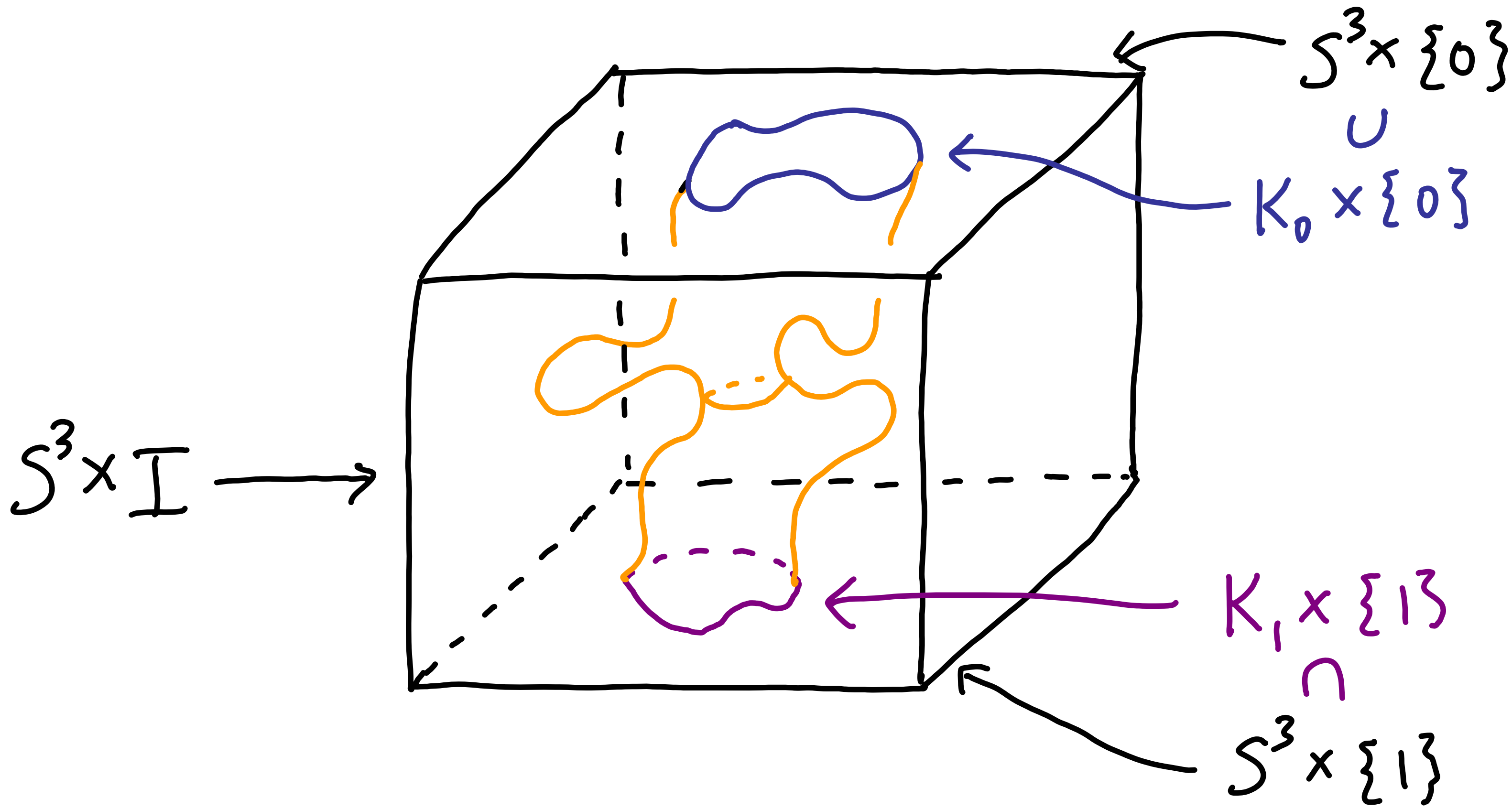
$$V_{\text{Trefoil}} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$



$$\rho_0(\text{Trefoil}) = -4/3 \neq 0$$

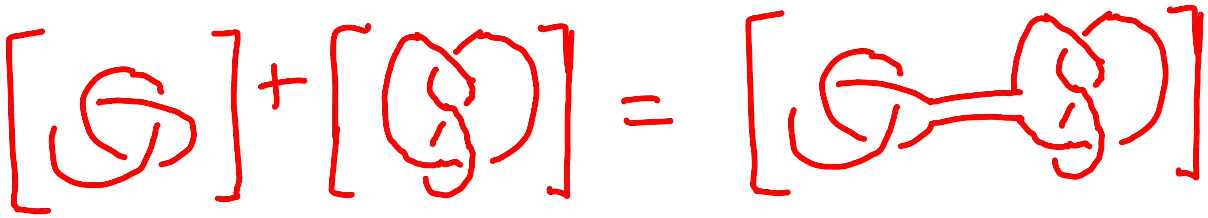
So Trefoil is <u>not</u> <u>slice</u>.

Def: Knots K_0 and K_1 are concordant if $K_0 \times \{0\}$ and $K_1 \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times I$.

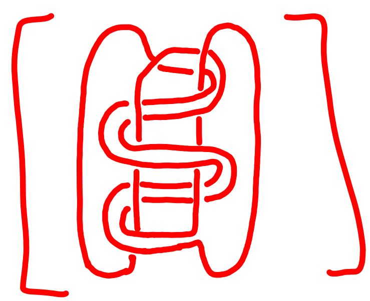


Defⁿ: $\mathcal{C} = \{\text{knots in } S^3\} / \text{concordance}$

- $\mathcal{C}$ is an abelian group under the operation connected sum of knots.

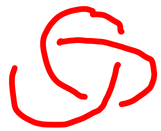
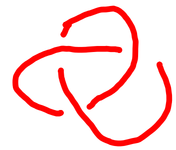


- $[K] = 0 \iff K$ is slice

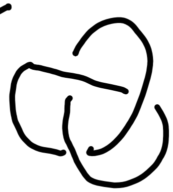
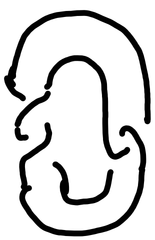


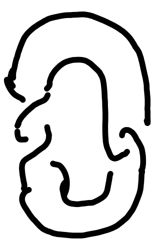
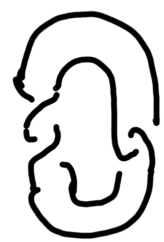
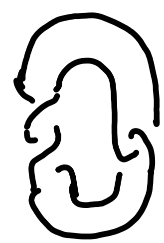
$= 0$

- The inverse of $[K]$ is $[\bar{K}]$ since $K \# \bar{K}$ is slice.

$-[$  $] = [$  $]$

Note:

- $[$  $] \neq 0$ since it is not slice
- $[$  $]$ is 2-torsion in $\mathcal{C}$ since

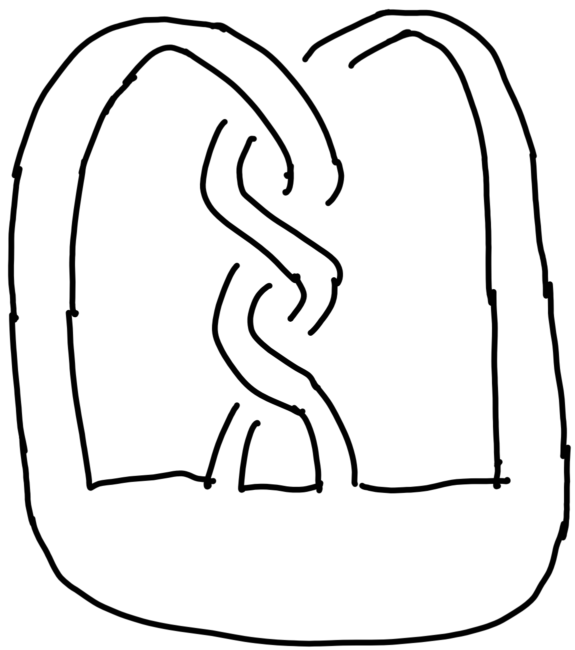
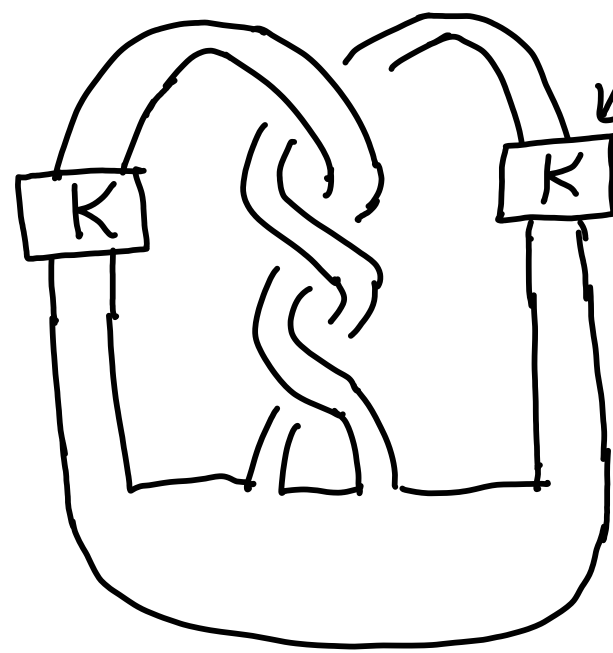
 is not slice but  $\#$  is slice.

- ~67, Milnor-Tristram used signatures to show $\mathcal{C}$ has infinite rank
- In late 60's Levine used invariants obtained from Seifert matrix (including signatures and Arf invariant) to define epimorphism

$$\mathcal{C} \xrightarrow{\pi} \text{Algebraic concordance group} \cong \mathbb{Z}^{\infty} \times \mathbb{Z}_2^{\infty} \times \mathbb{Z}_4^{\infty}$$

- $\ker \pi \overset{\text{def}}{=\!\!:}$ Algebraically slice knots

Ex:

$q_{46} =$  $\Longrightarrow$ 

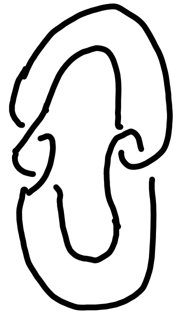
$= q_{46}(K)$

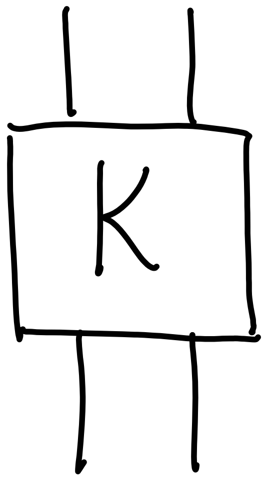
tie band in K

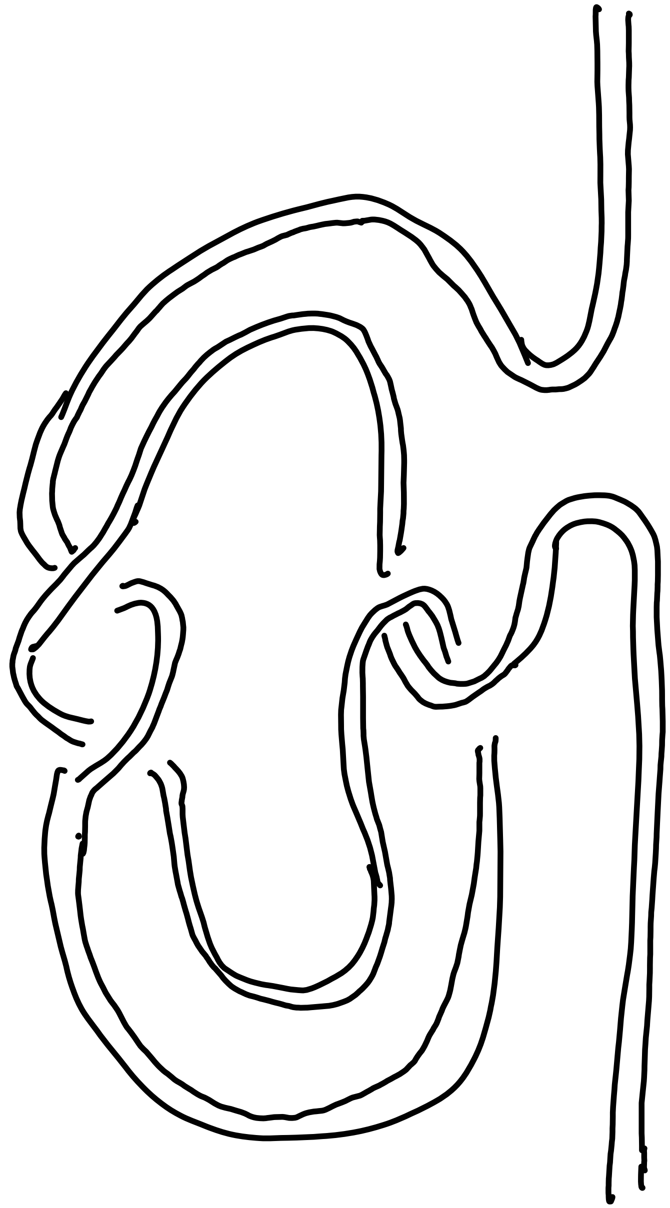
slice

algebraically slice

For K with certain non-vanishing signatures, Gilmer used Casson-Gordon invariants to show $q_{46}(K)$ is not slice.

If K = 



= 

“tie strings into knot K”

In 1997, Cochran-Orr-Teichner defined the (n)-solvable filtration of $\mathcal{C}$ ($n \in \mathbb{N}/2$)

$$0 = \left\{\begin{matrix}\text{slice}\\ \text{knots}\end{matrix}\right\} \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

- $\mathcal{F}_0$ = Arf invariant zero knots
- $\mathcal{F}_{0.5}$ = Algebraically slice knots
- $\mathcal{F}_{1.5} \subset$ knots with vanishing Casson-Gordon invariants.

If K is a knot,

$$M_K = \text{0-surgery on } K \quad (\text{closed 3-mfld})$$
$$= (S^3 - \text{nbhd}(K)) \cup \text{solid torus}$$

"closure" of $S^3 - \text{nbhd}(K)$.

If G = group $\Rightarrow$ derived series defined as

$$G^{(0)} := G$$
$$G^{(n)} := [G^{(n)}, G^{(n)}] \quad \text{where}$$

$$[A, B] = \{ aba^{-1}b^{-1} \mid a \in A, b \in B \}$$

Def A knot is (n)-solvable ($n \in \mathbb{N}$) if M_K (0-surgery on K) bounds a spin 4-mfld W (n-solution) s.t.

(1) $i_* : H_1(M_K) \xrightarrow{\cong} H_1(W)$

(2) $H_2(W)$ has a basis $\{f_i, g_i\}_{i=1}^{g}$ of embedded surfaces (w/ triv. normal bundle) all disjoint except $f_i \cdot g_i = 1$ (geometrically)

(3) $\pi_1(f_i), \pi_1(g_i) \subset \pi_1(W)^{(n)}$

- If $\pi_1(f_i) \subset \pi_1(W)^{(n+1)}$ as well then K is (n.5) solvable.

Note: If K is slice then

$M_K = \partial W$ where W is a spin 4-mfld s.t

(1) $i_*: H_1(M_K) \xrightarrow{\cong} H_1(W)$

(2) $H_2(W) = 0$

Hence slice knots are (n)-solvable $\forall n$.

Def: $K \in \mathcal{F}_n \Leftrightarrow K$ is (n)-solvable

Thm (n=0, ~67, Milnor-Tristram; n=1, ~81, Jiang; n=2, ~00, Cochran-Orr-Teichner)

For $n=0,1,2$, $\mathcal{F}_n/\mathcal{F}_{n.5}$ contains a $\mathbb{Z}^\infty$.

Thm (Livingston) $\mathcal{F}_1/\mathcal{F}_{1.5}$ contains a $\mathbb{Z}_2^\infty$.

Thm (Cochran Teichner ~02) For each $n \geq 0$,

rank $\mathcal{F}_n/\mathcal{F}_{n.5} \geq 1$.

Other work done at $\mathcal{F}_1/\mathcal{F}_{1.5}$ level by S. Friedl, T. Kim and P. Gilmer.

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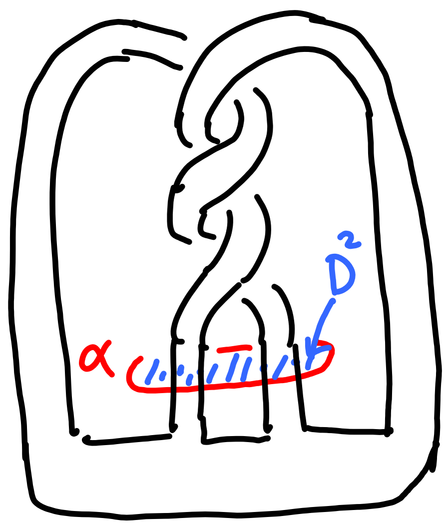
Thm (Cochran-H-Leidy, 06): For each $n \geq 0$, $\mathcal{F}_n / \mathcal{F}_{n.5}$ has infinite rank.

* In fact we can show our examples are linearly independent of Cochran-Teichner examples that give a $\mathbb{Z}$ in $\mathcal{F}_n / \mathcal{F}_{n.5}$.

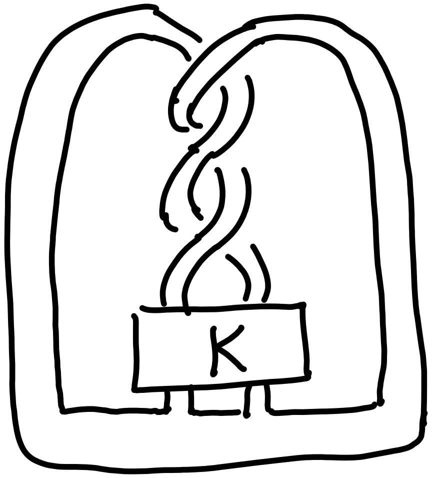
For this talk, I want to talk about special $\mathbb{Z}^\infty$ subgroups of $\mathcal{F}_n / \mathcal{F}_{n.5}$ associated to sequences of prime polynomials.

Iterated Doubling: How to create (n)-solvable knots.

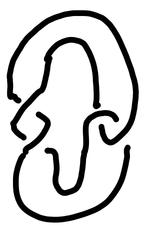
Let R be a slice knot, α a curve in $\pi_1(S^3-R)^{(1)}$ s.t. $\alpha = \partial D^2$ where D^2 = disk in S^3.

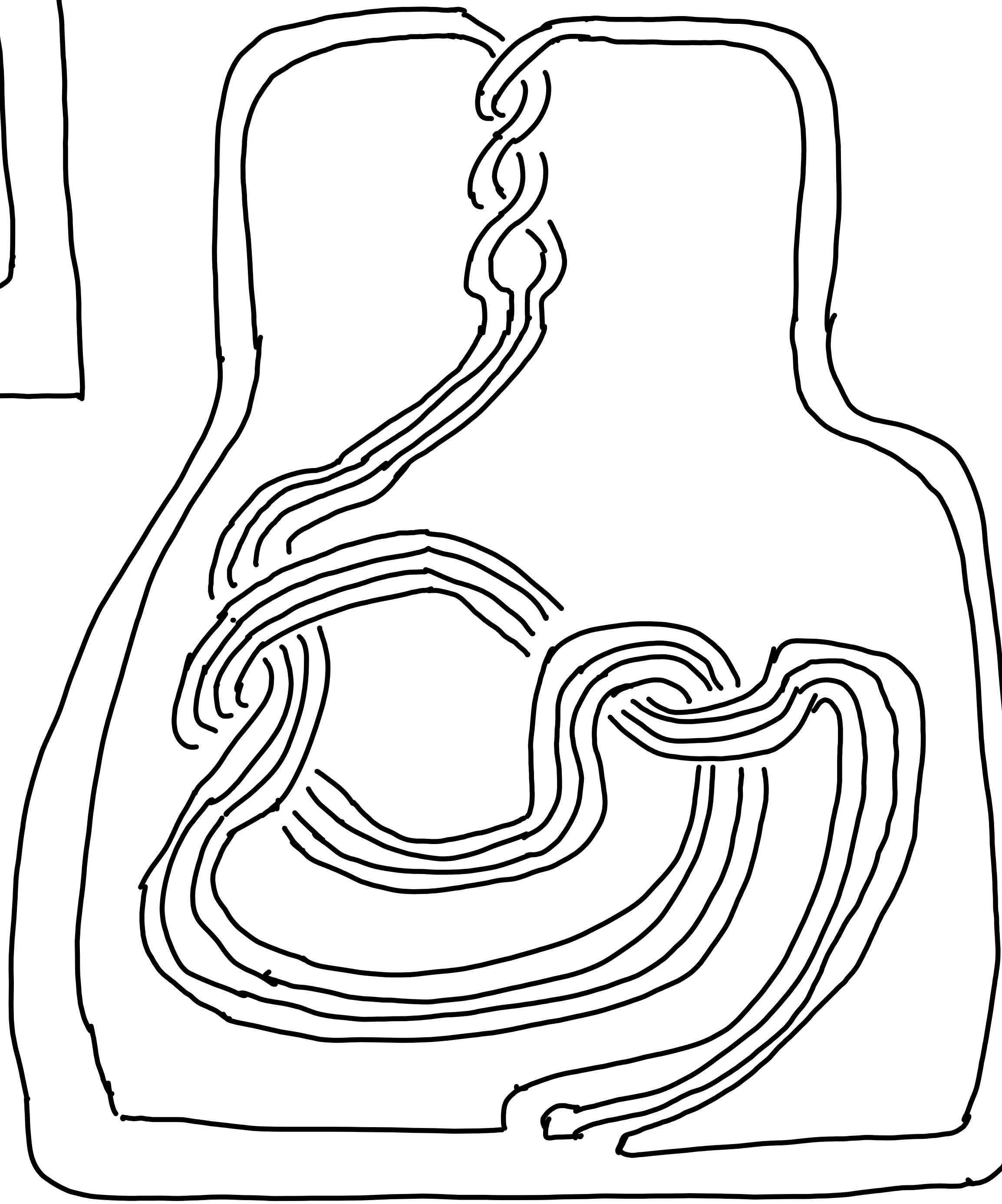
e.g. $R =$  (slice knot)

Tie strings that intersect D^2 into the knot $K \rightsquigarrow D_{(R,\alpha)}(K)$.

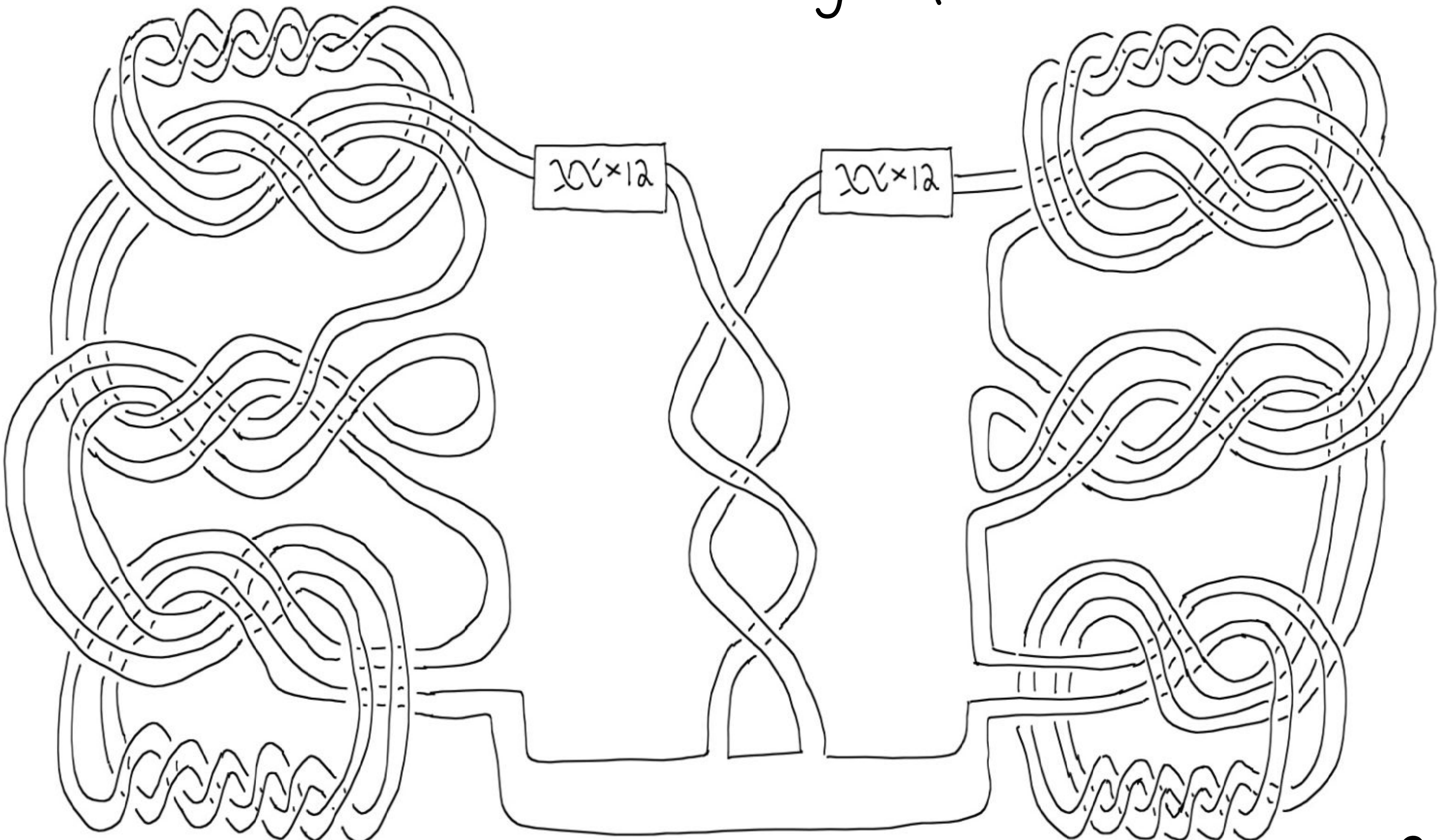
$D_{(R,\alpha)}(K) =$ 

Ex:

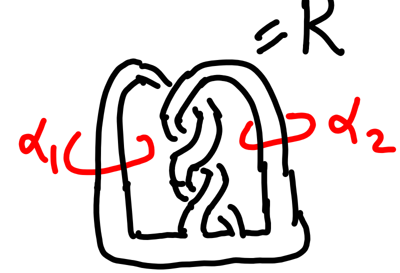
$D_{(R,\alpha)}\Big($  $\Big) =$



We can iterate the doubling operators.



example of $D_{(R,\alpha_1,\alpha_2)}(D_{(R,\alpha_1,\alpha_2)}(\text{trefoil}))$



Fact: If K is slice then $D_{(R,\alpha)}(K)$ is slice.

Thus we have functions (not homomorphisms)

$$\mathcal{C} \xrightarrow{D_{(R_1,\alpha_1)}} \mathcal{C} \xrightarrow{D_{(R_2,\alpha_2)}} \mathcal{C} \xrightarrow{D_{(R_3,\alpha_3)}} \mathcal{C} \longrightarrow \cdots$$

Q. Is $D_{(R,\alpha)}$ injective?

If so (for any (R,α)) then the knot concordance has a "fractal" structure.

Proposition: If $K \in \mathcal{F}_n$ then $D_{(R,\alpha)}(K) \in \mathcal{F}_{n+1}$.

Let K be any knot with $\mathrm{Arf}(K) = 0 \Rightarrow$
$K = (0)$ solvable $(K \in \mathcal{F}_0)$.

$\rightsquigarrow D_{(R_n,\alpha_n)}(\cdots(D_{(R_2,\alpha_2)}(D_{(R_1,\alpha_1)}(K)) \in \mathcal{F}_n$

However, in general, it is difficult to tell if such a knot is $(n+1)$-solvable.

To do this we use Cheeger-Gromov L^2 ρ-invts (L^2-signature defects).

Refine (n)-solvable filtration

For each sequence $\mathcal{P} = \{p_1(t), \dots, p_n(t)\}$, we associate a subgroup $\mathcal{F}_n^{\mathcal{P}}$ s.t.

$$\mathcal{F}_{n+1} \subset \mathcal{F}_{n+1}^{\mathcal{P}} \subset \mathcal{F}_n .$$

We show that for each $\mathcal{P}$, there is a $\mathbb{Z}^\infty$ subgroup of $\mathcal{F}_n/\mathcal{F}_{n+1}$ that survives in $\mathcal{F}_n/\mathcal{F}_{n+1}^{\mathcal{P}}$.

Example of Classical localization

Consider ring $\mathbb{Z}$ and a prime ideal $\langle p \rangle \subset \mathbb{Z}$.

If A is any module over $\mathbb{Z}$ (i.e. abelian gp)

we can localize A at the prime p:

Let $S = \{ n \in \mathbb{Z},\ n \neq 0 \text{ and } (p,n) = 1 \}$.

$$AS^{-1} = A \otimes_{\mathbb{Z}} \mathbb{Z}S^{-1} = A \otimes \{ m/n \mid (p,n) = 1 \}$$

Ex: $\boxed{p=2}$ • $A = \mathbb{Z}/3\mathbb{Z} \Rightarrow AS^{-1} = 0$ $\quad 1 \otimes 1 = 3 \otimes \frac{1}{3} = 0$

• $B = \mathbb{Z}/2\mathbb{Z} \Rightarrow BS^{-1} \cong \mathbb{Z}/2\mathbb{Z}$

• $\mathbb{Z}S^{-1} = \{ m/n \mid m \in \mathbb{Z},\ n = \text{odd} \}$

Non-commutative localization at a prime

Consider:

$Q = \{q_i\}$; $q_i \in \mathbb{Q}[t,t^{-1}]$, $q_i \neq \text{unit}, q_i \neq 0$

$A \triangleleft \Gamma$; A is abelian and Γ is a poly-torsion-free abelian group (i.e. $\mathbb{Z}\Gamma \subset$ field of fractions)

$\mathbb{Z} \longrightarrow A$; monomorphism

Prop: $S(Q) := \{ q_1(a_1) \cdots q_r(a_r) \mid q_i \in Q, a_i \in A, a_i \neq e \}$ is a right divisor set.

Hence we can invert the set $S = S(\mathbb{Q})$

$$\mathbb{Z}\Gamma \hookrightarrow \mathbb{Z}\Gamma \cdot S^{-1}$$

We would like to invert all "polynomials" that are "coprime" to a fixed $p \in \mathbb{Q}[t^{\pm 1}]$.

<u>Def:</u> We say $p(t)$ and $q(t)$ are <u>isogenous</u>, denoted $(\widetilde{p,q}) \neq 1$ if for some non-zero roots r_p of $p(t)$ and r_q of $q(t)$, and some $m, n \in \mathbb{Z} - \{0\}$, $r_p^n = r_q^m$.

Otherwise, we say they are <u>strongly coprime</u>, $(\widetilde{p,q}) = 1$.

Ex: $p(t) = t - 4 \quad q(t) = t^2 - 4.$

$(p,q) = 1$ since have no common root.

$(\widetilde{p,q}) \neq 1$ since $\underset{\substack{\parallel \\ \text{root of } q}}{2}^2 = \underset{\substack{\parallel \\ \text{root of } p}}{4}$

Ex: $p_k(t) = \big(kt - (k+1)\big)\big((k+1)t - k\big), \quad k \in \mathbb{Z}^+$

roots $\quad R_k = \left\{ \frac{k}{k+1}, \frac{k+1}{k} \right\}.$

$(\widetilde{p_k, p_\ell}) = 1$ when $k \neq \ell.$

Prop: $p, q \in \mathbb{Q}[t^{\pm 1}]$ (non-zero, nonunit)

$(\widetilde{p,q}) = 1 \iff$ for any f.g. free abelian group F, and nonzero $a, b \in F$ $p(a)$ is relatively prime to $p(b)$ in $\mathbb{Q}[F]$.

For $p(t)$ (non-unit, non-zero) define

$$S_p = S\left(\left\{ q \in \mathbb{Q}[t^{\pm 1}] \mid (\widetilde{p,q}) = 1 \right\}\right) \quad \left[\begin{matrix}\text{right divisor} \\ \text{set}\end{matrix}\right]$$

Def: If M is a (right) $\mathbb{Q}\Gamma$-module, then $MS_p^{-1} := M \otimes \mathbb{Q}\Gamma S_p^{-1}$ is <u>M localized at $p(t)$</u>.

Thm: For any $\overset{e}{\not{a}} \in A \triangleleft \Gamma$,

- $\mathbb{Q}\Gamma/p(a)\mathbb{Q}\Gamma \hookrightarrow \left(\mathbb{Q}\Gamma/p(a)\mathbb{Q}\Gamma\right) S_p^{-1}$

- $\left(\mathbb{Q}\Gamma/q(a)\mathbb{Q}\Gamma\right) S_p^{-1} = 0$ if $(p, \tilde{q}) = 1$.

We can use this localization to define a new "commutator series" assoccated to $P = \{p_1, \dots, p_n\}$

Recall $G^{(n+1)} = [G^{(n)}, G^{(n)}]$, $G^{(0)} = G$. [Derived Series]

where $[A,B] = \{aba^{-1}b^{-1} \mid a \in A, b \in B\}$

Let $P = \{p_1, \dots, p_n\}$. For each $1 \le k \le n$

define $G_P^{(1)} = \ker\left(G \longrightarrow \left(G/[G,G]\right)(\mathbb{Z}-\{0\})^{-1}\right)$ commutators + $\mathbb{Z}$-torsion

$$G_P^{(k+1)} = \ker\left(G_P^{(k)} \longrightarrow \frac{G_P^{(k)}}{[G_P^{(k)}, G_P^{(k)}]} \cdot S_{p_k}^{-1}\right)$$

module over $\mathbb{Q}[G/G_P^{(n)}]$.

Note: $G^{(k)} \subset G_P^{(k)}$

Def A knot is (n,P) solvable ($n\in\mathbb{N}$) if M_K (0-surgery on K) bounds a spin 4-mfld W (n-solution) s.t.

(1) $i_*: H_1(M_K) \xrightarrow{\cong} H_1(W)$

(2) $H_2(W)$ has a basis $\{f_i, g_i\}_{i=1}^{g}$ of embedded surfaces (w/ triv. normal bundle) all disjoint except $f_i \cdot g_i = 1$ (geometrically)

(3) $\pi_1(f_i), \pi_1(g_i) \subset \pi_1(W)_P^{(n)} \cap \pi(W)^{(n)}$

- $K \in \mathcal{F}_n^P \iff K$ is (n,P)-solvable

Theorem (Cochran-H-Leidy): For each $P = (p_1, \dots, p_n)$, there is a subgroup $\mathbb{Z}^\infty \cong Z_{\underline{P}} \subset \mathcal{F}_n / \mathcal{F}_{n+1}$.

such that if $Q = (q_1, \dots, q_n)$ is strongly coprime $\underline{P} = (p_1, \dots, p_n)$ [coordinate wise], then the image of $Z_{\underline{P}}$ under $\mathcal{F}_n / \mathcal{F}_{n+1} \xrightarrow{\pi_Q} \mathcal{F}_n / \mathcal{F}^Q_{n+1}$ is 0

and $\pi_P|_{Z_P} : Z_P \hookrightarrow \mathcal{F}_n / \mathcal{F}^P_{n+1}$ is a monomorphism.

Hence $Z_p \cap Z_q = 0$.

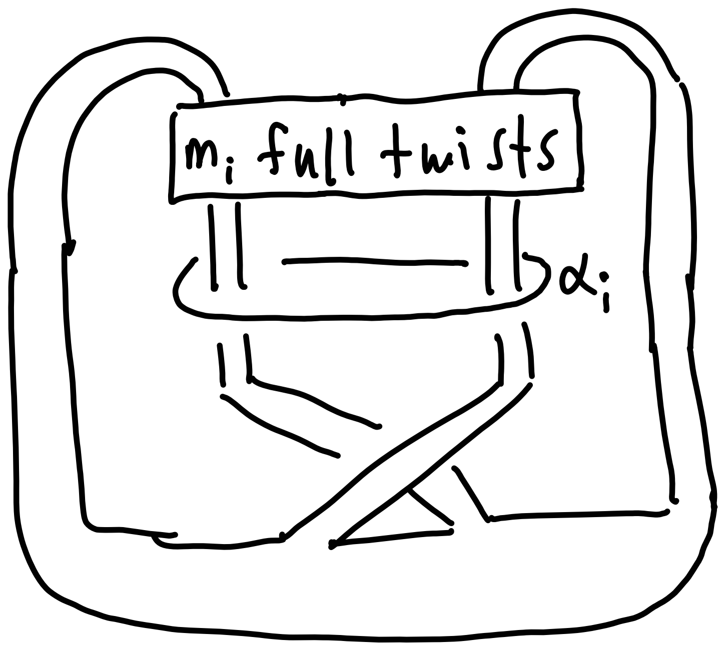
To construct examples:

Consider $p_k(t) = (kt - (1+k))((k+1)t - k)$ $k > 0$.

Recall $(\widetilde{p_k, p_\ell}) = 1$ for $k \neq \ell$.

Let $P(m_1, \dots, m_n) = (p_{m_1}, \dots, p_{m_n})$, $m_i \in \mathbb{Z}^+$.

Let $R_i =$



then

$\Delta_{R_i} = p_k(t)$

Let $\{K_j\}$ be a infinite set of knots with $\{\rho_0(K_j)\}_{j\in I}$ linearly independent (over $\mathbb{Z}$).

Define

$$\mathbb{K}^j(m_1,\dots,m_n) = D_{(R_n,\alpha_n)}(\cdots D_{(R_2,\alpha_2)}(D_{(R_1,\alpha_1)}(K_j))\cdots)$$

Then $$Z_{P(m_1,\dots,m_n)} = \{K^j_{(m_1,\dots,m_n)} \mid j\in I\} \cong \mathbb{Z}^\infty \subset \mathcal{F}_n/\mathcal{G}_{n+1}$$

and $Z_{P(m_1,\dots,m_n)} \cap Z_{P(s_1,\dots,s_n)}$ when $(m_1,\dots,m_n) \neq (s_1,\dots,s_n)$