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Lower order solvability of links

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Abstract

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The n-solvable filtration of the link concordance group, defined by Cochran, Orr, and Teichner in 2003, is a tool for studying smooth knot and link concordance that yields important results in low-dimensional topology. We focus on the first two stages of the n-solvable filtration, which are the classes $\mathcal{F}_0^m$, the class of 0-solvable links, and $\mathcal{F}_{0.5}^m$, the class of 0.5-solvable links. We introduce a new equivalence relation on links called 0-solve equivalence and establish both an algebraic and a geometric characterization 0-solve equivalent links. As a result, we completely characterize 0-solvable links and we give a classification of links up to 0-solve equivalence. We relate 0-solvable links to known results about links bounding gropes and Whitney towers in the 4-ball. We then establish a sufficient condition for a link to be 0.5-solvable and show that 0.5-solvable links must have vanishing Sato-Levine invariants.

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and most importantly, "to Him who is able to do immeasurably more than all we ask or imagine," *Eph 3:20*

This is not my accomplishment; it's yours. Thank you.

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Chapter 1

Introduction

1.1 Background

A *link* is an embedding $f : \bigsqcup_{i=1}^{m} S^1 \to S^3$ of an ordered, disjoint collection of oriented circles into the three-sphere. A *knot* is a link with only one component. Two links are *isotopic* if one can be smoothly deformed into the other through embeddings in S^3. Isotopy is a equivalence relation on links in S^3; by studying links as objects in 3-space, we actually study the isotopy classes of links. Links are very important in the larger context of low-diensional topology. For example, the fundamental theorem of Lickorish and Wallace, introduced in 1960, gives a description of closed, orientable 3-manifolds using a process called *surgery* on links in S^3.

Many open problems in low-dimensional topology center around the study of 4-manifolds. In the smooth category, even compact simply-connected 4-manifolds are unclassified. In the 1950's, Fox and Milnor introduced the notion of *concordance classes* of links. Link concordance is a 4-dimensional equivalence relation on links in S^3; studying link concordance can contribute greatly to the understanding of 4-manifolds. For example, Freedman and Quinn explain that "surgery is equivalent to the link slice problems, and a weak form of the embedding problem" [9].

In 1966, Fox and Milnor showed that concordance classes of knots form an abelian group under an operation called *connected sum*, called the *knot concordance group*, $\mathcal{C}$. Two knots K and K' are *concordant* if $K \times \{0\}$ and $K' \times \{1\}$ cobound a smoothly embedded annulus in $S^3 \times [0, 1]$. The identity element of this group is the equivalence class of the trivial knot. Any knot in this class is called *slice.* The knot concordance group has been well-studied since its introduction, but its structure is complex and remains largely unknown.

Here, we study the more general *(string) link concordance group*, $\mathcal{C}^m$, where m is the number of link components; when $m = 1$, this is the knot concordance group. As a tool for investigating the structure of $\mathcal{C}^m$, Cochran, Orr, and Teichner introduced the *n-solvable filtration* $\mathcal{F}^m_n, n \in \frac{1}{2}\mathbb{N}$ of $\mathcal{C}^m$ in 2003. The n-solvable filtration is an infinite sequence of nested subgroups of $\mathcal{C}^m$ and can be though of as an algebraic approximation to a link being slice. Elements of $\mathcal{F}^m_n$ are ordered, oriented links called *n-solvable* links.

While the n-solvable filtration of $\mathcal{C}^m$ has been studied since its inception, many of the existing results discuss quotients of the filtration. For example, Harvey shows that the quotients $\mathcal{F}^m_n/\mathcal{F}^m_{n+1}$ contain an infinitely generated subgroup [12]. Cochran and Harvey showed that the quotients $\mathcal{F}^m_n/\mathcal{F}^m_{n.5}$ contain an infinitely generated subgroup [5]. Cochran, Harvey, and Leidy [6] and Cha [2] have made notable contributions in this area.

In their seminal work on the n-solvable filtration, Cochran, Orr, and Teichner classified 0-solvable knots and 0.5-solvable knots. A parallel classification for links, however, remains unknown. This is the goal of our current work.

1.2 Summary of Results

Let $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$ be ordered, oriented, m-component links such that the pairwise linking numbers $lk(K_i, K_j)$ and $lk(K'_i, K'_j)$ vanish. We call this class of links $\mathcal{F}^m_{-0.5}$. We will establish an equivalence relation on the set $\mathcal{F}^m_{-0.5}$ called 0-solve equivalence, and we will discuss how 0-solve equivalence relates to the condition of 0-solvability for links. We will show that 0-solve equivalent links L and L' must have three specific algebraic link invariants in common. Two of these invariants are the Milnor's invariants $\bar{\mu}(ijk)$ and $\bar{\mu}(iijj)$. Milnor's family of concordance invariants measures "higher-order linking" among link components algebraically; each of these invariants are denoted by $\bar{\mu}$. The other algebraic invariant we will consider is the Arf invariant. Defined for knots, the Arf invariant is an algebraic $\mathbb{Z}_2$-valued concordance invariant that is computable by a variety of methods. We will also study a geometric move on link diagrams called the band-pass move. band-pass equivalence is a geometric equivalence relation on links, and we will see that we can use band-pass equivalence to study 0-solve equivalence. We establish a relationship among these conditions.

Theorem. 5.1. *For two ordered, oriented m-component links $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$ with vanishing pairwise linking numbers, the following* *conditions are equivalent:*

1. *L and L' are 0-solve equivalent,*

2. *L and L' are band-pass equivalent,*

3. $Arf(K_i) = Arf(K_i')$

   $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$

   $\bar{\mu}_L(iijj) \equiv \bar{\mu}_{L'}(iijj) \pmod{2}$ *for all* $i, j, k \in \{1, \ldots, m\}$.

As a corollary, we characterize the class $\mathcal{F}_0^m$ of 0-solvable links both algebraically and geometrically.

Corollary. 5.2. *For an ordered, oriented m-component link $L = K_1 \cup \ \ldots \ \cup K_m$, the following conditions are equivalent:*

1. *L is 0-solvable,*

2. *L is band-pass equivalent to the m-component unlink,*

3. $Arf(K_i) = 0$

   $\bar{\mu}_L(ijk) = 0$

   $\bar{\mu}_L(iijj) \equiv 0 \pmod{2}$.

Using the algebraic characterization in Theorem 5.1, we can then classify links up to 0-solve equivalence by giving an algorithm for choosing representatives of each 0-solve equivalence class of links. The following corollary describes the class of links up to 0-solve equivalence.

Corollary. 5.12. *For each m,*

$$\frac{\mathcal{F}^m_{-0.5}}{\mathcal{F}^m_0} \cong \mathbb{Z}_2^m \oplus \mathbb{Z}^{\binom{m}{3}} \oplus \mathbb{Z}_2^{\binom{m}{2}}$$

We then use the work of Conant, Schneiderman, and Teichner to broaden the scope of Theorem 5.1 to include applications to the study of gropes and Whitney towers. Gropes and Whitney towers are geometric objects that are used in the study of 4-manifolds. Our algebraic characterization of 0-solve equivalence gives a relationship between 0-solvable links, links bounding gropes, and links supporting Whitney towers.

Corollary. 6.5. *For an ordered, oriented, m-component link L, the following are equivalent.*

1. *L is 0-solvable.*
2. *L bounds disjoint, properly embedded gropes of class 2 in $\mathbb{B}^4$.*
3. *L bounds properly immersed disks admitting an order 2 Whitney tower in $\mathbb{B}^4$.*

Having an understanding of 0-solvable links, we will then turn our attention to the study of 0.5-solvability for links. We will again study the relationship between algebraic link invariants and 0.5-solvability as well as the relationship between 0.5-solvability of links and several geometric moves on link diagrams. We use two geometric equivalence relations on link diagrams called double-delta equivalence and double half-clasp pass equivalence; we show a relationship between these moves and 0.5-solvability.

Proposition. 7.2. *The double-delta move and the double half-clasp pass move both preserve 0.5-solvability.*

This tells us that, if a link L is double-delta equivalent or double half-clasp pass equivalent to the trivial link, then L is 0.5-solvable.

Finally, we give an algebraic condition on a link L that is necessary for L to be 0.5-solvable.

Theorem. 7.4. *For an ordered, oriented, m-component, 0.5-solvable link $L = K_1 \cup \ldots \cup K_m$, the Sato-Levine invariants $\bar{\mu}_L(iijj)$ vanish.*

1.3 Outline of Thesis

In Chapter 2, we introduce the (string) link concordance group, $\mathcal{C}^m$, and the n-solvable filtration $\{\mathcal{F}_n^m\}$ of $\mathcal{C}^m$. We give properties of n-solvable links and discuss the known classification of low stages of the n-solvable filtration of the knot concordance group $\mathcal{C}^1$.

In Chapter 3, we introduce machinery that we will use to study low stages of the n-solvable filtration of the link concordance group. We discuss Milnor's invariants and focus particularly on the Milnor's invariants $\bar{\mu}_L(iijj)$ and $\bar{\mu}_L(ijk)$. We also introduce several geometric pass moves on link diagrams that we will employ in later chapters.

In Chapter 4, we define a new notion of 0-solve equivalence for links. We establish several properties of 0-solve equivalence, including the relationship between 0-solve equivalence and 0-solvability.

In Chapter 5, we give two sets of necessary and sufficient conditions for two links to be 0-solve equivalent. This gives both an algebraic and a geometric characterization of 0-solve equivalent links. We use this characterization to classify links up to 0-solve equivalence.

In Chapter 6, we introduce the notion of gropes and Whitney towers, which are geometric tools for studying links and surfaces in 4-manifols. We demonstrate that links bounding gropes and Whitney towers are precisely 0-solvable links.

In Chapter 7, we investigate 0.5-solvability of links. We give geometric conditions on links that are sufficient for 0.5-solvability, and we show that a certain algebraic condition that is necessary for 0.5-solvability for links.

Chapter 2

Link Concordance and the n-Solvable Filtration

2.1 Knot and Link Concordance

A *knot* is a smooth embedding, $f : S^1 \hookrightarrow S^3$, of an oriented circle into the three-sphere. Two such embeddings, f and f' are *isotopic* if there is a smooth homotopy $F : S^1 \times [0,1] \to S^3$ such that $F \times \{0\} = f$ and $F \times \{1\} = f'$, where $F \times \{t\}$ is a smooth embedding for all t. That is, two knots are isotopic if one can be smoothly deformed into the other. With this notion of knot equivalence, we refer to a *knot*, K, meaning the isotopy class of an embedding $K : S^1 \hookrightarrow S^3$.

Two knots, K and J in S^3 are *concordant* if there exists a smooth embedding, $f : S^1 \times [0,1] \to S^3 \times [0,1]$, of an annulus into $S^3 \times [0,1]$, such that $f(S^1 \times \{0\}) = K \times \{0\}$ and $f(S^1 \times \{1\}) = J \times \{1\}$. Knot concordance is an equivalence relation on knots, and the collection of concordance classes of knots form a group under the operation of connected sum. This group is called the *knot concordance group*, $\mathcal{C}$, and it is known to be infinitely generated and abelian. The identity element of the knot concordance group is the class of knots that are concordant to the trivial knot. Such knots are called *slice*.

A *link* is a generalization of a knot in which we allow for more than one knotted oriented circle. More precisely, an ordered link is an embedding, $f : \bigsqcup_{i=1}^{m} S^1 \to S^3$, of m circles into the three-sphere. We say that the link has m *components*. We similarly consider isotopy classes of links in 3-space, which we denote $L = K_1 \cup \ldots \cup K_m$, where the K_i represent the individual components of the link L. It is natural to define *link concordance* as a generalization of knot concordance. Two links, L and L', are *concordant* if there exists a smooth embedding, $f : \bigsqcup_{i=1}^{m} S^1 \times [0,1] \to S^3 \times [0,1]$, such that $f(\bigsqcup_{i=1}^{m} S^1 \times \{0\}) = L \times \{0\}$ and $f(\bigsqcup_{i=1}^{m} S^1 \times \{1\}) = L' \times \{1\}$. Links which are concordant to the unlink are called *slice* links.

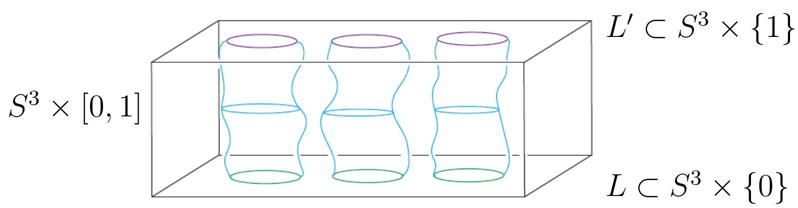


Figure 2.1: Link Concordance

In trying to assign a group structure on the concordance classes of oriented links, we quickly see that the operation of connected sum is not well-defined on links. To define an analogous group operation on links, we turn to the discussion of string links.

2.2 String Links and The Link Concordance Group

An m-component (pure) n - *string link* D, as defined by LeDimet in [15], is a proper, oriented submanifold of $\mathbb{D}^{n+2}$ that is homeomorphic to m disjoint copies of $\mathbb{D}^n$, $\{\mathbb{D}^n_i\}_{i=1}^m$, such that $D \cap \partial\mathbb{D}^{n+2} = \partial D$ is the standard trivial $(n-1)$-link U^{n-1}. We are only concerned with m-component 1-string links. These can be viewed as a generalization of an m-strand pure braid where we allow the strands to knot. Given any m-component string link D, we form the *closure* of D, denoted $\hat{D}$, by gluing the standard m-component trivial link to D along its boundary.

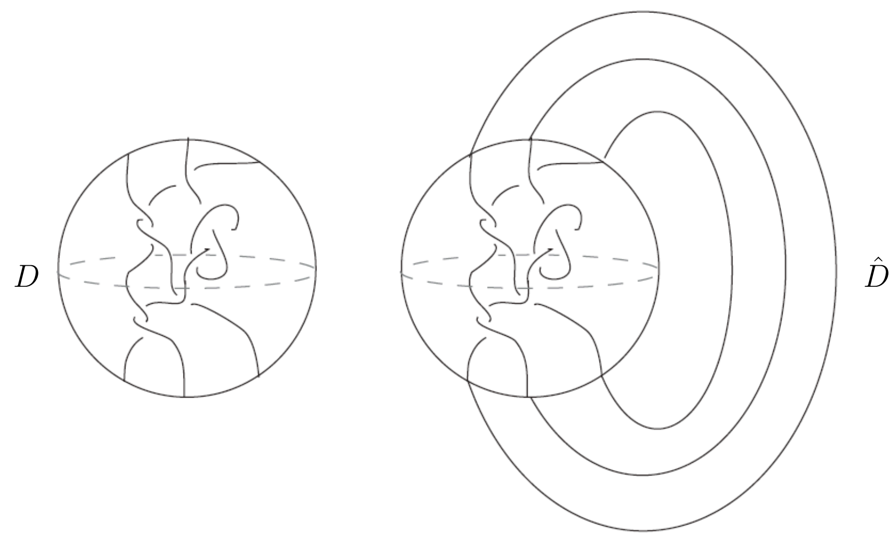


Figure 2.2: A 3-component string link D and its closure $\hat{D}$

As a result of Habegger and Lin, we can also form a string link from any link in S^3 by cutting along a carefully chosen ball. The following lemma specifies this relationship [11].

Lemma 2.1. [Habegger-Lin]*[11] For any link $L \in S^3$, there is a string link D such that $\hat{D}$ is isotopic to L.*

We can define a *concordance* between two m-component string links D and D'. We say that D and D' are *concordant* if there exists a proper smooth submanifold C of $\mathbb{D}^3 \times [0,1]$ such that C is homeomorphic to m disjoint copies of $\mathbb{D}^1 \times [0,1]$, where $C \cap (\mathbb{D}^3 \times \{0\}) = D$, $C \cap (\mathbb{D}^3 \times \{1\}) = D'$, and $C \cap (S^2 \times [0,1]) = U^0 \times [0,1]$. Using this definition, we see that there is a natural extension of C that gives a link concordance between the closures $\hat{D}$ and $\hat{D}'$.

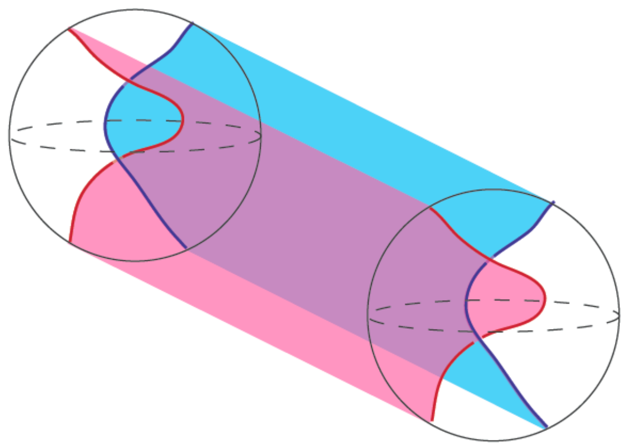


Figure 2.3: String link concordance

Unlike links, string links have a natural operation known as *stacking*. The set of concordance classes of m-component string links under stacking form a group $\mathcal{C}^m$, called the *string link concordance group* [15]. Furthermore, there is a group action on $\mathcal{C}^m$ such that, under closure, the orbits correspond to concordance classes of links [10]. In the case of $m = 1$, the group $\mathcal{C}^1$ corresponds exactly to the knot concordance group. For the case $m > 2$, it is well known that $\mathcal{C}^m$ is not abelian.

2.3 The n-Solvable Filtration

The link concordance group $\mathcal{C}^m$ is an infinitely generated group. One tool to study the structure of this group, defined by Cochran, Orr, and Teichner [7], is the *n-solvable filtration*, $\{\mathcal{F}_n^m\}$:

$$\{0\} \subset \cdots \subset \mathcal{F}_{n+1}^m \subset \mathcal{F}_{n.5}^m \subset \mathcal{F}_n^m \subset \cdots \subset \mathcal{F}_1^m \subset \mathcal{F}_{0.5}^m \subset \mathcal{F}_0^m \subset \mathcal{C}^m$$

Recall that a *filtration* of a group is a nested sequence of subgroups. The n-solvable filtration is indexed by the set $\frac{1}{2}\mathbb{N}$. For $k \in \frac{1}{2}\mathbb{N}$, $\mathcal{F}_k^m$ is the collection of *k-solvable* m-component links. It is important to note that the concordance class of m-component slice links is n-solvable for any n. In this way, the n-solvable filtration can be thought to "approximate" sliceness for links.

To define the condition of n-solvability, we must first establish several topological tools. Although we consider links $L = K_1 \cup \ldots \cup K_m$ in S^3, we have a very useful method of constructing different closed, oriented 3-manifolds based on L. We say that the 3-manifold M_L is the result of performing *0 framed surgery* on the link $L = K_1 \cup \ldots \cup K_m$ in S^3, where we define M_L by cutting and pasting.

$$M_L = (S^3 - N(L)) \cup_f [\bigsqcup_{i=1}^{m} (S^1 \times \mathbb{D}^2)_i]$$

Here, $N(L)$ refers to a tubular neighborhood of the link and f is the map that sends $\mu_i = (\{p_i\} \times \mathbb{D}^2)$ to the longitudes l_i of L, an untwisted copy of K_i on the boundary of $S^3 - N(L)$.

For a group G, we define the *derived series* $G^{(n)}$ of G recursively by $G^{(0)} := G$

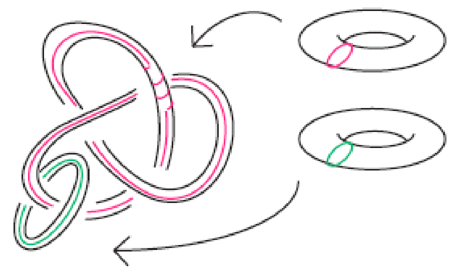


Figure 2.4: 0-Framed Surgery on a Link

and $G^{(n)} := [G^{(n-1)}, G^{(n-1)}]$, where $[G, G]$ refers to the commutator subgroup of G. We say that $G^{(n)}$ is the n^{th} *term of the derived series.*

Definition 2.2. We say that an m-component link $L = K_1 \cup \ldots \cup K_m$ is n*-solvable* for $n \in \mathbb{N}$, and we write $L \in \{\mathcal{F}_n^m\}$, if the manifold M_L obtained from performing 0-framed surgery on L in S^3 bounds a compact, smooth 4-manifold W under the following conditions:

1. $H_1(M_L) \cong \mathbb{Z}^m$, and the map induced by inclusion, $i_* : H_1(M_L) \to H_1(W)$ is an isomorphism on the first homology.

2. $H_2(W)$ has a basis consisting of compact, connected, embedded, oriented surfaces $\{L_i, D_i\}_{i=1}^r$ with trivial normal bundles, such that L_i intersects D_i transversely, exactly once, with positive sign, and otherwise, the surfaces are disjoint.

3. For all i, $\pi_1(L_i) \subset \pi_1(W)^{(n)}$ and $\pi_1(D_i) \subset \pi_1(W)^{(n)}$

The 4-manifold W is called an n*-solution* for L.

Additionally, we say that L is n*.5-solvable* if L is n-solvable and $\pi_1(L_i) \subset \pi_1(W)^{(n+1)}$.

Note that we consider a string link to be n-solvable if its closure is n-solvable.

2.4 Properties of n-Solvable Links

We establish properties of n-solvable links that will be useful in the following chapters.

Remark 2.3. *By definition, we require that if a link* $L = K_1 \cup \ldots \cup K_m$ *is* n*-solvable for any* $n \in \frac{1}{2}\mathbb{N}$*, that* $H_1(M_L) \cong \mathbb{Z}^m$*. Note that this is true if and only if the pairwise linking numbers between components of* L *vanish:* $lk(K_i, K_j) = 0$ *for* $1 \leq i < j \leq m$*.*

Henceforth, we will only consider links with this property, and we will denote the collection of concordance classes of m-component links with vanishing pairwise linking numbers as $\mathcal{F}^m_{-0.5} \subset \mathcal{C}^m$ for convenience.

Proposition 2.4. *For any* $n \in \frac{1}{2}\mathbb{N}$*, if a link* $L = K_1 \cup \ldots \cup K_m$ *is* n*-solvable, any sublink of* L *is also* n*-solvable.*

Proof. Note that, by induction and up to reordering the link, it suffices to show that the sublink $J = K_2 \cup \cdots \cup K_m \subset L$ created by removing the first link component is n-solvable. Let W be an n-solution for L. We form a new 4-manifold W', with $\partial W' = M_J$ by attaching a 0-framed 2-handle to ∂W with attaching sphere μ_1, the meridian of the link component $K_1 \subset L$. We wish to show that W' is an n-solution for J.

First, we observe that attaching the 2-handle to the element $[\mu_1] \in H_1(M_L) \cong H_1(W)$ kills the generator, and so we see that $H_1(M_J) \cong H_1(W') \cong \mathbb{Z}^{m-1}$ induced by inclusion.

Next, we observe that $H_2(W') \cong H_2(W)$ as the order of $[\mu_1] \in H_1(M_L)$ is infinite. Therefore, $H_2(W')$ has the desired generating set.

Finally, we observe that, for $\{L_i, D_i\}_{i=1}^r$ the basis for $H_2(W')$, we have that $\pi_1(L_i) \subset \pi_1(W)^{(n)} \subseteq \pi_1(W')^{(n)}$. Similarly, $\pi_1(D_i) \subset \pi_1(W)^{(n)} \subseteq \pi_1(W')^{(n)}$. Therefore, W' is an n-solution for J.

□

2.5 Known Classification of 0 and 0.5-Solvable Knots

In the case of knots, there is a complete classification of knots that are 0-solvable and knots that are 0.5-solvable. In this section, we give results from [7] classifying $\mathcal{F}_0^1$ and $\mathcal{F}_{0.5}^1$, the subgroup of 0-solvable knots and the subgroup of 0.5-solvable knots.

The *Arf invariant* of a knot K is a concordance invariant taking values in $\mathbb{Z}_2$. There are several equivalent ways of defining the Arf invariant.

Definition 2.5. For a knot $K \subset S^3$, a Seifert surface $\Sigma \subset S^3 - K$ for K, and for $\{a_i, b_i\}_{i=1}^g$ a symplectic basis for Σ, the Arf invariant $Arf(K)$ is the sum

$$Arf(K) = \sum_{i=1}^{g} lk(a_i, a_i^+) lk(b_i, b_i^+) (\text{mod} 2).$$

We may also compute the Arf invariant by considering the Alexander polynomial of a knot, $\Delta_K(t)$. It is known that $Arf(K) = 0 \Leftrightarrow \Delta_K(-1) \equiv \pm 1 \text{ (mod8)}$ [18]. Alternatively, it is known that the Arf invariant of a knot vanishes if and only if K is band-pass equivalent to the unknot [14]. We will discuss band-pass equivalence extensively in the following chapters.

In [7], Cochran, Orr, and Teichner give a classification of 0-solvable knots.

Theorem 2.6. [Cochran-Orr-Teichner][7] *A knot K is 0-solvable if and only if $Arf(K) = 0$.*

As we defined in section 2.1, a *slice* knot is a knot that is concordant to the trivial knot. An equivalent definition would be to say that a knot K is slice if K bounds a smoothly embedded disk $\Delta \subset \mathbb{B}^4$. It is known that, if K is a slice knot, there exists a Seifert matrix V for K where V is of the form $V = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, where A, B, and C are $g \times g$ matrices. The converse of this, however, is not true. There exist knots which are not slice, but have a Seifert matrix that has the form of a slice knot; these are called algebraically slice knots.

Definition 2.7. A knot K is called *algebraically slice* if there exists a Seifert matrix V for K such that $V = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$.

In [7], Cochran, Orr, and Teichner give a classification of 0.5-solvable knots.

Theorem 2.8. [Cochran-Orr-Teichner][7] *A knot K is 0.5-solvable if and only if K is algebraically slice.*

Chapter 3

Milnor's Invariants and Pass Moves

3.1 Milnor's $\bar{\mu}$-Invariants

In the 1950's, Milnor defined a classical family of link invariants called $\bar{\mu}$-invariants [16], [17]. These algebraic invariants are denoted $\bar{\mu}_L(I)$, where $I = i_1 i_2 \dots i_k$ is a word of length k and $i_j \in \{1, \dots, m\}$ refers to the j^{th} link component of L. The integer k is called the *length* of the Milnor invariant. Milnor's invariants are not, strictly speaking, link invariants; they have some indeterminacy resulting from the choice of meridians of the link. However, Habegger and Lin show that this indeterminacy corresponds exactly to the choice of ways of representing a link as the closure of a string link [11]. As such, Milnor's invariants are string link invariants whenever they are well defined. Furthermore, Milnor's invariants are concordance invariants, which makes them a useful tool in studying the link concordance group [1].

We will use two specific Milnor's invariants in the following chapter, $\bar{\mu}_L(iijj)$ and $\bar{\mu}_L(ijk)$. There are many ways of defining Milnor's invariants, but these specific two invariants can be thought of as higher order cup products. We give their definitions geometrically.

The Milnor's invariants $\bar{\mu}_L(iijj)$ for a link $L = K_1 \cup \ldots \cup K_m$ are also known as Sato-Levine invariants. Every 2-component sublink $K_i \cup K_j$ gives a Sato-Levine invariant $\bar{\mu}_L(iijj)$. We can compute this invariant by considering oriented Seifert surfaces Σ_i and Σ_j for K_i and K_j in the link exterior $S^3 - N(L)$. We may choose these surfaces in such a way that $\Sigma_i \cap \Sigma_j = \gamma \cong S^1$ [4]. Then, we push the curve γ off of one of the surfaces Σ_i in the positive normal direction to obtain a new curve γ^+. We define the Sato-Levine invariant to be $lk(\gamma, \gamma^+)$. This is a well-defined link invariant provided that the pairwise linking numbers of L all vanish, and it is equal to the Milnor's invariant $\bar{\mu}_L(iijj)$ [4].

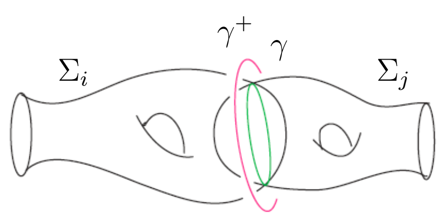


Figure 3.1: Computing $\bar{\mu}_L(iijj)$

The Milnor's invariant $\bar{\mu}_L(ijk)$ for a link $L = K_1 \cup \ldots \cup K_m$ also have a geometric definition [4]. Consider the sublink $J = K_i \cup K_j \cup K_k \subset L$ and let Σ_i, Σ_j, and Σ_k be oriented Seifert surfaces for K_i, K_j, and K_k. The intersection $\Sigma_i \cap \Sigma_j \cap \Sigma_k$ is a collection of points which are given an orientation induced by the outward normal on each Seifert surface. The count of these points up to sign gives us $\bar{\mu}_L(ijk)$ [4].

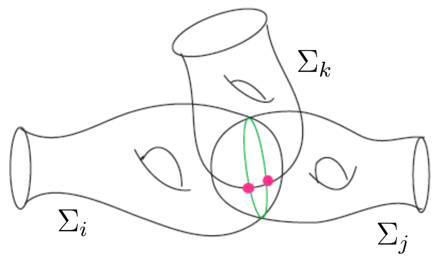


Figure 3.2: Computing $\bar{\mu}_L(ijk)$

3.2 Geometric Moves on Links

In the following chapters, we will employ several types of pass-moves on links. These are geometric moves performed locally on an oriented link diagram; coupled with isotopy, these moves each generate an equivalence relation on links.

Definition 3.1. A *band-pass move* on a link $L = K_1 \cup \ \ldots \ \cup K_m$ is the local move pictured in figure 3.3. We require that both strands of each band belong to the same link component. Links L and L' are *band-pass equivalent* if L can be transformed into L' through a finite sequence of band-pass moves and isotopy.

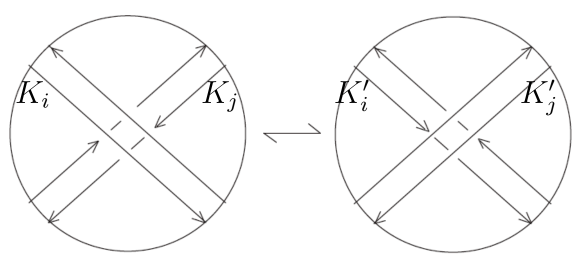


Figure 3.3: A band-pass move

Definition 3.2. A *clasp-pass move* on a link L is the local move in figure 3.4. Links

L and L' are *clasp-pass equivalent* if L can be transformed into L' through a finite sequence of clasp-pass moves and isotopy.

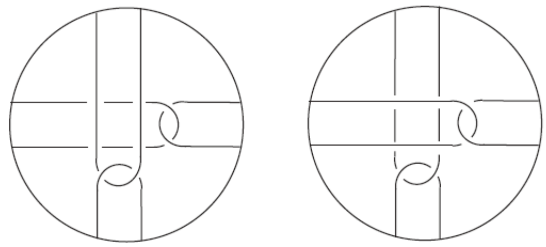


Figure 3.4: A clasp-pass move

Definition 3.3. A *double-delta move* on a link L is the local move in figure 3.5. We require that the strands of each band belong to the same link component, so that a double-delta move may involve no more than 3 distinct link components. Links L and L' are *double-delta equivalent* if L can be transformed into L' through a finite sequence of double-delta moves and isotopy.

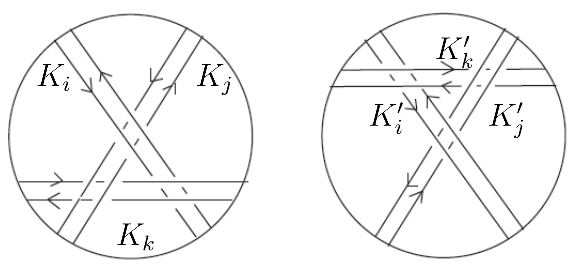


Figure 3.5: A double-delta move

Definition 3.4. A *double half-clasp pass move* on a link L is the local move in figure 3.6. We require that the strands of each band belong to the same component. Links L and L' are *double half-clasp pass equivalent* if L can be transformed into L' through a finite sequence of double half-clasp pass moves and isotopy.

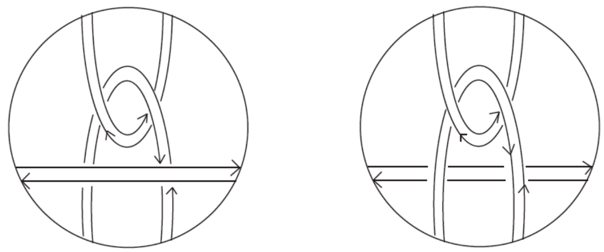


Figure 3.6: A double half-clasp pass move

Definition 3.5. We will also use a geometric move that is not a pass move. Pictured in figure 3.7 is a *double Borromean rings insertion* on a link. Here, we take three bands, where the strands of each band belong to the same link component, and insert two copies of the Borromean rings, preserving orientation. This geometric move may involve one, two, or three distinct link components.

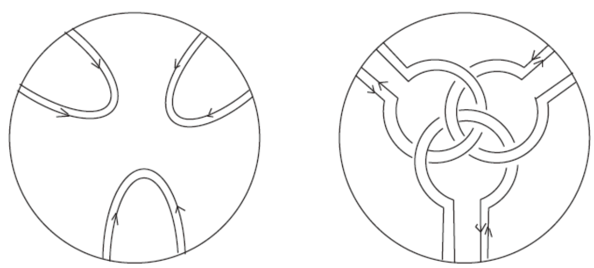


Figure 3.7: A double Borromean rings insertion

Chapter 4

0-Solve Equivalence

4.1 Defining 0-Solve Equivalence

In this chapter, we will define a new equivalence relation on the concordance classes of links called *0-solve equivalence*. We will then prove properties of 0-solve equivalence that will be helpful in the next chapter. Note that we only consider links $L = K_1 \cup \ldots \cup K_m$ such that the pairwise linking numbers vanish.

In order to define 0-solve equivalence, we first must establish several definitions. For a 4-manifold W^4, a *spin structure* on W is a choice of trivialization of the tangent bundle T_W over the 1-skeleton of W that can be extended over the 2-skeleton. A manifold endowed with a spin structure is called a *spin manifold*. It is known that W admits a spin structure if and only if the second Stiefel-Whitney class $w_2(W) = 0$. It is a result of Wu's that, for W a smooth, closed 4-manifold such that $H_1(W)$ has no 2-torsion, W is spin if and only if the intersection form Q_W on W is even (see [20] section 4.3).

We will define a notion of the *closure* of a 4-manifold that will be suitable to our needs.

Definition 4.1. Suppose that W is a 4-manifold such that $\partial W = M_L \bigsqcup -M_{L'}$. We define the *closure* of W, which we denote $\hat{W}$, to be the closed 4-manifold that is obtained from W by first attaching a 0-framed 2-handle to each meridinal curve in both M_L and $M_{L'}$. This results in a 4-manifold with boundary $S^3 \bigsqcup -S^3$, as depicted in the following figure.

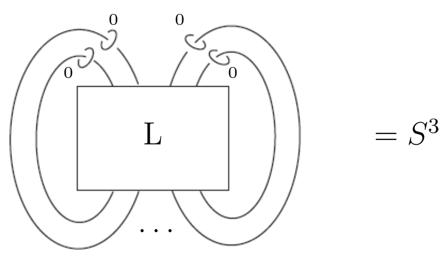


Figure 4.1: A link with "helper circles"

Now, we can attach a 4-handle to the S^3 boundary component and a 4-handle to the $-S^3$ boundary component, and we have formed the closed 4-manifold which we will call $\hat{W}$, as pictured in figure 4.2.

We now can give a definition of 0-solve equivalence on links, which we will use extensively in the following chapter.

Definition 4.2. Two ordered, oriented m-component links $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$ with vanishing pairwise linking numbers are *0-solve equivalent* if there exists a 4-manifold W with $\partial W = M_L \bigsqcup -M_{L'}$ such that the following conditions hold:

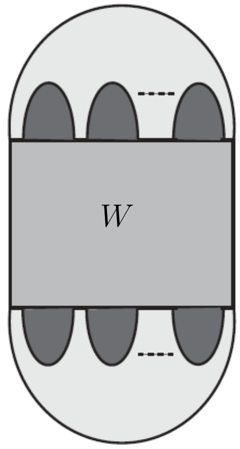


Figure 4.2: The closure of a 0-solve equivalence

1. The maps $i_* : H_1(M_L) \to H_1(W)$ and $j_* : H_1(M_{L'}) \to H_1(W)$ induced by inclusion are isomorphisms such that $i_*(\mu_i) = j_*(\mu_i')$, where μ_i denotes the meridian of the i^{th} link component.

2. $H_2(W, \partial W_-)$, where $\partial W_- = -M_{L'}$, is generated by oriented, embedded, connected pairs of surfaces $\{L_i, D_i\}_{i=1}^r$ with trivial normal bundles such that the surfaces are disjoint, with the exception that, for each i, L_i and D_i intersect transversely at exactly one point, oriented with positive direction.

3. $\hat{W}$ is a spin 4-manifold.

The manifold W is called a *0-solve equivalence* between L and L'.

Remark 4.3. *It is worthwhile to note that, for a 0-solve equivalence W,*

$$\frac{H_2(W)}{H_2(\partial W_-)} \cong H_2(W, \partial W_-)$$

Here, because the map induced by inclusion, $i_* : H_2(\partial W_-) \to H_2(W)$ is injective, we refer to $i_*(H_2(\partial W_-))$ by $H_2(\partial W_-)$; we see this result from considering the long exact sequence of the pair $(W, \partial W_-)$. We note that that $i_* : H_1(\partial W_-) \to H_1(W)$ is an isomorphism and that $H_3(W, \partial W_-) \cong H^1(W, \partial W_+) \cong Hom(H_1(W, \partial W_+), \mathbb{Z}) = 0$.

$$H_3(W, \partial W_-) \to H_2(\partial W_-) \xrightarrow{i_*} H_2(W) \twoheadrightarrow H_2(W, \partial W_-) \xrightarrow{0} H_1(\partial W_-) \xrightarrow{\cong} H_1(W) \longrightarrow$$

Furthermore, this portion of the sequence splits, and so $H_2(W) \cong H_2(W, \partial W_-) \oplus H_2(\partial W_-)$.

4.2 Properties of 0-Solve Equivalence

Proposition 4.4. *0-solve equivalence in an equivalence relation on links.*

Proof. 1. We consider the 4-manifold $W = M_L \times [0, 1]$. The first two conditions of 0-solve equivalence are trivial. We wish to show that the closure $\hat{W}$ has an even intersection form, and is therefore spin. The generators of $H_2(\hat{W})$ are $\{\hat{\Sigma}_i, F_i\}_{i=1}^m$, where $\Sigma_i \subset S^3 - N(L) \subset M_L$ is a Seifert surface for link component K_i, $\hat{\Sigma}_i$ is the closed surface $\Sigma_i \cup_{\ell_i} \mathbb{D}^2$ where ℓ_i is the 0-framing on K_i, and F_i is the core of the i^{th} added 2-handles attached to $\mu_i \times [0, 1] \subset M_L \times [0, 1]$. Each generator $\hat{\Sigma}_i$ and F_i has self-intersection zero. Thus, the intersection form $Q_{\hat{W}}$ is even, and so $\hat{W}$ is spin.

2. Let W be a 0-solve equivalence between L and L'. We flip W upside down to obtain W', where $\partial W'_- = -M_L$. We wish to show that W' is a 0-solve equivalence between L' and L. The first condition on the inclusion maps inducing isomorphisms on first homology is given, as W is a 0-solve equivalence.

We consider the long exact sequence of the pair $(W', \partial W'_+)$, coupled with the fact that $H_3(W', \partial W'_+) \cong H^1(W', \partial W'_-) = 0$:

$$\to H_3(W', \partial W'_+) \to H_2(\partial W'_+)$$

$$\xrightarrow{j_*} H_2(W') \to H_2(W', \partial W'_+) \xrightarrow{0} H_1(\partial W'_+) \xrightarrow{\cong}$$

This sequence splits since $H_2(W', \partial W'_+) \cong H^2(W', \partial W'_+) \cong 0$, and so $H_2(W') \cong H_2(W', \partial W'_+) \oplus j_*(H_2(\partial W'_+))$. We know that $H_2(W', \partial W'_+) = H_2(W, \partial W_-) =< L_i, D_i >_{i=1}^r \cong \mathbb{Z}^{2r}$, and $H_2(\partial W'_+) = \{a_i\} \cong \mathbb{Z}^m$. These generators $\{a_i\}_{i=1}^m$ are capped off Seifert surfaces and may be chosen to have self-intersection 0 and to be disjoint from the $\{L_i, D_i\}$. We have a similar exact sequence for the pair $(W', \partial W'_-)$,

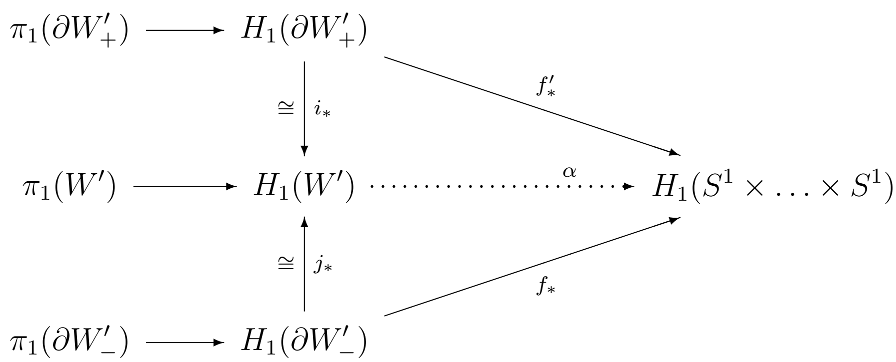
$$0 \to H_2(\partial W'_-) \xrightarrow{i_*} H_2(W') \twoheadrightarrow H_2(W', \partial W'_-) \xrightarrow{0} H_1(\partial W'_-) \xrightarrow{\cong} H_1(W')$$

Therefore, $\mathbb{Z}^{2(r+g)} \cong H_2(W') \cong H_2(W', \partial W'_-) \oplus H_2(\partial W'_-)$. Hence, the rank of $H_2(\partial W'_+)$ is the same as the rank of $H_2(\partial W'_-)$, and we may write $H_2(\partial W'_-) = \{a'_i\}_{i=1}^m$.

We choose oriented Seifert surfaces $\{\Sigma_i\}_{i=1}^m \subset S^3 - L$ for link components K_i; we similarly choose oriented Seifert surfaces $\{\Sigma'_i\}_{i=1}^m \subset S^3 - L'$ for link components K'_i. We let $\hat{\Sigma}_i$ and $\hat{\Sigma}'_i$ be closed surfaces in $\partial W'_-$ and $\partial W'_+$ Using the Thom-Pontryagin construction, we define maps $f_i : \partial W'_- \to S^1$ using the following construction. We take a produce neighborhood $\hat{\Sigma}_i \times [-1, 1]$ where the

+1 corresponds to the positive side of $\hat{\Sigma}_i$. We define the map $f_i : \hat{\Sigma}_i \times [-1,1] \to S^1$ by $(x,t) \mapsto e^{2\pi i t}$, and for $y \in \partial W'_-(\hat{\Sigma}_i \times [-1,1])$, we define $f_i(y) = -1$. We similarly define maps $F'_i : \partial W'_+ \to S^1$. Then, we consider the maps $f = f_1 \times \ldots \times f_m : \partial W'_+ \to \bigsqcup_{i=1}^m S^1$ and $f' = f'_1 \times \ldots \times f'_m : \partial W'_- \to \bigsqcup_{i=1}^m S^1$.

The maps f and f' induce isomorphisms on first homology, and we know that the inclusion maps $i_* : H_1(\partial W'_+) \to H_1(W')$ and $j_* : H_1(\partial W'_-) \to H_1(W')$ are both isomorphisms.



We wish to extend to a map $\alpha : H_1(W') \to H_1(S^1 \times \ldots \times S^1)$; this is possible because $f'_* \circ i_*^{-1} = f_* \circ j_*^{-1}$, by the Thom-Pontryagin construction and the fact that $i_*(\mu'_i) = j_*(\mu_i)$. Therefore, we can extend to the map $\alpha : H_1(W') \to H_1(S^1 \times \ldots \times S^1)$, and because $\pi_1(S^1 \times \ldots \times S^1) \cong \mathbb{Z}^m$ is abelian, we can also extend to the map $\bar{\alpha} : \pi_1(W') \to \pi_1(S^1 \times \ldots \times S^1)$. Then, knowing that the CW-complex $\bigsqcup_{i=1}^m$ is an Eilenberg-Maclane space $K(\mathbb{Z}^m, 1)$, we can extend the maps f, f' to a map $\bar{f} : W' \to S^1 \times \ldots \times S^1$. The preimage $\bar{f}^{-1}((0,\ldots,0,1,0,\ldots,0)) = M$ is a 3-submanifold of W' such that $\partial M = \hat{\Sigma}'_i \bigsqcup \hat{\Sigma}_i$. This tells us that $\hat{\Sigma}'_i$ and $\hat{\Sigma}_i$ are homologous, and so the generators $\{a_i\}$ and the generators $\{a'_i\}$ are homologous. Therefore, $H_2(W') \cong H_2(W', \partial W'_+) \oplus \{a'_i\}_{i-1}^m \cong H_2(W', \partial W'_-) \oplus \{a_i\}_{i-1}^m$, and so

$H_2(W', \partial W'_+) \cong H_2(W', \partial W'_-)$. Furthermore, $H_2(W, \partial W_-) = H_2(W', \partial W'_+)$ is generated by surfaces that intersect pairwise transversely exactly once. Therefore, we have shown that $H_2(W', \partial W'_-)$ also has such generators.

Lastly, we observe that $\hat{W} = \hat{W}'$. As $\hat{W}$ is spin, so is $\hat{W}'$.

3. Let W be a 0-solve equivalence between L and L' and let W' be a 0-solve equivalence between L' and L''. Let $V = W \cup_{M_{L'}} W'$. We wish to show that V is a 0-solve equivalence between L and L''. The first condition of 0-solve equivalence follows from the fact that W and W' are 0-solve equivalences. We consider a Mayor-Vietoris sequence for V,

$$\to H_3(V) \to H_2(W \cap V') \overset{i_*,-j_*}{\to} H_2(W) \oplus H_2(W') \twoheadrightarrow H_2(V) \overset{0}{\to} H_1(W \cap W') \to$$

$$H_1(W) \oplus H_1(W') \to H_1(V) \to$$

This tells us that $H_2(V) \cong \frac{H_2(W) \oplus H_2(W')}{< i_*(H_2(W \cap W')), -j_*(H_2(W \cap W')) >}$. Using the splitting of $H_2(W) \cong H_2(W, \partial W_-) \oplus H_2(\partial W_-)$, we observe that the inclusion $< i_*(H_2(W \cap W')), -j_*(H_2(W \cap W')) >$ maps isomorphically onto $H_2(\partial W_-)$ and trivially into $H_2(W, \partial W_-)$. Therefore, $H_2(V) \cong H_2(W, \partial W_-) \oplus H_2(W')$. Next, we consider the long exact sequence of the pair $(V, \partial V_-)$.

$$\to H_3(V, \partial V_-) \to H_2(\partial V_-) \to H_2(V) \twoheadrightarrow H_2(V, \partial V_-) \overset{0}{\to} H_1(\partial V_-) \overset{\cong}{\to} H_1(V) \to .$$

This tells us that $H_2(V, \partial V_-) \cong \frac{H_2(V)}{H_2(\partial V_-)}$. Coupled with the splitting $H_2(W') \cong H_2(W', \partial W'_-) \oplus H_2(\partial W'_-)$, we have that

$$H_2(V, \partial V_-) \cong \frac{H_2(W, \partial W_-) \oplus H_2(W', \partial W'_-) \oplus H_2(\partial W'_-)}{H_2(\partial V_-)}$$

$$\cong H_2(W, \partial W_-) \oplus H_2(W', \partial W'_-).$$

Therefore, $H_2(V, \partial V_-)$ has the desired generating set. Finally, we must show that $\hat{V}$ is a spin manifold. We show that the basis elements of $H_2(\hat{V})$ have even self-intersection.

We know that the closures $\hat{W}$ and $\hat{W'}$ are spin manifolds. The generators of $H_2(\hat{W})$ are the surfaces $\{L_i, D_i\}$ with self-intersection zero, capped Seifert surfaces for each link component, $\{\hat{\Sigma}_i\}$, and capped surfaces $\{\hat{S}_i\}$, where S_i is a surface in W realizing the null-homology between μ_i and μ'_i. Similarly, the generators of $H_2(\hat{W'})$ are the surfaces $\{L'_i, D'_i\}$ with self-intersection zero, capped Seifert surfaces for each link component, $\{\hat{\Sigma}'_i\}$, with self-intersection zero, and capped surfaces $\{\hat{S}'_i\}$, where S'_i is a surface in W' realizing the null-homology between μ'_i and μ''_i. Then, the basis elements for $H_2(\hat{V})$ are the surfaces $\{L_i, D_i\}, \{L'_i, D'_i\}, \{\hat{\Sigma}'_i\}$, and the capped surfaces $\{\hat{T}_i\}$, where $T_i = S_i \cup_{\mu'_i} S'_i$. Since the caps have no self-intersection, the self intersection $\hat{T}_i \cdot \hat{T}_i = S_i \cdot S_i + S'_i \cdot S'_i$, but because $\hat{W}$ and $\hat{W'}$ are spin, $S_i \cdot S_i$ and $S'_i \cdot S'_i$ are both even. Therefore, $\hat{V}$ has even intersection form and is spin.

□

Proposition 4.5. *If $L \sim_0 L'$ via a 0-solve equivalence W, and if V is a 0-solution for L', then L is 0-solvable, and $V' = W \bigsqcup_{M_{L'}} V$ is a 0-solution for L.*

Proof. The first condition of 0-solvability, that $i_* : H_1(M_L) \to H_1(V')$ is an isomorphism, follows from the fact that W is a 0-solve equivalence and that V is a 0-solution.

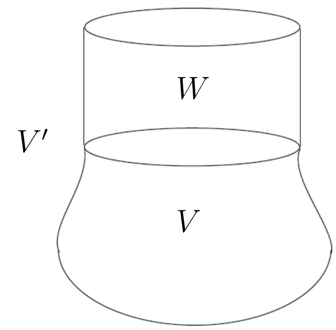


Figure 4.3: A 0-solution for L

We know that $H_2(V)$ is generated by surfaces $\{L_i, D_i\}_{i=1}^r$ that intersect pairwise once, transversely. We also know that $H_2(W, \partial W_-)$ is generated by surfaces $\{L'_i, D'_i\}_{i=1}^g$ that intersect pairwise once transversely. We wish to show the same about $H_2(V')$.

We consider the Mayer-Vietoris sequence for V'.

$$\to H_3(V') \to H_2(W \cap V) \overset{i_*, j_*}{\to} H_2(W) \oplus H_2(V) \to H_2(V') \overset{o}{\to} H_1(W \cap V) \to$$

$$H_1(W) \oplus H_1(V) \to$$

This gives us that $H_2(V') \cong \frac{H_2(W) \oplus H_2(V)}{< i_*(H_2(W \cap V)), j_*(H_2(W \cap V)) >}$. However, $H_2(W) \cong H_2(W, \partial W_-) \oplus H_2(\partial W_-)$, and the inclusion $< i_*(H_2(W \cap V)), j_*(H_2(W \cap V)) >$ maps isomorphically to the $H_2(\partial W_-)$ and maps trivially into $H_2(W, \partial W_-)$. Therefore, $H_2(V') \cong H_2(W, \partial W_-) \oplus H_2(V)$. This tells us that $H_2(V')$ has the proper generating set, and so V' is a 0-solution for L. □

Proposition 4.6. *For $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$ two m-component 0-solve equivalent links, the corresponding k-component sublinks $J = K_{i_1} \cup \cdots \cup K_{i_k}$ and $J' = K'_{i_1} \cup \cdots \cup K'_{i_k}$, where $i_j = i'_j \in \{1, \ldots, m\}$ are also 0-solve* *equivalent.*

Proof. Note that, by induction and reordering both links simultaneously, it suffices to show that the sublinks obtained by removing the first component of L and L' are 0-solve equivalent. Let $J = K_2 \cup \cdots \cup K_m$ and $J' = K_2' \cup \cdots \cup K_m'$.

Let W be the 4-manifold realizing the 0-solve equivalence between L and L'. We form a cobordism between M_J and $M_{J'}$ by attaching two 0-framed 2-handles to ∂W with attaching spheres μ_1 and μ_1', the meridians of the link components K_1 and K_1'. This yields a new 4-manifold, V, with $\partial V = M_J \bigsqcup -M_{J'}$. We wish to show that V is a 0-solve equivalence between J and J'.

By definition, $i_* : H_1(M_L) \hookrightarrow H_1(W)$ and $j_* : H_1(M_{L'}) \hookrightarrow H_1(W)$ are both isomorphisms. Moreover, $H_1(M_L) \cong < \mu_1, \ldots, \mu_m > \cong \mathbb{Z}^m$ and $H_1(M_L') \cong < \mu_1', \ldots, \mu_m' > \cong \mathbb{Z}^m$. Considering the effect on homology of attaching a 2-cell and the fact that $i_*(\mu_1) = j_*(\mu_1')$, we see that the maps induced by inclusion, $i_* : H_1(M_J) \to H_1(V)$ and $j_* : H_1(M_{J'}) \to H_1(V)$ are both isomorphisms, and it still holds that $i_*(\mu_k) = j_*(\mu_k)$ for $k \neq 1$.

We now consider how $H_2(V)$ differs from $H_2(W)$. By attaching a 2-handle to W along the curve μ_1, we do not change second homology, as $< i_*(\mu_1) >$ is infinite in $H_1(W)$. Then, attaching a 2-handle along the curve μ_1' creates an infinite cyclic direct summand in $H_2(V)$ generated by B, the oriented surface in V obtained by capping off the meridians μ_1 and μ_1' in W plus a surface in W realizing the null-homology between μ_i and μ_i'.

Next, we consider how $H_2(V, \partial V_-) \cong \frac{H_2(V)}{H_2(\partial V_-)}$ differs from $H_2(W, \partial W_-) \cong \frac{H_2(W)}{H_2(\partial W_-)}$.

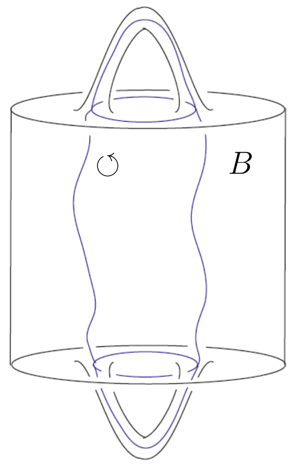


Figure 4.4: The surface $B \subset V$

Let Σ_1' be a Seifert surface for L_1' in $S^3 - N(L') \subset M_{L'}$, and let S_1' be a closed oriented surface in $M_{L'}$ obtained by capping off Σ_1' with the disk obtained from performing zero-framed surgery on L'. We notice that $S_1' \subset \partial W_-$ but S_1' is not in ∂V_-. From its construction, we can assume that B is an annulus near μ_i and μ_i'; therefore, we can assume that $B \cap M_{L'} = \mu_1'$, and so B intersects S_1' transversely at precisely one point. After possibly changing the orientation on B, we may assume that $B \cdot S_1' = +1$.

From Poincare duality, we have an intersection form $H_2(V, \partial V_-) \times H_2(V, \partial V_+) \to \mathbb{Z}$ where $(S_1', B) \mapsto +1$. Therefore, the class $[S_1'] \in H_2(V, \partial V_-)$ is nontrivial, where S_1' can be made disjoint from the generators of $H_2(W)$, and so $[S_1']$ is a generator of $H_2(V, \partial V_-)$.

We have that $H_2(V, \partial V_-) \cong H_2(W, \partial W_-) \oplus \mathbb{Z} \oplus \mathbb{Z}$ where the extra homology is generated by $\{[B], [S_1']\}$. We must show that $H_2(V, \partial V_-)$ has generators that intersect as in the definition of 0-solve equivalence.

So far, we have the the intersection matrix for $H_2(V, \partial V_-)$ looks like:

$$\begin{array}{c|cc|cccc}
 & S_1' & B & L_1 & D_1\dots & L_g & D_g \\
S_1' & 0 & 1 & 0 & 0\dots & 0 & 0 \\
B & 1 & \star & \gamma_1 & \gamma_2\dots & & \gamma_{2g} \\
\hline
L_1 & 0 & \gamma_1 & & & & \\
D_1 & 0 & \gamma_2 & & & & \\
\vdots & \vdots & \vdots & & A & & \\
L_g & 0 & & & & & \\
D_g & 0 & \gamma_{2g} & & & &
\end{array}$$

Here, A is the intersection matrix for $H_2(W, \partial W_-)$, $\{\gamma_i\}$ are integers representing the intersection of B with the generators of $H_2(W, \partial W_-)$, and $\star$ is an integer representing the self-intersection $B \cdot B$. We first seek to show that, for some choice of B, we have $\star = 0$.

Considering the way we constructed V, we observe that any self-intersection of B must occur in the interior of W. Moreover, B is a surface in $\hat{W}$, the closure of the 0-solve equivalence W. Because $\hat{W}$ is even, we know that $B \cdot B \equiv 0 \pmod{2}$. We introduce a change of basis to find a generator B_k that has trivial self-intersection. Suppose that $B \cdot B = 2k$ where $k < 0$. Then, let $S_1'^{+}$ be a push-off of the surface S_1'. Let α_1 be an arc in V from B to $S_1'^{+}$ that does not intersect the other generators $\{L_i, D_i\}$ of $H_2(V, \partial V_-)$. Let $N(\alpha_1)$ be a tubular ϵ-neighborhood of α_1, and define a new surface $B_1 = B \cup \partial N(\alpha_1) \cup S_1'^{+}$; then, in homology, $[B_1] = [B + S_1']$ Note that we can choose the arc so that the orientation on our new surface matches up with the original orientations on B and S_1'. This new surface B_1 also intersects S_1' exactly once transversely, and has self-intersection $B_1 \cdot B_1 = B \cdot B + S_1' \cdot S_1' + 2(B \cdot S_1') = B \cdot B + 2$. Since $B \cdot B = 2k$, by repeating this process k times, we will have a surface B_k that intersects S_1' exactly once and has self intersection zero. Note that, if $B \cdot B = 2k$ where $k > 0$, we would let $-S_1'$ be the surface S_1' with opposite orientation. We would then let α_1 be an arc from B to $-S_1'^{+}$ and we would then define B_1 as above. Under these conditions, the homology class $[B_1] = [B - S_1']$ and $B_1 \cdot B_1 = B \cdot B + 2(B \cdot - S_1') = B \cdot B - 2$. Therefore, we have that $[B_k] = [B \pm k S_1']$, and we use this change of basis for $H_2(V, \partial V_-)$, replacing B with B_k.

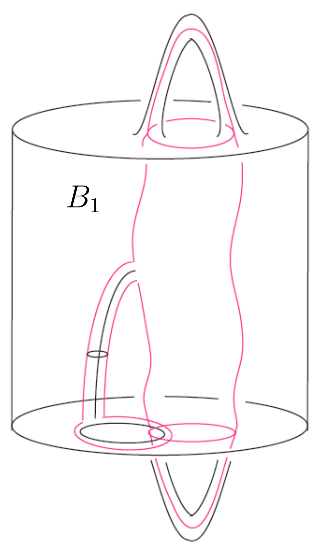


Figure 4.5: The new generator, B_1

Next, we consider the intersection of B_k with the original generators $\{L_i, D_i\}_{i=1}^{g}$ of $H_2(W, \partial W_-)$. Suppose that B_k intersects some generator $\Lambda \in \{L_i, D_i\}$ nontrivially. Since Λ can be chosen to be disjoint from S_1', we employ the same methods as above. We take an arc α_1 from Λ to $\pm S_1'$ that doesn't intersect B_k or the other generators and we consider the surface $\Lambda_1 = \Lambda \cup N(\alpha) \cup \pm S_1'^{+}$. Then, $\Lambda_1 \cdot B_k = \Lambda \cdot B_k \pm 1$.

By repeating this process enough times, we come up with a set of generators $\{S_1', B_k, \Lambda_1, \ldots, \Lambda_{2g}\}$ of $H_2(V, \partial V_-)$ with the intersection matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

□

These surfaces generate $H_2(V, \partial V_-)$ and they have the appropriate algebraic intersections. We can modify these surfaces by tubing to get a set of generators with the desired geometric intersection.

Finally, we note that $\hat{V}$, the closure of V, and $\hat{W}$, the closure of W are the same manifold by construction. Since $\hat{W}$ is spin, then $\hat{V}$ is spin, and so V is a 0-solve equivalence between J and J'.

Chapter 5

Classification of Links up to 0-Solve Equivalence

5.1 Statement of Classification Theorem

In this chapter, we give a classification of links up to 0-solve equivalence. From this, we give necessary and sufficient conditions for a link to be 0-solvable. These results are consequences of the following theorem, which relates the condition of 0-solve equivalence to band-pass equivalence and to the Arf and Milnor's invariants.

Theorem 5.1. *For two ordered, oriented m-component links $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$, the following conditions are equivalent:*

1. *L and L' are 0-solve equivalent,*
2. *L and L' are band-pass equivalent,*
3. *$Arf(K_i) = Arf(K'_i)$*

$\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$

$\bar{\mu}_L(iijj) \equiv \bar{\mu}_{L'}(iijj) \pmod{2}$.

As an immediate corollary to this theorem, if L' is the m-component unlink, we have the following result.

Corollary 5.2. *For an ordered, oriented m-component link $L = K_1 \cup \ldots \cup K_m$, the following conditions are equivalent:*

1. *L is 0-solvable,*
2. *L is band-pass equivalent to the m-component unlink,*
3. *$Arf(K_i) = 0$*

   $\bar{\mu}_L(ijk) = 0$

   $\bar{\mu}_L(iijj) \equiv 0 \pmod{2}$.

We will prove Theorem 5.1 in the following sections by first showing (2) $\Rightarrow$ (1), then showing that (1) $\Rightarrow$ (3), and finally, that (3) $\Rightarrow$ (2).

5.2 Proof of Theorem 5.1, Step 1

In this section, we will prove the first step of Theorem 5.1 by showing the following lemma.

Lemma 5.3. *If two ordered, oriented m-component links $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$ are band-pass equivalent, then L and L' are 0-solve equivalent.*

Proof. Because band-pass equivalence and 0-solve equivalence are both equivalence relations and thus transitive, we may assume that L and L' differ by a single band-pass move. Recall that we require the strands of each band to belong to the same link component.

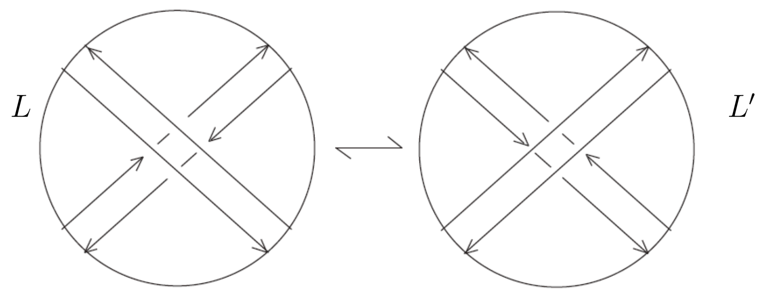


Figure 5.1: L and L' differ by a band-pass move.

We first consider the 4-manifold $M_L \times [0,1]$. Then, we attach two 0-framed 2-handles to $M_L \times \{1\}$ in the boundary of $M_L \times [0,1]$ along the attaching curves γ_i and γ_j pictured below.

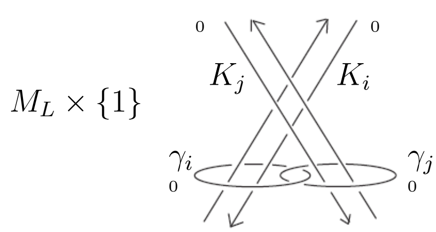


Figure 5.2: Attaching curves in $M_L \times [0,1]$

In the 3-manifold $M_L \times [0,1]$, we slide both strands of link component K_i over the 2-handle attached to γ_j and we slide both strands of link component K_j over the 2-handle attached to γ_i. Note that, in figure 5.3, the strands and attaching curves pictured each have a zero surgery coefficient; we perform the handle-slides in the closed manifold M_L.

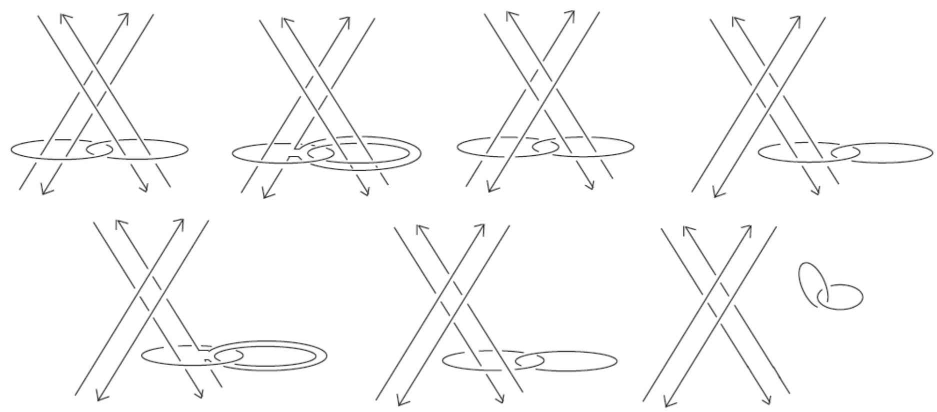


Figure 5.3: Performing a Band-Pass via Handleslides

The resulting 3-manifold, after sliding all four strands, is $M_{L'}$. Therefore, we consider the 4-manifold $W = M_L \times [0,1] \cup \{\text{two 2-handles}\}$, and we see that W is a cobordism between M_L and $-M_{L'}$. We wish to show that W is a 0-solve equivalence.

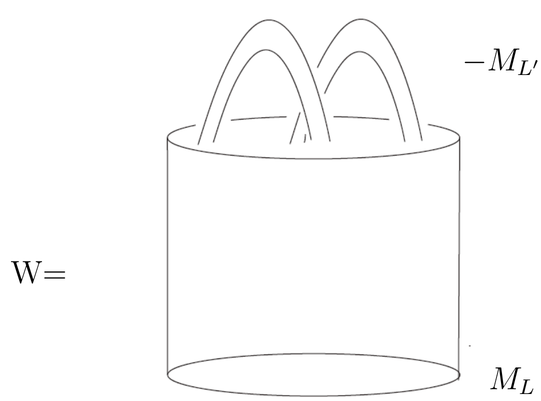


Figure 5.4: A cobordism between M_L and $-M_{L'}$

As the attaching curves γ_i and γ_j are null-homologous, attaching the 2-handles has no effect on H_1. Thus, we see that $i_* : H_1(M_{L'}) \to H_1(W)$ and $j_* : H_1(M_L) \to H_1(W)$

are isomorphisms.

Comparing the second homology of W to that of $M_L \times [0,1]$, we see that $H_2(W) \cong H_2(M_L \times [0,1]) \oplus \mathbb{Z} \oplus \mathbb{Z} \cong H_2(M_L) \oplus \mathbb{Z} \oplus \mathbb{Z}$ with the added homology generated by $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$, where $\hat{\Lambda}_i$ is the surface Λ_i pictured below capped off with a 2-cell from the attached 2-handles.

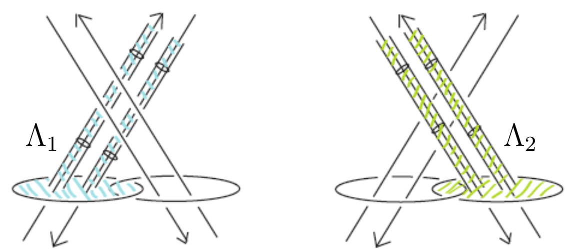


Figure 5.5: New Generators for $H_2(W)$

By pushing the interior of Λ_1 very slightly into $int(M_L \times [0,1]) \subset W$, we can assure that $\hat{\Lambda}_1 \cap \hat{\Lambda}_2 = \{p\}$, where p is a point which lies on one attaching curve. We choose orientations on $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$ to ensure that $\hat{\Lambda}_1 \cdot \hat{\Lambda}_2 = +1$. We can also assure that $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$ are disjoint from the generators of $H_2(M_L \times [0,1])$. Therefore, $H_2(W, \partial W_-) \cong \frac{H_2(W)}{H_2(\partial W_-)}$ is generated by $\{\hat{\Lambda}_1, \hat{\Lambda}_2\}$, which satisfies the second condition of being a 0-solve equivalence.

Finally, we must show that the closure $\hat{W}$ is a spin manifold. We will show that the intersection form $Q_{\hat{W}}$ is even. The generators of $H_2(\hat{W})$ are $\{\hat{\Sigma}_1, \ldots, \hat{\Sigma}_m, F_1, \ldots, F_m, \hat{\Lambda}_1, \hat{\Lambda}_2\}$, where $\hat{\Sigma}_i$ is an oriented Seifert surface for K_i, closed off with a disk in $M_L \times \{0\}$, F_i is the annulus $\mu_i \times [0,1]$ closed off with a disk from the process of closing W, and $\hat{\Lambda}_i$ are the surfaces mentioned above. We can assure that the $\hat{\Lambda}_i$ are disjoint from the F_i, and so the intersection matrix for $Q_{\hat{W}} = \bigoplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

These conditions show that W is a 0-solve equivalence between L and L'.

□

5.3 Proof of Theorem 5.1, Step 2

In this section, we will prove the second step of Theorem 5.1 using the following three lemmas.

Lemma 5.4. *Suppose that $L = K_1 \cup \ \ldots \ \cup K_m$ and $L' = K'_1 \cup \ \ldots \ \cup K'_m$ are two ordered, oriented, m-component, 0-solve equivalent links. Then, $Arf(K_i) = Arf(K'_i)$.*

Proof. By Proposition 4.6, K_1 and K'_i are 0-solve equivalent knots. By Proposition 4.5, either both K_i and K'_i are 0-solvable knots, or neither K_i nor K'_i are 0-solvable knots. If both K_i and K_i are 0-solvable knots, we know that $Arf(K_i) = Arf(K'_i) = 0$. If neither K_i nor K'_i are 0-solvable knots, we know that $Arf(K_i) = Arf(K'_i) = 1$. Either way, we must have that $Arf(K_i) = Arf(K'_i)$. □

Lemma 5.5. *Suppose that $L = K_1 \cup \ \ldots \ \cup K_m$ and $L' = K'_1 \cup \ \ldots \ \cup K'_m$ are two ordered, oriented, m-component, 0-solve equivalent links. Then, $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$.*

Proof. Consider the sublinks $J = K_i \cup K_j \cup K_k$ and $J' = K'_i \cup K'_j \cup K'_k$ of L and L', respectively. By Proposition 4.6, J and J' are 0-solve equivalent links; we wish to show that $\bar{\mu}_J(123) = \bar{\mu}_{J'}(123)$.

Let W be a 4-manifold realizing the 0-solve equivalence between J and J' From the 0-solve equivalence, we are given that $i_* : H_1(M_J) \to H_1(W)$ and $j_* : H_1(-M_{J'} \to H_1(W)$ are isomorphisms.

We choose oriented Seifert surfaces $\Sigma_1, \Sigma_2, \Sigma_3$ for components K_1, K_2, K_3 in $S^3 - J$ such that the pairwise intersections $\Sigma_i \cap \Sigma_j \cong S^1$. Similarly, we choose oriented Seifert surfaces $\Sigma_1', \Sigma_2', \Sigma_3'$ for link components K_1', K_2', K_3' in $S^3 - M_{-J'}$ with the same conditions.

In M_J and $M_{J'}$, each Seifert surface Σ_i and Σ_i' is capped off with a disk. Considering each disk with our chosen Seifert surface, we have established closed surfaces $\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3 \subset M_J$ and $\hat{\Sigma}_1', \hat{\Sigma}_2', \hat{\Sigma}_3' \subset M_{J'}$. Within each 3-manifold, the surfaces intersect pairwise in a circle. The intersections $\hat{\Sigma}_1 \cap \hat{\Sigma}_2 \cap \hat{\Sigma}_3 \subset M_J$ and $\hat{\Sigma}_1' \cap \hat{\Sigma}_2' \cap \hat{\Sigma}_3' \subset M_{J'}$ are each a collection of triple points.

Using the Thom-Pontryagin construction, we define maps $f_1 : M_J \to S^1$, $f_2 : M_J \to S^1$, and $f_3 : M_J \to S^1$ using the following construction. We take a product neighborbood $\hat{\Sigma}_i \times [-1, 1]$, where the $+1$ corresponds to the positive side of $\hat{\Sigma}_i$. We define the map $f_i : \hat{\Sigma}_i \times [-1, 1] \to S^1$ by $(x, t) \mapsto e^{2\pi i t}$, and for $y \in M_j - (\hat{\Sigma}_i \times [-1, 1])$, we define $f_i(y) = -1$. We similarly define maps $f_1' : -M_{J'} \to S^1$, $f_2' : -M_{J'} \to S^1$, and $f_3' : -M_{J'} \to S^1$. Then, we consider the maps $f = f_1 \times f_2 \times f_3 : M_J \to S^1 \times S^1 \times S^1$ and $f' = f_1' \times f_2' \times f_3' : M_{J'} \to S^1 \times S^1 \times S^1$. The maps π_1, π_2, and π_3 are the standard projection maps, so that $f_i = \pi_i \circ f$ and $f_i' = \pi_i \circ f'$.

The pre-image $f^{-1}((1, 1, 1)) = \hat{\Sigma_1} \cap \hat{\Sigma_2} \cap \hat{\Sigma_3}$ is a collection of isolated points $\{p_i\}_{i=1}^{k}$ in M_J; the pre-image $f'^{-1}((1, 1, 1)) = \hat{\Sigma_1}' \cap \hat{\Sigma_2}' \cap \hat{\Sigma_3}'$ is a collection of isolated points $\{p_i'\}_{i=1}^{k'}$ in $M_{J'}$. We also have a framing on the surfaces $\hat{\Sigma}_i$ and $\hat{\Sigma}_i'$ given by the normal

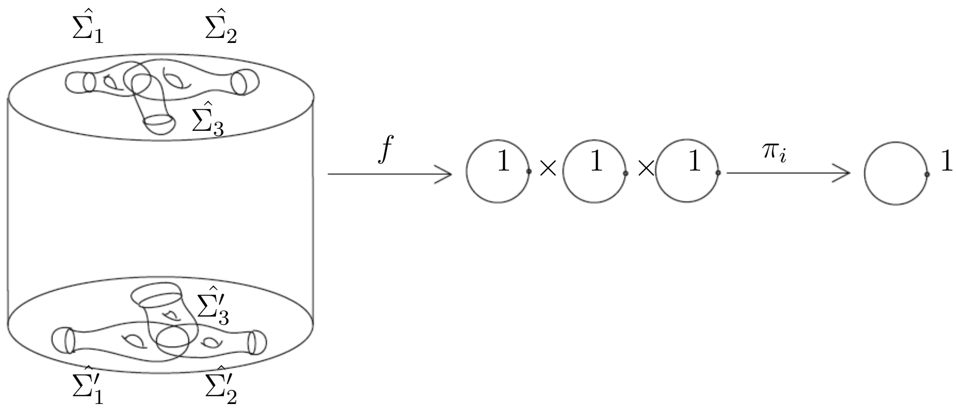
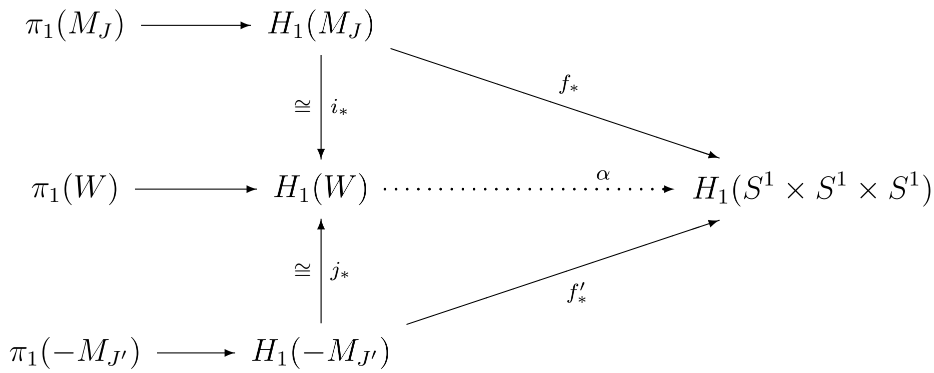


Figure 5.6: The Thom-Pontryagin Construction on M_J and $-M_{J'}$

direction to the tangent plane $T_x\hat{\Sigma}_i$ at each point $x \in \hat{\Sigma}_i$. Then, for each point $p_i \in f^{-1}((1,1,1))$ and for each point $p'_i \in f'^{-1}((1,1,1))$, we assign a sign of $+1$ if the orientation of p_i (respectively, p'_i) induced by the framings on the surfaces agrees with the orientation on M_J (respectively, $-M_{J'}$), and a sign of -1 otherwise. Then, the Milnor's invariant $\bar{\mu}_J(123) = \Sigma_{i=1}^{k}(-1)^{\epsilon_i}$, where $\epsilon_i = \pm 1$ is the sign of the point p_i. The Milnor's invariant $\bar{\mu}_{J'}(123) = \Sigma_{i=1}^{k}(-1)^{\epsilon'_i}$, where $\epsilon'_i = \pm 1$ is the sign of the point p'_i. We wish to show that these two quantities are equal.

The maps $f : M_J \to S^1 \times S^1 \times S^1$ and $f' : -M_{J'} \to S^1 \times S^1 \times S^1$ induce maps on first homology, $f_* : H_1(M_J) \to H_1(S^1 \times S^1 \times S^1)$ and $f'_* : H_1(-M_{J'}) \to H_1(S^1 \times S^1 \times S^1)$ that are isomorphisms. As W is a 0-solve equivalence, the inclusion maps i_* and j_* are also isomorphisms.



We wish to extend to a map $\alpha : H_1(W) \to H_1(S^1 \times S^1 \times S^1)$; this is possible because $f_* \circ i_*^{-1} = f'_* \circ j_*^{-1}$, by the Thom-Pontryagin construction and the fact that $i_*(\mu_i) = j_*(\mu_i)$. Therefore, we can extend to the map $\alpha : H_1(W) \to H_1(S^1 \times S^1 \times S^1)$, and because $\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3$ is abelian, we can also extend to the map $\bar{\alpha} : \pi_1(W) \to \pi_1(S^1 \times S^1 \times S^1)$. Then, knowing that the CW-complex $S^1 \times S^1 \times S^1$ is an Eilenberg-Maclane space $K(\mathbb{Z}^3, 1)$, we can extend the maps f, f' to a map $\bar{f} : W \to S^1 \times S^1 \times S^1$.

$\bar{f}^{-1}((1,1,1))$

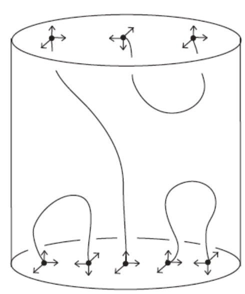


Figure 5.7: Framed Cobordism between $f^{-1}((1,1,1))$ and $f'^{-1}((1,1,1))$

The 1-manifold $\bar{f}^{-1}((1,1,1))$ in W is a framed cobordism between $f^{-1}((1,1,1))$ and $f'^{-1}((1,1,1))$. This tells us that $\bar{\mu}_J(123) = \Sigma_{i=1}^{k}(-1)^{\epsilon_i}$ and $\bar{\mu}_{J'}(123) = \Sigma_{i=1}^{k}(-1)^{\epsilon'_i}$ must be equal.

□

Lemma 5.6. *Suppose that $L = K_1 \cup \ \ldots \ \cup K_m$ and $L' = K_1' \cup \ \ldots \ \cup K_m'$ are two ordered, oriented, m-component, 0-solve equivalent links. Then, $\bar{\mu}_L(iijj) \equiv \bar{\mu}_{L'}(iijj)$ (mod2).*

Proof. Consider the sublinks $J = K_i \cup K_j$ and $J' = K_i' \cup K_j'$ of L and L', respectively. By Proposition 4.6, J and J' are 0-solve equivalent links; we wish to show that $\bar{\mu}_J(1122) \equiv \bar{\mu}_{J'}(1122)$ mod 2.

Let W be a 4-manifold realizing the 0-solve equivalence between J and J'. We know that $i_* : H_1(M_J) \xrightarrow{\cong} H_1(W) \cong \mathbb{Z}^2$ and $j_* : H_1(M_{J'}) \xrightarrow{\cong} H_1(W) \cong \mathbb{Z}^2$ are both isomorphisms. We can choose oriented Seifert surfaces Σ_1, Σ_2 for J_1 and J_2 in $S^3 - J$ such that $\Sigma_1 \cap \Sigma_2 = A \cong S^1$ [4]. Similarly, we choose oriented Seifert surfaces Σ_1', Σ_2' for J_1' and J_2' in $S^3 - J'$ such that $\Sigma_1' \cap \Sigma_2' = A' \cong S^1$.

In performing zero-surgery on the links J and J', we cap off the longitude of each link component with a disk. Considering each disk with our chosen Seifert surface, we have established closed surfaces $\hat{\Sigma}_1, \hat{\Sigma}_2 \subset M_J$ and $\hat{\Sigma}_1', \hat{\Sigma}_2' \subset M_{J'}$. By capping off the Seifert surfaces, we have not changed their intersections.

Using the Thom-Pontryagin construction, we define maps $f_1 : M_J \to S^1$ and $f_2 : M_J \to S^1$. We take a product neighborhood $\hat{\Sigma}_i \times [-1, 1]$, where $+1$ corresponds to the positive side of $\hat{\Sigma}_i$. We define $f_i : \hat{\Sigma}_i \times [-1, 1] \to S^1$ by $(x, t) \mapsto e^{2\pi i t}$, and for $y \in M_J - (\hat{\Sigma}_i \times [-1, 1])$, we define $f_i(y) = -1$. We similarly define maps $f_1' : -M_{J'} \to S^1$ and $f_2' : M_{J'} \to S^1$. We then consider the maps $f = f_1 \times f_2 : M_J \to S^1 \times S^1$ and $f' = f_1' \times f_2' : M_{J'} \to S^1 \times S^1$. The maps $\pi_1, \pi_2 : S^1 \times S^1 \to S^1$ are the standard projection maps, such that $\pi_i \circ f = f_i$.

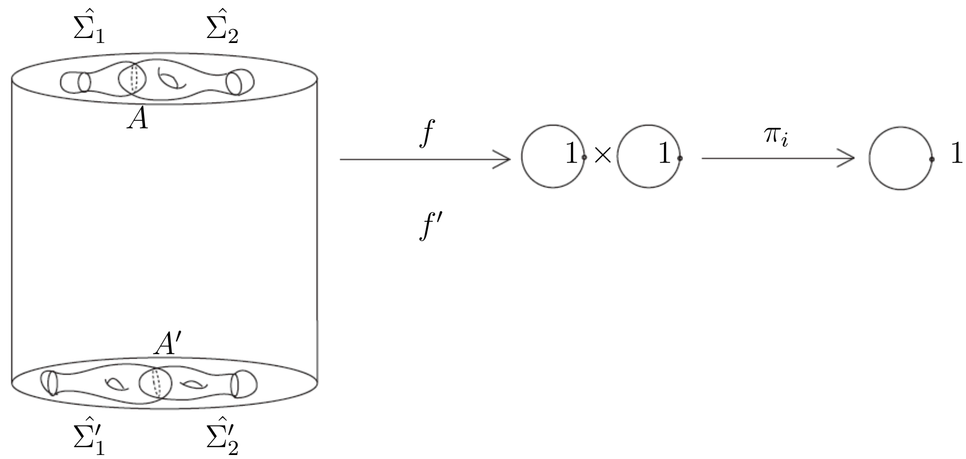
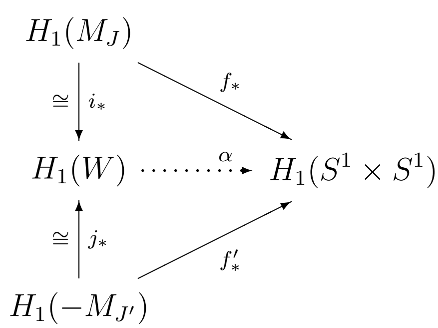


Figure 5.8: The Thom-Pontryagin Construction on M_J and $-M_{J'}$

The maps $f : M_J \to S^1 \times S^1$ and $f' : -M_{J'} \to S^1 \times S^1$ induce maps on first homology, $f_* : H_1(M_J) \to H_1(S^1 \times S^1)$ and $f'_* : H_1(-M_{J'}) \to H_1(S^1 \times S^1)$ that are isomorphisms. As W is a 0-solve equivalence, the inclusion maps i_* and j_* are also isomorphisms.



We wish to extend to a map $\alpha : H_1(W) \to H_1(S^1 \times S^1)$; this is possible because $f_* \circ i_*^{-1} = f'_* \circ j_*^{-1}$, by the Thom-Pontryagin construction and the fact that $i_*(\mu_i) = j_*(\mu_i)$. Therefore, we can extend to the map $\alpha : H_1(W) \to H_1(S^1 \times S^1)$, and because $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ is abelian, we can also extend to the map $\bar{\alpha} : \pi_1(W) \to \pi_1(S^1 \times S^1)$. Then, knowing that the CW-complex $S^1 \times S^1$ is an Eilenberg-Maclane space $K(\mathbb{Z}^2, 1)$, we can extend the maps f, f' to a map $\bar{f} : W \to S^1 \times S^1$.

The pre-image $\bar{f}^{-1}((1,1))$ is a surface in W with boundary $A \bigsqcup -A'$ in ∂W. In $S^1 \times S^1$, we consider D_p, an ϵ- neighborhood of the point $p = (1,1)$. Let $q = (e^{2\pi i \epsilon}, 1)$ be a point on the boundary of D_p. Then, $\bar{f}^{-1}(q)$ is a surface in W with boundary $B \bigsqcup -B'$, where B is just the curve A pushed off in the positive direction of $\hat{\Sigma}_1$ and B' is the curve A' pushed off in the positive direction of $\hat{\Sigma}'_1$. Furthermore, these surfaces $\bar{f}^{-1}(p)$ and $\bar{f}^{-1}(q)$ are disjoint in W, and if we consider a path $\gamma(t) = (e^{2\pi i \epsilon t}; 1), t \in [0,1]$, the pre-image $\bar{f}^{-1}(\gamma(t))$ is a 3-submanifold of W that is cobounded by the surfaces $\bar{f}^{-1}(p)$ and $\bar{f}^{-1}(q)$.

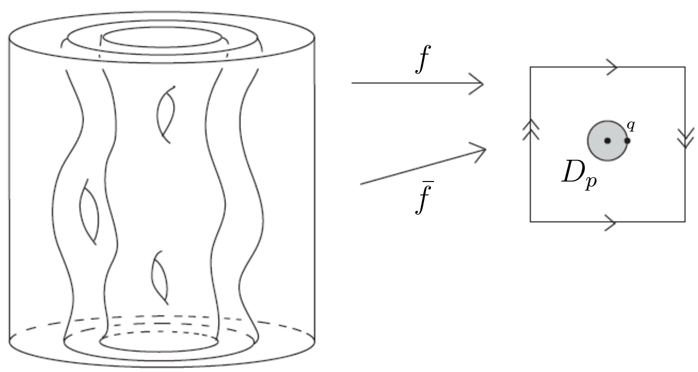


Figure 5.9: Surface pre-images under the map $\bar{f}$

The Sato-Levine invariants of J and J', as defined in chapter 3, are the quantities $\bar{\mu}_J(1122) = lk(A, B)$ and $\bar{\mu}_{J'}(1122) = lk(A', B')$. Therefore, we wish to show that $lk(A, B) \equiv lk(A', B')$ (mod2).

To show this, we construct the closure $\hat{W}$ from W by first attaching 0-framed 2-handles to the meridians μ_1 and μ_2 of link J and to meridians μ'_1 and μ'_2 of link J' and then attaching a 4-handle $\mathbb{B}^4$ to the S^3 boundary component and a 0-handle $\mathbb{B}^4$ to the $-S^3$ boundary component.

The curves A and B, when thought of in the boundary of the 4-manifold $W \cup_{\mu_1} (\mathbb{D}^2 \times \mathbb{D}^2) \cup_{\mu_2} (\mathbb{D}^2 \times \mathbb{D}^2)$, each bound surfaces in the 4-handle $\mathbb{B}^4$. We call these surfaces S and T. Similarly, the curves A' and B', when thought of in the boundary of the 4-manifold $W \cup_{\mu_1'} (\mathbb{D}^2 \times \mathbb{D}^2) \cup_{\mu_2'} (\mathbb{D}^2 \times \mathbb{D}^2)$, each bound surfaces in the 0-handle $\mathbb{B}^4$. We call these surfaces S' and T'. Noting that $\bar{\mu}_J(1122) = lk(A, B) = S \cdot T$ and $\bar{\mu}_{J'}(1122) = lk(A', B') = S' \cdot T'$, we observe that we wish to show that $S \cdot T + S' \cdot T' \equiv 0 \pmod 2$. We may also form closed surfaces in $\hat{W}$ by letting $\hat{S} = S \cup_A \bar{f}^{-1}(p) \cup_{A'} S'$ and $\hat{T} = T \cup_B \bar{f}^{-1}(q) \cup_{B'} T'$. Because $\bar{f}^{-1}(p) \cap \bar{f}^{-1}(q) = \emptyset$, the problem reduces to showing that $\hat{S} \cdot \hat{T} \equiv 0 \pmod 2$.

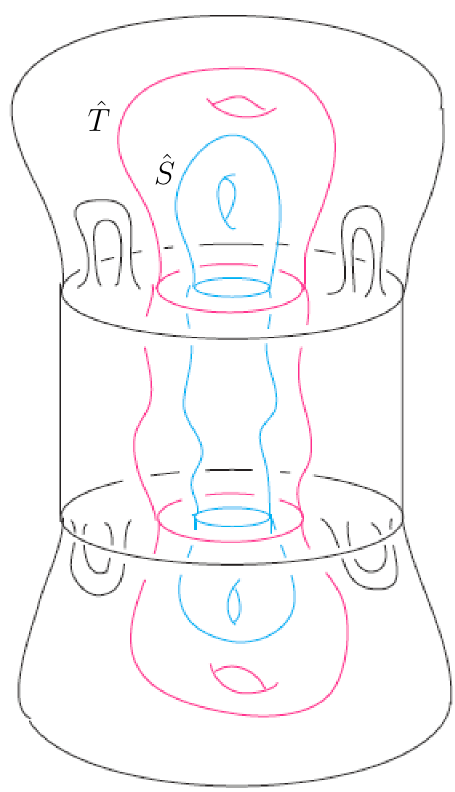


Figure 5.10: Surfaces $\hat{S}$ and $\hat{T}$ in $\hat{W}$

We observe that the surface $S \cup_A f^{-1}(\gamma(t)) \cup_B T \subset \mathbb{B}^4$ is a closed surface, and therefore is the boundary of a 3-chain. Similarly, the surface $S' \cup_{A'} f'^{-1}(\gamma(t)) \cup_{B'} T' \subset \mathbb{B}^4$ is a closed surface, and therefore is the boundary of a 3-chain. Piecing together these 3-chains, we see that $\hat{S}$ and $\hat{T}$ cobound a 3-chain, and so the homology classes $[\hat{S}] \in H_2(\hat{W}) = [\hat{T}] \in H_2(\hat{W})$ are equal. Finally, we recall that, as $\hat{W}$ is the closure of a 0-solve equivalence W, we require that $\hat{W}$ be spin, or equivalently, that the intersection form $Q_{\hat{W}}$ be even. Thus, we have that $\hat{S} \cdot \hat{T} = Q_{\hat{W}}([\hat{S}], [\hat{T}]) = Q_{\hat{W}}([\hat{S}], [\hat{S}])$ is even. This tells us that $\bar{\mu}_J(1122) \cong \bar{\mu}_{J'}(1122) \pmod 2$. □

5.4 Proof of Theorem 5.1, Step 3

In this section, we conclude the proof of Theorem 5.1 by showing the following lemma.

Lemma 5.7. *If $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$ are two ordered, oriented, m-component links such that*

1. $Arf(K_i) = Arf(K'_i)$,
2. $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$,
3. $\bar{\mu}_L(iijj) \equiv \bar{\mu}_{L'}(iijj) (\text{mod} 2)$,

then L and L' are band-pass equivalent.

Proof. We will use Taniyama and Yasuhara's result and proof of the following theorem [21].

Theorem 5.8. [Taniyama-Yasuhara] [21] *Let $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$ be ordered, oriented, m-component links. The following conditions are equivalent:*

1. L and L' are clasp-pass equivalent links,

2. $a_2(K_i) = a_2(K_i')$

$a_3(K_i \cup K_j) \equiv a_3(K_i' \cup K_j')$ (mod2)

and $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$,

3. $a_2(K_i) = a_2(K_i')$

$\mathrm{Arf}(K_i \cup K_j) = \mathrm{Arf}(K_i' \cup K_j')$

and $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$.

Here, a_j indicates the j^{th} coefficient of the Conway polynomial, as defined in [22]. Recall that a *clasp-pass move* is defined by the skein relation in the following figure. In particular, we note that a clasp-pass move is also a band-pass move, though the converse is not true.

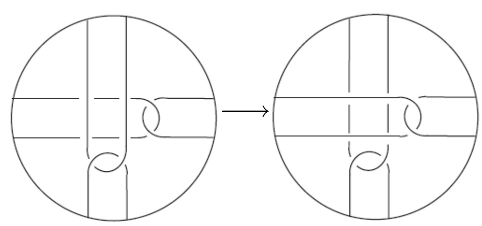


Figure 5.11: Clasp-pass move

Taniyama and Yasuhara prove the direction (2) $\Rightarrow$ (1). In the following proposition, we show that the three assumptions in Lemma 5.7, $Arf(K_i) = Arf(K_i')$, $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$, and $\bar{\mu}_L(iijj) \equiv \bar{\mu}_{L'}(iijj)$(mod2), are very similar to Taniyama and Yasuhara's assumptions that $a_2(K_i) = a_2(K_i')$, $a_3(K_i \cup K_j) \equiv a_3(K_i' \cup K_j')$ (mod2), and $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$.

Proposition 5.9. *Let $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$ be ordered, oriented, m-component links. The following conditions are equivalent:*

1. $a_2(K_i) \equiv a_2(K'_i) \pmod 2$

   $a_3(K_i \cup K_j) \equiv a_3(K'_i \cup K'_j) \pmod 2$

   $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$,

2. $Arf(K_i) = Arf(K'_i)$,

   $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$,

   $\bar{\mu}_L(iijj) \equiv \bar{\mu}_{L'}(iijj) \pmod 2$.

Proof. Again, here a_j refers to the j^{th} coefficient of the Conway polynomial of a link, $\Delta_L(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$, as defined in [22]. Thus, a_j refers to the coefficient of the term z^j in the Conway polynomial.

In [14], Kauffman shows that knots K_i and K'_i are band-pass equivalent $\Leftrightarrow a_2(K_i) \equiv a_2(K'_i) \pmod 2 \Leftrightarrow Arf(K_i) = Arf(K'_i)$.

In [3], Cochran writes the general Conway polynomial of a link L as $\Delta_L(z) = z^{m-1}(b_0 + b_2 z^2 + \cdots + b_{2n} z^{2n})$ where m is the number of components of L. The condition on links L and L' that $a_3(K_i \cup K_j) \equiv a_3(K'_i \cup K'_j) \pmod 2$ says that, for every 2-component sublink $J = K_i \cup K_j$ of L and $J' = K'_i \cup K'_j$ of L', the coefficient on the cubic term of $\Delta_J(z)$ is equivalent $\pmod 2$ to the cubic term of $\Delta_{J'}(z)$. Using Cochran's notation for a 2-component link, $\Delta_J(z) = z(b_0 + b_2 z^2 + \cdots + b_{2n} z^{2n})$. Therefore, the cubic term of the Conway polynomial has coefficient b_2, so $a_3(J) = b_2(J)$ and $a_3(J') = b_2(J')$. Then Corollary 4.2 in [3] gives that, for a 2-component link J, the coefficient $b_0(J) = -\bar{\mu}_J(12)$, and if $b_0(J) = 0$, then $b_2(J) = \bar{\mu}_J(1122)$.

Since we are assuming that all pairwise linking numbers of L and L' vanish, we have that $\bar{\mu}_J(1122) = b_2(J) = a_3(J)$ and $\bar{\mu}_{J'}(1122) = b_2(J') = a_3(J')$. Thus, the condition that $a_3(K_i \cup K_j) \equiv a_3(K'_i \cup K'_j) \pmod 2$ is equivalent to the condition that $\bar{\mu}_L(iijj) \equiv \bar{\mu}_{L'}(iijj)(\mathrm{mod} 2)$. □

To complete the proof of Lemma 5.7, we adapt Taniyama and Yasuhara's proof to show the following proposition.

Proposition 5.10. *For two ordered, oriented, m-component links $L = K_1 \cup \ldots \cup K_m$ and $L' = K'_1 \cup \ldots \cup K'_m$, if*

1. $a_2(K_i) \equiv a_2(K'_i) \bmod 2$,
2. $a_3(K_i \cup K_j) \equiv a_3(K'_i \cup K'_j) \bmod 2$, *and*
3. $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$,

then L and L' are band-pass equivalent.

Proof. Given that all pairwise linking numbers of L and L' vanish, Taniyama and Yasuhara show that L and L' are both delta-equivalent to the m-component unlink U^m, and therefore, that L and L' can both be obtained from U^m by inserting a sequence of Borromean rings, as pictured in the following skein relation.

We note that the feet of the Borromean rings insertion can attach to any part of the link. In this way, a Borromean rings insertion may involve one, two, or three different link components.

Definition 5.11. [Taniyama-Yasuhara]

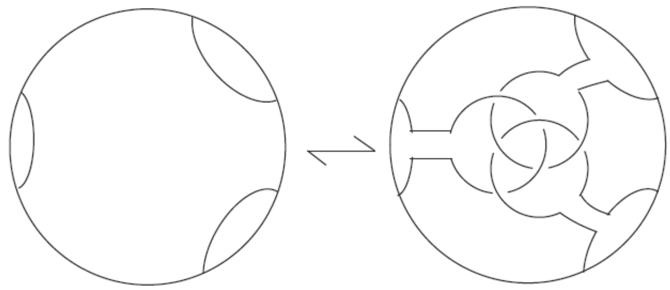
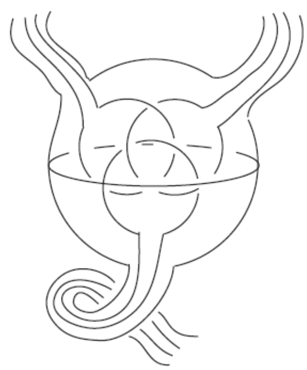


Figure 5.12: A Borromean Rings Insertion

A *Borromean chord* C is a neighborhood of a 3-ball containing a Borromean insertion and a neighborhood of its attaching bands.



We say that a chord C is of *type (i)* if each of the bands in the Borromean ring insertion attach to the i^{th} link component. C is of *type (ij)* if the bands attach to the i^{th} and j^{th} components, and C is of *type (ijk)* if the bands attach to the i^{th}, j^{th}, and k^{th} link components.

In [21] Lemma 2.5, Taniyama and Yasuhara show that any ordered, oriented, m-component link L is clasp-pass equivalent to an ordered, oriented, m-component link J, where J is formed from U^m by Borromean ring insertion, where J is of the following form.

1. Each Borromean chord of J of type (ijk) is contained in a 3-ball as illustrated in (a) or (b), and for each set of components i, j, k, not both (a) and (b) occur.

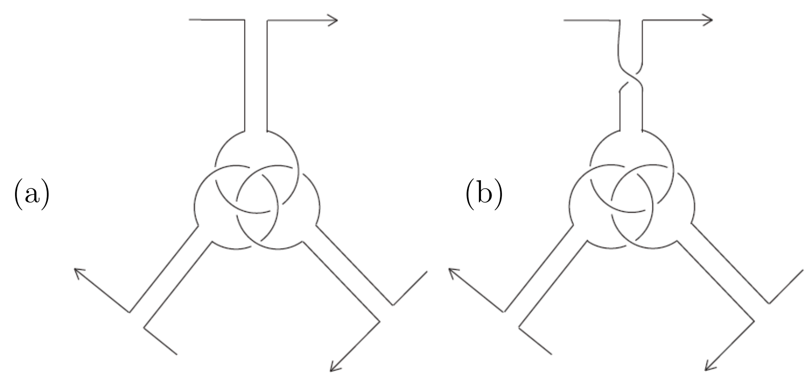


Figure 5.13: Borromean chords of type (ijk).

2. Each Borromean chord of J of type (ij) is contained in a 3-ball as illustrated in (c).

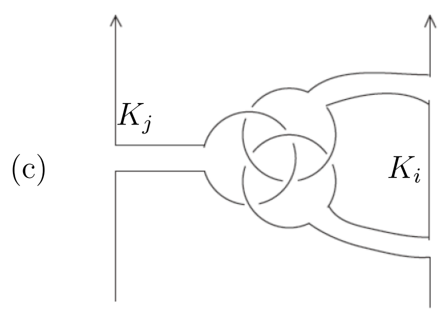


Figure 5.14: Borromean chords of type (ij)

Furthermore, Taniyama and Yasuhara show in Lemma 2.5 that any two Borromean chords of type (ij) cancel each other [21]. Therefore, for each $i < j \leq m$, we may have at most one Borromean chord of type (ij) as in (c).

3. Each Borromean chord of J of type (i) is contained in a 3-ball as illustrated in

(e) or (f), and for each component, not both (e) and (f) occur.

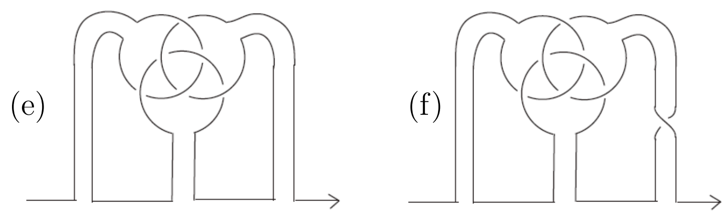


Figure 5.15: Borromean chords of type (i)

Therefore, let link L be clasp-pass equivalent to a link J of the above form, and let link L' be clasp-pass equivalent to a link J' of the above form. Using our assumptions on link L and L', we claim that J and J' are band-pass equivalent.

By calculation, $\bar{\mu}_L(ijk)$ is the signed number of Borromean chords of type (ijk), with sign $+1$ for chords as in figure (a) and sign -1 for chords as in figure (b). Given that $\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$, we know that link J and link J' have identical Borromean chords of type (ijk).

Cited by Taniyama and Yasuhara [21] and due to a result of Hoste [13], for a link $L = K_1 \cup \ldots \cup K_m$, $a_3(K_i \cup K_j) \equiv 0 (\text{mod} 2)$ if and only if there are an even number of Borromean chords of type (ij). Therefore, as we are assuming that $a_3(K_i \cup K_j) \equiv a_3(K'_i \cup K'_j) \pmod 2$, we know that for every choice of i and j, both J and J' have either one or zero Borromean chords of type (ij).

Links L and J are clasp-pass equivalent, and links L' and J' are clasp-pass equivalent. The coefficient a_2 is preserved under clasp-pass equivalence [21]. In figure 5.15, the closure of (e) is a trefoil knot, and the closure of (f) is the figure eight knot. The invariants a_2(trefoil) $= 1$ and a_2(figure eight) $= -1$. As the coefficient a_2 is additive under the connected sum of knots, the number of Borromean chords of type (i) in link J is given by $|a_2(K_i)|$, and the number of Borromean chords of type (i) in link J' is given by $|a_2(K_i')|$.

Since a_2(trefoil) $\equiv a_2$(figure eight) (mod2), (e) and (f) are band-pass equivalent [14]. Therefore, we may assume that all Borromean chords of both J and J' of type (i) are as illustrated in figure 5.15 (e). By enforcing the condition that $a_2(K_i) \equiv a_2(K_i')$ mod 2, we are assuming that links J and J' have the same parity of Borromean chords of type (i), all as in figure 5.15 (e). In Lemma 2.5, Taniyama and Yasuhara show that any two such Borromean chords will cancel (see in particular, [21] Figure 15). Therefore, for each i, we may assume that J and J' both have either one or zero Borromean chords of type (i).

Therefore, we see that links J and J' are band-pass equivalent. We have that $L \sim_{\text{Clasp-Pass}} J \sim_{\text{Band-Pass}} J' \sim_{\text{Clasp-Pass}} L'$. Recalling that a clasp-pass move is also a band-pass move, we conclude that L and L' are band-pass equivalent. □

□

5.5 Classifying Links up to 0-Solve Equivalence

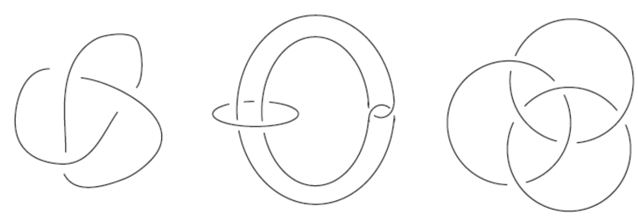
As a direct result of Theorem 5.1, the 0-solve equivalence class of a link $L = K_1 \cup \ldots \cup K_m$ is characterized by the three algebraic invariants, $Arf(K_i), \bar{\mu}_L(ijk)$, and $\bar{\mu}_L(iijj)$.

Corollary 5.12. *For each m,*

$$\frac{\mathcal{F}^m_{-0.5}}{\mathcal{F}^m_0} \cong \mathbb{Z}_2^m \oplus \mathbb{Z}^{\binom{m}{3}} \oplus \mathbb{Z}_2^{\binom{m}{2}}$$

Proof. Given a link $L = K_1 \cup \ \ldots \ \cup K_m$, each component K_i has an Arf invariant, $Arf(K_i) \in \mathbb{Z}_2$. Every 3-component sublink $J = K_i \cup K_j \cup K_k \subset L$ has a Milnor's invariant $\bar{\mu}_J(123) \in \mathbb{Z}$, and every 2-component sublink $J = K_i \cup K_j \subset L$ has a Sato-Levine invariant $\bar{\mu}_J(1122)(\text{mod}2) \in \mathbb{Z}_2$. □

We can use this corollary to choose representatives for each 0-solve equivalence class of m-component links. We note that the trefoil knot has Arf invariant 1. The Whitehead link has $\bar{\mu}_L(1122) = 1$. The Borromean rings have $\bar{\mu}_L(123) = \pm 1$. These links are pictured in the following figure.



$Arf(K) = 1$ $\bar{\mu}_L(1122) = 1$ $\bar{\mu}_L(123) = \pm 1$

Figure 5.16: Trefoil, Whitehead Link, and Borromean Rings

For each element $(a_1, \ldots, a_m, b_1, \ldots, b_{\binom{m}{3}}, c_1, \ldots, c_{\binom{m}{2}}) \in \mathbb{Z}_2^m \oplus \mathbb{Z}^{\binom{m}{3}} \oplus \mathbb{Z}_2^{\binom{m}{2}}$, we choose a link representative $J = J_1 \cup \cdots \cup J_m$ in the following way. The i^{th} component of the representative will be either the unknot or the trefoil knot, according to if a_i is 0 or 1. We order the triples of components (J_i, J_j, J_k) lexicographically. For the i^{th} triple (J_i, J_j, J_k), link components, we insert $|c_i|$ Borromean rings with sign corresponding to the sign of c_i. We also order the pairs of components (J_i, J_j) lexicographically.

For the i^{th} pair (J_i, J_j), components J_i and J_j form an unlink or a Whitehead link according to if c_i is 0 or 1.

For example, when $m = 2$, we have an eight element group, $\mathcal{C}^2/\mathcal{F}_0^2 \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_2$, which are represented by the eight two-component links in the following figure.

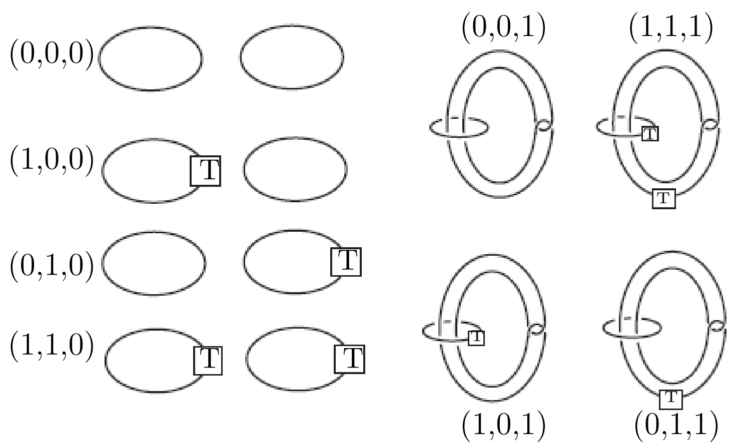


Figure 5.17: 2-component links up to 0-solve equivalence

The following figure shows the 3-component link representing $(0, 0, 0, 1, 1, 1, 1) \in \mathbb{Z}_2^3 \oplus \mathbb{Z} \oplus \mathbb{Z}_2^3$.

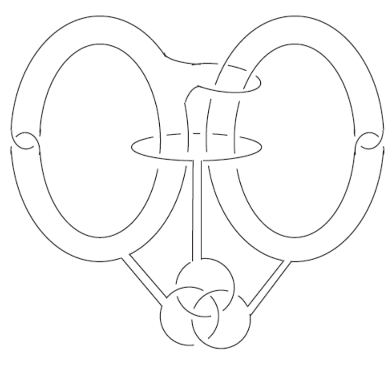


Figure 5.18: Example: A 3-component 0-solve equivalence class

Chapter 6

Gropes and Whitney Towers

Theorem 5.1 has a direct application to the study of *gropes* and *Whitney towers*. Gropes and Whitney towers are geometric objects that play an important role in the study of 4-manifolds. In this chapter, we introduce the notion of gropes and Whitney towers and give results due to Conant, Schneiderman, and Teichner that will expand the statement of Theorem 5.1.

6.1 Introducing Gropes and Whitney Towers

A *grope* is an oriented 2-complex created from joining oriented surfaces together in a prescribed way. Gropes have a natural complexity known as a *class*. We give a definition from [19].

Definition 6.1. A *grope* is a pair (2-complex, S^1) with a *class* $\in \mathbb{N}$. A class 1 grope is defined to be the pair (S^1, S^1). A class 2 grope $(S, \partial S)$ is a compact oriented connected surface S with a single boundary component. For $n > 2$, a class n grope is defined recursively. Let $\{\alpha_i, \beta_i\}_{i=1}^{g}$ be a symplectic basis for a class 2-grope S. For any $a_i, b_i \in \mathbb{N}$ such that $a_1 + b_1 = n$ and $a_i + b_i \geq n$, a class n grope is formed by attaching a class a_i grope to α_i and a class b_i grope to β_i.

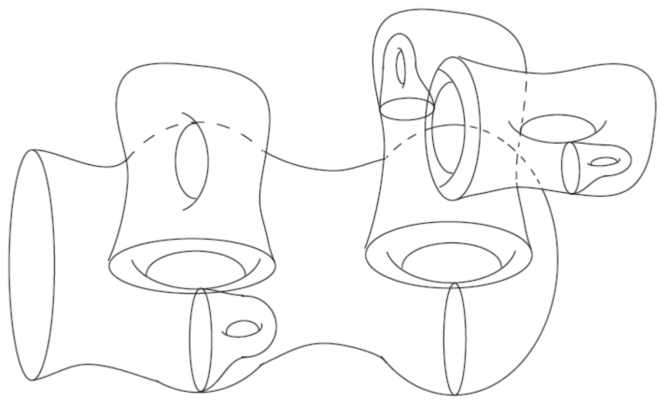


Figure 6.1: A grope of class 4

We will consider gropes in $\mathbb{B}^4$ with the boundary of the grope embedded in the $S^3 = \partial\mathbb{B}^4$. Therefore, the boundary of a grope will be a knot. We may then consider m disjointly embedded gropes, which together will bound an m-component link.

One reason that 4-manifolds are not well understood is that the *Whitney move* to eliminate intersections of immersed surfaces fails in four dimensions. To perform a Whitney move, we find a *Whitney disk* $W_{(I,J)}$ that pairs intersection points of two surface sheets I and J in a 4-manifold. We then change one surface, using $W_{(I,J)}$ as a guide. This is pictured in figure 6.2. In 4-dimensions, we can eliminate the intersection between sheets I and J, but we do so at the cost of possibly creating a new canceling pair of intersections, here between surface sheets I and K. We may then look for another Whitney disk, $W_{(I,K)}$ to eliminate the new intersection. For more detailed definitions and background on this subject, see [9].

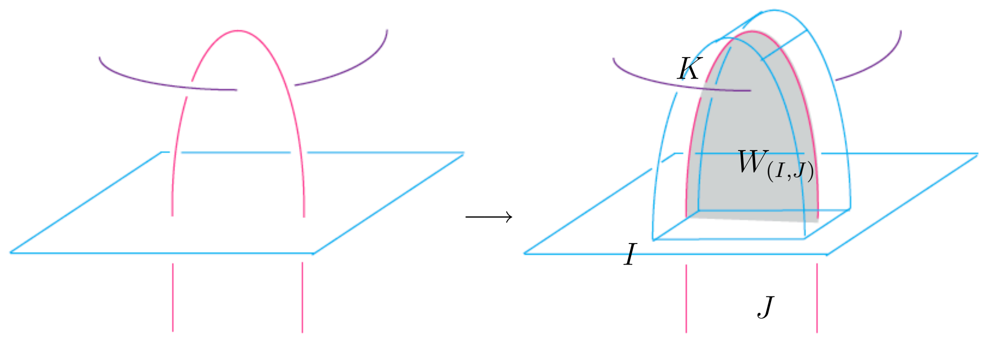


Figure 6.2: A Whitney disk and Whitney move

We give the following definition from [19]. An example of a Whitney tower is given in figure 6.3.

Definition 6.2.

- A *surface of order 0* in a 4-manifold X is a properly immersed surface. A *Whitney tower of order 0* in X is a collection of order 0 surfaces.
- The *order of a transverse intersection point* between a surface of order n and a surface of order m is $n + m$.
- A Whitney disk that pairs intersection points of order n is said to be a *Whitney disk of order* $(n + 1)$.
- For $n \geq 0$, a *Whitney tower of order* $(n + 1)$ is a Whitney tower W of order n together with Whitney disks pairing all order n intersection points of W. The interiors of these top order disks are allowed to intersect each other as well as allowed to intersect lower order surfaces.

We will consider Whitney towers in the 4-manifold $\mathbb{B}^4$. The sheets in our surfaces will all be disks, and we may then consider the boundaries of these disks in $\partial\mathbb{B}^4 = S^3$

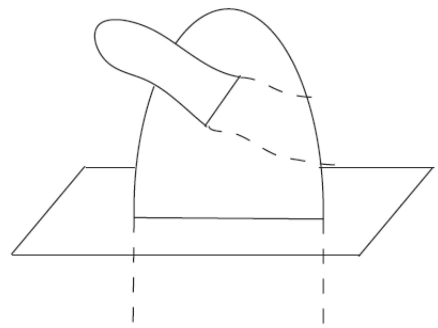


Figure 6.3: A Whitney tower

to be links in S^3.

6.2 Applications to Theorem 5.1

Schneiderman relates curves bounding gropes to curves supporting Whitney towers in the following theorem [19].

Theorem 6.3. [Schneiderman][19]

For any collection of embedded closed curves γ_i in the boundary of $\mathbb{B}^4$, the following are equivalent:

1. *$\{\gamma_i\}$ bound disjoint properly embedded class n gropes g_i in $\mathbb{B}^4$.*

2. *$\{\gamma_i\}$ bound properly immersed 2-disks $\mathbb{D}_i$ admitting an order $(n-1)$ Whitney tower W in $\mathbb{B}^4$.*

The following theorem, due to Conant, Schneiderman, and Teichner, relates Whitney towers to the algebraic link invariants used in Theorem 5.1 [8].

Theorem 6.4. [Conant-Schneiderman-Teichner][8]

A link L bounds a Whitney tower $\mathcal{W}$ of order n if and only if its Milnor invariants, Sato-Levine invariants, and Arf invariants vanish up to order n.

We may then combine these results with Theorem 5.1 to obtain the following result.

Corollary 6.5. *For an ordered, oriented, m-component link L, the following are equivalent.*

1. *L is 0-solvable.*
2. *L bounds disjoint, properly embedded gropes of class 2 in $\mathbb{B}^4$.*
3. *L bounds properly immersed disks admitting an order 2 Whitney tower in $\mathbb{B}^4$.*

Chapter 7

Properties of 0.5-Solvable Links

7.1 Geometric Moves Preserving 0.5-Solvability

In this section, we identify several geometric moves on links that preserve the condition of 0.5-solvability. First, we show the equivalence of three geometric link moves.

Proposition 7.1. *The double-delta move and the double half-clasp pass move are both equivalent to a double Borromean rings insertion.*

Proof. A double Borromean rings insertion is a geometric link move as pictured in figure 7.1. We require that the strands of all bands belong to the same link component.

One double-delta move will undo a double Borromean rings insertion, which we show in figure 7.2.

One double half-clasp pass move will also undo a double Borromean rings insertion, which we show in figure 7.3.

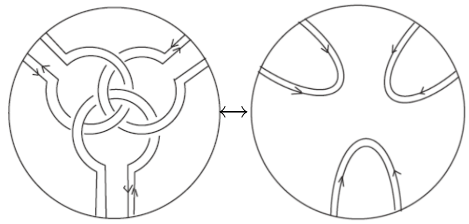


Figure 7.1: A double Borromean rings insertion

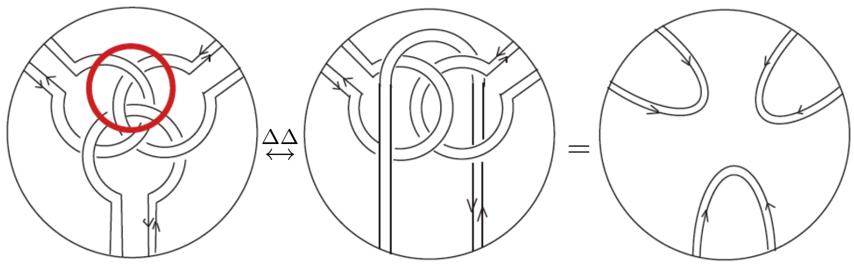


Figure 7.2: The double-delta move is equivalent to a double Borromean rings insertion

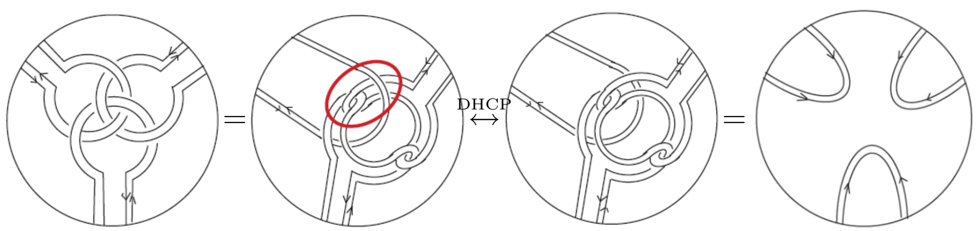


Figure 7.3: The double half-clasp pass move is equivelent to a double Borromean rings insertion

Therefore, the double-delta move, the double half-clasp pass move, and a double Borromean rings insertion are all equivalent moves.

□

Proposition 7.2. *The double half-clasp pass move (and thus, the double-delta move) preserves 0.5-solvability.*

Proof. Suppose that links $L = K_1 \cup \ \ldots \ \cup K_m$ and $L' = K'_1 \cup \ \ldots \ \cup K'_m$ differ by a double half-clasp pass move. Suppose furthermore that L is 0.5-solvable and that W is a 0.5-solution for L. We attach two 0-framed 2-handles $\mathbb{D}^2 \times \mathbb{D}^2$ to W along the attaching curves pictured in Figure 7.4 in $\partial W = M_L$.

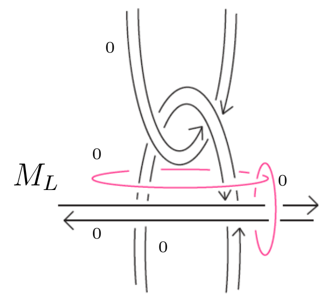


Figure 7.4: Attaching 2-handles to W.

We then perform the double half-clasp pass move by sliding strands of L over the attached handles. Note, that in figure 7.5, each of the strands and attaching curves pictured have an associated zero surgery coefficient; we perform the handle-slides in the closed 3-manifold M_L.

The resulting 4-manifold, $W' = W \cup \{\text{2-handles}\}$ has $\partial W' = M_{L'}$, and we wish to show that W' is a 0.5-solution for L'. First, we note that the attaching spheres of the 2-handles are null homologous in M_L, so attaching the 2-handles does not change first homology. As W is a 0.5-solution for L, the map induced by inclusion, $i_* : H_1(M_L) \to H_1(W)$, is an isomorphism, and so the map $j_* : H_1(M_{L'}) \to H_1(W')$ is also an isomorphism.

From attaching the 2-handles, $H_2(W') \cong H_2(W) \oplus \mathbb{Z} \oplus \mathbb{Z}$, where the extra homology is generated by surfaces $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$, where Λ_i are as pictured in Figure 7.6, and $\hat{\Lambda}_i$ is Λ_i closed off with a disk from the attached handles.

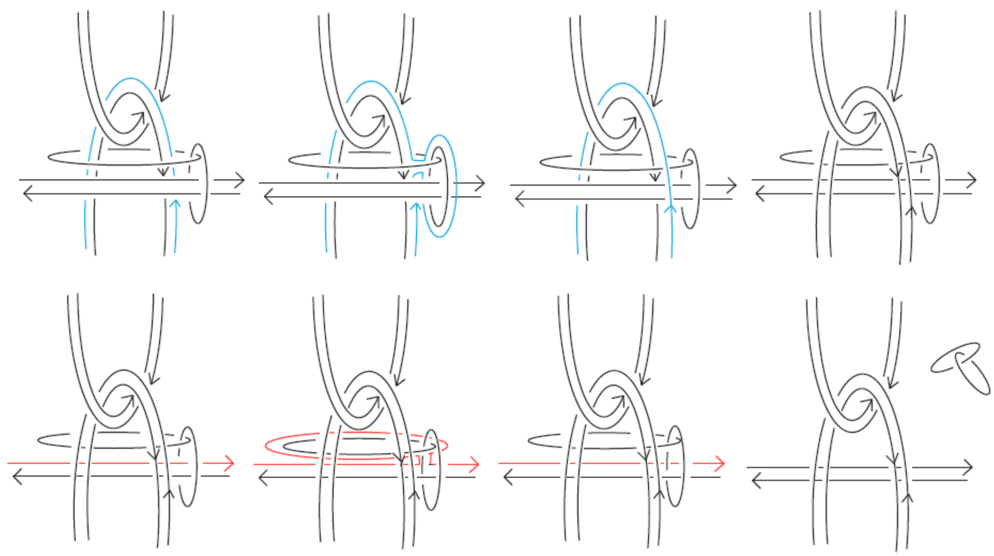


Figure 7.5: Performing a double half-clasp pass move via handleslides

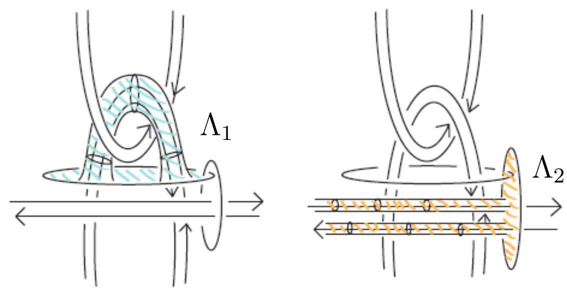


Figure 7.6: New generators for $H_2(W')$

By pushing $int(\Lambda_2)$ slightly into W, we see that $\hat{\Lambda}_1 \cap \hat{\Lambda}_2 = \partial\Lambda_2 \cap \Lambda_1$ is a single point, and we choose orientations on $\hat{\Lambda}_i$ such that $\hat{\Lambda}_1 \cdot \hat{\Lambda}_2 = +1$. This tells us that $H_2(W')$ has the generating set required for W' to be a 0.5-solution for L'. Finally, we note that the generators for $\pi_1(\hat{\Lambda}_1)$ are null homologous in $H_1(W')$, so $\pi_1(\hat{\Lambda}_1) \subset \pi_1(W')^{(1)}$. Therefore, W' is a 0.5-solution for $M_{L'}$.

This gives us the following corollary.

Corollary 7.3. *If an ordered, oriented, m-component link* $L = K_1 \cup \ \ldots \ \cup K_m$ *is* *double-delta equivalent to the m-component unlink, then L is 0.5-solvable.*

□

7.2 Sato-Levine Invariants of 0.5-Solvable Links

From Theorem 5.1, we know that 0-solvable links have even Sato-Levine invariants. In this section, we show that 0.5-solvable links must have vanishing Sato-Levine invariants.

Theorem 7.4. *For an ordered, oriented, m-component, 0.5-solvable link $L = K_1 \cup \ldots \cup K_m$, the Sato-Levine invariants $\bar{\mu}_L(iijj) = 0$.*

Proof. Let $J = K_i \cup K_j$ be a 2-component sublink of L. Then, J is 0.5-solvable. Let W be a 0.5-solution for J and let $\hat{W}$ be the closure of W obtained from attaching 0-framed 2-handles along the meridians μ_1 and μ_2 in $M_J = \partial W$ and then attaching a 4-handle $\mathbb{B}^4$. We choose oriented Seifert surfaces Σ_1 and Σ_2 for K_1 and K_2 such that $\Sigma_1 \cap \Sigma_2 = \gamma \cong S^1$, and we let $\hat{\Sigma}_i$ be the closure of Σ_i in M_J.

Using the Thom-Pontryagin construction, we define a map $f : \partial W \to S^1 \times S^1$. Let $\hat{\Sigma}_i \times [-1, 1]$ be a product neighborhood of $\hat{\Sigma}_i$ where the $+1$ corresponds to the positive side of $\hat{\Sigma}_i$. We define maps $f_1 : M_J \to S^1$ by $f_1(\hat{\Sigma}_1 \times \{t\}) = e^{2\pi i t}$, and for $y \in M_J - (\hat{\Sigma}_1 \times [-1, 1])$, $f_i(y) = -1$. We similarly define the map $f_2 : M_J \to S^1$, and we let $f = f_1 \times f_2 : M_J \to S^1 \times S^1$. The maps π_1 and π_2 are the standard projection maps, so that $f_i = \pi_i \circ f$.

The map $f : M_J \to S^1 \times S^1$ induces an isomorphism on first homology, $f_* :$

$H_1(M_J) \to H_1(S^1 \times S^1)$. As W is a 0.5-solution, the inclusion map i_* is also an isomorphism.

$$\begin{array}{ccccc} \pi_1(M_J) & \longrightarrow & H_1(M_J) & & \\ & & \cong \big\downarrow i_* & \overset{f_*}{\searrow} & \\ \pi_1(W) & \longrightarrow & H_1(W) & \overset{\alpha}{\cdots\cdots\rightarrow} & H_1(S^1 \times S^1) \end{array}$$

We can then extend to a map $\alpha : H_1(W) \to H_1(S^1 \times S^1)$, and because $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ is abelian, we can also extend to the map $\bar{\alpha} : \pi_1(W) \to \pi_1(S^1 \times S^1)$. Knowing that the CW-complex $S^1 \times S^1$ is an Eilenberg-Maclane space $K(\mathbb{Z}^2, 1)$, we can extend the map f to a map $\bar{f} : W \to S^1 \times S^1$. The pre-image $\bar{f}_i^{-1}(1) = M_i$ is a 3-submanifold of W such that $\partial M_i = \hat{\Sigma}_i$. Moreover, $M_1 \cap M_2 = \bar{f}^{-1}((1,1)) = F$ is a surface $F \subset W$ such that $\partial F = \gamma$. Furthermore, the curve γ bounds a disk S in the attached 4-handle $\mathbb{B}^4 \subset \hat{W}$, so we let $\hat{F} = F \cup_\gamma S$ be a closed surface in $\hat{W}$.

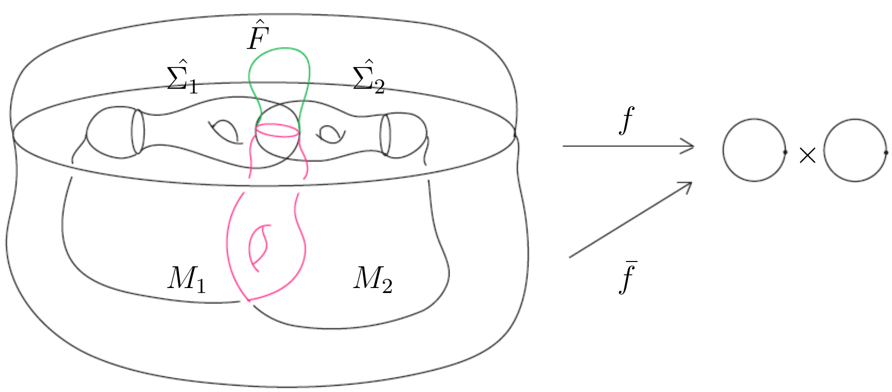


Figure 7.7: The Thom-Pontryagin construction on W

In $S^1 \times S^1$, we consider D_p, an ϵ-neighborhood of the point $p = (1, 1)$. Let $q = (e^{2\pi i \epsilon}, 1)$ be a point on the boundary of D_p. Then, $\bar{f}^{-1}(q)$ is a surface F^+ in

W with boundary γ^+, a push-off of the curve γ in M_J. F and $F^+$are disjoint, as they map to distinct points in $S^1 \times S^1$. Furthermore, if we consider a path $\alpha(t) = (e^{2\pi i \epsilon t}, 1), t \in [0,1]$, the pre-image $\bar{f}^{-1}(\alpha(t))$ is a 3-submanifold of W that is cobound by F and F^+. We may also consider the surface $\hat{F^+} \subset \hat{W}$, a closure of F^+ given by attaching a surface T in the 4-handle to F^+ along the curve γ^+. Then, $\bar{\mu}_J(1122) = lk(\gamma, \gamma^+) = S \cdot T = \hat{F} \cdot \hat{F^+}$, as surfaces $\hat{F}$ and $\hat{F^+}$ can only intersect in their caps in $\mathbb{B}^4$. We will use the following proposition to complete the proof.

Proposition 7.5. *For α a simple closed curve in W, the homology class $[\alpha] \in H_1(W)$ is given by the pair $(m_1, m_2) \in \mathbb{Z} \oplus \mathbb{Z}$ where $m_i = \alpha \cdot M_i$.*

Proof. $H_1(W) =< i_*(\mu_1), i_*(\mu_2) >$ is generated by the inclusion of the meridians of the link components of J. Thus, the homology class $[\alpha]$ can be written as $[\alpha] = m_1[\mu_1] + m_2[\mu_2] = (m_1, m_2)$. Recalling the Thom-Pontryagin construction, $\hat{\Sigma}_i = f_i^{-1}(1)$ and $M_i = \bar{f}_i^{-1}(1)$. We then see that the intersection $\mu_i \cdot M_j = \delta_{ij}$. Therefore, $\alpha \cdot M_i = m_i$. This says that the class $[\alpha] \in H_1(W)$ is given by the pair (m_1, m_2) such that $m_i = \alpha \cdot M_i$. □

Now, we consider the surfaces $\{L_i, D_i\}_{i=1}^r$ that generate $H_2(W)$, where $\pi_1(L_i) \subseteq \pi_1(W)^{(1)} = [\pi_1(W), \pi_1(W)]$. Choose the surfaces L_i to be transverse to M_1 and M_2. We may assume that $\{L_i\} \subset int(W)$. For each i, consider the intersection $L_i \cap F = L_i \cap M_1 \cap M_2$. The intersection $L_i \cap M_1 = b_i$, where b_i is some circle(s) on surface L_i. Because $\pi_1(L_i) \subseteq \pi_1(W)^{(1)}$, it must be true that $[b_i] = 0 \in H_1(W)$. By the proposition, we then must have that $b_i \cdot M_2 = 0$ for each i. This tells us that $L_i \cdot \hat{F} = 0$ for each i. We note that, as F and F^+ cobound a 3-manifold in W,

$L_i \cdot \hat{F}^{+} = 0$ as well.

We then can write the surfaces $\hat{F}$ and $\hat{F^{+}}$ in terms of the generators $\{L_i, D_i\}$ of $H_2(\hat{W})$. Let $\hat{F} = \sum_{i=1}^{r} (a_i L_i + b_i D_i)$ and let $\hat{F^{+}} = \sum_{i=1}^{r} (a_i^{+} L_i + b_i^{+} D_i)$. For each i, $0 = L_i \cdot \hat{F} = L_i \cdot \sum_{i=1}^{r} (a_i L_i + b_i D_i) = b_i$ and $0 = L_i \cdot \hat{F}^{+} = L_i \cdot \sum_{i=1}^{r} (a_i^{+} L_i + b_i^{+} D_i) = b_i^{+}$. Thus, all of the b_i's $= 0$ and all of the b_i^{+}'s $= 0$. Finally, we can conclude that $\bar{\mu}_J(1122) = \hat{F} \cdot \hat{F^{+}} = (\sum_{i=1}^{r} a_i L_i) \cdot (\sum_{i=1}^{r} a_i^{+} L_i)$. Recalling that $L_i \cdot L_j = 0$, this tells us that $\bar{\mu}_J(1122) = 0$. Therefore, if link $L = K_1 \cup \ldots \cup K_m$ is 0.5-solvable, its Sato-Levine invariants $\bar{\mu}_L(iijj)$ all vanish.

□

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