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Link Concordance and Groups

by

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Abstract

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This work concerns the study of link concordance using groups, both extracting concordance data from group theoretic invariants and determining the properties of group structures on links modulo concordance. Milnor's invariants are one of the more fundamental link concordance invariants; they are thought of as higher order linking numbers and can be computed using both Massey products (due to Turaev and Porter) and higher order intersections (due to Cochran). In this thesis, we generalize Milnor's invariants to knots inside a closed, oriented 3-manifold M. We call this the Dwyer number of a knot and show methods to compute it for null-homologous knots inside a family of 3-manifolds with free fundamental group. We further show Dwyer number provides the weight of the first non-vanishing Massey product in the knot complement in the ambient manifold. Additionally, we prove the Dwyer number detects knots K in M bounding smoothly embedded disks in specific 4-manifolds with boundary M which are not concordant to the unknot. This result further motivates our definition of a new link concordance group using the knotification construction of Ozsvàth and Szabò. Finally, we give a proof that the string link concordance group modulo its pure braid subgroup is non-abelian.

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For my sister Malak.

here is the deepest secret nobody knows
(here is the root of the root and the bud of the bud
and the sky of the sky of a tree called life;which grows
higher than soul can hope or mind can hide)
and this is the wonder that's keeping the stars apart

i carry your heart(i carry it in my heart)

---

e.e. cummings

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1.1 Introduction for non-mathematicians

Imagine standing in a flat, empty, rectangular field. You can walk from any place to any place, and the fastest way to do it would be a straight line. If you had to, and you had a smartphone to remind yourself how to do it, you could even figure out exactly how fast you were going while walking from one place on the field to another.

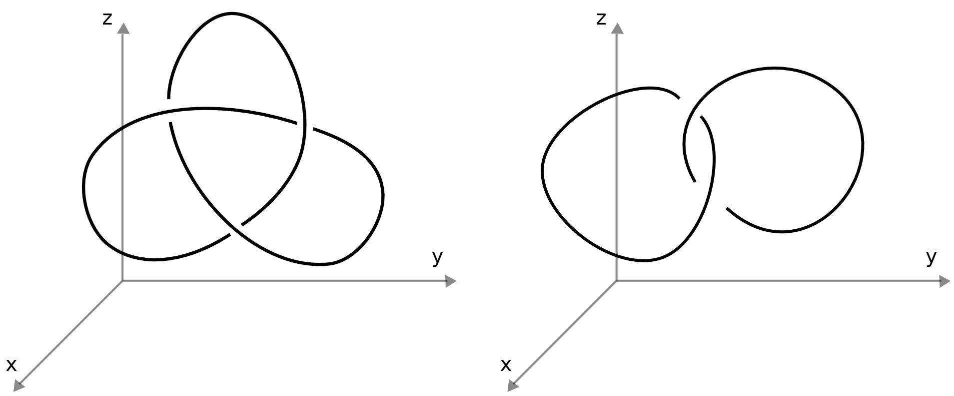
Now, imagine instead that you are standing in a maze. Without a bird's eye view, it would be very difficult to tell whether it was possible to walk from any point in the maze to any other, and what the quickest route would be. If you could understand more about the shape of the maze, even just a few hints, you would be better able to tell if you could walk from one point to another and whether you could do so in a reasonable amount of time.

The study of the shape of spaces is called topology. When we become concerned with measuring distances and angles, that is an example of geometry. Just like in the examples of an empty field and a maze, before you can understand how to move in a space or more generally use a space (geometry) you should understand its general features (topology). In fact, there is even evidence that our brains store information about our surroundings topologically [GPCI15].

Determining when two spaces have "the same shape" is difficult; this difficulty depends on the dimension of the spaces involved. Many properties can be proven for spaces of dimensions 5 and higher, while dimensions 1, 2, and 3 are can be understood well through other methods. 4-dimensional spaces can be thought of as the "bridge" between low-dimensional behavior and high-dimensional behavior and much about it is still unknown. Furthermore, we live in a (at least) 4-dimensional world: there are three spatial dimensions and one time dimension. Thus, understanding what

4-dimensional spaces are possible and how to identify them could prove helpful to better understanding the world around us.

It is perhaps a surprising fact that knot theory provides a useful tool for studying 3- and 4-dimensional spaces. A knot is a smooth, closed circle sitting in 3-dimensional Euclidean space, and an n-component link is a collection of n smooth, closed circles sitting in 3-dimensional space which do not intersect each other as in Figure 1.



(a) A knot.

(b) A 2-component link.

Figure 1: An example of a knot and link in 3-dimensional space.

A theorem of Lickorish and Wallace shows every 3-dimensional shape which is "nice" in some technical sense is obtained by a certain operation called surgery on a link; in this sense, studying knots and links can be viewed as studying something fundamental about 3-dimensional space. Moreover, we can algorithmically construct every 4-dimensional shape we could have a hope of doing geometry on using an operation called attaching 4-dimensional "handles" along a link [GS99]. In this way, we can organize relationships between 3- and 4-dimensional space using an algebraic structure called the knot concordance group and study it using tools from knot theory.

In this thesis, I develop a new tool called the Dwyer number to detect whether knots in a specific, more complicated family of 3-dimensional shapes are "the same" when considered up to a 4-dimensional relation called concordance. Furthermore, I analyze the properties of Dwyer number and use it to show there are infinitely many knots which are distinct up to this concordance relationship, but look as simple as possible when viewed in different, but related 4-dimensional shape. Additionally, I use more classical tools called Milnor's invariants (which the Dwyer number generalizes) in order to show that a different algebraic structure created from links is more complicated than previously thought.

1.2 Introduction for mathematicians

This thesis concerns low-dimensional topology; more specifically, this body of work is concerned with how algebraic structures associated to a topological space can classify the smooth topology of the space. Whether algebra can completely classify a space depends on dimension; while every n-manifold homotopy equivalent to the n-sphere S^n is homeomorphic to S^n due to work of [Fre82, Per03b, Per02, Per03a, Sma61], this homeomorphism cannot generally be upgraded to a diffeomorphism [FGMW10, Mil56]. In dimension 4, this problem is known as the smooth 4-dimensional Poincaré conjecture and remains one of the fundamental questions of low-dimensional topology. Many strange phenomena appear in the 4-dimensional world only; for example, $\mathbb{R}^n$ has exactly one smooth structure for $n \neq 4$ while $\mathbb{R}^4$ has infinitely many distinct smooth structures [GS99].

For this reason, this thesis is primarily about relationships between 3- and 4-dimensional topology. In particular, we use knot theory as a lens with which to approach this topic. Recall that a knot is a smooth (oriented) embedding of S^1 into S^3 and an n-component link is a smooth (oriented) embedding of n-disjoint copies of S^1 into S^3. A link can be associated to a 3-manifold by removing its normal bundle and gluing it back in with a specified framing (called Dehn surgery) and to a smooth 4-manifold by attaching handles; every 3-manifold and smooth 4-manifold arises in this way as outlined in [GS99]. Therefore knots and links provide a useful avenue to investigate such manifolds.

In order to use knots to understand smooth 4-manifold topology, we examine their equivalence classes up to a 4-dimensional relation called concordance. Two knots K_0 and K_1 are smoothly concordant if there is a smoothly embedded disjoint annulus in $S^3 \times I$ with boundary $K_0 \times \{0\}$ and $K_1 \times \{1\}$. A knot $K \subset S^3$ is slice if it bounds a smooth, properly embedded disk in B^4. This concordance perspective has useful implications for the study of 4-manifolds; for example, [Lev12] proves that showing the Whitehead double of the Borromean rings is not freely topologically slice would disprove the surgery conjecture for 4-manifolds with free fundamental group. Knots modulo concordance with the operation connected sum forms the knot concordance group $\mathcal{C}$ introduced by Fox and Milnor in [FM66].

The main goal of this work is to understand the interaction between 3- and 4-manifold topology using groups of knots and links modulo concordance. We examine the relationship between classical invariants derived from the quotients of an associated fundamental group by its lower central series and a modern construction from Heegaard Floer homology to better classify 3- and 4-manifold topology. Later, we extend the knot concordance group to a group of knots in connected sums of $S^2 \times S^1$. Finally, we show that a previous notion of concordance group of links called the string link concordance group is non-abelian even when its quotient is taken by the normal closure of the pure braid group, indicating that this group is more complex than previously thought.

1.3 Summary of results

Milnor's $\bar{\mu}$-invariants provide a way to classify the relationship between a link L in a homology 3-sphere M and the lower central series quotients of the fundamental group of the link complement; they can be contextualized as higher order linking numbers and have previously been generalized for a small class of homotopically nontrivial knots inside prime manifolds and Seifert fiber spaces [Hec11, Mil95]. $\bar{\mu}$-invariants are notoriously difficult to compute for even simple links; in practice, it is usually only possible to compute the first non-vanishing Milnor invariants. This subset of $\bar{\mu}$-invariants can be viewed as determining how deep the based homotopy classes of longitudes of L are in the lower central series of the link group $\pi_1(M \setminus \nu(L), *)$ where $\nu(L)$ is a regular neighborhood of L.

In order to capture similar higher order linking data contained in lower central series quotients for knots in arbitrary 3-manifolds, we defined a new concordance invariant which we call the Dwyer number and denote $D(K, \gamma)$ for a knot $K \subset M$ with M a closed, oriented 3-manifold and γ a specific simple closed curve in the same free homotopy class as K. $D(K)$ detects which elements of M can be represented by maps of special 2-complexes called half-gropes described in Section 3.2. $D(K)$ can be viewed as a generalization of the first non-vanishing Milnor's $\bar{\mu}$-invariant of a link. In particular, for $K \subset \#^i S^2 \times S^1$ with associated group $G = \pi_1(\#^i S^2 \times S^1 \setminus \nu(K), *)$, $D(K)$ detects how deep a based homotopy class of a longitude of K is in the lower central series G_q.

Definition 1.1. Let M be a oriented, closed 3-manifold and γ be a fixed smooth, embedded curve inside M. Let $[\gamma]$ be its corresponding free homotopy class. Let $K \subset M$ be a smooth knot with free homotopy class $[K] = [\gamma]$. Then the Dwyer number of K relative to γ is

$$D(K, \gamma) = \max \left\{ q \;\middle|\; \frac{\mathrm{H}_2(M \setminus \nu(K))}{\Phi_q(M \setminus \nu(K))} \cong \frac{\mathrm{H}_2(M \setminus \nu(\gamma))}{\Phi_q(M \setminus \nu(\gamma))} \right\}.$$

where $\Phi_q(X)$ denotes the subgroup of $H_2(X)$ consisting of element which can be represented by maps of special 2-complexes called class $q+1$ half gropes. A class

$q+1$ half-grope is constructed recursively out of layers of surfaces: the lowest stage (called the second stage for indexing reasons) is a an oriented surface. Exactly half of a symplectic basis for this surface themselves bound oriented surfaces, this process continues until we have q layers of surfaces.

This invariant is particularly simple for the case of $K \subset \#^l S^2 \times S^1$, which as we will see in Section 3.3.

Corollary 1.2. *Let $K \subset \#^l S^2 \times S^1$ be a null-homologous knot and γ be an unknot in $\#^l S^2 \times S^1$. In this case, denote $D(K,\gamma)$ by $D(K)$ and see that*

$$D(K) = max \; \{ \; q \;\; | \;\; \frac{H_q(\#^l S^2 \times S^1 \setminus K)}{\Phi_q(\#^l S^2 \times S^1 \setminus K} = 0 \; \}$$

Recall that for a knot K in S^3, K being slice in B^4 is equivalent to K being concordant to the unknot in $S^3 \times I$. In this thesis, we showed there is in an infinite family of knots in connected sums of $S^2 \times S^1$ which are slice in boundary connected sums of $S^2 \times D^2$ but are not concordant to the unknot (or each other) using our invariant $D(K)$. We further showed $D(K)$ is a new invariant carrying the properties enjoyed by the $\bar{\mu}$-invariants of links $L \subset S^3$. Notably, we showed $D(K)$ is an invariant of concordance of knots in $(\#^i S^2 \times S^1) \times I$ and exhibits the correspondence between Milnor's invariants and Massey products proved independently by Turaev in [Tur79] and Porter in [Por80].

Theorem 1.3. *If the Dwyer number of a null-homologous knot $K \subset \#^i S^2 \times S^1$ is q and $G = \pi_1(\#^n S^2 \times S^1 \setminus K, *)$, then*

1. *the longitude of K lies in G_q,*
2. *all non-vanishing Massey products of elements of $H^1(\#^i S^2 \times S^1 \setminus K); \mathbb{Z})$ are weight q or higher.*

Additionally, we used the Dwyer number to detect an entire family of knots in $\#^i S^2 \times S^1$ which bound disks in a particular 4-manifold, but are not concordant to the unknot as detailed in Section 3.3. This has applications to the definition of a new link concordance group as described in Section 3.4.

Theorem 1.4. *For each $i \in \mathbb{Z}_+$ there is a knot $J_i \subset \#^i S^2 \times S^1$ which bounds a smooth, properly embedded disk in $\natural^i S^2 \times D^2$ in but is not concordant to the trivial knot (even stably as in 3.36) .*

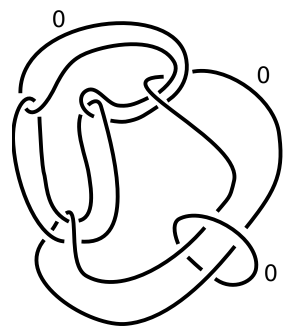


Figure 2: $J_3 \subset \#^3 S^2 \times S^1$

This theorem further motivated the definition of a new notion of link concordance group. Connected sum of knots does not extend naturally to links and defining a link concordance group is somewhat more complicated. In [Hos67], Hosokawa defined the group $\mathcal{H}$ of links modulo a specific type of cobordism using the operation of disjoint union instead; however, $\mathcal{H}$ is isomorphic to $\mathcal{C} \oplus \mathbb{Z}$ and therefore does not contain much more structure than the classical knot concordance group. Le Dimet in [LD88] as well as Donald and Owens in [DO12] defined groups by constructing multiple representatives for each link. Le Dimet's string link concordance group $\mathcal{C}(m)$ is not abelian as it contains the pure braid group as a subgroup, while the Donald-Owens group $\mathcal{L}$ is abelian[DO12, KLW98]. This indicates that some of their structure is determined by choice of group representative and not by inherent properties of links. In a collaboration with Matthew Hedden, we defined a new group using the knotification construction in [OS04a] taking an n-component link $L \subset S^3$ to a unique null-homologous knot $\kappa(L) \subset \#^{n-1} S^2 \times S^1$ called the knotification of L. The group operation is connected sum and we detail the group's construction in Section 3.4.

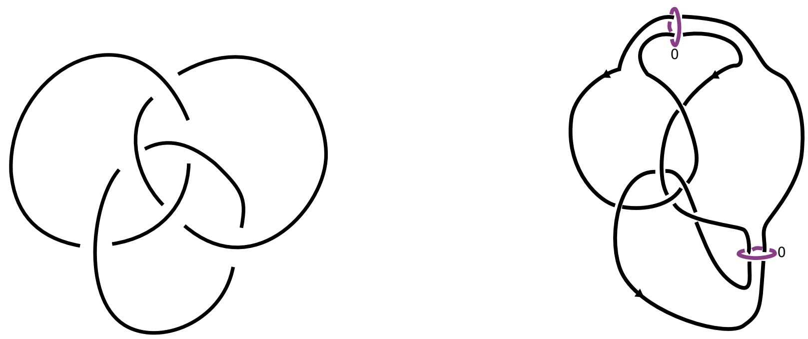


Figure 3: A link $L \subset S^3$ and a surgery diagram for its knotification $\kappa(L) \subset \#^3 S^2 \times S^1$.

Two knotified links are stably concordant they have concordant stabilizations in $(\#^i S^2 \times S^1) \times I$. A knotified link $\kappa(L) \subset \natural^i S^2 \times D^2$ is slice if it bounds a smooth, properly embedded disk in $\natural^i S^2 \times D^2$ as detailed in Section 3.3, and the zero elements in the group $\mathcal{C}^{\kappa(L)}$ are exactly slice knots. Though a knot $K \subset S^3$ is slice (trivial in $\mathcal{C}$) if and only if it is concordant to the unknot, we previously showed this is not true for knots in general 3-manifolds using Theorem 1.4. This theorem relies on the Dwyer number; we further showed the Dwyer number of a knotified link is bounded below by a function of the first nonvanishing Milnor invariant of the original link $L \subset S^3$

Theorem 1.5. *The Milnor invariants of a link $L \subset S^3$ give bounds on the Dwyer number $D(\kappa(L))$.*

As a result of this analysis, in order to construct the inverse of a knotified link, one must say two knotified links $\kappa(L_1)$ and $\kappa(L_2)$ are equivalent if the connected sum of their stabilizations $\kappa(L_1)$ and $\kappa(L_2)$ (with the reverse orientation) is slice. We call this stable slice equivalence.

Theorem 1.6. *The set of nullhomologous knots inside connected sums of $S^2 \times S^1$ modulo stable slice equivalence forms an abelian group $\mathcal{C}^{\mathcal{S}^2 \times \mathcal{S}^1}$ containing the set of* *knotifed links modulo stable slice equivalence $\mathcal{C}^{\kappa(L)}$ as a subgroup. Moreover, $\mathcal{C}^{\kappa(L)}$ contains the knot concordance group $\mathcal{C}$ as a subgroup and concordant links in S^3 become slice equivalent knotified links in $\mathcal{C}^{\kappa(L)}$.*

The knotification $\kappa(L) \subset \#^{n-1}S^2 \times S^1$ of an n-component link $L \subset S^3$ arises naturally in the seminal papers on Heegaard Floer homology [OS04a, OS08]. This theory is a rich source of useful but difficult to compute concordance invariants such as Ozsvàth and Szabò's τ-invariant [OS03], Ozsvàth, Stipsicz, and Szabò's Υ invariant [OSS17], and Jennifer Hom's ϵ invariant [Hom13] which have led to deep results about the knot concordance group. However, many of the link concordance invariants arising in this context factor through knotified links and therefore understanding these objects topologically may lead to important results in Heegaard Floer homology.

Finaly, in additional solo work we returned to previous notions of link concordance group and examined what structure in Le Dimet's string link concordance group $\mathcal{C}(m)$ is inherited from its subgroup of pure braids $\mathcal{P}(m)$ [HL90, LD88]. This quotient is difficult to study as the pure braid group is only normal in the case $m = 2$. Through carefully constructing examples and exploiting a specific type of Milnor invariant called the Sato-Levine invariant, we have shown the following.

Theorem 1.7. *The quotient of the group $\mathcal{C}(m)$ of string links on m strands by the normal closure of the pure braid group $\mathcal{P}(m)$ is non-abelian.*

1.4 Outline of thesis

Chapter 2 contains the necessary background information used in the results in this thesis. In Sections 2.1 and 2.2, we review the necessary background information on knot and link concordance and introduce the knot concordance group. In Section 2.3 there is further discussion on link concordance groups and a brief survey of the area. Section 2.4 is survey of Milnor's invariants starting from the original combinatorial group theory definition and ending with the intersection theory perspective from [Coc90] which we use in our results. Finally, Section 2.5 introduces a few of the main ideas from Heegaard Floer homology.

Chapter 3 contains the bulk of the results in this work. In Section 3.1 , we introduce a link concordance monoid using a construction from Heegaard Floer and introduce an invariant with which to study this monoid. Then in Section 3.2, we introduce a concordance invariant of knots in closed 3-manifolds which generalizes Milnor's invariants using subgroups of second homology generated by half-gropes of a certain height. In Section 3.3 we prove this invariant can be directly computed for null-homologous knots in connected sums of $S^2 \times S^1$ and exhibits important properties of Milnor's invariants. We further use it to show there is an infinite family of knots in these 3-manifolds which are slice in a certain bounding 4-manifold but are all distinct in concordance. Lastly, in Section 3.4 we use this result to construct a group from the monoid introduced in Section 3.1.

In Chapter 4, we discuss the relationship between string links and the indeterminacy of Milnor's invariants in Section 4.1. Then in Section 4.2, we prove the string link concordance group modulo the normal closure of the pure braid group is non-abelian using the Sato-Levine invariant (a specific type of Milnor invariant).

2 Background

2.1 Knot and link concordance in the 3-sphere

We begin by reminding the reader of the precise definitions of knots and links.

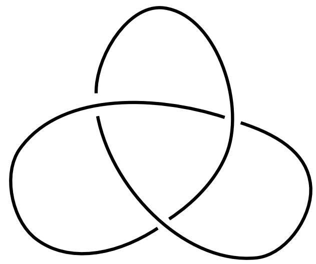


Figure 4: A non-trivial knot in S^3 with the fewest number of crossings.

Definition 2.1. A (smooth) knot in S^3 is a smooth embedding

$$f : S^1 \to S^3.$$

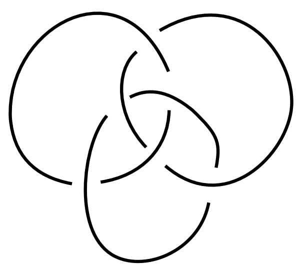


Figure 5: The Borromean rings in S^3

Definition 2.2. A (smooth) n-component link in S^3 is a smooth embedding

$$f : \bigsqcup_{i=1}^{n} S^1 \to S^3.$$

We will also describe distinguished curves in the link complement.

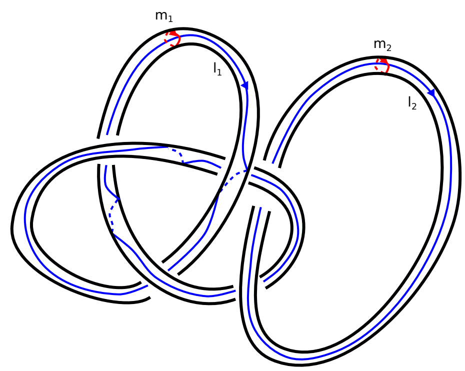


Figure 6: A link L with a meridian and longitude for each component.

Definition 2.3. Let L be an ordered, oriented n-component link in S^3 and L_i be its i^{th} component with $\nu(L_i)$ a regular neighborhood of this component.

- An i^{th} meridian of L is an oriented simple closed curve $m_i \in \partial(\nu(L_i))$ bounding a disk in $\nu(L_i)$.

- An i^{th} (0-framed) longitude of L is an oriented simple closed curve $l_i \in \partial(\nu(L_i))$ which generates $H_1(\nu(L_i))$, is trivial in $H_1(S^3 \setminus \nu(L))$, and has intersection number 1 with the chosen i^{th} meridian m_i.

Note that knots in an arbitrary 3-manifold are not necessarily null-homologous; thus, the concept of a 0-framing is only relevant for links with null-homologous components. As it will come up in later parts of the thesis, we will introduce the notion of surgery on a link.

Definition 2.4 (Dehn surgery along a link $L \subset M$)**.** Suppose we have a 3-manifold M and an oriented link $L = L_1 \cup ... \cup L_n$ in the interior of M. Let $N_1, ..., N_n$ be disjoint closed tubular neighborhoods of the N_i and choose a simple closed curve J_i in each ∂N_i. We then construct the 3-manifold

$$M' = (M \setminus (Int(N_1) \cup ... \cup Int(N_n))) \cup_h (N_1 \cup ... \cup N_n))$$

where h is a union of homeomorphisms $h_i : \partial N_i \to \partial N_i$ each of which takes a meridian μ_i of N_i into the specified J_i. This construction is unique; we need only specify the homotopy classes of J_i as per [Rol03].

Proposition 2.5. *[Rol03] If each component of L is null-homologous, then there is a preferred framing of the appropriate tubular neighborhoods N_i coming from pushing off into the surfaces which we call the 0-framing. This specifies meridians $\{\mu_i\}$ and longitudes $\{\lambda_i\}$ as in 2.3. In this case, we can write each curve J_i as*

$$h_*(\mu_i) = [J_i] = a_i\lambda_i + b_i\mu_i.$$

We can now uniquely label each component of L with the ratio $\frac{b_i}{a_i}$ (called the surgery coefficient of L_i) to specify the map h.

Two knots (and similarly two links) in the 3-sphere can be related by a 4-dimensional equivalence relation called concordance. We can contextualize this relation as being stronger than homotopy but weaker than ambient isotopy.

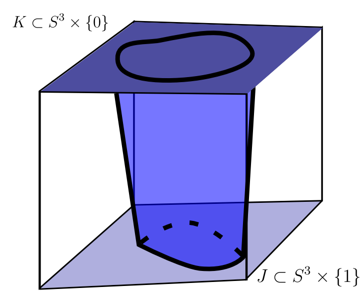


Figure 7: A schematic picture of a concordance between K and J in $S^3 \times I$.

Definition 2.6 (Knot Concordance)**.** Two knots K and J inside S^3 are smoothly concordant if there is a smooth, properly embedded annulus in $S^3 \times [0, 1]$ whose boundary is $K \times \{0\} \sqcup J \times \{1\}$.

Definition 2.7 (Link Concordance)**.** Two n-component links L_1 and L_2 in S^3 are smoothly concordant if their components are concordant by n disjoint smooth cylinders.

We can also consider a related class of knots and links.

Definition 2.8 (Slice Knot)**.** A knot $K \subset S^3$ is smoothly slice if bounds a smooth, properly embedded disk in B^4.

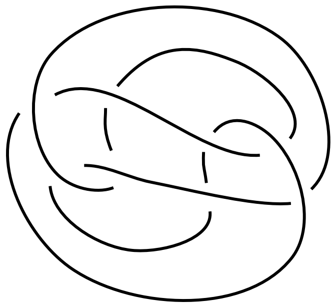
Definition 2.9 (Slice Link)**.** An n-component link L in S^3 is slice if it bounds n smooth, disjoint, properly embedded disks in B^4.

These concepts are related as a result of the following theorem.

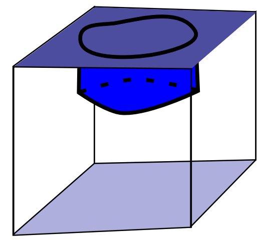
Proposition 2.10. *A knot $K \subset S^3$ is smothly slice if and only if it is smoothly concordant to the unknot, and an n-component link $L \subset S^3$ is smoothly slice if and only if it is smoothly concordant to the n-component unlink.*

As we will show in Section 3.3, this relationship between being concordant to an unknot (or link) and being slice is only true for knots and links in S^3. To conclude this section, we will sketch a brief example of how to construct a slice disk for a knot $K \subset S^3$. Sometimes we can see a way to construct a concordance to the unknot by taking one slice of $S^3 \times [0, 1]$ at a time; this happens when we can put a Morse function on the corresponding slice disk which has only minima. Such a disk is called a ribbon disk, and whether every slice knot bounds a ribbon disk is an open conjecture.

Example 2.11. First, consider the knot $K \subset S^3 \times \{0\}$. We see it bounds a cylinder $K \times [0, \epsilon]$ from Figure 8.



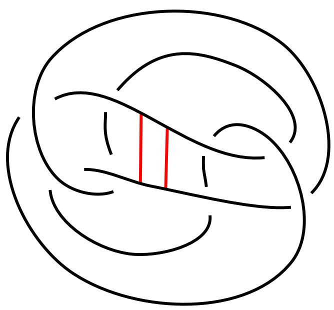
(a) A diagram of $K \subset S^3$



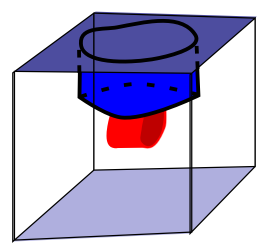
(b) A schematic picture of $K \times [0, \epsilon] \subset S^3 \times I$.

Figure 8: A cylinder bounded by the knot.

Now, attaching a band to $K \times \{\epsilon\}$ adds a band to the cylinder in $S^3 \times I$ as in Figure 9.



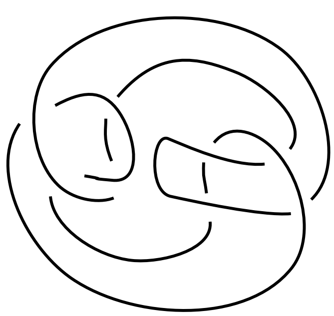
(a) Adding a band to $K \times \{\epsilon\}$.



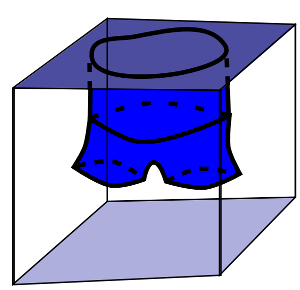
(b) A schematic picture of the resulting surface.

Figure 9: The result of attaching a band.

We further see from Figure 10 that we now have a 2-component unlink in this cross-section of $S^3 \times I$.



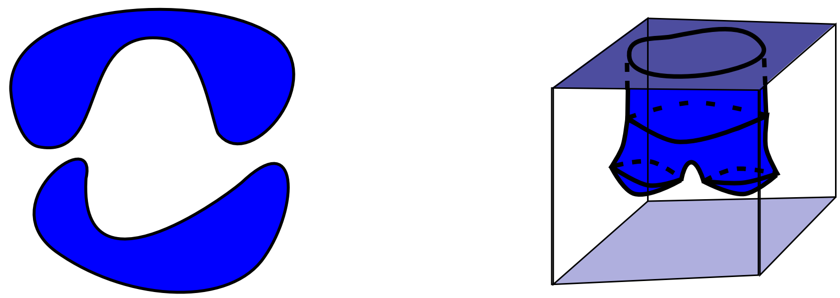
(a) An unlink in $S^3 \times \{\epsilon\}$.



(b) A schematic picture of a pair of pants.

Figure 10: A pair of pants in $S^3 \times I$ with K as a boundary component.

Finally, we can now cap off the unlink in $S^3 \times \{\epsilon\}$ with two disks, resulting in a ribbon disk in $S^3 \times I$ with two minima.



(a) Capping off the unlink with disks. (b) A schematic picture of a ribbon disk.

Figure 11: A properly embedded disk in B^4 with boundary K.

2.2 The knot concordance group

First observe that there is a binary operation on the set of knots.

Definition 2.12. Let K and J be oriented knots in the 3-sphere. The connected sum of K and J is the relative connected sum

$$K\#J = (S^3, K)\#(S^3, J)$$

Definition 2.13. The genus of a knot K is

$$g(K) = \min\{g(\Sigma) \mid \Sigma \text{ is an oriented surface in } S^3 \text{ with boundary } K\}.$$

We see that knots do not have inverses under this operation; this is because the genus of a knot is additive as detailed in [Rol03].

However, it turns out that once we take the quotient of the set of knots in S^3 by the concordance relation described in Section 2.1, we do get a group.

Proposition 2.14. *$K\# -\overline{K}$ is slice for every $K \subset S^3$.*

Proof. Consider the trivial concordance $K \times I \subset S^3 \times I$. Then remove an arc from $K \times \{0\}$ to $K \times \{1\}$, take a neighborhood of it, and remove the interior of this neighborhood. Finally, attach two parallel arcs from $K \times \{0\} - I$ to $K \times \{1\} - I$. The result is a disk Δ inside S^3 as the boundary of B^4. □

Theorem 2.15 (Fox-Milnor [FM66]). *The set of oriented knots in the* 3*-sphere modulo concordance with the connected sum operation forms a group called the knot concordance group* $\mathcal{C}$.

It is important to point out that this group is different than the group obtained from taking the set of oriented knots with connected sum and formally constructing inverses. This is called the Grothendeick group of the knot monoid and we will refer to it as $\mathcal{G}$. One way to see $\mathcal{C} \ncong \mathcal{G}$ is that every element of G is infinite order (again, because genus is additive), while Proposition 2.16 indicates $\mathcal{C}$ has elements of finite order.

Proposition 2.16. $\mathcal{C}$ *has the following properties*

- $\mathcal{C}$ *is abelian [FM66],*
- $\mathcal{C}$ *has elements of order* 2 *(such as negative amphichiral knots) [FM66]*
- $\mathcal{C}$ *surjects onto* $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ *[Lev69].*

Note that it is not known whether $\mathcal{C}$ contains elements of orders other than 2. It is also important to note that the kernel of the map $\mathcal{C} \to \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ is non-trivial [CG86]; it is called the subgroup of algebraically slice knots. In fact,

Theorem 2.17 (Jiang [Jia81]). *The group of algebraically slice knots contains a subgroup isomorphic to* $\mathbb{Z}^\infty$.

From this work, we can see the knot concordance group is extremely large and contains subgroups with very subtle properties such as the algebraically slice knots and the topologically slice knots. Recent work from Heegaard Floer homology using the smooth invariants τ, ϵ, and υ and the d-invariants of covers have led to important results in the study of topologically slice knots, for an overview, see [Hom17]. This group also admits the (n)-solvable, grope [COT03], and bipolar filtration [CHH13] indicating very complex structure is present in $\mathcal{C}$. There is also evidence $\mathcal{C}$ has a self-similar or "fractal"-like structure by using the grope quasi-metric by work in [CHP17].

2.3 The difficulty of defining a link concordance group

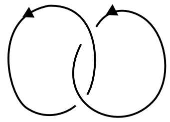
While it is natural to expect the knot concordance group would generalize to a link concordance group, it turns out this is not easy to do. The main difficulty lies in the fact that the connected sum of links is not well-defined, even for oriented, ordered links.

Example 2.18. Kyle Hayden pointed out the following counterexample to connected sum being defined for oriented, ordered links.

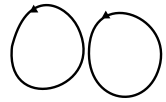
Taking the ordered, oriented "connected sum" of the Hopf link and unlink in Figure 12 results in the Hopf link whose complement is flat. However, using SnapPy one can show the result of the ordered, oriented "connected sum" of the Hopf link and the unlink in Figure 13 has hyperbolic complement, thus the two links are distinct.

There have been different attempts to get around this obstacle, first through changing the group operation.

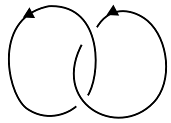
Definition 2.19 (Hosokawa Link Cobordism)**.** Let $L \subset S^3$ be a n-component link with a split sublink L' (i.e. there are $n - r$ mutually disjoint 3-balls each containing a component of L'). Order the components $L_1, ..., L_n$ so the components of L' are $L_r, ..., L_n$ for some r. Let B_1, ..., B_{n-r} be disjoint bands spanning L' lying on $S^3 \setminus L$.



(a) A Hopf link.

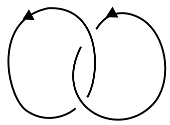


(b) An unlink.

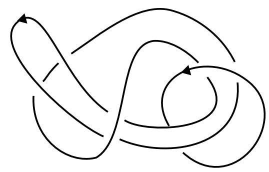


(c) A "connected sum" of the two.

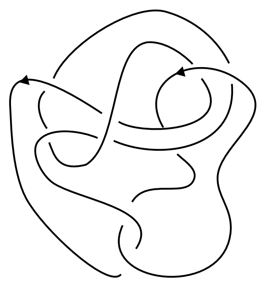
Figure 12: Hopf link"#" the trivial link with flat complement.



(a) A Hopf link.



(b) An unlink.



(c) A "connected sum" of the two.

Figure 13: Hopf link"#" the trivial link with hyperbolic complement.

Suppose that

$$[L_r] + ... + [L_n] + [\partial B_1] + ... + [\partial B_{n-r}] \in H_1(S^3 \setminus (L \setminus L')$$

is represented by a knot K_r. Let $\lambda = L_1 \cup ... \cup L_{r-1} \cup L'$ and $\lambda' = L_1 \cup ... \cup L_{r-1} \cup K_r$. We call replacing λ by λ' a Hosokawa fusion and its inverse a Hosokawa fission. If L_1 can be obtained from L_2 by a sequence of Hosokawa fusions and fissions, we say L_1 and L_2 are Hosokawa link cobordic.

Definition 2.20 (Hosokawa [Hos67]). $\mathcal{H} = (\frac{\text{n-component string links}}{\text{Hosokawa link cobordism}}, \sqcup)$

As it turns out, this group does not contain substantially different information from the classical knot concordance group.

Theorem 2.21 (Hosokawa [Hos67]). $\mathcal{H} \cong \mathcal{C} \oplus \mathbb{Z}$.

Another approach has been to change the underlying set.

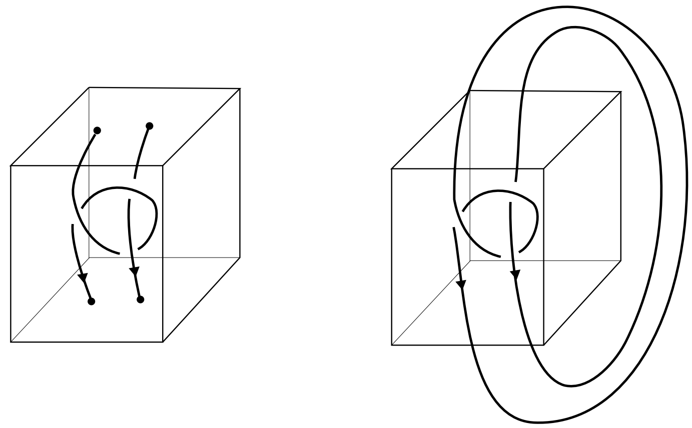


Figure 14: A string link in $D^2 \times I$ and its closure in S^3.

Definition 2.22 (n-component string link). Let D be the unit disk, I the unit interval, and $\{p_1, p_2, ..., p_k\}$ be n points in the interior of D. An n-component (pure) string link is a smooth proper embedding $\sigma : \bigsqcup_m I \to D \times I$ such that

$$\sigma|_{I_i(0)} = \{p_i\} \times \{0\}$$
$$\sigma|_{I_i(1)} = \{p_i\} \times \{1\}$$

A string link can be thought of as a generalization of a pure braid as in Section 4.2 and, for two string links, we can take their product by stacking them just like the group operation in the pure braid group. It can also be viewed as the result of choosing a basing disk for a link $L \subset S^3$ and removing a product of this disk with the unit interval from (S^3, L) as in [Ott11]. Furthermore, if we take the usual braid closure of a string link we obtain a link.

Definition 2.23. Two n-component string links σ_1 and σ_2 are concordant if there is a smooth embedding $H : \bigsqcup_m (I \times I) \to B^3 \times I$ which is transverse to the boundary such that

$$
\begin{aligned}
H|_{(\bigsqcup_m I \times \{0\})} =& \sigma_1 \\
H|_{(\bigsqcup_m I \times \{1\})} =& \sigma_2 \\
H|_{(\bigsqcup_m \partial I \times I)} =& j_0 \times id_I
\end{aligned}
$$

with $j_0 : \bigsqcup_m \partial I \to S^2$.

Now, we can define the group.

Definition 2.24 (Le Dimet [LD88])**.** $\mathcal{C}(n) = (\frac{\text{n-component string links}}{\text{string link concordance}}, \text{stacking})$

While $\mathcal{C}(1) \cong \mathcal{C}$, for larger n this group turns out to have an extremely different structure from $\mathcal{H}$ and, for that matter, from $\mathcal{C}$. In particular,

Theorem 2.25 (Le Dimet [LD88])**.** *$\mathcal{C}(n)$ is non-abelian for all n.*

It is important to note that lurking in these theorems is the fact that the choice of string link representative is not unique, even up to concordance. As detailed in [Ott11], point out that we can also obtain a string link representative for any link $L \subset S^3$ by choosing a basing disk as in [LD88]. It turns out that such a choice of basing is not in any sense unique.

Theorem 2.26 (Kirk-Livingston-Wang [KLW98])**.** *Each link $L \subset S^3$ has an infinite number of string link representatives which are all distinct in string link concordance.*

Therefore, it is not clear how much of the structure of $\mathcal{C}(n)$ can be better understood as a product of this indeterminacy. We examine this structure more carefully in Section 4.1.

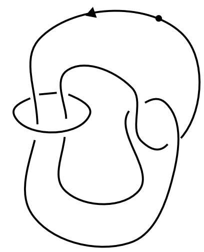


Figure 15: A partly oriented link in S^3

Definition 2.27. We call a link $L \subset S^3$

- Partly oriented if it has only one marked, oriented component,
- Marked oriented if each component is oriented and only one is marked

This allows us to define the connected sum # by only summing marked components. Note that this allows us to take the sum of links with different numbers of components.

Definition 2.28 (χ-concordance [DO12])**.** Let L_1 and L_2 be partly oriented or marked oriented links. We say L_1 and L_2 are χ-concordant if $-L_0 \# L_1$ bounds a smooth, properly embedded surface $F \subset B^4$ such that

- F is the disjoint union of a disk and some number of annuli and Möbuis bands;
- The boundary of the disk component of F is the marked component of $-L_0 \# L_1$;
- In the marked oriented case, we require F to also be oriented and for $-L_0 \# L_1$ to be the oriented boundary of F.

If L_1 and L_2 are both 1-component links (i.e. knots), this notion agrees with the previous definition of concordant knots.

Definition 2.29 (Donald-Owens [DO12])**.** We can now define two link concordance groups:

- $\mathcal{L} = (\frac{\text{Partly oriented links}}{\chi\text{-concordance}}, \#)$
- $\tilde{\mathcal{L}} = (\frac{\text{Marked, oriented links}}{\chi\text{-concordance}}, \#)$

Theorem 2.30 (Donald-Owens [DO12])**.** *Both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are abelian, both contain $\mathcal{C}$ as a summand, and each contains a $\mathbb{Z}_2$ summand and a $\mathbb{Z}^\infty$ subgroup in the complement of $\mathcal{C}$*

Thus we see this most recent definition of link concordance group in [DO12] shares properties of both previously defined types of link concordance groups; like $\mathcal{H}$ it is abelian, and like $\mathcal{C}(n)$, each link has a number of group representatives. Again, it displays quite different properties than previous notions of link concordance groups. Thus, in Section 3.4 we focus on defining a concordance group of links where each link has a unique representative and the group operation is still connected sum.

2.4 Milnor's invariants

A large portion of the work in this thesis has involved link concordance invariants called Milnor's invariants. While their original definition in [Mil57] is somewhat laborious, as we will see in this section they give us a natural way to detect subtle higher order linking data. Milnor's invariants arise in three contexts: 1) combinatorial group theory, 2) cohomology, and 3) intersection theory. The connection between the group theory and homotopy perspective using Massey products was conjectured by Milnor in his original definitions in [Mil57], but was not proven until much later by Turaev [Tur79] and independently by Porter [Por80]. Later, using work of Stein [Ste90], Cochran showed one can exploit the duality between Massey products and iterated intersections of surfaces in order to compute the first nonvanishing Milnor's invariants [Coc90]. In this section, we will survey definitions and major theorems about these invariants.

One of the most fundamental and useful tools used to study links in S^3 is linking number; intuitively, the linking number of two components of a link (or two knots) describes how difficult it is to pull the links apart. The linking number of two link components can be defined precisely in multiple ways; for the purposes of this thesis, we will define linking number as the following.

Definition 2.31 (Linking Number)**.** Let $J, K \subset S^3$ be smooth knots. Notice that $H_1(S^3 \setminus \nu(K)) \cong \mathbb{Z}$ and is generated by a meridian $[m]$ of K. The linking number of J and K is

$$\text{lk}(J, K) := [J] = n[m] \in H_1(S^3 \setminus \nu(K)).$$

In Figure 16, we see the linking number of the unlink is 0 and the linking number of the Hopf link is 1.

Figure 16: A trivial 2-component link (an unlink) and the Hopf link.

It is important to note that linking number is invariant under not only isotopy but also concordance [Cas75], and we will exploit this fact to motivate the study of more subtle linking data. While linking number is a useful invariant, it is not particularly sensitive. For example, Figure 17 shows the Whitehead link which has linking number 0 but is certainly not concordant to the unknot as later calculation will show.

Notice that for an arbitrary link in the 3-sphere, we can repackage the linking numbers between its components as a statement about the homology of the complement of the entire link.

Remark 2.32*.* If L is an n-component oriented link with L_i the 0-framed longitude of

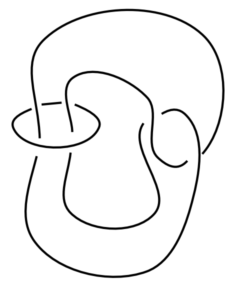


Figure 17: The Whitehead link.

the i^{th} component of L and $G = \pi_1(S^3 \setminus \nu(L), *)$, then

$$[L_i] = \Sigma_{i=1}^{n} \mathrm{lk}(L_i, L_j) \cdot x_i \in \mathrm{H}_1(S^3 \setminus \nu(L)) = G/[G, G]$$

where x_i represents the i^{th} meridian.

Note that the fundamental group of a link complement is not itself a concordance invariant; however, as the previous fact shows we can extract concordance data from it. From this perspective, one might wonder if there is concordance information detected by other quotients of the fundamental group. Theorem 2.33 indicates there is. Recall that the lower central series of a group G is defined recursively by $G_1 = G$ and $G_{n+1} = [G, G_n]$.

Theorem 2.33 (Casson [Cas75])**.** *If L_1 and L_2 are concordant links in S^3 whose groups are G and H, then G/G_q and H/H_q are isomorphic for all q.*

As a result, we have an entire family of nilpotent groups which can tell us important information about concordance. Furthermore, we have the following important result on nilpotent quotients of groups.

Theorem 2.34 (Stallings' Integral Theorem [Sta65])**.** *Let $\varphi : A \to B$ be a homomorphism inducing an isomorphism on $H_1(-;\mathbb{Z})$ and an epimorphism on $H_2(-;\mathbb{Z})$. Then, for each n, ϕ induces an isomorphism $A/A_n \cong B/B_n$.*

In order to use these theorems to detect whether a link is non-trivial, we must understand how the structure of the group quotients relate to the topology we are using them to study. The following lemma is well known, but we include it here along with its proof to motivate later definitions.

Lemma 2.35. *Let L be an n-component smooth link in S^3 with fundamental group $\pi_1(S^3 \setminus \nu(L), *)$.*

1. *If L has only one component (i.e. L is a knot), then $G/G_q \cong \mathbb{Z}$ for all q.*
2. *If L is the n-component unlink, then G is a free group generated by the meridians of L.*
3. *If L is an n-component slice link and F is a free group on n letters, then $G/G_q \cong F/F_q$ for all q.*

Proof.

1. Consider the homomorphism $f : \mathbb{Z} \to \pi_1(S^3 \setminus L, *)$ defined by sending the generator 1 of $\mathbb{Z}$ to a meridian of L. Since L is a knot, this map induces isomorphisms on homology. Thus, by Theorem 2.34, f induces isomorphisms on the lower central series quotients.
2. This is a straightforward calculation using the Seifert-van Kampen theorem.
3. Since L is a slice link, it is concordant to the n-component unlink. By Theorem 2.33, and part 2 of the lemma, G/G_q is isomorphic to F/F_q for all q.

□

This lemma first tells us that the nilpotent quotients of the fundamental group of knot will not tell us any new concordance information, as we might expect as there is nothing for the knot to “link” with in the first place. Therefore, these lower central series quotients will only be useful for studying links (at least when working in the

3-sphere, as we will see later). Secondly, this lemma indicates that if we want to determine if an n-component link is non-trivial in concordance, we should examine whether the lower central series quotients of its link group are isomorphic to the lower central series quotients of a corresponding free group. One difficulty here is that even the lower central series quotients of free groups on $n > 1$ letters become unwieldy quite quickly. Thankfully, we can detect whether these quotients are isomorphic using a clever group presentation.

Theorem 2.36 (Milnor [Mil57])**.** *Let $L \subset S^3$ be an n-component link and $G = \pi_1(S^3 \setminus \nu(L), *)$. For $q \geq 1$, we have*

$$\frac{G}{G_q} \cong \langle x_1, ..., x_n \mid [x_i, R_q(l_i)], 1 \leq i \leq n, F_q \rangle$$

where x_i is represented by a ith meridian of L, l_i is represented by an ith longitude of L, F is the free group on the letters $x_1, ..., x_n$, and $R_q(l_i) = l_i$ mod G_q.

We can interpret this presentation geometrically as per Turaev's proof in [Tur79]. First, it is a standard fact that the meridians of a link $L \subset S^3$ normally generate the fundamental group of the complement of L in S^3. Therefore, the first step is to choose appropriate meridians for L. In [Tur79], the author chooses a basing of each link component in order to specify these meridians. He then uses these meridians to generate a free group, and then taks its quotient by the relations coming from boundary tori (the $[x_i, R_q(l_i)]$) and the q-fold iterated commutators. Furthermore, this presentation indicates that for an arbitrary link $L \subset S^3$ the obstruction to the nilpotent quotients of the link group being isomorphic to the nilpotent quotients of the free group lies in the words $[x_i, R_q(l_i)]$.

In summary, it is clear that for any $q \in \mathbb{Z}$ with $q \geq 2$, we have

$$\frac{G}{G_q} \cong \frac{F}{F_q} \iff [x_i, R_q(l_i)] \in F_q \text{ for each } i \iff R_q(l_i) \in F_{q-1}.$$

Recall that the entire point of this line of reasoning was to determine whether a given n-component link $L \subset S^3$ was concordant to the n-component unknot, and thus we have reduced this question to determining whether, for each link component L_i, a representative of its longitude l_i modulo G_q (as a word in the free group on n letters) lies in F_{q-1}. Given such a word, this question is more tractable thanks to the following results of Magnus.

Theorem 2.37 (Magnus [Mag35])**.** *There is an embedding of the free group F with generators $x_1, ..., x_n$ into the ring of formal integral power series in the noncommuting variables $X_1, ..., X_n$. This embedding M is defined by $x_i \mapsto 1 + X_i$ for all $i \in \{1, ..., n\}$ and $x_i^{-1} \mapsto 1 - X_i + X_i^2 - X_i^3 + ...$ for all $i \in \{1, ..., n\}$. For a word $w \in F$, we say $M(w)$ is its Magnus expansion.*

We can now detect exactly when a word lies in a certain term of the lower central series of the free group.

Theorem 2.38 (Magnus [Mag35])**.** *Let F be the free group generated by $x_1, ..., x_n$. The reduced word w lies in F_q if and only if all of the non-zero terms of Magnus expansion $M(w) - 1$ are order n or higher.*

Note that these statements can be reformulated in terms of Fox calculus; this perspective is useful when trying to compute a specific coefficient of an expansion [Fox53]. Recall that the goal of this section was to introduce a notion of higher order linking number; finally, all of the ingredients to do such a thing are now present.

Definition 2.39 (Milnor [Mil57])**.** Let $L \subset S^3$ be an oriented, ordered link and let $G = \pi_1(S^3 \setminus \nu(L), *$ be the fundamental group of its complement. The Milnor invariants of L are integers $\overline{\mu}(i_1, ..., i_k)$ each corresponding to a multi-index $(i_1, ..., i_k)$ where $i_j \in \{1, ..., n\}$. Let l_{i_k} be the i_k^{th} longitude of L and let $R_k(l_{i_k})$ be its image in G/G_k. Expressed in the generators found in Theorem 2.36, this group element corresponds to a word w in meridians $x_1, ..., x_n$. The Magnus expansion of this word is $M(w) = 1 + \Sigma_I \epsilon_I X^I$ where the sum is taken over all possible multi-indices $I = (j_1, ..., j_m)$ and X^I is shorthand for $X^{j_1} ... X^{j_m}$. Then,

$$\overline{\mu}(i_1, ..., i_k) = \epsilon_I(R_k(l_{i_k})).$$

This integer is well-defined if all the Milnor's invariants of order less than k are 0, otherwise, this integer is defined to be the residue class modulo

$$\Delta = \gcd\{\overline{\mu}(\tilde{I})\}$$

where $\tilde{I}$ is obtained from $i_1, ..., i_k$ by removing one index and cyclically permuting the other indices.

Notice that Milnor's invariants are only defined modulo certain Milnor's invariants of smaller weight. Therefore, in this section (and the majority of this thesis) we are primarily concerned with the first Milnor's invariants which are non-zero. In Section 4.1, we will briefly address how to get around this indeterminacy using string links in $D^2 \times I$ instead of links $L \subset S^3$ as in [Lev87]. This indeterminacy is exactly the indeterminacy in choosing a basing for a link as in Turaev's proof of Theorem 2.36.

Corollary 2.40. *For $L \subset S^3$ an oriented, ordered link, $\overline{\mu}(ij) = lk(L_j, L_i)$.*

Proof. Consider the class l in $G = \pi_1(S^3 \setminus \nu(L), *)$ representing the longitude of L_i and notice that G/G_2 is exactly $H_1(S^3 \setminus \nu(L)$, so by Remark 2.32 we have that l is represented by $\Pi_{k=1}^n lk(L_k, L_i) x_k$ (written multiplicatively). The Magnus expansion of this word is $1 + \Sigma_{k=1}^{n} lk(L_k, L_i)X_k + \textit{higher order terms}$ and thus we have the result. □

Before illustrating other ways to compute these invariants, we first observe the following facts.

Theorem 2.41 (Casson [Cas75]). *All $\overline{\mu}_L(I)$ are concordance invariants.*

Moreover, a subset of $\overline{\mu}_L(I)$ are invariants up to an even stronger relation. Recall that two links L_1 and $L_2 \in S^3$ are link homotopic if there is a homotopy from L_1 to L_2 such that the images of the components of L_1 remain disjoint for all time.

Theorem 2.42 (Milnor [Mil54]). *The $\overline{\mu}_L(I)$ with non-repeating indices are link homotopy invariants.*

Furthermore, we know what happens to these invariants when we connect two links with any band.

Theorem 2.43 (Cochran [Coc90]). *The first non-zero Milnor invariant is additive under band sum.*

We also have the following theorems giving a topological interpretation of vanishing Milnor's invariants.

Proposition 2.44 (Cochran [Coc90]). *The $\overline{\mu}$-invariants of L vanish if and only if the components of any link L', formed by adding parallel push-offs of the components of L, bound pairwise disjoint immersed 2-disks in the 4-ball.*

Recall that a boundary link is a link whose components bound disjoint Seifert surfaces; the following theorem shows such links are invisible to these invariants.

Theorem 2.45. *If $L \subset S^3$ is a boundary link, all its Milnor's invariants vanish. Moreover, any interior band sum (defined in Section 3.3) of a boundary link has vanishing Milnor's invariants by [Coc90].*

However, for many years it was extremely difficult to detect anything about links whose Milnor's invariants all vanished. In fact, it was not known until [CO93] that there were links with vanishing Milnor's invariants which are not concordant to boundary links.

Recall from the beginning of this section that Milnor's invariants can be interpreted using cohomology; this was conjectured by Milnor but remained an open problem for many years. As we are primarily concerned with the first non-zero Milnor invariant, we have stated the following result for this case. For further details on dealing with this indeterminacy, see [Por80].

Theorem 2.46 (Turaev [Tur79], Porter [Por80]). *Let $L \subset S^3$ be an oriented, n-component link and let L_i refer to its i^{th} component. Let $u_i \in H^1(S^3 \setminus L_i)$ be the Alexander dual of the generator of $H_1(L_i)$ determined by the orientation of L_i. For $i, j \in \{1, ..., n\}$, set $\gamma_{i,j}$ equal to the Lefschetz dual of the element in $H_1(S^3, L_i \cup L_j)$ determined by a path from L_i to L_j.*

Let $(l_1, ..., l_p)$ be a sequence of integers with $l_k \in \{1, ..., n\}$. If all Milnor invariants of L of weight less than p are 0, then the Massey product $\langle u_{l_1}, ..., u_{l_p} \rangle$ in $S^3 \setminus L$ is defined and

$$\langle u_{l_1}, ..., u_{l_p} \rangle = (-1)^p \overline{\mu}_L(l_1, ..., l_p) \gamma_{l_1, l_p}.$$

Notice this means we can recover the appropriate Milnor invariants by evaluating $\langle u_{l_1}, ..., u_{l_p} \rangle$ on the corresponding boundary components of $S^3 \setminus \nu(L)$. This perspective can be quite useful for certain applications, however, in this thesis we will use the dual picture as in [Coc90]. Recall that Massey products are generalized cup products; just as cup products are dual to intersections, it turns out that Massey products are dual to higher order intersections. Cochran's work in [Coc90] allows us to compute Milnor's invariants using surface intersections and pushoffs as we will see in an example.

Definition 2.47. Let m be a positive integer and F be the free group generated by

$\{x_1, ..., x_m\}$. The set of n-bracketings B_n is given inductively by:

1. $B_1 = \{x_1, ..., x_m\}$

2. $B_n = \{(\sigma, \omega) \mid \sigma \in B_k,\ \omega \in B_{n-k},\ 1 \leq k \leq n-1.$

An n-bracketing β is an element of B_n and its weight $w(\beta)$ is said to be n. We say a subset of $\cup B_n$ is coherent if $(\omega, \sigma) \in S)$ implies $w \in S$ and $\sigma \in S$.

Definition 2.48 (Surface system of length n [Coc90])**.** A surface system of length n for L is a pair $((C), (V))$ of sets satisfying:

1. There exists a coherent subset S of $\cup B_i$ such that if $w(\sigma) < n$, then $\sigma \in S$,

2. $\mathcal{V} = \{V(\sigma) \subset S^3 \setminus L \mid \sigma \in S\}$ where the $V(\sigma)$ are compact, oriented surfaces (or the empty set) which intersect transversely and $V(\alpha, \beta) = -V(\beta, \alpha)$ (i.e. reverses orientation),

3. $\mathcal{C} = \{c(\sigma) \subset S^3 \mid \sigma \in S\}$ where $c(x_i)$ is the i^{th} 0-framed longitude of L and $c(\beta, \alpha) = V(\beta) \cap V(\alpha)$. Note by convention, $V(\alpha) \cap V(\alpha) = \emptyset$,

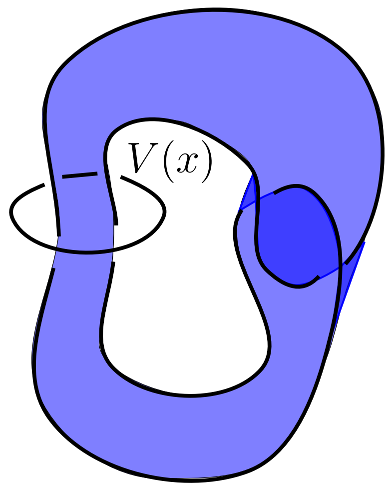
4. Let $c^+(\beta, \alpha)$ denote a pushoff of $c(\beta, \alpha)$ with respect to $V(\beta)$ and $V(\alpha)$ such that the interior of the annulus spanning c and c^+ misses all elements of C. Define $c^+(x_i) = c(x_i)$ and note that the pushoffs must satisfy $c^+(\beta, \alpha) = -c^+(\alpha, \beta)$,

5. $\partial V(\sigma) = c^+(\sigma)$,

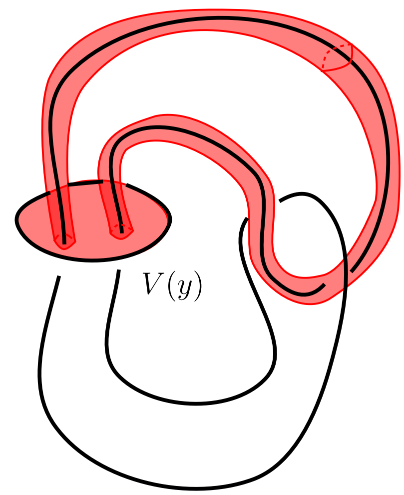
6. Suppose $w(\alpha) + w(\beta) \leq n$. Then $c(\sigma) \cap V(\beta)$ is empty unless $c(\sigma) \subset V(\beta)$.

We can see this idea is dual to a defining system for a Massey product.

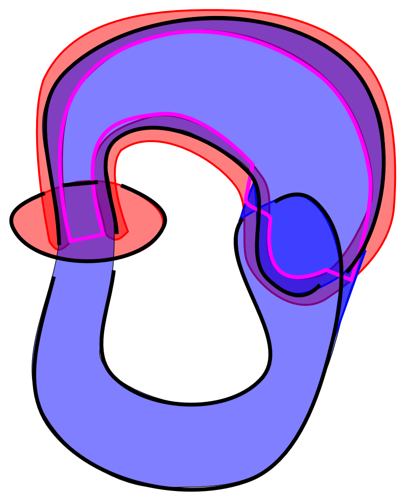
Example 2.49. For the Whitehead link, we can construct the surface system in Figure 18.



(a) A surface $V(x)$ bounded by $c(x)$.



(b) A surface $V(y)$ bounded by $c(y)$.



(c) $V(y)$ and $V(x)$ intersecting in $c(x, y)$.

Figure 18: A length 2 surface system $(\{V(x), V(y)\}, \{c(x), c(y), c(x, y)\})$.

Now we have a surface system, but we need a dual notion of evaluating the Massey product in order to get integers, and for this we need to take linking numbers.

Definition 2.50. A linking of weight n for $(\mathcal{C}, \mathcal{V})$ is $lk(c(\alpha), c^+(\beta))$ where $w(\alpha) + w(\beta) = n$.

We will further state a loose definition of Cochran, for details see [Coc90].

Definition 2.51. A system of weight n for L is a system $(\mathcal{S}, \mathcal{V}, \mathcal{C})$ of some length for

L which is "big enough" so that all potential linkings of weight less than or equal to n "are defined" and all linkings of weight less than n are 0.

Finally, this intersection picture allows us to reformulate what was previously stated for Milnor's invariants. First,

Corollary 2.52 (Cochran [Coc90]). *If a system of weight n exists for $L \subset S^3$ such that all n-linkings are zero, then the longitudes of L lie in G_n.*

Moreover,

Proposition 2.53 (Cochran [Coc90]). *If a complete system of weight n exists for L then for $i_1 \neq i_n$,*

$$\overline{\mu}_L(i_1 ... i_n) = (-1)^n \Sigma lk(c(\alpha), c^+(\beta))$$

where $lk(c(\alpha), c^+(\beta))$ are minimal n-linkings. The index set is in natural correspondence with the set of all bracketings of $x_{i_1} ... x_{i_n}$.

Finally, we will sketch a computation of the Milnor invariants of the Whitehead link.

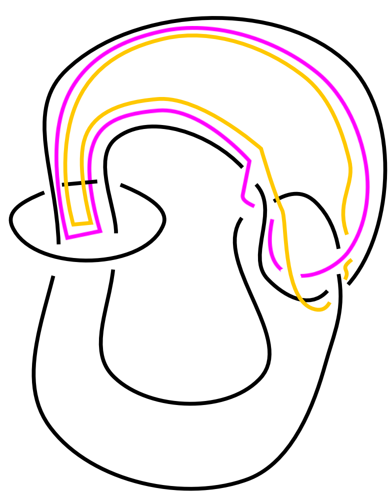


Figure 19: $c(x, y)$ together with its pushoff.

Example 2.54. From Figure 18, we see that

$$lk(c(x), c^+(y)) = 0$$
$$lk(c(x), c^+(x, y)) = 0$$
$$lk(c(y), c^+(x, y)) = 0$$
$$lk(c(x, y), c^+(x, y)) = 1$$

so

$$\overline{\mu}_L(1212) = lk(c(x, y), c^+(x, y)) = 1$$

In practice, this surface system method gives a faster way to compute first non-zero Milnor invariant of a link. Additionally, the perspective of surfaces allowed Cochran to prove the following realization theorem which we will use to great effect in Section 3.3.

Theorem 2.55 (Cochran [Coc90])**.** *Let $\mathcal{S}$ be the set of all links of at least m components. Given any integer $m \geq 2$ and any sequence $I = i_1 i_2 ... i_n$ where $i_j \in \{1, ..., m\}$,*

1. *There is an algorithm (polynomial in $\mid I \mid$)) calculating a non-negative integer.*

$$\delta(I) = \begin{cases} 0, & \text{if } \{\mid \overline{\mu}_L(I) \mid\mid L \in \mathcal{S}\} = \{0\} \\ min(\{\mid \overline{\mu}_L(I) \mid\mid L \in \mathcal{S}\} \setminus 0), & \text{otherwise.} \end{cases} \tag{2.1}$$

2. *There is an algorithm to construct an m-component Brunnian link L all of whose $\overline{\mu}$-invariants of weight less than $\mid I \mid$ are zero and such that $\overline{\mu}_L(I) = \delta(I)$. This algorithm begins with either the Hopf link or the Whitehead link, proceeds by Bing doubling, and ends with band sums.*

We end this section by referencing more recent work relating Milnor's invariants to Whitney towers.

Definition 2.56 (Whitney disk)**.** Let A and B be submanifolds of X intersecting in points x and y. A Whitney disk from x to y is a is a smooth map $W : D^2 \to X$ where

$$W(S^1 \cap \{\mathrm{Re}(z) \geq 0\}) \subset A,$$

$$W(S^1 \cap \{\mathrm{Re}(z) \leq 0\}) \subset B,$$

$$W(-i) = \mathbf{x}, \text{ and } W(i) = \mathbf{y}.$$

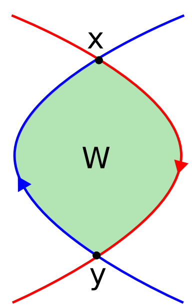


Figure 20: A Whitney disk from **x** to **y**.

We can view a Whitney disk as pairing up oppositely oriented intersection points. Using these disks, we can build Whitney towers.

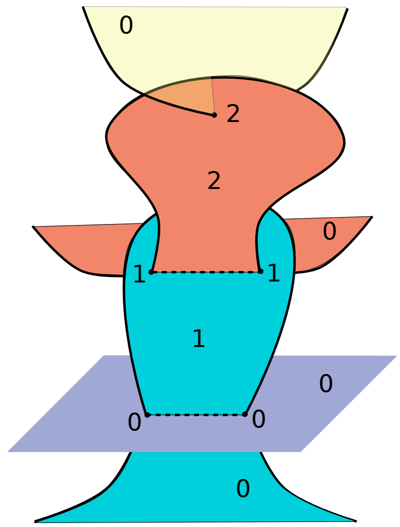


Figure 21: An order 2 Whitney tower with orders of surfaces, Whitney disks, and intersection points clearly labeled.

Definition 2.57. An order n Whitney tower W inside a 4-manifold X is defined recursively

1. An order 0 surface $\Sigma \subset X^4$ is connected, oriented, and properly immersed. W is order 0 if it is a collection of order 0 surfaces.

2. If Σ_1 and Σ_2 are transverse with orders $Ord(\Sigma_1) = n$, $Ord(\Sigma_2) = m$, then $Ord(p) = n + m$ for intersection points $p \in \Sigma_1 \cap \Sigma_2$.

3. A Whitney disk pairing order n intersections has order $n + 1$

4. $Ord(W) = n$ if it is an order $n-1$ Whitney tower with immersed Whitney disks pairing all order $n-1$ intersection points.

These towers W can be recorded using the intersection tree invariant $\tau_n(W)$ (not to be confused with the Heegaard Floer τ-invariant introduced in Section 3.1). Furthermore,

Theorem 2.58. *(Conant-Schneiderman-Teichner [CST12]) If a link $L \subset S^3$ bounds a Whitney tower W of order n, then all $\bar{\mu}_L(I)$ of weight $\mid I \mid < n$ vanish. Moreover, all $\bar{\mu}_L(I)$ of weight n can be computed from the tree-valued invariant $\tau_n(W)$ described in [CST12].*

2.5 Heegaard Floer homology basics

The Heegaard Floer homolgy of a 3-manifold Y include HF^{∞}, HF^{-}, HF^{+}, and $\widehat{HF}(Y)$ by Ozsvàth and Szabò in [OS04b] and independently by Rasmussen [Ras03]. The main idea behind their construction is to examine a pointed Heegaard diagram for a 3-manifold and use this data to construct certain Lagrangian submanifolds determined by the attaching curves inside a particular symplectic manifold specified by the Heegaard surface. Then, the homology groups using chain complexes whose chain groups are collections of intersection points of these Lagrangians and whose differentials involve maps of pseudo-holomorphic disks connecting these intersection points.

Though these groups are invariants of 3-manifolds, a null-homologous knot in a 3-manifold induces filtrations on the previously mentioned complexes and from these we can obtain knot invariants up to different types of knot equivalence [OS04a]. For knots in S^3, we can extract the concordance invariant τ from the complex $\widehat{CF}$ using this filtration as detailed in [OS03].

In this thesis, we will specifically use the complexes $CFK^{\infty}(L)$ and $\widehat{CFK}(L)$ and the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ as these choices are sufficient for our purposes. Using the knot filtration induced by knotified links in $\#^{n-1}S^2 \times S^1$, we can define invariants τ_i of links in S^3 using a construction of Hedden in [Hed08] which are isotopy invariants of knotified links detecting when the filtration level is large enough to capture a specific element Θ_i in $\widehat{HF}(\#^{n-1}S^2 \times S^1)$.

First, we will sketch the construction of knot Floer homology for null-homologous knots inside $\#^l S^2 \times S^1$ for any l for completeness. In order to do this, we will briefly describe how to compute the Floer homology of an arbitrary 3-manifold Y and then detail how a null-homologous knot K inside $Y = \#^l S^2 \times S^1$ us a filtration on $CFK^\infty(K)$ which induces a filtration on $\widehat{CFK}(K)$.

We can define a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ for an oriented 3-manifold Y in the following way. Σ is an oriented genus g surface, $\boldsymbol{\alpha} = \alpha_1, ..., \alpha_g$ and $\boldsymbol{\beta} = \beta_1, ..., \beta_g$ are sets of pairwise disjoint, homologically linearly independent, tranverse, simple closed curves on Σ. These sets of curves specify a decomposition of Y into handlebodies U_α and U_β glued together along Σ.

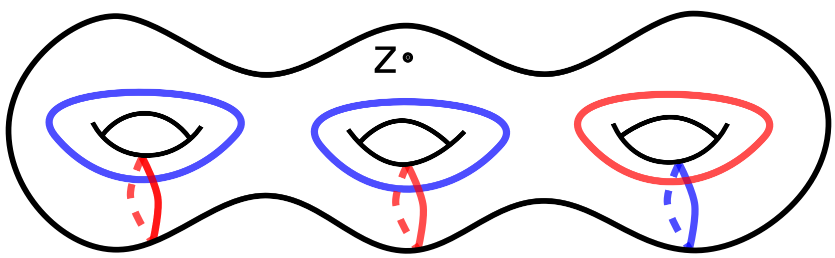


Figure 22: A pointed Heegaard diagram for S^3 with red $\boldsymbol{\alpha}$-curves and blue $\boldsymbol{\beta}$-curves.

Consider the g-fold symmetric product $\mathrm{Sym}^g(\Sigma)$ (a symplectic 2g-manifold) with two distinguished Lagrangian tori

$$\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g \text{ and } \mathbb{T}_\beta = \beta_1 \times ... \times \beta_g$$

which can be made transverse by an isotopy of the attaching curves in Σ. Roughly, the Heegaard Floer homology of $\#^l S^2 \times S^1$ is the Lagrangian Floer homology of these tori inside $\mathrm{Sym}^g(\Sigma)$ over different coefficient rings as $\mathbb{F}[U]$-modules. The generators are the unordered tuples of intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathrm{Sym}^g(\Sigma)$. This generating set is partitioned by Spin^C-structures on Y which can be thought of from the Turaev perspective as homology classes of vector fields on Y. The differentials of these chain complexes count pseudo-holomorphic maps of disks with certain boundary conditions connecting these intersection points.

More specifically, the chain complex $CF^\infty(Y)$ is freely generated over the Laurent polynomial ring $\mathbb{F}[U, U^{-1}]$ by points in $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that each α and each β curve appear exactly once. Each such generator $\mathbf{x}$ is assigned a Spin^C structure $\mathfrak{s}_w(\mathbf{x})$ depending on the basepoint w using the map in [OS04b], and two generators correspond to the same Spin^C if and only if they can be connected by a Whitney disk.

Recall a Whitney disk from $\mathbf{x}$ to $\mathbf{y}$ in $\mathrm{Sym}^g(\Sigma)$ (just like in Section 2.4) is a smooth map $W : D^2 \to \mathrm{Sym}^g(\Sigma)$ where

$$W(S^1 \cap \{\mathrm{Re}(z) \geq 0\}) \subset \mathbb{T}_\alpha,$$

$$W(S^1 \cap \{\mathrm{Re}(z) \leq 0\}) \subset \mathbb{T}_\beta,$$

$$W(-i) = \mathbf{x}, \text{ and } W(i) = \mathbf{y}.$$

For any pair of unordered tuples of intersection points $\mathbf{x}$ and $\mathbf{y}$, let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the space of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$. To each such disk ϕ there is an associated moduli space of psuedo-holomorphic representatives of ϕ called $\mathcal{M}(\phi)$ with a corresponding integer $\mu(\phi)$ called the Maslov index of ϕ which we can think of as the "expected dimension" of $\mathcal{M}(\phi)$. Note that we can concatenate disks in the way you might expect and that Maslov index is additive. Moreover, for a point $s \in \Sigma$, we can also define the geometric multiplicity of ϕ at s as

$$n_s(\phi) = \{s\} \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma).$$

Now, we can define the boundary map on the above chain complex by

$$\partial^{\infty}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})|\mu(\phi)=1\}} \big(\#\big(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\big) \bmod 2\big) U^{n_w(\phi)}\mathbf{y}.$$

Morally, this differential is the sum over all intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ of the mod 2 count of points in the quotient space $\frac{\mathcal{M}(\phi)}{\mathbb{R}}$. The power of U keeps track of the multiplicity of ϕ at the basepoint w. While there are admissibility requirements of the diagram for this construction to work as stated for manifolds with positive first Betti number, for these details we we refer the reader to [OS04b].

Now, the complex $CF^{\infty}(Y)$ splits along $\mathrm{Spin}^C(Y) \cong H_2(Y;\mathbb{Z})$ using the map

$$\mathfrak{s}_w : \{\mathbb{T}_\alpha \cap \mathbb{T}_\beta\} \to \mathrm{Spin}^C(Y)$$

described in [OS04b]. Each summand corresponding to $\mathfrak{s} \in \mathrm{Spin}^C(Y)$ is generated by intersection points $\mathbf{x}$ with $\mathfrak{s}_w(x) = \mathfrak{s}$ and relatively $\mathbb{Z}/(\mathfrak{d}(\mathfrak{s})\mathbb{Z})$-graded by the Maslov grading (also called homological grading) defined by

$$M(U^i\mathbf{x}) - M(U^j\mathbf{y}) = \mu(\phi) - 2n_w(\phi) - 2i + 2j \bmod \mathfrak{d}(\mathfrak{s}).$$

This grading is independent of choice of $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ and defined modulo

$$\mathfrak{d}(\mathfrak{s}) = \mathrm{gcd}_{\xi\in H^2(Y;\mathbb{Z})}\langle c_1(\mathfrak{s}), \xi\rangle.$$

It is important to note that multiplication by U lowers Maslov grading by 2. Now, the literature also defines the complex $CF^{\infty}(Y)$ as the chain complex generated by

$[x, i]$ with differential

$$\partial^\infty(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})|\mu(\phi)=1\}} \big(\#(\frac{\mathcal{M}(\phi)}{\mathbb{R}}) \bmod 2\big)[\mathbf{y}, i - n_w(\phi)]$$

and U-action defined to be $U[x, i] = [x, i - 1]$. These two notions of $CF^\infty(Y)$ are isomorphic with isomorphism given by the map sending U^i to $[\mathbf{x}, -i]$; either complex can be taken as the definition of $CF^\infty(Y)$.

Much of the power of this package of invariants comes from subcomplexes of $CF^\infty(Y)$; we will use $\widehat{CF}(Y)$ which can be defined as the kernel of the U-action. Explicitly, its corresponding differential is ,

$$\hat{\partial}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})|\mu(\phi)=1,\ n_w(\phi)=0\}} \big(\#(\frac{\mathcal{M}(\phi)}{\mathbb{R}}) \bmod 2\big)\mathbf{y}.$$

These $\mathbb{F}[U]$-modules $HF^\infty(Y)$ and $\widehat{HF}(Y)$ are 3-manifold invariants. When we have a nullhomologous knot K inside Y, we can use a doubly-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ to obtain an additional filtration on the complexes $CF^\infty(Y)$ and $\widehat{CF}(Y)$ and we can then use these filtered complexes to construct invariants of null-homologous knots $K \subset Y$. The knot Floer homology $HFK^\infty(K)$ and $\widehat{HFK}(K)$ is corresponding associated graded.

We can recover a knot K from this diagram by connecting w to z inside $\Sigma \setminus \boldsymbol{\alpha}$ to get an oriented arc a and then connecting z to w inside $\Sigma \setminus \boldsymbol{\beta}$ to get an oriented arc b. Then, we push a into handlebody U_α and b into handlebody U_b. The original knot can now be expressed as $K = a \cup b$.

In this thesis, we are primarily concerned with nullhomologous knots inside $\#^l S^2 \times S^1$ (for any $l \in \mathbb{Z}$) and thus we will only describe the construction of knot Floer homology for this special case. As proved in [OS04b], the Floer homology of these 3-manifolds is supported in a single Spin^C structure $\mathfrak{s}_0$ where $c_1(\mathfrak{s}_0) = 0$ and the corresponding homology is

$$HF^{\infty}_{*}(\#^{l} S^2 \times S^1) \cong H_*(\mathbb{T}^l) \otimes \mathbb{F}[U, U^{-1}]$$
$$\widehat{HF}_{*}(\#^{l} S^2 \times S^1) \cong H_*(\mathbb{T}^l)$$

where $\mathbb{T}^l$ is the l-dimensional torus. In this case, the Maslov grading is a $\mathbb{Z}$-grading. To give a feel for how the computation of these invariants go, we have sketched the argument for the case $\widehat{HF}_{*}(\#^{2} S^2 \times S^1)$.

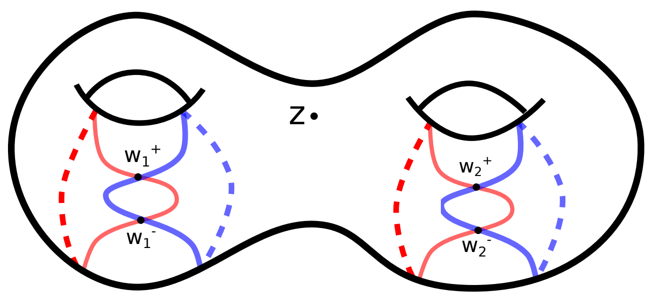


Figure 23: A pointed Heegaard diagram for $\#^2 S^2 \times S^1$.

Example 2.59. Note that we follow the argument of Section 9.1 of [OS04b], though our situation is slightly different as we are working over $\mathbb{F} = \mathbb{Z}_2$ instead of $\mathbb{Z}$ and thus do not need to be as concerned with orientation. From the diagram, we can see there are four total intersection points between α-curves and β-curves: w_1^+, w_1^-, w_2^+, and w_2^-. Thus, the generators of $\widehat{CF}(\#^2 S^2 \times S^1)$ are the unordered pairs with one component from each α and β curve

$$\{(w_1^+, w_2^+), (w_1^+, w_2^-), (w_1^-, w_2^+), (w_1^-, w_2^-).$$

By the argument in [OS04b], these generators have relative gradings

$$M((w_1^+, w_2^+), (w_1^-, w_2^-)) = 2$$
$$M((w_1^+, w_2^+), (w_1^+, w_2^-)) = 1$$
$$M((w_1^+, w_2^-), (w_1^-, w_2^+)) = 0$$
$$M((w_1^+, w_2^-), (w_1^-, w_2^-)) = 1.$$

For reasons which will soon become clear when knots enter the story, we will require the gradings to be symmetric and thus,

$$M(w_1^+, w_2^+) = 1$$
$$M(w_1^+, w_2^-) = 0 = M(w_1^-, w_2^+)$$
$$M(w_1^-, w_2^-) = -1.$$

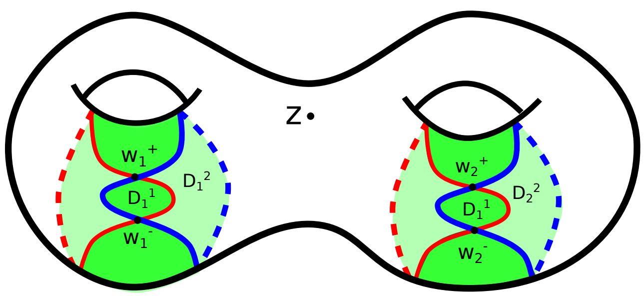


Figure 24: The disks D_1^1, D_1^2, D_2^1, and D_2^2.

Moreover, by [OS04b] we can see the four disks $D_1^1, D_1^2 \in \pi(w_1^+, w_1^-)$ and $D_2^1, D_2^2 \in$

$\pi(w_2^+, w_2^-)$ give us the differential

$$\partial(w_1^+, w_2^+) = (w_1^-, w_2^+) + (w_1^-, w_2^+) + (w_1^+, w_2^-) + (w_1^+, w_2^-) = 0$$
$$\partial(w_1^+, w_2^-) = (w_1^-, w_2^-) + (w_1^-, w_2^-) = 0$$
$$\partial(w_1^-, w_2^+) = (w_1^-, w_2^-) + (w_1^-, w_2^-) = 0$$
$$\partial(w_1^-, w_2^-) = 0$$

Therefore

$$\widehat{HF}_*(S^2 \times S^1) \cong \mathbb{F}_{(-1)} \oplus \mathbb{F}^2_{(0)} \oplus \mathbb{F}_{(1)}$$

Now, we obtain additional structure on $\widehat{HF}(\#^l S^2 \times S^1$ and $HF^\infty(\#^l S^2 \times S^1)$ when we have a null-homologous knot $K \subset \#^l S^2 \times S^1$. More specifically, K induces filtrations on these homologies by way of Seifert surfaces. Let $[F] \in H_2(\#^l S^2 \times S^1; \mathbb{Z})$ be the homology class of a Seifert surface for K. In [OS04a], Ozsváth and Szabó define a map

$$\underline{\mathfrak{s}} : \{\mathbb{T}_\alpha \cap \mathbb{T}_\beta\} \to \mathrm{Spin}^C((\#^l S^2 \times S^1)_0(K)) \cong \mathrm{Spin}^C(\#^l S^2 \times S^1) \times \mathbb{Z}$$

sending each intersection point to a Spin^C-structure on the 3-manifold obtained by 0-surgery along K. In the case where K is null-homologous, the group $\mathrm{Spin}^C((\#^l S^2 \times S^1)_0(K))$ is isomorphic to $\mathrm{Spin}^C(\#^l S^2 \times S^1) \times \mathbb{Z}$. Every class $\mathfrak{t} \in \mathrm{Spin}^C((\#^l S^2 \times S^1)_0(K))$ corresponds to the element $(\mathfrak{s}, \frac{1}{2}\langle c_1(\mathfrak{s}), [\widehat{F}]\rangle)$ where $\mathfrak{s}$ is obtained by restricting $\mathfrak{t}$ to $\#^l S^2 \times S^1 \setminus \nu(K)$ and then extending uniquely to $\#^l S^2 \times S^1$. In the second projection, $\widehat{F}$ is the above Seifert surface F capped off in $(\#^l S^2 \times S^1)_0(K)$ by a meridianal disk of the surgery torus.

As stated before, every generator of $HF^\infty(\#^l S^2 \times S^1)$ corresponds to the same Spin^C-structure $\mathfrak{s}_0$ on $\#^l S^2 \times S^1$ (in particular this is the Spin^C-structure whose first

Chern class is 0) so we can view the isomorphism $\text{Spin}^C((\#^l S^2 \times S^1)_0(K)) \cong \mathbb{Z}$ as a bijection from relative Spin^C-structures on 0-surgery along K to $\mathbb{Z}$ and identify the two sets.

In this way we can identify $\underline{\mathfrak{s}}(\mathbf{x})$ with the integer m with $\frac{1}{2}\langle c_1(\underline{\mathfrak{s}}(\mathbf{x})), [\widehat{F}]\rangle = m$ and refer to it as $\mathfrak{t}_m$ (in the literature some refer to this structure as merely "m"). In general $\mathfrak{t}_m$ does not depend on the homology class $[\widehat{F}]$ [OS04a]. In the case where K is the knotification of a link $L \subset S^3$, this class will depend on the relative homology class of a connected Seifert surface for L again by [OS04a].

The knot Floer complex, $CFK^\infty(\#^l S^2 \times S^1, [F], K, \mathfrak{s})$, is generated (as a $\mathbb{F}$-vector space) by triples $[\mathbf{x}, i, j]$ satisfying the constraint

$$\mathfrak{t}_m + (i-j)\text{PD}(\mu) = \mathfrak{t}_0$$

where $\mathfrak{t}_0$ is the unique Spin^C-structure on $(\#^l S^2 \times S^1)_0(K)$ extending $\mathfrak{s}_0$ with $\langle c_1(\mathfrak{t}_0), [\hat{F}]\rangle = 0$ and $\text{PD}(\mu) \in H^2((\#^l S^2 \times S^1)_0(K); \mathbb{Z})$ is the Poincaré dual to the meridian of K. In this setting, addition refers to the action of $H^2((\#^l S^2 \times S^1)_0(K); \mathbb{Z})$ on $\text{Spin}^C((\#^l S^2 \times S^1)_0(K))$ as detailed in [OS04b].

This complex has differential

$$\partial^\infty(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) | \mu(\phi) = 1\}} \big(\#\big(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\big) \text{ mod } 2\big)[\mathbf{y}, i - n_w(\phi), j - n_z(\phi)],$$

and is a $\mathbb{Z}[U]$-module where U acts by $U[\mathbf{x}, i, j] = [\mathbf{x}, i-1, j-1]$. When K is the knotification of a link $L \subset S^3$, this is the same complex as $CFK^\infty(L)$ described in [OS04a].

We can define a partial order on $\mathbb{Z} \oplus \mathbb{Z}$ by $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$ and, with this ordering, can see that $CFK^\infty(\#^l S^2 \times S^1, [F], K, \mathfrak{s})$ is a $\mathbb{Z} \oplus \mathbb{Z}$-filtered complex with the filtration defined on generators by

$$\mathcal{F} : CFK^\infty \to \mathbb{Z} \oplus \mathbb{Z}$$
$$[\mathbf{x}, i, j] \longrightarrow (i, j)$$

since $\mathcal{F}(\partial^\infty([\mathbf{x}, i, j]) \leq \mathcal{F}[\mathbf{x}, i, j]$ for all $[\mathbf{x}, i, j]$.

Fix $\mathfrak{t} \in \text{Spin}^C((\#^l S^2 \times S^1)_0(K))$ extending $\mathfrak{s} \in \text{Spin}^C(\#^l S^2 \times S^1)$. The forgetful map sending $[\mathbf{x}, i, j]$ to $[\mathbf{x}, i]$ induces an isomorphism of chain complexes $g : CFK^\infty(\#^l S^2 \times S^1, [F], K, \mathfrak{t}) \to CF^\infty(\#^l S^2 \times S^1, \mathfrak{s})$. Moreover, since $CF^\infty(\#^l S^2 \times S^1)$ is only supported in the Spin^C-structure $\mathfrak{s}_0$ as stated above, $CFK^\infty(\#^l S^2 \times S^1, [F], K, \mathfrak{t})$ has trivial homology unless $\mathfrak{t}$ extends $\mathfrak{s}_0$. Thus, we can suppress the Spin^C-structure from our notation and write $CFK^\infty(\#^l S^2 \times S^1, [F], K)$.

By previous discussion, this also means that the map on relative Spin^C structures is an isomorphism to $\mathbb{Z}$, since $HFK^\infty(\#^l S^2 \times S^1, K, F)$ splits along such Spin^C-structures, we have

$$HFK^\infty(\overset{l}{\#} S^2 \times S^1, K, F) = \bigoplus_{m \in \mathbb{Z}} HFK^\infty(\overset{l}{\#} S^2 \times S^1, K, F, \mathfrak{t}_m)$$

where each summand is isomorphic as a $\mathbb{F}[U, U^{-1}]$-module (though with shifted gradings). It is significant to also mention that this non-trivial isomorphism between $HFK^\infty(\#^l S^2 \times S^1, K, F)$ and $CF^\infty(\#^l S^2 \times S^1)$ is $\mathbb{F}[U]$-equivariant and gives a filtration on $CF^\infty(\#^l S^2 \times S^1)$ by taking $[\mathbf{x}, i] \cong U^{-i}\mathbf{x}$ to the image of $g^{-1}([\mathbf{x}, i])$ under the projection $[\mathbf{x}, i, j] \to j$. This is often referred to as the Alexander filtration (though there are other equivalent formulations of it) and we can view $HFK^\infty(K)$ as the associated graded corresponding to this filtration.

Often, $HFK^\infty(K)$ is constructed on a plane with the algebraic filtration (the filtration coming from CF^∞ corresponding to the U-action) as the x-axis and the

Alexander filtration as the y-axis.

Analogously to how we defined $\widehat{CF}(Y)$ earlier, we can restrict $CFK^{\infty}(\#^l S^2 \times S^1, [F], K)$ to the subcomplex $CFK^{0,*}(Y)$ generated by elements in the kernel of the U-action (i.e. elements with $i = 0$). The filtration on $CF^{\infty}(\#^l S^2 \times S^1)$ coming from K as described previously now induces a filtration on $\widehat{CF}(\#^l S^2 \times S^1)$ and the associated graded, called $\widehat{HFK}(\#^l S^2 \times S^1, [F], K))$, admits an extra $\mathbb{Z}$-grading called the Alexander grading.

In particular, denote by $\mathcal{F}(\#^l S^2 \times S^1, K, m)$ the subcomplex of $\widehat{CF}(\#^l S^2 \times S^1)$

$$\mathcal{F}(\#^l S^2 \times S^1, K, m) := CFK^{\infty}\{i = 0, j \leq m\}.$$

This filtration does not depend on the homology class $[\widehat{F}]$ by work in [OS04a] and [Hed08].

One of the more motivating facts from this theory is that the knot Floer homology of a link categorifies the Alexander-Conway polynomial of a link (when the grading is symmetric as in the earlier example). More precisely, when K is the knotification of an n-component link $L \subset S^3$, the Euler characteristic of $\widehat{HFK}(\#^l S^2 \times S^1, K, F)$ is related to the Alexander-Conway polynomial of L, $\Delta_L(t)$ by the formula

$$\sum \chi(\widehat{HFK}(\#^l S^2 \times S^1, K)) \cdot t^i = (t^{-1/2} - t^{1/2})^{n-1} \Delta_L(t).$$

It is worth mentioning that what we are describing in this section is the knot Floer homology of a link in S^3; there is a notion of link Floer homology of a link defined in [OS08] which has the same underlying complex as the knot Floer complex of the knotified link in sums of $S^{\times} S^1$. However, one must be careful about the details, as the knot Floer homology described here is doubly-pointed while link Floer homology has two basepoints for every link component. One major difference is that the link

Floer complex is filtered by a lattice, the details of which are discussed in [OS08]. We can think of the knot Floer homology of a link as categorifying the single variable Alexander polynomial while the link Floer homology categorifies the multivariable Alexander polynomial.

As our work is concerned with concordance, it is useful to note that these different flavors of Heegaard Floer homology have the structure of a $(3+1)$-TQFT in the sense that a 4-dimensional Spin^C cobordism $(W, \mathfrak{t})$ between manifolds Y_1 and Y_2 induces a map

$$F^{\circ}_{W,\mathfrak{t}} : HF^{\circ}(Y_1, \mathfrak{s}_1) \to HF^{\circ}(Y_2, \mathfrak{s}_2)$$

where $\circ$ can denote any of the above types of Heegaard Floer homology and $\mathfrak{t}\mid_{Y_i} = \mathfrak{s}_i$. In fact, these cobordism maps are compositions of maps corresponding to handle attachment in a handle decomposition of W which is detailed in [OS04a] and [OS03] and which affect the knot filtrations in a predictable way thanks to work on the knot filtration and "sufficiently large" integral surgeries along a knot K; we will use this feature to great effect later.

3 Knotified Links and Concordance Invariants from Group Theory

3.1 A link concordance monoid and Heegaard Floer homology

In Section 2.3, we detailed how defining a concordance group of links in S^3 is challenging due to the fact that the connected sum of links is not well defined. We then outlined previous definitions of link concordance group: the Hosokawa group $\mathcal{H}$ [Hos67] of links with the operation of disjoint union, Le Dimet's string link concordance group $\mathcal{C}(n)$ [LD88] for which we will go into greater detail in Section 4.1, and the Donald-Owens groups $\mathcal{L}$, $\mathcal{L}_0$ of marked links [DO12].

As $\mathcal{H} \cong \mathcal{C} \oplus \mathbb{Z}$ where $\mathcal{C}$ is the usual knot concordance group, it is clear that linking data is perhaps better captured by the groups $\mathcal{C}(n)$, $\mathcal{L}$, and $\mathcal{L}_0$. These groups are quite interesting in their own right; however, each group enlarges the set of links to give multiple representatives in each group for every link $L \subset S^3$. In fact, each n-component link in the 3-sphere has an infinite number of representatives in $\mathcal{C}(n)$ [KLW98] and n different representatives in $\mathcal{L}$ and $\mathcal{L}_0$.

These groups have extremely different structures; in particular, $\mathcal{C}(n)$ is non-abelian as it contains the pure braid group on n strands as a subgroup while both $\mathcal{L}$ and $\mathcal{L}_0$ are abelian. Thus, one of the goals of this thesis is to define a notion of link concordance group where each link has a unique representative. Ideally, such a group would provide insight into how much of the structure of previously defined link groups $\mathcal{C}(n)$, $\mathcal{L}$, and $\mathcal{L}_0$.

To get around this problem, in a collaboration with Matthew Hedden at Michigan State University we defined a new concordance group of links using the modern lens of Heegaard Floer homology. Note that all work in Subsections 3.1 and 3.4 is joint with Hedden. Instead of links in S^3 we will use a construction from [OS04b] to obtain a knot inside the connected sum of some number of copies of $S^1 \times S^2$ for every link in S^3.

Definition 3.1 (Knotified Link [OS04a])**.** Let L be an m-component link in S^3. The knotified version of this link is denoted $\kappa(L)$, lies in $\#^{m-1} S^1 \times S^2$, and is obtained in the following way:

Fix $m-1$ embedded copies of S^0 inside L labeled $\{p_i, q_i\}$ so that if each p_i and q_i are identified, the resulting quotient of L is a connected graph. Now, view each pair as the feet of a 4-dimensional 1-handle we can attach to B^4, and note that looking at the boundary of the result we get L inside $\overset{m-1}{\#} S^1 \times S^2$. We can now band sum the components of L together inside the boundary of these 1-handles to get a knot $\kappa(L)$ inside $\overset{m-1}{\#} S^1 \times S^2$. We call $\kappa(L)$ the knotification of the link L. This operation is well-defined.

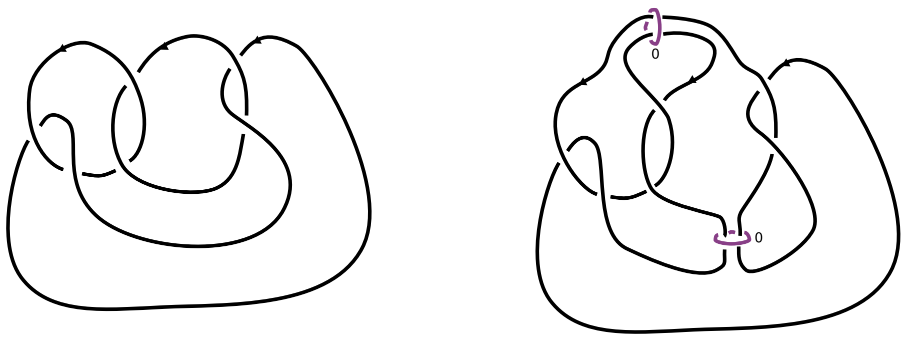


Figure 25: A link $L \subset S^3$ and its knotification $\kappa(L) \subset S^1 \times S^2 \# S^1 \times S^2$.

In this thesis, a "knotified link" will refer to any knot in any connected sum of $S^1 \times S^2$ obtained through this process. As we will see, this allows us to define a binary operation. Notice that for knots K and J inside an arbitrary, oriented 3-manifold Y, the connected sum of K and J can always be defined via

$$K \# J := (Y, K) \# (Y, J)$$

though the resulting knot is now in the 3-manifold $Y \# Y$. This operation is well defined; however, we must take care to define the underlying set so this operation is closed. In the case of knots in S^3, $S^3 \# S^3$ is homeomorphic to S^3 and thus we were not concerned with this issue. For $Y \neq S^3$, we have the following definitions.

Definition 3.2 (Stabilization of K)**.** Let K be a knot in $\#^l S^1 \times S^2$. Isotope K so it lies away from a small ball. Choose an unknot in this ball and perform 0-surgery on it to get K' inside $\#^{l+1} S^1 \times S^2$, call this a stabilization of K.

Definition 3.3 (Destabilization of K)**.** Let K be a knot in $\#^l S^1 \times S^2$ such that $\kappa(L)$ can be isotoped to lie away from a $S^1 \times S^2$ factor. Choose a simple closed curve representing a generator of the homology of this $S^1 \times S^2$ and perform 0-surgery on it. The result is a knot K' in $\#^{l-1} S^1 \times S^2$, which we call a destabilization of K.

With this language we can now define a monoid.

Proposition 3.4. *Let* $D_i = \{\text{knots } K \subset \#^i S^1 \times S^2\}$*. Then* $\mathfrak{D} := \left(\varinjlim D_i, \#\right)$ *is a monoid.*

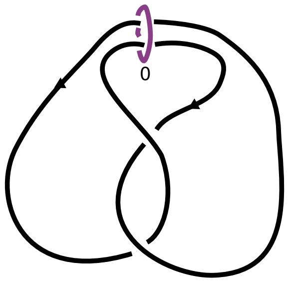
Proof. From the above discussion, connected sum of knotified links is closed on this set. Moreover, the trivial knot in S^3 (viewed as a connected sum of 0 copies of $S^1 \times S^2$) still functions as an identity element. □

While in this work we specifically focus on knots in $\#^l S^2 \times S^1$ for some l which are constructed from a link in S^3 in the following sense, there is nothing inherent to the group construction which limits us to this subset so we state the most general case first for the sake of completeness.

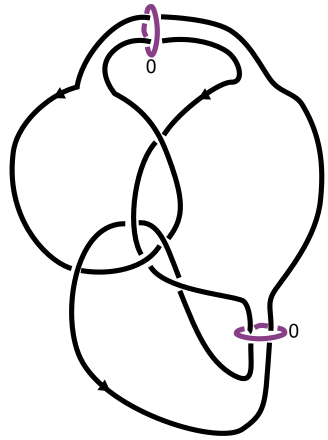
Example 3.5. κ(Hopf link) $\subset S^1 \times S^2$, κ(Borromean rings) $\subset \#^2 S^1 \times S^2$, and their connected sum κ(Hopf link) $\#$ κ(Borromean rings) $\subset \#^3 S^1 \times S^2$ are shown in figure 26.

Proposition 3.6. *Let* $C_i = \{\text{knotified links and stabilizations of knotified links}\}$ *inside* $\#^i S^1 \times S^2\}$*. Then* $\mathfrak{C} := \left(\varinjlim C_i, \#\right)$ *is a sub-monoid of* $\mathfrak{D}$*.*

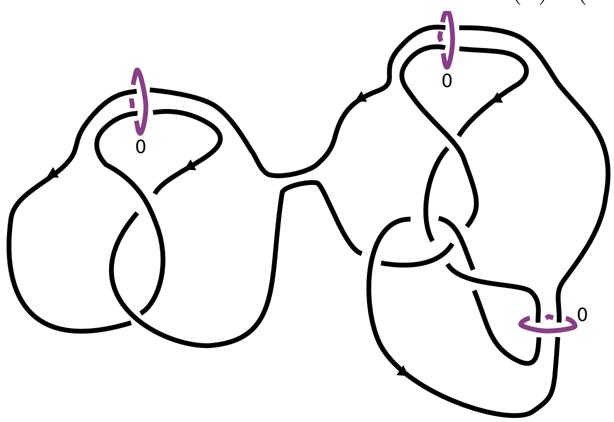
Proof. It is clear that the set of knotified links and stabilizations of them is closed under connected sum as we can isotope such a knot in order to find a separating S^2 and attach arcs to the band intersecting this S^2 to construct two (possibly stabilized) knotified links. □



(a) κ(Hopf link)



(b) κ(Borromean rings)



(c) κ(Hopf link) # κ(Borromean rings)

Figure 26: An example of the connected sum of two knots.

Now, we have constructed a monoid whose elements are constructed from links in S^3. As a brief digression, recall that we were led to think about these knotified links using a construction from Heegaard Floer homology. By approaching the problem of defining a link concordance group from this angle, we may be able to use the package of invariants from Heegaard Floer homology to lend insight about the group. We have briefly outlined the relevant ideas from this theory in Section 2.5. For a more detailed overview of Heegaard Floer, there are survey papers by Ozsváth and Szabó [OS17], Hom [Hom17], and Manolescu [Man14].

One of the major concordance invariants to arise from this Heegaard Floer package is the τ- invariant defined by Ozsváth and Szabó in [OS03]. Among many other significant results, τ is homomorphism from the classical knot concordance group to $\mathbb{Z}$, it gives a lower bound on the 4-ball genus of a knot, and results in an alternate proof of the Milnor conjecture for torus knots.

Its definition was generalized by Hedden in [Hed08] and it is this definition we will use in our work. Let I_m be the map on homology induced by the inclusion

$$\iota_m : \mathcal{F}(\#^l S^2 \times S^1, K, m) \hookrightarrow \widehat{CF}(\#^l S^2 \times S^1).$$

Definition 3.7 (Hedden [Hed08])**.** Let $[x] \in \widehat{HF}(\#^l S^2 \times S^1)$ be a non-zero Floer homology class. The τ-invariant corresponding to $[x]$ is

$$\tau_{[x]}(\#^l S^2 \times S^1, K) := \min\{m \in \mathbb{Z} \mid [x] \subset \text{Im } I_m\}$$

Theorem 3.8 (Hedden [Hed08])**.** *$\tau_{[x]}(\#^l S^2 \times S^1, K)$ is an invariant of $(\#^l S^2 \times S^1, K)$.*

Moreover, the original τ-invariant from [OS03] can be thought of as a special case of $\tau_{[x]}$ when $l = 0$. Note that there is one non-zero element of $\widehat{HF}(S^3)$ and thus there is only one corresponding τ invariant for knots in this 3-manifold.

Note that $\widehat{HF}(S^3) \cong \mathbb{F}$ and thus there is only one possible τ-invariant (the one originally defined in [OS03]. This invariant is useful for studying the classical knot concordance group because,

Theorem 3.9 (Ozsváth and Szabó [OS03])**.** *Let W be a smooth, oriented 4-manifold with $B_2^+(W) = 0 = b_1(W)$ and $\partial W = S^3$. If Σ is any smooth, properly embedded surface with boundary $\partial\Sigma = K$ in W, then we have the following inequality:*

$$2\tau(K) + \mid [\Sigma] \mid + [\Sigma] \cdot [\Sigma] \leq 2g(\sigma).$$

Theorem 3.10 (Properties of τ [OS03]). • *τ is additive under connected sum;*

- *$\tau : \mathcal{C} \to \mathbb{Z}$ is a homomorphism;*
- *$\mid \tau(K) \mid \leq g^*(K)$ where $g^*(K)$ is the genus of a minimal smooth, properly embedded surface in B^4 with boundary K. This is called the 4-ball genus of K.*

Though we get a τ invariant for every non-zero element of $\widehat{HF}(\#^l S^2 \times S^1)$, as we are concerned with concordance and the 4-dimensional implications of these invariants we will restrict to the following family of $\tau_{[x]}$.

Fix a non-negative integer l. The 3-manifold $\#^l S^2 \times S^1$ can be viewed as the boundary of l diferent possible 4-manifolds $W_i = \natural^i S^1 \times B^3 \natural^{l-i} S^2 \times D^2$ for $i \in \{1, ..., l\}$. Removing a B^3 from the interior of W_i gives a different smooth, Spin^C cobordism from S^3 to $\#^l S^2 \times S^1$ and each W_i can be understood as the result of attaching i 4-dimensional 1-handles and $l - i$ 4-dimensional 2-handles to $\#^l S^2 \times S^1 \times I$.

Thus, each W_i has a corresponding map on the Floer homology; namely,

$$\widehat{F}_i : \widehat{HF}(S^3) \to \widehat{HF}(\#^l S^2 \times S^1)$$

sending the generator $[x] \in \widehat{HF}(S^3)$ to a non-zero element $\Theta_i \in \widehat{HF}(\#^l S^2 \times S^1)$. Because these maps merely correspond to handle attachment, we can predict exactly which element Θ_i will be based on which bounding 4-manifold we are concerned with.

Definition 3.11. Let $K \in \#^l S^2 \times S^1$ be a nullhomologous knot. Then we define

$$\tau_i := \min\{m \in \mathbb{Z} \mid \Theta_i \in \text{Im } I_m\}.$$

If $L \subset S^3$, we can also define

$$\tau_i(L) := \tau_i(\kappa(L)).$$

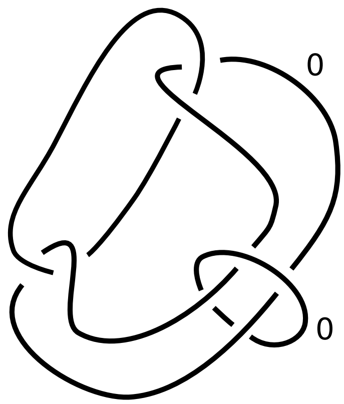


Figure 27: The so-called Borromean knot $B \subset \#^2 S^2 \times S^1$.

Example 3.12. Recall from Example 2.59 the Heegaard Floer homology of $\#^2 S^2 \times S^1$ is

By Proposition 9.2 in [OS04a], the hat flavor of knot Floer homology of $B \subset \#^2 S^2 \times S^1$ is

$$\widehat{HF}_*(S^2 \times S^1) \cong \mathbb{F}_{(-1)} \oplus \mathbb{F}^2_{(0)} \oplus \mathbb{F}_{(1)}$$

and from Proposition 9.2 [OS04a] we can gather that

$$\widehat{HFK}(\#^2 S^2 \times S^1, k, m) \cong \begin{cases} \mathbb{F}_{(1)}, & \text{if } m = 1 \\ \mathbb{F}^2_{(0)}, & \text{if } m = 0 \\ \mathbb{F}_{(-1)}, & \text{if } m = -1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Now, we see that $\#^2 S^2 \times S^1$ is the boundary of three different 4-manifolds:

$$W_0 = S^2 \times D^2 \natural S^2 \times D^2$$
$$W_1 = B^3 \times S^1 \natural S^2 \times D^2$$
$$W_2 = B^3 \times S^1 \natural B^3 \times S^1$$

which correspond to adding 4-dimensional 1 and 2-handles to $(\#S^2 \times S^1) \times I$. Thanks to [OS04b], we see that

$$\begin{aligned}
\tau_0 &= \min\{m \in \mathbb{Z} \mid (w_1^-, w_2^-) \in \text{Im } I_m\} \\
&= -1 \\
\tau_1 &= \min\{m \in \mathbb{Z} \mid (w_1^+, w_2^-) \in \text{Im } I_m \text{ or } (w_1^-, w_2^+) \in \text{Im } I_m\} \\
&= 0 \\
\tau_2 &= \min\{m \in \mathbb{Z} \mid (w_1^+, w_2^+) \in \text{Im } I_m\} \\
&= 1
\end{aligned}$$

Unpublished work of Hedden indicates that these τ-invariants provide a bound on the genera of surfaces bounded by a null-homologous knot in $\#^l S^2 \times S^1$. In our ongoing work together we plan to use these invariants to construct homomorphisms from the link concordance group we will define in Section 3.4.

In this thesis, we use these Heegaard Floer constructions as a starting point to understand a group structure on the set of links in the 3-sphere modulo concordance. Recall that Section 2.3 illustrated that defining such a group is difficult because the connected sum of links in S^3 is not well defined. Due to [OS04b], we now have a monoid of knots in sums of $S^2 \times S^1$ for which the connected sum is defined. Moreover, we have a package of powerful, gauge theoretic invariants to study these knots with.

However, in order to understand these objects and what they have to do with links we must first understand what is topologically happening to links when we knotify them. Furthermore, we should understand how the concordance of knots in $\#^l S^2 \times S^1$ differs from the concordance of knots and links in S^3. For that, we turn to group theory.

3.2 Concordance data in the lower central series

In Section 2.4 we detected subtle higher order linking data for a link in the 3-sphere using the lower central series quotients of the fundamental group of the link complement. It is natural to ask whether this approach could detect concordance information for knots and links in other 3-manifolds. As this thesis will demonstrate, the nilpotent quotients of a knot complement in more complicated 3-manifolds prove to be quite useful. In Section 2.4, we introduced a theorem of Casson which can in fact be generalized by the following result.

Proposition 3.13 (Cappell-Shaneson [CS73])**.** *Let M be a closed, oriented 3-manifold and $K_1 \subset M$ be a nullhomologous knot. If K_1 is topologically concordant (by a concordance $C \cong S^1 \times I$) to another knot K_2 inside of $M \times I$, then the inclusion maps*

$$\iota_i : (M \times \{n_i\}) \setminus \nu(K_i) \to (M \times I) \setminus C, \; with n_i = 0, 1$$

induce isomorphisms on homology with $\mathbb{Z}[\pi_1(M)]$-coefficients.

Combining Theorem 3.13 with Theorem 2.34 leads us to to note,

Corollary 3.14. *If K_1 and K_2 are smooth knots inside a closed, oriented 3-manifold M with $G = \pi_1(M \setminus \nu(K_1)$ and $H = \pi_1(M \setminus \nu(K_2)$, then $G/G_q \cong H/H_q$ for all q.*

Therefore, we might think to use the same construction as in Section 2.4. However, this approach does not work for knots and links in arbitrary 3-manifolds as we can no longer use the combinatorial group theory tools from Theorems 2.37 and 2.38. In this thesis, we constructed an invariant using this nilpotent group quotients using a somewhat different approach. However, we will see in Section 3.3 that this invariant shares many of the useful properties of Milnor's invariants. Note that other generalizations of Milnor's invariants are those of Miller [Mil95] for knots in certain non-trivial homotopy classes of Seifert fiber spaces and that of Heck [Hec11] for non-trivial knots in prime manifolds.

First we must go back to the original proof of Theorem 2.34 and examine why the condition relating maps on group homology to nilpotent quotients of the respective groups is sufficient. Stallings proved this theorem using the following sequence.

Theorem 3.15 (Stallings [Sta65])**.** *For a group G with normal subgroup N there is a natural exact sequence*

$$H_2(G) \to H_2\left(\frac{G}{N}\right) \to \frac{N}{[G,N]} \to H_1(G) \to H_1\left(\frac{G}{N}\right) \to 0$$

.

In the case that $N = G_n$, we have

$$H_2(G) \to H_2\left(\frac{G}{G_n}\right) \to \frac{G_n}{G_{n+1}} \to H_1(G) \to H_1\left(\frac{G}{G_n}\right) \to 0.$$

Note that Theorem [Sta65] follows from a clever application of Theorem 3.15. This sequence indicates the significance of the kernel of the map $H_2(G) \to H_2(G/G_n)$. This is called the n^{th} Dwyer subgroup of G, and we denote it by $\Phi_n(G)$. By [CH08], we can also think of this subgroup from a topological perspective.

Definition 3.16 (Grope)**.** A half-grope in X is a surface $f : \Sigma \to X$ (with Σ oriented) such that, for some standard symplectic basis of curves for Σ

$$\{\, a_i, b_i \mid 1 \leq i \leq \text{ g}(\Sigma) \,\}$$

we have

$$f_*([a_i]) \subset \pi_1(X)_n.$$

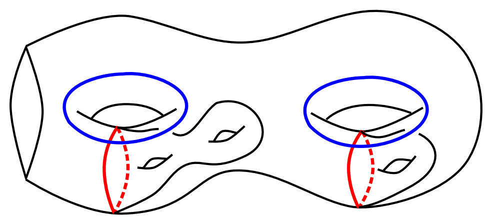


Figure 28: A class 3 half-grope.

We often abuse notation and identify a half grope with its image. Additionally, note that Σ is allowed to have boundary. Intuitively, we can view half gropes as a topological repackaging of the lower central series of the fundamental group of the space. With that in mind, we can reformulate the definition of this subgroup in the following way. For a more extensive discussion of the connection between gropes, lower central series quotients, and Milnor's invariants, see [FT95].

Definition 3.17 (Cochran-Harvey [CH08])**.** Let X be a topological space. Define the n^{th} Dwyer subgroup of X as the set

$$\Phi_n(X) = \{x \in H_2(X) \mid x \text{ can be represented a class } n+1 \text{ half-grope}\}.$$

This is a subgroup of $H_2(X)$ and, for a group G, notice $\Phi_n(K(G,1)) = \Phi_n(G)$ as originally defined Dwyer .

Recall that Theorem 2.34 provided a sufficient condition for all lower central series quotients of a group to be isomorphic. The Dwyer subgroup allows us to strengthen this result.

Theorem 3.18 (Dwyer's Integral Theorem [Dwy75]). *Let $\varphi : A \to B$ be a homomorphism that induces an isomorphism on $H_1(-;\mathbb{Z})$. Then for any positive integer n the following are equivalent:*

1. *φ induces an isomorphism $A/A_n \cong B/B_n$*
2. *φ induces an epimorphism from $H_2(A;\mathbb{Z})/\Phi_n(A)$ to $H_2(B;\mathbb{Z})/\Phi_n(B)$*
3. *φ induces an isomorphism from $H_2(A;\mathbb{Z})/\Phi_n(A)$ to $H_2(B;\mathbb{Z})/\Phi_n(B)$ and a monomorphism from $H_2(A;\mathbb{Z})/\Phi_{n+1}(A)$ to $H_2(B;\mathbb{Z})/\Phi_{n+1}(B)$*

In Section 2.4, we saw Milnor's invariants give us concordance invariants for a link $L \subset S^3$ detecting when the nilpotent quotients of $\pi_1(S^3 \setminus \nu(L)$ are isomorphic to the nilpotent quotients of a free group with appropriate rank. Now, we can use Theorem 3.18 to construct a concordance invariant for an arbitrary knot $K \subset M$ where M is a closed, oriented 3-manifold using similar principles.

Definition 3.19. Let M be a oriented, closed 3-manifold and γ be a fixed smooth, embedded curve inside M. Let $[\gamma]$ be its corresponding free homotopy class. Let $K \subset M$ be a smooth knot with free homotopy class $[K] = [\gamma]$. Then the Dwyer number of K relative to γ is

$$D(K,\gamma) = \max \left\{ q \;\middle|\; \frac{\mathrm{H}_2(M \setminus \nu(K))}{\Phi_q(M \setminus \nu(K))} = \frac{\mathrm{H}_2(M \setminus \nu(\gamma))}{\Phi_q(M \setminus \nu(\gamma))} \right\}$$

It is not difficult to see why this invariant is well defined; by work of Massey [Mas81] on generalizing Alexander duality, the homology groups of the complement of a smooth, embedded curve in M are well defined. Moreover, since the Dwyer subgroups are nested and corresponding maps are natural, it is not possible for

$$\frac{\mathrm{H}_2(M \setminus \nu(K))}{\Phi_r(M \setminus \nu(K))} \cong \frac{\mathrm{H}_2(M \setminus \nu(\gamma))}{\Phi_r(M \setminus \nu(\gamma))}$$

but

$$\frac{\mathrm{H}_2(M \setminus \nu(K))}{\Phi_s(M \setminus \nu(K))} \ncong \frac{\mathrm{H}_2(M \setminus \nu(\gamma))}{\Phi_s(M \setminus \nu(\gamma))}$$

for $s \geq r$.

Moreover, $D(K, \gamma)$ is a concordance invariant just like the Milnor invariants it was inspired by.

Theorem 3.20. *$D(K, \gamma)$ is an invariant of concordance in $M \times I$.*

Proof. Let K be a smooth, oriented knot in M in the same free homotopy class as γ and let $G = \pi_1(M \setminus \nu(K))$. If J is another smooth, oriented knot in M in the same smooth concordance class as K with fundamental group $N = \pi_1(M \setminus \nu(J))$, it follows immediately that K and J are freely homotopic and therefore $D(J, \gamma)$ is well-defined. By Proposition 3.13 and Theorem 2.34, $G/G_q \cong N/N_q$ for all q since K and J are concordant. Now, by Theorem 3.18 this implies $H_2(G)/\Phi_q(G) \cong H_2(N)/\Phi_q(N)$ for all q.

This almost gives us the result; what remains to be shown is that these quotient groups can be computed in terms of homology of the corresponding spaces instead of homology of their groups. Recall that we can construct a $K(G, 1)$ by attaching cells of dimension 3 and higher to $(M \setminus \nu(K)$. Furthermore, the homology groups of G are the homology groups of this $K(G, 1)$, thus, we have the following exact sequence

$$\pi_2(M \setminus \nu(K)) \to H_2(M \setminus \nu(K)) \to H_2(G) \to 0.$$

Now, consider the Dwyer subgroup $\Phi_n(M \setminus \nu(K)) \subset H_2(M \setminus \nu(K))$ and notice that every element of $\pi_2(M \setminus \nu(K))$ maps to an element inside $\Phi_n(M \setminus \nu(K))$ since a 2-sphere is a half-grope of arbitrary large class. Therefore, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & \pi_2(M \setminus \nu(K)) & \searrow & & & & & \\
 & & \downarrow & & & & & & \\
1 & \longrightarrow & \Phi_q(M \setminus \nu(K)) & \longrightarrow & H_2(M \setminus \nu(K)) & \longrightarrow & \frac{H_2(M\setminus\nu(K))}{\Phi_q(M\setminus\nu(K))} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Phi_q(G) & \longrightarrow & H_2(G) & \longrightarrow & \frac{H_2(G)}{\Phi_q(G)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

and thus

$$\frac{H_2(G)}{\Phi_q(G)} \cong \frac{H_2(M \setminus \nu(K))/\mathrm{Im}(\pi_2(M \setminus \nu(K)))}{\Phi_q(M \setminus \nu(K))/\mathrm{Im}(\pi_2(M \setminus \nu(K)))} \cong \frac{H_2(M \setminus \nu(K))}{\Phi_q(M \setminus \nu(K))}$$

by the first and third isomorphism theorems.

□

The utility of this approach is that it allows us to do computations involving the homology of the fundamental group of a knot complement using iterated surfaces and to geometrically realize these computations.

3.3 A generalization of Milnor's invariants

In special cases, it turns out this invariant is straightforward to work with and has direct connections to Milnor's invariants for links. In this thesis, we will focus on the case of null-homologous knots $K \subset \#^l S^2 \times S^1$ which has applications to the study of link concordance in S^3 as detailed in Section 3.1; we will address other cases in future work. Results in this section about the concordance of knots in $\#^l S^2 \times S^1 \times I$

will indicate how different this case is from that of knots in S^3. Furthermore, these results will motivate the definition of a new link concordance group in Section 3.4.

Corollary 3.21. *Let $K \subset \#^l S^2 \times S^1$ be a null-homologous knot and γ be an unknot in $\#^l S^2 \times S^1$. In this case, denote $D(K,\gamma)$ by $D(K)$ and see that*

$$D(K) = max\ \{\ q\ \mid\ \frac{H_q(\#^l S^2 \times S^1 \setminus K)}{\Phi_q(\#^l S^2 \times S^1 \setminus K)} = 0\ \}$$

Proof. A simple Seifert-van Kampen argument shows $\pi_1(M \setminus \nu(\gamma), *)$ is a free group F on l generators. Since $H_2(F) = 0$, the corollary follows. □

As we will see, the case for nullhomologous knots in $\#^l S^2 \times S^1$ is relevant to defining a group structure on links in the 3-sphere modulo concordance. Morally, we can view Milnor's invariants for links $L \subset S^3$ as detecting how deep the longitudes of L lie in the lower central series of the link group. In a similar fashion, the Dwyer number for nullhomologous knots in $\#^l S^2 \times S^1$ detects how deep the 0-framed longitude of $K \subset \#^l S^2 \times S^1$ is in the lower central series of the knot group (and thus how "close" the group quotients of the knot group are to the group quotients of the corresponding free group). Similarly, it also detects the length of the first non-zero Massey product in the cohomology of the complement of the knot K in an analogous way to the celebrated theorem of Turaev [Tur79] and independently Porter [Por80] relating Milnor's invariants of links in S^3 to Massey products in the link complement. The case of $K \subset \#^l S^2 \times S^1$ may serve as a model for the computation of $D(K,\gamma)$ in more general situations as we will explore in later work.

To compute $D(K)$ we find the following lemma useful.

Proposition 3.22. *Let K be a nullhomologous knot in $\#^l S^2 \times S^1$ and $\nu(K)$ be a tubular neighborhood of it. Then*

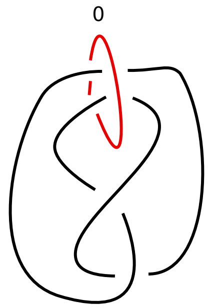


Figure 29: A nullhomologous knot in $S^1 \times S^2$.

$$H_0(\overset{l}{\#} S^2 \times S^1 \setminus \nu(K)) = \mathbb{Z}$$

$$H_1(\overset{l}{\#} S^2 \times S^1 \setminus \nu(K)) = \mathbb{Z}^{l+1}$$

$$H_2(\overset{l}{\#} S^2 \times S^1 \setminus \nu(K)) = \mathbb{Z}^{l}$$

$$H_n(\overset{l}{\#} S^2 \times S^1 \setminus \nu(K)) = 0 \textit{ for } n \geq 3$$

Moreover, $H_1(\overset{l}{\#} S^2 \times S^1 \setminus \nu(K)) =< \mu, d_1, ..., d_l >$ *where* μ *is a meridian of* K *and* $d_1, ..., d_l$ *generate* $H_1(\overset{l}{\#} S^2 \times S^1)$. $H_2(\overset{l}{\#} S^2 \times S^1 \setminus \nu(K)) = \mathbb{Z}^l$ *is generated by surfaces consisting of the* S^2 *generators of* $H_2(\#^l S^2 \times S^1)$ *together with the appropriate tubes along* K.

Proof. Let $L = (K, L_1, ..., L_l)$ as described above. Consider a Mayer-Vietoris sequence in reduced homology with $A = \nu_\epsilon(K)$ and $B = \#^l S^2 \times S^1 \setminus \nu_{\epsilon/2}(K)$. Then $A \cup B = \#^l S^2 \times S^1$ and $A \cap B$ deformation retracts to $\partial\nu(K)$. Note that $\mathrm{H}_1(\#^l S^2 \times S^1) = \mathbb{Z}^l$ is generated by meridians d_i, $1 \leq i \leq l$ of unlink $L' = (L_1, ..., L_l)$. $\mathrm{H}_2(\#^l S^2 \times S^1) = \mathbb{Z}^l$ is generated by embedded 2-spheres e_i, $1 \leq i \leq l$ such that each sphere e_i is the union of the disk Σ_i bounded by L_i in S^3 and the disk D_i bounded by L_i in the surgery solid torus.

Now, $\partial_* : \mathrm{H}_3(\#^l S^2 \times S^1) \to \mathrm{H}_2(\partial\nu(K))$ is clearly an isomorphism from the definition of this map, so we have the following long exact sequence:

$$0 \to \tilde{\mathrm{H}}_2(\overset{l}{\#} S^2 \times S^1 \setminus \nu(K)) \to \mathbb{Z}^l \to \mathbb{Z}^2 \to \mathbb{Z} \oplus \tilde{\mathrm{H}}_1(\overset{l}{\#} S^2 \times S^1 \setminus \nu(K)) \to \mathbb{Z}^l \to 0$$

Since the last term is a projective module, there is a splitting map $s : \mathbb{Z}^l \to \mathbb{Z} \oplus \tilde{\mathrm{H}}_1(\#^l S^2 \times S^1 \setminus \nu(K))$ and thus by exactness $\mathbb{Z} \oplus \tilde{\mathrm{H}}_1(\#^l S^2 \times S^1 \setminus \nu(K)) \cong \mathbb{Z}^l \oplus im(f : \mathbb{Z}^2 \to \mathbb{Z} \oplus \tilde{\mathrm{H}}_1(\#^l S^2 \times S^1 \setminus \nu(K)))$ where f sends meridian μ to $(0, -j_*(\mu))$ and longitude λ to $(\lambda, 0)$ since K is nullhomologous. Hence $\mathbb{Z} \oplus \tilde{\mathrm{H}}_1(\#^l S^2 \times S^1 \setminus \nu(K)) \cong \mathbb{Z}^l \oplus \mathbb{Z} \oplus < -j_*(\mu) > \cong \mathbb{Z}^{l+1} \oplus < -j_*(\mu) >$.

Now we should determine the order of $-j_*(\mu) \in \tilde{\mathrm{H}}_1(\#^l S^2 \times S^1 \setminus \nu(K))$. If $d(-j_*(\mu)) = 0$, then Ker $(\tilde{\mathrm{H}}_1(\partial \nu(K)) \to \mathbb{Z} \oplus \tilde{\mathrm{H}}_1(\#^l S^2 \times S^1 \setminus \nu(K))$, which is equal to $d\mathbb{Z} = \mathrm{Im}\ (\tilde{\mathrm{H}}_2(\#^l S^2 \times S^1) \cong \mathbb{Z}^l \to H_1(\nu(K)) \cong \mathbb{Z}^2)$.

Since K is nullhomologous, viewing K as a 1-component sublink of L as above gives $lk(K, L_i) = 0$ which implies that the algebraic intersection $K \cap \Sigma_i = 0$. From this surgery perspective of $K \subset \#^l S^2 \times S^1$, it is clear that the geometric intersection $K \cap D_i = 0$. Thus, $\nu(K)$ intersects generator $e_i = [\Sigma_i \cup_{L_i} D_i]$ in pairs of oppositely oriented meridians of K (or does not intersect e_i at all) . So Im $(\mathbb{Z}^l \to \mathbb{Z}^2) = 0$ and therefore $-j_*(\mu)$ is infinite order.

Now, we have shown $\mathbb{Z} \oplus \tilde{\mathrm{H}}_1(\#^l S^2 \times S^1 \setminus \nu(K)) \cong \mathbb{Z}^{l+2}$ and thus $\tilde{\mathrm{H}}_1(\overset{l}{\#} S^2 \times S^1 \setminus \nu(K)) \cong \mathbb{Z}^{l+1}$. Finally, by exactness we see that im $(\tilde{\mathrm{H}}_2(\#^l S^2 \times S^1 \setminus \nu(K)) \to \mathbb{Z}^l) = \mathbb{Z}^l$ and therefore $\tilde{\mathrm{H}}_2(\#^l S^2 \times S^1 \setminus \nu(K)) \cong \mathbb{Z}^l$ □

Recall from Section 2.4 that, in order to define Milnor's invariants, we exploited a specific presentation of the nilpotent quotients of the fundamental group of the link complement in S^3. We obtain a similar presentation in this context.

Lemma 3.23. *Let $K \subset \#^l S^2 \times S^1$ be a nullhomologous knot with 0-framed longitude l and $G = \pi_1(\#^n S^1 \times S^2 \setminus \nu(K), *)$ the fundamental group of a neighborhood of its* *complement. There is a homomorphism* $\varphi : F \to G$ *where* F *is the free group on* $l+1$ *letters and a word* $R_q(l)$ *with* $\varphi(R_q(l)) \cong l$ *mod* G_q *such that* G/G_q *has the following presentation:*

$$\langle x, a_1, ..., a_n \mid [x, R_q(l)], F_q \rangle$$

where F_q *are the weight* q *commutator relations and each* a_i *generates the* i^{th} $\mathbb{Z}$ *summand in* $H_1(\#^l S^2 \times S^1)$*.*

Proof. Notice that $K \subset \#^m S^1 \times S^2$ can be viewed as the result of 0-surgery on a sublink L' of the ordered, oriented $m+1$ component link $L = (K, U_1, U_2, ..., U_m) \subset S^3$. Here H will denote the group $\pi_1(S^3 \setminus L, *)$, and by a theorem of Milnor in [Mil57] the quotients H/H_q have the presentation

$$\langle x, a_1, ..., a_m \mid [x, R_q(l_1)], [a_1, R_q(l_2)], ..., [a_n, R_q(l_{m+1})], F_q \rangle$$

where l_i is the i^{th} longitude of L and, for the meridian homomorphism $\phi : F \to H$, $R_q(l_i)$ is a word such that $\phi(R_q(l_i)) \cong l_i$ mod H_q. Since $K \subset \#^m S^1 \times S^2$ is obtained from $L \subset S^3$ by killing longitudes $l_2, ..., l_{m+1}$ we can see

$$G \cong H/\langle l_i \mid 2 \leq i \leq m+1 \rangle.$$

The quotient map $p : H \twoheadrightarrow G$ induces surjections $p_q : H/H_q \twoheadrightarrow G/G_q$. Since the kernel of p is generated by longitudes l_i, $2 \leq i \leq m+1$ and each $l_i \cong R_q(l_i)$ mod H_q we see that $p_q(R_q(l_i)) = 1$ and G/G_q is presented by

$$\langle x, a_1, ..., a_n \mid [x, R_q(l)], F_q \rangle.$$

□

This allows us to quickly obtain the following result, showing that $D(K)$ also detects how deep the longitude of K is in the lower central series of the fundamental group of its complement.

Theorem 3.24. *If $K \subset \#^n S^2 \times S^1$ is null-homologous and $D(K) > q$, then the 0-framed longitude of K lies in G_{q-1} where $G = \pi_1(\#^n S^2 \times S^1 \setminus K, *)$.*

Proof. Consider the homomorphism $\varphi : F = \langle x, a_1, ..., a_n \rangle \to G$ sending x to a meridian of K and each a_i to a loop homotopic to the i^{th} copy of $S^1 \times \{0\}$. This homomorphism clearly induces isomorphisms $H_1(F) \to H_1(G) \cong \mathbb{Z}^{n+1}$. $D(K) > q$ implies $\mathrm{H}_2(\#^l S^2 \times S^1 \setminus \nu(K))/\Phi_q(\#^l S^2 \times S^1 \setminus \nu(K)) = 0$. As in the proof that $D(K)$ is a concordance invariant, this implies $H_2(G)/\Phi_q(G) = 0$ and therefore by Dwyer's theorem [Dwy75] φ induces an isomorphism $F/F_q \to G/G_q$.

By lemma 3.23, we also know

$$\frac{G}{G_q} \cong \langle x, a_1, ..., a_n \mid [x, R_q(l)], F_q \rangle$$

and for it to be isomorphic to F/F_q, $[x, R_q(l)] \in F_q$. Thus, $[x, R_q(l)]$ is trivial in F/F_q. Since x is a generator of F, this implies $R_q(l) \in F_{q-1}$ and thus is trivial in F/F_{q-1}. However, by properties of lower central series quotients we know $F/F_q \cong G/G_q$ implies $F/F_{q-1} \cong G/G_{q-1}$ so $\varphi(R_q(l))$ is trivial in G/G_{q-1}. Recall $\varphi(R_q(l)) = la$ where $a \in G_q$. By the third isomorphism theorem, $G/G_{q-1} \cong (G/G_q)/(G_{q-1}/G_q)$. The image of a longitude l in this quotient is in the same class as la (as they differ by an element of G_q). Thus, the residue class of a longitude $l \in G$ inside G/G_{q-1} is mapped under this isomorphism to the class represented by la which is trivial. Thus, $l \in G_{q-1}$. □

We further see that this Dwyer number can detect the weight of the first non-vanishing Massey product in the complement of nullhomologous $K \subset \#^l S^2 \times S^1$.

Proposition 3.25. *If $D(K) = q$ then all non-vanishing Massey products in $H^*(\#^l S^2 \times S^1 \setminus \nu(K))$ are weight $\geq q$.*

Proof. This follows directly from an argument in the proof of proposition 6.8 in [CGO01]. More specifically, if $D(K) = q$ then the knot group quotients G/G_i are isomorphic to the free group quotients F/F_q by [Dwy75] and this isomorphism is induced by the meridional map $F \to G$. This is exactly the group quotient criteria used in [CGO01] in the context of k-surgery equivalent manifolds and we can see that by their proof, all Massey products of weight less than q in $H^*(\#^l S^2 \times S^1 \setminus \nu(K))$ vanish and there is a Massey product of weight q in this cohomology ring which is nonzero. □

Despite the fact that the definition of $D(K)$ looks quite unrelated to that of Milnor's invariants defined in Section 2.4, the previous results show $D(K)$ is a concordance invariant which detects how deep a knot's longitude is in the lower central series of its knot group and detects the weight of the first non-vanishing Massey product in the knot complement. These are exactly the properties of Milnor's invariants that make them so useful. It turns out this is no coincidence; as we will see, the Dwyer number of a knot in $\#^l S^2 \times S^1$ is directly related to the Milnor's invariants of an associated link in S^3. We can in fact use the Milnor's invariants of this associated link to compute the Dwyer number. The following lemma lays the groundwork to do this.

Lemma 3.26. *Let $L = (K, U_1, ..., U_l) \subset S^3$ be an $l+1$-component link such that the l-component sublink $L' = (U_1, ..., U_l)$ is the unlink and 0-surgery on U gives a nullhomologous knot K in $\#^l S^2 \times S^1$. If all Milnor invariants $\mu_L(I)$ of weight $|I| < q$ are trivial, then $D(K) \geq q$.*

Proof. Let $m_1, ..., m_l$ be meridians of L'. Since L' is an unlink, the longitudes u_i of its components bound disjoint disks and thus so do 0-framed pushoffs u_i', call the disks bounded by these pushoffs $\{D_i\}$. Once we perform 0-surgery on U, these longitude pushoffs also bound disjoint disks Δ_i inside the surgery tori. Clearly $\{S_i^2 = D_i \cup \Delta_i\}$ is a generating set for $H_2(\#^l S^2 \times S^1)$.

By a theorem of Milnor in [Mil57], $\mu_L(I) = 0$ for $|I| < q$ implies each longitude $\lambda_1, ..., \lambda_l$ of L' is the boundary of the image of a class $q+1$ half grope $f_i : \Sigma_i \to S^3 \setminus L$. Notice that we can extend each of these maps to $\bar{f}_i : \Sigma_i \cup D^2 \to S_0^3(L') = \#^l S^2 \times S^1$ where $\Sigma_i \cup D^2$ is the class $q+1$ half grope whose first stage is a closed surface since $\{\lambda_i\}$ bound disjoint disks in $S_0^3(L') \setminus (S^3 \setminus L)$. Notice that the images of these gropes are disjoint from knot K.

Let F_i be the image of the first stage of $\bar{f}_i$. Consider the homology class $[F_i] \in$ $\mathrm{H}_2(\#^l S^2 \times S^1)$. We know from intersection theory that

$$[F_i] = \sum_{j=1}^{1} n_j [S_j^2] = \sum_{j=1}^{1} [F_i] \cdot [m_j][S_j^2] = [S_i^2]$$

This is because all pairwise linking numbers between components of L are 0 (since these linking numbers are just the weight 1 Milnor invariants of L), and by construction, $[F_i] \cdot [m_j] = lk(U_i, U_j) = \delta_{ij}$.

Since $\mathrm{H}_2(\#^l S^2 \times S^1 \setminus \nu(K)) \cong \mathbb{Z}^l$ by Proposition 3.22 generators are pullbacks under this isomorphism induced by inclusion of generators of $H_2(\#^l S^2 \times S^1)$. Since the F_i can be pulled back to $\#^l S^2 \times S^1$, by the above argument we can see they generate $H_2(\#^l S^2 \times S^1 \setminus \nu(K))$.

□

Equipped with this lemma, we can now prove the following theorem allowing us to compute the the Dwyer number of a knot using the Milnor's invariants of an associated link. For details on how to compute Milnor's invariants for links in S^3, refer to Section 2.4.

Theorem 3.27. *Let $L = (K, U_1, ..., U_n)$ be an ordered, oriented link in S^3 such that the sublink $U = (U_1, ..., U_n)$ is an unlink and 0-surgery on U results in a nullhomologous knot $K' \subset \#^n S^1 \times S^2$. If $\bar{\mu}_L(I) = 0$ for $|I| < q$ and, for some multi-index $J = (1, ..., \iota_q)$ of length q, $\bar{\mu}_L(I) \neq 0$, then $D(K') = q$*

Proof. Let $H = \pi_1(S^3 \setminus \nu(L))$ and $G = \pi_1(\#^n S^1 \times S^2 \setminus \nu(K))$. Notice since $\#^n S^1 \times S^2$ is constructed via attaching 2 and 3 cells to $S^3 \setminus \nu(L)$, we have a surjection $\varphi : H \to G$ whose kernel is generated by the longitudes of U.

Let $F = \langle x_1, ..., x_{n+1} \rangle$ be a free group. By our condition on the Milnor invariants of L, a result of Milnor [Mil57] shows the map $h : F \to H$ sending x_i to a meridian of the i^{th} component of L induces an isomorphism $F/F_q \cong H/H_q$. As shown in lemma 3.26, our assumption also implies $D(K') \geq q$. Dwyer's theorem then gives us that the map $g : F \to G$ sending x_1 to the meridian of K' and x_i to the image under ϕ of a meridian of U_{i-1} for $2 \geq i \geq n + 1$ induces an isomorphism $F/F_q \cong H/H_q$.

Again by work of Milnor [Mil57], the conditions on $\bar{\mu}_L(I)$ give us that the based homotopy class of a 0-framed longitude λ_K of K does not lie in H_q. In other words, such a class is nontrivial in H/H_q. By construction, we have the following commutative diagram.

$$\begin{array}{ccc} H/H_q & \xrightarrow{\varphi_q} & G/G_q \\ & {}^{h_q}\nwarrow \qquad \swarrow^{g_q} & \\ & F/F_q & \end{array}$$

For a 0-framed longitude λ_K, we see $\varphi(\lambda_K)$ is a 0-framed longitude of K'. From our diagram, we see $H/H_q \cong G/G_q$ by the isomorphism $g_q \circ h_q$ and thus $\varphi_q(\lambda_K)$ is nontrivial in G/G_q. Therefore $\phi(\lambda_K)$ is not in G_q. Note that any other 0-framed longitude of K' will be homotopic to a conjugate of $\phi(\lambda_K)$; since G_q is normal, this means no 0-framed longitude of K' is in G_q and by Theorem **??**, $D(K) < q + 1$. □

We see this theorem gives us a way to construct many nullhomologous knots which are not concordant to each other (or the unknot) using the realization theorems of Tim Cochran in [Coc90] which allow us to construct links in S^3 whose first non-zero Milnor's invariants are a specific weight q.

Proposition 3.28. *For each $i \in \mathbb{Z}_{\geq 3}$, there is a knot $J_i \subset \#^l S^2 \times S^1$ for some l with $D(J_i) = i$.*

Proof. Consider the oriented Hopf link inside S^3. To get J_3, double one component of it to get the Borromean rings and perform 0-surgery on two components as shown in Figure 33. Since the Borromean rings have first nonzero μ-invariant $\mu(123)$, by Theorem 3.27 $D(J_3) = 3$. Iterate this Bing-doubling process on the doubled Hopf link as shown in figure 33 and perform 0-surgery on the sublink leaving out exactly one new component after this doubling procedure.

Notice that the underlying link in the surgery diagram is the result of the "Bing doubling along a tree" procedure described in [Coc90] and thus the resulting link from doubling a single component of J_i to get J_{i+1} has a non-vanishing Milnor invariant of weight $i+1$. Additionally, all Milnor invariants of smaller weight vanish, and therefore by Theorem 3.27 $D(J_i) = i$ for all $i \geq 3$.

□

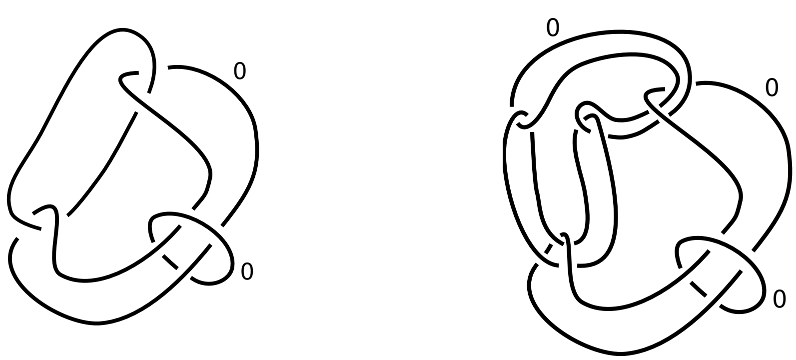


Figure 30: $J_3 \subset \#^2 S^2 \times S^1$ and $J_4 \subset \#^3 S^2 \times S^1$.

Note that it is often possible to get a lower bound on the Dwyer number directly by constructing embedded gropes which generate homology. In this way, if we have a fixed K for which $D(K)$ is known, we can sometimes directly show a specific J is not concordant to it by demonstrating the appropriate embedded grope as in Figure 31. Pictured is a knot in $\#^2 S^2 \times S^1$ with an embedded class 4 half grope and an embedded S^2, together they generate $H_2(\#^2 S^2 \times S^1 \setminus K$ and thus K is not concordant to J_3. Note that this lower bound is not sharp; from Theorem 3.27 and Section 2.4 we know the underlying link in Figure 31 has a non-vanishing Milnor invariant of weight 8 and thus $D(K) = 8$.

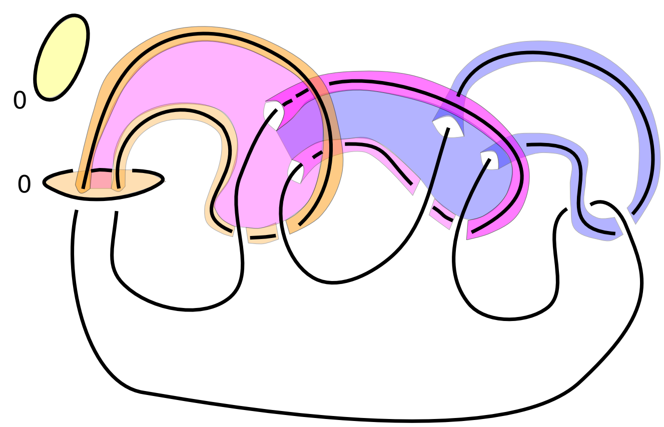


Figure 31: An embedded class 4 half grope in $\#^2 S^2 \times S^1 \setminus K$.

Recall our original intention behind studying the lower central series quotients of these knot groups was not only to better understand knot concordance in $\#^l S^2 \times S^1$, but to understand what happens to links $L \subset S^3$ when they are knotified as in Section 3.1.

Though there have been other generalizations of Milnor's invariants to knots and links in specific types of 3-manifolds, other formulations have not provided a similar relationship between concordances, lower central series quotients, and Massey products as given by Milnor's invariants. Recall our original intention behind studying the lower central series quotients of these knot groups was not only to better understand knot concordance in $\#^l S^2 \times S^1$, but to understand what happens to links $L \subset S^3$ when they are knotified as in Section 3.1. We can now use this work to predict properties of knotified links based on the Milnor invariants of a link in S^3.

Definition 3.29 (Interior band sum [Coc90])**.** Given a link L of m components and disjoint, oriented, embedded bands $b_1, ..., b_k$ in S^3 homeomorphic to $[0,1] \times [-1,1]$ whose intersections with L are along the initial and terminal arcs $b_i(\{0,1\} \times [-1,1])$ which lie in L and have the opposite orientation, a new (polygonal) link can be defined by deleting the collection of arcs $\{0,1\} \times [-1,1]$ and replacing them with $[0,1] \times \{-1,1\}$.

This definition allows us to use a slight restatement of a theorem of Cochran from [Coc90] which we will use in calculations. The restatement also corrects an indexing error in the original proof.

Theorem 3.30 (Cochran [Coc90])**.** *Suppose that $b(L)$ is an interior band sum involving k bands. If the first non-vanishing $\bar{\mu}$-invariant of L is weight $\leq r$ $(r < \infty)$, then the first non-vanishing $\bar{\mu}$-invariant of $b(L)$ is weight greater than $\left\lfloor \frac{r}{(k+1)} \right\rfloor$.*

We can now prove the following bound on the Dwyer number of a knotified link.

Proposition 3.31. *Let $L \subset S^3$ be an n-component link whose first nonzero Milnor invariant $\overline{\mu}_L(I)$ is weight q for some positive integer q. Then $D(\kappa(L)) \geq \left\lceil \frac{q-1}{n} \right\rceil$.*

Proof. The knotification of L can also be constructed using the following process. First, take the disjoint union of L with an $n-1$-component unlink U and perform an interior band sum on the sublink $L \subset L \sqcup U$ with $n-1$ bands, one through each component of the added unlink. We will call the resulting link J. Notice that U is unchanged by this procedure and L has fused to become a one-component sublink which we call L'. We finally arrive at the knotification by performing 0-surgery on the sublink of L' corresponding to U. By the above theorem of Cochran, the first nonzero $\bar{\mu}_J$ is weight $r+1$. The result then follows from Theorem 3.27. □

We conclude this section by illustrating the following properties of $D(K)$.

Proposition 3.32. *For any nullhomologous knot $K \subset \#^l S^2 \times S^1$ for any l,*

$$3 \leq D(K) \leq \infty$$

and each integer $q \in [3, \infty)$ is realized by at least one $K \#^l S^2 \times S^1$ for each l.

Proof. Since K is nullhomologous, from Lemma 3.23 we can see that the only relation in G/G_q not coming from a product of simple commutators in F_q is $[x, R_q(l)]$ where $R_q(l)$ is freely homotopic to a 0-framed longitude of K modulo F_q and is therefore in F_2. Thus, $[x, R_q(l)] \in F_3$ and $D(K) \geq 3$. If K is the unknot, $\#^l S^2 \times S^1 \setminus \nu(K)$ is homeomorphic to the connected sum of $\#^l S^2 \times S^1$ with a solid torus. Thus every generator of $\mathrm{H}_2(\#^l S^2 \times S^1 \setminus \nu(K))$ can be represented by a map of a half grope of arbitrary class. □

We also see that our invariant has applications to recent work of Celoria in [Cel18].

Definition 3.33. A knot K inside a 3-manifold Y is local if there is an embedded $B^3 \subset Y$ such that $K \subset B^3$. Similarly, a link L inside a 3-manifold Y is local if there is an embedded $B^3 \subset Y$ such that $L \subset B^3$

Definition 3.34 (Celoria [Cel18])**.** Two knots K_0 and K_1 in a 3-manifold Y are almost concordant $K_0 \sim_{ac} K_1$ if there are local knots K_0' and K_1' in Y such that $K_0 \# K_0'$ is concordant to $K_1 \# K_1'$ in $Y \times I$.

Now, we can see the following.

Proposition 3.35. *Fix $l \in \mathbb{Z}_{\geq 0}$ and let K be a null-homologous knot in $\#^l S^2 \times S^1$. Then $D(K)$ is an invariant of almost concordance in $\#^l S^2 \times S^1$.*

Proof. There is a surgery diagram for K as an $l+1$-component link in S^3 with 0-surgery performed on l of the components (note this diagram is certainly not unique). By Theorem 3.27, $D(K_i)$ is exactly the weight of the first non-vanishing Milnor invariant of this link in S^3 which we will call L. By abusing notation, let K' also refer to the image of the local knot inside S^3. We then see that infecting the component of L corresponding to K after surgery by the knot K'_i results in a link L'_i with all of the same Milnor invariants as L_i by Proposition 4.2 in [Ott11]. Finally, it is clear that there is a 0-surgery on an l-component sublink of L'_i gives the sum $K\#K' \subset \#^l S^2 \times S^1$ and thus applying Theorem 3.27 again we have $D(K) = D(K\#K')$ when K' is a local knot. □

Therefore Theorem 3.28 also gives us a knot in $\#^i S^2 \times S^1$ for each i which not almost-concordant in $\#^i S^2 \times S^1$ to the unknot in the sense of [Cel18].

3.4 The group of knots in $\#^l S^2 \times S^1$

In Section 2.2, we described how to construct the knot concordance group from the set of knots in S^3 with the operation of connected sum by instead looking at residue classes of knots modulo concordance. Then, in Section 2.3, we illustrated how links in S^3 modulo concordance do not form a group in the same way as connected sum is not well defined. Later in Section 3.1, we were motivated by constructions from Heegaard Floer homology to examine knotifications of links, which are themselves knots in connected sums of $S^2 \times S^1$. Finally, in Section 3.3 we developed an invariant called the Dwyer number in order to study knots in connected sums of $S^2 \times S^1$. This is a concordance invariant which shares many of the useful properties of Milnor's invariants of knots in S^3; additionally, we showed the Milnor's invariants of a link give bounds on the Dwyer number of the knotifiation of the link. In this section, which is joint with Matthew Hedden, we will use this information to define a new link concordance group where each link has a unique representative and the group operation is still connected sum.

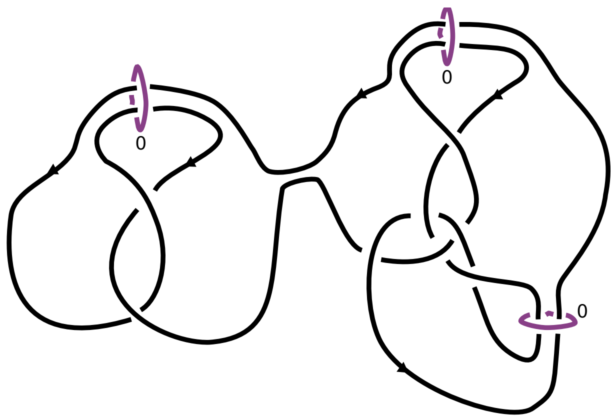


Figure 32: κ(Hopf link) # κ(Borromean rings)

Recall from Section 3.1 that the connected sum of knotified links is well-defined. However, the ambient manifold the knotified links lie in is changing, and thus in Section3.1 we defined notions of stabilization and destabilization. In Section 2.2, we saw the set of knots in S^3 with the operation of connected sum is not a group since knots up to isotopy do not have inverses; however, taking the quotient of the knot monoid by concordance fixes this problem. In order to use concordance to turn our monoids $\mathfrak{C}$ and $\mathfrak{D}$ from Proposition 3.4 into a group, we must account for the fact that the ambient manifold our knots lie in is changing.

Definition 3.36 (Stable Concordance)**.** Two knots $K_1 \subset \#^n S^2 \times S^1$ and $K_2 \subset \#^m S^2 \times S^1$ are (smoothly) stably concordant if they cobound a smoothly embedded annulus in $(\#^l S^2 \times S^1) \times I$ for some non-negative integer l.

We can see that concordance implies stable concordance and thus this notion is compatible with our previous understanding of concordance for the special case of knots in S^3.

Proposition 3.37. *Stable concordance is an equivalence relation on the monoid* $\mathfrak{D}$ *(and subsequently on* $\mathfrak{C}$*).*

Proof. The relation is obviously reflexive and symmetric. Consider a stable concordance C_1 between K_1 and K_2 in $\#^i S^2 \times S^1 \times I$ and a stable concordance C_2 between K_2 and K_3 in $\#^j S^2 \times S^1 \times I$. Now, stabilize $\max(i,j) - j$ times "along the concordance." More precisely, stabilize each $\{t_i\} \times \#^n S^2 \times S^1$ $\max(i,j) - j$ times (away from C_1) to get a concordance C_1' in $I \times \#^{\max(i,j)} S^2 \times S^1$. Similarly, stabilize each $\{t_2\} \times \#^i$ $\max(i,j) - i$ times away from C_2 to get a concordance C_2' in $I \times \#^{\max(i,j)} S^2 \times S^1$. Then, "stack" these concordances to get a new concordance $C_1' \cup C_2'$ between K_1 and K_3. □

Constructing the inverse of a knot $K \subset S^3$ inside $\mathcal{C}$ relies on the fact that K is concordant to the unknot in S^3 if and only if K bounds a smooth, properly embedded disk inside B^4 (i.e. if K is slice in B^4) as detailed in Section 2.2. This equivalence does not hold in manifolds other than S^3, as we will soon show by combining results from Section 3.3 with the following definitions. First, we must have the following proposition to justify using the Dwyer number to study these stabilized objects.

Proposition 3.38. *The Dwyer number* $D(K)$ *is a stable concordance invariant for null-homologous knots* $K \subset \#^l S^2 \times S^1$*.*

Proof. We proved in Section 3.3 that Dwyer number is invariant under concordance. Thus, we only need to show it is invariant under stabilization and destabilization. Assume $D(K) = q$. This means that every homology class in $\mathrm{H}_2(\#^l S^2 \times S^1 \setminus K)$ bounds an at least class q half grope in $\#^l S^2 \times S^1$, call these gropes Σ_i.

Stabilization: Choose a small ball in $\#^l S^2 \times S^1 \setminus \{\cup_i \Sigma_i\}$ and stabilize inside of this ball to get $K' \in$. Notice that $\mathrm{H}_2(\#^{l+1} S^2 \times S^1 \setminus K') \cong \mathrm{H}_2(\#^l S^2 \times S^1 \setminus K) \oplus \mathbb{Z}$ where the new summand is generated by an embedded S^2 inside the new copy of $S^2 \times S^1$ from the stabilization. Since S^2 is a half-grope of arbitrary height, stabilization cannot lower the Dwyer number.

Since $D(K) = q$, there is however an element of $\mathrm{H}_2(\#^n S^2 \times S^1 \setminus \nu(K))$ which does not bound a $q+1$ half grope. It is clear that stabilization will not allow this generator to bound a grope of higher height. More precisely, stabilization introduces another S^2 generator to the homology and therefore the only way it could change existing homology classes is by taking the sum with another S^2 sphere, which will not raise the Dwyer number of the stabilized knot.

Destabilization: If K can be destabilized, this means K can be isotoped away from some j^{th} $S^2 \times S^1$ summand. We can find a meridian m_j in this summand which does not intersect any Σ_i for $i \neq j$. Performing 0-surgery on m_j gives K' in $\#^{n-1} S^2 \times S^1$, we see that the elements of $\mathrm{H}_2(\#^{n-1} S^2 \times S^1 \setminus K') \cong \mathbb{Z}^{l-1}$ all still bound gropes of height at least q.

It remains to be shown that destabilization will also not allow a generating set where every generator bounds a grope of height $q+1$ or higher. Assume $D(K) < q$ and K can be destabilized. Thus, there is an element $x \in \mathrm{H}_2(\#^n S^2 \times S^1 \setminus \nu(K))$ which cannot bound a class $q+1$ half-grope. Note that destabilizing K involves isotoping K to be away from an embedded $S^2 \times S^1 \setminus B^3$. The second homology of this submanifold is generated by an embedded S^2 (which is a half-grope of arbitrary height) and therefore destabilizing will not change our original class x. After destabilizing, by abuse of notation we see $x \in \mathrm{H}_2(\#^{l-1} S^2 \times S^1 \setminus \nu(K))$ still does not bound a class $q+1$ half grope.

□

The moral here is that stabilizing and destabilizing a knot $K \subset \#^l S^2 \times S^1$ will only add and remove spheres to elements of the second homology, which will not affect a symplectic basis of any of the representatives of these homology classes. Now that we understand how to use Dwyer number in the context of stable concordance, we can finally exploit its properties to understand how concordance and sliceness differ for general 3-manifolds.

Definition 3.39. A null-homologous knot $K \subset \#^l S^2 \times S^1$ is (stably) slice if (some stabilization of) it bounds a smooth, properly embedded disk in $\natural^l S^2 \times D^2$ for a non-negative integer m.

This definition is motivated by the argument in the following lemma; if we had chosen another 4-manifold to view K as slice in (namely, a boundary sum of both $S^2 \times D^2$ and $S^1 \times B^3$) we would be able to perform surgery away from such a null-homologous disk and obtain a disk inside a boundary sum of only $S^2 \times D^2$. This definition of a slice knot has also been used in [DNPR18] in the case when $l = 1$.

It is important to note that in typical generalizations of sliceness it is required that such a disk be null-homologous. In this case, since $H_2\big(\natural^{2l} S^2 \times D^2, \partial\big(\natural^{2l} S^2 \times D^2\big)\big) = 0$ all properly embedded disks are null-homologous.

Definition 3.40. Let $K \subset \#^l S^2 \times S^1$ be an oriented, null-homologous knot. Define $\overline{K}$ as K with the reverse orientation, and $-K$ as the knot constructed in the following way. Represent $K \subset \#^l S^2 \times S^1$ by a surgery diagram for the link $K \sqcup U$ where U is an l-component unknot with 0 surgery coefficient. Then, let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the orientation reversing diffeomorphism $(x, y, z) \mapsto (x, y, -z)$. This extends uniquely to S^3 and then to the 0-surgery $F : \#^l S^2 \times S^1 \to \#^l S^2 \times S^1$. Define $-K$ as the image $F(-K)$ (note that any other $-K$ obtained in this way will be isotopic to this one). Diagrammatically, this is the same thing as switching all the crossings of K.

Lemma 3.41. *$K \# -\overline{K}$ is slice.*

Proof. The proof is similar to the proof in Section 2.2 for knots in the 3-sphere. Consider the trivial concordance $(\#^l S^2 \times S^1, K) \times I$. Then, just as before for knots

$K \subset S^3$, remove an arc from $K \times \{0\}$ to $K \times \{1\}$, take a neighborhood of it, and remove the interior of this neighborhood. Finally, attach two parallel arcs from $(K \times \{0\}) - I$ to $(K \times \{1\}) - I$. A straighforward handle decomposition argument shows this results in a disk in $\natural^l(B^3 \times S^1 \natural^l S^2 \times D^2)$ with boundary $K\# - \overline{K}$ in $\#^{2l} S^2 \times S^1$. We can perform surgery on the $B^3 \times S^1$ factors to get $K\# - \overline{K}$ bounding a disk $\Delta \subset \natural^{2l} S^2 \times D^2$. □

Additionally, modifying the same construction results in the following statement.

Lemma 3.42. *If K and J are stably concordant, $K\# - \overline{J}$ is (stably) slice.*

Proof. Note that any concordance between knots $K \subset \#^l S^2 \times S^1$ and $J \subset \#^l S^2 \times S^1$ inside $(\#^l S^2 \times S^1) \times I$ for some l can be smoothly perturbed to contain a straight arc $t \times \{\text{point}\}$. By a similar argument to the proof of the previous lemma, we obtain a smooth, null-homologous disk inside $\natural^{2l} S^2 \times D^2$ whose boundary is $K\# - \overline{J}$. □

If a knot is slice in $\#^l S^2 \times S^1$, it is clear from the definition that it is stably slice. In fact, it is easy to construct other examples of stably slice knots.

Lemma 3.43. *Let $K \subset \#^l S^2 \times S^1$. Consider a surgery diagram for this knot in $\#^l S^2 \times S^1$; in particular, there is a diagram such that $\#^l S^2 \times S^1$ is obtained through 0-surgery on an l-component unlink U. Let V be an unlink whose i^{th} component is the i^{th} meridian of U. The result of 0-surgery on V is S^3 and the image of K is some knot K'. If K' is slice in B^4, then K is stably slice.*

Proof. Consider $K' \subset S^3 = \partial(B^4)$ and let Δ be a slice disk for K'. Attach l 0-framed 2-handles to B^4 along the link U. Notice that the image of Δ is a disk $\Delta' \subset \natural^l S^2 \times D^2$ with boundary K. □

Not all of these knots will necessarily be knotified links; the following proposition allows us to construct slice knotified links.

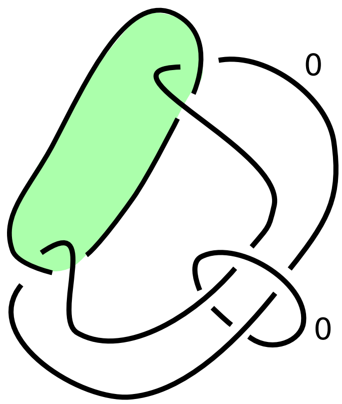
Proposition 3.44. *If some fusion of an n-component link $L \subset S^3$ is slice then $\kappa(L) \subset \#^{n-1} S^2 \times S^1$ is slice.*

Proof. Let $L \subset S^3$ be a link such that some fusion of it is slice. Recall that a fusion of a link is an internal band sum between different components of L. Knotify L using these fusion bands. Now, consider $\kappa(L) \times I$ inside $(\#^{n-1} S^2 \times S^1) \times I$. Attach $n-1$ 0-framed 2-handles to $(\#^{n-1} S^2 \times S^1) \times I$ along meridians of $(\#^{n-1} S^2 \times S^1) \times \{1\}$ in order to get a new manifold W with boundary $\#^{n-1} S^2 \times S^1 \sqcup S^3$. We can see that $\kappa(L) \times \{1\}$ is a slice knot inside S^3 by construction so we can cap off W with a B^4 and $\kappa(L) \times \{1\}$ with a disk Δ inside this B^4. The result is a properly embedded disk $\kappa(L) \times I \sqcup \Delta$ inside $\natural^{n-1} S^2 \times D^2$ with boundary $\kappa(L)$. □

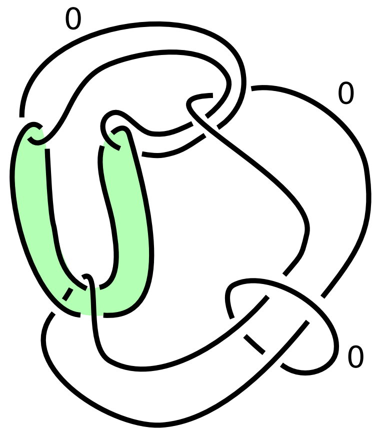
Equipped with this proposition, we can now further examine the relationship between a knot being (stably) slice and a knot being (stably) concordant to the unknot.

Theorem 3.45. *There are infinitely many $K_i \subset \#^i S^2 \times S^1$, each of which is slice and not concordant (even stably) to the unknot.*

Proof. Let K_1 be the knotified Hopf link. By a calculation in 3.28, since the Dwyer number defined in Section 3.3 $D(K_1)$ is finite (it is equal to 4) K_1 is not concordant (even stably) to the unknot. Moreover, by Proposition 3.44 it is slice. Now, consider the oriented Hopf link inside S^3. To get K_2, double one component of it to get the Borromean rings and perform 0-surgery on two components as shown in the figure 33. This knot is slice by Lemma 3.4 and as shown in the following section $D(K) = 3$ and therefore it is not (even stably) concordant to the unknot. Iterate this process as shown in figure 33 to get K_i for $i > 2$. The resulting knots are clearly slice by Lemma 3.4 and $D(K_i) = i + 1$ thus they are distinct in (stable) concordance from the unknot and for that matter from each other. □



(a) Bing double and surger once.



(b) Bing double and surger twice.

Figure 33: Examples of K_2 and K_2 with their slice disks.

What we now see is that taking the quotient of the monoid in definition 3.6 by even stabilized concordance is not enough to define inverses the way we could for knots in S^3. This motivates the more general definition of (stable) slice equivalence.

Definition 3.46 (Stable Slice Equivalence)**.** Two null-homologous knots $K_1 \subset \#^l S^2 \times S^1$ and $K_2 \subset \#^l S^2 \times S^1$ are slice equivalent if $K_1 \# -\overline{K}_2$ is slice in $\natural^l S^2 \times D^2$. More generally, two null-homologous knots $K_1 \subset \#^n S^2 \times S^1$ and $K_2 \#^m S^2 \times S^1$ are stably slice equivalent if there are stabilizations $K_1' \subset \#^l S^2 \times S^1$ and $K_2' \subset \#^l S^2 \times S^1$ such that $K_1' \# -\overline{K}_2'$ is slice in $\natural^l S^2 \times D^2$.

Note that one could also consider this equivalence as first taking the quotient of the set of null-homologous knots by stable concordance, then declaring slice knots to be trivial.

Lemma 3.47. *Stable slice equivalence is an equivalence relation.*

Proof. It is symmetric because of Lemma 3.41. It is clearly reflexive since, if $K_1' \# -\overline{K}_2'$ bounds a smooth, properly embedded, null-homologous disk Δ in $\natural^l S^2 \times D^2$ then reversing the orientation of $(\natural^l S^2 \times D^2, K_1' \# -\overline{K}_2')$ sends Δ to the oppositely oriented disk with boundary $K_2'\# - \overline{K}_1'$. Proving transitivity relies on this stabilization property. Assume $K_1\# - \overline{K}_2$ bounds a smooth, properly embedded, null-homologous disk Δ_1 in $\natural^{l_1} S^2 \times D^2$ and $K_2\# - \overline{K}_3$ bounds a smooth, properly embedded, null-homologous disk Δ_2 in $\natural^{l_2} S^2 \times D^2$. Note that $K_2\# - \overline{K}_2$ is slice and thus bounds another disk Δ_3 inside $\natural^{l_3} S^2 \times D^2$. Reverse the orientation of this disk.

We can take the boundary connected sum of $\natural^{l_1} S^2 \times D^2 \ \natural^{l_2} S^2 \times D^2 \natural^{l_3} S^2 \times D^2$ while taking care to glue disks Δ_1, Δ_2, and Δ_3 together along subsets of their boundaries so that $\overline{K}_2 \setminus$ an arc $\subset \Delta_3$ is glued to $-\overline{K}_2 \setminus$ an arc $\subset \Delta_1$ and $-K_2 \setminus$ an arc $\subset \Delta_3$ is glued to $K_2 \setminus$ an arc $\subset \Delta_2$. After smoothing corners, the result of this process is a disk $\Delta_1 \cup \Delta_3 \cup \Delta_2$ inside $\natural^{l_1+l_2+l_3} S^2 \times D^2$ whose boundary is the connected sum of some stabilizations of K_1 and $-\overline{K}_3$, thus stable slice equivalence is transitive. □

Theorem 3.48 (Stable Knot Concordance Group)**.** *The monoid $\mathfrak{D}$ modulo stable slice equivalence is a group $\mathcal{C}^{\mathfrak{S}^2 \times \mathfrak{S}^1}$ which we call the stable knot concordance group in $S^2 \times S^1$.*

Proof. This follows immediately from Lemma 3.41. □

Note there is nothing inherent to $\#^l S^2 \times S^1$ about this definition (or even the theorems justifying it), and it would be reasonable to construct the stable knot concordance group of an arbitrary 3-manifold. One could perhaps further examine these structures by focusing on the stable knot concordance groups of prime 3-manifolds and studying maps between concordance groups on manifolds and the concordance groups on the components of their prime decompositions. However, these are questions for later work; in this thesis we are primarily concerned with the stable concordance group of knots in $S^2 \times S^1$ due to its application to the study of link concordance in S^3.

Figure 34: How to "cut" a band of $\kappa(L)$ by attaching a band.

Corollary 3.49 (Knotified Link Concordance Group)**.** *The monoid $\mathfrak{C}$ modulo stable slice equivalence is a group $\mathcal{C}^{\kappa(L)}$ which we call the knotified link concordance group.*

Proof. From Definition 3.1 we clearly see that for any knotified link $\kappa(L) \subset \mathfrak{C}$ we can cut some number of bands as in Figure 34 to recover L as an n-component local link. We can then map L inside the appropriate embedded 3-ball to S^3 to obtain $L \subset S^3$. Note that this band cutting procedure is not unique even up to isotopy Now, consider $\kappa(L_1) \subset \#^{l_1} S^2 \times S^1$ and $\kappa(L_2) \subset \#^{l_2} S^2 \times S^1$ where L_1 is an $l_1 + 1$ component link in S^3 and L_2 is an $l_2 + 1$ component link in S^3. The resulting connected sum is $\kappa(L_1)\#\kappa(L_2) \subset \#^{l_1+l_2} S^2 \times S^1$. It is clear that we can cut $l_1 + l_2 + 1$ bands from $\kappa(L_1)\#\kappa(L_2)$ in order to obtain a local link $L' \subset \#^{l_1+l_2} S^2 \times S^1$; these are exactly the bands we had to cut to obtain L_1 and L_2 together with an additional band used to connect sum $\kappa(L_1)$ and $\kappa(L_2)$. Again, we can now map L' inside its embedded B^3 diffeomorphically by some map φ to S^3 to get a link $\varphi(L')$. Therefore $\kappa(L_1)\#\kappa(L_2) = \kappa(\varphi(L')$. □

This group $\mathcal{C}^{\kappa(L)}$ (and thus clearly $\mathcal{C}^{\mathfrak{S}^2 \times \mathfrak{S}^1}$) contains the classical knot concordance group as we can see by the following argument.

Proposition 3.50. *The knot concordance group $\mathcal{C}$ injects into $\mathcal{C}^{\kappa(L)}$.*

Proof. We begin with an argument similar to that of Davis, Nagel, Park, and Ray [DNPR18]. Let $K \subset S^3$ be a knot. Without loss of generality, we can view K as lying inside some 3-ball B_1 and let $\iota : B_1 \to \#^n S^2 \times S^1$ be an inclusion. Assume $\iota(K)$ is slice . Now, perturb $\iota(B_1)$ to be disjoint from another embedded 3-ball B^2 and let $Y = \#^l S^2 \times S^1 \setminus B_2$. Consider the universal cover $p : \tilde{Y} \to Y$ of this 3 manifold. By lifting properties, $\iota(B_1)$ lifts to disjoint 3-balls $\iota(B_1) \subset \tilde{Y}$ and therefore ιK lifts to a link $L := p^{-1}(\iota K)$, each component of which is ambient isotopic to K inside $\iota(B_1) \subset \tilde{Y}$. By a result of Boden and Nagel in [BN17] with a refinement stated by Nagel, Orson, Powell, and Park [NOPP], there is a smooth embedding $\varphi : \tilde{Y} \to B^3$.

We can see that $p \times Id : \tilde{Y} \times I \to Y \times I$ is also a cover and recall from the proof of 3.41 that $Y \times I$ is diffeomorphic to $\natural^l S^1 \times D^3 \natural^l S^2 \times D^2$. Notice that the boundary of this 4-manifold is $\#^{2l} S^2 \times S^1$ and moreover that we can perform surgery on it l times to obtain the 4-manifold $\natural^l S^1 \times D^3 \natural^l S^2 \times D^2$ without changing the boundary (and in particular, without changing the knot K lying in this boundary).

Since K is slice in $\natural^l S^2 \times D^2$,, there is an ambient isotopy of K so that it lies in a $\natural^l S^2 \times D^2$ summand of $\natural^{2l} S^2 \times D^2$ and bounds a smooth, null-homologous disk in this summand. Now, we can perform surgery again on the remaining factors of this manifold and the result will be $\natural^l S^1 \times D^3 \natural^l S^2 \times D^2$, this time with a disk Δ in it whose boundary is isotopic to K. A handle argument shows this ambient isotopy extends to a diffeomorphism of $\natural^l S^1 \times D^3 \natural^l S^2 \times D^2$ which we will call Ψ. Now, $\Psi^{-1}(\Delta)$ is a smooth, properly embedded disk in $\natural^l S^1 \times D^3 \natural^l S^2 \times D^2$ whose boundary is K and which lifts to disjoint disks $(p \times Id)^{-1}(\Psi^{-1}(\Delta)$ with boundary L.

Finally, pick a specific lift $\tilde{D}$ of $\Psi^{-1}(\Delta)$. This induces a choice of lift $\iota(\tilde{K})$ for $\iota(K)$. Notice that $\varphi \times I : \tilde{Y} \times I \to B^3 \times I \cong B^4$ is a smooth embedding taking $\partial(\tilde{Y} \times I)$ to $\partial(\to B^3 \times I) \cong S^3$, thus $\varphi \times I(\tilde{D})$ is a smooth, properly embedded disk in B^4 whose boundary is isotopic to K.

□

Thus we have constructed a notion of a link concordance group which is independent of the choice of link representative and can be studied using both classical tools developed from group theory and more modern tools from Heegaard Floer homology. This project is ongoing; in future work, we hope to better understand this group. One approach is to construct homomorphisms from this group to the integers using the τ_i-invariants defined in Section 3.1 which we are currently pursuing.

4 String Link Concordance and the Pure Braid Group

4.1 Milnor's invariants and string links

As stated in Section 2.4, the ambiguity in defining Milnor's invariants for links $L \subset S^3$ is exactly the ambiguity in choosing a basing disk for the link as in Definition 2.22. For completeness, we include this brief section outlining the interaction between string links and Milnor's invariants. Work of Habegger and Lin in [HL90] and [HL98] revolutionized the study of Milnor's invariants by classifying string links up to link homotopy and understanding this classification using group actions. In this section, we will take the perspective of Levine [Lev87] to briefly summarize how the Milnor invariants of string links can be computed in general without ambiguity.

Definition 4.1. We have similar definitions of meridians $\{\mu_i\}$ and longitudes $\{\lambda_i\}$ for string links $S \subset D^2 \times I$ as we did for links $L \subset S^3$.

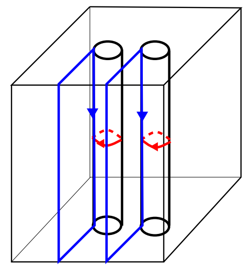


Figure 35: Meridians and longitudes for a 2-component string link.

Refer to $\widehat{S}$ as the string link closure of S as in 2.3. Finally, let $\pi = \pi_1(D^2 \times I \setminus S, *)$, where L is an n-strand string link.

Proposition 4.2. *If π, $\{\mu_i\}$, and $\{\lambda_i\}$ are as above, then*

1. *$H_1(\pi)$ is free abelian of rank m with basis the homology classes of $\{\mu_i\}$;*

2. *π is normally generated by $\{\mu_i\}$;*

3. *π has a presentation of deficiency m;*

4. *$H_2(\pi) = 0$.*

Corollary 4.3. *Let $\mu : F \to \pi$ the map from the free group on n letters to π. Then $f_q : F/F_q \to \pi/\pi_q$ is an isomorphism for all q.*

Notice that this proposition tells us that for string links, the nilpotent quotients of the fundamental group of any string link only depend on the number of components. Recall that Milnor's invariants first told us at what stage the nilpotent quotients of the fundamental group of a link complement stopped being isomorphic to that of the unlink with the appropriate number of components. Then, if we could get around the ambiguity, they gave us integers to distinguish links with isomorphic q^{th} quotient. With string links, we see that we always have isomorphic quotients and thus can compare string links directly in the following way.

Definition 4.4. The nilpotent completion of a group G is $\widetilde{G} = \lim_{q} \frac{G}{G_q}$.

Therefore, notice that for all string links we have $\widetilde{\pi} \cong \widetilde{F}$. We can now define the Milnor's invariants of a string link.

Definition 4.5 (J. Levine [Lev87])**.** Let $S \subset D^2 \times I$ be an n-strand string link with longitudes $\{\lambda_i\}$. Let $\{\overline{\lambda}_i\}$ be their images under the composite map $\pi \to \widetilde{\pi} \cong \widetilde{F}$. We can refer to $\{\overline{\lambda}_i\}$ as the Milnor invariants of S.

While this may seem strange as we introduced sets of integers in Section 2.4 and now we have group elements in the nilpotent completion of the free groups, we should see that this is merely a reformulation of Milnor's definition which avoids the Magnus expansion. Recall that Milnor's invariants defined in Section 2.4 were integers because we used results in [Mag35] to quantify the longitudes of a link using the Magnus embedding since we did not know the group quotients were isomorphic. In the case of string links, since these quotients are always isomorphic we can directly look at the images of the longitudes. In fact, we have the following.

Proposition 4.6 (Well known, proven in [Lev87])**.** *Let $L \subset S^3$ be a link with group G and $\{\mu_i, \lambda_i\}$ be the image in G of a set of meridians and longitudes for a string link $S \subset D^2 \times I$ whose closure is L. Let $f : F \to G$ be the homomorphism sending each free generator x_i to the image of μ_i for $1 \leq i \leq n$ and $\{\overline{\mu}\}$ be the so-called Milnor invariants of string link S. Then the following are equivalent.*

1. *$\lambda_i \in G_q$ for all $1 \leq i \leq n$.*
2. *f induces an isomorphism $F/F_{q+1} \to G/G_{q+1}$.*
3. *$\overline{\lambda_i} \in \widetilde{F}_q$ for all $1 \leq i \leq n$.*
4. *The image of $\overline{\lambda_i}$ under the canonical projection $\widetilde{F} \to F/F_q$ is zero for all $1 \leq i \leq n$.*

4.2 The non-abelian nature of string link concordance

In Section 4.1, we discussed how the ambiguity in choosing a string link representative of a link is intrinsically tied to the ambiguity in basing a link (and thus the ambiguity in Milnor's invariants). Therefore, perhaps some of the structure in the string link concordance group is more a byproduct of the ambiguity than a byproduct of linking.

One of the goals of this thesis is to understand how much of the structure of several different notions of link concordance groups is a byproduct of the way the group elements are represented. This is in stark contrast to the case of knots in S^3, for which each oriented knot has a unique representative.

Recall that one of the unique properties of $\mathcal{C}(n)$ is that it is nonabelian [LD88]. This is because it contains the pure braid group $\mathcal{P}(n)$ as a subgroup; therefore, it is not clear whether this non-abelian property is merely inherited from the pure braid group. This question is made far more difficult by the following fact.

Theorem 4.7 (Kirk-Livingston-Wang [KLW98]). *$\mathcal{P}(n)$ is not a normal subgroup of $\mathcal{C}(n)$ for $n > 2$.*

To this end, we proved the following theorem.

Theorem 4.8. *$\frac{\mathcal{C}(n)}{Ncl(\mathcal{P}(n))}$ is non-abelian for all n.*

Proof. First notice that the Milnor invariant $\mu_P(iijj) = 0$ (this invariant is called the Sato-Levine invariant insert reference), $i \neq j$ for any pure braid $P \in \mathcal{P}(n)$; this is because the Sato-Levine invariant only depends on a 2-component sublink P' of P (namely, the link whose components are the i^{th} and j^{th} components of P). Note that P' is a 2-strand braid and thus the only nonzero $\mu_{P'}(I)$ is exactly the linking number $\mu_{P'}(12)$.

Now, let $b \in Ncl(\mathcal{P}(n))$ with all pairwise linking numbers zero for reasons which will soon be clear. Then $b = \Pi_{l=1}^{m} s_l p_l s_l^{-1}$ where $p_l \in \mathcal{P}(n)$ and $s_l \in \mathcal{C}(n)$ for all l.

Now, by Theorem 2.43 we see that $\mu_{\Pi_{l=1}^{m} s_l p_l s_l^{-1}}(iijj) = \Sigma_1^m \mu_{s_l p_l s_l^{-1}}(iijj)$.

Again, $\mu_{s_l p_l s_l^{-1}}(iijj)$ will depend only on the 2-component sublinks of each string link $s_l p_l s_l^{-1}$ made up of the i^{th} and j^{th} components of $s_l p_l s_l^{-1}$. Each of these sublinks will be a product of the i^{th} and j^{th} components of s_l, p_l, and s_l^{-1}. We will show what happens for a specific l. Call these components L, P', and L^{-1}

Figure 36: Pushing the string link L along the full twists P.

As is clear in Figure 36, P is just a product of full twists since it is a 2-strand braid and thus we can slide L along the twist region to obtain the word PLL^{-1}. We see this string link is concordant to P and thus $\mu_{s_l p_l s_l^{-1}}(iijj) = \mu_P(iijj) = 0$. Finally, this tells us $\mu(iijj)_b = 0$.

What we have now shown is that any $b \in Ncl(\mathcal{P}(n))$ with vanishing pairwise linking numbers also has vanishing Sato-Levine invariant. Hence to prove the theorem it will suffice to show a specific commutator in $\mathcal{C}(2)$ has a non-vanishing Sato-Levine invariant. Since this link has pairwise linking number 0, there is no indeterminacy in computing the $\mu_{\widehat{C}}(1122)$ where $\widehat{C}$ is the closure. We will use the surface system method outlined in Section 2.4.

The link C in Figure 37 has $\mu_C(iijj) = 4$ using the surfaces in 38. □

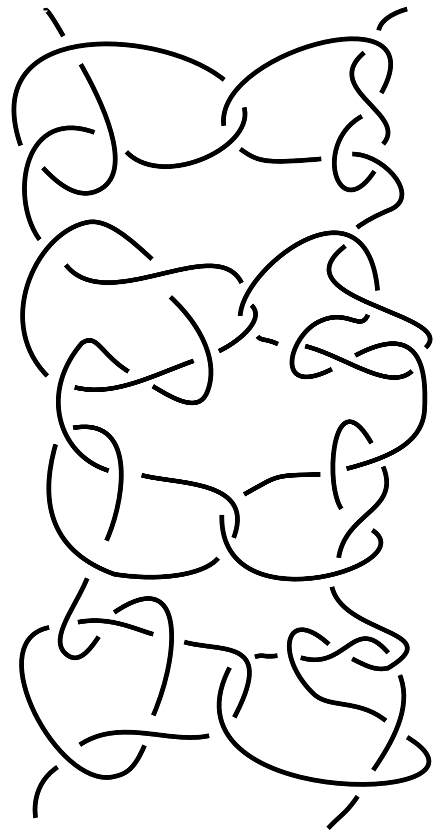
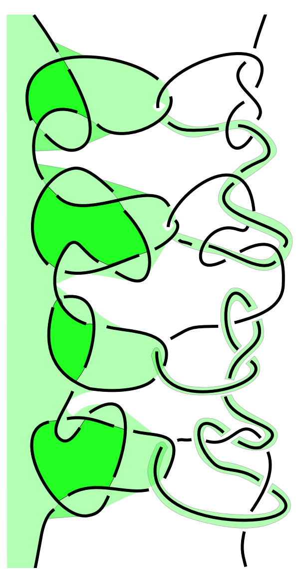
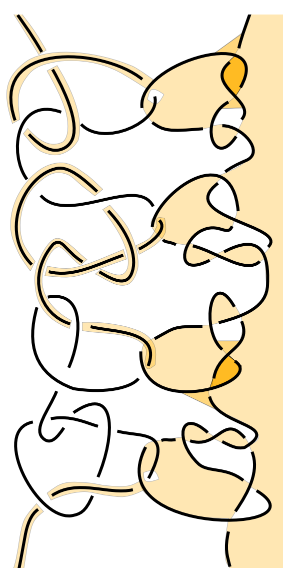


Figure 37: $C \in [\mathcal{C}(2), \mathcal{C}(2)]$ with $\mu_C(iijj) = 4$.

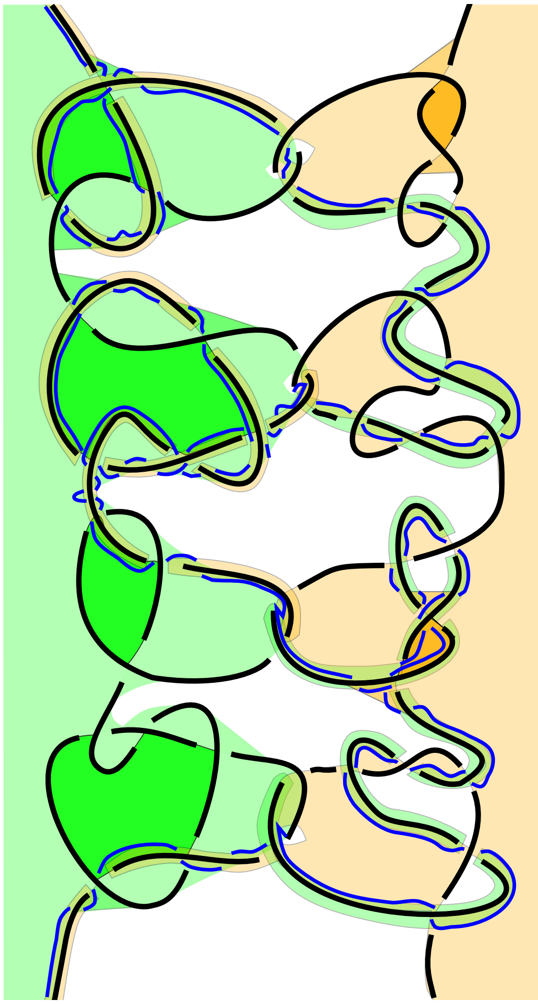


(a) A surface $V(x)$.

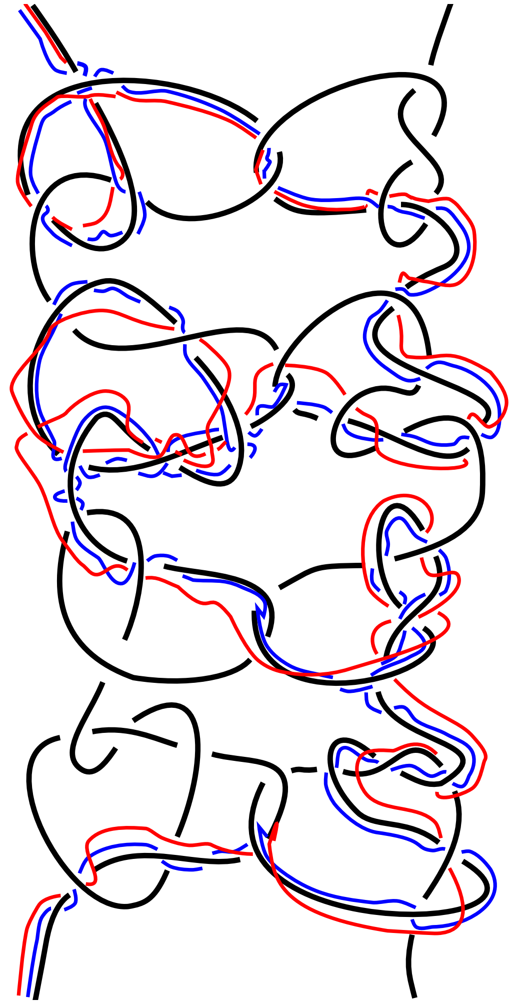


(b) A surface $V(y)$.

Figure 38: The first stage of a surface system for $\widehat{C}$.



(a) The curve $c(x,y) = V(y) \cap V(x)$.



(b) $c(x,y)$ and its pushoff $c^{+}(x,y)$.

Figure 39: Computing $c(x,y)$ and the linking $lk(c(x,y), c^{+}(x,y)) = 4$.

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