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Derivatives of Genus One and Three Knots

by

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Abstract

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A derivative L of an algebraically slice knot K is an oriented link disjointly embedded in a Seifert surface of K such that its homology class forms a basis for a metabolizer H of K. For genus one knots, we produce a new example of a smoothly slice knot with non-slice derivatives. Such examples were first discovered by Cochran and Davis. In order to do so, we define an operation on a homology B^4 that we call an n-twist annulus modification. Further, we give a new construction of smoothly slice knots and exotically slice knots via n-twist annulus modifications. For genus three knots, we show that the set $S_{K,H} = \{\bar{\mu}_L(123) - \bar{\mu}_{L'}(123) \mid L, L' \in \mathfrak{d}K/\mathfrak{d}H\}$ contains $n \cdot \mathbb{Z}$, where $\mathfrak{d}K/\mathfrak{d}H$ is the set of all the derivatives associated with a metabolizer H and n is an integer determined by a Seifert form of K and a metabolizer H. As a corollary, we show that it is possible to realize any integer as the Milnor's triple linking number of a derivative of the unknot on a fixed Seifert surface and with a fixed metabolizer.

Acknowledgments

I want to thank my advisors, Tim Cochran and Shelly Harvey, for introducing me to the world of knot concordance. Without both of your supervision and constant help this dissertation would not have been possible.

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CHAPTER 1

Introduction

1. Background

A k–component link L is the isotopy class of an ordered, oriented embedding $\bigsqcup_k S^1 \to S^3$ and a knot K is simply a 1–component link. Knot theory is deeply connected to the study of 3– and 4–manifold topology. For instance, Lickorish and Wallace [**Lic62, Wal60**] proved that any closed, orientable, connected 3–manifold may be obtained from S^3 by performing a certain type of operation (called Dehn surgery) on a link.

1.1. Knot concordance and link concordance. Instead of classifying knots up to isotopy, if we consider certain weaker equivalence relations called concordances, there are more connections between 4–manifold topology and knot theory.

Two links $L_1 \hookrightarrow S^3 \times \{0\}$ and $L_2 \hookrightarrow S^3 \times \{0\}$ are concordant if they cobound a collection of smooth properly embedded annuli in $S^3 \times [0, 1]$. Concordance gives an equivalence relation on the set of links. Further, concordance classes of knots have an interesting structure. Under the connected sum operation, concordance classes of knots forms an abelian group called the knot concordance group, denoted $\mathcal{C}$. A knot K is called smoothly slice if it represents the identity element in $\mathcal{C}$, or equivalently if it bounds a smooth properly embedded 2-disk D^2, called a smooth slice disk, in B^4. Moreover, if K bounds a smoothly embedded 2-disk D^2 in a smooth oriented 4-manifold M that is homeomorphic to the standard B^4 but not necessarily diffeomorphic to the standard B^4, we call K exotically slice in M and D^2 an exotic slice disk. Note that we can define a radial function on B^4 and by a small isotopy of D^2, one can ensure that the radial function restricts to a Morse function on D^2. If a knot K bounds a smoothly embedded 2-disk D^2 in the standard B^4 where there are no local maxima of the radial function restricted to D^2, we call K a ribbon knot. A link L is a smoothly slice link if each component of L bounds a smoothly embedded disjoint 2-disk D^2 in the standard B^4.

1.2. Derivatives of knots. Any knot in S^3 bounds a Seifert Surface F. From F, we can define a Seifert form $\beta_F : H_1(F) \times H_1(F) \to \mathbb{Z}$, which is defined by $\beta_F([x],[y]) = \mathrm{lk}(x, y^+)$, where x is union of simple closed curves on F that represents $[x]$, y^+ is positive push off of union of simple closed curves on F that represents $[y]$, and lk denotes linking number. It was proven by Levine [**Lev69**, Lemma 2] that if K is a smoothly slice knot then β_F is metabolic for any Seifert surface F for K, i.e. there exists a metabolizer $H = \mathbb{Z}^{\frac{1}{2}\operatorname{rank} H_1(F)}$, a direct summand of $H_1(F)$, such that β_F vanishes on H. We call a knot algebraically slice knot if it has a metabolic Seifert form. Then a link $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{\frac{1}{2}\operatorname{rank} H_1(F)}$ disjointly embedded in a surface F where its homology classes form a basis for H is called a derivative of K (see Figure 1.1). Note that we can define a derivative of a knot for any algebraically slice knot.

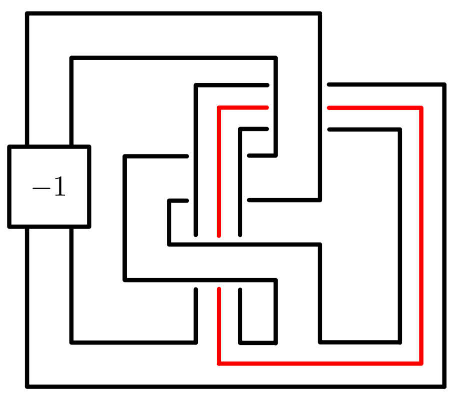

FIGURE 1.1. A derivative of 6_1 knot

2. Results

In this thesis, we study derivatives of genus one and three knots. First, we focus on derivatives of knots with genus one Seifert surfaces. In order to do so, we will introduce an operation on homology B^4 which we call an n-twist annulus modification in Chapter 2.1. Then we will fix a smoothly slice knot and use this modification under a certain condition (see Chapter 2.1), to obtain an infinite family of exotically slice knots. The basic idea for this condition, which is called ℓ-nice, is that there exist a smooth proper embedding of an annulus in B^4 such that the result of Dehn surgery along the interval is homeomorphic to B^4. In addition, if there is a smooth slice disk disjoint from the annulus, then the image of the knot under the modification is exotically slice (see Figure 1.2). The technique we use here is similar to the technique that was used in [**CD15**] to construct a smoothly slice knot with non-slice derivatives.

$B^4 = B^3 \times [0,1] =$

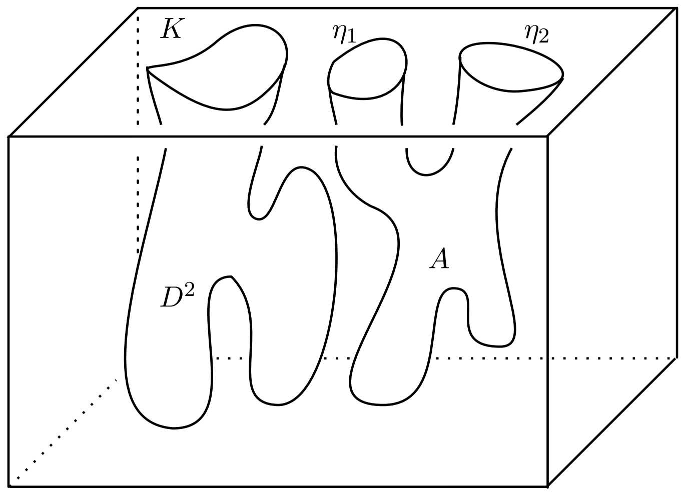

FIGURE 1.2. Schematic picture of the assumptions of Theorem 2.3.

THEOREM 2.3. Let K be a smoothly slice knot, ϕ_A be ℓ-nice, and $\eta_1 \cup -\eta_2$ be the boundary of A. Suppose there exists a smoothly embedded slice disk for K in the complement of A. Then $\frac{n\ell+1}{n}$ Dehn surgery on η_1 followed by $\frac{n\ell-1}{n}$ Dehn surgery on η_2 produces an exotically slice knot $K_{(\phi_A,n)} \subset S^3$.

If we restrict our condition further (see Chapter 2.2), which is called ℓ-standard, we can use annulus modifications to obtain a infinite family of smoothly slice knots. The condition is that the smooth proper embedding of an annulus is isotopic to the standard annulus, denoted A_ℓ and shown in Figure 2.4.

THEOREM 2.6. Let K be a smoothly slice knot, ϕ_A be ℓ-standard, and $\eta_1 \cup -\eta_2$ be the boundary of A. Suppose there exists a smoothly embedded slice disk for K in the complement of A. Then $\frac{n\ell+1}{n}$ Dehn surgery on η_1 followed by $\frac{n\ell-1}{n}$ Dehn surgery on η_2 produces a smoothly slice knot $K_{(\phi_A,n)} \subset S^3$.

The main application for n-twist annulus modifications is to produce a smoothly slice genus one knot with non-slice derivatives. If a knot K has a derivative which is a smoothly slice link, then K is smoothly slice (see Figure 1.3). A natural question is whether the converse holds. This was asked by Kauffman in 1982, for genus one knots.

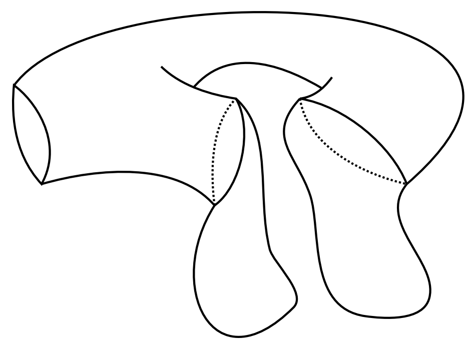

FIGURE 1.3. Surgery on derivative gives us smoothly slice disk

CONJECTURE 1.1 ([**Kau87**][**Kir95**, N1.52]). If K is a smoothly slice knot and F is a genus one Seifert surface for K then there exists a derivative γ on F such that γ is a smoothly slice knot.

A lot of evidence supporting this conjecture was found by Casson, Cooper, Gilmer, Gordon, Livingsont, Litherland, Cochran, Orr, Teichner, Harvey, Leidy and others [**Gil83, Lit84, Gil93, GL92a, GL92b, GL13, CHL10, COT04**]. However, the conjecture is false: Cochran and Davis recently constructed a smoothly slice knot K where neither of its derivatives is smoothly slice in [**CD15**]. Surprisingly both of the derivatives have non-zero Arf invariant. In particular, they are not algebraically slice. In this thesis we present a different example of a smoothly slice knot with non-slice derivatives. We obtain this example by using an n-twist annulus modification.

THEOREM 3.1. Let R_1 be a knot described in Chapter 3.1. Then R_1 is a smoothly slice knot with non-slice derivatives on a Seifert surface.

We have few more applications of n-twist annulus modifications. An annulus twist (see Chapter 3.2) is an operation on S^3 which was used in [**Oso06**]. Osoinach used annulus twists to produce 3-manifolds that can be obtained by same coefficient Dehn surgery on an infinite family of distinct knots. In particular, Osoinach showed that if a knot K has an orientation preserving annulus presentation (see Chapter 3.2), 0-surgery on K_n, where K_n is a knot obtained by an n-fold annulus twist on K, is diffeomorphic to 0-surgery on K. Thus, if K is a smoothly slice knot with an orientation preserving annulus presentation, K_n is exotically slice for any integer n [**CFHH13**, Proposition 1.2]. This was also pointed out in [**AJOT13**], where they use annulus twists to produce an infinite family of distinct framed knots which represents a diffeomorphic 4-manifold. In this thesis, we will reprove the statement with slightly stronger assumptions, using n-twist annulus modifications. More precisely, we show that K_n is exotically slice if K is a ribbon knot in Proposition 3.2.

In fact, in [**AT16**] Abe and Tange showed that if K is a ribbon knot with an annulus presentation with $+1$ or -1 framing (see Chapter 3.2), K_n is smoothly slice. In this thesis, we use n-twist annulus modifications to show this statement is true for a very specific case, namely when K is 8_{20} knot (see Proposition 3.3). Also, in [**AT16**] they show that an n-fold annulus twist on 8_{20} are ribbon knot when $n \geq 0$, but it is still not known whether other slice knots obtained by an annulus twist are ribbon knots.

Moreover, we consider n-twist annulus modifications on general annuli. More precisely, we no longer require the link $\eta_1 \cup \eta_2$ to be isotopic to L_ℓ (see Figure 2.2), which is one of the requirement for ϕ_A to be either ℓ-nice or ℓ-standard. By using these general annuli, we show that any exotic slice disk can be obtained by an annulus modification performed on some exotic slice disk bounding the unknot as follows.

THEOREM 3.6. Given an exotically slice knot K with an exotic 4-ball M and an exotic slice disk D, there exists an exotic 4-ball M' and there exists an exotic disk D' in M' for the unknot in $\partial M' = S^3$, such that (M, D) arises from (M', D') via a -1–twist annulus modification on some annulus $A' \subseteq M' - D'$.

For the second half of the thesis we study derivatives of knots with genus three Seifert surfaces. Note that Conjecture 1.1 can be easily generalized to higher genus knots as follows.

CONJECTURE 1.2. If K is a smoothly slice knot and F is a genus g Seifert surface for K then there exists a derivative $L = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_g$ on F such that L is a smoothly slice link.

It is still an open problem whether Kauffman's conjecture is true for knots with genus greater than one. Hence it is natural to study derivatives of knots with higher genus Seifert surfaces. In this thesis, we focus on knots which have genus three Seifert surfaces and study the behavior of its derivatives. In particular, we are interested in calculating their Milnor's triple linking number (see Chapter 4.2 for the precise definition). Also, note that by definition, if a link L is a derivative of a knot, then L has pairwise linking number zero. Since the Milnor's triple linking number generalizes the ordinary linking number, it is natural to ask if we can understand all the possible Milnor's triple linking numbers of derivatives once we fix a knot and its metabolizer. To this end, we study the set

$$S_{K,H} = \{\bar{\mu}_L(123) - \bar{\mu}_{L'}(123) \mid L, L' \in \mathfrak{d}K/\mathfrak{d}H\},$$

where $\mathfrak{d}K/\mathfrak{d}H$ is the set of all the derivatives associated with a metabolizer H and $\bar{\mu}_L(123)$ and $\bar{\mu}_{L'}(123)$ are the Milnor's triple linking number links L and L' respectively.

THEOREM 5.3. Let K be an algebraically slice knot with genus three Seifert surface F. Suppose H is a metabolizer of K and $\{b_1, b_2, b_3\}$ is a basis for H. Extend $\{b_1, b_2, b_3\}$ to a symplectic basis for $H_1(F)$ and let $M = \begin{pmatrix} B & A \\ A^\top - \mathrm{Id} & 0 \end{pmatrix}$ be the resulting Seifert matrix. Then

$$S_{K,H} \supseteq (\det(A - \mathrm{Id}) - \det(A)) \cdot \mathbb{Z}.$$

It is still an open problem whether $S_{K,H}$ is in fact equal to $(\det(A - \mathrm{Id}) - \det(A)) \cdot \mathbb{Z}$ or not. As a corollary of Theorem 5.3, we show that even when we choose the knot to be the simplest possible knot, Milnor's triple linking number of derivatives can become very complicated.

Corollary 5.3. Any integer can be realized as the Milnor's triple linking number of a derivative of the unknot on a fixed Seifert surface and with a fixed metabolizer.

3. Outline of thesis

We start by introducing annulus modifications in Chpater 2. In Chapter 3, we give few applications of annulus modifications. As a main application we find an example of a smoothly slice knot with non-slice derivatives. In Chapter 4, we state the precise definition of Milnor's triple linking number and study the effect of string link infection on Milnor's triple linking number. In Chapter 5, we show that the behavior Milnor's triple linking number of derivatives is quite complex.

CHAPTER 2

Annulus modifications

1. The technique

In this section, we discuss the technique for constructing new slice knots from a fixed smoothly slice knot and slice disk. Let M be a smooth compact 4-manifold with non–empty boundary, and assume that M is an integer homology B^4. Let $\phi_A : S^1 \times [0,1] \hookrightarrow M$ be a smooth proper embedding of an annulus and denote $\mathrm{Im}(\phi_A|_{S^1\times\{0\}}) = \eta_1$, $\mathrm{Im}(\phi_A|_{S^1\times\{1\}}) = \eta_2$, and $\mathrm{Im}(\phi_A) = A$. Further assume $\eta_1 \cup \eta_2$ is contained in some three ball in ∂M so it makes sense to talk about the linking number of η_1 and η_2. Then let $\ell = \mathrm{lk}(\eta_1, \eta_2)$ and n be any integer. Note that ϕ_A can be extended to a smooth proper embedding $\phi_{N(A)} : S^1 \times D^2 \times [0,1] \hookrightarrow M$, where $\mathrm{Im}\,(\phi_{N(A)}) = N(A)$ and $N(A)$ is a tubular neighborhood of A. Further, we choose $\phi_{N(A)}$ so that $S^1 \times \{1\} \times \{0\}$ is identified with preferred longitude of η_1. Let λ_1 and λ_2 be preferred longitudes for η_1 and η_2 respectively, and let μ_1 and μ_2 be meridians of η_1 and η_2 respectively. Then $\phi_{N(A)}$ gives the following identifications:

- $S^1 \times \{1\} \times \{0\} \subseteq N(A)$ is identified with λ_1.
- $\{1\} \times \partial D^2 \times \{0\} \subseteq N(A)$ is identified with μ_1.
- $S^1 \times \{1\} \times \{1\} \subseteq N(A)$ is identified with $\lambda_2 + 2\ell\mu_2$.
- $\{1\} \times \partial D^2 \times \{1\} \subseteq N(A)$ is identified with $-\mu_2$.

Let $\mathrm{Aut}(S^1 \times S^1)$ be the set of isotopy classes of diffeomorphisms from $S^1 \times S^1$ to itself. Recall that there is a bijective correspondence between $\mathrm{Aut}(S^1 \times S^1)$ and $GL(2,\mathbb{Z})$ [**Rol76**, Chapter 2]. Then let ρ_n be the element in $\mathrm{Aut}(S^1 \times S^1)$ which corresponds to $\begin{pmatrix} 1 & n \\ l & n\ell+1 \end{pmatrix}$, that is, ρ_n sends the longitude to the longitude plus ℓ times the meridian and the meridian to n times the longitude plus $n\ell + 1$ times the meridian. Using $\phi_{N(A)}|_{S^1\times\partial D^2\times[0,1]}$ and ψ_n, we define our modification on M.

Let $f_n := \phi_{N(A)}\big|_{S^1\times\partial D^2\times[0,1]}\circ\psi_n$; note that f_n is a diffeomorphism from $S^1\times\partial D^2\times[0,1]$ to itself. Then let $M_{(\phi_A,n)} = (M - N(A)) \cup_{f_n} S^1 \times [0,1] \times D^2$. We will call $M_{(\phi_A,n)}$ *the n-twist annulus modification on M at ϕ_A*. This modification is just doing Dehn surgery along the interval. This is similar to a logarithmic transformation.

We will perform annulus modifications on B^4 to get new smoothly slice knots and new exotically slice knots. First, we describe the basic idea. Fix a smoothly slice knot K with a smooth slice disk D^2 in the 4-ball B^4 from now on. Let E be an smoothly embedded 4-manifold in $B^4 \setminus N(D^2)$ (see Figure 2.1). Let $\widetilde{B}$ be a new manifold obtained by taking out E and glueing it back differently to the complement. Let $\widetilde{K} \subseteq \partial\widetilde{B}$ be the image of K in the modified manifold. Since $\widetilde{K}$ bounds a smoothly embedded disk in $\widetilde{B}$, if $\widetilde{B}$ is homeomorphic to B^4, then the resulting new knot $\widetilde{K} \subseteq \partial\widetilde{B}$ is exotically slice, and if $\widetilde{B}$ is diffeomorphic to B^4, then the resulting new knot $\widetilde{K} \subseteq \partial\widetilde{B}$ is smoothly slice. In our case E is going to be a tubular neighborhood of a proper smooth embedding of an annulus.

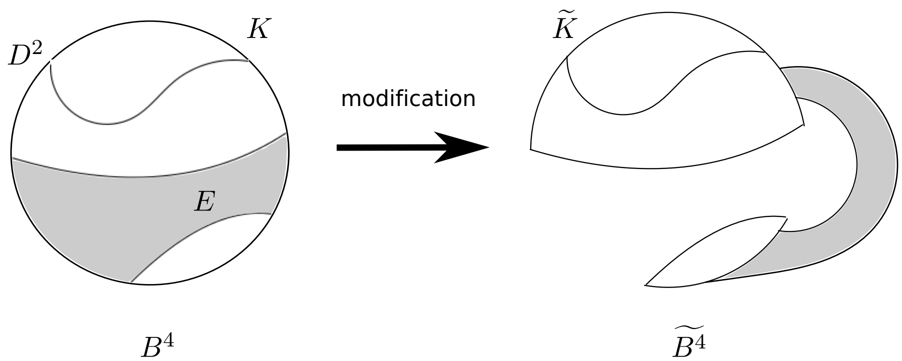

FIGURE 2.1. Schematic of basic idea of the modification

To be more precise about the technique we will need some definitions.

DEFINITION 2.1. Let $\phi_A : S^1 \times [0,1] \hookrightarrow B^4$ be a smooth proper embedding of an annulus with $\mathrm{Im}(\phi_A|_{S^1\times\{0\}}) = \eta_1$, $\mathrm{Im}(\phi_A|_{S^1\times\{1\}}) = \eta_2$, $\mathrm{Im}(\phi_A) = A$, $\ell = \mathrm{lk}(\eta_1,\eta_2)$, and $\eta_1 \cup \eta_2 \subseteq S^3 - K$. We will say ϕ_A is *ℓ-nice* if it satisfies the following:

(1) The link $\eta_1\cup\eta_2$ is isotopic to L_ℓ in S^3, where L_ℓ is the two component link described by black curves in Figure 2.2.

(2) $[c] = id \in \pi_1(B^4 - N(A))$, where c is the knot disjoint from L_ℓ described by the red curve in Figure 2.2 and μ_1 is a meridian of η_1 as above.

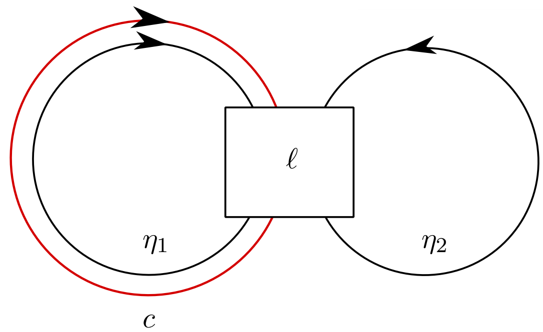

FIGURE 2.2. The link $L_\ell = \eta_1 \cup \eta_2$, where the box containing ℓ represents the ℓ full twists, and the knot c.

REMARK 2.2.

(1) *Condition* (2) *from Definition 2.1 is technical condition imposed to make sure that the resulting manifold after performing an n-twist annulus modification at ϕ_A is simply connected.*

(2) *When the link $\eta_1 \cup \eta_2$ is isotopic to L_0 in S^3, the condition* (2) *from Definition 2.1 is automatically satisfied. Note that $[c]$ represents trivial element in $\pi_1(S^3 - N(\eta_1 \cup \eta_2))$, hence it represents trivial element in $\pi_1(B^4 - N(A))$.*

(3) *When the link $\eta_1 \cup \eta_2$ is isotopic to L_ℓ in S^3, $\frac{n\ell+1}{n}$ Dehn surgery on η_1 followed by $\frac{n\ell-1}{n}$ Dehn surgery on η_2 produces S^3. This can be easily seen by series of Rolfsen twists, which is described in Figure 2.3.*

When ϕ_A is ℓ-nice, performing an n-twist annulus modification on B^4 along at ϕ_A gives us the following main theorem.

THEOREM 2.3. *Let K be a smoothly slice knot, ϕ_A be ℓ-nice, and $\eta_1 \cup -\eta_2$ be the boundary of A. Suppose there exists a smoothly embedded slice disk for K in the complement of A. Then $\frac{n\ell+1}{n}$ Dehn surgery on η_1 followed by $\frac{n\ell-1}{n}$ Dehn surgery on η_2 produces an exotically slice knot $K_{(\phi_A,n)} \subset S^3$.*

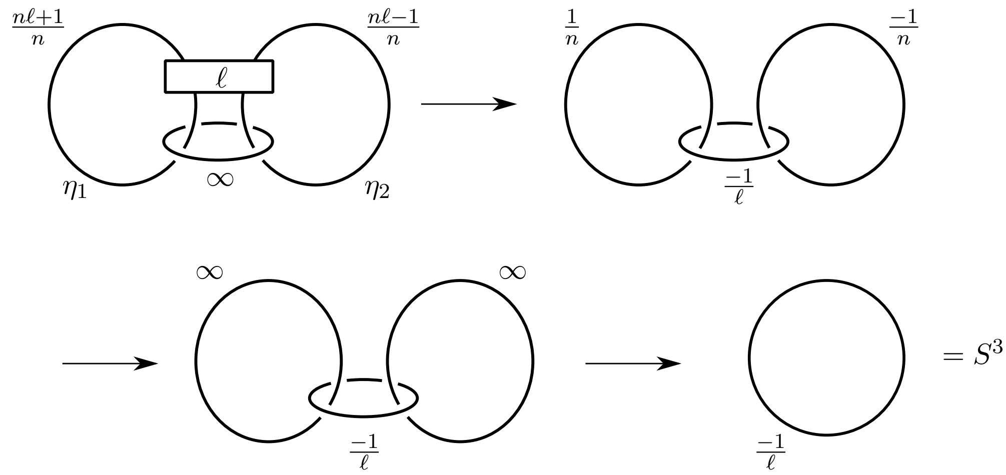

FIGURE 2.3. The first arrow is given by a $-\ell$ Rolfsen twist on the ∞ circle. The second arrow is given by a $-n$ Rolfsen twist on η_1 and n a Rolfsen twist on η_2. The third arrow is given by deleting the components with coefficient ∞.

PROOF. We will perform an n-twist annulus modification on B^4 at ϕ_A to get $B^4_{(\phi_A,n)}$. It is easy to check that $B^4_{(\phi_A,n)}$ is a homology B^4 by using a Mayer-Vietoris sequence. We omit this detail.

We need to show that $B^4_{(\phi_A,n)}$ is simply connected. We will use Seifert-van Kampen theorem to see $B^4_{(\phi_A,n)}$ is simply connected. First we apply it to B^4 to get the following equations, where i_1 is natural inclusion of $S^1 \times \partial D^2 \times [0,1]$ into $B^4 - N(A)$ and i_2 is natural inclusion of $S^1 \times \partial D^2 \times [0,1]$ into $N(A)$.

$$
\begin{aligned}
\{\text{id}\} = \pi_1(B^4) &= \frac{\pi_1(B^4 - N(A)) * \pi_1(N(A))}{< (i_1)_*([\mu_1]) = (i_2)_*([\mu_1]), (i_1)_*([\lambda_1]) = (i_2)_*([\lambda_1]) >} \\
&= \frac{\pi_1(B^4 - N(A))}{< (i_1)_*([\mu_1]) >} \text{ (Since, } \pi_1(N(A)) = \mathbb{Z} \text{ is generated by } (i_2)_*([\lambda_1]).)
\end{aligned}
$$

Hence we have $\pi_1(B^4 - N(A))/ < (i_1)_*([\mu_1]) >= \{\text{id}\}$. Now we apply Seifert-van Kampen theorem to $B^4_{(\phi_A,n)}$. Recall that f_n is a diffeomorphism from $S^1 \times \partial D^2 \times [0,1]$ to itself defined as above and let $y = [(i_2)_*([\mu_1])]$ be a generator of $\pi_1(N(A)) = \mathbb{Z}$.

$$
\begin{aligned}
\pi_1(B^4_{(\phi_A,n)}) &= \frac{\pi_1(B^4 - N(A)) * \pi_1(N(A))}{<(i_1)_*([\mu_1]) = (i_2 \circ {f_n}^{-1})_*([\mu_1]), (i_1)_*([\lambda_1]) = (i_2 \circ {f_n}^{-1})_*([\lambda_1])>} \\
&= \frac{\pi_1(B^4 - N(A)) * <y>}{<(i_1)_*([\mu_1]) = y^{-n}, (i_1)_*([\lambda_1]) = y^{n\ell+1}>} \\
&= \frac{\pi_1(B^4 - N(A))}{<((i_1)_*([\mu_1])^{\ell} \cdot (i_1)_*([\lambda_1]))^n \cdot (i_1)_*([\mu_1])>} \\
&= \frac{\pi_1(B^4 - N(A))}{<([c])^n \cdot (i_1)_*([\mu_1])>} \text{ (Since, } [c] = (i_1)_*([\mu_1])^{\ell} \cdot (i_1)_*([\lambda_1]).) \\
&= \frac{\pi_1(B^4 - N(A))}{<(i_1)_*([\mu_1])>} \text{ (By the condition (2) in Definition 2.1.)} \\
&= \{\text{id}\} \text{ (By observation above.)}
\end{aligned}
$$

This shows $B^4_{(\phi_A,n)}$ is simply connected as we needed.

What is now left to do is to understand what happens on the boundary. Notice that $\partial B^4_{(\phi_A,n)}$ is the result of Dehn surgeries on η_1 and η_2, since we are simply removing two solid tori from ∂B^4 and glueing them back differently. Hence it is enough to calculate the coefficient on both curves to specify the boundary. We are using f_n to glue $S^1 \times D^2 \times [0,1]$ to $B^4 - N(A)$, so we have the following identifications:

- $S^1 \times \{1\} \times \{0\}$ is identified with $\lambda_1 + \ell\mu_1$.
- $\{1\} \times \partial D^2 \times \{0\}$ is identified with $n\lambda_1 + (n\ell + 1)\mu_1$.
- $S^1 \times \{1\} \times \{1\}$ is identified with $\lambda_2 + \ell\mu_2$.
- $\{1\} \times \partial D^2 \times \{1\}$ is identified with $n\lambda_2 + (n\ell - 1)\mu_2$.

Recall that λ_1 and λ_2 are the preferred longitudes of η_1 and η_2 respectively, and μ_1 and μ_2 are meridians of η_1 and η_2 respectively. So that the meridian of $S^1 \times D^2 \times \{0\}$ is identified with $n\lambda_1 + (n\ell+1)\mu_1$ and the meridian of $S^1 \times D^2 \times \{1\}$ is identified with $n\lambda_2 + (n\ell-1)\mu_2$. This shows that $\frac{n\ell+1}{n}$ is the coefficient for η_1 and $\frac{n\ell-1}{n}$ for η_2 which implies $\partial B^4_{(\phi_A,n)}$ is the top left picture in Figure 2.3. By Remark 2.2 (3), $\partial B^4_{(\phi_A,n)}$ is S^3.

Thus by the 4-dimensional topological Poincaré conjecture we can conclude that $B^4_{(\phi_A,n)}$ is homeomorphic to B^4 [**Fre84**, Theorem 1.6], which implies that $K_{(\phi_A,n)}$ is exotically slice for any integer n. $\square$

2. Special case

In this section, we will discuss a special case of Chapter 2.1, which guarantees that the resulting manifold $B^4_{(\phi_A,n)}$ is diffeomorphic to B^4.

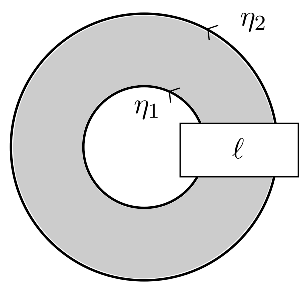

FIGURE 2.4. The annulus A_ℓ

DEFINITION 2.4. Let $\phi_{A_\ell} : S^1 \times [0,1] \hookrightarrow B^4$ be a smooth proper embedding of an annulus with $\mathrm{Im}(\phi_{A_\ell}) = A_\ell$ where A_ℓ is obtained by pushing in the interior of the annulus, described in Figure 2.4. We will call ϕ_A *ℓ-standard* if ϕ_A is ℓ-nice and if, in addition, the annulus A is smoothly isotopic through proper embeddings to A_ℓ.

REMARK 2.5. *When the link $\eta_1 \cup \eta_2$ is isotopic to link L_ℓ and if it bounds an annulus A, smoothly isotopic through proper embeddings to A_ℓ, then the condition* (2) *from Definition 2.1 is automatically satisfied. Note that the curve c bounds a smoothly embedded disk in $B^4 - N(A)$ which is described in Figure 2.5. Hence* $[c]$ *represents a trivial element in* $\pi_1(B^4 - N(A))$ *and we see that the condition is satisfied.*

Then we have the following theorem. Note that this is an analogue of Theorem 3.1 in [**CD15**].

THEOREM 2.6. *Let K be a smoothly slice knot, ϕ_A be ℓ-standard, and $\eta_1 \cup -\eta_2$ be the boundary of A. Suppose there exists a smoothly embedded slice disk for K in the complement of A. Then $\frac{n\ell+1}{n}$ Dehn surgery on η_1 followed by $\frac{n\ell-1}{n}$ Dehn surgery on η_2 produces a smoothly slice knot $K_{(\phi_A,n)} \subset S^3$.*

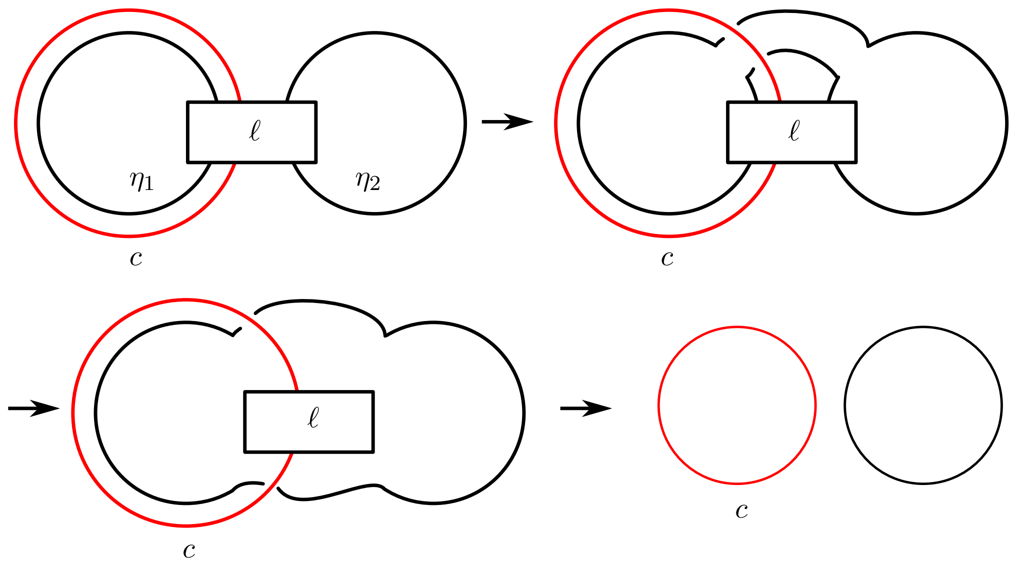

FIGURE 2.5. Top left figure is the link $\eta_1 \cup \eta_2$ and the knot c. The second picture is obtained by performing band sum between η_1 and η_2. The third and fourth pictures are obtained by isotopy of the black curve. Note that the knot c becomes completely disjoint from the curve representing a cross section of the annulus A after the band sum between η_1 and η_2.

PROOF. By Theorem 2.3, the only thing that we need to show is that, for any integer n, $B^4_{(\phi_A,n)}$ is diffeomorphic to the standard B^4 and not just homeomorphic when ϕ_A is ℓ-standard.

Note that if ϕ_A and $\phi_{A'}$ are smooth proper embedding of annuli into B^4 that are smoothly isotopic through proper embeddings, then $B^4_{(\phi_A,n)}$ is diffeomorphic to $B^4_{(\phi_{A'},n)}$. This follows from the ambient isotopy theorem [**Hir94**, Chapter 8, Theorem 1.3].

We first show $B^4_{(\phi_A,n)}$ is diffeomorphic to the standard B^4 when ϕ_A is 0-standard. We can think of B^4 as $B^3 \times [0,1]$, so we have a smooth proper embedding of an annulus $U \times [0,1] \subseteq B^3 \times [0,1] = B^4$, where U is the unknot. Then observe $U \times [0,1] \subseteq B^4$ is isotopic to A_0; one could visualize this by pulling the boundary of $U \times [0,1]$ to $\partial B^3 \times [0,1]$. Hence we can conclude that $B^4_{(\phi_{A_0},n)}$ is diffeomorphic to $B^3_{\frac{1}{n}} \times [0,1] = B^3 \times [0,1] = B^4$, where $B^3_{\frac{1}{n}}$ is $\frac{1}{n}$ Dehn surgery along the unknot. Thus, $B^4_{(\phi_A,n)}$ is diffeomorphic to the standard B^4 for any integer n when ϕ_A is 0-standard.

For $\ell \neq 0$, we need to define one more modification. Let M be be a compact integer homology B^4 and $\phi_D : D^2 \hookrightarrow M$ be a smooth proper embedding of a disk. Let D denote $\mathrm{Im}(\phi_D)$. Then carve out a tubular neighborhood of D and attach a 2-handle along the meridian of ∂D with framing ℓ. We will call this a *ℓ-disk modification on M at ϕ_D* and denote the resulting manifold as $D_\ell(M)$.

Let D_0 be a disk which is obtained by pushing in the interior of the smoothly embedded disk in S^3 to the standard B^4. Note that if D is a proper embedding of a disk in B^4 and if it is isotopic through proper embedding to D_0, then a ℓ-disk modification on B^4 is simply adding a canceling 1-handle / 2-handle pair, which does not change the 4-manifold.

We will modify two particular disjoint proper smooth embeddings $\phi_{\widetilde{A}}$ and $\phi_{\widetilde{D}}$ which are described in Figure 2.6. To be more precise about $\phi_{\widetilde{A}}$ and $\phi_{\widetilde{D}}$, each cross section of $\mathrm{Im}(\phi_{\widetilde{A}})$ is the unknot and each cross section of $\mathrm{Im}(\phi_{\widetilde{D}})$ is an fixed unknotted arc that goes through the unknot. Let $\widetilde{A}$ denote $\mathrm{Im}(\phi_{\widetilde{A}})$ and $\widetilde{D}$ denote $\mathrm{Im}(\phi_{\widetilde{D}})$. We will do two modifications on B^4: an n-twist annulus modification along $\phi_{\widetilde{A}}$ and an ℓ-disk modification at $\phi_{\widetilde{D}}$. Note that the order of these modifications does not matter so we have $\widetilde{D}_\ell(B^4_{(\phi_{\widetilde{A}},n)}) = \widetilde{D}_\ell(B^4)_{(\phi_{\widetilde{A}},n)}$ (see Figure 2.7).

$B^4 = B^3 \times [0,1] =$

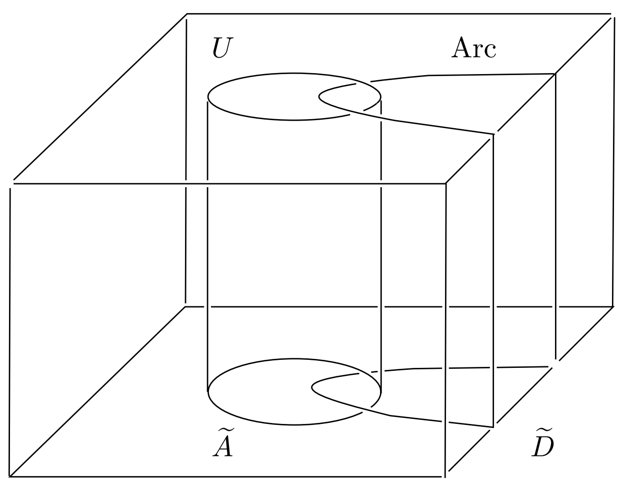

FIGURE 2.6. Annulus $\widetilde{A}$ and disk $\widetilde{D}$

We now describe the result of each modification from Figure 2.7.

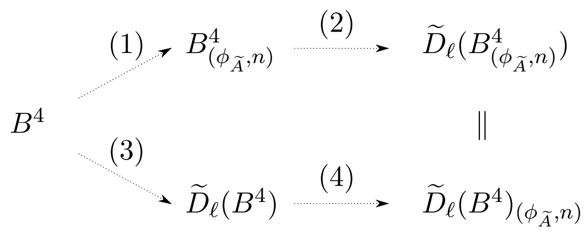

FIGURE 2.7. Two modifications on B^4. (1) and (4) are obtained by an n-twist annulus modification along $\phi_{\widetilde{A}}$. (2) and (3) are obtained by an ℓ-disk modification at $\phi_{\widetilde{D}}$.

(1) For the modification (1), note that $\phi_{\widetilde{A}}$ is isotopic to A_0. In that case, we have shown already that $B^4_{(\phi_{\widetilde{A}},n)}$ is diffeomorphic to the standard B^4.
(2) For the modification (2), Dehn surgery at each level does not change the Arc. Thus $\widetilde{D} \subseteq B^4_{(\phi_{\widetilde{A}},n)} \cong B^4$ and $D_0 \subseteq B^4$ are smoothly isotopic through proper embeddings, which implies that $\widetilde{D}_\ell(B^4_{(\phi_{\widetilde{A}},n)})$ is diffeomorphic to the standard B^4.
(3) For the modification (3), $\widetilde{D} \subseteq B^4$ and $D_0 \subseteq B^4$ are smoothly isotopic through proper embeddings, which implies that $\widetilde{D}_\ell(B^4)$ is diffeomorphic to the standard B^4.
(4) For the modification (4), recall that $\widetilde{D}_\ell(B^4_{(\phi_{\widetilde{A}},n)}) = \widetilde{D}_\ell(B^4)_{(\phi_{\widetilde{A}},n)}$. $\widetilde{D}_\ell(B^4)_{(\phi_{\widetilde{A}},n)}$ is diffeomorphic to the standard B^4 by (2).

Before performing any modifications on B^4, we can isotope $\widetilde{A}$ to $\partial B^4 = S^3$ away from $\widetilde{D}$. We can visualize this (see Figure 2.6) by pushing $\widetilde{A}$ in to $\partial B^3 \times [0,1]$ to the right. By abuse of notation, let $\widetilde{A}$ denote the image of $\widetilde{A}$ under an ℓ-disk modification at $\phi_{\widetilde{D}}$. After the modification, $\widetilde{A} \subseteq \widetilde{D}_\ell(B^4)$ is smoothly isotopic through proper embeddings to the annulus in S^3 that is described in the right hand side of Figure 2.8. Hence $\widetilde{A} \subseteq \widetilde{D}_\ell(B^4) \cong B^4$ is smoothly isotopic through proper embeddings to $A_\ell \subseteq B^4$, which was described in the beginning of the section. Then we can conclude that $B^4_{(\phi_{A_\ell},n)}$ is diffeomorphic to $\widetilde{D}_\ell(B^4)_{(\phi_{\widetilde{A}},n)}$, which is diffeomorphic to the standard B^4. Hence $B^4_{(\phi_{A_\ell},n)}$ is diffeomorphic to the standard B^4 for all integers n which concludes the proof. $\square$

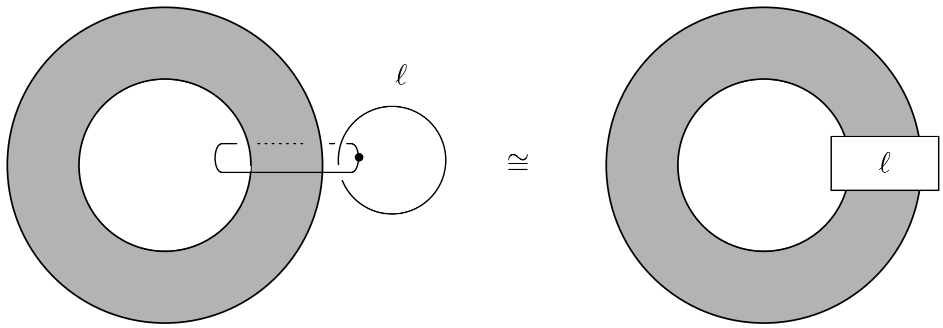

FIGURE 2.8. Image of $\widetilde{A}$ under (3) from Figure 2.7

We end this section by using a result of Scharlemann and a result of Livingston to find a sufficient criterion for 0-nice ϕ_A to be 0-standard.

THEOREM 2.7. [**Sch85**, Main Theorem] *Suppose that η_1 and η_2 are knots in S^3 which form a split link and that a certain band sum of η_1 and η_2 yields the unknot. Then η_1 and η_2 are each unknotted and the band sum is connected sum.*

THEOREM 2.8. [**Liv82**, Theorem 4.2] *Let F_1 and F_2 be orientable surfaces embedded in S^3, bounding the unlink. F_1 and F_2 are isotopic through proper embeddings if and only if F_1 and F_2 are homeomorphic.*

Let $h : B^4 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 \leq 1\} \to \mathbb{R}$ where $h(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ and let ϕ_A be a proper smooth embedding of an annulus in B^4. We may assume that the restriction map $h \circ \phi_A$ is a Morse function by applying a small isotopy on ϕ_A. By abuse of notation we will refer to critical points of $h \circ \phi_A$ as critical points of ϕ_A. Then we have the following corollary.

COROLLARY 2.9. *Let K be a smoothly slice knot and let ϕ_A be 0-nice. If ϕ_A has one critical point of index zero and one critical point of index one then ϕ_A is 0-standard. Hence if there exists a smoothly embedded slice disk for K in the complement of A, $\frac{1}{n}$ Dehn surgery on η_1 and $\frac{-1}{n}$ Dehn surgery on η_2 produces a smoothly slice knot $K_{(\phi_A,n)} \subseteq S^3$.*

PROOF. By Theorem 2.6 it is enough to show that ϕ_A is 0-standard. In other words it would be enough to show ϕ_A and ϕ_{A_0} are smoothly isotopic through proper embeddings, when ϕ_A has one critical point of index zero and one critical point of index one.

Since η_1 and η_2 form a two component unlink, they bound smoothly embedded disks D_1 and D_2 respectively. A critical point of index one corresponds to a band sum between η_1 and η_2 which can be isotoped into S^3; we will call this band B. Let η_0 be the resulting knot after doing the band sum. A critical point of index zero corresponds to a disk bounded by η_0 which also could be isotoped into S^3 hence η_0 is the unknot. We will call this disk D_0.

By Theorem 2.7 [**Sch85**, Main Theorem], B is connected sum, and hence B does not intersect D_1 and D_2. Thus we have two disks D_0 and $D_1 \cup B \cup D_2$ in S^3 bounded by η_0. Since any two disks bounded by same curve in S^3 can be isotoped into each other, we can isotope D_0 into $D_1 \cup B \cup D_2$ and then push it slightly off $D_1 \cup B \cup D_2$, so that they are disjoint. This gives an annulus that is cobounded by η_1 and η_2, namely $D_0 \cup B$.

Thus we have isotoped ϕ_A into S^3. By Theorem 2.8 [**Liv82**, Theorem 4.2] there is only one isotopy class of embedding of an annulus into S^3 which sends the boundary components to the unlink. We see that ϕ_A and ϕ_{A_0} are smoothly isotopic through proper embeddings. Then we can apply Theorem 2.6 to conclude our proof. $\square$

CHAPTER 3

Applications of annulus modifications

1. An example of a slice knot with non–slice derivatives

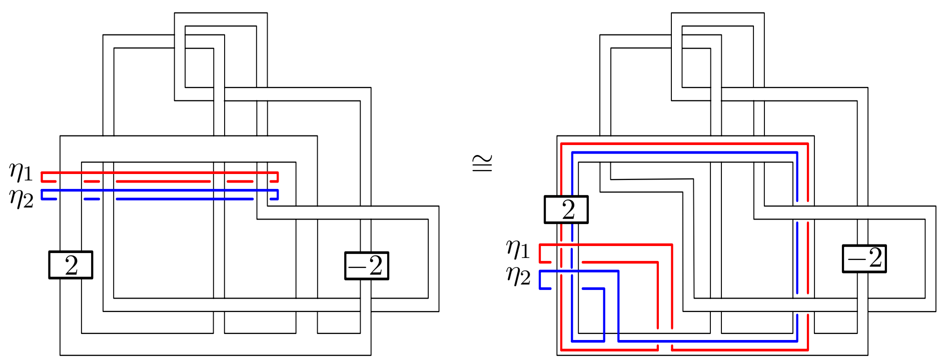

FIGURE 3.1. The knot R and the two component unlink η_1, and η_2 ($\cong$ means isotopic)

Let R be the knot shown in Figure 3.1 and let η_1, η_2 be the 2-component unlink in the complement of R. Note that the core of a second band is a derivative of R and in fact it is the Stevedore's knot, which implies that R is smoothly slice. Figure 3.2 describes a slice disk D^2 for R together with an annulus A, as we now describe in detail. Black curves on the last picture in Figure 3.2 is the $(2,0)$-cable of Stevedore's knot which is concordant to $(2,0)$-cable of the unknot which means it is a slice link. Hence we have three component slice link on the last picture of Figure 3.2 and we have slice disks for this link. With these disks black curves in Figure 3.2 describe slice disk D^2 for R and red curves in Figure 3.2 describe ϕ_A such that $\mathrm{Im}(\phi_A) = A$. This completes the description of the disk D^2 and the annulus A.

It is easy to see that ϕ_A is 0-nice, since there was no intersection between black curves and red curves in Figure 3.2. Further, ϕ_A has one critical point of index zero and one critical point of index one since there was only one band sum between η_1 and η_2. Let R_1 be a knot obtained by 1 surgery on η_1 and -1 surgery on η_2, then by applying Corollary 2.9 we can conclude that R_1 is a smoothly slice knot. Now we show that the knot R_1 is an example of a slice knot with non-slice derivatives. Note that this is an analogue of Proposition 5.2 in [**CD15**].

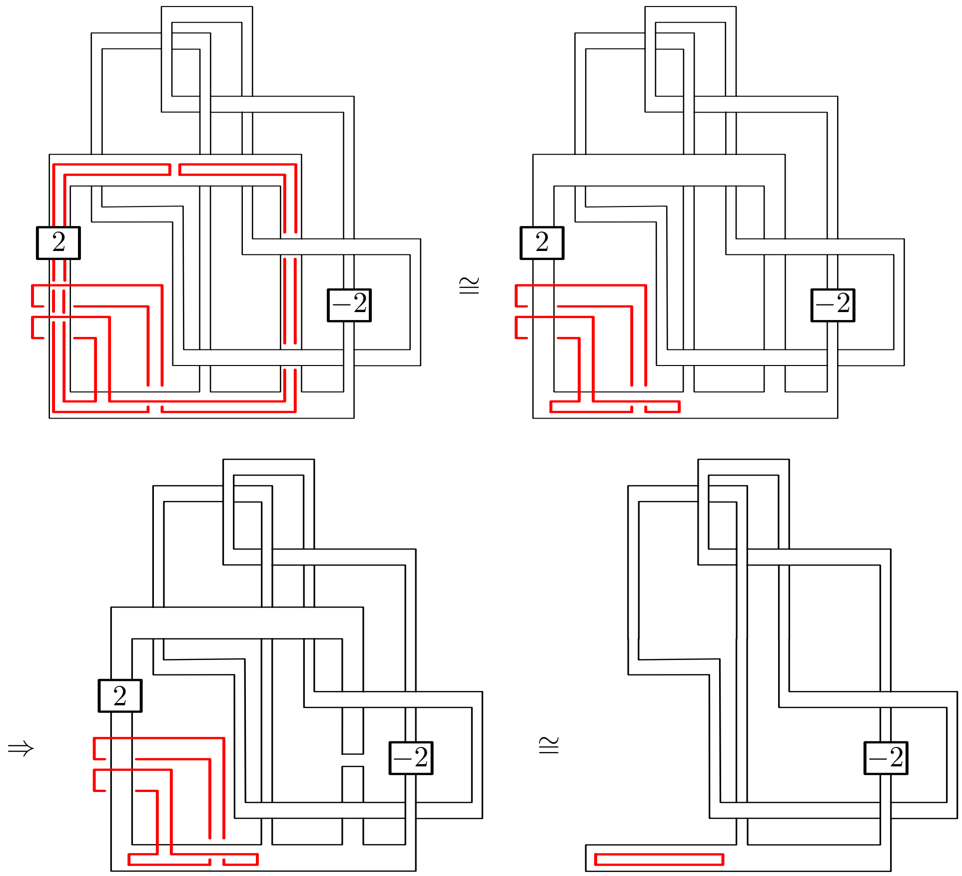

FIGURE 3.2. The first figure is obtained from the right side of Figure 3.1 by adding a band between η_1 and $-\eta_2$. The third figure is obtained by performing a ribbon move on R ($\cong$ means isotopic)

THEOREM 3.1. *Let R_1 be the knot described as above. Then R_1 is a smoothly slice knot with non-slice derivatives on a Seifert surface.*

PROOF. By Corollary 2.9, R_1 is a smoothly slice, so it is enough to show that R_1 has non-slice derivatives.

Let F be the Seifert surface of R described in Figure 3.3. Let x_1 and x_2 be the cores of the bands of F. Then $\{[x_1], [x_2]\}$ is a basis for $H_1(F)$ and the Seifert matrix with respect to $\{[x_1], [x_2]\}$ is $M = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$, where $M = (m_{i,j}) = \mathrm{lk}(x_i, {x_j}^+)$ and ${x_j}^+$ is push off of x_j in positive direction.

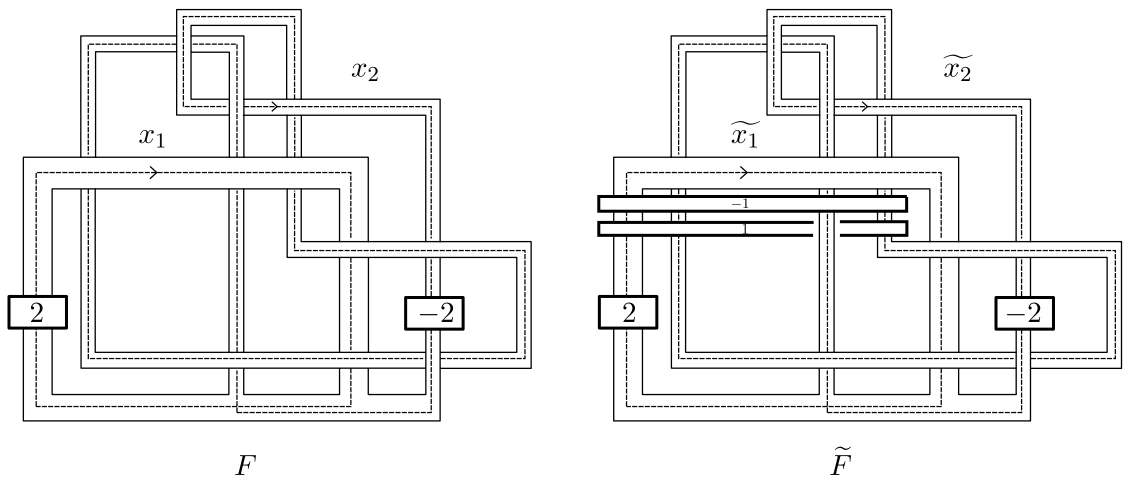

FIGURE 3.3. Seifert surface F for R and Seifert surface $\widetilde{F}$ for R_1

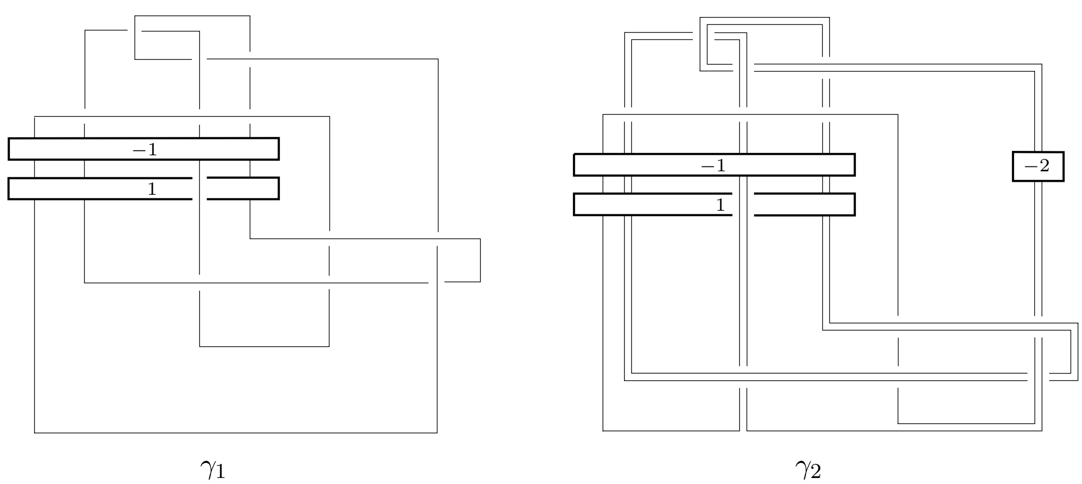

FIGURE 3.4. Non-slice derivatives of R_1

Let $\widetilde{F}$ be the Seifert surface for R_1 obtained from F by doing a 1-twist Annulus modification on ϕ_A (see Figure 3.3). Let $\widetilde{x_1}$ and $\widetilde{x_2}$ be the cores of bands of $\widetilde{F}$. Then $\{[\widetilde{x_1}], [\widetilde{x_2}]\}$

is a basis for $H_1(\widetilde{F})$. The Seifert matrix with respect to $\{[\widetilde{x_1}], [\widetilde{x_2}]\}$ is $\widetilde{M} = \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix}$, where $\widetilde{M} = (\widetilde{m_{i,j}}) = \mathrm{lk}(\widetilde{x}_i, \widetilde{x}_j{}^+)$ and $\widetilde{x}_j{}^+$ is push off of $\widetilde{x}_j$ in positive direction. This implies that the derivative curves for R_1 are γ_1 and γ_2 where $[\gamma_1] = [\widetilde{x_1}] + [\widetilde{x_2}]$ and $[\gamma_2] = [\widetilde{x_1}] - 2[\widetilde{x_2}]$, shown in Figure 3.4.

We calculate the Alexander polynomial for each derivative curve, $\Delta_{\gamma_1}(t) = 4 - 9t + 4t^2$, and $\Delta_{\gamma_2}(t) = 1 + t + 3t^2 - 11t^3 + 3t^4 + t^5 + t^6$. This implies that γ_1 and γ_2 are not smoothly slice knot since $\Delta_{\gamma_1}(-1) = \Delta_{\gamma_2}(-1) = 17$ which is not a square of an odd prime. Note also that this implies that the curves γ_1 and γ_2 are not even algebraically slice. □

2. Annulus modifications and annulus twists

We will first recall the definition of an oriented annulus presentation of a knot which was first introduced by Osoinach in [**Oso06**]. Detailed discussion could be found in [**AJOT13**] or [**AT16**].

Let $\phi_B : S^1 \times [0,1] \hookrightarrow S^3$ be a smooth embedding of an annulus with $\mathrm{Im}(\phi_B) = B$, and let c be a ε framed unknot in S^3 where $\varepsilon \in \{+1, -1\}$. These are described in the Figure 3.5. Let $\phi_b : [0,1] \times [0,1] \hookrightarrow S^3$ be a smooth embedding of a band with following properties:

- $\mathrm{Im}(\phi_b) = b$
- $b \cap \partial B = \mathrm{Im}(\phi_b|_{\partial[0,1]\times[0,1]})$
- $b \cap \mathring{B}$ only has ribbon singularities.
- $b \cap c = \emptyset$
- $B \cup b$ is orientable.

Then we say a knot K admits an oriented annulus presentation (B, b, ε) if $(\partial B - b \cap \partial B) \cup \mathrm{Im}(\phi_b|_{[0,1]\times\partial[0,1]})$ is isotopic to K after ε Dehn surgery on c. The right side of Figure 3.5 shows that the 8_{20} knot admits an oriented annulus presentation.

Suppose a knot K has an annulus presentation (B, b, ε) and let B' be an annulus which is obtained by slightly pushing B into the interior of B, which is described in the Figure 3.6. Let $\partial B' = \eta_1 \cup -\eta_2$, and n be some integer, then $\frac{-n\varepsilon+1}{n}$ Dehn surgery on η_1 and $\frac{-n\varepsilon-1}{n}$

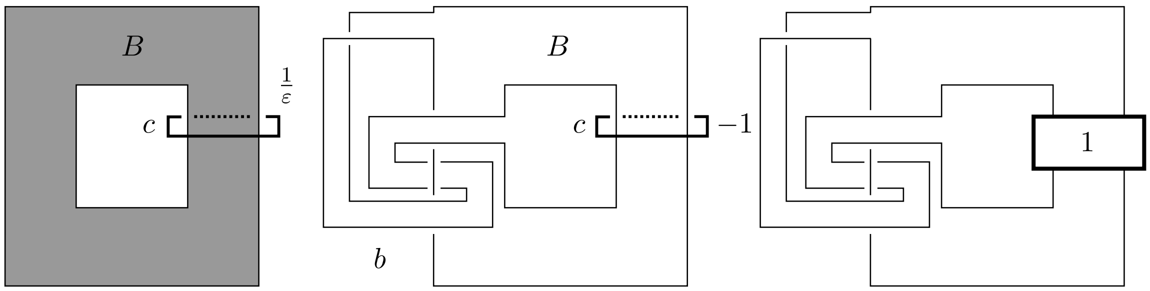

FIGURE 3.5. The 8_{20} knot admits an oriented annulus presentation $(B, b, -1)$

Dehn surgery on η_2 is called n-fold annulus twist on K defined by Osoinach in [**Oso06**] and the resulting knot will be denoted as $K_{(B,b,\varepsilon),n}$. Note $\text{lk}(\eta_1, \eta_2) = -\varepsilon$.

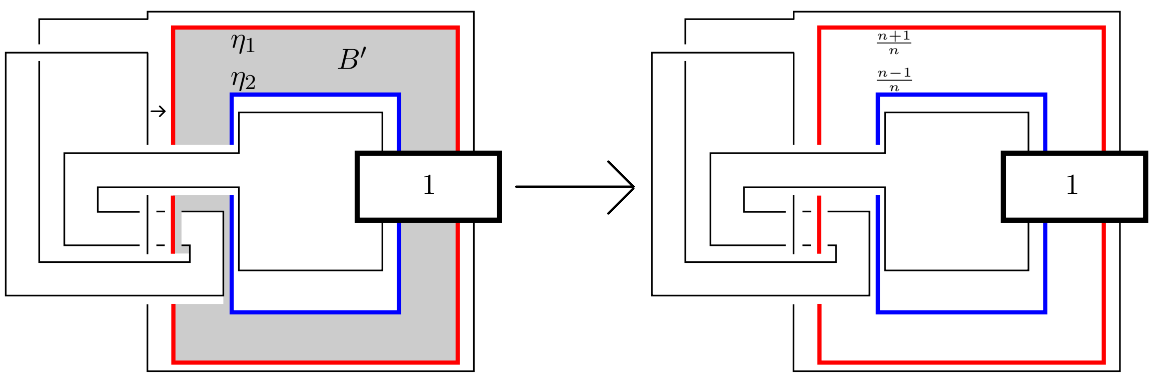

FIGURE 3.6. B' on the left, and n-fold annulus twist on B' on the right for the 8_{20} knot

In [**Oso06**, Thm 2.3], Osoinach showed that 0 framed Dehn surgery on K is homeomorphic to 0 framed Dehn surgery on $K_{(B,b,\varepsilon),n}$ for any integer n. In particular, by [**CFHH13**, Proposition 1.2], if K is smoothly slice then $K_{(B,b,\varepsilon),n}$ is exotically slice, which was also observed in [**AJOT13**]. We will reprove this statement, with slightly stronger assumptions, using annulus modifications instead of [**Oso06**, Thm 2.3].

PROPOSITION 3.2. *Let K be a ribbon knot with the annulus presentation (B, b, ε). Then $K_{(B,b,\varepsilon),n}$ is exotically slice.*

PROOF. We will use B', η_1, and η_2, as described above and in Figure 3.6. Note that by Theorem 2.3 it will be enough to show that there exists a smooth proper embedding

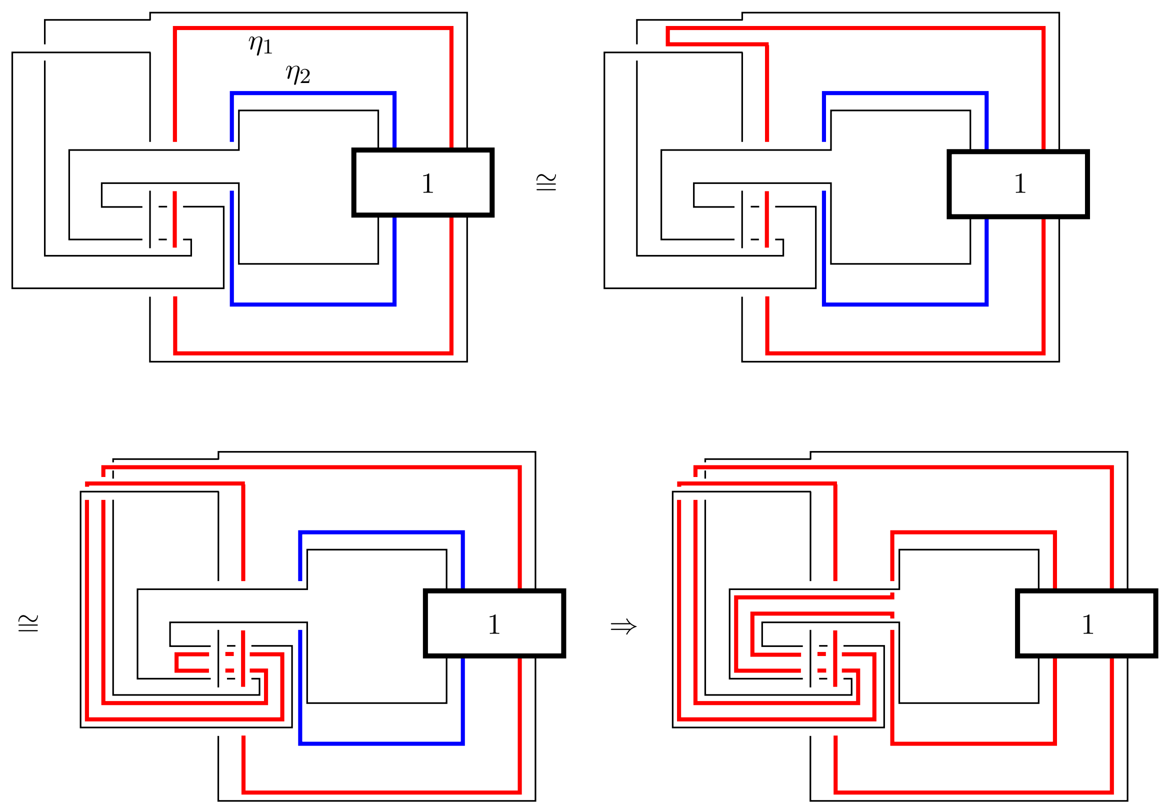

Figure 3.7. Adding a band between η_1 and η_2. The band is following the band b.

$\phi_A : S^1 \times [0,1] \hookrightarrow B^4$ such that $\text{Im}(\phi_A|_{S^1\times\{0\}}) = \eta_1$, $\text{Im}(\phi_A|_{S^1\times\{1\}}) = \eta_2$ and ϕ_A is $(-\varepsilon)$-nice. We can find ϕ_A by performing a band sum as in Figure 3.7.

The resulting link after performing a band sum is the $(2,0)$ cable of K, which is a slice link, so we can cap off each component. Then we obtain a slice disk for K and a smooth proper embedding of an annulus ϕ_A which is disjoint from the slice disk. This guarantees the condition (1) from the Definition 2.1.

Further, we assumed that K is a ribbon knot. Then by construction of the annulus A we know that A does not contain any local maxima. This implies we have a surjective map induced by inclusion: $i_* : \pi_1(S^3 - N(\eta_1 \cup \eta_2)) \twoheadrightarrow \pi_1(B^4 - N(A))$. Since $\pi_1(S^3 - N(\eta_1 \cup \eta_2)) = \mathbb{Z}^2$, $\pi_1(B^4 - N(A))$ is an abelian group. Hence $\pi_1(B^4 - N(A)) = H_1(B^4 - N(A)) = \mathbb{Z}$ which is generated by $[\mu_1] \in H_1(B^4 - N(A))$, where μ_1 is a meridian of η_1. Then it is easy to check $[c]$ represents trivial element in $\pi_1(B^4 - A)$ hence ϕ_A satisfies condition (2) from the Definition 2.1. This implies that ϕ_A is $(-\varepsilon)$-nice and concludes the proof. □

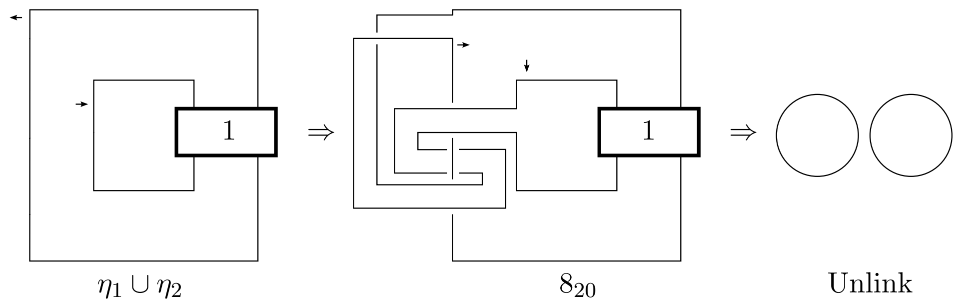

FIGURE 3.8. The second figure is obtained by adding a band between η_1 and η_2 as in Figure 3.7. The third figure is obtained by doing a ribbon move on 8_{20} as in Figure 3.10. Arrows indicate where the band comes out and get attached to the other component.

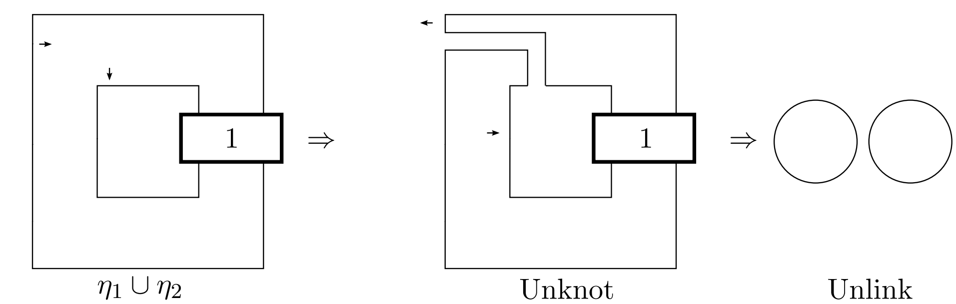

FIGURE 3.9. The second figure is obtained by adding a band between η_1 and η_2 as in Figure 3.10. The third figure is obtained by doing a ribbon move on the unknot as in Figure 3.7. Arrows indicate where the band comes out and get attached to the other component. Note that these two annuli described in Figure 3.8 and 3.9 are isotopic.

Further, in [**AT16**], Abe and Tange showed that if K is a ribbon knot admitting an annulus presentation (B, b, ε) where ε is 1 or -1, then $K_{(B,b,\varepsilon),n}$ is smoothly slice for any integer n. We will reprove this in a very specific case, namely when K is 8_{20} knot, using annulus modifications.

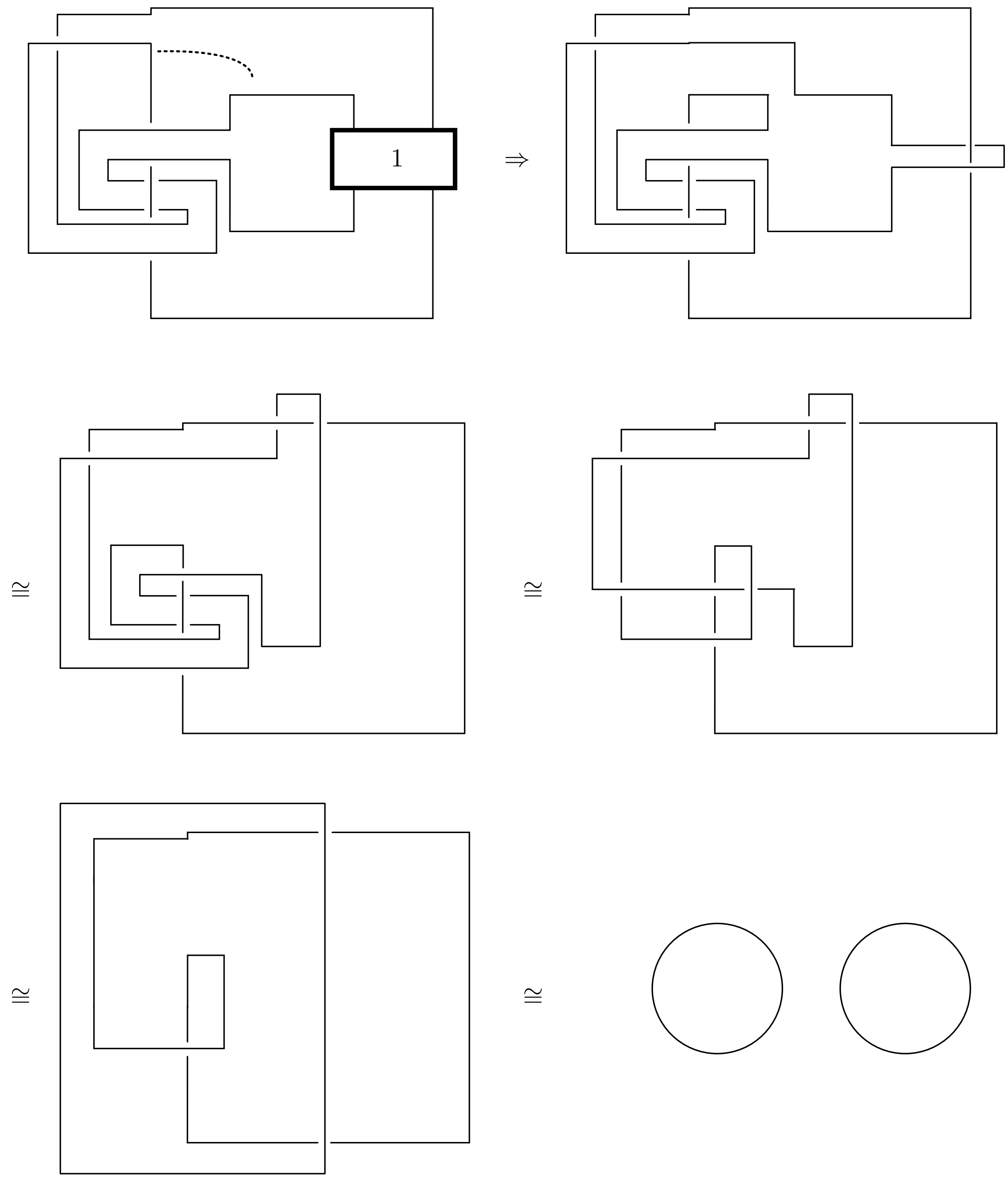

FIGURE 3.10. Performing a ribbon move on 8_{20} gives a two component unlink. The second figure is obtained by performing a ribbon move along the dotted line. The rest of the figures are obtained by isotopies.

PROPOSITION 3.3. *Let K be the 8_{20} knot with annulus presentation $(B, b, -1)$ as in Figure 3.5. Then $K_{(B,b,-1),n}$ is smoothly slice for any integer n.*

Proof. Let ϕ_A be the smooth proper embedding of an annulus described in the proof of Proposition 3.2. By Theorem 2.6, it is enough to show that ϕ_A is 1-standard. In other words it is enough to show that ϕ_A is isotopic to ϕ_{A_1} which was described in Chapter 2.2. Note that we can find a ribbon disk for K by attaching a band as in Figure 3.10.

The embedding ϕ_A is isotopic to the embedding defined by the switching the order of the two band sums in the movie description of the annulus, as shown in Figure 3.9. Note that the link becomes the unknot after the first band sum (see Figure 3.9). Using Scharlemann's corollary in [**Sch85**, Corollary page 127] we can isotope what remains of the annulus, which is a ribbon disk with two local minima for the unknot that arises after the first band sum, to be a standard disk. This implies that ϕ_A is isotopic to ϕ_{A_1} as needed. □

3. Annulus modifications on general annuli

In this section, we will consider more general annuli that we can apply annulus modifications on. We will restrict our attention to the topological category, since we are considering general annuli. More precisely, we will perform an n-twist annulus modification to a smooth compact 4-manifold M where M is homeomorphic to B^4, but not necessarily diffeomorphic to the standard B^4. When the resulting manifold after an n-twist annulus modification is homeomorphic to B^4, the resulting knot will be exotically slice but not necessarily smoothly slice. We restate the Theorem 2.3 for the general case.

Theorem 3.4. *Let K be an exotically slice knot, bounding a smoothly embedded 2-disk D^2 in M where M is homeomorphic to B^4. Let $\eta_1 \cup \eta_2$ be an oriented link in $S^3 - K$ and let $\phi_A : S^1 \times [0,1] \hookrightarrow M$ be a smooth proper embedding of an annulus with $\mathrm{Im}(\phi_A|_{S^1\times\{0\}}) = \eta_1$, $\mathrm{Im}(\phi_A|_{S^1\times\{1\}}) = \eta_2$, $\mathrm{Im}(\phi_A) = A$, $\ell = \mathrm{lk}(\eta_1, \eta_2)$, and $A \cap D^2 = \emptyset$. Suppose $M_{(\phi_A,n)}$, the n-twist annulus modification on M at ϕ_A, is homeomorphic to B^4 for an integer n. Then $\frac{n\ell+1}{n}$ Dehn surgery on η_1 followed by $\frac{n\ell-1}{n}$ Dehn surgery on η_2 will produce an exotically slice knot $K_{(\phi_A,n)} \subseteq S^3$.*

It is a natural question to ask if there exists a smooth proper embedding of an annulus $\phi_A : S^1 \times [0,1] \hookrightarrow M$ where $M_{(\phi_A,n)}$ is homeomorphic to B^4 for non-zero n, while $\eta_1 \cup \eta_2$ is not isotopic to L_ℓ (see Figure 2.2). The following proposition gives us plenty of examples of such smooth proper embedding of an annuli.

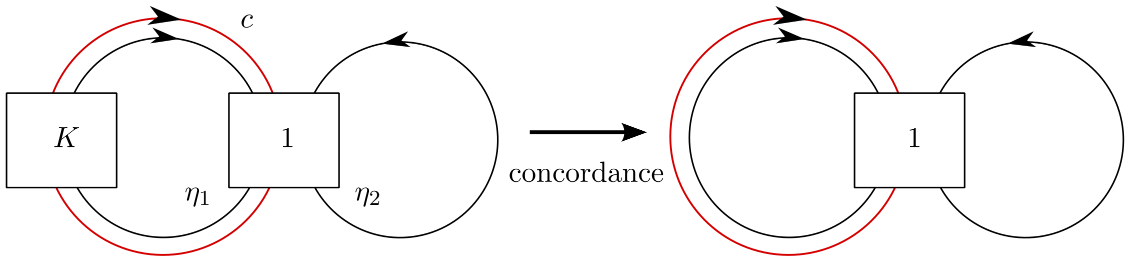

FIGURE 3.11. Positive Hopf link with a knot K tied up in the first component.

PROPOSITION 3.5. *Let $\eta_1 \cup \eta_2$ be the positive Hopf link with a knot K tied up in the first component η_1 (see Figure 3.11). If K is exotically slice in M, then there exists a smooth proper embedding of an annulus $\phi_A : S^1 \times [0,1] \hookrightarrow M$ with $\mathrm{Im}(\phi_A|_{S^1\times\{0\}}) = \eta_1$, $\mathrm{Im}(\phi_A|_{S^1\times\{1\}}) = \eta_2$ such that $M_{(\phi_A,1)}$ is homeomorphic to B^4.*

PROOF. Since η_1 is exotically slice in M, $\eta_1 \cup \eta_2$ is concordant to a positive Hopf link in M minus a B^4. Hence we can simply cap off the concordance with an annulus that positive Hopf link bounds in S^3, to achieve a smooth proper embedding of an annulus $\phi_A : S^1 \times [0,1] \hookrightarrow M$ with $\mathrm{Im}(\phi_A|_{S^1\times\{0\}}) = \eta_1$, $\mathrm{Im}(\phi_A|_{S^1\times\{1\}}) = \eta_2$. Now we need to show that $M_{(\phi_A,1)}$ is homeomorphic to B^4.

It is easy to check that $M_{(\phi_A,1)}$ is a homology B^4 by using a Mayer-Vietoris sequence. In order to see that $M_{(\phi_A,1)}$ is simply connected we can use the same argument from Remark 2.5. By using the concordance described in Figure 3.11, the curve c from Figure 3.11 bounds a smoothly embedded disk in $M - N(A)$ for the same reason from Remark 2.5 (see Figure 2.5). Therefore, $[c]$ represents the trivial element in $\pi_1(M - N(A))$ which implies that $\pi_1(M_{(\phi_A,1)}) = \{\mathrm{id}\}$ by applying the same argument from the proof of Theorem 2.3.

Lastly, we see that on the boundary we have 2–Dehn surgery on η_1 followed by 0–Dehn surgery on η_2. Then we can use η_2 as a helper circle to undo crossings of η_1 to get a positive Hopf link with coefficient 2 on one component and 0 on the other component. Hence we have that $\partial M_{(\phi_A,1)}$ is S^3. Then again by the 4-dimensional topological Poincaré conjecture we can conclude that $M_{(\phi_A,1)}$ is homeomorphic to B^4 [**Fre84**, Theorem 1.6]. $\square$

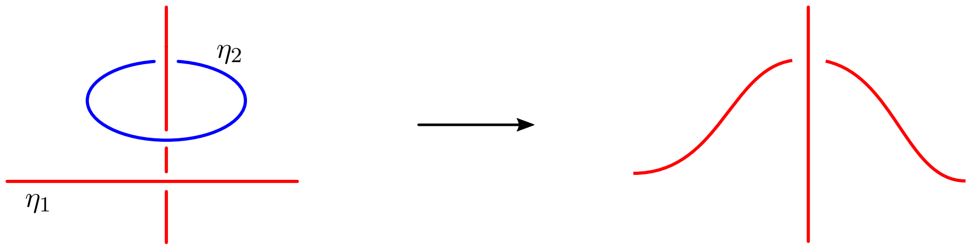

FIGURE 3.12. Crossing change on η_1 obtained by performing a band sum between η_1 and η_2. The second figure is obtained by adding a band between η_1 and η_2.

Note that for a link $\eta_1 \cup \eta_2$ isotopic to the negative Hopf link with a knot K tied up in the first component η_1, we can also apply the same argument if K is exotically slice in M. The only difference is that the linking number of η_1 and η_2 is negative one. Hence we have a smooth proper embedding of an annulus $\phi_A : S^1 \times [0,1] \hookrightarrow M$ with $\mathrm{Im}(\phi_A|_{S^1\times\{0\}}) = \eta_1$ and $\mathrm{Im}(\phi_A|_{S^1\times\{1\}}) = \eta_2$, where $M_{(\phi_A,-1)}$ is homeomorphic to B^4. In fact there are more examples of such links. For instance, let $\eta_1 \cup \eta_2$ be the Hopf link with a knot K tied up in the first component η_1 and suppose that K is an unknotting number one knot. Note that we can obtain the unknot from $\eta_1 \cup \eta_2$ by performing a band sum between η_1 and η_2 (see Figure 3.12). Then by the similar argument from the proof of Proposition 3.5, it is easy to see that there exists a smooth proper embedding of an annulus $\phi_A : S^1 \times [0,1] \hookrightarrow M$ with $\mathrm{Im}(\phi_A|_{S^1\times\{0\}}) = \eta_1$ and $\mathrm{Im}(\phi_A|_{S^1\times\{1\}}) = \eta_2$, where either $M_{(\phi_A,1)}$ or $M_{(\phi_A,-1)}$ is homeomorphic to B^4. By using these general annuli we have the following theorem, which tells us that any exotically slice knot can be obtained by the image of the unknot in the boundary of a smooth 4-manifold homeomorphic to B^4 after an annulus modification from the unknot.

THEOREM 3.6. *Given an exotically slice knot K with an exotic 4-ball M and an exotic slice disk D, there exists an exotic 4-ball M' and there exists an exotic disk D' in M' for the unknot in $\partial M' = S^3$, such that (M, D) arises from (M', D') via a -1–twist annulus modification on some annulus $A' \subseteq M' - D'$.*

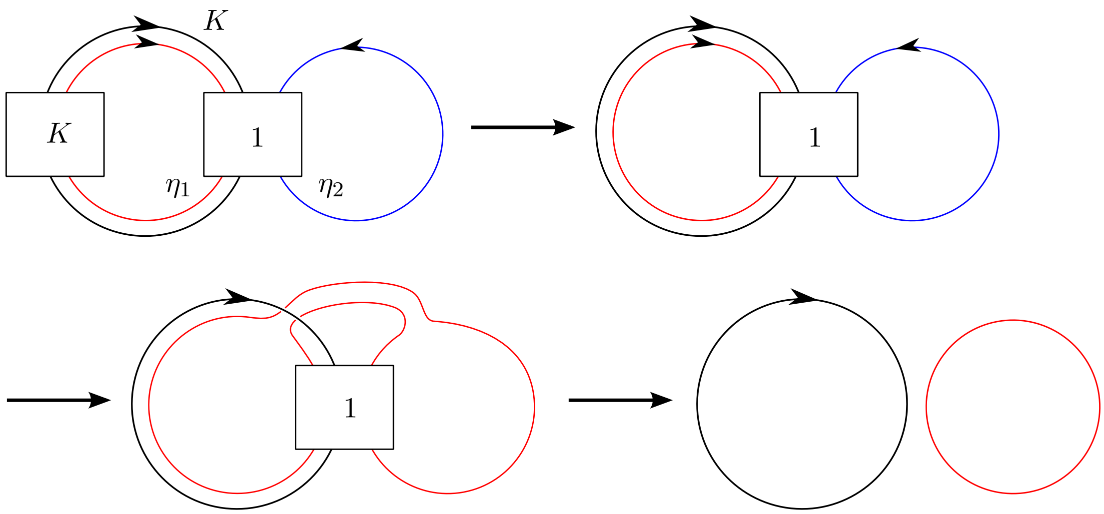

FIGURE 3.13. A movie picture of a smoothly embedded 2-disk D and a smooth proper embedding of an annulus ϕ_A. The second figure is obtained by a concordance in exotic $B^3 \times [0,1]$. The third figure is obtained by adding a band between the red curve and the blue curve. The fourth figure is obtained by isotopy of the red curve.

PROOF. Note that it is enough to show that there exist a smoothly embedded 2-disk D^2 in M which bounds K and a smooth proper embedding of an annulus $\phi_A : S^1 \times [0,1] \hookrightarrow M$ with $\text{Im}(\phi_A|_{S^1\times\{0\}}) = \eta_1$, $\text{Im}(\phi_A|_{S^1\times\{1\}}) = \eta_2$, $\text{Im}(\phi_A) = A$, $A \cap D^2 = \emptyset$ and $1 = \text{lk}(\eta_1, \eta_2)$, so that $M_{(\phi_A,1)}$ is homeomorphic to B^4 and $K_{(\phi_A,1)}$ is the unknot, since we can simply perform a -1-twist annulus modification on $M_{(\phi_A,1)}$ with A', the image of A under the modification. Let D^2 be a slice disk for K and $\phi_A : S^1 \times [0,1] \hookrightarrow M$ be the annulus described in Figure 3.13. Then $M_{(\phi_A,1)}$ is homeomorphic to B^4, since ϕ_A is the same annulus as the one used in the proof of Proposition 3.5. In addition by performing handle slides, isotopies and Rolfsen twists, we see that $K_{(\phi_A,1)}$ is the unknot as needed (see Figure 3.14). □

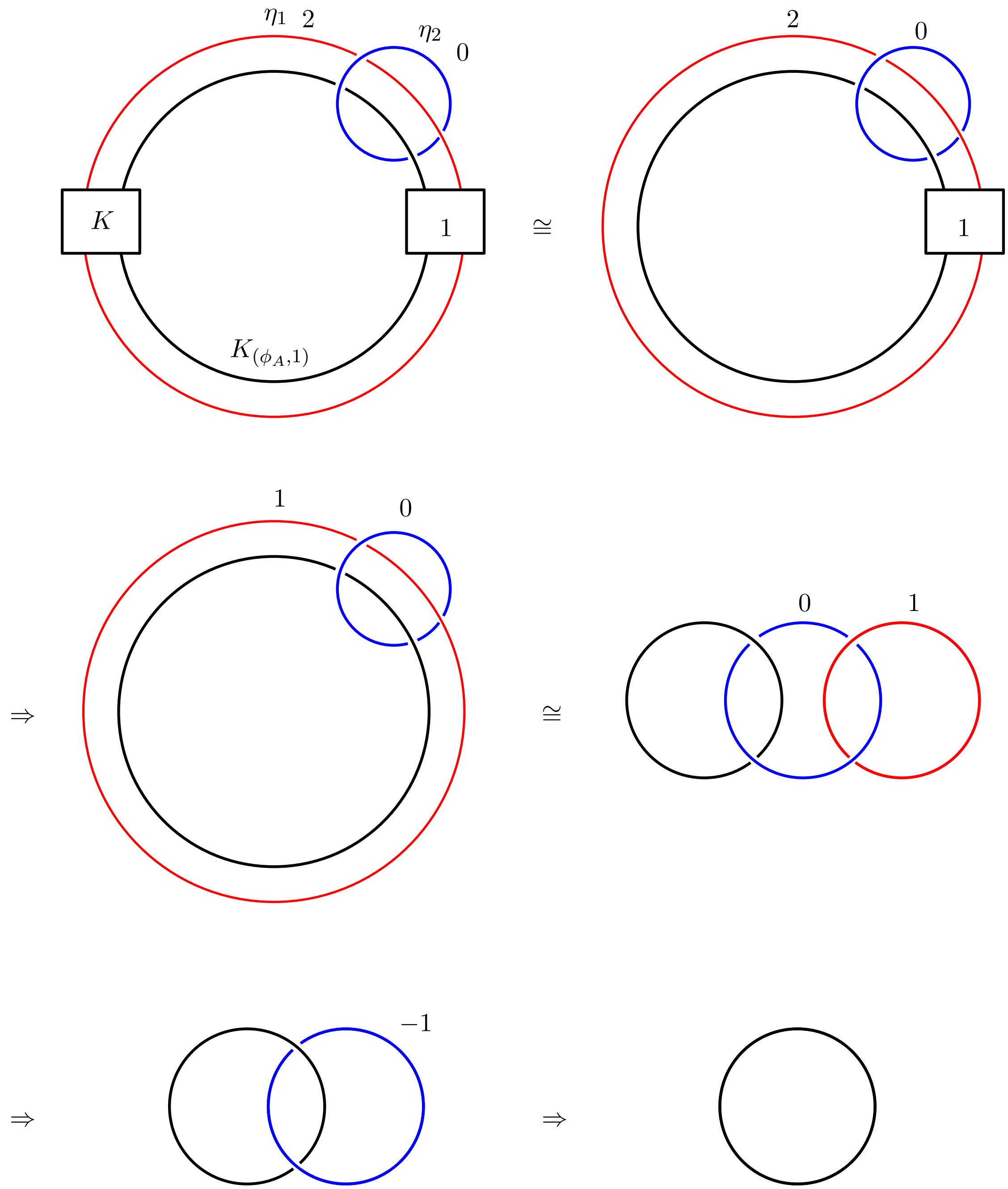

FIGURE 3.14. The first figure describes $K_{(\phi_A,1)}$. The second figure is obtained by performing handle slides. The rest of the figures are obtained by performing Rolfsen twists and isotopies.

CHAPTER 4

Milnor's triple linking number and string link infections

1. Band forms and Seifert matrices

Let K be an algebraically slice knot and let F be its Seifert surface. Then we have a Seifert form $\beta_F : H_1(F) \times H_1(F) \to \mathbb{Z}$. Since K is an algebraically slice knot, there exists a metabolizer $H = \mathbb{Z}^{\frac{1}{2}\operatorname{rank} H_1(F)}$, a direct summand of $H_1(F)$, such that β_F vanishes on H. Let $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{\frac{1}{2}\operatorname{rank} H_1(F)}$ be a derivative of K associated with H. For each i, let $b_i = [\gamma_i]$ in $H_1(F)$, and extend $\{b_1, b_2, \cdots, b_{\frac{1}{2}\operatorname{rank} H_1(F)}\}$ to a symplectic basis $\{a_1, \cdots, a_{\frac{1}{2}\operatorname{rank} H_1(F)}, b_1, \cdots, b_{\frac{1}{2}\operatorname{rank} H_1(F)}\}$ where a_i is an intersection dual of b_i for each i. This gives us a disk band form for the Seifert surface F as in Figure 4.1. From this we get a Seifert Matrix

$$M = \begin{pmatrix} B & A \\ A - \mathrm{Id} & 0 \end{pmatrix}$$

for K with respect to a basis $\{a_1, \cdots, a_{\frac{1}{2}\operatorname{rank} H_1(F)}, b_1, \cdots, b_{\frac{1}{2}\operatorname{rank} H_1(F)}\}$, where A and B are $\frac{1}{2}\operatorname{rank} H_1(F)$ by $\frac{1}{2}\operatorname{rank} H_1(F)$ integer matrices and Id is a $\frac{1}{2}\operatorname{rank} H_1(F)$ by $\frac{1}{2}\operatorname{rank} H_1(F)$ identity matrix.

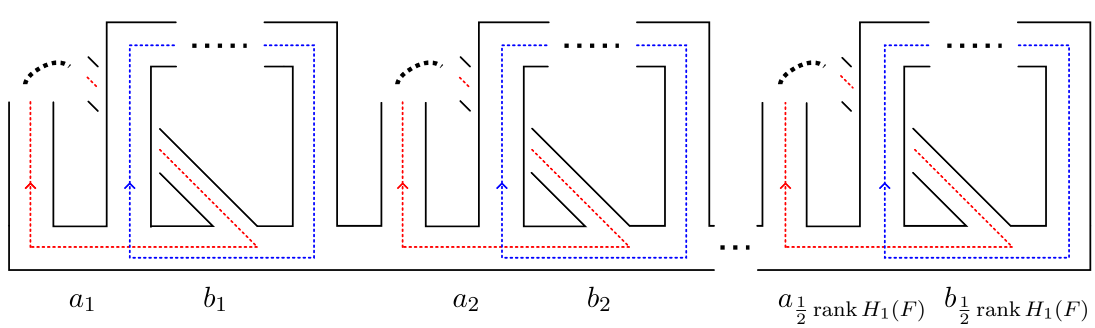

FIGURE 4.1. Disk-band form of a Seifert surface F for K

2. Milnor's triple linking number

In this section, we will recall some definitions and properties of Milnor's triple linking number (for more general setting and detailed discussion see [**Mil54**], [**Mil57**], [**Orr89**]). We will focus on 3-component links $L = L_1 \cup L_2 \cup L_3$ with $\operatorname{lk}(L_i, L_j) = 0$ for $i, j \in \{1, 2, 3\}$ where $i \neq j$. Let μ_1, μ_2 and μ_3 be meridians of L_1, L_2 and L_3 respectively and let λ_1, λ_2 and λ_3 be longitudes of L_1, L_2 and L_3 respectively. Let $\pi = \pi_1(S - L)$ and $\mathfrak{F}$ be a free group generated by x_1, x_2 and x_3. For a group G, the nth lower central series of G is denoted as G_n and defined inductively by $G_1 = G$ and $G_i = [G, G_{i-1}]$ for $i > 1$. Note that since $\operatorname{lk}(L_i, L_j) = 0$ for $i, j \in \{1, 2, 3\}$ where $i \neq j$, $[\lambda_i] \in \pi_2$ for all $i \in \{1, 2, 3\}$. Hence by [**Mil57**] we have an isomorphism $\mathfrak{F}/\mathfrak{F}_3 \to \pi/\pi_3$, which is induced by $\overset{3}{\vee} S^1 \to S^3 - L$ where each of the circle from the wedge gets map to each of the meridian of the link L, *i.e.* x_1 is identified with $[\mu_1]$, x_2 is identified with $[\mu_2]$, and x_3 is identified with $[\mu_3]$. Hence from now on we will identify $\mathfrak{F}/\mathfrak{F}_3$ with π/π_3.

Recall that the Magnus representation $\phi : \mathfrak{F} \to P$, where P is the power series ring with non-commutative variable a_1, a_2 and a_3, is defined by $\phi(x_i) = 1 + a_i$ and $\phi(x_i^{-1}) = 1 - a_i + a_i^2 - a_i^3 + \cdots$ for $i \in \{1, 2, 3\}$. Magnus proved that for any $y \in \mathfrak{F}$, $y \in \mathfrak{F}_k$ if and only if $\phi(y) = 1 + \sum c_i \cdot w_i$ where c_i is an integer, w_i is a word in a_1, a_2 and a_3, and length of w_i is greater than equal to k for all i (see [**MKS04**]).

We now define Milnor's triple linking number $\bar{\mu}_L(123)$ of a link L as the coefficient of $a_1 \cdot a_2$ in $\phi([\lambda_3]) \in P$. Notice that by Magnus's work, $[\lambda_3] \in \pi_3$ if and only if $\bar{\mu}_L(123) = 0$. It is also known that $\mathfrak{F}_2/\mathfrak{F}_3 \simeq \pi_2/\pi_3$ is a finitely generated free abelian group with a basis $\{[x_1, x_2], [x_1, x_3], [x_2, x_3]\}$ (see [**Hal50**]). Then it is not hard to see that $\bar{\mu}_L(123)$ is equal to n_1 where $p([\lambda_3]) = [x_1, x_2]^{n_1} \cdot [x_1, x_3]^{n_2} \cdot [x_2, x_3]^{n_3} \in \mathfrak{F}_2/\mathfrak{F}_3 = \pi_2/\pi_3$ where $p : \pi_2 \to \pi_2/\pi_3$ is a projection map.

We will recall some properties of lower central series, which helps to compute $\bar{\mu}_L(123)$.

Proposition 4.1 ([**MKS04**]). *Let a, b, c be elements of a group G. Then*

(1) $[G_n, G_m] \subseteq G_{n+m}$ *for any positive integers n and m*

(2) $[a, b \cdot c] = [a, b] \cdot [a, c] \pmod{G_3}$

(3) $[a \cdot b, c] = [a, c] \cdot [b, c] \pmod{G_3}$

From Proposition 4.1 (1) we have the following proposition.

Proposition 4.2 ([**Coc90**]). *Let γ be a loop in $S^3 - L$ and let γ_1 and γ_2 be basings of γ. If either $[\gamma_1]$ or $[\gamma_2]$ is in π_2, then $[\gamma_1] = [\gamma_2]$ in $\pi_2/\pi_3 = \mathfrak{F}_2/\mathfrak{F}_3$.*

Proof. For some $g \in \pi$ we have $\gamma_1 = g \cdot \gamma_2 \cdot g^{-1} = g \cdot \gamma_2 \cdot g^{-1} \cdot \gamma_2^{-1} \cdot \gamma_2 = [g, \gamma_2] \cdot \gamma_2$. We can conclude the statement since $[g, \gamma_2] \in \pi_3$ by Proposition 4.1 (1). □

This tells us that as long as any basing of a loop γ in $S^3 - L$ is contained in π_2 we do not need to specify the basing. Using Proposition 4.1 (2), (3) we have the following immediate proposition.

Proposition 4.3. *Let $\mathfrak{F}$ be a free group generated by x_1, x_2 and x_3 and let w_1, w_2 be elements of $\mathfrak{F}$. For $i \in \{1, 2, 3\}$, let n_i denote the exponent sum of x_i's occurring in w_1, and let m_i denote the exponent sum of x_i's occurring in w_2. Then $[w_1, w_2] = [x_1, x_2]^{(n_1 m_2 - n_2 m_1)} \cdot [x_1, x_3]^{(n_1 m_3 - n_3 m_1)} \cdot [x_2, x_3]^{(n_2 m_3 - n_3 m_2)}$ in $\mathfrak{F}_2/\mathfrak{F}_3$.*

Proof. By Proposition 4.1 (2), (3) we can expand out $[w_1, w_2]$. Then the statement follows immediately. □

There is a nice geometric interpretation of Milnor' triple linking number of a link L introduced by Cochran in [**Coc90**]. Given L as above, suppose F_1, F_2 and F_3 are Seifert surfaces for L_1, L_2 and L_3 respectively. Further, for $i, j \in \{1, 2, 3\}$ assume $F_i \cap L_j = \phi$ when $i \neq j$ and adjust so that triple intersection points are isolated. Then Milnor's triple linking number $\bar{\mu}_L(123)$ is the number of triple intersection points of F_1, F_2 and F_3 counted with signs. The sign of an intersection $p \in F_1 \cap F_2 \cap F_3$ is positive if and only if $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ agrees with standard orientation of S^3 where $\overrightarrow{v_i}$ is a normal vector to F_i at p, for $i \in \{1, 2, 3\}$. This tells us that $\bar{\mu}_L(\sigma(123)) = sign(\sigma) \cdot \bar{\mu}_L(123)$ for $\sigma \in S_3$ and changing the orientation of a component of L changes the sign of Milnor' triple linking number.

3. Infection by string links

In this section, we will recall definitions of a string link and a string link infection. The following definitions are from [**JKP14**].

Definition 4.4. (String Link)

(1) An r-multi-disk $\mathbb{E}$ is an oriented disk D^2 with r ordered embedded open disks $E_1, E_2, \cdots, E_r$. We have pairwise disjoint paths $\Gamma_1, \Gamma_2, \cdots, \Gamma_r$ such that $\Gamma_i(0) \in \partial E_i$ and $\Gamma_i(1) \in \partial \mathbb{E}$ (see Figure 4.2).

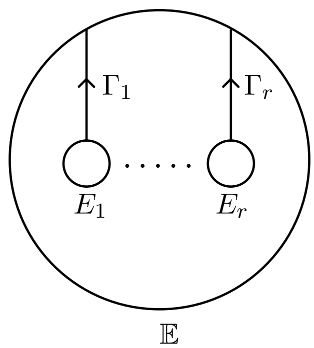

Figure 4.2. An r-multi-disk $\mathbb{E}$

(2) Let p_i be the fixed point in the interior of E_i for each $i = 1, 2, \cdots, r$. An r-component string link is a smooth proper embedding $J : \bigsqcup_{i=1}^{r} p_i \times I \to D^2 \times I$ such that $J(p_i \times \{0\}) = p_i \times \{0\} \in D^2 \times \{0\}$ and $J(p_i \times \{1\}) = p_i \times \{1\} \in D^2 \times \{1\}$ for each $i = 1, 2, \cdots, r$. We denote $\nu(J) : \bigsqcup_{i=1}^{r} p_i \times I \to D^2 \times I$ as a tubular neighborhood of J. We abuse notation and use J and $\nu(J)$ as image of J and image of $\nu(J)$ respectively.

(3) For $i = 1, 2, \cdots, r$, the meridian of the ith component of J is the simple closed curve, up to ambient isotopy, on the boundary of $\nu(J)(E_i \times I)$, which has the linking number 1 with the ith component of J. For $i = 1, 2, \cdots, r$, let $\delta_i : I \to \partial E_i \times I$ be a 0-framed parallel of the ith component such that $\delta_i(0) = \Gamma_i(0) \times \{1\} \in \mathbb{E} \times I$ and $\delta_i(1) = \Gamma_i(0) \times \{1\} \in \mathbb{E} \times I$. Then the longitude l_i of the ith component of J is the concatenation of arcs as follows: $l_i = \delta_i \cup (\Gamma_i \times \{1\}) \cup (\Gamma_i(1) \times I) \cup (-\Gamma_i \times \{0\})$.

The following definitions are also from [**JKP14**].

Definition 4.5. (Infection by a string link) Let L be a link in S^3 and J be an r-component string link in $D^2 \times I$.

(1) Let $\mathbb{E}$ be an r-multi disk, then an embedding $\varphi : \mathbb{E} \to S^3$ is a proper r-multi disk in (S^3, L) if $\varphi(\mathbb{E})$ intersects L only in $\varphi(E_i)$ transversely for $i = 1, 2, \cdots, r$.

(2) Let $\mathbb{E}_\varphi$ be the image of $\mathbb{E}$ under φ and let E_φ be the image of $E_1 \cup E_2 \cup \cdots \cup E_r$ under φ. We define the link $S(L, J, \varphi)$ to be the image of the link L under the following homeomorphism (for detailed diskussion for this homeomorphism see [**CFT09**]):

$$
\begin{aligned}
&(S^3 \setminus (int(\mathbb{E}_\varphi \setminus E_\varphi) \times I)) \cup ((D^2 \times I) \setminus \nu(J)) \\
=&(S^3 \setminus (\mathbb{E}_\varphi \times I)) \cup (((D^2 \times I) \setminus \nu(J)) \cup (\overline{\mathbb{E}_\varphi} \times I)) \\
\cong& S^3.
\end{aligned}
$$

We call the resulting link $S(L, J, \varphi)$ multi-infection of L by J along $\mathbb{E}_\varphi$. In Figure 4.3, we present an example of $S(L, J, \varphi)$ for a particular L, J and $\mathbb{E}_\varphi$.

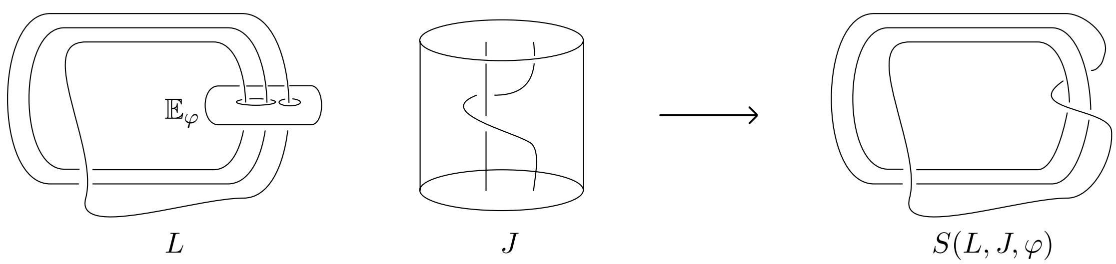

Figure 4.3. An infection by a string link

4. Geometric moves on knots

4.1. Geometric moves on knots and links. In this section, we will recall definitions of a double delta move and a double Borromean rings insertion move and see how they are related. The following definitions are from [**Mar13**].

Definition 4.6. (Double delta move) A double delta move on a knot K is the local move described in Figure 4.4.

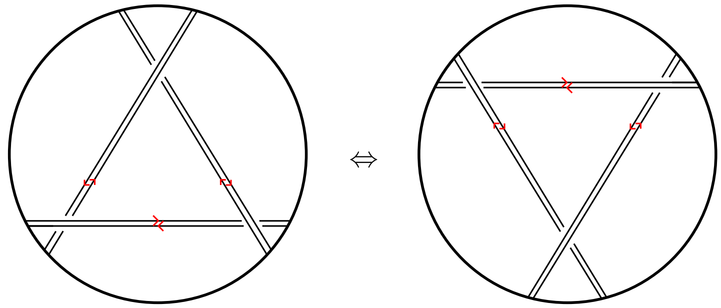

FIGURE 4.4. A double delta move

DEFINITION 4.7. (Double Borromean rings insertion move) A double Borromean rings insertion move on a knot K is the local move described in Figure 4.5.

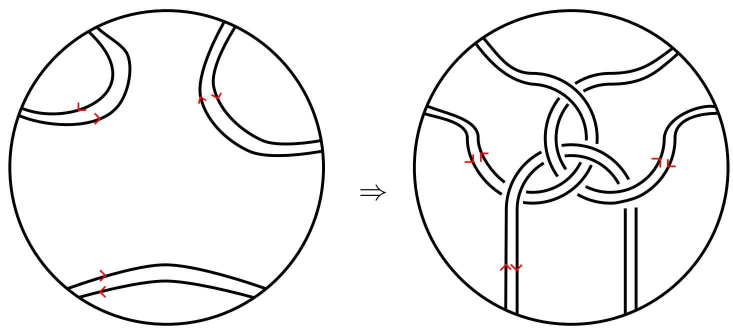

FIGURE 4.5. A double Borromean rings insertion move

REMARK 4.8.

(1) *Note that a double Borromean rings insertion move is a special case of a string link infection* $S(L, J, \varphi)$ *where* $\varphi : \mathbb{E} \to S^3$ *is given as in Figure 4.6, and* $\widehat{J}$ *is a Borromean rings.*

(2) *A double delta move can be achieved by a double Borromean rings insertion move which is explained in Figure 4.7.*

Let K_1 and K_2 be knots with Seifert surfaces F_1 and F_2 respectively. Suppose that the Seifert matrices with respect to F_1 and F_2 coincide. Then Naik and Stanford [**NS03**] proved that it is possible to alter F_1 only by applying double delta moves on the bands of F_1 to become F_2. Then by the Remark 4.8 (2) we can get to F_2 from F_1 only by applying

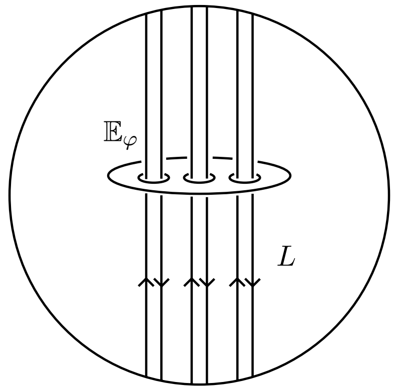

FIGURE 4.6. A double Borromean rings insertion move is a special case of a string link infection.

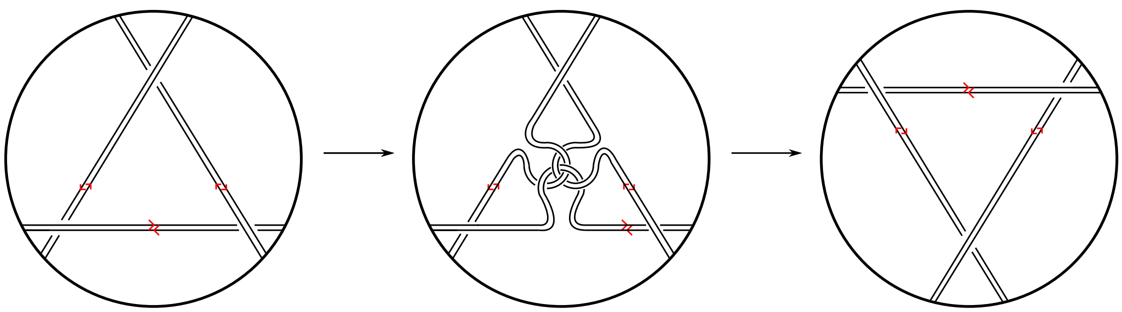

FIGURE 4.7. A double delta move achieved by a double Borromean rings insertion move.

double Borromean rings insertion moves on the band of F_1. We summarize this observation as follows.

PROPOSITION 4.9. *Let K_1 and K_2 be knots with Seifert surfaces F_1 and F_2 respectively. Suppose that the Seifert matrices with respect to F_1 and F_2 coincide. Then F_1 can be altered only by double Borromean rings insertion moves on the bands of F_1 to become F_2.*

5. The effect of string link infection on Milnor's triple linking number

We first recall a simplified version of a lemma from [**JKP14**].

Lemma 4.10 ([**JKP14**, Lemma 4.1]). *Let $L = \gamma_1 \cup \gamma_2 \cup \gamma_3$ be an oriented three component link with pairwise linking number zero, and let J be an oriented three component string link whose closure $\widehat{J}$ has pairwise linking number zero. Let $\varphi : \mathbb{E} \to S^3$ be a proper 3-multi disk in (S^3, L) such that for each $i \in \{1, 2, 3\}$, $\varphi(E_i)$ only intersects L at γ_i. Denote the algebraic intersection number between $\varphi(E_i)$ and γ_j by n_i^j for $i, j \in \{1, 2, 3\}$. Then $S(L, J, \varphi)$ has zero pairwise linking number and*

$$\bar{\mu}_{S(L,J,\varphi)}(123) = \bar{\mu}_{\widehat{J}}(123)(\sum_{\sigma \in S_3} sign(\sigma) n_1^{\sigma(1)} n_2^{\sigma(2)} n_3^{\sigma(3)}) + \bar{\mu}_L(123).$$

We will change the assumptions from Lemma 4.10 slightly for the purpose of this thesis. To be more precise, we will omit the assumption that $\varphi(E_i)$ only intersects L at γ_i for each $i \in \{1, 2, 3\}$. Although the proof from [**JKP14**] goes through with these weaker assumptions, for completeness we will present the proof.

Lemma 4.11. *Let $L = \gamma_1 \cup \gamma_2 \cup \gamma_3$ be an oriented three component link with pairwise linking number zero, and let J be an oriented three component string link whose closure $\widehat{J}$ has pairwise linking number zero. Let $\varphi : \mathbb{E} \to S^3$ be a proper 3-multi disk in (S^3, L). Denote the algebraic intersection number between $\varphi(E_i)$ and γ_j by n_i^j for $i, j \in \{1, 2, 3\}$. Then $S(L, J, \varphi)$ has zero pairwise linking number and*

$$\bar{\mu}_{S(L,J,\varphi)}(123) = \bar{\mu}_{\widehat{J}}(123)(\sum_{\sigma \in S_3} sign(\sigma) n_1^{\sigma(1)} n_2^{\sigma(2)} n_3^{\sigma(3)}) + \bar{\mu}_L(123).$$

Proof. It is straightforward to see that $S(L, J, \varphi)$ has pairwise linking number zero, since $\widehat{J}$ has pairwise linking number zero. Asumme that $\varphi(E_i)$ intersects the link L transversely. Let α_i^j be the number of positive intersections and β_i^j be the number of negative intersections between $\varphi(E_i)$ and γ_j for $i, j \in \{1, 2, 3\}$. Let J' be the oriented string link obtained from J by taking $\alpha_i^1 + \alpha_i^2 + \alpha_i^3$ many parallel copies of the ith component of J with the same orientation and $\beta_i^1 + \beta_i^2 + \beta_i^3$ many parallel copies of the ith component of J with the opposite orientation for $i \in \{1, 2, 3\}$. Then we can consider $S(L, J, \varphi)$ as the result of performing several exterior band sums between L and $\widehat{J'}$, the closure of J' (see Figure 4.8).

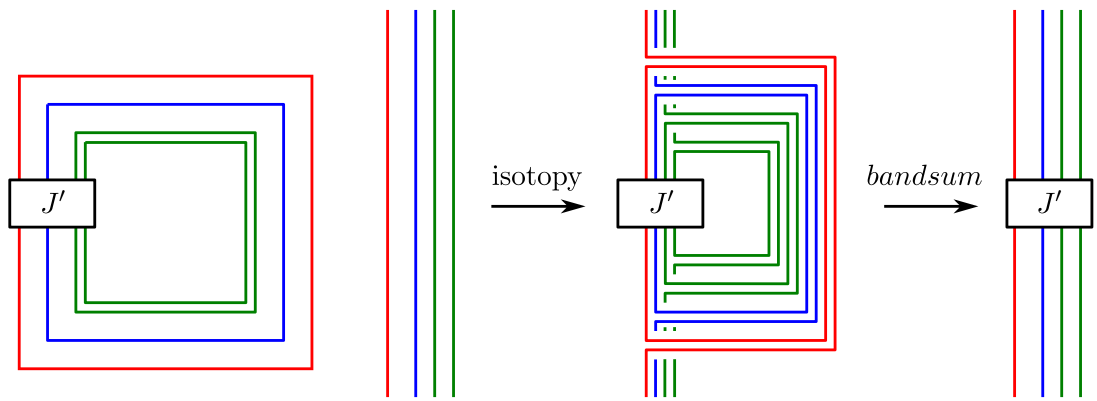

FIGURE 4.8. Band sums between L and $\widehat{J'}$, the closure of J'.

We will label each component of J' with index $\{1, 2, \cdots, \sum_{i,j}(\alpha_i^j + \beta_i^j)\}$ and we define a set map $h : \{1, 2, \cdots, \sum_{i,j}(\alpha_i^j + \beta_i^j)\} \to \{1, 2, 3\}$, where for $n \in \{1, 2, \cdots, \sum_{i,j}(\alpha_i^j + \beta_i^j)\}$, $h(n)$ is the index of the component of the link $L = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where nth component of J' is being banned summed to. It is known that the first non-vanishing Milnor invariant is additive under the exterior band sum (see [**Coc90**]). Hence we have the following equation.

$$\bar{\mu}_{S(L,J,\varphi)}(123) = \sum_{\{I' \subset \{1,2,\cdots,\sum_{i,j}(\alpha_i^j+\beta_i^j)\} \mid h(I')=\{1,2,3\}, |I'|=3\}} sign(h(I'_1)h(I'_2)h(I'_3)) \cdot \bar{\mu}_{\widehat{J}}(123) + \bar{\mu}_L(123)$$

where I'_i is ith component of I' for $i = 1, 2, 3$ and we are considering $(h(I'_1)h(I'_2)h(I'_3))$ as an element of S_3.

Recall that reversing the orientation of a component of a link changes the sign of the Milnor's triple linking number (see Chapter 4.2). Also, note that n_j^i, the algebraic intersection number between $\varphi(E_i)$ and γ_j, is equal to $\alpha_i^j - \beta_i^j$. Combining these with the above equation, we have the following desired equation:

$$\bar{\mu}_{S(L,J,\varphi)}(123) = \bar{\mu}_{\widehat{J}}(123) \cdot \sum_{\sigma \in S_3} (sign(\sigma) \cdot (\alpha_1^{\sigma(1)}\alpha_2^{\sigma(2)}\alpha_3^{\sigma(3)} - \beta_1^{\sigma(1)}\alpha_2^{\sigma(2)}\alpha_3^{\sigma(3)} - \alpha_1^{\sigma(1)}\beta_2^{\sigma(2)}\alpha_3^{\sigma(3)} -$$

$$\alpha_1^{\sigma(1)}\alpha_2^{\sigma(2)}\beta_3^{\sigma(3)} + \beta_1^{\sigma(1)}\beta_2^{\sigma(2)}\alpha_3^{\sigma(3)} + \alpha_1^{\sigma(1)}\beta_2^{\sigma(2)}\beta_3^{\sigma(3)} + \beta_1^{\sigma(1)}\alpha_2^{\sigma(2)}\beta_3^{\sigma(3)} -$$

$$\beta_1^{\sigma(1)}\beta_2^{\sigma(2)}\beta_3^{\sigma(3)})) + \bar{\mu}_L(123)$$

$$= \bar{\mu}_{\widehat{J}}(123) \cdot \sum_{\sigma \in S_3} (sign(\sigma) \cdot (n_1^{\sigma(1)}\alpha_2^{\sigma(2)}\alpha_3^{\sigma(3)} - \alpha_1^{\sigma(1)}\beta_2^{\sigma(2)}n_3^{\sigma(3)} - n_1^{\sigma(1)}\alpha_2^{\sigma(2)}\beta_3^{\sigma(3)} +$$

$$\beta_1^{\sigma(1)}\beta_2^{\sigma(2)}n_3^{\sigma(3)})) + \bar{\mu}_L(123)$$

$$= \bar{\mu}_{\widehat{J}}(123) \cdot \sum_{\sigma \in S_3} (sign(\sigma) \cdot (n_1^{\sigma(1)}\alpha_2^{\sigma(2)}n_3^{\sigma(3)} - n_1^{\sigma(1)}\beta_2^{\sigma(2)}n_3^{\sigma(3)})) + \bar{\mu}_L(123)$$

$$= \bar{\mu}_{\widehat{J}}(123) \cdot \sum_{\sigma \in S_3} (sign(\sigma) \cdot (n_1^{\sigma(1)}n_2^{\sigma(2)}n_3^{\sigma(3)})) + \bar{\mu}_L(123).$$

□

CHAPTER 5

Milnor's triple linking number and derivatives of genus three knots

In this section, we will be focusing on algebraically slice knots which have a genus three Seifert surface and study the behavior of its derivatives in terms of Milnor's triple linking number. We will first prove two lemmas which will be useful. Let K be an algebraically slice knot with a genus three Seifert surface F. Let $H = \mathbb{Z}^3$ be a metabolizer for K and let $\{b_1, b_2, b_3\}$ be a basis for H. We define a set of integers $S_{K,H,\{b_1,b_2,b_3\}}$ as follows:

$$S_{K,H,\{b_1,b_2,b_3\}} = \{\bar{\mu}_L(123) - \bar{\mu}_{L'}(123) | L, L' \in \mathfrak{d}K/\mathfrak{d}H_{\{b_1,b_2,b_3\}}\}$$

where $\mathfrak{d}K/\mathfrak{d}H_{\{b_1,b_2,b_3\}}$ is the set of all the derivatives $L = \gamma_1 \cup \gamma_2 \cup \gamma_3$ associated with a metabolizer H such that $[\gamma_1] = b_1, [\gamma_2] = b_2,$ and $[\gamma_3] = b_3$.

LEMMA 5.1. *Let K and $\widetilde{K}$ be algebraically slice knots, and suppose F and $\widetilde{F}$ are genus three Seifert surfaces for K and $\widetilde{K}$ respectively. Let H and $\widetilde{H}$ be metabolizers for K and $\widetilde{K}$ respectively and let $\{b_1, b_2, b_3\}$ and $\{\widetilde{b_1}, \widetilde{b_2}, \widetilde{b_3}\}$ be bases for H and $\widetilde{H}$ respectively. If there exist symplectic bases $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and $\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{b_1}, \widetilde{b_2}, \widetilde{b_3}\}$ for $H_1(F)$ and $H_1(\widetilde{F})$ respectively such that the Seifert matrices arising from these bases coincide, then $S_{K,H,\{b_1,b_2,b_3\}} = S_{\widetilde{K},\widetilde{H},\{\widetilde{b_1},\widetilde{b_2},\widetilde{b_3}\}}$.*

PROOF. Suppose $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and $\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{b_1}, \widetilde{b_2}, \widetilde{b_3}\}$ are symplectic bases for $H_1(F)$ and $H_1(\widetilde{F})$ respectively such that the Seifert matrices arising from these bases coincide. Using these bases we can get disk-band form for F and $\widetilde{F}$ (see Chapter 4.1). By Proposition 4.9, we can get to $\widetilde{F}$ from F by performing double Borromean rings insertion moves on the bands of F, since K and $\widetilde{K}$ have the same Seifert matrix. Also, since we can think of double Borromean rings insertion move as a special case of a string link infection (see Remark 4.8 (1)), we can apply Lemma 4.11 to our situation. Suppose $L = \gamma_1 \cup \gamma_2 \cup \gamma_3$

and $L' = \gamma_1' \cup \gamma_2' \cup \gamma_3'$ are in $\mathfrak{d}K/\mathfrak{d}H_{\{b_1,b_2,b_3\}}$. If we perform one double Borromean rings insertion move on bands of F, we get two links $S(L, J, \varphi)$ and $S(L', J, \varphi)$, where $\widehat{J}$ is a Borromean rings. By Lemma 4.11, for $i, j \in \{1, 2, 3\}$, Milnor's triple linking number of $S(L, J, \varphi)$ only depends on the algebraic intersection number between $\varphi(E_i)$ and γ_j and Milnor's triple linking number of $S(L', J, \varphi)$ only depends on the algebraic intersection number between $\varphi(E_i)$ and γ_j'. Since $[\gamma_j] = [\gamma_j'] \in H_1(F)$ for $j = 1, 2, 3$, we know that they have the same algebraic intersection number with $\varphi(E_i)$ for $i = 1, 2, 3$. Therefore, we can conclude $\bar{\mu}_L(123) - \bar{\mu}_{L'}(123) = \bar{\mu}_{S(L,J,\varphi)}(123) - \bar{\mu}_{S(L',J,\varphi)}(123)$, i.e. a double Borromean rings insertion move does not change the difference of Milnor's triple linking number of two derivatives. By applying more double Borromean rings insertion moves, we can achieive $\widetilde{K}$. Using the same argument, we know that the difference of their Milnor's triple linking number after all the double Borromean rings insertion moves is still $\bar{\mu}_L(123) - \bar{\mu}_{L'}(123)$. Therefore we can conclude that $S_{K,H,\{b_1,b_2,b_3\}} \subseteq S_{\widetilde{K},\widetilde{H},\{\widetilde{b_1},\widetilde{b_2},\widetilde{b_3}\}}$. We get the other inclusion by simply switching the roles of K and $\widetilde{K}$. □

Further, we define a set of integers $S_{K,H}$ as follows:

$$S_{K,H} = \{\bar{\mu}_L(123) - \bar{\mu}_{L'}(123) | L, L' \in \mathfrak{d}K/\mathfrak{d}H\}$$

where $\mathfrak{d}K/\mathfrak{d}H$ is the set of all the derivatives associated with a metabolizer H. We show that the set $S_{K,H}$ is in fact equal to $S_{K,H,\{b_1,b_2,b_3\}} \cup -S_{K,H,\{b_1,b_2,b_3\}}$ for any basis $\{b_1, b_2, b_3\}$ for H. (i.e. Choice of a basis does not matter once we pick a metabolizer for K.)

Lemma 5.2. *Let K be an algebraically slice knot with genus three Seifert surface. Suppose H is a metabolizer of K and $\{b_1, b_2, b_3\}$ is a basis for H. Then*

$$S_{K,H} = S_{K,H,\{b_1,b_2,b_3\}} \cup -S_{K,H,\{b_1,b_2,b_3\}}.$$

Proof. Note that $-S_{K,H,\{b_1,b_2,b_3\}} = S_{K,H,\{-b_1,b_2,b_3\}}$. Therefore $S_{K,H} \supseteq S_{K,H,\{b_1,b_2,b_3\}} \cup -S_{K,H,\{b_1,b_2,b_3\}}$. For the other direction, we suppose $\gamma_1 \cup \gamma_2 \cup \gamma_3$ is a derivative of K associated with H. Take a parallel copy of γ_2 and perform a band sum on the Seifert surface F from γ_1 to the parallel copy of γ_2 (see Figure 5.1). Let γ_1' be the resulting knot after the band sum. Also, let $\gamma_2' = \gamma_2$ and $\gamma_3' = \gamma_3$. We claim that $\bar{\mu}_{\gamma_1\cup\gamma_2\cup\gamma_3}(123) = \bar{\mu}_{\gamma_1'\cup\gamma_2'\cup\gamma_3'}(123)$. Recall from Chapter 4.2 that there is a nice geometric interpretation of Milnor's triple linking number which is counting the number of triple intersection points of Seifert surfaces with sign. Let F_1, F_2 and F_3 be Seifert surfaces for γ_1, γ_2 and γ_3 respectively, where $F_i \cap \gamma_j = \phi$ for $i \neq j$ and F_1, F_2 and F_3 have isolated triple intersection points. We will take a parallel copy of F_2 which bounds the parallel copy of γ_2 and take a boundary–connected sum with F_1 along the band which was used to perform band sum between γ_1 and the parallel copy of γ_2 (see Figure 5.1). Let F_1' be the resulting surface which bounds γ_1'. Also, let $F_2' = F_2$ and $F_3' = F_3$. Then notice that we have not introduced any new triple intersection points between F_1', F_2' and F_3', since parallel copy of F_2 does not intersect F_2.

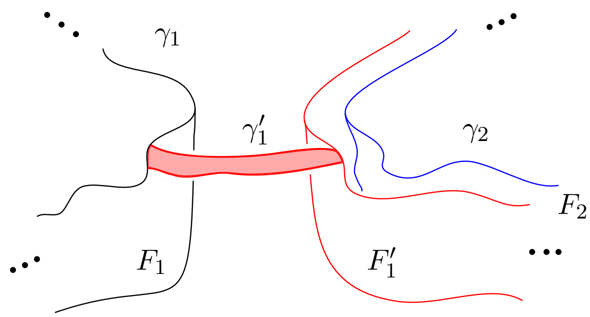

FIGURE 5.1. Taking a parallel copy of γ_2 and performing a band sum with γ_1.

However, it is not guaranteed that $F_i' \cap \gamma_j' = \phi$ for $i \neq j$, and also it is not guaranteed that F_1' is a Seifert surface for γ_1', since the band could have went through it self. We can fix this by altering F_1'. When the band goes through F_1' we will take out two disks from F_1' and attach a cylinder which connects the two circles as in Figure 5.2. When the band goes through either F_2' or F_3', we will perform tubing as in Figure 5.3. After all the alterations, it is guaranteed that $F_i' \cap \gamma_j' = \phi$ for $i \neq j$ and F_1' is a Seifert surface for γ_1'. Since, we have not introduced any new triple intersection points, we have $\bar{\mu}_{\gamma_1\cup\gamma_2\cup\gamma_3}(123) = \bar{\mu}_{\gamma_1'\cup\gamma_2'\cup\gamma_3'}(123)$ as desired.

Note that for any two derivative $\gamma_1 \cup \gamma_2 \cup \gamma_3$ and $\gamma_1' \cup \gamma_2' \cup \gamma_3'$ associated to H, we can perform several moves as above (i.e. taking a parallel copy of one component and performing a band sum with other components) with possibly changing orientations to get

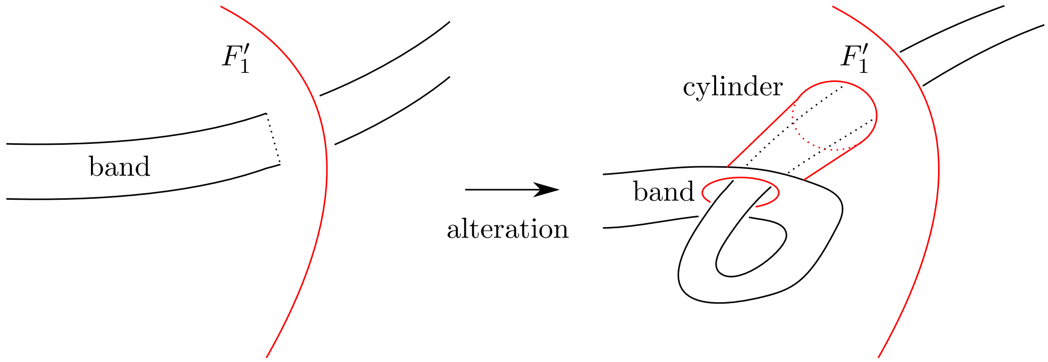

FIGURE 5.2. Alteration when F_1' intersects the band

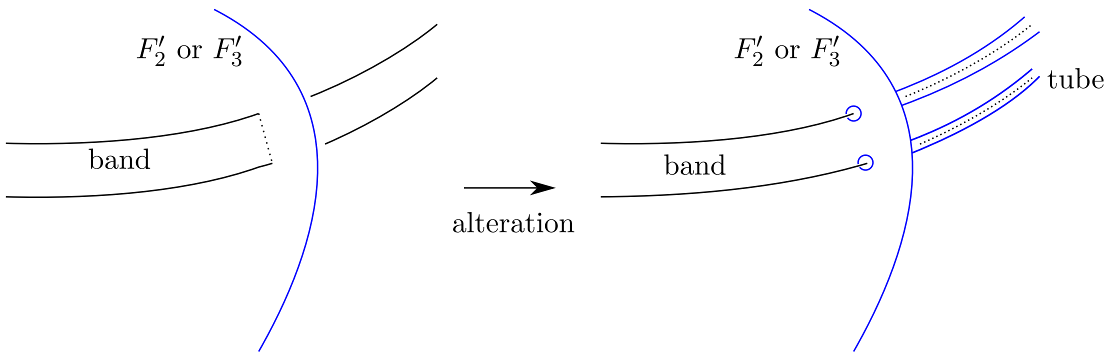

FIGURE 5.3. Alteration when F_2' or F_3' intersects the band

from one to the other. While performing such moves we observed that up to change of sign, their Milnor's triple linking numbers do not change. Hence we can conclude that $S_{K,H} \subseteq S_{K,H,\{b_1,b_2,b_3\}} \cup -S_{K,H,\{b_1,b_2,b_3\}}$ as needed. □

Now, we state the main theorem of this chapter.

THEOREM 5.3. *Let K be an algebraically slice knot with genus three Seifert surface F. Suppose H is a metabolizer of K and $\{b_1, b_2, b_3\}$ is a basis for H. Extend $\{b_1, b_2, b_3\}$ to a symplectic basis for $H_1(F)$ and let $M = \begin{pmatrix} B & A \\ A^\top - \mathrm{Id} & 0 \end{pmatrix}$ be the resulting Seifert matrix. Then*

$$S_{K,H} \supseteq S_{K,H,\{b_1,b_2,b_3\}} \supseteq (\det(A - \mathrm{Id}) - \det(A)) \cdot \mathbb{Z}.$$

Proof. Let K be a given algebraically slice knot. By Lemma 5.1 we can assume that K is the knot described in Figure 5.12 at the end of this section, which has the simplest Seifert surface F with the Seifert matrix M. Let $\gamma_1 \cup \gamma_2 \cup \gamma_3$ be a three component link that is described in Figure 5.12. Note that $\gamma_1 \cup \gamma_2 \cup \gamma_3$ is a derivative of K associated with H. Since $\gamma_1 \cup \gamma_2 \cup \gamma_3$ is the unlink, $\bar{\mu}_{\gamma_1\cup\gamma_2\cup\gamma_3}(123) = 0$

We will produce links $L_n \in \mathfrak{d}K/\mathfrak{d}H_{\{b_1,b_2,b_3\}}$ such that $\bar{\mu}_{L_n}(123) = n \cdot (\det(A - \mathrm{Id}) - \det(A))$ for each positive integer n. Let $\{a_1, b_1, a_2, b_2, a_3, b_3\}$ be a symplectic basis for $H_1(F)$ and let

$$A = \begin{pmatrix} a & x_1 & y_1 \\ x_2 & b & z_1 \\ y_2 & z_2 & c \end{pmatrix}.$$

Note that $\det(A - \mathrm{Id}) - \det(A) = ((a-1)(b-1)(c-1) - abc + x_1x_2 + y_1y_2 + z_1z_2)$.

We will start with the case when $n = 1$. Let $\gamma_{1,1} \cup \gamma_{1,2} \cup \gamma_{1,3}$ be the embedding of circles on the surface $\widetilde{F_1}$ as described in Figure 5.13, 5.14, and 5.15 at the end of this section. we denote $\gamma'_{1,i}$ as the intersection dual of $\gamma_{1,i}$ for $i = 1, 2, 3$. Let $\phi : \widetilde{F_1} \to F$ be the obvious map which sends the core of ith band of $\widetilde{F_1}$ to the core of the ith band of F. By abusing notation we denote the image of $\gamma_{1,i}$ as $\gamma_{1,i}$ and the image of $\gamma'_{1,i}$ as $\gamma'_{1,i}$, for $i = 1, 2, 3$. Note that for $i = 1, 2, 3$, $[\gamma'_{1,i}] = a_i$ and $[\gamma_{1,i}] = b_i$ in $H_1(F)$ and $\gamma_{1,1} \cup \gamma_{1,2} \cup \gamma_{1,3} \in \mathfrak{d}K/\mathfrak{d}H_{\{b_1,b_2,b_3\}}$. Let L_1 be the link $\gamma_{1,1} \cup \gamma_{1,2} \cup \gamma_{1,3} \subset S^3$ on the Seifert surface F.

We will show that $\bar{\mu}_{L_1}(123) = ((a-1)(b-1)(c-1) - abc + x_1x_2 + y_1y_2 + z_1z_2)$. Note that $\bar{\mu}_{L_1}(123)$ is equal to $\bar{\mu}_{\gamma_{1,2}\cup\gamma_{1,3}\cup\gamma_{1,1}}(123)$ by definition, so we will compute $\bar{\mu}_{\gamma_{1,2}\cup\gamma_{1,3}\cup\gamma_{1,1}}(123)$ instead. Let π be $\pi_1(S^3 - L_1)$, $\lambda_{1,1}, \lambda_{1,2}, \lambda_{1,3}$ be longitudes of $\gamma_{1,1}, \gamma_{1,2}, \gamma_{1,3}$ respectively, and $\mu_{1,1}, \mu_{1,2}, \mu_{1,3}$ be meridians of $\gamma_{1,1}, \gamma_{1,2}, \gamma_{1,3}$ respectively. Then $[\lambda_{1,1}] \in \pi_2$, since $\mathrm{lk}(\gamma_{1,i}, \gamma_{1,j}) = 0$, for $i \neq j$. Hence by Proposition 4.2, it is not necessary to specify the basing of $\lambda_{1,1}$, for the calculation of $\bar{\mu}_{\gamma_{1,2}\cup\gamma_{1,3}\cup\gamma_{1,1}}(123)$. Suppose $\lambda_{1,1}$ bounds a Seifert surface F_1 which does not intersect $\gamma_{1,2}$ and $\gamma_{1,3}$. For $i = 1, 2, \cdots, k_1$, where k_1 is the genus of F_1, let φ_i and ψ_i be the core of the $2i-1$th band and the $2i$th band of F_1 respectively (see Figure 5.4). For $i = 1, 2, \cdots, k_1$, let $c_i = [\varphi_i]$ and $d_i = [\psi_i]$ in $\pi_1(S^3 - L_1)$, then notice that $[\lambda_{1,1}] = \prod_{i=1}^{k_1} [c_i, d_i] \in \pi_2/\pi_3$. Then by Proposition 4.3 we only need to calculate the exponent sum of $[\mu_{1,2}]$ and $[\mu_{1,3}]$ occurring in c_i and d_i, for all $i = 1, 2, \cdots, k_1$. Notice that basing of φ_i and ψ_i does not matter since we are only interested in their exponent sum of $[\mu_{1,2}]$ and $[\mu_{1,3}]$.

We will find a Seifert surface F_1 for $\lambda_{1,1}$ which does not intersect $\gamma_{1,2}$ and $\gamma_{1,3}$. Let F_1 be the obvious surface which bounds $\lambda_{1,1}$ (see Figure 5.5) and we push it to the negative direction of F. The problem with this surface is that it intersects $\gamma_{1,1}, \gamma_{1,2}$, and $\gamma_{1,3}$. In order to fix this problem we need to perform several modifications, which are similar to the modifications performed in the proof of Lemma 5.2 on F_1.

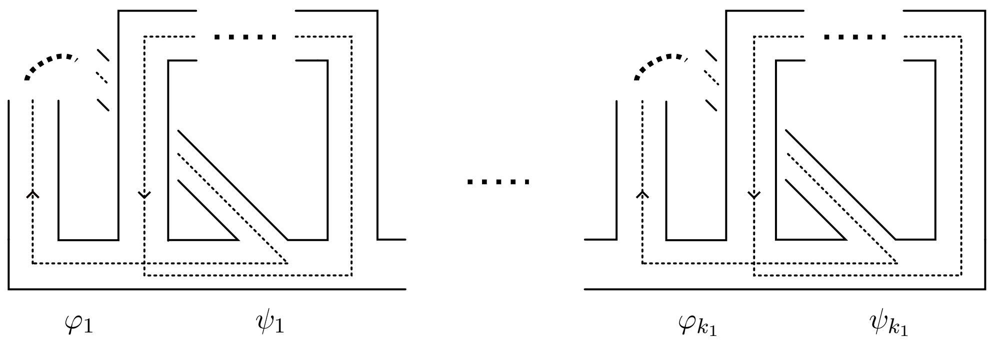

Figure 5.4. Label of cores of the bands of the Seifert surface F_1 for $\lambda_{1,1}$

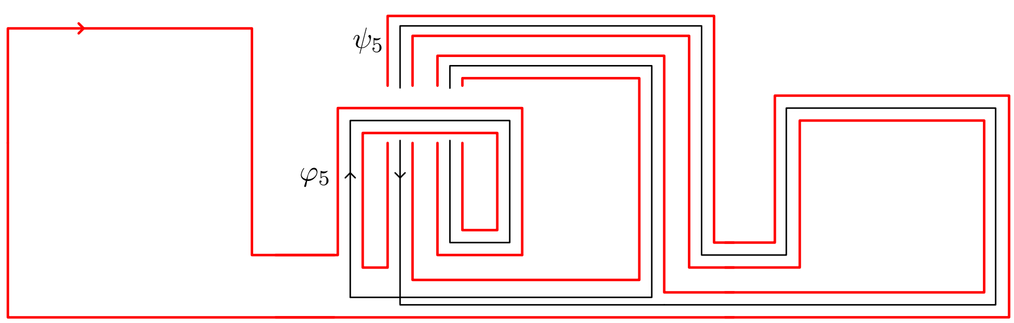

Figure 5.5. The Seifert surface F_1 before the modifications and cores of two bands φ_5 and ψ_5

F_1 intersects $\gamma_{1,1}$ only when the third band or the fifth band wraps around the second band. In this case, we will drill out two disks from F_1 and connect them using a cylinder as in Figure 5.6 to modify the surface F_1. Note that whenever we do this we can use φ

in Figure 5.6 as one of the core of the band. Since it has zero exponent sum of $[\mu_{1,2}]$ and $[\mu_{1,3}]$ occurring in $[\varphi]$, we do not have to worry about the case when F_1 intersects $\gamma_{1,1}$ for the calculation of $\bar{\mu}_{L_1}(123)$.

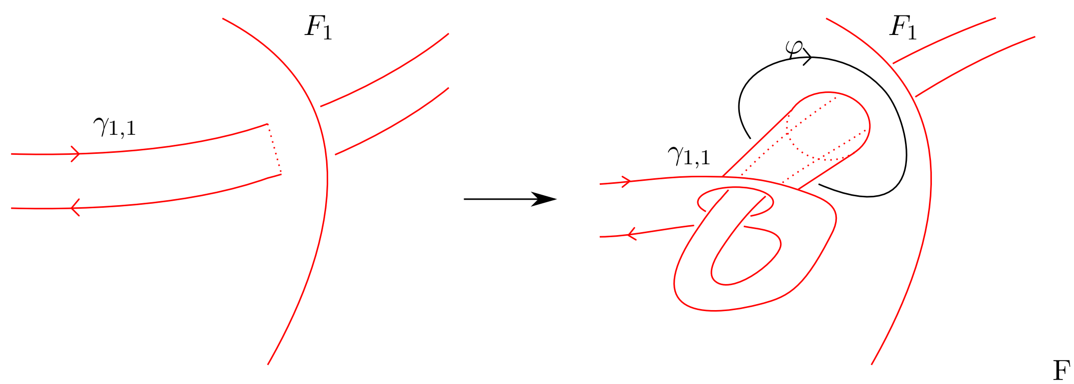

FIGURE 5.6. A modification of F_1 when $\gamma_{1,1}$ intersects F_1

When the first band goes through the surface F_1, we tube along the $\gamma_{1,2}$ and $\gamma_{1,3}$, where F_1 intersects $\gamma_{1,2}$ and $\gamma_{1,3}$, as in Figure 5.7. We will let $\varphi_1, \varphi_2, \psi_1, \psi_2$ be the circles described in Figure 5.7. We need to calculate the exponent sum of $[\mu_{1,2}]$ and $[\mu_{1,3}]$ occurring in $[\varphi_1], [\varphi_2], [\psi_1], [\psi_2]$. For $[\varphi_1]$, it has one exponent sum of $[\mu_{1,2}]$ and for $[\varphi_2]$, it has one exponent sum of $[\mu_{1,3}]$. Therefore, we only need to calculate the exponent sum of $[\mu_{1,3}]$ for $[\psi_1]$, and $[\mu_{1,2}]$ for $[\psi_2]$. Note that we can think of ψ_1 as positive push off of a circle on the Seifert surface F which takes $-a_3$ value in $H_1(F)$ and we can think of ψ_2 as negative push off of a circle on the Seifert surface F which takes $a_2 + b_1 + b_2 + b_3$ value in $H_1(F)$. Then ψ_1 has $(c-1)$ exponent sum of $[\mu_{1,3}]$ and ψ_2 has $-b$ exponent sum of $[\mu_{1,2}]$ by the following calculations :

$$\begin{aligned}
\mathrm{lk}(\psi_1, \gamma_{1,3}) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}^T \\
&= -(c-1) \\
\mathrm{lk}(\psi_2, \gamma_{1,2}) &= \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T \\
&= b.
\end{aligned}$$

Then in total we have $-(c-1)-b$ exponent sum of $[[\mu_{1,2}],[\mu_{1,3}]]$ in $[\varphi_1,\psi_1]\cdot[\varphi_2,\psi_2]\in\pi_2/\pi_3$, by Proposition 4.3. Since the first band goes through the surface $(a-1)$ times in total, we have $(a-1)(-(c-1)-b)$ effect on the Milnor's triple linking number.

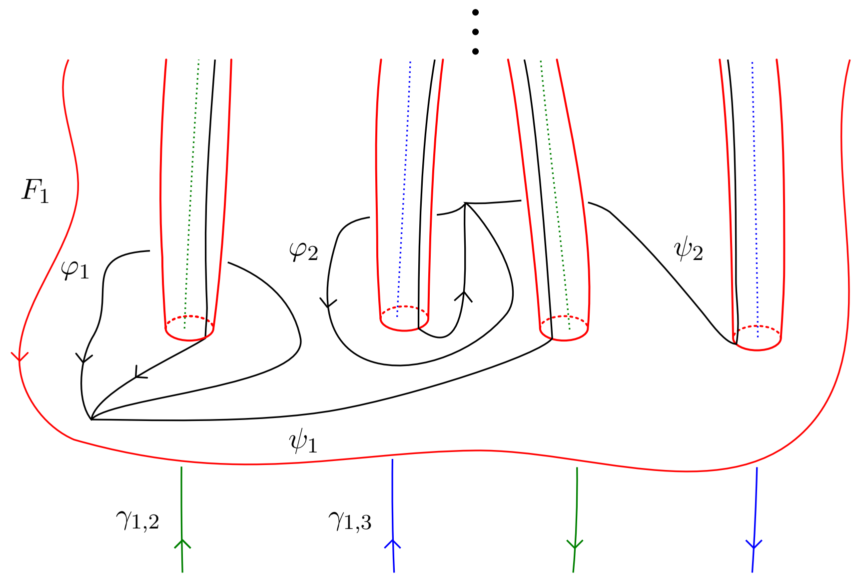

FIGURE 5.7. ψ_1,φ_1,ψ_2, and φ_2

For the case when the third band goes through the surface F_1, we use the same method. We tube along the $\gamma_{1,3}$ as before and let φ_3,ψ_3 be the circles described in Figure 5.8. Same as before we calculate the exponent sum of $[\mu_{1,2}]$ and $[\mu_{1,3}]$ occurring in $[\varphi_3]$ and $[\psi_3]$ respectively. $[\varphi_3]$ only has one exponent sum of $[\mu_{1,3}]$, so for $[\psi_3]$ we only need to calculate the exponent sum of $[\mu_{1,2}]$. We can think of ψ_3 as positive push off of a circle on the Seifert surface F which takes $-a_1+b_1-b_3$ value in $H_1(F)$, hence ψ_3 has $-x_1$ exponent sum of $[\mu_{1,2}]$ by the following calculations :

$$\begin{aligned}\operatorname{lk}(\psi_3,\gamma_{1,2}) &= \begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}\cdot M\cdot\begin{pmatrix}-1 & 0 & 0 & 1 & 0 & -1\end{pmatrix}^T\\ &= -x_1\end{aligned}$$

Then again by Proposition 4.3, in total we have x_1 exponent sum of $[[\mu_{1,2}],[\mu_{1,3}]]$ in $[\varphi_3,\psi_3]\in\pi_2/\pi_3$. Since the third band goes through the surface x_2 times in total, we have x_1x_2 effect on the Milnor's triple linking number.

We use the same method for the case when the fifth band goes through the surface. We tube along the $\gamma_{1,2}$ as before and let φ_4, ψ_4 be the circles described in Figure 5.8. $[\varphi_4]$ only has one exponent sum of $[\mu_{1,2}]$, so for $[\psi_4]$ we only need to calculate the exponent sum of $[\mu_{1,3}]$. We can think of ψ_4 as positive push off of a circle on the Seifert surface F which takes $a_1 - b_1 + b_3$ value in $H_1(F)$ hence ψ_4 has y_1 exponent sum of $[\mu_{1,3}]$ by the following calculations :

$$\begin{aligned}\mathrm{lk}(\psi_3, \gamma_{1,2}) =& \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}^T \\ =& y_1\end{aligned}$$

Then again by Proposition 4.3, in total we have y_1 exponent sum of $[[\mu_{1,2}], [\mu_{1,3}]]$ in $[\varphi_4, \psi_4] \in \pi_2/\pi_3$. Since the third band goes through the surface y_2 times in total, we have $y_1 y_2$ effect on the Milnor's triple linking number.

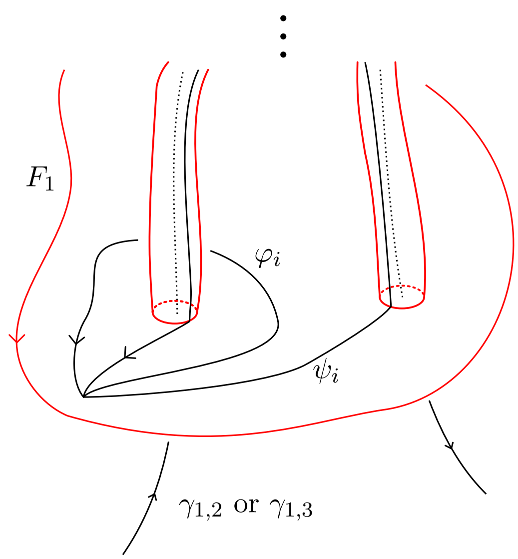

FIGURE 5.8. ψ_i and φ_i for $i = 3, 4$

Let φ_5 and ψ_5 be the cores of the two bands of F_1 before the modifications (see Figure 5.5). We have pushed the surface F_1 towards the negative directions of F, so we can think of φ_5 as negative push off of a circle on the Seifert surface F which takes $a_2 + b_2$ value in $H_1(F)$ and we can think of ψ_5 as negative push off of a circle on the Seifert surface F

which takes $-a_3 - b_2$ value in $H_1(F)$. Then φ_5 has b exponent sum of $[\mu_{1,2}]$ and z_1 exponent sum of $[\mu_{1,3}]$, and ψ_5 has $-z_2$ exponent sum of $[\mu_{1,2}]$ and $-c$ exponent sum of $[\mu_{1,3}]$ by the following calculations:

$$\begin{aligned}
\operatorname{lk}(\varphi_5, \gamma_{1,2}) =& \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T \\
=& b \\
\operatorname{lk}(\varphi_5, \gamma_{1,3}) =& \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T \\
=& z_1 \\
\operatorname{lk}(\psi_5, \gamma_{1,2}) =& \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T \\
=& - z_2 \\
\operatorname{lk}(\psi_5, \gamma_{1,3}) =& \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T \\
=& - c.
\end{aligned}$$

Using Proposition 4.3, in total we have $-bc + z_1 z_2$ exponent sum of $[[\mu_{1,2}], [\mu_{1,3}]]$ in $[\varphi_5, \psi_5] \in \pi_2/\pi_3$. If we total all the effects we get $(a-1)(-(c-1)-b) + x_1x_2 + y_1y_2 - bc + z_1z_2 = (a-1)(b-1)(c-1) - abc + x_1x_2 + y_1y_2 + z_1z_2$ as desired.

Now for integer n greater than 1, we will produce a link $L_n \in \mathfrak{d}K/\mathfrak{d}H_{\{b_1,b_2,b_3\}}$ such that $\bar{\mu}_{L_n}(123) = n\cdot((a-1)(b-1)(c-1) - abc + x_1x_2 + y_1y_2 + z_1z_2)$. We describe $\widetilde{F_n}$ and embedding of circles $\gamma_{n,1} \cup \gamma_{n,2} \cup \gamma_{n,3}$ on $\widetilde{F_n}$. In order to do so we start with $\gamma_{1,1} \cup \gamma_{1,2} \cup \gamma_{1,3}$ embedded in $\widetilde{F_1}$ and we make some modifications to it. We use Figure 5.13, 5.14, and 5.15 to describe the modifications. Let $\widetilde{F_n}$ be the same surface as $\widetilde{F_1}$. For Figure 5.13 and Figure 5.15, we take n parallel copies of the green curve without changing anything else. For Figure 5.14 we alter the core of the bands which bound $\gamma_{1,1}$. The cores wrap around the third and the fourth band of $\widetilde{F_n}$ as described in Figure 5.9. Further inside the red band we will make some alterations as in Figure 5.10. We denote the red circle, the green circle, and the blue circle on $\widetilde{F_n}$ by $\gamma_{n,1}, \gamma_{n,2}$, and $\gamma_{n,3}$ respectively. Also, we denote $\gamma'_{n,i}$ as the intersection dual of $\gamma_{n,i}$, for $i = 1, 2, 3$. For the convenience of the reader we present $\gamma_{2,1}, \gamma_{2,2}, \gamma_{2,3}$ in Figure 5.16, 5.17, and 5.17 at the end of this section. As before let $\phi_n : \widetilde{F_n} \to F$ be the obvious map which sends the core of ith band of $\widetilde{F_n}$ to the core of the ith band of F. By abusing notation we denote the image of $\gamma_{n,i}$ under ϕ_n as $\gamma_{n,i}$ and the image of $\gamma'_{n,i}$ under ϕ_n as $\gamma'_{n,i}$ for $i = 1, 2, 3$. Then it is easy to check that for $i = 1, 2, 3$, $[\gamma'_{n,i}] = a_i$ and $[\gamma_{n,i}] = b_i$ in $H_1(F)$ and $\gamma_{n,1} \cup \gamma_{n,2} \cup \gamma_{n,3} \in \mathfrak{d}K/\mathfrak{d}H_{\{b_1,b_2,b_3\}}$. Let L_n be the link $\gamma_{n,1} \cup \gamma_{n,2} \cup \gamma_{n,3} \subset S^3$ on the Seifert surface F. We denote the obvious surface that $\gamma_{n,1}$ bounds by F_n.

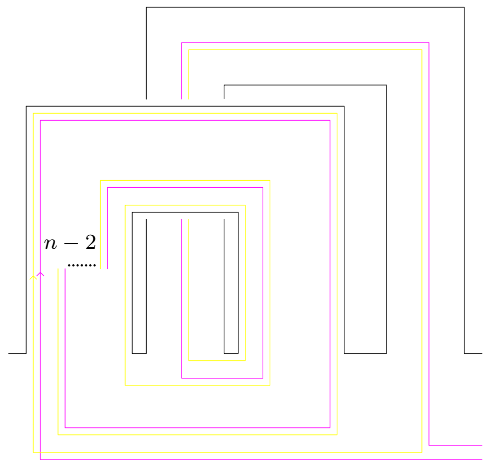

FIGURE 5.9. The yellow curve represents core of the first band φ_{n,k_n} and the purple curve represents core of the second band ψ_{n,k_n}.

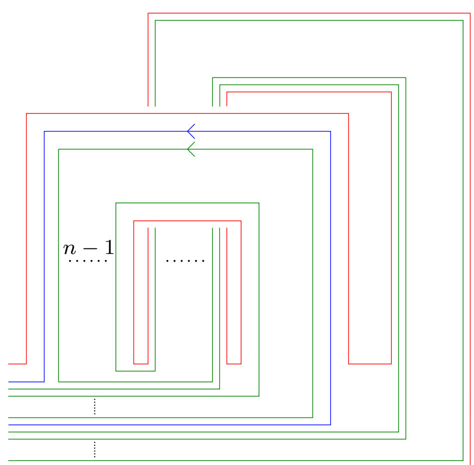

FIGURE 5.10. The green curve represents $\gamma_{n,2}$ and the blue curve represents $\gamma_{n,3}$ inside two bands which bounds $\gamma_{n,1}$.

For the computation of $\bar{\mu}_{L_n}(123)$ we omit some of the details since they are very similar to the case when $n = 1$. We modify F_n so that it does not intersect $\gamma_{n,1}, \gamma_{n,2}$, and $\gamma_{n,3}$.

As before, we will not worry about the case when F_n intersects $\gamma_{n,1}$. When the first band goes through F_n, since we took n parallel copies of the green curve, the total effect by the green curve on Milnor's triple linking number should be multiplied by n. Let $\psi_{n,2}$ denote the blue curve. Note that the blue curve was denoted as ψ_2 for the case when $n = 1$. Since $\psi_{n,2}$ represents $n \cdot a_2 + b_1 + b_2 + b_3$ in $H_1(F)$, we have the following calculation:

$$\begin{aligned} \mathrm{lk}(\psi_{n,2}, \gamma_{1,2}) = & \begin{pmatrix} 0 & n & 0 & 1 & 1 & 1 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T \\ = & n \cdot b. \end{aligned}$$

This tells us that the total effect by the blue curve on Milnor's triple linking number should be multiplied by n also. Hence in total we have $n \cdot (a-1)(-(c-1)-b)$ effect on the Milnor's triple linking number. It is easy to see that the effect of the third band and the fifth band going through the surface F_n also needs to be multiplied by n, hence we have $n \cdot x_1 x_2 + n \cdot y_1 y_2$.

Let φ_{n,k_n} and ψ_{n,k_n} be the cores of two bands of F_n before any modifications, where k_n is the genus of F_n. Again, we can think of φ_{n,k_n} as negative push off of a circle on the Seifert surface F which takes $n \cdot a_2 + b_2$ value in $H_1(F)$ and we can think of ψ_{n,k_n} as negative push off of a circle on the Seifert surface F which takes $-(n-1) \cdot a_2 - b_2 - a_3$ value in $H_1(F)$. Hence φ_{n,k_n} has $n \cdot b$ exponent sum of $[\mu_{1,2}]$ and $n \cdot z_1$ exponent sum of $[\mu_{1,3}]$ and ψ_{n,k_n} has $-(n-1) \cdot b - z_2$ exponent sum of $[\mu_{1,2}]$ and $-(n-1) \cdot z_1 - c$ exponent sum of $[\mu_{1,3}]$ by the following calculations:

$$\mathrm{lk}(\varphi_{n,k_n}, \gamma_{1,2}) = \begin{pmatrix} 0 & n & 0 & 0 & 1 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T$$
$$= n \cdot b$$
$$\mathrm{lk}(\varphi_{n,k_n}, \gamma_{1,3}) = \begin{pmatrix} 0 & n & 0 & 0 & 1 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T$$
$$= n \cdot z_1$$
$$\mathrm{lk}(\psi_{n,k_n}, \gamma_{1,2}) = \begin{pmatrix} 0 & -(n-1) & -1 & 0 & -1 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T$$
$$= -(n-1) \cdot b - z_2$$
$$\mathrm{lk}(\psi_{n,k_n}, \gamma_{1,3}) = \begin{pmatrix} 0 & -(n-1) & -1 & 0 & -1 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T$$
$$= -(n-1) \cdot z_1 - c.$$

In total, using Proposition 4.3 we have $n \cdot (-bc + z_1 z_2)$ exponent sum of $[[\mu_{1,2}], [\mu_{1,3}]]$ in $[\varphi_{n,k_n}, \psi_{n,k_n}] \in \pi_2/\pi_3$. If we total all the effects, we get $n \cdot ((a-1)(b-1)(c-1) - abc + x_1 x_2 + y_1 y_2 + z_1 z_2)$ as desired.

Even though we have not covered the case when n is a negative integer, this can be easily handled. On F, imagine we are sliding the third and the fourth band to the right so they pass the fifth and the sixth band. Then imagine mapping F_n to it so that now the green circle represents b_3 in $H_1(F)$ and the blue circle represents b_2 in $H_1(F)$. We can follow the same calculations and since the roles of the green circle and the blue circle have been switched, we get the desired equation for the negative integers. □

Remark 5.4. *Note that the set of integers* $(\det(A - \mathrm{Id}) - \det(A)) \cdot \mathbb{Z}$ *was obtained by choosing a symplectic basis on* $H_1(F)$*. Suppose that we picked a different symplectic basis* $\{\widetilde{a_1}, \widetilde{a_2}, \widetilde{a_3}, \widetilde{b_1}, \widetilde{b_2}, \widetilde{b_3}\}$ *such that* $\{\widetilde{b_1}, \widetilde{b_2}, \widetilde{b_3}\}$ *forms a basis for* H*. Using the new basis we get a Seifert matrix*

$$\widetilde{M} = \begin{pmatrix} \widetilde{B} & P_2^\top A P_1 \\ P_1^\top (A^\top - Id) P_2 & 0 \end{pmatrix},$$

where P_1 *and* P_2 *are change of basis matrices. Note that we have* $\det(P_1^\top(A^\top - Id)P_2) - \det(P_2^\top A P_1) = \pm\det(A^\top - Id) - \det(A)$*, so we can conclude the set* $(\det(A - \mathrm{Id}) - \det(A)) \cdot \mathbb{Z}$ *does not depend on the choice of a basis.*

We have the first corollary which immediately follows from Theorem 5.3. This corollary tells us that even the derivatives of the unknot have complicated Milnor's triple linking number.

Corollary 5.5. *Let* U *be the unknot with a Seifert surface* F *described in Figure 5.11. For* $i = 1, 2, 3$*, let* α_i *be the core of the* $(2i-1)$*th band and* β_i *be the core of the* $(2i)$*th band. Let* $H := \mathrm{span}([\beta_1], [\beta_2], [\beta_3])$*, then* $S_{U,H} = \mathbb{Z}$*.*

Proof. Let $a_i = [\alpha_i] \in H_1(F)$ and $b_i = [\beta_i] \in H_1(F)$ for $i = 1, 2, 3$. Then we have a Seifert matrix

$$M = \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix}$$

with respect to the basis $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. Let $H = \mathrm{span}(b_1, b_2, b_3)$ be a metabolizer, then by Theorem 5.3, we have $S_{U,H} \supseteq (0 \cdot 0 \cdot 0 - 1 \cdot 1 \cdot 1) \cdot \mathbb{Z} = \mathbb{Z}$ as desired. □

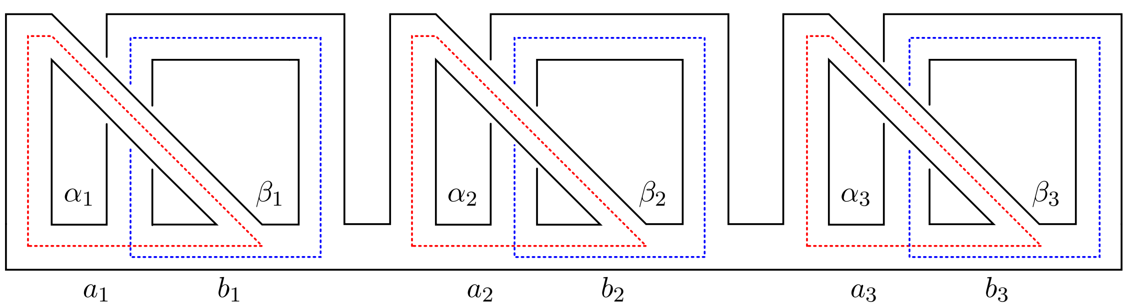

Figure 5.11. Disk-band form of a Seifert surface F for the unknot U

We have one more immediate corollary from Theorem 5.3 which tells us that for a knot which is the connected sum of three genus one algebraically slice knots, it has at least one derivative with non-zero Milnor's triple linking number.

Corollary 5.6. *Let* K_1, K_2, K_3 *be algebraically slice knots with genus one Seifert surfaces. Then* $K = K_1 \# K_2 \# K_3$ *has a derivative* $\gamma_1 \cup \gamma_2 \cup \gamma_3$ *where* $\bar{\mu}_{\gamma_1 \cup \gamma_2 \cup \gamma_3}(123) \neq 0$*.*

Proof. It is enough to show that there exist a metabolizer H such that $S_{K,H} \supseteq m \cdot \mathbb{Z}$ where $m \neq 0$.

Let F_1, F_2, and F_3 be genus one Seifert surfaces for K_1, K_2, and K_3 respectively. For $i = 1, 2, 3$, let $M_i = \begin{pmatrix} d_i & e_i \\ e_i - 1 & 0 \end{pmatrix}$ be a Seifert matrix with respect to a symplectic basis $\{a_i, b_i\}$ of $H_1(F_i)$, so that $H_i := \text{span}(b_i)$ is a metabolizer for K_i. For $i = 1, 2, 3$, let $n_i = gcd(2e_i - 1, -d_i), x_i = \frac{2e_i - 1}{n_i}$, and $y_i = \frac{-d_i}{n_i}$, then there exist a pair of integers (z_i, w_i) such that $-x_i w_i + z_i y_i = 1$, since $gcd(x_i, y_i) = 1$.

Then for $i = 1, 2, 3$, we have the following calculations :

$$\begin{aligned}
\begin{pmatrix} z_i & w_i \end{pmatrix} \cdot \begin{pmatrix} d_i & e_i \\ e_i - 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_i \\ y_i \end{pmatrix} &= \frac{1}{n_i} \cdot \begin{pmatrix} z_i & w_i \end{pmatrix} \cdot \begin{pmatrix} d_i(2e_i - 1) - e_i d_i \\ (e_i - 1)(2e_i - 1) \end{pmatrix} \\
&= \frac{1}{n_i} \cdot (d_i(2e_i - 1)z_i - e_i d_i z_i + (e_i - 1)(2e_i - 1)w_i) \\
&= \frac{1}{n_i} \cdot (-d_i z_i - (2e_i - 1)w_i + e_i(d_i z_i + (2e_i - 1)w_i)) \\
&= \frac{1}{n_i} \cdot (n_i - n_i e_i) \\
&= 1 - e_i
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} x_i & y_i \end{pmatrix} \cdot \begin{pmatrix} d_i & e_i \\ e_i - 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} z_i \\ w_i \end{pmatrix} &= \frac{1}{n_i} \cdot \begin{pmatrix} 2e_i - 1 & -d_i \end{pmatrix} \cdot \begin{pmatrix} d_i z_i + e_i w_i \\ (e_i - 1)z_i \end{pmatrix} \\
&= \frac{1}{n_i} \cdot (d_i z_i(2e_i - 1) + e_i w_i(2e_i - 1) - (e_i - 1)z_i d_i) \\
&= \frac{1}{n_i} \cdot (e_i(d_i z_i + w_i(2e_i - 1))) \\
&= \frac{1}{n_i} \cdot (-n_i e_i) \\
&= -e_i
\end{aligned}$$

$$\begin{pmatrix} x_i & y_i \end{pmatrix} \cdot \begin{pmatrix} d_i & e_i \\ e_i - 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \frac{1}{{n_i}^2} \cdot \begin{pmatrix} 2e_i - 1 & -d_i \end{pmatrix} \cdot \begin{pmatrix} d_i(2e_i - 1) - e_i d_i \\ (e_i - 1)(2e_i - 1) \end{pmatrix}$$
$$= \frac{1}{{n_i}^2} \cdot (d_i(2e_i - 1)^2 - e_i(2e_i - 1)d_i - d_i(e_i - 1)(2e_i - 1))$$
$$= \frac{d}{n_i} \cdot (4e_i^2 - 4e_i + 1 - 2e_i^2 + e_i - 2e_i^2 + 3e_i - 1)$$
$$= \frac{1}{n_i} \cdot (0)$$
$$= 0.$$

Therefore, for $i = 1, 2, 3$, $\widetilde{H_i} := \text{span}(x_i a_i + y_i b_i)$ is also a metabolizer for K_i and we have a Seifert matrix $\widetilde{M_i} = \begin{pmatrix} * & -e_i + 1 \\ -e_i & 0 \end{pmatrix}$ for K_i with respect to a symplectic basis $\{z_i a_i + w_i b_i, x_i a_i + y_i b_i\}$. Hence we can change basis so that the Seifert matrix takes value of either $\begin{pmatrix} * & e_i \\ e_i - 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} * & -e_i + 1 \\ -e_i & 0 \end{pmatrix}$, for $i = 1, 2, 3$. Therefore, we can assume $|e_i| > |e_i - 1|$ for each $i = 1, 2, 3$, which implies that $|e_1| \cdot |e_2| \cdot |e_3| > |e_1 - 1| \cdot |e_2 - 1| \cdot |e_3 - 1|$.

Now, let $F = F_1 \natural F_2 \natural F_3$ be a Seifert surface for K and let H be a metabolizer generated by $\{b_1, b_2, b_3\}$. Then by Theorem 5.3, we have $S_{K,H} \supseteq ((e_1 - 1)(e_2 - 1)(e_3 - 1) - e_1 e_2 e_3) \cdot \mathbb{Z}$ where $((e_1 - 1)(e_2 - 1)(e_3 - 1) - e_1 e_2 e_3) \neq 0$ as desired. $\square$

We will end this section with few remarks.

Remark 5.7.

(1) *Note that Corollary 5.5 is a special case of Corollary 5.6, where K_1, K_2, K_3 are the unknots.*

(2) *For the proof of Corollary 5.6, we only needed the assumption that K has the same Seifert matrix as the connected sum of three genus one algebraically slice knots.*

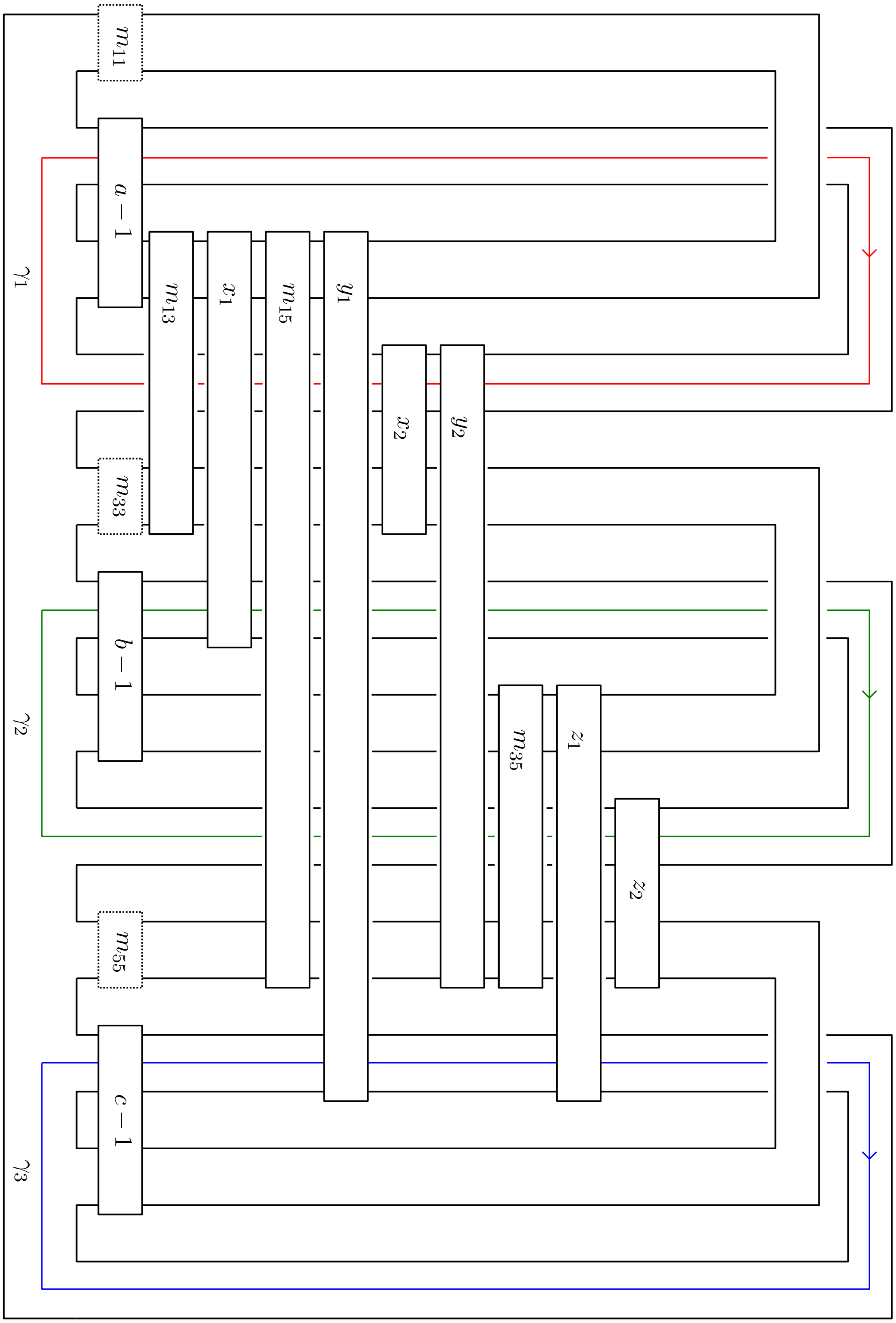

FIGURE 5.12. Knot K with the simplest Seifert surface and the derivative $\gamma_1 \cup \gamma_2 \cup \gamma_3$ associated with H. (Dotted box represents full twists and solid box represents full twists between two bands with no twist on each bands.)

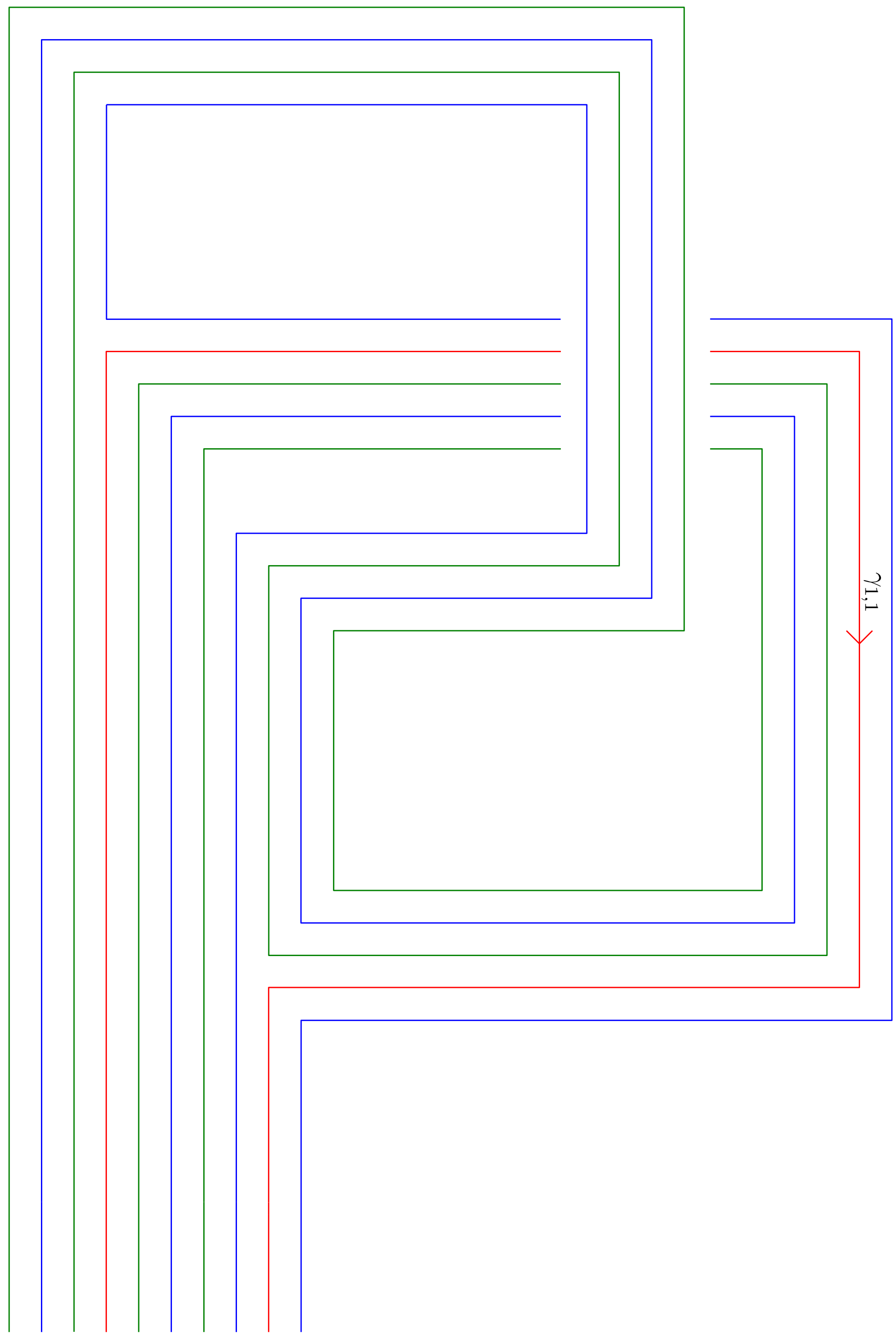

FIGURE 5.13. Link $L_1 = \gamma_{1,1} \cup \gamma_{1,2} \cup \gamma_{1,3}$ embedded in $\widetilde{F_1}$. The surface $\widetilde{F_1}$ is omitted, as its placement is evident in the diagram.

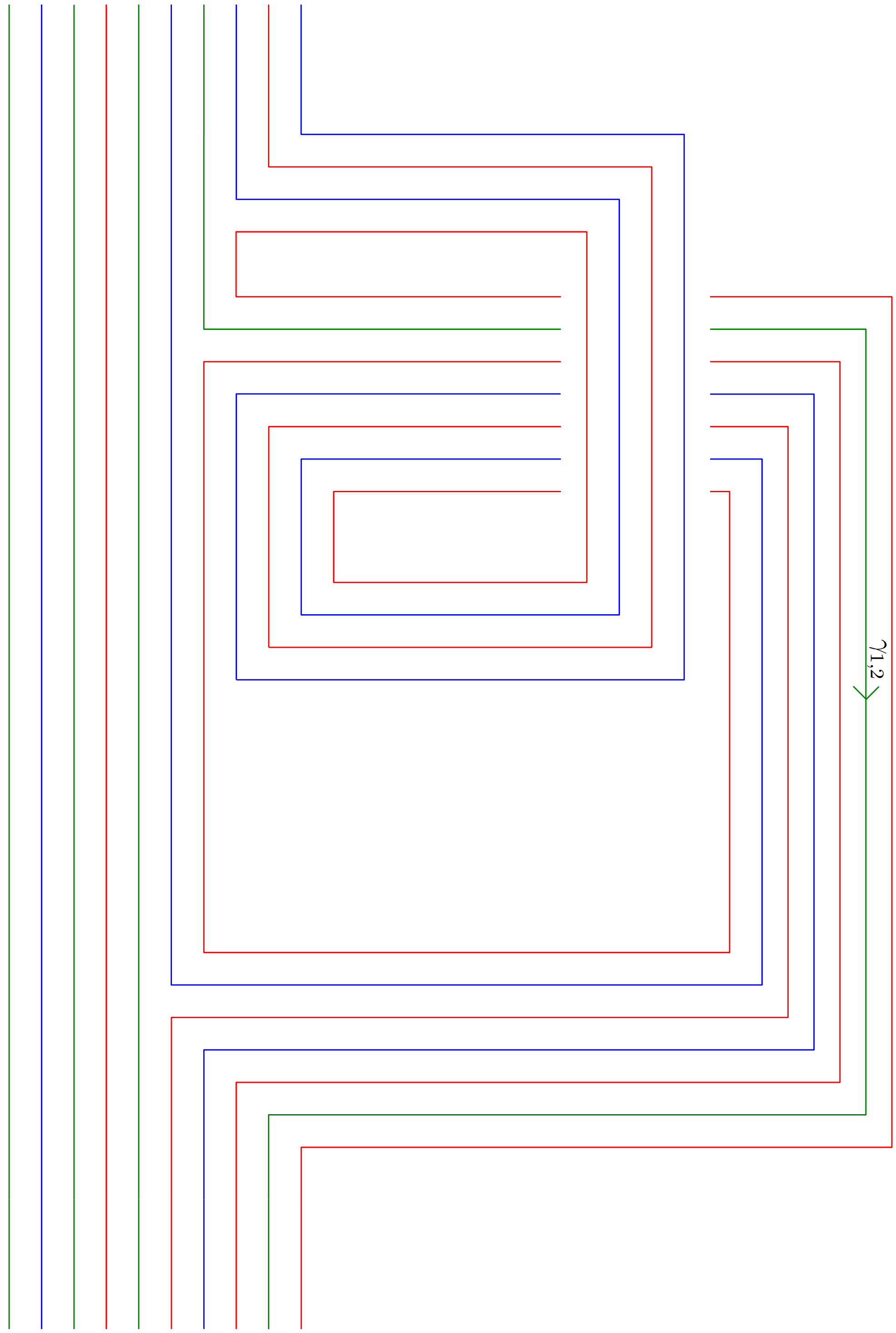

FIGURE 5.14. Link $L_1 = \gamma_{1,1} \cup \gamma_{1,2} \cup \gamma_{1,3}$ embedded in $\widetilde{F_1}$. The surface $\widetilde{F_1}$ is omitted, as its placement is evident in the diagram.

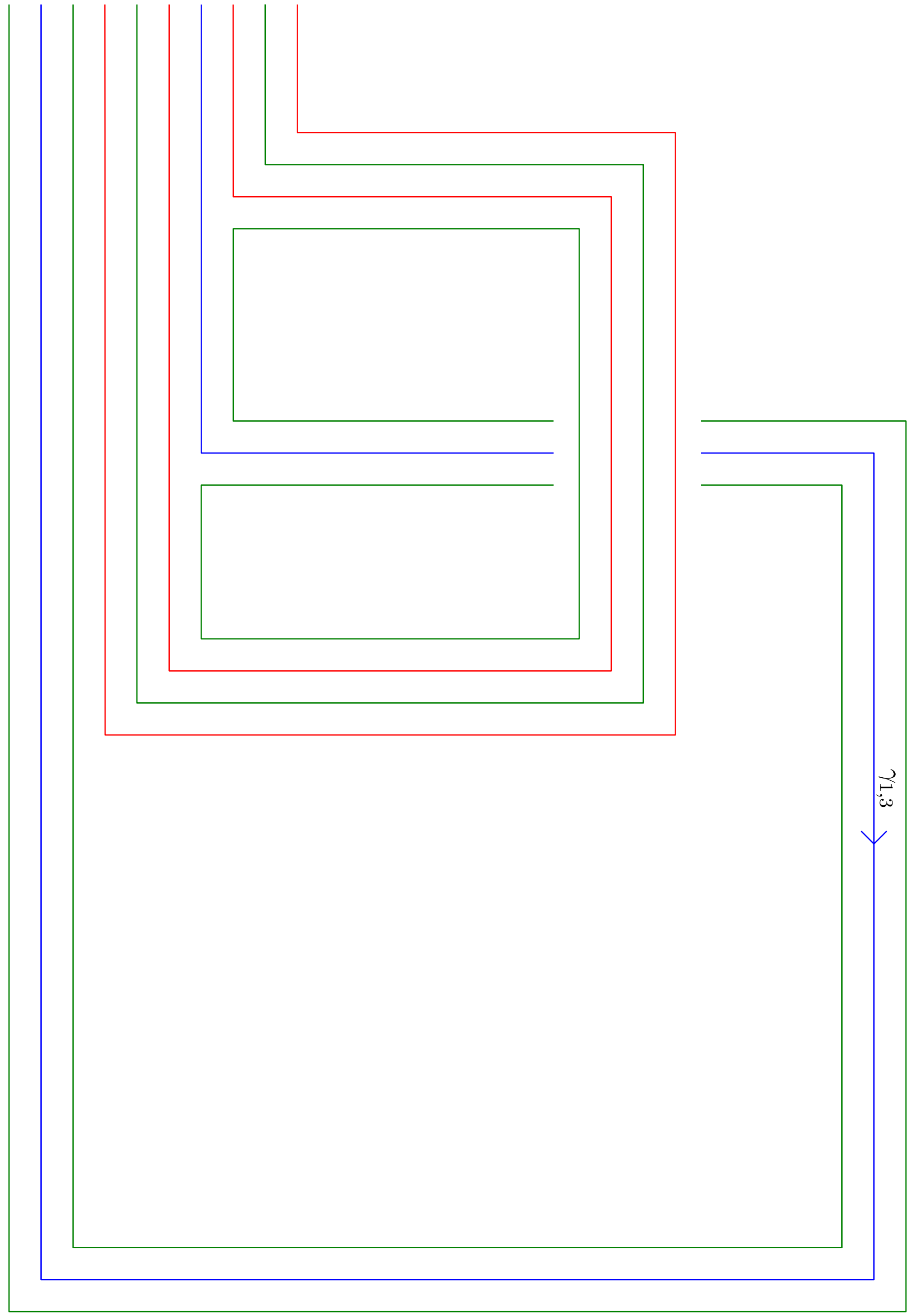

FIGURE 5.15. Link $L_1 = \gamma_{1,1} \cup \gamma_{1,2} \cup \gamma_{1,3}$ embedded in $\widetilde{F_1}$. The surface $\widetilde{F_1}$ is omitted, as its placement is evident in the diagram.

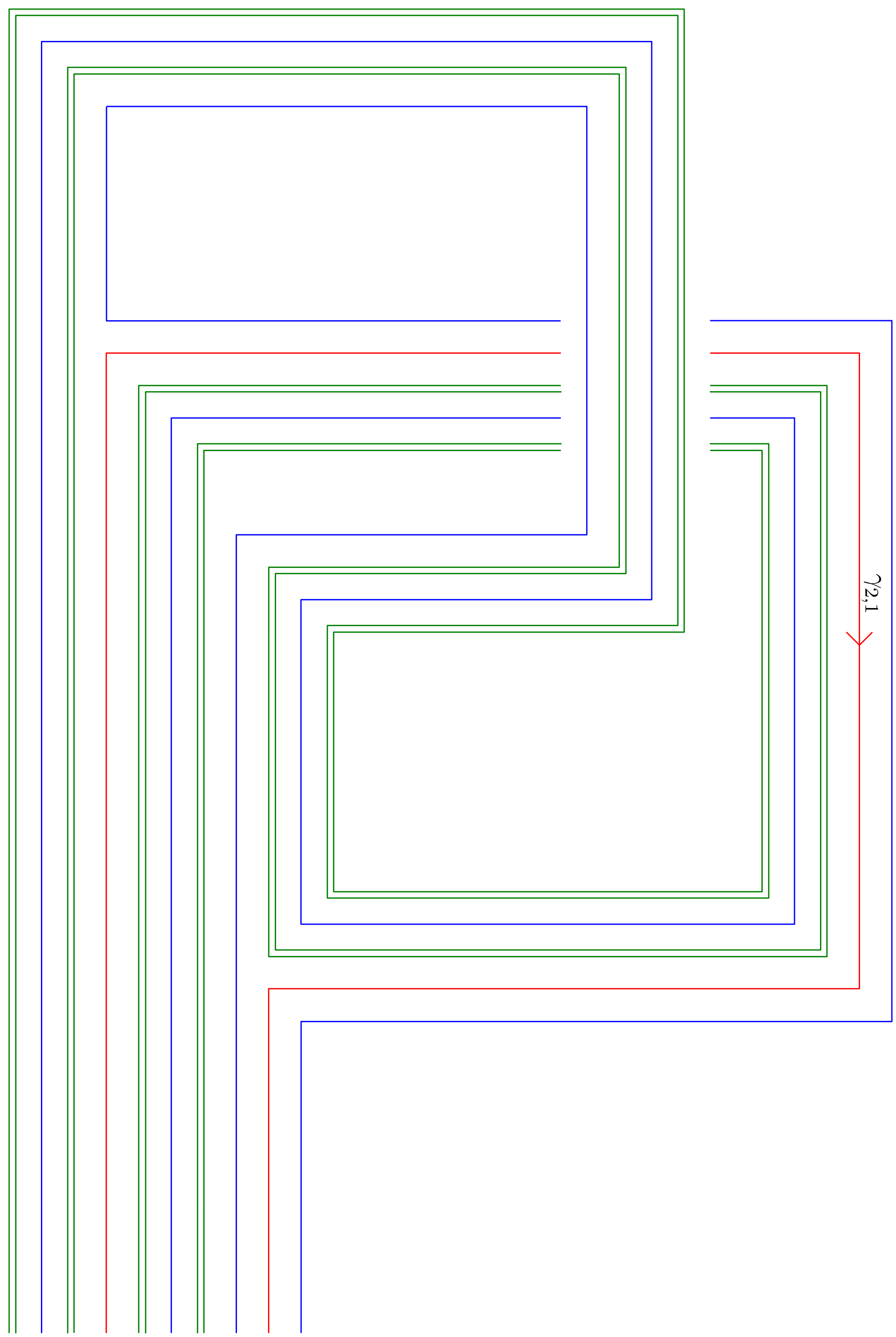

FIGURE 5.16. Link $L_2 = \gamma_{2,1} \cup \gamma_{2,2} \cup \gamma_{2,3}$ embedded in $\widetilde{F_2}$. The surface $\widetilde{F_2}$ is omitted, as its placement is evident in the diagram.

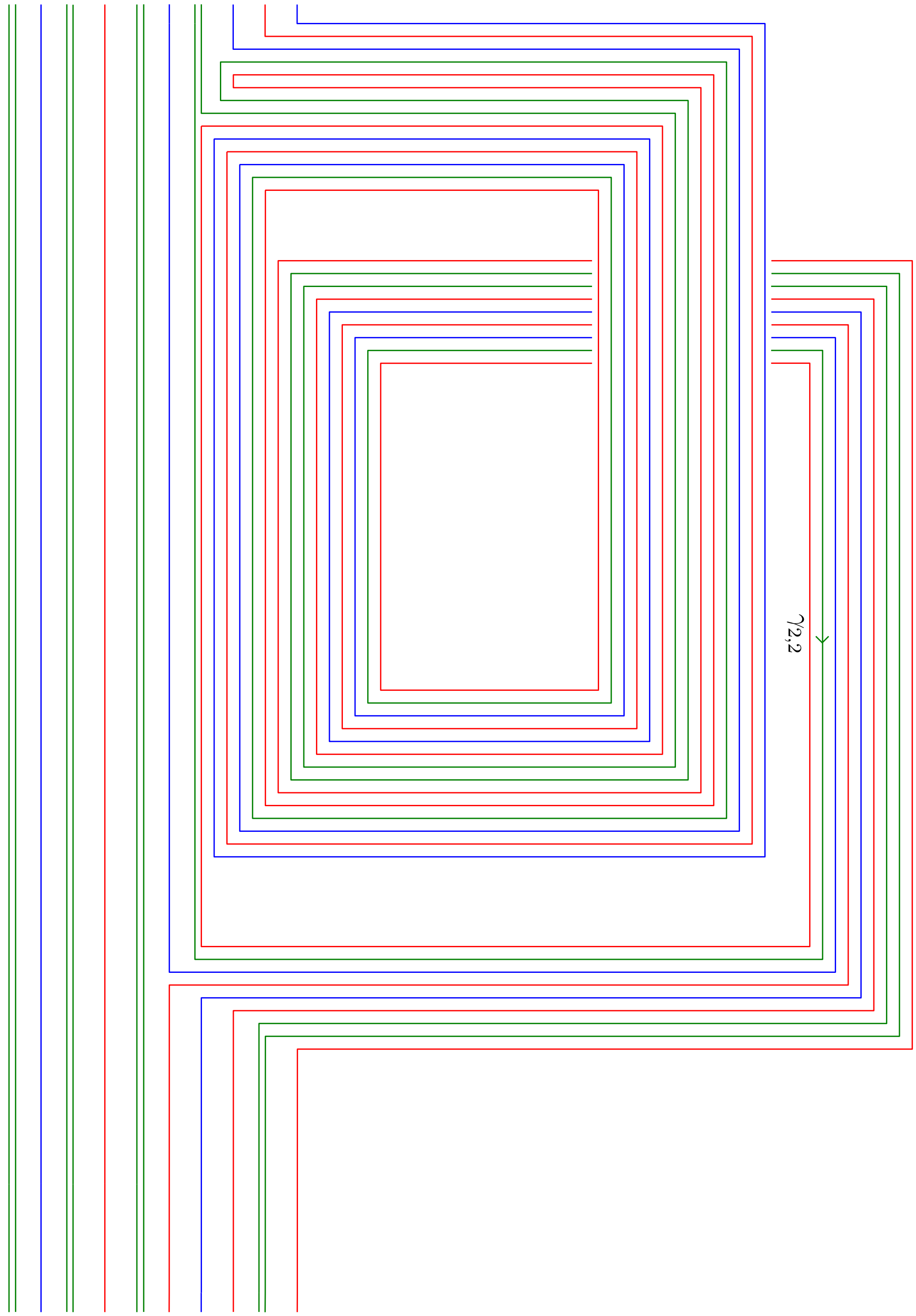

FIGURE 5.17. Link $L_2 = \gamma_{2,1} \cup \gamma_{2,2} \cup \gamma_{2,3}$ embedded in $\widetilde{F_2}$. The surface $\widetilde{F_2}$ is omitted, as its placement is evident in the diagram.

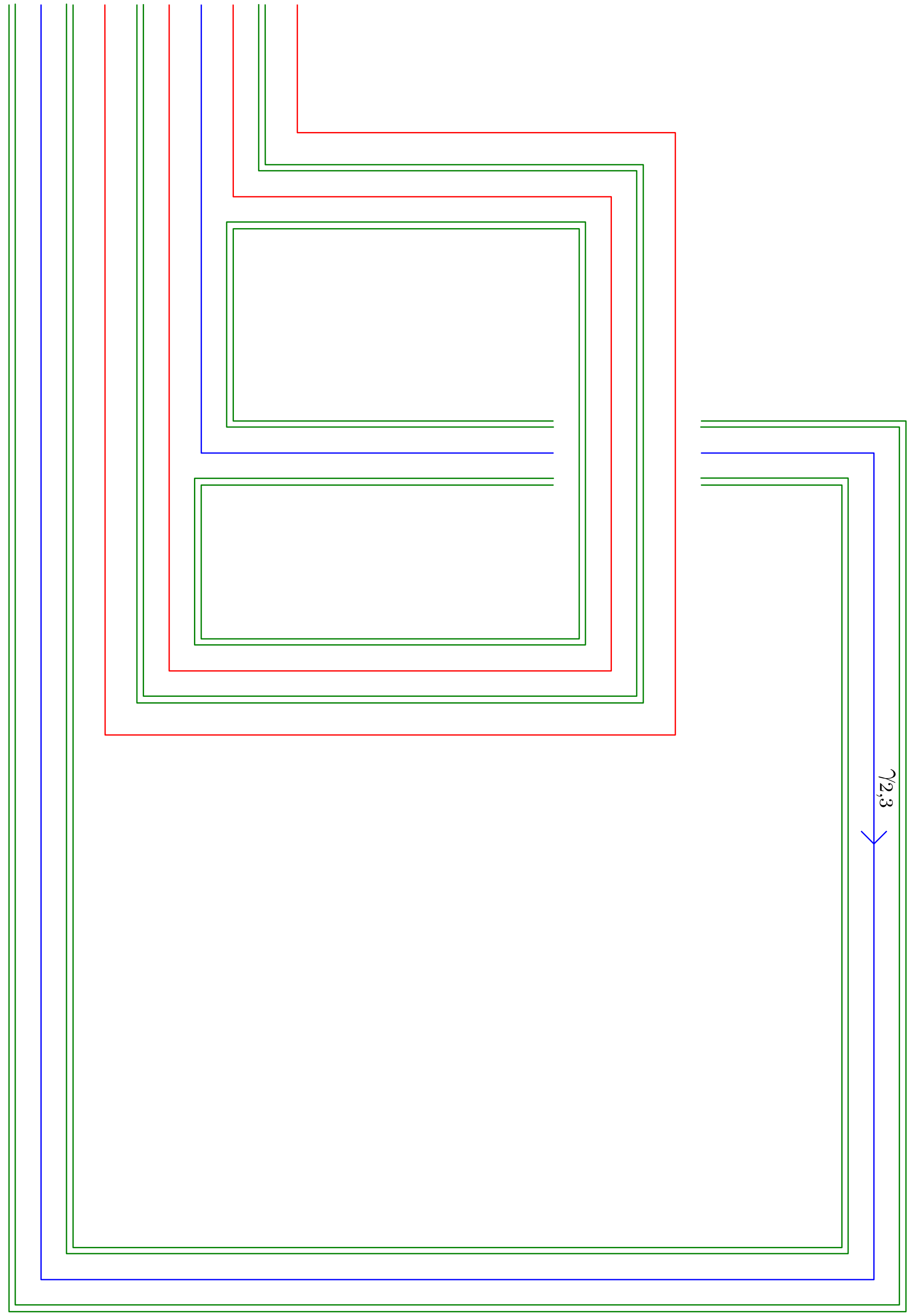

FIGURE 5.18. Link $L_2 = \gamma_{2,1} \cup \gamma_{2,2} \cup \gamma_{2,3}$ embedded in $\widetilde{F_2}$. The surface $\widetilde{F_2}$ is omitted, as its placement is evident in the diagram.

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