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Lower Order Solvability, Seifert Forms, and Blanchfield Forms of Links

by

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Abstract

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We define and study specific generalizations of Seifert forms and Blanchfield forms to links and study their relationships with lower order solvability and with each other. We define Seifert $\mathbb{Z}$-surfaces for links with pairwise linking numbers zero and prove that if a link is 0.5-solvable then every Seifert $\mathbb{Z}$-surface has a metabolizer. We use this result to determine that Arf invariants and Milnor's invariants are not sufficient to classify 0.5-solvable links. We define nonsingular localized Blanchfield forms for links with pairwise linking numbers zero and build on work of Cochran-Orr-Teichner and Cochran-Harvey-Leidy to show that 1-solvability implies each of these Blanchfield forms are hyperbolic. We also define Blanchfield forms on the infinite cyclic covers of the exterior of a link with pairwise linking numbers zero and build on work of Friedl-Powell to prove that in a special case, a Seifert $\mathbb{Z}$-surface having a metabolizer implies the Blanchfield form is hyperbolic. There are well known definitions of boundary Seifert surfaces and multivariable Blanchfield forms for boundary links. We define a boundary metabolizer for a boundary Seifert surface, which is more restrictive than the usual definition of a metabolizer, and prove that the existence of a boundary metabolizer implies both 0.5-solvability and that the multivariable Blanchfield form is hyperbolic.

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Chapter 1

Introduction

A *knot* is a smooth embedding of the circle S^1 into the 3-dimensional sphere S^3. A knot is a special case of a *link*. An *m-component link* is a smooth embedding of m disjoint copies of S^1 into S^3. We use [Rol76] and [Lic97] as references for classical knot theory. These objects are intimately connected to the study of 3- and 4-dimensional manifolds. For example, every closed orientable connected 3-manifold and every smooth 4-manifold can be built with the starting data of a link and integers assigned to each component [Lic62],[Wal60],[Kir78].

A slice knot was originally defined as a cross-section, or slice, of a sphere embedded in 4-dimensional space [FM66]. Slice knots arise in the study of complex hypersurfaces, are related to the failure of the Whitney trick in 4 dimensions, and allow us to give the set of knots a group structure, yielding the *knot concordance group* $\mathcal{C}$. However slice knots are difficult to detect and there is no algorithm to do so. Thus, an important problem in knot theory is approximating sliceness and finding obstructions to sliceness.

One such approximation is algebraic sliceness. Every knot bounds some orientable surface called a *Seifert surface*. To this surface one can associate a matrix called a *Seifert matrix*. If it is congruent to a matrix with a half rank block of zeros, it is said to have a *metabolizer* and the original knot is called *algebraically slice.* All slice knots are algebraically slice and this condition was completely classified by Levine.

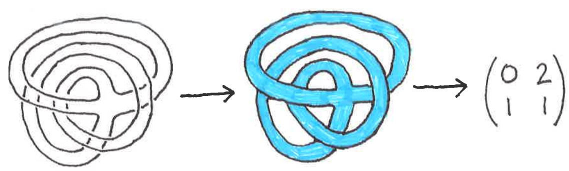

Figure 1.1: A knot, an associated Seifert surface and Seifert matrix [Rol76]. Since the matrix has a half rank block of zeros, the knot is algebraically slice.

One obstruction to sliceness is the *Blanchfield form*, a linking form on the infinite cyclic cover $\widetilde{X}$ of the knot complement. That is, it is a pairing on the first homology group $H_1(\widetilde{X})$, viewed as a $\mathbb{Z}[t^{\pm 1}]$ module. The precise definition is rather technical, but fortunately for knots it is easily computable from Seifert matrices. The Blanchfield form is *hyperbolic* if there exists a totally isotropic submodule $P \subset H_1(\widetilde{X})$, that is $P = P^{\perp}$ with respect to the Blanchfield form. Since the Blanchfield form may be represented by a matrix, finding P reduces to the linear algebra problem of finding a congruent matrix with a half rank block of zeros. By [Kea75], a knot is algebraically slice if and only if its Blanchfield form is hyperbolic.

Refining the notion of algebraic sliceness, the *n-solvable filtration* on the knot concordance group defined by [COT03] gives successively finer approximations of sliceness.

$$\cdots \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

The farther down in the filtration the knot lives, the closer it is to being slice. Unfortunately n-solvability is itself difficult to check so it is important to try to find easily computable algebraic invariants that obstruct or detect when a knot is n-solvable. A knot is 0-solvable if and only if the *Arf invariant* vanishes. A knot is 0.5-solvable if and only if it is algebraically slice and if and only if it has a hyperbolic Blanchfield form. If a knot is 1.5-solvable, then all Casson-Gordon invariants vanish.

There are analogous definitions of sliceness and n-solvability for links, and again we can think of n-solvability as measuring how close a link is to being slice. 0-solvability of links is equivalent to Arf invariants and several *Milnor's invariants* vanishing [Mar], and there is a list of Milnor's invariants that vanish when a link is 0.5-solvable [MO]. Since 0-solvability is classified using Milnor's invariants, we may ask if there is a complete list of Milnor's invariants such that if they all vanish, the Arf invariants vanish, and all components are 0.5-solvable knots, then the link must be 0.5-solvable. However using my results I was able to find a counterexample.

Corollary 3.18. *There exists an example of a 2-component boundary link whose components are unknots that is not* 0.5*-solvable.*

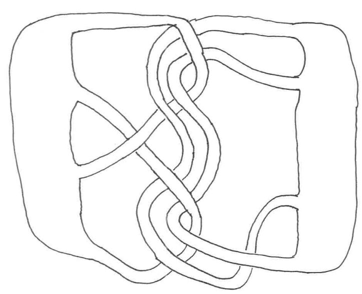

Figure 3.8: A boundary link with unknotted components that is not 0.5-solvable, drawn as the boundary of two disk-band surfaces.

The example link's components are unknots so the Arf invariants vanish. Moreover, the components are 0.5-solvable knots. It is a boundary link hence all Milnor's invariants vanish. Therefore Milnor's invariants are not enough to classify 0.5-solvability. Instead I return to the classification of 0.5-solvability of knots for inspiration, and look at generalizing Seifert forms and Blanchfield forms to links.

In this thesis we study specific generalizations of Seifert forms and Blanchfield forms to links and their relationships to each other and to lower order solvability.

1.1 Summary of Results

First we define and study a specific generalization of Seifert forms to links. We define Seifert $\mathbb{Z}$-surfaces Σ_φ, for m-component links with pairwise linking numbers zero, associated to epimorphisms $\varphi : \mathbb{Z}^m \to \mathbb{Z}$. These surfaces are similar to the Seifert surfaces for multilinks studied by Eisenbud, Neumann, and Cimasoni [EN85][Cim04]. We also define closed Seifert $\mathbb{Z}$-surfaces $\widehat{\Sigma}_\varphi$. We define four Seifert $\mathbb{Z}$-forms $\widehat{\theta}_\varphi, \theta_\varphi, \theta^+_\varphi, \theta^-_\varphi$, each with associated Seifert $\mathbb{Z}$-matrices $\widehat{A}_\varphi, A_\varphi, A^+_\varphi, A^-_\varphi$, corresponding to each Seifert $\mathbb{Z}$-surface Σ_φ. The definitions of $\theta^\pm_\varphi$ and $A^\pm_\varphi$ coincide with the definitions of Seifert matrices for multilinks given by Cimasoni [Cim04].

We prove the following theorem, which is analogous to the result for knots that if a knot is 0.5 solvable, then it is algebraically slice.

Theorem 3.17. *If L is 0.5-solvable, then for every φ, every closed Seifert $\mathbb{Z}$-surface $\widehat{\Sigma}_\varphi$ for L associated to φ has a metabolizer.*

We obtain Corollary 3.18 by applying Theorem 3.17 to the link in Fig. 3.8.

Next we turn our attention to Blanchfield forms. Cochran, Orr, and Teichner generalize the classical Blanchfield form for knots to higher order linking forms for knots on generalized Alexander modules and to linking forms for compact connected oriented 3-manifolds with first Betti number 1 [COT03]. Leidy defines linking forms for any closed connected oriented 3-manifold, and in particular for the zero surgery manifold for a link.

Using methods from [COT03], [CHL09],[CHL08] show that if a link L is 1-solvable, then any localized Blanchfield form has a submodule $P \subset P^\perp$. We show that if the localization is a PID, then additionally $P = P^\perp$ and thus the Blanchfield form is hyperbolic.

We choose specific localizations R_φ which are PIDs, and obtain the following corollary.

Corollary 4.12. *If L is 1-solvable, then $\mathcal{B}\ell^M_{R_\varphi}$ is hyperbolic for all φ.*

We can also define a Blanchfield form $\mathcal{B}\ell^X_{\Lambda_\varphi}$ on the infinite cyclic cover X_φ of the link complement and a Blanchfield form $\mathcal{B}\ell^M_{\Lambda_\varphi}$ on the infinite cyclic cover M_φ of the zero-surgery M of the link. By [FP17], if $H_1(X_\varphi)$ or $H_1(M_\varphi)$ is torsion, then the corresponding Blanchfield form may be calculated in terms of an intersection pairing on the surface Σ_φ or $\widehat{\Sigma}_\varphi$. We use this result to prove that if $H_1(X_\varphi)$ is torsion, the Blanchfield form $\mathcal{B}\ell^X_{\Lambda_\varphi}$ may be computed in terms of Seifert $\mathbb{Z}$-matrices, and in the special case that $\varphi = (1, \ldots, 1)$, we show the following.

Corollary 5.6. *If L has pairwise linking numbers zero and $H_1(X_\varphi)$ is torsion, then $\mathcal{B}\ell^X_{\Lambda_\varphi}$ is represented by the matrix $\left(tA_\varphi - A_\varphi^T\right)$*

Theorem 5.7. *If L has pairwise linking numbers zero and $H_1(X_\varphi)$ is torsion, and a Seifert $\mathbb{Z}$-surface Σ_φ for L has a metabolizer, then $\mathcal{B}\ell^X_{\Lambda_\varphi}$ is hyperbolic.*

We also generalized Friedl and Powell's result to the non-torsion case, but have yet to translate this into Seifert $\mathbb{Z}$-matrices.

Finally we turn to the special case of boundary links. We define what it means for a boundary Seifert surface to have a metabolizer and prove the following.

Theorem 6.3. *Let $L = K_1 \cup \cdots \cup K_m$ be an m-component boundary link. If there exists a boundary Seifert surface $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_m$ for L that has a metabolizer then L is 0.5-solvable.*

Theorem 6.4. *Let L be a boundary link and suppose there exists a boundary Seifert surface Σ for L that has a metabolizer. Then the Blanchfield form $\mathcal{B}\ell$ on $H_1(X;\mathbb{Z}\Gamma)$ is hyperbolic.*

Chapter 2

Link Concordance and n-Solvability

A *knot* is the image of a smooth embedding $S^1 \hookrightarrow S^3$. We require knots to be oriented, with orientation induced by the standard counter-clockwise orientation on S^1. We consider knots up to *isotopy*. That is, two embeddings $f_0, f_1 : S^1 \hookrightarrow S^3$ determine equivalent knots if they are homotopic through smooth embeddings preserving orientation. Thus a knot K refers to the isotopy class of an embedding $f : S^1 \hookrightarrow S^3$ where $f(S^1) = K$, and two knots are called isotopic or equivalent if one can be smoothly deformed into the other.

Two knots K_0 and K_1 are called *concordant* if they cobound an annulus smoothly embedded in $S^3 \times [0,1]$. That is, if there exists a smooth embedding $f : S^1 \times [0,1] \to S^3 \times [0,1]$ such that $f(S^1 \times 0)$ is a copy of K_0 lying in $S^3 \times 0$, and $f(S^1 \times 1)$ is a copy of K_1 lying in $S^3 \times 1$. Related to concordance is the idea of slice knots. A knot in S^3 is called *slice* if it bounds a disk smoothly and properly embedded in the 4-dimensional ball B^4. It turns out that for knots in S^3, being concordant to the unknot is equivalent to being slice, and two knots K and J are concordant if and only if $K\# -J$ is slice, where $K\# -J$ is the connected sum of K and $-J$, the knot obtained from J by switching all crossings and reversing the orientation.

Knot concordance is an equivalence relation on the set of knots. The set of con-

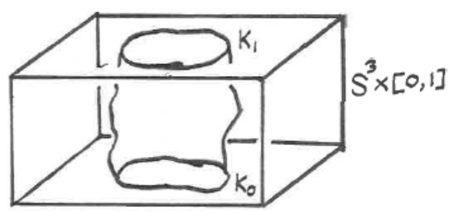

Figure 2.1: A knot concordance

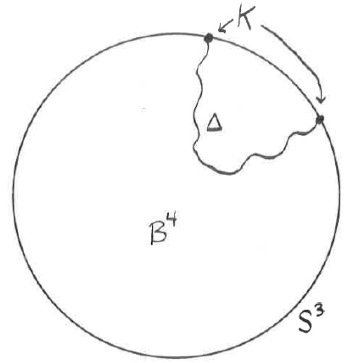

Figure 2.2: A slice knot K with slice disk Δ

cordance classes of knots together with the operation of connected sum forms a group called the *knot concordance group*, denoted $\mathcal{C}$. This group is known to be infinitely generated and abelian. The identity element is the concordance class of the unknot, which is the set of slice knots.

An *m-component link* is a disjoint union of m knots, which are referred to as its link components. That is, an m-component link is the image of a smooth embedding $\bigsqcup_{i=1}^m S^1 \hookrightarrow S^3$. We also require links to be oriented, again with orientation induced by the standard counter-clockwise orientation on S^1. By choosing an ordering of the link components we get an *m-component ordered link* $L = K_1 \cup \cdots \cup K_m$, the image of a smooth embedding $f : \bigsqcup_{i=1}^m S_i^1 \hookrightarrow S^3$, where S_i^1 denotes the ith copy of S^1, such that $f(S_i^1) = K_i$ for each i. We also consider links up to isotopy. Note that a knot is a link with a single component.

Similarly, two m-component ordered links $L_0 = K_1^0 \cup \cdots \cup K_m^0$ and $L_1 = K_1^1 \cup \cdots \cup K_m^1$ are called *concordant* if they cobound m disjoint annuli smoothly embedded in $S^3 \times [0,1]$. That is, if there exists a smooth embedding $f : \bigsqcup_{i=1}^m S_i^1 \times [0,1] \to S^3 \times [0,1]$ such that $f\left(\bigsqcup_{i=1}^m S_i^1 \times 0\right)$ is a copy of L_0 lying in $S^3 \times 0$ and $f\left(\bigsqcup_{i=1}^m S_i^1 \times 1\right)$ is a copy of L_1 lying in $S^3 \times 1$, and for each i the restriction of f to $S_i^1 \times [0,1]$ gives a concordance between K_i^0 and K_i^1. Two m-component links are concordant if for some choice of orderings they are concordant as m-component ordered links. An m-component link

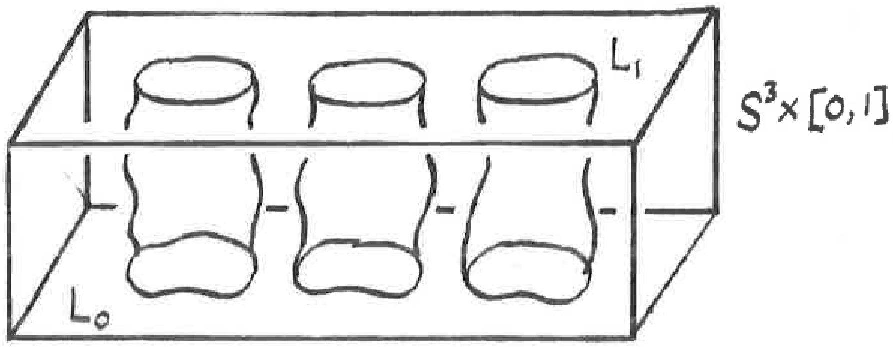

Figure 2.3: A link concordance

in S^3 is called *slice* if it bounds m disjoint disks smoothly and properly embedded in B^4, and an m-component link is slice if and only if it is concordant to the m-component unlink.

Concordance is an equivalence relation on the set of m-component links, but connected sum is not well-defined for links, so we do not naturally get groups.

2.1 The n-Solvable Filtration on the Knot Concordance Group

In 2003, Cochran, Orr, and Teichner defined the n-solvable filtration $\{\mathcal{F}_n\}$ on the knot concordance group $\mathcal{C}$ [COT03]:

$$\{0\} \cdots \subset \mathcal{F}_{n+1} \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

We delay the definition of n-solvability to Section 2.2 where we define it for links. The n-solvable filtration is indexed by the set $\frac{1}{2}\mathbb{N}$. For each n, we have that n-solvability is a concordance invariant and the set of concordance classes of n-solvable knots forms a subgroup $\mathcal{F}_n$ of $\mathcal{C}$. As the word filtration suggests, these subgroups are nested; that is, $(n+1)$-solvability implies $n.5$-solvability implies n-solvability and so on.

Slice knots are n-solvable for all $n \in \frac{1}{2}\mathbb{N}$ and as n approaches infinity, we may think of n-solvable knots as successively finer approximations of slice knots. For each $n \in \mathbb{N}$ there exist n-solvable knots that are not $n.5$-solvable, and all Casson-Gordon invariants vanish for 1.5-solvable knots. The n-solvable filtration was groundbreaking in that it could detect infinitely many classes of knots that are not smoothly slice, while previously known smooth concordance invariants are captured in the lower orders of the filtration.

Theorem 2.1 (Cochran-Orr-Teichner [COT03])**.** *A knot K is 0-solvable if and only if it has trivial Arf invariant.*

Theorem 2.2 (Kearton [Kea75], Cochran-Orr-Teichner [COT03])**.** *For a knot K, the following are equivalent:*

1. *K is* 0.5*-solvable.*
2. *K is algebraically slice.*
3. *The Blanchfield form for K is hyperbolic.*

Kearton proved that (2) is equivalent to (3) and Cochran, Orr, and Teichner proved that (1) implies (2). The fact that (2) implies (1) is known but not written down so we provide a proof here. For this proof we need the following definition and proposition by Martin.

Definition 2.3 (Martin [Mar13])**.** A *double-delta move* on a link L is the local move shown in Figure 2.4. We require that the strands of each band belong to the same link component, so that a double-delta move may involve no more than 3 distinct link components. Links L and L' are called *double-delta equivalent* if L can be transformed into L' through a finite sequence of double-delta moves and isotopy.

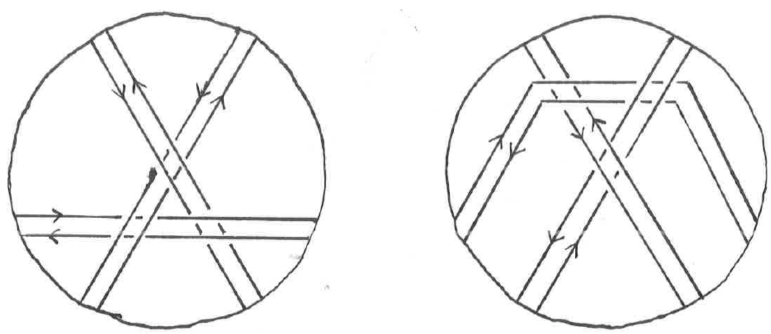

Figure 2.4: A double-delta move

Note that elsewhere in the literature no restriction is made on the strands of the bands, and a double-delta move may involve up to 6 distinct link components. This restriction is important for the proof of the following proposition.

Proposition 2.4 (Martin [Mar13])**.** *The double-delta move preserves* 0.5*-solvability.*

Thus if a link L is double-delta equivalent to a 0.5-solvable link L', then L is also 0.5-solvable. Now we are ready to prove (2) implies (1).

Proposition 2.5. *If a knot K is algebraically slice, then K is* 0.5*-solvable.*

Proof. Since K is algebraically slice, it has a Seifert surface Σ and a Seifert matrix $A = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$. Isotope Σ into disk-band form such that the cores of half the bands represent a basis $\{a_1, \ldots, a_g\}$ for the metabolizer, as shown in Figure 2.5. The curves $a_1, \ldots, a_g$ form a link J with pairwise linking numbers zero. Two links have the same sets of pairwise linking numbers if and only if they are equivalent under delta moves [MN89]. Hence J is delta equivalent to an unlink $J' = a_1' \cup \cdots \cup a_g'$. The set of delta moves that transform J into J' corresponds to a set of delta moves on the bands of Σ, transforming Σ into a new surface Σ', and this corresponds to a set of double-delta moves on the boundary of the bands transforming K into K'. Hence K is double-delta equivalent to K' which bounds Σ'. Now that J' is an unlink, we may cut open Σ' along J' and then capping with $2g$ disks as shown in Figure 2.6. Thus K' is a ribbon

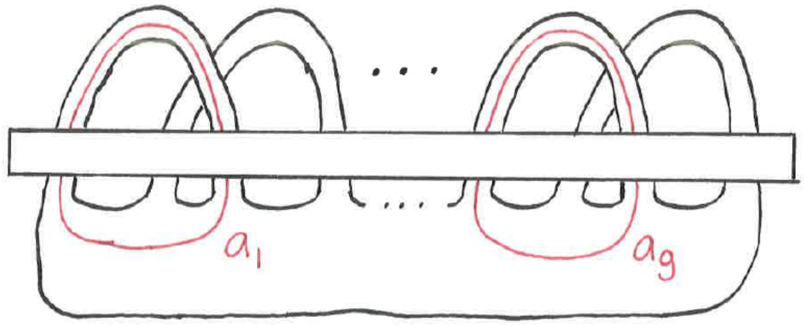

Figure 2.5: Σ in disk-band form with metabolizer represented by $a_1, \ldots, a_g$. The box is a string link on the bands.

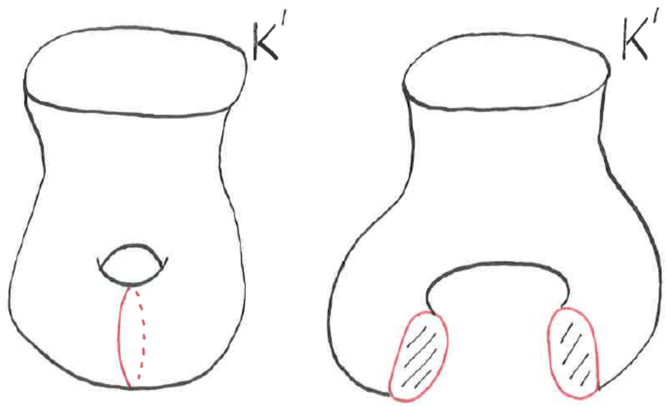

Figure 2.6: An example of cutting and capping along J' for a genus 1 surface

knot. Now we have the K is doubled delta equivalent to a ribbon knot. Double-delta moves preserve 0.5-solvability [Mar13]. Therefore K is 0.5-solvable. $\square$

2.2 n-Solvability of Links

Cochran, Orr, and Teichner also defined n-solvability for links [COT03]. As for knots, n-solvability is a concordance invariant, and $(n+1)$-solvability implies $n.5$-solvability implies n-solvability, etc. Additionally, if a link is n-solvable, then every sublink is also n-solvable. Slice links are n-solvable for all $n \in \frac{1}{2}\mathbb{N}$, so again as n approaches infinity we may think of n-solvable links as successively finer approximations of slice links. Before we give the definition of n-solvability, we recall the definitions of the

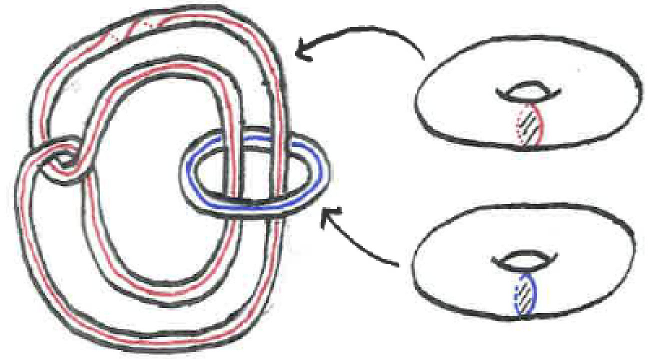

Figure 2.7: The zero surgery manifold for the Whitehead link

zero surgery manifold for a link, and the derived series of a group.

Given an m-component link $L = K_1 \cup \cdots \cup K_m \subset S^3$, we obtain the *zero surgery manifold* M_L, a closed, connected, oriented 3-manifold, by deleting a tubular neighborhood of L from S^3 and gluing in m solid tori so that the meridians of the solid tori are glued to the longitudes of the link components. That is, $M_L = (S^3 - N(L)) \cup_f (\bigsqcup_{i=1}^m S^1 \times D^2)$, where $N(L)$ denotes a tubular neighborhood of L and the attaching map f sends the meridian $\{p\} \times \partial D^2$ of the ith solid torus to the longitude of the ith link component, an untwisted copy of K_i on the boundary of $S^3 - N(L)$.

Given a group G, the *derived series* $G^{(n)}$ of G is defined recursively by $G^{(0)} = G$, $G^{(1)} = [G, G]$, the commutator subgroup of G, and in general for $n \geq 1$, $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$.

Definition 2.6. An m-component link L is called *n-solvable* for $n \in \mathbb{Z}_{\geq 0}$ and $L \in \mathcal{F}_n^m$ if the zero surgery manifold M_L bounds a compact, oriented, smooth 4-manifold W such that the following is satisfied:

1. W is an H_1-bordism. That is, the map on first homology induced by inclusion, $H_1(M_L; \mathbb{Z}) \to H_1(W; \mathbb{Z})$, is an isomorphism.

2. $H_2(W; \mathbb{Z})$ has a basis represented by embedded surfaces $\{L_i, D_i\}_{i=1}^r$ with trivial

normal bundles, such that L_i and D_i intersect transversely and geometrically exactly once, and otherwise the surfaces are disjoint. Hence the intersection form for W with respect to this basis looks like $\bigoplus_{i=1}^{r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. For each i, $\pi_1(L_i) \subset \pi_1(W)^{(n)}$ and $\pi_1(D_i) \subset \pi_1(W)^{(n)}$.

Then W is called an n-*solution* for L.

An m-component link L is called $n.5$-*solvable* and $L \in \mathcal{F}^m_{n.5}$ if L is n-solvable with n-solution W and additionally $\pi_1(L_i) \subset \pi_1(W)^{(n+1)}$ for each i. Then W is called an $n.5$-*solution* for L.

We will focus on lower order solvability. When $n = 0$, the third condition is trivially satisfied. When $n = 1$, it means that the surfaces lift to the universal abelian cover of W. Thus, a link L is 0.5-solvable if there exists a 0-solution W for L such that the surfaces L_i lift to the universal abelian cover of W. A link L is 1-solvable if there exists a 0-solution W for L such that both the surfaces L_i and D_i lift to the universal abelian cover of W.

Recall that for knots 0 and 0.5-solvability is classified. In 2013 Martin classified 0-solvability of links using Arf invariants and Milnor's invariants, which we will not define here.

Theorem 2.7 (Martin [Mar13],[Mar])**.** *An m-component link $L = K_1 \cup \cdots \cup K_m$ is* 0-*solvable if and only if the following conditions hold:*

1. Arf$(K_i) = 0$.

2. $\overline{\mu}_L(ijk) = 0$.

3. $\overline{\mu}_L(iijj) \equiv 0$ mod 2.

Martin also found an additional necessary condition for 0.5-solvability of links using Milnor's invariants.

Theorem 2.8 (Martin-Otto [Mar13],[MO])**.** *If an m-component link L is* 0.5*-solvable, then the Sato-Levine invariants $\overline{\mu}_L(iijj) = 0$.*

One might ask if 0.5-solvability of links can be classified using Arf invariants and Milnor's invariants. However, this is not possible, even for 2-component links. In Corollary 3.18 we find an example of a 2-component boundary link with unknotted components that is not 0.5-solvable. The unknotted components guarantee that the Arf invariants vanish, and moreover that the components are 0.5-solvable knots, and the fact that it is a boundary link guarantees that all Milnor's invariants vanish. Therefore this example shows that Arf invariants and Milnor's invariants are not enough to classify 0.5-solvability of links.

Chapter 3

Seifert Forms

A *Seifert surface* for a knot K is a bicollared compact connected orientable surface Σ smoothly embedded in S^3 such that $\partial\Sigma = K$. Seifert surfaces exist for all knots, in fact, there is an algorithm called *Seifert's algorithm* for constructing a Seifert surface given any knot diagram. Given a Seifert surface Σ for a knot K, we define the *Seifert form* $\theta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ by $\theta(x, y) = \ell k(x^+, y)$, where $\ell k(\cdot, \cdot)$ is the usual linking number in S^3, and x^+ is the positive push-off of x, that is, the curve in $S^3 - \Sigma$ obtained by pushing x in the positive normal direction off of Σ and into its bicollar. Any matrix A representing θ with respect to some basis of $H_1(\Sigma)$ is called a *Seifert matrix* for K.

Given a Seifert surface Σ for a knot K, a *metabolizer* for Σ is a half-rank direct summand H of $H_1(\Sigma)$ on which the Seifert form θ vanishes. That is, for all $x, y \in H$ we have $\theta(x, y) = 0$. Equivalently, Σ has a metabolizer if for some choice of basis for $H_1(\Sigma)$ there is a corresponding Sefiert matrix A of the form $A = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$, where each block is square.

There are several ways to generalize Seifert matrices to links. A classical Seifert surface for a link L is simply a bicollared compact connected orientable surface Σ

smoothly embedded in S^3 such that $\partial\Sigma = L$. Classical Seifert surfaces for links may be constructed by applying Seifert's algorithm to a link diagram, and then connected the surface as needed with tubes.

We define Seifert $\mathbb{Z}$-surfaces for links with pairwise linking numbers zero. These surfaces turn out to be similar to the "Seifert surfaces for multilinks" studied by Eisenbud, Neumann and Cimasoni [EN85][Cim04]. Classical Seifert surfaces for links are special cases of Seifert $\mathbb{Z}$-surfaces.

3.1 Seifert $\mathbb{Z}$-Surfaces, Forms, and Matrices

Let X be a smooth, compact, connected, oriented 3-manifold that is either closed or has toroidal boundary and let $\Gamma = H_1(X)$. Choose a primitive $\varphi \in H^1(X) \cong \mathrm{Hom}(H_1(X), \mathbb{Z})$ so that we may consider φ to be an epimorphism $\varphi : \Gamma \to \mathbb{Z}$. Since S^1 is a $K(\mathbb{Z}, 1)$, we have $\mathrm{Hom}(\Gamma, \mathbb{Z}) \cong [X, S^1]$, the set of homotopy classes of continuous maps $X \to S^1$, and any continuous map may be approximated by a homotopic smooth map, there exists a smooth map $f : X \to S^1$ such that the induced map on first homology $f_* : H_1(X) \to H_1(S^1)$ is exactly equal to φ. Pull back a regular point to obtain a smoothly embedded, compact, oriented surface we'll call Σ_φ. If X is closed then so is Σ_φ and if X has toroidal boundary, then Σ_φ has boundary and is properly embedded. If we pull back a neighborhood of the regular point we obtain a bicollar for Σ_φ. The surface Σ_φ is called a surface *dual* to the primitive class $\varphi \in H^1(X)$.

Now let $X = X_L = S^3 - N(L)$ be the exterior of an m-component link $L = K_1 \cup \cdots \cup K_m$. Recall that $N(L) = N(K_1) \cup \cdots \cup N(K_m)$ is a tubular neighborhood of L and is homeomorphic to the disjoint union of m solid tori. Define the *ith meridian of* L, or the *meridian of* K_i, as the simple closed curve μ_i embedded in the torus $\partial N(K_i)$ such that the inclusion of μ_i into $N(K_i)$ is homotopically trivial and $\ell k(K_i, \mu_i) = 1$ Define the *ith longitude of* L, or the *longitude of* K_i, as the simple closed curve λ_i embedded in the torus $\partial N(K_i)$ such that λ_i and μ_i intersect exactly once geometrically and the linking number $\ell k(K_i, \lambda_i) = 0$. Both μ_i and λ_i are unique up to isotopy on the torus.

For a link exterior X_L, $\Gamma = H_1(X) \cong \mathbb{Z}^m$ and is generated by $\{\mu_1, \dots, \mu_m\}$, the set of meridians of L. Then a homomorphism $\varphi : \Gamma \to \mathbb{Z} = \langle t \rangle$ is given by $\varphi(\mu_i) = t^{k_i}$ for each $i = 1, \dots, m$ where each $k_i \in \mathbb{Z}$. The map is completely determined by the integers k_i, so we will write $\varphi = (k_1, \dots, k_m)$. In order for the map to be an epimorphism and thus represent a primitive class in $H^1(X)$, we must have $\gcd(k_1, \dots, k_m) = 1$.

Definition 3.1. Let $X_L = S^3 - N(L)$ be the exterior of an m-component link $L = K_1 \cup \dots \cup K_m$ with pairwise linking numbers zero and let Σ_φ be a surface dual to a primitive class $\varphi = (k_1, \dots, k_m)$, so Σ_φ is compact, oriented, bicollared, and smoothly and properly embedded in $X_L = S^3 - N(L)$. The surface Σ_φ is called a *Seifert $\mathbb{Z}$-surface for L associated to φ* if the following conditions are satisfied.

1. Σ_φ is connected.
2. If $k_i \neq 0$, then Σ_φ has exactly $|k_i|$ boundary components on $\partial N(K_i)$, oriented such that $\partial \Sigma_\varphi = k_i \lambda_i$ in $H_1(\partial N(K_i))$. See Figure 3.1
3. If $k_i = 0$, then $\Sigma_\varphi \cap N(K_i)$ is empty. Σ_φ misses $N(K_i)$ completely.

Note that these surfaces are similar to those studied in [EN85] [Cim04].

Theorem 3.2 (Eisenbud-Neumann, Cimasoni [EN85][Cim04])**.** *For any link L with pairwise linking numbers zero and any primitive class $\varphi \in H^1(X_L)$, there exists a Seifert $\mathbb{Z}$-surface for L associated to φ.*

That there exists a dual surface satisfying conditions (2) and (3) follows from a lemma in [EN85] and that there exists such a dual surface that is connected follows from a lemma in [Cim04].

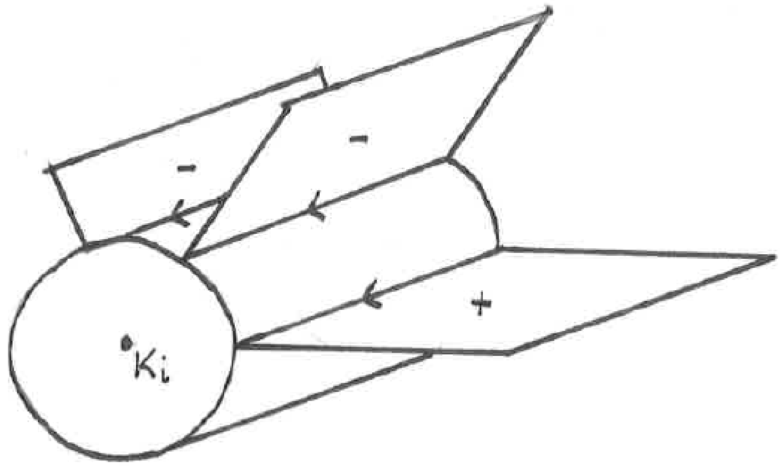

Figure 3.1: A Seifert $\mathbb{Z}$-surface meeting $N(K_i)$ for $|k_i| = 3$

Let $X = X_L = S^3 - N(L)$ be the exterior of a link L with pairwise linking numbers zero, and let Σ_φ be a Seifert $\mathbb{Z}$-surface for L associated to φ. Define the associated *Seifert $\mathbb{Z}$-form* to be the pairing $\theta_\varphi : H_1(\Sigma_\varphi) \times H_1(\Sigma_\varphi) \to \mathbb{Z}$ by $\theta_\varphi(x, y) = \ell k(x^+, y)$, where $\ell k(\cdot, \cdot)$ is the usual linking number in S^3, and x^+ is the positive push-off of x. That is, since Σ_φ has a bicollar, we actually have $N(\Sigma_\varphi) \cong \Sigma_\varphi \times [-1, 1]$ properly embedded in X_L, where Σ_φ is identified with $\Sigma_\varphi \times 0$. If we abuse notation and let x be a curve on Σ_φ representing the homology class x, then $x \times [-1, 1]$ is embedded in $\Sigma_\varphi \times [-1, 1]$ and let x^+ be $x \times 1$.

We may also define Seifert $\mathbb{Z}$-forms $\theta^\pm_\varphi : H_1(\Sigma_\varphi) \times H_1(\Sigma_\varphi \cup N(L)) \to \mathbb{Z}$ by $\theta^+_\varphi(x, y) = \ell k(x^+, y)$ and $\theta^-_\varphi(x, y) = \ell k(x^-, y)$, where x^- is the negative push-off of x. These forms are analagous to those defined in [Cim04].

The definition of n-solvability of L depends on the zero surgery manifold M_L and not the link exterior X_L. So it would be helpful to define Seifert $\mathbb{Z}$-forms for M_L.

Recall that for an m-component link L, $H_1(X_L)$ is generated by the meridians $\mu_1, \ldots, \mu_m$ and the homology class of a simple closed curve c in X_L is given by $\sum_{j=1}^m \ell k(c, K_j)\mu_j$. Recall that $M_L = X_L \cup_f \bigsqcup_{i=1}^m S^1 \times D^2$ where the attaching map f sends the meridian $\{p\} \times \partial D^2$ of the ith solid torus to λ_i, the ith longitude of L. It is easily computable by a Mayer-Vietoris sequence that the effect on homology of gluing in these solid tori is killing the longitudes of L. So $H_1(M) = H_1(X_L)/\langle \lambda_1, \ldots, \lambda_m \rangle$. In $H_1(X_L)$, the ith longitude is given by $\lambda_i = \sum_{j=1}^{m} \ell k(\lambda_i, K_j)\mu_j = \sum_{j \neq i} \ell k(K_i, K_j)\mu_j$. Thus $H_1(M_L) = H_1(X_L)$ if and only if L has pairwise linking numbers zero.

Given a primitive $\varphi = (k_1, \ldots, k_m)$ and a Seifert $\mathbb{Z}$-surface $\Sigma_\varphi \subset X_L = S^3 - N(L)$ for a link L with pairwise linking numbers zero, let $\widehat{\Sigma}_\varphi$ denote the closed surface embedded in the zero surgery manifold M_L obtained by capping off the boundary components of Σ_φ with disks coming from the zero surgery, as shown in Figure 3.2, and call it a *closed Seifert $\mathbb{Z}$-surface for L associated to φ*. Since L has pairwise

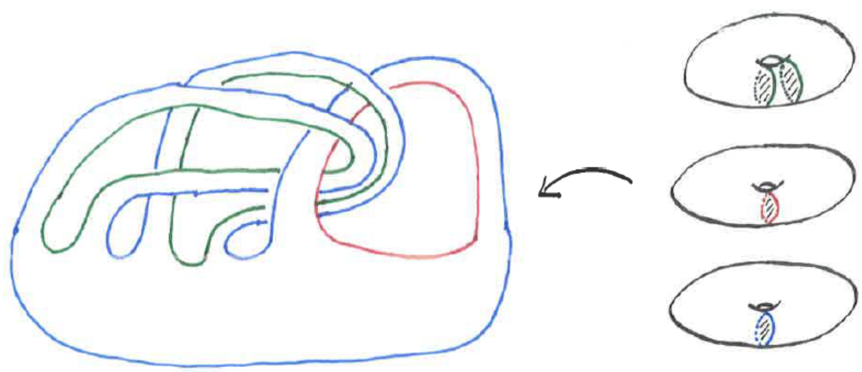

Figure 3.2: An example of a closed Seifert $\mathbb{Z}$-surface for a 3-component link with $\varphi = (2, 1, 1)$

linking numbers all zero, $H_1(M_L) = H_1(X_L)$ so the class $\varphi \in H^1(X_L)$ represented as an epimorphism $H_1(X_L) \to \mathbb{Z}$ with $\varphi(\mu_i) = t^{k_i}$ translates directly to an epimorphism $H_1(M_L) \to \mathbb{Z}$ with $\varphi(\mu_i) = t^{k_i}$ representing a primitive class $\varphi \in H^1(M_L)$. To check that $\widehat{\Sigma}_\varphi$ is a surface dual to $\varphi \in H^1(M_L)$, we need to check that the intersection number of $\widehat{\Sigma}_\varphi$ with each meridian μ_i equals k_i.

We wish to define a Seifert type pairing $H_1(\widehat{\Sigma}_\varphi) \times H_1(\widehat{\Sigma}_\varphi) \to \mathbb{Z}$. Note that linking number is only defined for simple closed curves in S^3, but $\widehat{\Sigma}_\varphi$ lives in S^3 except for finitely many disks, call them $D_1, \ldots, D_{h-1}$, which are contractible. So given a simple closed curve c in $\widehat{\Sigma}_\varphi$, we can isotope c off each disk D_i so that it lies entirely in Σ_φ, and thus in S^3. Choose a triangulation of $\widehat{\Sigma}_\varphi$ so that it restricts to a triangulation of D_i for each $i = 1, \ldots, h-1$. Recall that $Z_1(Y) \subseteq C_1(Y)$ is the free abelian group generated by the 1-cycles of Y.

Definition 3.3. Define a mapping $Z_1(\widehat{\Sigma}_\varphi) \to Z_1(\Sigma_\varphi)$ by $c \mapsto \check{c}$ where $\check{c}$ is obtained from c by eliminating intersection with the disks D_i in the following way. If c intersects D_i in an arc a, replace a with an arc d on ∂D_i such that $\partial d = \partial a$, as in Figure 3.3. If c intersects D_i in a circle c', then first choose an arc d on ∂D_i such that $\partial d \neq 0$. The circle c' is nullhomotopic in $\widehat{\Sigma}_\varphi$ so replace it with the 1-cycle $d - d$, as in Figure 3.4.

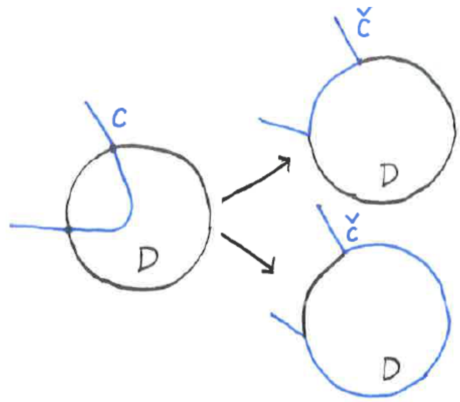

Figure 3.3: Pushing an arc off of D_i

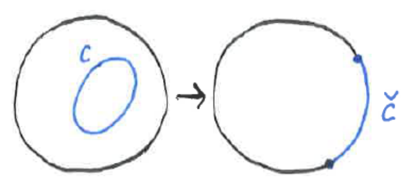

Figure 3.4: Pushing a circle off of D_i

Note that the mapping $c \mapsto \check{c}$ is not well defined on homology when the surface has more than one boundary component. Consider Figure 3.5. Σ is a genus 2 surface with 2 boundary components and $\widehat{\Sigma}$ is the surface with the boundary components filled in with disks. The red and blue simple closed curves do not intersect the disks, so the mapping will not change them. It is clear that they are homologous in $\widehat{\Sigma}$, but are not homologous in Σ.

Proposition 3.4. *When L has pairwise linking numbers zero, the pairing $H_1(\widehat{\Sigma}_\varphi) \times H_1(\widehat{\Sigma}_\varphi) \to \mathbb{Z}$ defined by $(x, y) \mapsto \ell k(\check{x}^+, \check{y})$ is well-defined.*

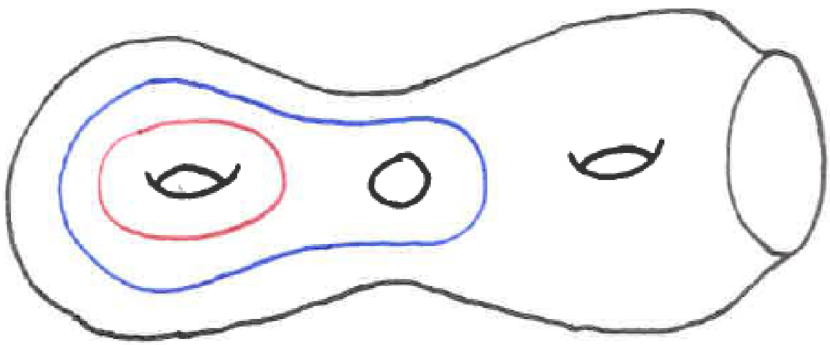

Figure 3.5: The mapping $c \mapsto \check{c}$ is undefined.

Proof. Let $\{a_1, \ldots, a_{2g}\}$ be a basis for $H_1(\widehat{\Sigma}_\varphi)$ and let $\{a_1, \ldots, a_{2g}\} \cup \{b_1, \ldots, b_{h-1}\}$ be a basis for $H_1(\Sigma_\varphi)$. Suppose that $x = x'$ and $y = y'$ in $H_1(\widehat{\Sigma}_\varphi)$. Then under the mapping they can only differ in $H_1(\Sigma_\varphi)$ by boundary components. So $\check{x} - \check{x}' = \sum_{i=1}^{h-1} p_i b_i$ and $\check{y} - \check{y}' = \sum_{j=1}^{h-1} q_j b_j$ for some integers p_i, q_i. Then by the bilinearity of the linking number, and by the fact that the mapping $x \mapsto x^+$ is an isomorphism, we obtain the following.

$$\begin{aligned} \ell k\left(\check{x}^+, \check{y}\right) - \ell k\left(\check{x}'^+, \check{y}'\right) &= \ell k\left((\check{x} - \check{x}')^+, \check{y} - \check{y}'\right) \\ &= \ell k\left(\sum_{i=1}^{h-1} p_i b_i^+, \sum_{j=1}^{h-1} q_j b_j\right) \\ &= \sum_{i=1}^{h-1}\sum_{j=1}^{h-1} p_i q_j \ell k\left(b_i^+, b_j\right) \end{aligned}$$

This equals 0, since the pairwise linking numbers of L are all 0. $\square$

Define the Seifert $\mathbb{Z}$-form $\widehat{\theta}_\varphi : H_1(\widehat{\Sigma}_\varphi) \times H_1(\widehat{\Sigma}_\varphi) \to \mathbb{Z}$ by $\widehat{\theta}_\varphi(x, y) = \ell k(\check{x}^+, \check{y})$. A matrix $\widehat{A}_\varphi$ representing $\widehat{\theta}_\varphi$ is called a *Seifert $\mathbb{Z}$-matrix for L associated to the closed Seifert $\mathbb{Z}$-surface* $\widehat{\Sigma}_\varphi$.

Definition 3.5. Let L be an m-component link with pairwise linking numbers zero and let $\varphi = (k_1, \dots, k_m)$ such that $1 \leq \ell \leq m$ entries of the list are nonzero. Let Σ_φ be a Seifert $\mathbb{Z}$-surface for L associated to φ with genus g and h boundary components. Let θ_φ, θ_φ^+, and θ_φ^- be the associated Seifert $\mathbb{Z}$-forms. Let B be an ordered basis for $H_1(\Sigma_\varphi)$ and let C be an ordered basis for $H_1(\Sigma_\varphi \cup N(L))$ satisfying the following conditions.

1. The first $2g$ elements of B form a basis for $H_1(\widehat{\Sigma}_\varphi)$.
2. The remaining $h-1$ elements of B are homology classes of boundary components of Σ_φ.
3. The first $2g$ elements of C form a basis for $H_1(\widehat{\Sigma}_\varphi)$.
4. The next $m-1$ elements of C are homology classes of longitudes of L.

Then the matrix A_φ representing θ_φ with respect to the basis B, and the matrices $A_\varphi^\pm$ respectively representing $\theta_\varphi^\pm$ with respect to the bases B and C, are called *Seifert $\mathbb{Z}$-matrices for L* associated to the Seifert $\mathbb{Z}$-surface Σ_φ.

Note that when $\varphi = (1, \dots, 1)$, the definition of the matrix A_φ is near identical to Gee's "ordered Seifert matrices" [Gee08].

Suppose L is an m-component link with pairwise linking numbers zero and Σ_φ is a Seifert $\mathbb{Z}$-surface for L, with genus g and h boundary components. Then $H_1(\Sigma_\varphi)$ has rank $r = 2g + h - 1$. Let $B = \{a_1, \dots, a_{2g}\} \cup \{b_1, \dots, b_{h-1}\}$ be a basis.

Proposition 3.6. *A Seifert $\mathbb{Z}$-matrix A_φ has the form of a block matrix*

$$A_\varphi = \begin{pmatrix} A & C^T \\ C & 0 \end{pmatrix}$$

where the $2g \times 2g$ block $A = \widehat{A}_\varphi$ is a Seifert $\mathbb{Z}$-matrix for $\widehat{\Sigma}_\varphi$.

Proof. The sublist of B given by $B' = \{a_1, \ldots, a_{2g}\}$ is a basis for $H_1(\widehat{\Sigma}_\varphi)$ so θ_φ restricted to the subgroup of $H_1(\Sigma_\varphi)$ generated by B' is exactly $\widehat{\theta}_\varphi$. Then ijth entry of A is given by $\theta_\varphi(a_i, a_j) = \widehat{\theta}_\varphi(a_i, a_j)$, so $A = \widehat{A}_\varphi$, a Seifert $\mathbb{Z}$-matrix for $\widehat{\Sigma}_\varphi$.

The $(h-1) \times 2g$ block C has entries given by $\theta_\varphi(b_i, a_j) = \ell k(b_i^+, a_j)$. But since the b_j are boundary components, they do not intersect any of the a_i, so $\ell k(b_i^+, a_j) = \ell k(b_i, a_j)$. Then since linking number is symmetric, the other off-diagonal block is given by C^T, the transpose of C.

The bottom right $(h-1)\times(h-1)$ block depends only on the boundary components. Since the boundary components are disjoint and do not intersect, the ijth entry of this block is given by $\theta_\varphi(b_i, b_j) = \ell k(b_i^+, b_j) = \ell k(b_i, b_j)$. Each b_i is represented by a curve on $\partial N(K_k)$, for some k, that is homotopic to the kth longitude λ_k in $\partial N(K_k)$, which is in turn homotopic to K_k in S^3. For $i \neq j$, say that b_i is homotopic to K_k and b_j is homotopic to K_ℓ. If $k \neq \ell$, then $\ell k(b_i, b_j) = \ell k(K_k, K_\ell) = 0$, since the pairwise linking numbers of L are all zero. If $k = \ell$, then $\ell k(b_i, b_j) = 0$ since b_i and b_j are homotopic. Now for the case when $i = j$. Let b_h be the homology class of the hth boundary component of Σ_φ. Then the sum $\sum_{i=1}^{h} b_i = 0 \in H_1(\Sigma_\varphi)$ since it is represented by the boundary of Σ_φ. Then for each $1 \leq i \leq h-1$ we have the following.

$$
\begin{aligned}
0 &= \theta_\varphi\left(b_i, \sum_{j=1}^{h} b_j\right) \\
&= \sum_{j=1}^{h} \theta_\varphi(b_i, b_j) \\
&= \sum_{j \neq i} \theta_\varphi(b_i, b_j) + \theta_\varphi(b_i, b_i) \\
&= \theta_\varphi(b_i, b_i)
\end{aligned}
$$

Since we can use the same logic as for the $i \neq j$ case to show that $\theta_\varphi(b_i, b_h) = 0$. $\square$

Similarly, we can partition the Seifert $\mathbb{Z}$-matrices $A_\varphi^\pm$ into block matrices. The proof is very similar, so is omitted.

Proposition 3.7. *Seifert $\mathbb{Z}$-matrices $A_\varphi^\pm$ have the form of block matrices*

$$A_\varphi^+ = \begin{pmatrix} A & D & E^+ \\ C & 0 & F^+ \end{pmatrix} \qquad\qquad A_\varphi^- = \begin{pmatrix} A^T & D & E^- \\ C & 0 & F^- \end{pmatrix}$$

where the $2g \times 2g$ block $A = \widehat{A}_\varphi$ is a Seifert $\mathbb{Z}$-matrix for $\widehat{\Sigma}_\varphi$. Additionally, if A_φ is a Seifert $\mathbb{Z}$-matrix for L with respect to the same basis B used for $A_\varphi^\pm$, and A_φ is partitioned into a block matrix as in the statement of Proposition 3.6, then the blocks A and C are the same as those in A_φ.

3.2 The Infinite Cyclic Covers

Let X be a compact, connected, oriented 3-manifold that is either closed or has toroidal boundary. Choose a primitive $\varphi \in H^1(X) \cong \operatorname{Hom}(H_1(X), \mathbb{Z})$, so that we may consider φ to be an epimorphism $H_1(X) \to \mathbb{Z}$, and let Σ_φ be a surface dual to φ. Let X_φ be the regular infinite cyclic cover of X associated to the kernel of the composition $\pi_1(X) \xrightarrow{ab} H_1(X;\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z}$, where the first map is abelianization. The group of covering transformations $\operatorname{Deck}(X_\varphi) = \pi_1(X)/\ker(\varphi \circ ab) = H_1(X)/\ker\varphi$. Since φ is an epimorphism, $\operatorname{Deck}(X_\varphi) = \mathbb{Z}$. So X_φ is indeed an infinite cyclic cover. Let Λ_φ denote the group ring $\mathbb{Z}[\Gamma/\ker\varphi] \cong \mathbb{Z}[t^{\pm 1}]$. Although the rings Λ_φ are all isomorphic to each other, the modules $H_1(X_\varphi) = H_1(X;\Lambda_\varphi)$ are not in general isomorphic, since the coefficients are twisted by φ.

Now we will construct X_φ. Let $N = \Sigma_\varphi \times I$ be the bicollar of the surface, and let $Y = X - N^\circ$ be the complement. Let $\Sigma^\pm = \Sigma_\varphi \times \{\pm 1\}$, and let $\iota_\pm :$ $H_1(\Sigma_\varphi;\mathbb{Z}) \cong H_1(\Sigma^\pm;\mathbb{Z}) \to H_1(Y;\mathbb{Z})$. Take infinitely many copies Y_i and N_i, and glue them together as in Figure 3.6 so that $\Sigma_i^+ \subset N_i$ is identified with $\Sigma_i^+ \subset Y_i$ and $\Sigma_i^- \subset N_i$ is identified with $\Sigma_{i-1}^- \subset Y_{i-1}$. We claim that the space we have constructed

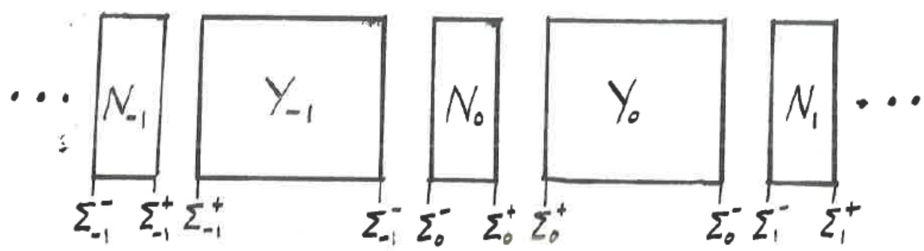

Figure 3.6: Gluing instructions for constructing X_φ

is the infinite cyclic cover X_φ, and we use the convention that in the total space X_φ, we have Σ_i^- lies in Y_{i-1}.

When we identify Σ_φ with its unique lift Σ_0 to X_φ and similarly identify Y with Y_0, the maps $\iota_\pm$ induce maps $\iota_+ : H_1(\Sigma_0;\mathbb{Z}) \cong H_1(\Sigma_0^+;\mathbb{Z}) \to H_1(Y_0;\mathbb{Z})$ and $\iota_- : H_1(\Sigma_0;\mathbb{Z}) \cong H_1(\Sigma_0^-;\mathbb{Z}) \to H_1(Y_{-1};\mathbb{Z})$.

Let $\iota_Y : H_1(Y_0;\mathbb{Z}) \to H_1(X_\varphi;\mathbb{Z})$ be the map induced by inclusion. Let $\iota_\Sigma : H_1(\Sigma_0;\mathbb{Z}) \to H_1(X_\varphi;\mathbb{Z})$ be the composition $\iota_\Sigma = \iota_Y \circ \iota_+$ and ι_Σ extends to a map on $H_1(N_0;\mathbb{Z})$.

We look at the Mayer-Vietoris sequence for the pair $(\sqcup_{i\in\mathbb{Z}} N_i, \sqcup_{i\in\mathbb{Z}} Y_i)$. Their intersection is the disjoint union of $\sqcup_{i\in\mathbb{Z}}\Sigma_i^+$ and $\sqcup_{i\in\mathbb{Z}}\Sigma_i^-$ and their union is X_φ. We get the long exact sequence with $\mathbb{Z}$ coefficients:

$$\cdots \xrightarrow{\partial_{k+1}} H_k\left(\sqcup\Sigma_i^+\right) \oplus H_k\left(\sqcup\Sigma_i^-\right) \xrightarrow{f_k} H_k\left(\sqcup N_i\right) \oplus H_k\left(\sqcup Y_i\right) \xrightarrow{g_k} H_k(X_\varphi) \xrightarrow{\partial_k} \cdots$$

Since Σ_φ, $\Sigma^\pm$, N, and Y are all in the kernel of φ, their homology groups with Λ coefficients twisted by φ have the form $H_k(*;\Lambda_\varphi) \cong H_k(*;\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda \cong H_k(\sqcup_{i\in\mathbb{Z}}(*)_i)$, and $H_k(X;\Lambda_\varphi) = H_k(X_\varphi;\mathbb{Z})$. So we may relabel the terms of the sequence and we move from $(*)_i$ to $(*)_{i+1}$ by multiplying by t. Then we have the following long exact sequence with Λ coefficients.

$$\cdots \xrightarrow{\partial_{k+1}} H_k\left(\Sigma^+\right) \oplus H_k\left(\Sigma^-\right) \xrightarrow{f_k} H_k\left(N\right) \oplus H_k\left(Y\right) \xrightarrow{g_k} H_k(X) \xrightarrow{\partial_k} \cdots$$

This also allows the maps $\iota_\pm$, ι_Σ and ι_Y to induce maps on the homology groups with Λ coefficients twisted by φ.

The map f_k is given by inclusion into the first component and negative inclusion into the second. Since $\Sigma^\pm$ is already in N, this part of the map is just identity. Also for Σ^+, inclusion into Y is just identity. But for Σ^-, we have that Σ_i^- lies in Y_{i-1}, so this inclusion is multiplication by t^{-1}. So we have $f_k : (x, y) \mapsto (x + y, -x - yt^{-1})$.

Since $\Sigma^\pm$ are pushoffs of Σ and N is a thickening of Σ, we have that $H_k(\Sigma^\pm)$ and $H_k(N)$ are isomorphic to $H_k(\Sigma)$. So we get the long exact sequence

$$\cdots \xrightarrow{\partial_{k+1}} H_k\left(\Sigma\right) \oplus H_k\left(\Sigma\right) \xrightarrow{f_k} H_k\left(\Sigma\right) \oplus H_k\left(Y\right) \xrightarrow{g_k} H_k(X) \xrightarrow{\partial_k} \cdots$$

Now f_k is still the identity map into the first component, but when we map into $H_k(Y)$, we must first include into $\Sigma^\pm$ and then into Y. This is exactly the maps $\iota_\pm$. So now we have $f_k : (x, y) \mapsto (x + y, -\iota_+(x) - \iota_-(y)t^{-1})$. We also have $g_k : (x, y) \mapsto \iota_\Sigma(x) + \iota_Y(y)$.

Let's take a closer look at f_0. Choose a basepoint x_0 on Σ. Since Σ is connected, x_0 generates $H_0(\Sigma; \mathbb{Z})$, so it also generates $H_0(\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda = H_0(\Sigma; \Lambda)$ as a Λ module. Since Y is path-connected $\iota_-(x_0)$ is homologous to $\iota_+(x_0)$ and we can say $\iota_+(x_0)$ generates $H_0(Y; \Lambda)$. Then a general element of $H_0(\Sigma; \Lambda) \oplus H_0(\Sigma; \Lambda)$ is $(x_0 r, x_0 s)$ for some r, s in Λ, and the map f_0 becomes $(x_0 r, x_0 s) \mapsto (x_0(r + s), \iota_+(x)(-r - t^{-1}s))$.

Suppose that $f_0(x_0 r, x_0 s) = 0$. Then we must have $r + s = 0$ and $-r - t^{-1}s = 0$. So $s = -r$ and so the second equality becomes $r(-t^{-1} + 1) = 0$. Since Λ has no zero divisors and $-t^{-1} + 1$ is not 0, we must have $r = 0$ and thus $s = 0$. Therefore f_0 is injective.

Then since the sequence is exact, we must have $\partial_1 = 0$. So we get the following exact sequence with Λ coefficients.

$$H_2(X) \xrightarrow{\partial_2} H_1(\Sigma) \oplus H_1(\Sigma) \xrightarrow{f_1} H_1(\Sigma) \oplus H_1(Y) \xrightarrow{g_1} H_1(X) \longrightarrow 0$$

Now we wish to get rid of the extra $H_1(\Sigma)$ terms.

Lemma 3.8. *There exist maps ∂_2', f_1', and g_1' so that $f_1'(x) = (\iota_+ - \iota_- t^{-1})(x)$, the second component of $f_1(-x, x)$, and the following sequence is exact.*

$$H_2(X) \xrightarrow{\partial_2'} H_1(\Sigma) \xrightarrow{f_1'} H_1(Y) \xrightarrow{g_1'} H_1(X) \longrightarrow 0$$

Proof. First let's define ∂_2'. By exactness of the original sequence, the image of ∂_2 equals the kernel of f_1, which is the set of pairs $(-x, x)$ such that $(\iota_+ - \iota_- t^{-1})(x) = 0$. So for any $z \in H_2(X)$, we have $\partial_2(z) = (-x, x)$ for some $x \in H_1(\Sigma)$, and $f_1'(x) = 0$. Hence we define $\partial_2'(z) = x$. Then by construction, the image of ∂_2' equals the kernel of f_1'.

Now we define $g_1'(y) = g_1(0, y) = \iota_Y(y)$. Then y is in the kernel of g_1' if and only if $(0, y)$ is in the kernel of g_1 which is the image of f_1 by the exactness of the original sequence. But $f_1(w, x) = (0, y)$ for some w, x in $H_1(\Sigma)$ if and only if $w + x = 0$, so $w = -x$. Thus y equals the second component of $f_1(-x, x)$ so $y = f_1'(x)$. Therefore the kernel of g_1' equals the image of f_1'.

Now we need only show that g_1' is surjective. Let z be an element of $H_1(X)$. By exactness of the original sequence we know g_1 is surjective so $z = g_1(x, y)$ for some x in $H_1(\Sigma)$ and y in $H_1(Y)$. Also by exactness, the kernel of g_1 equals the image of f_1, so we have $0 = g_1(f_1(x, 0)) = g_1(x, -\iota_+(x))$. Thus we have $z = g_1(x, y) - 0 = g_1(x, y) - g_1(x, -\iota_+(x)) = g_1(0, y + \iota_+(x)) = g_1'(y + \iota_+(x))$, therefore g_1' is surjective. $\square$

So now we have the exact sequence

$$H_2(X;\Lambda_\varphi) \longrightarrow H_1(\Sigma;\Lambda_\varphi) \xrightarrow{\iota_+ - \iota_- t^{-1}} H_1(Y;\Lambda_\varphi) \xrightarrow{\iota_Y} H_1(X;\Lambda_\varphi) \longrightarrow 0$$

This tells us that for X a compact, connected, oriented 3-manifold that is closed or has toroidal boundary, the first homology of the infinite cyclic cover corresponding to $\pi_1(X) \to H_1(X) \xrightarrow{\varphi} \mathbb{Z}$ is given by the cokernel of the map $\iota_+ - \iota_- t^{-1}$.

We are of course particularly interested in the cases where X is a link exterior or the zero surgery manifold of a link. The following theorem is a special case of a theorem of Cimasoni, with slight modification. We provide the proof because our proof is more detailed.

Theorem 3.9 (Cimasoni [Cim04])**.** *When $X = X_L$ is the exterior of a link L with pairwise linking numbers all zero, and $A_\varphi^\pm$ are Seifert $\mathbb{Z}$-matrices for L, the right Λ_φ-module $H_1(X_L;\Lambda_\varphi)$ is presented by the matrix $tA_\varphi^+ - A_\varphi^-$.*

Proof. Since $H_1(\Sigma;\Lambda_\varphi) = H_1(\Sigma;\mathbb{Z}) \otimes_\mathbb{Z} \Lambda$, and $H_1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^r$ for some r by choosing a suitable basis, we have $H_1(\Sigma;\Lambda_\varphi) \cong \Lambda^r$. Similarly, $H_1(Y;\Lambda_\varphi) \cong \Lambda^s$ for some s. Then the map $\iota_+ - \iota_- t^{-1}$ may be represented by a matrix P, which serves as a presentation matrix for $H_1(X;\Lambda_\varphi)$, and we get $H_1(X;\Lambda_\varphi) \cong \Lambda^s/\Lambda^r P$.

Let L be an m-component link with pairwise linking numbers zero, let Σ_φ be a Seifert $\mathbb{Z}$-surface for L, and let θ_φ be the associated Seifert $\mathbb{Z}$-form. Order the link $L = K_1 \cup \cdots \cup K_m$, which induces orderings on the sets of meridians and longitudes of L and on the abbreviation $\varphi = (k_1, \ldots, k_m)$, so that the components K_i with $k_i = 0$ are at the end of the list. Let $L' = K_1 \cup \cdots \cup K_\ell$ be the sublink of L consisting of the ℓ components for which $k_i \neq 0$ and let $L^0 = K_{\ell+1} \cup \cdots \cup K_m$ be the sublink of L consisting of the $m - \ell$ components for which $k_i = 0$. It is possible that $L' = L$ and L^0 is empty. According to our assumptions, the Seifert $\mathbb{Z}$-surface Σ_φ will miss $N(L^0)$

completely, and will have at least one boundary component on the neighborhood of each of the components of L'.

The Seifert $\mathbb{Z}$-surface Σ_φ has $h = |k_1| + \cdots + |k_\ell|$ boundary components, and suppose that it has genus g. So $H_1(\Sigma_\varphi)$ has rank $r = 2g + h - 1$. Let $a_1, \ldots, a_{2g}$ denote the homology classes of the part of the surface with genus. That is, the set $\{a_1, \ldots, a_{2g}\}$ forms a basis for the first homology group of the closed surface obtained from Σ_φ by capping off the boundary components with disks. Let $b_{i,1}, \ldots, b_{i,k_i}$ denote the homology classes of the boundary components of Σ_φ that lie on $\partial N(K_i)$. Each b_{ij} is homotopic to the ith longitude λ_i. Then $H_1(\Sigma_\varphi)$ is generated by the basis $B = \{a_1, \ldots, a_{2g}\} \cup \{b_{i,1}, \ldots, b_{i,k_i}\}_{i=1}^{\ell} - \{b_{\ell,k_\ell}\}$.

Now we'll look at $H_1(Y)$. We have $Y = X - N(\Sigma_\varphi) = S^3 - N(L) - N(\Sigma_\varphi)$ and $N(L) = N(L' \sqcup L^0$. We have that Σ_φ is disjoint from $N(L^0)$ so we have $Y = S^3 - (N(\Sigma_\varphi \cup L') \sqcup N(L^0))$. Then by Alexander duality,

$$\begin{aligned} H_1(Y) &= H_1(S^3 - (N(\Sigma_\varphi \cup L') \sqcup N(L^0)) \\ &\cong H^1(N(\Sigma_\varphi \cup L') \sqcup N(L^0)) \\ &\cong H_1(N(\Sigma_\varphi \cup L') \sqcup N(L^0)) \\ &\cong H_1(N(\Sigma_\varphi \cup L')) \oplus H_1(N(L^0)). \end{aligned}$$

We know $H_1(N(L^0))$ is generated by the longitudes of the components of L^0, so this part of $H_1(Y)$ is generated by the meridians of L^0 and is free of rank $s = m - \ell$.

Now we will compute $H_1(N(\Sigma_\varphi \cup L'))$. Choose a basepoint x_0 in the interior of Σ_φ and for each boundary component $b_{i,j}$, choose a path $\beta_{i,j}$ connecting x_0 to $b_{i,j}$. Let $U = N(\Sigma_\varphi)$ and let $V = N(L') \cup N(\{\beta_{i,j}\})$. Then $U \cup V = N(\Sigma_\varphi \cup L')$ and $U \cap V$ deformation retracts to a connected graph consisting of the circles $\{b_{i,j}\}$ and the paths $\{\beta_{i,j}\}$. Its fundamental group is the free group on the generators $\{\beta_{i,j} b_{i,j} \beta_{i,j}^{-1}\}$ and

$H_1(U \cap V)$ is the free abelian group on the generators $\{b_{i,j}\}$.

V also deformation retracts to a connected graph, this one consisting of the longitudes λ_i and the paths $\beta_{i,j}$. As V deformation retracts, the boundary components $b_{i,j}$ all retract to the longitude λ_i, or λ_i^{-1}, depending on the sign on k_i. Then if $|k_i| > 1$, the paths $\beta_{i,j}$ create extra loops. For $j = 1, \ldots, k_i - 1$ let $c_{i,j} = \beta_{i,j}\beta_{i,j+1}^{-1}$. Then $\Pi_1(V)$ is the free group generated by $\{\beta_{i,1}\lambda_i\beta_{i,1}^{-1}, \beta_{i,1}\beta_{i,2}^{-1}, \beta_{i_2}\beta_{i_3}^{-1}, \ldots, \beta_{i,k_i-1}\beta_{i,k_i}^{-1}\}_{i=1}^{\ell}$, and $H_1(V)$ is the free abelian group generated by $\{\lambda_i, c_{i,1}, c_{i,2}, \ldots, c_{i,k_i-1}\}_{i=1}^{\ell}$. Now we'll look at the Mayer-Vietoris sequence on U and V. Since $U \cap V$ is connected, $\partial_1 = 0$. We have the exact sequence

$$\begin{aligned} H_1(U \cap V) &\to H_1(\Sigma_\varphi) \oplus H_1(N(L') \cup N(\{\beta_{i,j}\})) \to H_1(N(\Sigma_\varphi \cup L')) \to 0 \\ b_{i,j} &\mapsto (b_{i,j}, -\lambda_i^{\pm 1}) \end{aligned}$$

Hence $H_1(N(\Sigma_\varphi \cup L'))$ is the free abelian group of rank $r = 2g+h-1$ generated by $\{a_1, \ldots, a_{2g}\} \cup \{\lambda_1, \ldots, \lambda_{\ell-1}\} \cup \{c_{i,1}, \ldots, c_{i,k_i-1}\}_{i=1}^{\ell}$, and thus $H_1(\Sigma_\varphi \cup N(L))$ is generated by the basis $C = \{a_1, \ldots, a_{2g}\} \cup \{\lambda_1, \ldots, \lambda_{\ell-1}, \lambda_{\ell+1}, \ldots, \lambda_m\} \cup \{c_{i,1}, \ldots, c_{i,k_i-1}\}_{i=1}^{\ell}$

Thus $H_1(Y)$ is free abelian of rank $r+s$ and has the dual basis $C^* = \{\alpha_1, \ldots, \alpha_{2g}\} \cup \{\mu_1, \ldots, \mu_{\ell-1}, \mu_{\ell+1}, \ldots, \mu_m\} \cup \{\gamma_{i,1}, \ldots, \gamma_{i,k_i-1}\}_{i=1}^{\ell}$. By dual we mean that α_i links a_i exactly once and does not link any other basis element of $H_1(N(\Sigma_\varphi \cup L')) \oplus H_1(N(L^0))$, $\gamma_{i,j}$ links $c_{i,j}$ exactly once, and so on.

Then an element y of $H_1(Y)$ may be written as

$$y = \sum_{i=1}^{2g} \ell k(y, a_i)\alpha_i + \sum_{\substack{i=1 \\ i \neq \ell}}^{m} \ell k(y, \lambda_i)\mu_i + \sum_{i=1}^{\ell} \sum_{j=1}^{k_i-1} \ell k(y, c_{i,j})\gamma_{i,j}.$$

Lemma 3.10. *The maps $i_\pm : H_1(\Sigma_\varphi) \to H_1(Y)$ are represented respectively by the matrices $A_\varphi^\pm$.*

Proof. We'll check that A^+_φ represents i_+ on the basis B for $H_1(\Sigma_\varphi)$. Then the proof for A^-_φ representing i_- is similar. For simplicity, rename the elements of the bases B, C and C^* so that $B = \{b_1, \dots, b_r\}$, $C = \{c_1, \dots, c_s\}$, and $C^* = \{c_1^*, \dots, c_s^*\}$, but keep that C^* is a dual basis to C in the sense that $\ell k(c_i^*, c_j) = \delta_{ij}$, so we still have that an element $y \in H_1(Y)$ is given by $\sum_{j=1}^s \ell k(y, c_j)c_j^*$. Then the ijth entry of A^+_φ is given by $\ell k(b_i^+, c_j)$ For each $i = 1, \dots, r$, the generator b_i is given by the row vector $\vec{e_i}$ which has a 1 in the ith place and 0s elsewhere. Then $\vec{e_i}A^+_\varphi$ picks out the ith row of A, $(\ell k(b_i^+, c_1), \dots, \ell k(b_i^+, c_s))$, and this row vector represents $\sum_{j=1}^s \ell k(b_i^+, c_j)c_j^= b_i^+ = i_+(b_i)$ in $H_1(Y)$. Thus $i_+(b_i)$ is represented by $\vec{e_i}A^+_\varphi$. Then for any $x \in H_1(\Sigma_varphi)$, we have $x = \sum_{i=1}^r x_i b_i$, so x is represented by the row vector $\vec{x} = (x_1, \dots, x_r)$ and $i_+(x)$ is represented by the row vector $\vec{x}A^+_\varphi$. Therefore A^+_φ represents i_+. □

The presentation matrix P for $H_1(X_L; \Lambda_\varphi)$ represents the map $i_+ - i_- t^{-1}$. Thus P is given by $A^+_\varphi - A^-_\varphi t^{-1}$. Multiply by t and use the commutativity of Λ_φ to get $tA^+_\varphi - A^-_\varphi$. □

Note that when $\varphi = (1, \dots, 1)$, we have that $H_1(\Sigma_\varphi) = H_1(\Sigma_\varphi \cup N(L))$. Then $A^+_\varphi = A_\varphi$ and $A^-_\varphi = A^T_\varphi$ and we obtain the following corollary where the presentation matrix is analagous to the one for knots.

Corollary 3.11. *When $X = X_L$ is the exterior of a link L with pairwise linking numbers all zero, and A_φ is a Seifert $\mathbb{Z}$-matrix for L associated to $\varphi = (1, \dots, 1)$, the right Λ_φ-module $H_1(X_L; \Lambda_\varphi)$ is presented by the matrix $tA_\varphi - A^T_\varphi$.*

Now we turn to the case when $X = M_L$ is the zero surgery manifold of a link L.

Theorem 3.12. *When M is the zero surgery manifold for a link L with pairwise linking numbers all zero, and $\widehat{A}_\varphi$ is a Seifert $\mathbb{Z}$-matrix for L associated to a closed Seifert $\mathbb{Z}$-surface $\widehat{\Sigma}_\varphi$ dual to a primitive class $\varphi \in H^1(M)$, the right Λ_φ-module $H_1(M; \Lambda_\varphi)$*

is presented by the block matrix

$$\Psi_\varphi = \begin{pmatrix} t\widehat{A}_\varphi - \widehat{A}_\varphi^T & (t-1)D & tE^+ - E^- \\ C & 0 & F^+ \\ G & 0 & H \end{pmatrix}$$

where the blocks $C, D, E^\pm, F^+$ *are the same as those found in the the block matrix decomposition of the matrices* $A_\varphi^\pm$ *in the statement of Proposition 3.7.*

Proof. Let $X = S^3 - N(L)$ be the exterior of a link L and let M be the zero surgery manifold of L. Let Σ_φ be a Seifert $\mathbb{Z}$-surface for L associated to a primitive class $\varphi \in H^1(X)$ and let $\widehat{\Sigma}_\varphi$ be the corresponding closed Seifert surface embedded in M, which is dual to $\varphi \in H^1(M)$. Let $Y = X - \Sigma_\varphi = S^3 - (\Sigma_\varphi \cup N(L))$ and let $\widehat{Y} = M - \widehat{\Sigma}_\varphi$. Let $\iota_\pm : H_1(\Sigma_\varphi) \overset{\cong}{\to} H_1(\Sigma_\varphi^\pm) \to H_1(Y)$ and let $\hat{\iota}_\pm : H_1(\widehat{\Sigma}_\varphi) \overset{\cong}{\to} H_1(\widehat{\Sigma}_\varphi^\pm) \to H_1(\widehat{Y})$ and let $\iota_Y : H_1(Y) \to H_1(X)$ and $\hat{\iota}_Y : H_1(\widehat{Y}) \to H_1(M)$ be the maps induced by inclusion. We know we have an exact sequence

$$H_1(\widehat{\Sigma}_\varphi; \Lambda_\varphi) \xrightarrow{\hat{\iota}_+ - \hat{\iota}_- t^{-1}} H_1(\widehat{Y}; \Lambda_\varphi) \xrightarrow{\hat{\iota}_{\widehat{Y}}} H_1(M; \Lambda_\varphi) \longrightarrow 0$$

and we know that $H_1(\widehat{\Sigma}_\varphi; \Lambda_\varphi) = H_1(\widehat{\Sigma}_\varphi) \otimes_{\mathbb{Z}} \Lambda_\varphi$ and $H_1(\widehat{Y}; \Lambda_\varphi) = H_1(\widehat{Y}) \otimes_{\mathbb{Z}} \Lambda_\varphi$. Using the same language as in the proof of Theorem 3.9, $H_1(\widehat{\Sigma}_\varphi)$ has rank $r = 2g$ and is generated by the set $\{a_1, \ldots, a_{2g}\}$. Let's compute $H_1(\widehat{Y})$.

Lemma 3.13. *For* $\varphi = (k_1, \ldots, k_m)$*, we have*

$$H_1(M_L - \widehat{\Sigma}_\varphi) = H_1(X_L - \Sigma_\varphi) / \langle b_1^+, \ldots, b_h^+, \lambda_{\ell+1}, \ldots, \lambda_m \rangle$$

where $b_1, \ldots, b_h$ *are the boundary components of* Σ_φ *and* $\lambda_{\ell+1}, \ldots, \lambda_m$ *are the longitudes of the sublink* L^0 *of* L *consisting of the* $m - \ell$ *components for which* $k_i = 0$*.*

Proof. $\widehat{Y} = M - N(\widehat{\Sigma}_\varphi) = (X_L - N(\Sigma_\varphi)) \cup V$, where $V = (\bigsqcup_{i=1}^m S^1 \times D^2)$ minus some meridianal disks (used to cap the boundary components of Σ_φ to create $\widehat{\Sigma}_\varphi$.) So V is the disjoint union of solid tori and 3-dimensional 2-handles. Precisely,

$$\widehat{Y} = M - N(\widehat{\Sigma}_\varphi) = \left(S^3 - N(\Sigma_\varphi \cup L)\right) \cup_f \left(\left[\bigsqcup_{i=1}^{\ell} \bigsqcup_{j=1}^{|k_i|} [-1,1] \times D^2 \right] \cup \left[\bigsqcup_{i=1}^{m-\ell} S^1 \times D^2 \right] \right)$$

where the attaching map f takes $\{-1\} \times \partial D^2$ of the ijth copy of $[-1,1] \times D^2$ to b_{ij}^+ and takes $\{+1\} \times \partial D^2$ of the ijth copy of $[-1,1] \times D^2$ to $b_{i,j+1}^-$, where we use the convention that $b_{i,|k_i|+1} = b_{i,1}$, and takes the meridian $\{p\} \times \partial D^2$ of the ith solid torus to the $(\ell + i)$th longitude $\lambda_{\ell+i}$.

The intersection of $X - N(\Sigma_\varphi)$ with V, the space on the right in the above equation, is exactly $\partial N(L) - N(\partial\Sigma_\varphi) = (\partial N(L') - N(\partial\Sigma_\varphi)) \sqcup \partial N(L^0)$.

Just as in the calculation of $H_1(M)$, gluing in the solid tori kills off the longitudes $\lambda_{\ell+1}, \ldots, \lambda_m$. Recall that b_{ij}^+ is homotopic to $b_{i,j+1}^-$. Then copies of $[-1,1] \times D^2$ are just 3-dimensional 2-handles attached along b_{ij}^+, which kills the b_{ij}^+. For simplicity, relabel the basis elements. Now $H_1(\widehat{\Sigma}_\varphi)$ has basis $A = \{a_i\}_{i=1}^{2g}$, $H_1(\Sigma_\varphi)$ has basis $B = A \cup \{b_i\}_{i=1}^{h-1}$, $H_1(N(\Sigma_\varphi \cup L)$ has basis $C = A \cup \{\lambda_i\}_{\substack{1 \le i \le m \\ i \neq \ell}} \cup \{c_i\}_{i=1}^{h-\ell}$ and $H_1(S^3 - N(\Sigma_\varphi \cup L))$ has dual basis $C^* = \{\alpha_i\}_{i=1}^{2g} \cup \{\mu_i\}_{\substack{i=1,\ldots,m \\ i \neq \ell}} \cup \{\gamma_i\}_{i=1}^{h-\ell}$. Then

$$H_1(M - N(\widehat{\Sigma}_\varphi)) = H_1(S^3 - N(\Sigma_\varphi \cup L)) / \langle b_1^+, \ldots, b_h^+, \lambda_{\ell+1}, \ldots, \lambda_m \rangle.$$

□

An element $y \in H_1(S^3 - N(\Sigma_\varphi \cup L))$ may be written as

$$y = \sum_{j=1}^{2g} \ell k(y, a_j)\alpha_j + \sum_{\substack{1 \le j \le m \\ j \neq \ell}} \ell k(y, \lambda_j)\mu_j + \sum_{j=1}^{h-\ell} \ell k(y, c_j)\gamma_j.$$

Since L has pairwise linking numbers zero, $\ell k(\lambda_i, \lambda_j)$ and $\ell k(b_i^+, \lambda_k)$ will be 0 for all i, j. So each μ_i, $i \neq \ell$, generates a copy of $\mathbb{Z}$ in $H_1(M - N(\widehat{\Sigma}_\varphi))$. So we are left with

$$H_1(M - N(\widehat{\Sigma}_\varphi)) = \mathbb{Z}^{2g+h-1+m-\ell}/\mathbb{Z}^{h-1+m-\ell}Q = \mathbb{Z}^{m-1} \oplus \mathbb{Z}^{2g+h-\ell}/\mathbb{Z}^{h-1+m-\ell}Q'$$

with generating set $\{\mu_i\}_{\substack{1\leq i\leq m \\ i\neq\ell}} \cup \{a_i\}_{i=1}^{2g} \cup \{c_i\}_{i=1}^{h-\ell}$ and where Q is given by the block matrix

$$Q = \begin{pmatrix} C & 0 & F \\ G & 0 & H \end{pmatrix}$$

with $(h-1) \times 2g$ block $C = \big(\ell k(b_i^+, a_j)\big)$, $(h-1) \times (h-\ell)$ block $F = \big(\ell k(b_i^+, c_j)\big)$, $(m-\ell) \times 2g$ block $G = (\ell k(\lambda_i, a_j))$, and $(m-\ell) \times (h-\ell)$ block $H = (\ell k(\lambda_i, c_j))$. Note that the blocks C and F will be identical to the blocks C and F^+ in the matrix A_φ^+ in the statement of Proposition 3.7.

Lemma 3.14. *The maps $\hat{\imath}_\pm : H_1(\widehat{\Sigma}_\varphi) \to H_1(M - N(\widehat{\Sigma}_\varphi))$ are represented respectively by the block matrices $\widehat{A}_\varphi^\pm$ where*

$$\widehat{A}_\varphi^+ = \begin{pmatrix} \widehat{A}_\varphi & D & E^+ \end{pmatrix} \qquad\qquad \widehat{A}_\varphi^- = \begin{pmatrix} \widehat{A}_\varphi^T & D & E^- \end{pmatrix}$$

and by represent, we mean that for $x \in H_1(\widehat{\Sigma}_\varphi)$, the vector $x\widehat{A}_\varphi^\pm$ gives $\hat{\imath}_\pm(x)$ in terms of the generators of $H_1(M - N(\widehat{\Sigma}_\varphi))$. Essentially, $\widehat{A}_\varphi^\pm$ are composed of the first rows of blocks of $A_\varphi^\pm$. See Proposition 3.7.

Proof. Consider the following diagram (which does not commute) of groups of 1-cycles.

$$\begin{array}{ccccc} Z_1(\widehat{\Sigma}_\varphi) & \xrightarrow{\cong} & Z_1(\widehat{\Sigma}_\varphi^+) & \hookrightarrow & Z_1(M - N(\widehat{\Sigma}_\varphi)) \\ \downarrow & & & & \uparrow \\ Z_1(\Sigma_\varphi) & \xrightarrow{\cong} & Z_1(\Sigma_\varphi^+) & \hookrightarrow & Z_1(S^3 - N(\Sigma_\varphi \cup L)) \end{array}$$

The top row gives the map $\hat{\iota}_+ : Z_1(\widehat{\Sigma}_\varphi) \to Z_1(M - N(\widehat{\Sigma}_\varphi))$ and the bottom row gives the map $\iota_+ : Z_1(\Sigma_\varphi) \to Z_1(S^3 - N(\Sigma_\varphi \cup L))$. The hooked arrows denote inclusions. The mapping $Z_1(\widehat{\Sigma}_\varphi) \to Z_1(\Sigma_\varphi)$ is given by $c \mapsto \check{c}$. See Definition 3.3. We wish to show that for all $c \in Z_1(\widehat{\Sigma}_\varphi)$, $\hat{\iota}_+(c)$ is homologous in $M - N(\widehat{\Sigma}_\varphi \cup L)$ to $\iota_+(\check{c})$, the 1-cycle obtained by traveling from $Z_1(\widehat{\Sigma}_\varphi)$ to $Z_1(M - N(\widehat{\Sigma}_\varphi \cup L))$ along the lower path.

Starting with $c \in Z_1(\widehat{\Sigma}_\varphi)$ and traveling along the upper path, we first get c^+ the push off of c in $\widehat{\Sigma}_\varphi^+ \subset \partial N(\widehat{\Sigma}_\varphi)$, then we include into $M - N(\widehat{\Sigma}_\varphi)$. The 1-cycle does not change, it is still c^+, now called $\hat{\iota}_+(c)$. Traveling along the lower path we first get $\check{c}$, which is homologous to c in $\widehat{\Sigma}_\varphi$. Next we get $(\check{c})^+$ which is clearly homologous to c^+ in $\widehat{\Sigma}_\varphi^+$. So $(\check{c})^+$ is homologous to c^+ in $\partial N(\widehat{\Sigma}_\varphi)$, and thus also in $\partial N(\widehat{\Sigma}_\varphi \cup L)$. Now we include into $S^3 - N(\Sigma_\varphi \cup L)$, and $(\check{c})^+$ is unchanged, but renamed $\iota_+(\check{c})$. $\iota_+(\check{c})$ and $\hat{\iota}_+(c)$ are still homologous in $\partial N(\widehat{\Sigma}_\varphi \cup L)$. Therefore they are homologous in $M - N(\widehat{\Sigma}_\varphi)$. Therefore the diagram commutes in homology.

However, since the mapping $c \mapsto \check{c}$ is not well-defined on homology, there is a question as to whether the bottom route is well defined on homology. Let $a = \sum_{i=1}^{2g} n_i a_i$ be a homology class in $H_1(\widehat{\Sigma}_\varphi)$. Then since $\check{a}$ must be homologous to a in Σ_φ, we must have $\check{a} = a + b$ in $H_1(\Sigma_\varphi)$ where $b = \sum_{i=1}^{h-1} p_i b_i$. Then $(\check{a})^+ = a^+ + b^+$ in $H_1(\Sigma_\varphi^+)$ and $H_1(S^3 - N(\Sigma_\varphi \cup L))$. Now the map induced by inclusion on homology $H_1(S^3 - N(\Sigma_\varphi \cup L)) \to H_1(M - N(\widehat{\Sigma}_\varphi))$ is a quotient map. Recall that $H_1(M - N(\widehat{\Sigma}_\varphi))$ has the same generating set as $H_1(S^3 - N(\Sigma_\varphi \cup L))$, but some elements, including b_i^+ for $i = 1, \ldots, h$, are killed off. Thus in $H_1(M - N(\widehat{\Sigma}_\varphi))$, $a^+ + b^+ = a^+$. So the bottom route is the well-defined mapping $a \mapsto \iota_+(a)$, viewed as an element of $H_1(M - N(\widehat{\Sigma}_\varphi))$.

Therefore $\hat{\iota}_+(a) = \iota_+(a)$, viewed as an element of $H_1(M - N(\widehat{\Sigma}_\varphi))$. Similarly, $\hat{\iota}_-(a) = \iota_-(a)$. By Lemma 3.10, the maps $\iota_\pm$ are represented by the matrices $A_\varphi^\pm$. Restricting to the basis elements $\{a_1, \ldots, a_{2g}\}$, which form a basis for $H_1(\widehat{\Sigma}_\varphi)$, we obtain the desired matrices $\widehat{A}^{\pm}_{\varphi}$. $\square$

Since $H_1(\widehat{\Sigma}_{\varphi};\Lambda_{\varphi}) = H_1(\widehat{\Sigma}_{\varphi})\otimes_{\mathbb{Z}}\Lambda_{\varphi}$ and $H_1(M-N(\widehat{\Sigma}_{\varphi});\Lambda_{\varphi}) = H_1(M-N(\widehat{\Sigma}_{\varphi}))\otimes_{\mathbb{Z}} \Lambda_{\varphi}$, the maps $\hat{\imath}_{\pm} : H_1(\widehat{\Sigma}_{\varphi}) \to H_1(M - N(\widehat{\Sigma}_{\varphi}))$ extend as expected to maps on the corresponding right Λ_{φ}-modules, and these maps are also represented by $\widehat{A}^{\pm}_{\varphi}$.

Then the map $\hat{\imath}_+ - \hat{\imath}_- t^{-1} : H_1(\widehat{\Sigma}_{\varphi};\Lambda_{\varphi}) \to H_1(M - N(\widehat{\Sigma}_{\varphi});\Lambda_{\varphi})$ is represented by the matrix $\widehat{A}^{+}_{\varphi} - \widehat{A}^{-}_{\varphi}t^{-1}$. For some $x \in H_1(\widehat{\Sigma}_{\varphi};\Lambda_{\varphi})$, the vector $\widehat{A}^{+}_{\varphi} - \widehat{A}^{-}_{\varphi}t^{-1}$ gives $(\hat{\imath}_+ - \hat{\imath}_- t^{-1})(x)$ in terms of the generators of $H_1(M - N(\widehat{\Sigma}_{\varphi});\Lambda_{\varphi})$.

For simplicity, let $Y = S^3 - N(\Sigma_{\varphi} \cup L)$ and let $\widehat{Y} = M - N(\widehat{\Sigma}_{\varphi})$. Let $q : H_1(\partial\Sigma_{\varphi} \cup L) \to H_1(Y)$ be the map represented by the matrix Q which gives a presentation of $H_1(\widehat{Y})$. Let ι_Y and $\hat{\imath}_Y$ be the maps induced by inclusion. Let $\hat{p} = \hat{\imath}_+ - \hat{\imath}_- t^{-1}$ and let p be the map represented by the matrix $P = \widehat{A}^{+}_{\varphi} - \widehat{A}^{-}_{\varphi}t^{-1}$. Consider the following commutative diagram with exact rows, where all homology is with coefficients in Λ_{φ}.

$$\begin{tikzcd}
H_1(\partial\Sigma_{\varphi} \cup L) \arrow[r, "q"] & H_1(Y) \arrow[r, "\iota_Y"] & H_1(\widehat{Y}) \arrow[r] \arrow[d, equal] & 0 & \\
 & H_1(\widehat{\Sigma}_{\varphi}) \arrow[u, "p"] \arrow[r, "\hat{p}"] & H_1(\widehat{Y}) \arrow[r, "\hat{\imath}_Y"] & H_1(M) \arrow[r] & 0
\end{tikzcd}$$

This encodes the information that $H_1(\widehat{Y}) = \mathrm{cok} q$, $H_1(M) = \mathrm{cok}\hat{p}$, and $\hat{p} = \iota_Y \circ p$. Now define the map $\psi : H_1(\widehat{\Sigma}_{\varphi}) \oplus H_1(\partial\Sigma_{\varphi} \cup L) \to H_1(Y)$ by $\psi(x, y) = p(x) + q(y)$. We claim that the following sequence is exact.

$$H_1(\widehat{\Sigma}_{\varphi}) \oplus H_1(\partial\Sigma_{\varphi} \cup L) \xrightarrow{\psi} H_1(Y) \xrightarrow{\hat{\imath}_Y \circ \iota_Y} H_1(M) \longrightarrow 0$$

The map $\hat{\imath}_Y \circ \iota_Y$ is clearly surjective as it is the composition of two surjective maps.

Now we show that $\text{im}\psi \subset \ker(\hat{\iota}_Y \circ \iota_Y)$. Let $x \in H_1(\widehat{\Sigma}_\varphi)$ and let $y \in H_1(\partial\Sigma_\varphi \cup L)$.

$$\begin{aligned}
\hat{\iota}_Y \circ \iota_Y \circ \psi(x,y) &= \hat{\iota}_Y \circ \iota_Y \left(p(x) + q(y)\right) \\
&= \hat{\iota}_Y \left(\iota_Y \left(p(x)\right) + \iota_Y \left(q(y)\right)\right) \\
&= \hat{\iota}_Y \left(\iota_Y \left(p(x)\right)\right) \\
&= \hat{\iota}_Y \left(\hat{p}(x)\right) \\
&= 0
\end{aligned}$$

since $\iota_Y \circ q = 0$ by exactness of the top row of the diagram, and $\hat{\iota}_Y \circ \hat{p} = 0$ by exactness of the bottom row. This proves the claim. Thus $H_1(M;\Lambda_\varphi)$ equals the cokernel of ψ, which is represented by the block matrix

$$\Psi_\varphi = \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} \widehat{A}_\varphi - \widehat{A}_\varphi^T t^{-1} & D\left(1 - t^{-1}\right) & E^+ - E^- t^{-1} \\ C & 0 & F^+ \\ G & 0 & H \end{pmatrix}$$

Therefore $H_1(M;\Lambda_\varphi)$ is presented by the matrix Ψ_φ. $\square$

3.3 Seifert $\mathbb{Z}$-Forms and 0.5-Solvability

Definition 3.15. Let L be a link with pairwise linking numbers 0. Let Σ_φ be a Seifert $\mathbb{Z}$-surface for L with associated Seifert $\mathbb{Z}$-form θ_φ on $H_1(\Sigma_\varphi)$. Let $\widehat{\Sigma}_\varphi$ be the corresponding closed Seifert $\mathbb{Z}$-surface with associated Seifert $\mathbb{Z}$-form $\widehat{\theta}_\varphi$ on $H_1(\widehat{\Sigma}_\varphi)$.

1. A *metabolizer for* $\widehat{\Sigma}_\varphi$ is a direct summand $\widehat{H} \subset H_1(\widehat{\Sigma}_\varphi)$ of half rank such that $\widehat{\theta}_\varphi(x,y) = 0$ for all $x, y \in \widehat{H}$. That is, $\widehat{\Sigma}_\varphi$ has a metabolizer if there exists a

Seifert $\mathbb{Z}$-matrix $\widehat{A}_\varphi$ associated to $\widehat{\Sigma}_\varphi$ of the form

$$\widehat{A}_\varphi = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}.$$

Note that this coincides with the usual definition of a metabolizer.

2. A *strong metabolizer* for Σ_φ is a direct summand $H \subset H_1(\Sigma_\varphi)$ on which θ_φ vanishes, and such that $H = \widehat{H} \cup B$, where $\widehat{H} \subset H_1(\widehat{\Sigma}_\varphi)$ is a metabolizer for $\widehat{\Sigma}_\varphi$ and B is the submodule of $H_1(\Sigma_\varphi)$ generated by the boundary components of Σ_φ. That is, there exists a corresponding Seifert $\mathbb{Z}$-matrix of the form

$$A_\varphi = \begin{pmatrix} \widehat{A}_\varphi & C^T \\ C & 0 \end{pmatrix} = \left(\begin{array}{cc|c} * & * & * \\ * & 0 & 0 \\ \hline * & 0 & 0 \end{array}\right)$$

We will use this definition in Theorem 5.7.

Theorem 3.16. *Suppose a link L (with pairwise linking numbers zero) has a Seifert $\mathbb{Z}$-surface associated to $\varphi = (1, \dots, 1)$ such that the corresponding closed Seifert $\mathbb{Z}$-surface has a metabolizer. Then L is double-delta equivalent to a weakly slice link, that is, a link that bounds a surface of genus 0 in the 4-ball.*

Proof. This will be very similar to the proof of Theorem 2.5. Let Σ be a Seifert $\mathbb{Z}$-surface for L associated to $\varphi = (1, \dots, 1)$ such that its corresponding closed Seifert $\mathbb{Z}$-surface $\widehat{\Sigma}$ has a metabolizer. Let $a_1, \dots, a_g$ be elements of $H_1(\Sigma)$ such that when they are included into $H_1(\widehat{\Sigma})$ they form a basis for the metabolizer. Since $\varphi = (1, \dots, 1)$, the boundary of Σ is exactly equal to L, so we can isotope Σ so that it is in disk-band form as in Figure 3.7 with the metabolizer curves $a_1, \dots, a_g$ labeled. The curves $a_1, \dots, a_g$ form a link J with pairwise linking numbers zero. Two links have the same

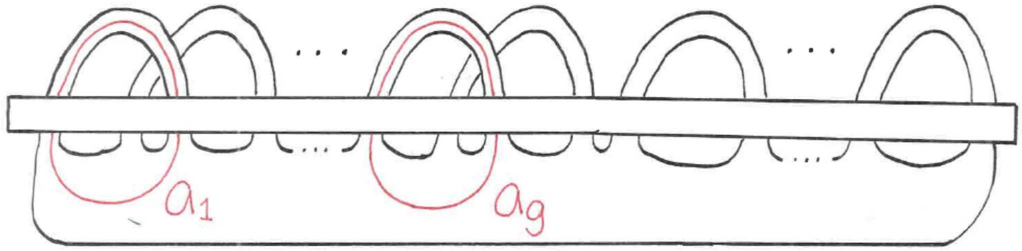

Figure 3.7: Σ in disk-band form with metabolizer represented by $a_1, \ldots, a_g$, where the box contains a string link on the bands

sets of pairwise linking numbers if and only if they are equivalent under delta moves [MN89]. So J is delta equivalent to the unlink. Note that the curves a_i form the cores of bands that involve only one link component. So as we perform delta moves on J to transform it into an unlink $J' = a_1' \cup \cdots \cup a_g'$, we are performing delta moves on the bands and transforming the Seifert surface Σ into some surface Σ', and we are performing double-delta moves on L and transforming it into some link L'. Note that double delta moves preserve pairwise linking numbers, so L' still has pairwise linking numbers zero. Note also that Σ' is a Seifert surface for L'. Now as in the proof of Theorem 2.5, we cut Σ' along the curves $a_1', \ldots, a_g'$ and cap with $2g$ disks. We now have a genus 0 surface with boundary L' and at most ribbon intersections. Pushing the surface into the 4-ball, we can remove the ribbon intersections. Thus L' is weakly slice. $\square$

Note that if L' were slice, or at least 0.5-solvable, then we could conclude that L is 0.5-solvable, by Proposition 2.4.

Theorem 3.17. *Let L be a 0.5-solvable link with 0.5 solution W, and $\partial W = M$ the zero surgery manifold for L. Let $\widehat{\Sigma}_\varphi \subset M$ be the closed Seifert $\mathbb{Z}$-surface for L associated to a primitive class $\varphi \in H^1(M)$. Then $\widehat{\Sigma}_\varphi$ has a metabolizer.*

Proof. First we'll show there exists a 3-manifold $R \subset W$ with $\partial R = \widehat{\Sigma}_\varphi$. Corresponding to the epimorphism $\varphi : H_1(M) \to \mathbb{Z}$, we have a map $f : M \to S^1$ with

$f^{-1}(1) = \widehat{\Sigma}_\varphi$ and $f_* = \varphi$. We will extend this map to all of W step by step using the cells of some CW cell decomposition of W.

The 1-skeleton of the CW complex is just a graph. Choose a maximal tree T in W that contains a maximal tree in ∂W. Extend f over all of T in an arbitrary way. For any 1-cell σ not in T, we get a 1-cycle c that is the union of σ and a 1-chain in T that connects the ends of σ. Define f on σ so that $[f(c)]$ is the image under the composition $H_1(W) \xleftarrow{\cong} H_1(M) \xrightarrow{\varphi} H_1(S^1)$. This extends f to the 1-skeleton of W.

For any 2-cell d in W, its boundary is 0 in $H_1(W)$ so $[f(\partial d)] = 0$ in $H_1(S^1)$, so f is null-homotopic on ∂d, so f extends over d. This extends f to the 2-skeleton of W. Finally, we can extend f over all the 3- and 4-cells since any map from the boundary of an n-cell to S^1 is nullhomotopic for $n \geq 3$, since S^1 is aspherical.

Now we have a map $f : W \to S^1$. We can make it transverse to $1 \in S^1$. Then $R = f^{-1}(1)$ is an oriented 3-manifold properly embedded in W, with a bicollar, and $\partial R = \widehat{\Sigma}_\varphi$. The key to this working is that W is an H_1-bordism.

Since W is a 0.5 solution for L, it has a 0.5 Lagrangian represented by disjoint (1)-surfaces L_i. We can choose the L_i to be transverse to R. Next we'll show that we can modify R, without changing its boundary, to make it disjoint from these surfaces.

R and L_1 intersect in a 1-manifold C. L_1 is a (1) surface, so $\pi_1(L_i) \subset [\pi_1(W), \pi_1(W)]$. So L_1 lifts to the universal abelian cover of W, and therefore all infinite cyclic covers as well. So L_1 lifts to W_φ, the infinite cyclic cover of W corresponding to the extended map $\varphi : H_1(W) \to \mathbb{Z}$, where $\varphi = f_*$, and so does C. We can construct W_φ by cutting along R and gluing infinitely many copies of $W - R$ together. When we cut W along R, we are cutting F_1 along C. C must be separating. Otherwise, when we cut along C, L_1 is connected so the cover of L_1 in W_φ will be a single connected piece. But L_1 lifts to W_φ so there must be infinitely many copies of L_1 in W_φ. Therefore L_1 must be disconnected when we cut along C. Thus C is nullhomologous in L_1, so bounds a nested collection of subsurfaces in L_1. The L_i have trivial normal bundles by definition, thus so does each subsurface. Remove each circle of intersection in C by removing its trivial regular neighborhood in R, $S^1 \times \mathbb{D}^2$, and replacing it with the circle bundle in W over the subsurface. Since all the L_i are disjoint we can remove the intersection circles from L_1 without adding additional intersection circles to the rest of the L_i. Do this for each L_i.

Now we have a 3-manifold R properly embedded in W with $\partial R = \widehat{\Sigma}_\varphi$ and R is disjoint from all the surfaces L_i in the 0.5 Lagrangian for W. Let

$$H = \ker\left(H_1(\widehat{\Sigma}_\varphi) \to H_1(R)/TH_1(R)\right).$$

We'll show that H is a metabolizer for $\widehat{\Sigma}_\varphi$.

We begin by defining a closed 4-manifold $\hat{W}$. Let X_L be another 4-manifold with boundary M_L, constructed by attaching 0-framed 2-handles to L in $S^3 = \partial B^4$. Construct $\hat{W}$ by gluing W and X_L together along their common boundary M_L. We claim that $H_2(\hat{W}) = H_2(W)$.

Consider the following Meyer-Vietoris sequence.

$$H_2(M_L) \to H_2(X_L) \oplus H_2(W) \to H_2(\hat{W}) \to H_1(M_L) \to H_1(X_L) \oplus H_1(W)$$

Since X_L consists of only a 0-handle and 2-handles, we have $H_1(X_L) = 0$. So the last map in the sequence simplifies to the map induced by inclusion, $H_1(M_L) \to H_1(W)$, which is an isomorphism since W is an H_1-bordism. So the sequence reduces to

$$H_2(M_L) \xrightarrow{(i_X, -i_W)} H_2(X_L) \oplus H_2(W) \to H_2(\hat{W}) \to 0.$$

Therefore $H_2(\hat{W}) = H_2(X_L) \oplus H_2(W)/\text{image}(i_X, -i_W)$.

Now consider the long exact sequence of the pair (W, M_L).

$$H_2(M_L) \xrightarrow{i_W} H_2(W) \to H_2(W, M_L) \to H_1(M_L) \to H_1(W)$$

Again, since W is an H_1-bordism, the last map is an isomorphism. So by exactness, the second to last map is the zero map, so the second map is a surjection. By Poincare duality and the universal coefficent theorem, $H_2(W, M_L) = H^2(W) = H_2(W)/TH_2(W) \oplus H_1(W)$. But $H_1(W) = H_1(M_L) = \mathbb{Z}^m$, and $H_2(W)$ is also torsion-free, so $H_2(W, M_L) \cong H_2(W)$, so they have the same rank. Note that an epimorphism between finitely generated free abelian groups of the same rank is an isomorphism. So again by exactness, we have i_W is the zero map.

Now consider the long exact sequence of the pair (X_L, M_L). Since $H_1(X_L) = 0$, we have

$$H_2(M_L) \xrightarrow{i_X} H_2(X_L) \to H_2(X_L, M_L) \to H_1(M_L) \to 0.$$

Every group in this sequence is isomporphic to $\mathbb{Z}^m$. We have $H_2(X_L)$ is generated by Seifert surfaces for the link components of L union the cores of the added 2-handles, so equals $\mathbb{Z}^m$, and $H_1(X_L) = 0$, so $H_2(X_L, M_L) = H^2(X_L) = H_2(X_L)$. So the map $H_2(X_L, M_L) \to H_1(M_L)$ is a surjection between finitely generated abelian groups of the same rank, so is an isomorphism. Then by exactness the previous map in the sequence is the zero map and thus i_X is another surjection between finitely groups of the same rank, hence an isomorphism.

Now we have that i_X is an isomorphism and i_W is trivial, so the image of $(i_X, -i_W)$ is simply $H_2(X_L)$. Thus $H_2(\hat{W}) \cong H_2(W)$, as claimed.

Let $[x], [y] \in H$. The classes $[x], [y]$ may be represented by simple closed curves x, y embedded in $\widehat{\Sigma}_\varphi \subset M_L \subset X_L$, and $\widehat{\Sigma}_\varphi$ lies in S^3, minus a neighborhood of L, except for the added disks coming from the surgery tori. Since the disks are contractible, we can push x, y off of these disks without changing their homology classes, so that they lie entirely in $S^3 = \partial B^4$, the 0-handle of X_L.

Since $[x] \in H$, we have $[x] \in TH_1(R)$, so for some $m \neq 0$, $m[x] = 0$ in $H_1(R)$. So there exists a 2-chain $c_x \in C_2(R) \subset C_2(W) \subset C_2(\hat{W})$ with $\partial c_x = mx$. Similarly, we have $c_y \in C_2(R)$ with $\partial c_y = ny$ for some $n \neq 0$. Using the bicollar of R, which restricts to a bicollar of $\widehat{\Sigma}_\varphi$, we can push c_y off of R, and simultaneously push ny off of $\widehat{\Sigma}_\varphi$, to obtain the 2-chain $c_y^+ \in C_2(W) \subset C_2(\hat{W})$ with $\partial c_y^+ = ny^+$.

Since we arranged for x,y to lie entirely in $S^3 - N(L) \subset S^3$, mx and ny^+ also lie entirely in S^3. Now B^4 is contractible so mx and ny^+ bound some 2-chains d_x and d_y^+ in $B^4 \subset X_L \subset \hat{W}$.

So $\partial(c_x - d_x) = mx - mx = 0$, so $c_x - d_x$, and similarly $c_y^+ - d_y^+$, are 2-cycles in $\hat{W}$.

By definition of a 0.5-solution, $H_2(W)$ has basis $\{[L_i], [D_i]\}$ and intersection form $\bigoplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By above, $H_2(\hat{W}) \cong H_2(W)$. Adding X_L to W to create the closed manifold $\hat{W}$ had no effect on the second homology. So $H_2(\hat{W})$ is still generated by the homology classes of the embedded surfaces $L_i, D_i \subset W \subset \hat{W}$, so the intersection form also does not change.

Then $[c_x - d_x] = \sum a_j[L_j] + b_j[D_j]$ and $[c_x - d_x] \cdot [L_i] = b_i$. But $c_x - d_x \in R \cup M_L$ which is disjoint from all the L_i, so we get that all b_j are zero and $[c_x - d_x]$ is a linear combination of the $[L_i]$. So we can add copies of the L_i to c_x to make $[c_x - d_x] = 0$ in $H_2(\hat{W})$, and still have $\partial c_x = mx$.

Thus we have

$$0 = 0 \cdot [c_y^+ - d_y^+] = [c_x - d_x] \cdot [c_y^+ - d_y^+],$$

which equals the total intersection number of constituent 2-cells of $c_x - d_x$ with constituent 2-cells of $c_y^+ - d_y^+$. So by slight abuse of notation, we have

$$0 = [c_x - d_x] \cdot [c_y^+ - d_y^+] = (c_x \cdot c_y^+) - (c_x \cdot d_y^+) - (d_x \cdot c_y^+) + (d_x \cdot d_y^+).$$

Now c_x is a 2-chain in R plus some copies of the L_i and c_y^+ is a pushoff of a 2-chain c_y in R. We have that c_y is disjoint from all the L_i, thus so is its pushoff c_y^+, and additionally c_y^+ is disjoint from R. Hence $c_x \cdot c_y^+ = 0$.

Remember c_x and c_y^+ lie entirely in $W \subset \hat{W}$ and d_x and d_y^+ lie entirely in $X_L \subset \hat{W}$. So the only way that c_x and d_y^+ can intersect is on their boundaries. But their boundaries are 1-dimensional and so generically will not intersect in a 4-manifold. Thus $c_x \cdot d_y^+ = 0$ and similarly $d_x \cdot c_y^+ = 0$.

Now we must have $d_x \cdot d_y^+ = 0$. Recall that x and y and hence y^+ were pushed off of the capping disks coming from the 0-surgery, so that they lie entirely in $S^3 \subset M_L$, and d_x and d_y^+ live entirely in $B^4 \subset X_L \subset \hat{W}$. Thus by one definition of linking number we have $\mathrm{lk}(x, y^+) = d_x \cdot d_y^+ = 0$. Therefore we have $V(x, y) = 0$, as desired.

H is half rank, or at least contains a half rank subspace.

We know by the usual half lives, half dies duality argument that the kernel of $i : H_1(\widehat{\Sigma}_\varphi; \mathbb{Q}) \to H_1(R; \mathbb{Q})$ is half rank. We also may choose a basis for $H_1(\widehat{\Sigma}_\varphi; \mathbb{Z})$ so that the first g basis elements lie in $\ker(i)$ when mapped to $H_1(\widehat{\Sigma}_\varphi; \mathbb{Q})$. See for instance [Lic97] Lemma 8.15 and Corollary 8.16. So these basis elements must also lie in H. □

Using this theorem, we find an example of a boundary link with 2 unknotted components that is not 0.5-solvable. The unknotted components guarantee that the Arf invariants vanish, and the fact that it is a boundary link guarantees that all Milnor's invariants vanish. Therefore this example shows that Milnor's invariants are not enough to classify 0.5-solvability of links.

Corollary 3.18. *There exists an example of a 2-component boundary link whose components are unknots that is not* 0.5*-solvable.*

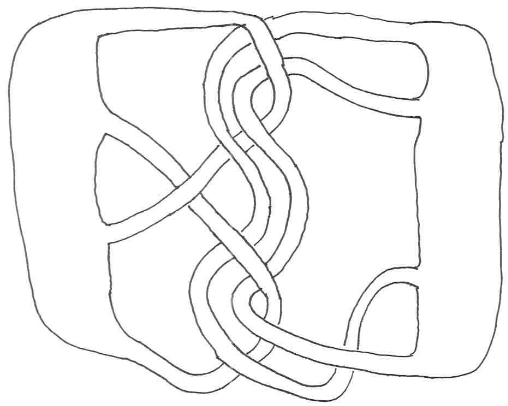

Figure 3.8: A boundary link with unknotted components that is not 0.5-solvable, drawn as the boundary of two disk-band surfaces.

Proof. Let L be the link in Fig. 3.8 and choose $\varphi = (1,1)$. Then an associated Seifert $\mathbb{Z}$-surface $\Sigma_{(1,1)}$ is composed of the obvious disk-band surfaces in Fig. 3.8, connected by a tube. Choosing the basis for $H_1(\widehat{\Sigma}_{(1,1)})$ given by curves going through the core of each band, we compute the corresponding Seifert $\mathbb{Z}$-matrix

$$\widehat{A}_{(1,1)} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

We compute that $\widehat{A}_{(1,1)} + (\widehat{A}_{(1,1)})^T$ has signature -2. Therefore there is no change of basis such that $\widehat{A}_{(1,1)}$ is congruent to a matrix of the form $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$, therefore $\widehat{\Sigma}_{(1,1)}$ cannot have a metabolizer. Thus by Theorem 3.17, the link cannot be 0.5-solvable. $\square$

Chapter 4

Blanchfield Forms

Let K be a knot in S^3 and let $X = S^3 - K$, its complement, and let M be its zero surgery manifold. Let $\widetilde{X}$ be the universal abelian cover of X, and $\widetilde{M}$ the universal abelian cover of M. Note that the universal abelian covers are the infinite cyclic covers, since $H_1(X) = H_1(M) = \mathbb{Z}$.

Recall that $\Lambda = \mathbb{Z}[t^{\pm 1}]$. The first homology group of the infinite cyclic cover $H_1(\widetilde{X}) = H_1(X;\Lambda)$ is a Λ-module called the *Alexander module* for K. For knots, the Alexander module is always torsion.

The *classical Blanchfield form*, also called a *linking form*, for a knot K is a non-singular sesquilinear pairing

$$\mathcal{B}\ell^X : H_1(X;\Lambda) \times H_1(X;\Lambda) \longrightarrow \mathbb{Q}(t)/\Lambda$$

defined as follows. Let $x, y \in H_1(X;\Lambda)$. Since the Alexander module is torsion, there exists a Laurent polynomial $p(t) \in \Lambda$ such that $p(t)x = 0$ in homology, so $p(t)x$ bounds a 2-chain D in $\widetilde{X}$. Then define

$$\mathcal{B}\ell^X(x,y) = \frac{1}{p(t)} \sum_{i=-\infty}^{\infty} \left(D \cdot t^i y\right) t^{-i}$$

up to elements of Λ, where $(D \cdot t^i y)$ is the usual intersection form in $\widetilde{X}$.

By nonsingular we mean that the mapping $H_1(X;\Lambda) \to \overline{\mathrm{Hom}_\Lambda(H_1(X;\Lambda), \mathbb{Q}(t)/\Lambda)}$ given by $y \mapsto \mathcal{B}\ell^X(\cdot, y)$ is an isomorphism.

It is a well-known result of Kearton and Trotter that the classical Blanchfield form for a knot may be represented by the matrix $(1-t)\left(tA - A^T\right)^{-1}$, where A is a Seifert matrix for K [Kea75][Tro73][FP17].

We could have also defined the classical Blanchfield form for knots on the Λ-module $H_1(\widetilde{M}) = H_1(M;\Lambda)$. However, for knots, $H_1(X;\Lambda) \cong H_1(M;\Lambda)$ and the two Blanchfield forms $\mathcal{B}\ell^X$ and $\mathcal{B}\ell^M$ are isomorphic.

Higher order Alexander modules and higher order linking forms for knots and for closed 3-manifolds with first Betti number 1 were introduced in [COT03] and further developed in [Coc04] and [Lei06]. Higher order Alexander modules for 3-manifolds in general were defined and invesitgated in [Har05]. In [Lei], Leidy defines higher order linking forms for any closed connected oriented 3-manifold.

4.1 Blanchfield Forms for any 3-Manifold

Before we define Blanchfield forms for any closed connected oriented 3-manifold, we need to define PTFA groups and the homology of PTFA covering spaces.

We take the following definition of PTFA groups from [COT03].

Definition 4.1. A group Γ is called *poly-torsion-free abelian* (PTFA) if it admits a normal series $< 1 >= G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = \Gamma$ such that the factors G_{i+1}/G_i are torsion-free abelian.

Remark 4.2. Let Γ be a PTFA group. We will need the following facts.

1. The group ring $\mathbb{Z}\Gamma$ is an Ore domain, and therefore it is possible to define its right ring of fractions, which we denote $\mathcal{K}$ [Pas77].

2. $\mathcal{K}$ is naturally a right $\mathcal{K}$-module and a $\mathbb{Z}\Gamma$-bimodule.

3. $\mathcal{K}$ is flat as a left $\mathbb{Z}\Gamma$-module. That is, the functor $- \otimes_{\mathbb{Z}\Gamma} \mathcal{K}$ is exact [Ste75].

4. Every module over $\mathcal{K}$ is a free module [Ste75] and such modules have a well-defined rank $\mathrm{rk}_{\mathcal{K}}$ which is additive on short exact sequences [Coh85].

5. Let R be any localization $\mathbb{Z}\Gamma \subseteq R \subseteq \mathcal{K}$ and let M be a right R-module. Define the *rank of* M by $\mathrm{rk}_R(M) = \mathrm{rk}_{\mathcal{K}}(M \otimes_R \mathcal{K})$. M is a torsion R-module if and only if $M \otimes_R \mathcal{K} = 0$ [Ste75], that is if and only if M has rank 0. In general, the set of torsion elements of M is a submodule, denoted TM, which equals the kernel of the inclusion $M \to M \otimes_R \mathcal{K}$. [Coc04].

Now we turn to the homology of covering spaces, viewed as homology with twisted coefficients. We take the following definition from [Har05], with the restriction that ϕ be an epimorphism added for simplicity.

Definition 4.3. Suppose that X has the homotopy type of a connected CW-complex, Γ is a PTFA group, and $\phi : \pi_1(X) \to \Gamma$ is an epimorphism. Let X_Γ denote the regular connected covering space of X associated to the kernel of ϕ. If $A \subset X$ is a subcomplex there is an induced coefficient system on A, $\phi \circ i_* : \pi_1(A) \to \Gamma$, and we get a regular cover (X_Γ, A_Γ) covering (X, A). Let M be a $\mathbb{Z}\Gamma$-bimodule. The *equivariant homology and cohomology of* X *and* (X, A) are defined below, and are well-known to be isomorphic to the homology and cohomology of X and (X, A) with coefficient system induced by ϕ [Whi78]. Let

$$H_*(X; M) = H_*(C_*(X_\Gamma) \otimes_{\mathbb{Z}\Gamma} M) \text{ and}$$

$$H_*(X, A; M) = H_*(C_*(X_\Gamma, A_\Gamma) \otimes_{\mathbb{Z}\Gamma} M) \text{ as a right } \mathbb{Z}\Gamma\text{-modules and let}$$

$$H^*(X; M) = H_*(\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*(X_\Gamma), M)) \text{ and}$$

$$H^*(X, A; M) = H_*(\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*(X_\Gamma, A_\Gamma), M)) \text{ as a left } \mathbb{Z}\Gamma\text{-modules.}$$

Remark 4.4. We take the following facts about equivariant homology and cohomology from [COT03] and [Coc04].

1. Note that $H_*(X;\mathbb{Z}\Gamma)$ is exactly $H_*(X_\Gamma)$ as a right $\mathbb{Z}\Gamma$-module. Moreover, if M is flat as a left $\mathbb{Z}\Gamma$-module then $H_*(X;M) \cong H_*(X_\Gamma) \otimes_{\mathbb{Z}\Gamma} M$. In particular this holds for $M = \mathcal{K}$ by Remark 4.2.

2. If X is a compact, oriented n-manifold then by Poincaré duality, $H_p(X;M)$ is isomorphic to $\overline{H^{n-p}(X,\partial X;M)}$, which is just $H^{n-p}(X,\partial X;M)$ made into a right $\mathbb{Z}\Gamma$-module using the involution on the group ring.

3. There exists a universal coefficient spectral sequence which collapses to the usual Universal Coefficient Theorem for coefficients in a principal ideal domain (in particular for $\mathcal{K}$). Hence $H^n(X;\mathcal{K}) \cong \mathrm{Hom}_{\mathcal{K}}(H_n(X;\mathcal{K}),\mathcal{K})$.

Now we are ready to give Leidy's definition. Given any left R-module M, let $\overline{M}$ denote the usual associated right R-module resulting from the involution of R. Given any right R-module M, let $M^\# = \overline{\mathrm{Hom}_R(M;\mathcal{K}/R)}$.

Theorem 4.5 (Leidy [Lei])**.** *Suppose that M is a closed, connected, oriented 3-manifold, Γ is a PTFA group and $\phi : \pi_1(M) \to \Gamma$ is an epimorphism. Let R be any Ore localization $\mathbb{Z}\Gamma \subseteq R \subseteq \mathcal{K}$. There exists a linking form $\mathcal{B}\ell_R^M : TH_1(M;R) \to (TH_1(M;R))^\#$.*

Proof. We wish to define $\mathcal{B}\ell_R^M$ as the map fitting in the following commutative diagram on the next page.

Here B is the Bockstein homomorphism from the homology Bockstein sequence arising from the short exact sequence $0 \to R \to \mathcal{K} \to \mathcal{K}/R \to 0$. Also PD is Poincare duality, κ is the Kroenecker evaluation map, and $\iota^\sharp$ is the dual of the map $\iota : TH_1(M;R) \to H_1(M;R)$.

$$\begin{tikzcd}
H_2(M;\mathcal{K}/R) \arrow[r, "B"] \arrow[d, "PD"] & TH_1(M;R) \arrow[dddl, bend left, "\mathcal{B}\ell_R^M"] \\
\overline{H^1(M;\mathcal{K}/R)} \arrow[d, "\kappa"] & \\
(H_1(M;R))^\sharp \arrow[d, "\iota^\sharp"] & \\
(TH_1(M;R))^\sharp &
\end{tikzcd}$$

Clearly there is a question as to whether $\mathcal{B}\ell_R^M$ is well defined. Consider the homology Bockstein sequence.

$$H_2(M;\mathcal{K}) \xrightarrow{\psi} H_2(M;\mathcal{K}/R) \xrightarrow{B} H_1(M;R) \to H_1(M;\mathcal{K})$$

Since $\mathcal{K}$ is a flat R-module, $H_1(M;\mathcal{K}) = H_1(M;R) \otimes_R \mathcal{K}$, so the kernel of the rightmost map above is exactly $TH_1(M;R)$. Then by exactness $TH_1(M;R) = \operatorname{im} B \cong \operatorname{cok}\psi$. Thus $\mathcal{B}\ell_R^M$ is well-defined if $\operatorname{im}\psi \subset \ker\left(\iota^\sharp \circ \kappa \circ PD\right)$. Consider the following commutative diagram.

$$\begin{tikzcd}
H_2(M;\mathcal{K}) \arrow[r, "\psi"] \arrow[d, "PD"] & H_2(M;\mathcal{K}/R) \arrow[d, "PD"] \\
\overline{H^1(M;\mathcal{K})} \arrow[r] \arrow[d, "\kappa"] & \overline{H^1(M;\mathcal{K}/R)} \arrow[d, "\kappa"] \\
\overline{\mathrm{Hom}_R(H_1(M;R),\mathcal{K})} \arrow[r] \arrow[d, "\iota^\sharp"] & \overline{\mathrm{Hom}_R(H_1(M;R),\mathcal{K}/R)} \arrow[d, "\iota^\sharp"] \\
\overline{\mathrm{Hom}_R(TH_1(M;R),\mathcal{K})} \arrow[r] & \overline{\mathrm{Hom}_R(TH_1(M;R),\mathcal{K}/R)}
\end{tikzcd}$$

Since $\mathcal{K}$ is a torsion-free R-module, $\mathrm{Hom}_R(TH_1(M;R),\mathcal{K}) = 0$. Thus $\operatorname{im}\psi \subset \ker\left(\iota^\sharp \circ \kappa \circ PD\right)$ so $\mathcal{B}\ell_R^M$ is well-defined.

We can also define the corresponding pairing $\mathcal{B}\ell_R^M(x,y) = \mathcal{B}\ell_R^M(x)(y)$. $\square$

Note that this theorem can easily be extended to define Blanchfield forms $\mathcal{B}\ell_R^X$ on $H_1(X;R)$ where X is a compact connected oriented manifold with boundary.

$$\begin{tikzcd}
H_2(X;\mathcal{K}/R) \arrow[r, "B"] \arrow[d] & TH_1(X;R) \arrow[dddd, bend left, "\mathcal{B}\ell_R^X"] \\
H_2(X,\partial X;\mathcal{K}/R) \arrow[d, "PD"] & \\
\overline{H^1(X;\mathcal{K}/R)} \arrow[d, "\kappa"] & \\
(H_1(X;R))^\sharp \arrow[d, "\iota^\sharp"] & \\
(TH_1(X;R))^\sharp &
\end{tikzcd}$$

Classical Blanchfield forms for knots are always nonsingular, and in [COT03] it is proven that linking forms for closed, connected, oriented 3-manifolds M with $\beta_1(M) = 1$ on generalized Alexander modules $H_1(M;R)$ are nonsingular when R is a PID. In this case, it is sufficient to show that the Kronecker evaluation map is an isomorphism. This is because when $\beta_1(M) = 1$, the generalized Alexander module $H_1(M;R)$ is always torsion, so the Bockstein map is an isomorphism, and there is no need for the map $\iota^{\#}$. When we remove the condition that $\beta_1(M) = 1$, there are additional sources of singularity.

Proposition 4.6. *If R is a PID, then the Blanchfield form $\mathcal{B}\ell_R^M$ on $TH_1(M;R)$ is nonsingular.*

Proof. When R is a PID the UCSS collapses and we get the usual short exact sequence

$$0 \to \operatorname{Ext}^1_R(H_0(M;R),\mathcal{K}/R) \to H^1(M;\mathcal{K}/R) \xrightarrow{\kappa} \operatorname{Hom}_R(H_1(M;R),\mathcal{K}/R) \to 0$$

But $\mathcal{K}$ and $\mathcal{K}/R$ is are injective R-modules, since they are clearly divisible and by

[Ste75] a divisible module over a PID is injective. Thus $\mathrm{Ext}^1_R(H_0(M;R),\mathcal{K}/R) = 0$ and κ is an isomorphism. Similarly, $\kappa : H^1(M;\mathcal{K}) \to \mathrm{Hom}_R(H_1(M;R),\mathcal{K})$ is an isomorphism.

So now the only sources of singularity are B and $\iota^\sharp$. The inclusion $\iota : TH_1(M;R) \to H_1(M;R)$ is injective and $\mathcal{K}/R$ is an injective module, so $\mathrm{Hom}_R(-,\mathcal{K}/R)$ is an exact functor. Thus $\iota^\sharp$ is surjective. So $TH_1(M;R)^\sharp \cong H_1(M;R)^\sharp/\ker\iota^\sharp$.

Then since PD and κ are isomorphisms, and we already know that B is a surjection and $\ker B = \mathrm{im}\,\psi$, we have the following composition of isomorphisms.

$$TH_1(M;R) \xleftarrow{B} \frac{H_2(M;\mathcal{K}/R)}{\mathrm{im}\,\psi} \xrightarrow{PD} \frac{\overline{H^1(M;\mathcal{K}/R)}}{\mathrm{im}(PD\circ\psi)} \xrightarrow{\kappa} \frac{H_1(M;R)^\#}{\mathrm{im}(\kappa\circ PD\circ\psi)}$$

Thus if we can show that $\ker\iota^\# = \mathrm{im}(\kappa\circ PD\circ\psi)$ then we are done. By commutativity of the diagram, since the lower left corner is 0, it is clear that $\mathrm{im}(\kappa\circ PD\circ\psi) \subset \ker\iota^\#$. Since the diagram is commutative and the κ maps are isomorphisms in both columns, it is enough to show that $\ker\iota^\sharp$ is in the image of the third horizontal map ψ. Let $f \in \ker\iota^\sharp$. So $f : H_1(M;R) \to \mathcal{K}/R$ and is zero on $TH_1(M;R)$. Since R is a PID and $H_1(M;R)$ is finitely generated, we have that $H_1(M;R) = R^k \oplus TH_1(M;R)$ and let each copy of R be generated by x_i. For each $i = 1,\ldots,k$, choose a representative $y_i \in \mathcal{K}$ so that $f(x_i) = y_i + R \in \mathcal{K}/R$. Define $g : H_1(M;R) \to \mathcal{K}$ by $g(x_i) = y_i$ and g vanishes on $TH_1(M;R)$. Then $\psi(g) = f$. $\square$

We are most interested in the case where $M = M_L$ is the zero surgery manifold of a link L. The following is a special case of Lemma 3.5 in [CHL09] and Lemma 3.8 in [CHL08].

Theorem 4.7 (Cochran-Harvey-Leidy [CHL08][CHL09])**.** *Let $n \in \mathbb{N}$. Suppose L is n-solvable with zero surgery manifold M and n-solution W and $\phi : \pi_1(M) \to \Gamma$ is a nontrivial coefficient system that extends to $\pi_1(W)$ and Γ is a PTFA group with*

$\Gamma^{(n)} = 1$. *Suppose also that R is an Ore localization $\mathbb{Z}\Gamma \subset R \subset \mathcal{K}$. Let P be the kernel of $j : TH_1(M; R) \to TH_1(W; R)$. Then $P \subset P^{\perp}$ with respect to the Blanchfield form $\mathcal{B}\ell_R^M$ on $TH_1(M; R)$.*

Proof. We need the following lemmas.

Lemma 4.8 (Cochran-Harvey-Leidy [CHL08][CHL09])**.** *Assuming the hypotheses of Theorem 4.7, the following is exact.*

$$TH_2(W, M; R) \xrightarrow{\partial} TH_1(M; R) \xrightarrow{j} TH_1(W; R)$$

Lemma 4.9 (Cochran-Harvey-Leidy [CHL08][CHL09])**.** *There is a Blanchfield type form $\mathcal{B}\ell_R^{rel}$ such that the following diagram is commutative up to sign.*

$$\begin{tikzcd}
TH_2(W, M; R) \arrow[r, "\partial"] \arrow[d, "\mathcal{B}\ell_R^{rel}"] & TH_1(M; R) \arrow[d, "\mathcal{B}\ell_R^{M}"] \\
TH_1(W; R)^{\sharp} \arrow[r, "j^{\sharp}"] & TH_1(M; R)^{\sharp}
\end{tikzcd}$$

The definition of $\mathcal{B}\ell_R^{rel} : TH_2(W, M; R) \to TH_1(W; R)^{\sharp}$ is almost identical to that of $\mathcal{B}\ell_R^{M}$.

$$\begin{tikzcd}
H_3(W, M; \mathcal{K}/R) \arrow[r, "B"] \arrow[d, "PD"] & TH_2(W, M; R) \arrow[dddl, bend left, "\mathcal{B}\ell_R^{rel}"] \\
H^1(W; \mathcal{K}/R) \arrow[d, "\kappa"] & \\
(H_1(W; R))^{\sharp} \arrow[d, "\iota^{\sharp}"] & \\
(TH_1(W; R))^{\sharp} &
\end{tikzcd}$$

Similarly we could define $\mathcal{B}\ell_R^{inv} : TH_1(W; R) \to TH_2(W, M; R)$.

$$
\begin{tikzcd}
H_2(W;\mathcal{K}/R) \arrow[r, "B"] \arrow[d, "PD"] & TH_1(W;R) \arrow[dddl, bend left, "\mathcal{B}\ell_R^{inv}"] \\
H^2(W,M;\mathcal{K}/R) \arrow[d, "\kappa"] & \\
(H_2(W,M;R))^\sharp \arrow[d, "\iota^\sharp"] & \\
(TH_2(W,M;R))^\sharp &
\end{tikzcd}
$$

Then we have the following commutative diagram, the top row of which is exact.

$$
\begin{tikzcd}
TH_2(W,M;R) \arrow[r, "\partial"] \arrow[d, "\mathcal{B}\ell_R^{rel}"] & TH_1(M;R) \arrow[r, "j"] \arrow[d, "\mathcal{B}\ell_R^{M}"] & TH_1(W;R) \arrow[d, "\mathcal{B}\ell_R^{inv}"] \\
TH_1(W;R)^\sharp \arrow[r, "j^\sharp"] & TH_1(M;R)^\sharp \arrow[r, "\partial^\sharp"] & TH_2(W,M;R)^\sharp
\end{tikzcd}
$$

Note if R is a PID, then both $\mathcal{B}\ell_R^{rel}$ and $\mathcal{B}\ell_R^{inv}$ are nonsingular. This will be near identical to the proof that $\mathcal{B}\ell_R^M$ is nonsingular when R is a PID.

Now we can prove the theorem. First we'll show that $P \subset P^\perp$.

Let $a \in P$. By exactness, $P = \ker j = \operatorname{im} \partial$, so there exists an $A \in TH_2(W,M;R)$ with $\partial A = a$. So $\mathcal{B}\ell(a) = \mathcal{B}\ell(\partial A) = \mathcal{B}\ell \circ \partial(A)$. And by commutativity of the above diagram, $\mathcal{B}\ell \circ \partial(A) = j^\sharp \circ \mathcal{B}\ell^{rel}(A) = j^\sharp(\mathcal{B}\ell^{rel}(A)) = (\mathcal{B}\ell^{rel}(A)) \circ j$. So for all $b \in P$, $\mathcal{B}\ell(a)(b) = (\mathcal{B}\ell^{rel}(A) \circ j)(b) = \mathcal{B}\ell^{rel}(A)(j(b)) = \mathcal{B}\ell^{rel}(A)(0) = 0$, since $b \in \ker j$. Therefore $P \subset P^\perp$. $\square$

When R is a PID, we can say more. The following proof is near identical to the proof for the case $\beta_1(M) = 1$ in [COT03].

Proposition 4.10. *Assume the hypotheses of Theorem 4.7. Additionally, if R is a PID, then $P = P^\perp$.*

Proof. Consider the monomorphism $j : TH_1(M;R)/P \to TH_1(W;R)$. We noted above that $\mathcal{K}/R$ is an injective R-module when R is a PID, so $\mathrm{Hom}_R(-,\mathcal{K}/R)$ is an exact functor, so we see that $j^\sharp : (TH_1(W;R))^\sharp \to (TH_1(M;R)/P)^\sharp$ is an epimorphism.

Now for $a \in P^\perp$, we have $\mathcal{B}\ell(a)(b) = 0$ for all $b \in P$ so $\mathcal{B}\ell(a)$ descends to an element of $(TH_1(M;R)/P)^\sharp$. Thus $\mathcal{B}\ell(a)$ is in the image of $j^\sharp$. Recall that since R is a PID, we have both $\mathcal{B}\ell_R^M$ and $\mathcal{B}\ell_R^{rel}$ are isomorphisms. So we can pull an element in the preimage of $\mathcal{B}\ell(a)$ up to $A \in TH_2(W,M;R)$, and then by commutativity of the diagram we have $\mathcal{B}\ell_R^M(\partial A) = \mathcal{B}\ell(a)$ and thus $\partial A = a$ so $a \in P$

Alternatively, if $\mathcal{B}\ell^{inv}$ is nonsingular, we also get $P^\perp \subset P$. Let $a \in P^\perp$. Then $\mathcal{B}\ell^{inv}(j(a)) = (\mathcal{B}\ell^{inv} \circ j)(a) = (\partial^\sharp \circ \mathcal{B}\ell)(a) = \partial^\sharp(\mathcal{B}\ell(a)) = \mathcal{B}\ell(a) \circ \partial$. Then for all $B \in TH_2(W,M;R)$, we have $\mathcal{B}\ell^{inv}(j(a))(B) = (\mathcal{B}\ell(a) \circ \partial)(B) = \mathcal{B}\ell(a)(\partial B) = 0$ since $\partial B \in P$. Thus $\mathcal{B}\ell^{inv}(j(a)) = 0$ in $TH_2(W,M;R)^\sharp$. Then since $\mathcal{B}\ell^{inv}$ is nonsingular, we must have $j(a) = 0$, thus $a \in P$. $\square$

4.2 Blanchfield Forms from the Universal Abelian Cover

We wish to define Blanchfield forms for the zero surgery manifold M for links. To be able to use the results of the previous section we need a PTFA group Γ, an epimorphism $\phi : \pi_1(M) \to \Gamma$ that extends to $\pi_1(W)$ for n-solutions W when L is n-solvable, and we need an Ore localization $\mathbb{Z}\Gamma \subseteq R \subseteq \mathcal{K}$ that is a PID.

We first look at the universal abelian cover $\widetilde{M} \xrightarrow{p} M$, the regular cover of M corresponding to the subgroup $G^{(1)} = [G,G]$, the commutator subgroup of $G = \pi_1(M)$. The deck group of this cover is $G/G^{(1)} = H_1(M)$. If we restrict ourselves to the class of m-component links for which the pairwise linking numbers are all zero, then $H_1(M) = H_1(X) = \mathbb{Z}^m$, which is trivially a PTFA group. (Use the normal series

$\langle 1 \rangle \triangleleft \mathbb{Z}^m$.) Otherwise, $H_1(M)$ may have torsion.

For now, let L be an m-component link with pairwise linking numbers zero. Let M be the zero surgery manifold of L and let $\Gamma = H_1(M) = \mathbb{Z}^m$. For our coefficient system, we simply use abelianization $\pi_1(M) \to H_1(M) = \Gamma$. Since Γ is abelian, $\Gamma^{(1)} = 1$, so we can only satisfy the hypotheses of Theorem 4.7 for 1-solvable links. Suppose that L is 1-solvable and suppose that W is a 1-solution. The coefficient system $\pi_1(M) \to H_1(M) = \Gamma$ clearly extends to $\pi_1(W)$, since we also have an abelianization $\pi_1(W) \to H_1(W)$ and since W is a 1-solution it is an H_1-bordism so $H_1(W) \cong H_1(M) = \Gamma$. Now all we need is a suitable Ore localization $\mathbb{Z}\Gamma \subseteq R \subseteq \mathcal{K}$ where R is a PID.

Definition 4.11. Let $\Gamma = H_1(M)$. Choose a primitive $\varphi \in H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbb{Z})$, so that we can consider φ to be an epimorphism $\Gamma \to \mathbb{Z}$. Then define the localization

$$\mathcal{R}_\varphi = \mathbb{Z}\Gamma \left(\mathbb{Z}[\ker\varphi] - 0\right)^{-1}.$$

For $\varphi \neq 0$, the short exact sequence $0 \to \ker\varphi \to \Gamma \to \mathbb{Z} \to 0$ splits. Once we choose a splitting $s : \mathbb{Z} \to \Gamma$, this induces an isomorphism $\mathbb{Z}\Gamma \cong \mathbb{Z}[\ker\varphi][t^{\pm 1}]$. Note that we must have $\ker\varphi \cong \mathbb{Z}^{m-1}$. Then if we let $\mathbb{K}_\varphi$ denote the fraction field of $\mathbb{Z}[\ker\varphi]$, we get an isomorphism $\mathcal{R}_\varphi \cong \mathbb{K}_\varphi[t^{\pm 1}]$, which is well known to be a PID. Thus $\mathcal{R}_\varphi$ is actually a principal ideal domain.

This type of localization is discussed in more generality in [COT03] and [Har05], where the rings are skew Laurent polynomial rings. In our case $\mathbb{K}_\varphi$ is truly a field and not a skew field.

Combining Proposition 4.6, Theorem 4.7, and Proposition 4.10 with the above we obtain the following.

Corollary 4.12. *Suppose that L is a link with pairwise linking numbers zero and M*

is the zero surgery manifold for L. For any primitive $\varphi \in H^1(M)$, the Blanchfield form $\mathcal{B}\ell^M_{\mathcal{R}_\varphi}$ on $TH_1(M;\mathcal{R}_\varphi)$ is nonsingular. Additionally, if L is 1-solvable, then $\mathcal{B}\ell^M_{\mathcal{R}_\varphi}$ is hyperbolic.

4.3 Blanchfield Forms from the Infinite Cyclic Covers

Let L be a link with pairwise linking numbers zero. Let $X = S^3 - N(L)$ be its exterior and M be its zero surgery manifold. Let $\Gamma = H_1(X) = H_1(M)$. Recall from Section 3.2 that for each primitive $\varphi \in H^1(X) = H^1(M)$ we obtain an infinite cyclic cover X_φ of X, the regular connected cover corresponding to the composition $\pi_1(X) \to H_1(X) = \Gamma \xrightarrow{\varphi} \mathbb{Z}$, and an infinite cyclic cover M_φ of M, the regular connected cover corresponding to the composition $\pi_1(M) \to H_1(M) = \Gamma \xrightarrow{\varphi} \mathbb{Z}$. Recall further that we define $\Lambda_\varphi = \mathbb{Z}[\Gamma/\ker\varphi] \cong \mathbb{Z}[t^{\pm 1}]$.

It will be convenient to localize Λ_φ to a PID. Define the localization

$$\mathcal{Q}_\varphi = \Lambda_\varphi(\mathbb{Z}-0)^{-1} = \mathbb{Z}[\Gamma/\ker\varphi](\mathbb{Z}-0)^{-1} = \mathbb{Q}[\Gamma/\ker\varphi] \cong \mathbb{Q}[t^{\pm 1}].$$

We now wish to define a linking form $\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}$ on $TH_1(X;\mathcal{Q}_\varphi)$.

For $M = \mathcal{Q}_\varphi$, $\mathcal{K}$, or $\mathcal{K}/\mathcal{Q}_\varphi$, we define the map $\kappa : \mathrm{Hom}_{\mathcal{Q}_\varphi}(\overline{C_*(X;\mathcal{Q}_\varphi)}, M) \to \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_*(X;\mathcal{Q}_\varphi), M)}$ so that $\kappa(f)$ is the map $\sigma \mapsto \overline{f(\sigma)}$. This induces an isomorphism of $\mathcal{Q}_\varphi$ modules $\kappa : H^i(X;M) \to H_i(\overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_*(X;\mathcal{Q}_\varphi), M)})$. We also have a map $\mathrm{ev} : H_i(\overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_*(X;\mathcal{Q}_\varphi), M)}) \to \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(H_i(C_*(X;\mathcal{Q}_\varphi)), M)}$ so that $\mathrm{ev}\circ\kappa$ is the Kronecker evaluation map. We let BS denote the Bockstein map, discussed earlier when we first defined the Blanchfield form, and PD denote Poincare duality. Let $j^\#$ be the map dual to the inclusion $j : TH_1(X;\mathcal{Q}_\varphi) \to H_1(X;\mathcal{Q}_\varphi)$.

The following definition is adapted from the work of Friedl and Powell [FP17]. All chain, homology, and cohomology groups have coefficients in $\mathcal{Q}_\varphi$ unless otherwise specified.

$$
\begin{tikzcd}
TH_1(X) \arrow[r] \arrow[dddr, bend right, "\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}"'] & TH_1(X,\partial X) \arrow[r, "\mathrm{PD}"] & TH^2(X) & H^1(X;\mathcal{K}/\mathcal{Q}_\varphi) \arrow[l, two heads, "\mathrm{BS}"'] \arrow[d, "\kappa"] \\
 & & & H_1(\overline{\mathrm{Hom}(C_*(X),\mathcal{K}/\mathcal{Q}_\varphi)}) \arrow[d, "\mathrm{ev}"] \\
 & & & \overline{\mathrm{Hom}(H_1(X),\mathcal{K}/\mathcal{Q}_\varphi)} \arrow[d, "j^\#"] \\
 & & & \overline{\mathrm{Hom}(TH_1(X),\mathcal{K}/\mathcal{Q}_\varphi)}
\end{tikzcd}
$$

Similarly to what we did before, we can show that the Blanchfield form is well defined using the following diagram that is commutative up to sign.

$$
\begin{tikzcd}
H_2(X;\mathcal{K}) \arrow[r] \arrow[d] & H_2(X;\mathcal{K}/\mathcal{Q}_\varphi) \arrow[r, two heads, "\mathrm{BS}"] \arrow[d] & TH_1(X) \arrow[d] \\
H_2(X,\partial X;\mathcal{K}) \arrow[r] \arrow[d, "\mathrm{PD}"] & H_2(X,\partial X;\mathcal{K}/\mathcal{Q}_\varphi) \arrow[r, two heads, "\mathrm{BS}"] \arrow[d, "\mathrm{PD}"] & TH_1(X,\partial X) \arrow[d, "\mathrm{PD}"] \\
H^1(X;\mathcal{K}) \arrow[r] \arrow[d, "\kappa"] & H^1(X;\mathcal{K}/\mathcal{Q}_\varphi) \arrow[r, two heads, "\mathrm{BS}"] \arrow[d, "\kappa"] & TH^2(X) \arrow[d, "\kappa"] \\
H_1(\overline{\mathrm{Hom}(C_*(X),\mathcal{K})}) \arrow[r] \arrow[d, "\mathrm{ev}"] & H_1(\overline{\mathrm{Hom}(C_*(X),\mathcal{K}/\mathcal{Q}_\varphi)}) \arrow[r, two heads, "\mathrm{BS}"] \arrow[d, "\mathrm{ev}"] & TH_2(\overline{\mathrm{Hom}(C_*(X),\mathcal{Q}_\varphi)}) \\
\overline{\mathrm{Hom}(H_1(X),\mathcal{K})} \arrow[r] \arrow[d] & \overline{\mathrm{Hom}(H_1(X),\mathcal{K}/\mathcal{Q}_\varphi)} \arrow[d, "j^\#"] & \\
0 \arrow[r] & \overline{\mathrm{Hom}(TH_1(X),\mathcal{K}/\mathcal{Q}_\varphi)} &
\end{tikzcd}
$$

Chapter 5

Calculating Blanchfield Forms

5.1 Blanchfield Forms from Intersection Forms on Surfaces

Let L be a link with pairwise linking numbers zero and let $X = S^3 - N(L)$ be its exterior. Choose a primitive element $\varphi \in H^1(X)$ and let Σ_φ be a Seifert $\mathbb{Z}$-surface for L corresponding to φ. Let $Y = X - \Sigma_\varphi$. Recall from Section 3.2 that we have the following exact sequence.

$$H_2(X;\Lambda_\varphi) \longrightarrow H_1(\Sigma_\varphi;\Lambda_\varphi) \xrightarrow{\iota_+ - \iota_- t^{-1}} H_1(Y;\Lambda_\varphi) \xrightarrow{\iota_Y} H_1(X;\Lambda_\varphi) \longrightarrow 0$$

Recall that $H_1(\Sigma_\varphi;\Lambda_\varphi) = H_1(\Sigma_\varphi) \otimes \Lambda_\varphi$ and $H_1(Y;\Lambda_\varphi) = H_1(Y) \otimes \Lambda_\varphi$. Then if $H_1(\Sigma_\varphi)$ has rank r and $H_1(Y)$ has rank s, we get that $H_1(\Sigma_\varphi;\Lambda_\varphi) \cong \Lambda_\varphi^r$ and $H_1(Y;\Lambda_\varphi) \cong \Lambda_\varphi^s$, and then the map $\iota_+ - \iota_- t^{-1}$ may be represented by a matrix P. So we have

$$H_2(X;\Lambda_\varphi) \longrightarrow \Lambda_\varphi^r \xrightarrow{P} \Lambda_\varphi^s \longrightarrow H_1(X;\Lambda_\varphi) \longrightarrow 0$$

and $H_1(X;\Lambda_\varphi) \cong \Lambda_\varphi^s/\Lambda_\varphi^r P$. We wish to find a presentation matrix for $TH_1(X;\Lambda_\varphi)$, but this will be easier if we localize to $\mathcal{Q}_\varphi$ coefficients.

Note that $\mathcal{Q}_\varphi$ is a flat left Λ_φ module and $-\otimes_{\Lambda_\varphi}$ is an exact functor. Since $\mathcal{Q}_\varphi$ is a PID, we can diagonalize P, and $H_1(X;\mathcal{Q}_\varphi)$ decomposes into the direct sum of its torsion and torsion-free submodules. Let D be a $d \times d$ diagonal matrix with nonzero diagonal entries such that P diagonalizes into the block matrix

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

(If we are unable to get a diagonal with nonzero entries, then P is equivalent to 0 as a presentation matrix, and $H_1(X;\mathcal{Q}_\varphi)$ is free.)

Then $H_1(X;\mathcal{Q}_\varphi) \cong {\mathcal{Q}_\varphi}^s/{\mathcal{Q}_\varphi}^r P \cong {\mathcal{Q}_\varphi}^d/{\mathcal{Q}_\varphi}^d D \oplus {\mathcal{Q}_\varphi}^{s-d}$. So by a change of basis for $H_1(\Sigma;\mathcal{Q}_\varphi)$ and $H_1(Y;\mathcal{Q}_\varphi)$ we get the following diagram with exact rows.

$$\begin{tikzcd}
H_1(\Sigma;\mathcal{Q}_\varphi) \arrow[r, "\iota_+ - \iota_- t^{-1}"] \arrow[d, "\cong"] & H_1(Y;\mathcal{Q}_\varphi) \arrow[r, "\iota_Y"] \arrow[d, "\cong"] & H_1(X;\mathcal{Q}_\varphi) \arrow[r] \arrow[d, equal] & 0 \\
{\mathcal{Q}_\varphi}^r \arrow[r, "P"] \arrow[d, "\cong"] & {\mathcal{Q}_\varphi}^s \arrow[r] \arrow[d, "\cong"] & H_1(X;\mathcal{Q}_\varphi) \arrow[r] \arrow[d, "\cong"] & 0 \\
{\mathcal{Q}_\varphi}^d \arrow[r, "D"] & {\mathcal{Q}_\varphi}^d \arrow[r] & TH_1(X;\mathcal{Q}_\varphi) \arrow[r] & 0 \\
\bigoplus & \bigoplus & \bigoplus & \\
{\mathcal{Q}_\varphi}^{r-d} \arrow[r, "0"] & {\mathcal{Q}_\varphi}^{s-d} \arrow[r] & H_1(X;\mathcal{Q}_\varphi)/TH_1(X;\mathcal{Q}_\varphi) \arrow[r] & 0
\end{tikzcd}$$

Additionally, since D is a diagonal matrix with nonzero entries, and $\mathcal{Q}_\varphi$ has no zero divisors, D must be injective.

Define $\operatorname{diag}(\Sigma)$ to be the submodule of $H_1(\Sigma;\mathcal{Q}_\varphi)$ that is isomorphic to the domain of D. Similarly define $\operatorname{diag}(Y)$ to be the submodule of $H_1(Y;\mathcal{Q}_\varphi)$ that is isomorphic to the codomain of D. Then since D is injective, $\iota_+ - \iota_- t^{-1}$ must be injective when restricted to $g(\Sigma)$. Thus we have the following exact sequence:

$$0 \longrightarrow \operatorname{diag}(\Sigma) \xrightarrow{\iota_+ - \iota_- t^{-1}} \operatorname{diag}(Y) \xrightarrow{\iota_Y} TH_1(X;\mathcal{Q}_\varphi) \longrightarrow 0$$

Now when we localize to the fraction field $\mathcal{K} = \mathbb{Q}(t)$ we get an isomorphism.

$$0 \longrightarrow \operatorname{diag}(\Sigma) \otimes_{\mathcal{Q}_\varphi} \mathcal{K} \xrightarrow{\iota_+ - \iota_- t^{-1}} \operatorname{diag}(Y) \otimes_{\mathcal{Q}_\varphi} \mathcal{K} \xrightarrow{\iota_Y} 0$$

The above isomorphism is important for the calculation of the Blanchfield form.

It is clear that $\operatorname{diag}(Y) = \iota_Y^{-1}(TH_1(X;\mathcal{Q}_\varphi))$. Ideally we would also have $\operatorname{diag}(\Sigma) = \iota_\Sigma(TH_1(X;\mathcal{Q}_\varphi)$. Unfortunately this is not true, so we define the submodule $g(\Sigma) = \operatorname{diag}(\Sigma) \cap \iota_\Sigma^{-1}(TH_1(X;\mathcal{Q}_\varphi))$, the "good" part of $H_1(\Sigma;\mathcal{Q}_\varphi)$. Now we have that $\iota_\Sigma(g(\Sigma)) \subset TH_1(X;\mathcal{Q}_\varphi)$ and since for all $x \in H_1(\Sigma;\mathcal{Q}_\varphi)$, we have $\iota_Y(\iota_+(x)) = \iota_\Sigma(x)$ in $H_1(X;\mathcal{Q}_\varphi)$, we also have $\iota_+(g(\Sigma)) \subset \operatorname{diag}(Y)$.

Note that the above analysis not only makes sense when X is the exterior of a link, but for any compact connected oriented 3-manifold that is either closed or has toroidal boundary.

The following theorem and the subsequent lemmas and proofs are adapted and modified from the work of Friedl and Powell [FP17].

Theorem 5.1. *Let X be a compact, connected and oriented 3-manifold that is either closed or has toroidal boundary, together with a primitive class $\varphi \in H^1(X)$. Let Σ be a connected surface dual to φ. Then for any $v, w \in g(\Sigma) \subset H_1(\Sigma;\mathcal{Q}_\varphi)$ we have*

$$\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}(\iota_\Sigma(v), \iota_\Sigma(w)) = -(\iota_+ - \iota_- t^{-1})^{-1}(\iota_+(v)) \cdot_\Sigma w$$

where $\iota_+(v)$ lies in diag$(Y) \subset$ diag$(Y) \otimes_{\mathcal{Q}_\varphi} \mathcal{K}$ and $\cdot_\Sigma$ denotes the sequilinear intersection pairing $H_1(\Sigma) \otimes_{\mathbb{Z}} \mathcal{K} \times H_1(\Sigma) \otimes_{\mathbb{Z}} \mathcal{K} \to \mathcal{K}$.

Recall the definition of Blanchfield form $\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}$ from Section 4.3. We first define maps Υ and Ω by the following commutative diagram, where all chain, homology, and cohomology groups have coefficients in $\mathcal{Q}_\varphi$ unless otherwise specified.

$$\begin{tikzcd}
TH_1(X) \arrow[r] & TH_1(X,\partial X) \arrow[r, "\mathrm{PD}"] & TH^2(X) \arrow[d, "\kappa"] & H^1(X;\mathcal{K}/\mathcal{Q}_\varphi) \arrow[l, twoheadrightarrow, "\mathrm{BS}"'] \arrow[d, "\kappa"] \\
g(\Sigma) \arrow[u, "\iota_\Sigma"] \arrow[rr, bend right, ":=\Upsilon"] & & TH_2(\overline{\mathrm{Hom}(C_*(X),\mathcal{Q}_\varphi)}) \arrow[dddr, bend right, ":=\Omega"] & H_1(\overline{\mathrm{Hom}(C_*(X),\mathcal{K}/\mathcal{Q}_\varphi)}) \arrow[l, twoheadrightarrow, "\mathrm{BS}"] \arrow[d, "\mathrm{ev}"] \\
& & & \overline{\mathrm{Hom}(H_1(X),\mathcal{K}/\mathcal{Q}_\varphi)} \arrow[d, "j^\#"] \\
& & & \overline{\mathrm{Hom}(TH_1(X),\mathcal{K}/\mathcal{Q}_\varphi)} \arrow[d, "\iota_\Sigma^\#"] \\
& & & \overline{\mathrm{Hom}(g(\Sigma),\mathcal{K}/\mathcal{Q}_\varphi)}
\end{tikzcd}$$

We will now analyze the map Υ. We define the following maps and conventions.

Pick a CW structure for Σ and equip $\Sigma \times [-1,1]$ with the corresponding product CW structure. Extend this to a CW structure for Y and thus for X.

For a chain complex C_*, denote the cycles by Z_* and the boundaries by B_*. Denote the projection map $Z_* \to H_*$ by p.

The short exact sequence $0 \to Z_1(\Sigma) \to C_1(\Sigma) \xrightarrow{\partial} B_0(\Sigma) \to 0$ splits since $B_0(\Sigma)$ is a submodule of the free $\mathbb{Z}$-module $C_0(\Sigma)$ and hence is also free. Choose a splitting $b : C_1(\Sigma) \to Z_1(\Sigma)$.

Identify $C_k(\Sigma \times I, \Sigma \times \{\pm 1\})$ with the free $\mathbb{Z}$-module generated by the open product k-cells and let $\times I : C_k(\Sigma) \to C_{k+1}(\Sigma \times I, \Sigma \times \{\pm 1\})$ be the chain isomorphism induced by mapping each k-dimensional cell in Σ to its corresponding open $(k+1)$-dimensional open product cell.

For convenience, we identify Σ with its unique lift Σ_0 in X_φ. Let $\iota_{\Sigma\times I} : C_k(\Sigma \times$

$I, \Sigma \times \{\pm 1\}) \to C_k(X_\varphi)$ be the inclusion map of $\mathcal{Q}_\varphi$-modules. It sends an open product cell in $\Sigma \times I$ to the same cell in X_φ. This induces an inclusion map of $\mathcal{Q}_\varphi$-modules $\iota_{\Sigma\times I} : C_k(\Sigma \times I, \Sigma \times \{\pm 1\}; \mathcal{Q}_\varphi) \to C_k(X; \mathcal{Q}_\varphi)$.

Let $c : C_2(X_\varphi) \to C_2(\Sigma \times I, \Sigma \times \{\pm 1\})$ be the chain map of $\mathcal{Q}_\varphi$-modules which is the identity on the open product cells and zero otherwise. This also induces a map of $\mathcal{Q}_\varphi$-modules $c : C_2(X; \mathcal{Q}_\varphi) \to C_2(\Sigma \times I, \Sigma \times \{\pm 1\}; \mathcal{Q}_\varphi)$. Note that c is a splitting of the inclusion $\iota_{\Sigma\times I}$.

Denote the intersection pairing on $H_1(\Sigma_\varphi; \mathbb{Z})$ by $\cdot_\Sigma$. This extends to a pairing on $H_1(\Sigma_\varphi; \mathbb{Q})$ and we may extend this to a hermitian pairing

$$\begin{aligned} H_1(\Sigma_\varphi; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{Q}_\varphi \times H_1(\Sigma_\varphi; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{Q}_\varphi &\to \mathcal{Q}_\varphi \\ (v \otimes p, w \otimes q) &\mapsto \overline{p}(v \cdot_\Sigma w) q \end{aligned}$$

and since $H_1(\Sigma_\varphi; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{Q}_\varphi \cong H_1(\Sigma_\varphi; \mathcal{Q}_\varphi)$, this extends to a hermitian pairing on $H_1(\Sigma_\varphi; \mathcal{Q}_\varphi)$.

Now we define a map ϕ_w by the following commutative diagram with coefficients in $\mathcal{Q}_\varphi$ unless otherwise specified.

$$\begin{tikzcd} & & & & & \mathcal{Q}_\varphi \\ H_1(\Sigma) \arrow[urrrrr, bend left, "v \to -v \cdot_\Sigma w"] & Z_1(\Sigma) \arrow[l, two heads, "p"'] \arrow[r, hook] & C_1(\Sigma) \arrow[l, bend right, "b"'] \arrow[r, "\times I", "\cong"'] & C_2(\Sigma \times I, \Sigma \times \{\pm 1\}) \arrow[rr, hook, "\iota_{\Sigma\times I}"'] & & C_2(X) \arrow[u, ":= \phi_w"'] \arrow[ll, bend right, "c"'] \end{tikzcd}$$

Lemma 5.2. *For any w in $g(\Sigma)$, the homomorphism $\phi_w : C_2(X; \mathcal{Q}_\varphi) \to \mathcal{Q}_\varphi$ defined by the above commutative diagram represents $\Upsilon(w)$ in $TH_2(\overline{Hom_{\mathcal{Q}_\varphi}(C_*(X; \mathcal{Q}_\varphi), \mathcal{Q}_\varphi)})$.*

Proof. Consider the diagram of $\mathcal{Q}_\varphi$-modules in Fig. 5.1, where the homology, cohomology, and chain groups all have $\mathcal{Q}_\varphi$-coefficients.

Most of the diagram commutes by Friedl-Powell. See [FP17] for details. We have

$$
\begin{tikzcd}
 & & & \mathrm{Hom}(H_1(\Sigma), \mathcal{Q}_\varphi) \arrow[r] & \overline{\mathrm{Hom}(H_1(\Sigma), \mathcal{Q}_\varphi)} \\
g(\Sigma) \arrow[r, hook] \arrow[d, "\cong"', "i"] & H_1(\Sigma) \arrow[r] \arrow[d, "\cong"', "i"] \arrow[urr, bend left, "w \mapsto (v \mapsto -v \cdot_\Sigma w)"] & H_1(\Sigma, \partial\Sigma) \arrow[r, "-\mathrm{PD}"] \arrow[d, "\cong"', "i"] & H^1(\Sigma) \arrow[r, "\kappa", "\cong"'] \arrow[u, "\mathrm{ev}", "\cong"'] & H_1(\overline{\mathrm{Hom}(C_*(\Sigma), \mathcal{Q}_\varphi)}) \arrow[u, "\mathrm{ev}", "\cong"'] \\
i(g(\Sigma)) \arrow[r, hook] \arrow[dd, "\iota_{\Sigma \times I}"] & H_1(\Sigma \times I) \arrow[r] \arrow[d, "\iota_{\Sigma \times I}"] & H_1(\Sigma \times I, \partial\Sigma \times I) \arrow[r, "\mathrm{PD}"] \arrow[d, "\iota_{\Sigma \times I}"] & H^2(\Sigma \times I, \Sigma^\pm) \arrow[r, "\kappa", "\cong"'] \arrow[u, "(\times I)^*", "\cong"'] \arrow[d, "c^*"] & H_1(\overline{\mathrm{Hom}(C_*(\Sigma \times I, \Sigma^\pm), \mathcal{Q}_\varphi)}) \arrow[u, "(\times I)^*", "\cong"'] \arrow[d, "c^*"] \\
 & H_1(X) \arrow[r] & H_1(X, \partial X) \arrow[r, "\mathrm{PD}"] & H^2(X) \arrow[r, "\kappa", "\cong"'] & H_2(\overline{\mathrm{Hom}(C_*(X), \mathcal{Q}_\varphi)}) \\
TH_1(X) \arrow[rr] \arrow[ur, hook] & & TH_1(X, \partial X) \arrow[r, "\mathrm{PD}"] \arrow[u, hook] & TH^2(X) \arrow[r, "\kappa", "\cong"'] \arrow[u, hook] & TH_2(\overline{\mathrm{Hom}(C_*(X), \mathcal{Q}_\varphi)}) \arrow[u, hook]
\end{tikzcd}
$$

Figure 5.1: Commutative Diagram for Lemma 5.2

added the leftmost column and bottom row, and it is easy to show that these new squares commute.

We obtain the homology class of ϕ_w if we start with w in $g(\Sigma)$ and travel along the top row, passing through the map $w \mapsto -(-) \cdot_\Sigma w$, then traveling down along the rightmost column to $H_2(\overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_*(X;\mathcal{Q}_\varphi),\mathcal{Q}_\varphi)})$.

If we start with w in $g(\Sigma)$ and then instead travel down the leftmost column, then to the bottom right along the bottom row, we obtain $\Upsilon(w)$. Then since the diagram commutes, when we include $\Upsilon(w)$ into the module $H_2(\overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_*(X;\mathcal{Q}_\varphi),\mathcal{Q}_\varphi)})$, we must find that $\Upsilon(w) = [\phi_w]$. So $[\phi_w]$ must be torsion, and ϕ_w represents $\Upsilon(w)$. $\square$

We will now analyze the map Ω. Again, to do so we first define some more maps.

Since $H_1(\Sigma;\mathbb{Z})$ is a free $\mathbb{Z}$-module, the short exact sequence $0 \to B_1(\Sigma) \to Z_1(\Sigma) \xrightarrow{p} H_1(\Sigma) \to 0$ is split and we may choose a splitting $a : H_1(\Sigma;\mathbb{Z}) \to Z_1(\Sigma;\mathbb{Z})$. This extends to a splitting with $\mathbb{Q}$ coeffcients, which extends to a splitting $a : H_1(\Sigma;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{Q}_\varphi \to Z_1(\Sigma;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{Q}_\varphi$, which in turn induces a splitting $a : H_1(\Sigma;\mathcal{Q}_\varphi) \to Z_1(\Sigma;\mathcal{Q}_\varphi)$.

Consider the following exact sequence.

$$0 \to Z_2(X;\mathcal{Q}_\varphi) \to C_2(X;\mathcal{Q}_\varphi) \xrightarrow{\partial} Z_1(X;\mathcal{Q}_\varphi) \xrightarrow{p} H_1(X;\mathcal{Q}_\varphi) \to 0$$

The kernel of p is $B_1(X;\mathcal{Q}_\varphi)$ which is the image of ∂. We would like to have a similar sequence with $TH_1(X;\mathcal{Q}_\varphi)$, so that this term will vanish when we localize to $\mathcal{K}$-coefficients. We have the short exact sequence

$$0 \to B_1(X;\mathcal{Q}_\varphi) \xrightarrow{i} Z_1(X;\mathcal{Q}_\varphi) \xrightarrow{p} H_1(X;\mathcal{Q}_\varphi) \to 0.$$

$B_1(X;\mathcal{Q}_\varphi)$ and $Z_1(X;\mathcal{Q}_\varphi)$ are both free $\mathcal{Q}_\varphi$-modules since $\mathcal{Q}_\varphi$ is a PID and they are submodules of $C_1(X;\mathcal{Q}_\varphi) = C_1(\widetilde{X}) \otimes_{\mathcal{Q}_\varphi} \mathcal{Q}_\varphi$ which is a free $\mathcal{Q}_\varphi$-module since $C_1(\widetilde{X})$

is a free $\mathbb{Z}$-module. Say $B_1(X;\mathcal{Q}_\varphi) \cong {\mathcal{Q}_\varphi}^r$ and $Z_1(X;\mathcal{Q}_\varphi) \cong {\mathcal{Q}_\varphi}^s$. Then the map $i : B_1(X;\mathcal{Q}_\varphi) \to Z_1(X;\mathcal{Q}_\varphi)$ may be represented by a matrix, and this matrix may be diagonalized so we get $H_1(X;\mathcal{Q}_\varphi) \cong {\mathcal{Q}_\varphi}^s / i({\mathcal{Q}_\varphi}^r) \cong {\mathcal{Q}_\varphi}^\ell / {\mathcal{Q}_\varphi}^\ell D \oplus {\mathcal{Q}_\varphi}^{s-\ell}$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B_1(X;\mathcal{Q}_\varphi) & \xrightarrow{i} & Z_1(X;\mathcal{Q}_\varphi) & \xrightarrow{p} & H_1(X;\mathcal{Q}_\varphi) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & {\mathcal{Q}_\varphi}^r & \xrightarrow{i} & {\mathcal{Q}_\varphi}^s & \xrightarrow{p} & H_1(X;\mathcal{Q}_\varphi) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & {\mathcal{Q}_\varphi}^\ell & \xrightarrow{D} & {\mathcal{Q}_\varphi}^\ell & \xrightarrow{p} & TH_1(X;\mathcal{Q}_\varphi) & \longrightarrow & 0 \\
 & & \bigoplus & & \bigoplus & & \bigoplus & & \\
0 & \longrightarrow & {\mathcal{Q}_\varphi}^{r-\ell} & \xrightarrow{0} & {\mathcal{Q}_\varphi}^{s-\ell} & \xrightarrow{p} & H_1(X;\mathcal{Q}_\varphi)/TH_1(X;\mathcal{Q}_\varphi) & \longrightarrow & 0
\end{array}
$$

Of course, since i is injective and D is injective, this tells us that ${\mathcal{Q}_\varphi}^{r-\ell} = 0$. So $B_1(X;\mathcal{Q}_\varphi) \cong {\mathcal{Q}_\varphi}^\ell$, so we get the short exact sequence

$$0 \to B_1(X;\mathcal{Q}_\varphi) \xrightarrow{i} p^{-1}(TH_1(X;\mathcal{Q}_\varphi)) \xrightarrow{p} TH_1(X;\mathcal{Q}_\varphi) \to 0.$$

The kernel of p is $B_1(X;\mathcal{Q}_\varphi)$ which is the image of ∂. Thus we have the following exact sequence.

$$0 \to Z_2(X;\mathcal{Q}_\varphi) \to C_2(X;\mathcal{Q}_\varphi) \xrightarrow{\partial} p^{-1}(TH_1(X;\mathcal{Q}_\varphi)) \xrightarrow{p} TH_1(X;\mathcal{Q}_\varphi) \to 0$$

Then when we localize to $\mathcal{K}$-coefficients we get the short exact sequence

$$0 \to Z_2(X;\mathcal{K}) \to C_2(X;\mathcal{K}) \xrightarrow{\partial} p^{-1}(TH_1(X;\mathcal{Q}_\varphi)) \otimes_{\mathcal{Q}_\varphi} \mathcal{K} \to 0$$

and since $\mathcal{K}$ is a field the sequence is split. Hence we may choose a splitting $d : p^{-1}(TH_1(X;\mathcal{Q}_\varphi)) \otimes_{\mathcal{Q}_\varphi} \mathcal{K} \to C_2(X;\mathcal{K})$ of the boundary map.

Let $j : p^{-1}(TH_1(X;\mathcal{Q}_\varphi)) \to Z_1(X;\mathcal{Q}_\varphi)$ be the inclusion map.

Finally, given $\psi \in \mathrm{Hom}_{\mathcal{Q}_\varphi}(C_2(X;\mathcal{Q}_\varphi),\mathcal{Q}_\varphi)$, denote the corresponding homomorphism in $\mathrm{Hom}_{\mathcal{K}}(C_2(X;\mathcal{K}),\mathcal{K})$ by $\psi^{\mathcal{K}}$.

Lemma 5.3. *The homomorphism*

$$\begin{aligned} TH_2(\overline{Hom_{\mathcal{Q}_\varphi}(C_*(X;\mathcal{Q}_\varphi),\mathcal{Q}_\varphi)}) &\to \overline{Hom_{\mathcal{Q}_\varphi}(g(\Sigma),\mathcal{K}/\mathcal{Q}_\varphi)} \\ \psi &\mapsto \psi^{\mathcal{K}}(d\circ\iota_\Sigma\circ a) \end{aligned}$$

is precisely the homomorphism Ω.

Proof. Consider the diagram of $\mathcal{Q}_\varphi$-modules in Fig. 5.2, where all homology and chain groups have $\mathcal{Q}_\varphi$ coefficients unless otherwise specified.

It is easy to show that the diagram commutes. The zig zag path in the upper left corner is the Bockstein homomorphism, and since it is a surjection onto the torsion part of the homology module, we know that a lift to $\overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_1(X;\mathcal{Q}_\varphi),\mathcal{K})}$ exists, if we start with a cocycle ψ in $\overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_2(X;\mathcal{Q}_\varphi),\mathcal{Q}_\varphi)}$ such that $[\psi]$ is torsion.

Start with such a cocycle ψ. If we take the uppermost route down to the bottom right, we have the map Ω. On the other hand, if we take the lowermost route, we end up with $\psi^{\mathcal{K}}\circ d\circ\iota_\Sigma\circ a$. Since the diagram commutes, these must be equal. $\square$

Consider the inclusion $g(\Sigma)\otimes_{\mathcal{Q}_\varphi}\mathcal{K}\to H_1(\Sigma;\mathcal{Q}_\varphi)\otimes_{\mathcal{Q}_\varphi}\mathcal{K} = H_1(\Sigma;\mathcal{K})$. Since $\mathcal{K}$ is a field, the map splits. We choose a splitting $e: H_1(\Sigma;\mathcal{K})\to g(\Sigma)\otimes_{\mathcal{Q}_\varphi}\mathcal{K}$ so that e is the identity on elements of $H_1(\Sigma;\mathcal{K})$ that lie in $g(\Sigma)\otimes_{\mathcal{Q}_\varphi}\mathcal{K}$, and 0 otherwise. We similarly choose a splitting that we will also call $e: Z_1(\Sigma;\mathcal{K})\to p^{-1}(g(\Sigma))\otimes_{\mathcal{Q}_\varphi}\mathcal{K}$ so that $e(x)$ is the identity if $p(x)\in g(\Sigma)\otimes_{\mathcal{Q}_\varphi}\mathcal{K}$ and 0 otherwise. Then the square

$$\begin{tikzcd} H_1(\Sigma;\mathcal{K}) \arrow[d, shift left, "e"] & Z_1(\Sigma;\mathcal{K}) \arrow[l, two heads, "p"'] \arrow[d, shift left, "e"] \\ g(\Sigma)\otimes_{\mathcal{Q}_\varphi}\mathcal{K} \arrow[u, hook, shift left] & p^{-1}(g(\Sigma))\otimes_{\mathcal{Q}_\varphi}\mathcal{K} \arrow[l, two heads, "p"] \arrow[u, hook, shift left] \end{tikzcd}$$

$$
\begin{tikzcd}
 & \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_1(X),\mathcal{K})} \arrow[rr] \arrow[dr, "i^*"] \arrow[dd, "\partial^*"] & & \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_1(X),\mathcal{K}/\mathcal{Q}_\varphi)} \arrow[dr, "i^*"] & \\
 & & \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(Z_1(X),\mathcal{K})} \arrow[rr] \arrow[d, "j^*"] & & \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(Z_1(X),\mathcal{K}/\mathcal{Q}_\varphi)} \arrow[d, "j^*"] \\
\overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_2(X),\mathcal{Q}_\varphi)} \arrow[r] \arrow[dr, "\psi\mapsto\psi^{\mathcal{K}}"'] & \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(C_2(X),\mathcal{K})} \arrow[d, "="] & \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(p^{-1}(TH_1(X)),\mathcal{K})} \arrow[rr] \arrow[d, "="] & & \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(p^{-1}(TH_1(X)),\mathcal{K}/\mathcal{Q}_\varphi)} \arrow[d, "\iota_\Sigma^*"] \\
 & \overline{\mathrm{Hom}_{\mathcal{K}}(C_2(X;\mathcal{K}),\mathcal{K})} \arrow[r, bend right, "d^*"'] & \overline{\mathrm{Hom}_{\mathcal{K}}(p^{-1}(TH_1(X))\otimes_{\mathcal{Q}_\varphi}\mathcal{K},\mathcal{K})} \arrow[l, "\partial^*"'] & & \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(p^{-1}(g(\Sigma)),\mathcal{K}/\mathcal{Q}_\varphi)} \arrow[d, "a^*"] \\
 & & & & \overline{\mathrm{Hom}_{\mathcal{Q}_\varphi}(g(\Sigma),\mathcal{K}/\mathcal{Q}_\varphi)}
\end{tikzcd}
$$

Figure 5.2: Commutative Diagram for Lemma 5.3

commutes.

We are now ready to give a proof of Theorem 5.1. Recall that for Σ, Y, or X, we have the projection map $p : Z_1 \to H_1$. Consider the following diagram of $\mathcal{Q}_\varphi$-module homomorphisms in Fig. 5.3.

The top right map $\phi_w^{\mathcal{K}}$ comes from tensoring up the map $\phi_w : C_2(X; \mathcal{Q}_\varphi) \to \mathcal{Q}_\varphi$ we defined earlier. The top triangle commutes by the definition of $\phi_w^{\mathcal{K}}$. It is straightforward to check all other squares and triangles commute, with any choice of maps indicated.

Now let $v, w \in g(\Sigma)$. We write

$$z = (e \circ b \circ (\times I)^{-1} \circ c \circ d \circ \iota_\Sigma \circ a)(v) \in p^{-1}(g(\Sigma)) \otimes_{\mathcal{Q}_\varphi} \mathcal{K},$$

so z is the element obtained from v by taking the long path on the right. By the previous two lemmas, the definition of $\phi_w^{\mathcal{K}}$, and the commutativity of the top left square we have

$$\begin{aligned}
\mathcal{B}\ell(\iota_\Sigma(w))(\iota_\Sigma(v)) &= \Omega(\Upsilon(w))(v) \\
&= \Omega(\phi_w)(v) \\
&= (\phi_w^{\mathcal{K}} \circ d \circ \iota_\Sigma \circ a)(v) \\
&= -(p \circ b \circ (\times I)^{-1} \circ c \circ d \circ \iota_\Sigma \circ a)(v) \cdot_\Sigma w \\
&= -(p \circ e \circ b \circ (\times I)^{-1} \circ c \circ d \circ \iota_\Sigma \circ a)(v) \cdot_\Sigma w \\
&= -p(z) \cdot_\Sigma w.
\end{aligned}$$

Since the big rectangle commutes, going all the way around it gives the identity map so

$$\iota_Y \circ (\iota_+ - \iota_- t^{-1}) \circ e \circ b \circ (\times I)^{-1} \circ c \circ d((\iota_\Sigma \circ a)(v)) = (\iota_\Sigma \circ a)(v)$$

$$
\begin{tikzcd}
 & & & & \mathcal{K}/\mathcal{Q}_\varphi \\
 & & & & \mathcal{K} \arrow[u] \\
H_1(\Sigma;\mathcal{K}) \arrow[urrrr, bend left, "v \mapsto -v \cdot_\Sigma w"] \arrow[d, bend left, "e"] & Z_1(\Sigma;\mathcal{K}) \arrow[l, two heads, "p"'] \arrow[r, hook] \arrow[d, bend left, "e"] & C_1(\Sigma;\mathcal{K}) \arrow[r, "\times I", "\cong"'] \arrow[l, bend right, "b"'] & C_2(\Sigma \times I, \Sigma^\pm;\mathcal{K}) \arrow[r, hook, "\iota_{\Sigma \times I}"'] & C_2(X;\mathcal{K}) \arrow[u, "\phi_w^{\mathcal{K}}"] \arrow[l, bend right, "c"'] \arrow[ddd, two heads, "\partial"'] \\
g(\Sigma) \otimes \mathcal{K} \arrow[u, hook] \arrow[d, hook] & p^{-1}(g(\Sigma)) \otimes \mathcal{K} \arrow[l, two heads, "p"'] \arrow[u, hook] \arrow[d, hook] & & & \\
\mathrm{diag}(\Sigma) \otimes \mathcal{K} \arrow[d, "\cong"', "\iota_+ - \iota_- t^{-1}"] & p^{-1}(\mathrm{diag}(\Sigma)) \otimes \mathcal{K} \arrow[l, two heads, "p"'] \arrow[d, "\iota_+ - \iota_- t^{-1}"] & & & \\
\mathrm{diag}(Y) \otimes \mathcal{K} & p^{-1}(\mathrm{diag}(Y)) \otimes \mathcal{K} \arrow[l, two heads, "p"'] \arrow[rrr, hook, "\iota_Y"] & & & p^{-1}(TH_1(X)) \otimes \mathcal{K} \arrow[uuu, bend right, "d"'] \\
g(\Sigma) \otimes \mathcal{K} \arrow[u, "\iota_+"] \arrow[r, hook, bend right, "a"'] & p^{-1}(g(\Sigma)) \otimes \mathcal{K} \arrow[l, two heads, "p"'] \arrow[u, hook, "\iota_+"'] & & & \\
g(\Sigma) \arrow[u, hook] \arrow[r, hook, bend right, "a"'] & p^{-1}(g(\Sigma)) \arrow[l, two heads, "p"'] \arrow[u, hook] \arrow[uurrr, hook, "\iota_\Sigma"] & & &
\end{tikzcd}
$$

Figure 5.3: Commutative Diagram for Theorem 5.1

hence $(\iota_Y \circ (\iota_+ - \iota_- t^{-1}))(z) = (\iota_\Sigma \circ a)(v)$. Since the bottom triangle commutes, $\iota_\Sigma = \iota_Y \circ \iota_+$, so we have $(\iota_Y \circ (\iota_+ - \iota_- t^{-1}))(z) = (\iota_Y \circ \iota_+ \circ a)(v)$. Thus since ι_Y is a monomorphism we have $(\iota_+ - \iota_- t^{-1})(z) = (\iota_+ \circ a)(v)$.

Then since the bottom two squares commute we have $p((\iota_+ - \iota_- t^{-1})(z)) = \iota_+(v)$. Since the next square up commutes, this is the same as $(\iota_+ - \iota_- t^{-1})(p(z)) = \iota_+(v)$, and since $\iota_+ - \iota_- t^{-1} : \mathrm{diag}(\Sigma) \otimes_{\mathcal{Q}_\varphi} \mathcal{K} \to \mathrm{diag}(Y) \otimes_{\mathcal{Q}_\varphi} \mathcal{K}$ is an isomorphism, we have $p(z) = (\iota_+ - \iota_- t^{-1})^{-1}(\iota_+(v))$. Note that $(\iota_+ - \iota_- t^{-1})^{-1}(\iota_+(v))$ is in $\mathrm{diag}(\Sigma)$ but it may not lie in $g(\Sigma)$.

Combining these observations we obtain that

$$\mathcal{B}\ell(\iota_\Sigma(w))(\iota_\Sigma(v)) = -p(z) \cdot_\Sigma w = -(\iota_+ - \iota_- t^{-1})^{-1}(\iota_+(v)) \cdot_\Sigma w.$$

5.2 Blanchfield Forms from Seifert Forms

The following lemma is adapted from a lemma of Harvey in [Har05].

Lemma 5.4. *The torsion submodule $TH_1(X; \mathcal{Q}_\varphi)$ is contained in the image of the map $\iota_\Sigma : H_1(\Sigma; \mathcal{Q}_\varphi) \to H_1(X; \mathcal{Q}_\varphi)$.*

Proof. Let $x \in TH_1(X; \mathcal{Q}_\varphi)$ such that $x \neq 0$. Because x is torsion, there exists a $p(t) \in \mathcal{Q}_\varphi$ such that $xp(t) = 0$. We may assume that $p(t) = 1 + tc_1 + \cdots + t^q c_q$ for $q \geq 0$ and $c_i \in \mathbb{Q}$, since $\mathcal{Q}_\varphi \cong \mathbb{Q}[t^{\pm 1}]$ and $xp(t) = 0$ if and only if $xp(t)u = 0$ for any unit $u \in \mathcal{Q}_\varphi$. Note that we actually have $q > 0$, since if $q = 0$ then $p(t) = 1$ and $x = 0$, but we assumed $x \neq 0$.

Since ι_Y is surjective, $x = \iota_Y(y)$ for some $y \in H_1(Y; \mathcal{Q}_\varphi)$. Then $\iota_Y(yp(t)) = \iota_Y(y)p(t) = xp(t) = 0$ so $yp(t)$ is in the kernel of ι_Y which equals the image of $\iota_+ - \iota_- t^{-1}$. Thus $yp(t) = (\iota_+ - \iota_- t^{-1})(\sigma)$ for some $\sigma \in H_1(\Sigma; \mathcal{Q}_\varphi)$. Since $H_1(\Sigma; \mathcal{Q}_\varphi) \cong H_1(\Sigma; \mathbb{Q}) \otimes_\mathbb{Q} \mathcal{Q}_\varphi$ and $H_1(Y; \mathcal{Q}_\varphi) \cong H_1(Y; \mathbb{Q}) \otimes_\mathbb{Q} \mathcal{Q}_\varphi$ we may write $\sigma = \sum_{i \in \mathbb{Z}} \alpha'_i \otimes t^i$

and $y = \sum_{i\in\mathbb{Z}} \beta'_i \otimes t^i$ where $\alpha'_i \in H_1(\Sigma;\mathbb{Q})$ and $\beta'_i \in H_1(Y;\mathbb{Q})$ and only finitely many α'_i, β'_i are nonzero.

Let $-r$ be the least integer such that β'_{-r} is nonzero. That is, $y = \sum_{i\in\mathbb{Z}} \beta'_i \otimes t^i = \beta'_{-r} \otimes t^{-r} + \cdots + \beta'_{-r+n} \otimes t^{-r+n}$. Let $\beta_i = \beta'_{-r+i}$. Then $yt^r = \sum_{i=0}^n \beta_i \otimes t^i$, so $y = \sum_{i=0}^n \beta_i \otimes t^{i-r}$. One can easily check that $yt^r p(t) = (\iota_+ - \iota_- t^{-1})(\sigma t^r)$. Let $\alpha_i = \alpha'_{i-r}$ and then $\sigma t^r = \sum_{i\in\mathbb{Z}} \alpha'_i \otimes t^{i+r} = \sum_{i\in\mathbb{Z}} \alpha'_{i-r} \otimes t^i = \sum_{i\in\mathbb{Z}} \alpha_i \otimes t^i$.

So we have

$$yt^r p(t) = \sum_{i=0}^{n}(\beta_i \otimes t^i)\sum_{j=0}^{q} t^j c_j = \sum_{k=0}^{n+q}\sum_{i+j=k}(\beta_i c_j \otimes t^{i+j}) = \sum_{k=0}^{n+q}\left(\sum_{i+j=k}\beta_i c_j\right)\otimes t^k$$

and

$$\begin{aligned}
(\iota_+ - \iota_- t^{-1})(\sigma t^r) &= \sum_{i\in\mathbb{Z}}(\iota_+ - \iota_- t^{-1})(\alpha_i \otimes t^i)\\
&= \sum_{i\in\mathbb{Z}} \iota_+(\alpha_i \otimes t^i) - \iota_-(\alpha_i \otimes t^i)t^{-1}\\
&= \sum_{i\in\mathbb{Z}} \iota_+(\alpha_i) \otimes t^i - \iota_-(\alpha_i) \otimes t^{i-1}\\
&= \sum_{i\in\mathbb{Z}} \iota_+(\alpha_i) \otimes t^i - \sum_{i\in\mathbb{Z}} \iota_-(\alpha_i) \otimes t^{i-1}\\
&= \sum_{i\in\mathbb{Z}} \iota_+(\alpha_i) \otimes t^i - \sum_{i\in\mathbb{Z}} \iota_-(\alpha_{i+1}) \otimes t^{i}\\
&= \sum_{i\in\mathbb{Z}} \iota_+(\alpha_i) \otimes t^i - \iota_-(\alpha_{i+1}) \otimes t^{i}\\
&= \sum_{i\in\mathbb{Z}} (\iota_+(\alpha_i) - \iota_-(\alpha_{i+1})) \otimes t^{i}
\end{aligned}$$

and since $yt^r p(t) = (\iota_+ - \iota_- t^{-1})(\sigma t^r)$, for each $0 \le k \le n+q$ we have

$$\sum_{i+j=k} \beta_i c_j = \iota_+(\alpha_k) - \iota_-(\alpha_{k+1}).$$

Then since $p(t)$ is monic, $c_0 = 1$, so for $0 \le k \le n$ we have

$$\beta_k = \iota_+(\alpha_k) - \iota_-(\alpha_{k+1}) - \sum_{i+j=k, i<k} \beta_i c_j.$$

Now we'll show by induction that $\iota_Y(\beta_k \otimes t^{k-r})$ is in the image of ι_Σ for each $0 \le k \le n$. Since $y = \sum_{k=0}^{n} \beta_k \otimes t^{k-r}$ and $\iota_Y(y) = x$, this will complete the proof.

First recall that $\iota_\Sigma = \iota_Y \circ \iota_+$ and by the exactness of the sequence $\iota_Y \circ (\iota_+ - \iota_- t^{-1}) = 0$. Thus $\iota_\Sigma = \iota_Y \circ \iota_+ = \iota_Y \circ \iota_- t^{-1}$. Then for any $z \in H_1(\Sigma; \mathcal{Q}_\varphi)$, we have $\iota_Y \circ \iota_-(z) = \iota_Y \circ \iota_- t^{-1}(zt) = \iota_\Sigma(zt)$.

Thus for $0 \le k \le n$, we have

$$\begin{aligned}
\iota_Y((\iota_+(\alpha_k) - \iota_-(\alpha_{k+1})) \otimes t^{k-r}) &= \iota_Y(\iota_+(\alpha_k \otimes t^{k-r})) - \iota_Y(\iota_-(\alpha_{k+1} \otimes t^{k-r})) \\
&= \iota_\Sigma(\alpha_k \otimes t^{k-r}) - \iota_\Sigma(\alpha_{k+1} \otimes t^{k-r+1}) \\
&= \iota_\Sigma(\alpha_k \otimes t^{k-r} - \alpha_{k+1} \otimes t^{k-r+1})
\end{aligned}$$

Hence $\iota_Y(\beta_0 \otimes t^{-r}) = \iota_Y((\iota_+(\alpha_0) - \iota_-(\alpha_1)) \otimes t^{-r}) = \iota_\Sigma(\alpha_0 \otimes t^{-r} - \alpha_1 \otimes t^{-r+1}) \in \mathrm{im}(\iota_\Sigma)$. Now fix $0 \le k \le n$ and assume that $\iota_Y(\beta_i \otimes t^{i-r}) \in \mathrm{im}(\iota_\Sigma)$ for each $0 \le i \le k-1$, so there exists $z_i \in H_1(\Sigma; \mathcal{Q}_\varphi)$ such that $\iota_Y(\beta_i \otimes t^{i-r}) = \iota_\Sigma(z_i)$. Then

$$\begin{aligned}
\iota_Y(\beta_k \otimes t^{k-r}) &= \iota_Y \left(\left(\iota_+(\alpha_k) - \iota_-(\alpha_{k+1}) - \sum_{i+j=k, i<k} \beta_i c_j \right) \otimes t^{k-r} \right) \\
&= \iota_Y \left((\iota_+(\alpha_k) - \iota_-(\alpha_{k+1})) \otimes t^{k-r} \right) - \sum_{i+j=k, i<k} \iota_Y \left(\beta_i c_j \otimes t^{k-r} \right) \\
&= \iota_\Sigma \left(\alpha_k \otimes t^{k-r} - \alpha_{k+1} \otimes t^{k-r+1} \right) - \sum_{i+j=k, i<k} \iota_Y \left(\beta_i \otimes t^{i-r} \right) \left(c_j t^j \right) \\
&= \iota_\Sigma \left(\alpha_k \otimes t^{k-r} - \alpha_{k+1} \otimes t^{k-r+1} \right) - \sum_{i+j=k, i<k} \iota_\Sigma \left(z_i \right) \left(c_j t^j \right) \\
&= \iota_\Sigma \left(\alpha_k \otimes t^{k-r} - \alpha_{k+1} \otimes t^{k-r+1} - \sum_{i+j=k, i<k} z_i c_j t^j \right) \in \mathrm{im}(\iota_\Sigma)
\end{aligned}$$

□

When $H_1(X;\mathcal{Q}_\varphi)$ is torsion, $g(\Sigma) = H_1(\Sigma;\mathcal{Q}_\varphi)$, $\operatorname{diag}(Y) = H_1(Y;\mathcal{Q}_\varphi)$, and Theorem 5.1 simplifies to the main theorem in [FP17]. Then for $X = S^3 - N(L)$ the exterior of a link, we can relate the Blanchfield forms $\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}$ to the Seifert $\mathbb{Z}$-matrices A_φ and $A^\pm_\varphi$.

Theorem 5.5. *Let L be a link with pairwise linking numbers zero, let $X = S^3 - N(L)$ be its exterior, and let A_φ, A^+_φ, and A^-_φ be Seifert $\mathbb{Z}$-matrices for L with respect to the primitive class $\varphi \in H^1(X)$ and suppose that $H_1(X;\mathcal{Q}_\varphi)$ is torsion. Suppose that A_φ is an $r \times r$ matrix, and $A^\pm_\varphi$ are $r \times s$ matrices. Then the Blanchfield form $\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}$ on $H_1(X;\mathcal{Q}_\varphi)$ is isomorphic to the pairing*

$$\begin{aligned} {\mathcal{Q}_\varphi}^s/{\mathcal{Q}_\varphi}^r\left(A^+_\varphi - A^-_\varphi t^{-1}\right) \times {\mathcal{Q}_\varphi}^s/{\mathcal{Q}_\varphi}^r\left(A^+_\varphi - A^-_\varphi t^{-1}\right) &\to \mathcal{K}/\mathcal{Q}_\varphi \\ (v,w) &\mapsto vA^+_\varphi\left(A^+_\varphi - A^-_\varphi t^{-1}\right)^{-1}\left(A_\varphi - A^T_\varphi\right)\overline{w}^T \end{aligned}$$

Proof. By Lemma 5.4, $H_1(X;\mathcal{Q}_\varphi) \in \operatorname{im}(\iota_\Sigma)$. This means that the defintion of the Blanchfield form in [FP17] is defined for every $x, y \in H_1(X;\mathcal{Q}_\varphi)$. That is, for every such x, y, there exists $v, w \in H_1(\Sigma_\varphi;\mathcal{Q}_\varphi)$ such that

$$\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}(x,y) = \mathcal{B}\ell^X_{\mathcal{Q}_\varphi}(\iota_\Sigma(v), \iota_\Sigma(w)) = -(\iota_+ - \iota_- t^{-1})^{-1}(\iota_+(v)) \cdot_{\Sigma_\varphi} w.$$

Recall that the maps $\iota_\pm$ are represented by the matrices $A^\pm_\varphi$. The intersection form for any bicollared surface in S^3 is given by $v \cdot w = \ell k(v, w^+) - \ell k(w, v^+)$ [Rol76]. Since $\ell k(v, w^+) = \ell k(v^-, w)$ and $\ell k(w, v^+) = \ell k(v^+, w)$, the intersection form for Σ_φ is represented by the matrix $A^T_\varphi - A_\varphi$. By this we mean that if v and w are represented by $1 \times r$ row vectors, then $v \cdot_{\Sigma_\varphi} w = v\left(A^T_\varphi - A_\varphi\right)\overline{w}^T$. The bar over w is only necessary once we extend the intersection form to a pairing over $H_1(\Sigma_\varphi;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{Q}_\varphi$. Now

$$\begin{aligned}
\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}(\iota_\Sigma(v), \iota_\Sigma(w)) &= -(\iota_+ - \iota_- t^{-1})^{-1}(\iota_+(v)) \cdot_{\Sigma_\varphi} w \\
&= -(\iota_+ - \iota_- t^{-1})^{-1}(\iota_+(v)) \left(A_\varphi^T - A_\varphi\right) \overline{w}^T \\
&= \iota_+(v) \left(A_\varphi^+ - A_\varphi^- t^{-1}\right)^{-1} \left(A_\varphi - A_\varphi^T\right) \overline{w}^T \\
&= vA_\varphi^+ \left(A_\varphi^+ - A_\varphi^- t^{-1}\right)^{-1} \left(A_\varphi - A_\varphi^T\right) \overline{w}^T
\end{aligned}$$

□

Corollary 5.6. *Let L be a link with pairwise linking numbers zero and let A_φ be a Seifert $\mathbb{Z}$-matrix of size $r \times r$ for L with respect to the primitive class $\varphi = (1, \ldots, 1) \in H^1(X_L)$ and suppose that $H_1(X; \mathcal{Q}_\varphi)$ is torsion. Then the Blanchfield form $\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}$ on $H_1(X; \mathcal{Q}_\varphi)$ is isomorphic to the pairing*

$$\begin{aligned}
{\mathcal{Q}_\varphi}^r / {\mathcal{Q}_\varphi}^r \left(tA_\varphi - A_\varphi^T\right) \times {\mathcal{Q}_\varphi}^r / {\mathcal{Q}_\varphi}^r \left(tA_\varphi - A_\varphi^T\right) &\to \mathcal{K}/\mathcal{Q}_\varphi \\
(v, w) &\mapsto -v(t-1) \left(tA_\varphi - A_\varphi^T\right)^{-1} \overline{w}^T
\end{aligned}$$

Proof. Recall that when $\varphi = (1, \ldots, 1)$, $A_\varphi^+ = A_\varphi$ and $A_\varphi^- = A_\varphi^T$. Then for $v, w \in$

$H_1(\Sigma_\varphi; \mathcal{Q}_\varphi)$ represented as $r \times 1$ row vectors, we have

$$\begin{aligned}
\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}(\iota_\Sigma(v), \iota_\Sigma(w)) &= -vA_\varphi \left(A_\varphi - A_\varphi^T t^{-1}\right)^{-1} \left(A_\varphi - A_\varphi^T\right) \overline{w}^T \\
&= -vA_\varphi \left(A_\varphi - A_\varphi^T t^{-1}\right)^{-1} \left(\left(A_\varphi - A_\varphi^T t^{-1}\right) + \left(A_\varphi^T t^{-1} - A_\varphi^T\right)\right) \overline{w}^T \\
&= -vA_\varphi \left(A_\varphi - A_\varphi^T t^{-1}\right)^{-1} \left(A_\varphi - A_\varphi^T t^{-1}\right) \overline{w}^T \\
&\qquad - vA_\varphi \left(A_\varphi - A_\varphi^T t^{-1}\right)^{-1} \left(A_\varphi^T t^{-1} - A_\varphi^T\right) \overline{w}^T \\
&= -vA_\varphi \overline{w}^T - vA_\varphi \left(A_\varphi - A_\varphi^T t^{-1}\right)^{-1} \left(A_\varphi^T t^{-1} - A_\varphi^T\right) \overline{w}^T \\
&= -vA_\varphi \left(A_\varphi - A_\varphi^T t^{-1}\right)^{-1} \left(A_\varphi^T t^{-1} - A_\varphi^T\right) \overline{w}^T \\
&= -vA_\varphi \left(A_\varphi - A_\varphi^T t^{-1}\right)^{-1} \left(t^{-1} - 1\right) A_\varphi^T \overline{w}^T \\
&= -vA_\varphi \left(A_\varphi - A_\varphi^T t^{-1}\right)^{-1} \left(t^{-1} - 1\right) \left(\overline{wA_\varphi}\right)^T \\
&= -v \left(A_\varphi - A_\varphi^T t^{-1}\right)^{-1} \left(t^{-1} - 1\right) \overline{w}^T \\
&= v \left(tA_\varphi - A_\varphi^T\right)^{-1} (t-1) \overline{w}^T
\end{aligned}$$

where we only need equality modulo $\mathcal{Q}_\varphi$. $\square$

Theorem 5.7. *Let L be an m-component link with pairwise linking numbers zero and let $\varphi = (1, \ldots, 1)$ represent a primitive class in $H^1(X_L)$. Suppose further that $H_1(X; \mathcal{Q}_\varphi)$ is torsion. Suppose that there exists a Seifert Z-surface for L with respect to φ that has a strong metabolizer. Then the Blanchfield form $\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}$ on $H_1(X; \mathcal{Q}_\varphi)$ is hyperbolic.*

Proof. Suppose that Σ is a Seifert $\mathbb{Z}$-surface for L with respect to $\varphi = (1, \ldots, 1)$ with a strong metabolizer $H = \widehat{H} \cup B \subseteq H_1(\Sigma)$, where $\widehat{H}$ is a metabolizer for the corresponding closed Seifert $\mathbb{Z}$-surface $\widehat{\Sigma}$ and B is generated by the boundary components of Σ. Since $\varphi = (1, \ldots, 1)$, the boundary of Σ is exactly the longitudes of L. So let $\{a_{g+1}, \ldots, a_{2g}\} \cup \{\lambda_1, \ldots, \lambda_{m-1}\}$ be a basis for H, and extend it to a basis $\{a_1, \ldots, a_{2g}\} \cup \{\lambda_1, \ldots, \lambda_{m-1}\}$ of $H_1(\Sigma)$. Let A be a Seifert $\mathbb{Z}$-matrix for L and Σ

with respect to this basis, so we have $A = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$, where the upper left block is of size $g \times g$ and the lower right block is a size $(g+m-1) \times (g+m-1)$ zero matrix. We'll call this "metabolizer form."

By Corollary 5.6, the Blanchfield form $\mathcal{B}\ell^X_{\mathcal{Q}_\varphi}$ on $H_1(X; \mathcal{Q}_\varphi)$ is represented by the matrix $(t-1)\left(tA - A^T\right)^{-1}$, which we'll call $\mathcal{B}\ell$. First we'll show that $\mathcal{B}\ell$ has a block of zeros.

Multiplying A by t does not affect the block of zeroes, nor does taking its transpose. Subtracting two matrices in metabolizer form also preserves the form. So $S = tA - A^T$ is in metabolizer form. Now we invert S.

Since $H_1(X; \mathcal{Q}_\varphi)$ is torsion, S is indeed invertible over $\mathcal{K}$. Recall that $S^{-1} = \frac{1}{\det S} S^{\text{adj}}$, where S^{adj} is the adjugate matrix, or classical adjoint matrix, of S, so we need only check that S^{adj} is in metabolizer form. Recall that the ij entry of S^{adj} is given by $(-1)^{i+j} \det(M_{ji})$, where M_{ij} is the $(2g+m-2) \times (2g+m-2)$ matrix obtained from S by deleting the ith row and jth column. Notice that when $i, j \leq g$, M_{ij} has the form $M_{ij} = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ where the upper left block is size $(g-1) \times (g-1)$ and the lower right block is a $(g+m-1) \times (g+m-1)$ block of zeros. We'll use the following lemma to show that such a matrix always has determinant 0.

Lemma 5.8. *Suppose that A is a matrix of size $r \times r$ where $r = m + n$ and $m > n$. Suppose further that A decomposes as a block matrix $A = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ where the upper left block is a $m \times m$ block of zeros and D is a $n \times n$ block. Then* $\det A = 0$.

Proof. Recall that $\det A = \sum_{\sigma \in S_r} \left(\text{sgn}(\sigma) \prod_{i=1}^r a_{i,\sigma(i)}\right)$, where S_n is the group of permutations of the set $\{1, \ldots, r\}$ and a_{ij} is the ijth entry of A. We'll show that $\prod_{i=1}^r a_{i,\sigma(i)} = 0$ for all permutations σ. Let σ be an arbitrary permutation. By assumption, if $i \leq m$ and $j \leq m$ then $a_{ij} = 0$. There are m numbers $j \leq m$ and

$r = m+n$ numbers i and remember that $m > n$. Then by pigeonhole principle, there exists at least one number $i \leq m$ such that $\sigma(i) = j \leq m$. Then $a_{i,\sigma(i)} = 0$ and thus $\prod_{i=1}^{r} a_{i,\sigma(i)} = 0$. This works for all σ, therefore $\det A = 0$. □

For all $m \geq 1$, we have $g+m-1 > g-1$. Then by the lemma, when $i, j \leq g$, we have $\det M_{ij} = 0$. Thus S^{-1} is of the form $S^{-1} = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$ where the upper left block is a $g \times g$ block of zeros and the lower right block is of size $(g+m-1) \times (g+m-1)$. We'll call this "reverse metabolizer form."

Finally, multiplying by $t-1$ preserves the block of zeros, so $\mathcal{B}\ell$ is also in reverse metabolizer form. Now we'll use this to show that $\mathcal{B}\ell$ is hyperbolic.

$H_1(X; \mathcal{Q}_\varphi)$ is generated by the set $\{\alpha_1, \ldots, \alpha_{2g}, \mu_1, \ldots, \mu_{m-1}\}$. Let P be the submodule of $H_1(X; \mathcal{Q}_\varphi)$ generated by $\{\alpha_1, \ldots, \alpha_g\}$. Since $\mathcal{B}\ell$ has reverse metabolizer form, we easily see that $\mathcal{B}\ell(x, y) = 0$ for all $x, y \in P$. Thus $P \subset P^\perp$.

Let $x = \sum_{i=1}^{2g} x_i\alpha_i + \sum_{i=1}^{m-1} x'_i\mu_i \in P^\perp$. Then $\mathcal{B}\ell(x, \alpha_i) = 0$ for all $i \leq g$. Let T be the $(2g+m-1) \times g$ matrix whose columns are the column vectors representing the basis elements of P, and let V be the lower left block of $\mathcal{B}\ell$, so $\mathcal{B}\ell = \begin{pmatrix} 0 & * \\ V & * \end{pmatrix}$, and V is of size $(g+m-1) \times g$. Then the product matrix $\mathcal{B}\ell T = \begin{pmatrix} 0 \\ V \end{pmatrix}$ and $x\mathcal{B}\ell T = \vec{0}$. Let $y = \sum_{i=g+1}^{2g} x_i\alpha_i + \sum_{i=1}^{m-1} x'_i\mu_i$. Then $yV = \vec{0}$. Since $\det \mathcal{B}\ell \neq 0$, we must have $\det V \neq 0$. Hence $\vec{0} = \vec{0}V^{\mathrm{adj}} = yVV^{\mathrm{adj}} = y \det V$ implies that $y = 0$. Thus for $g+1 \leq i \leq 2g$, $x_i = 0$ and for $1 \leq i \leq m-1$, $x'_i = 0$, so $x = \sum_{i=1}^{g} x_i\alpha_i \in P$. Hence $P^\perp \subset P$, so $P = P^\perp$. Therefore the Blanchfield form $\mathcal{B}\ell$ is hyperbolic.

□

Chapter 6

Algebraically Slice Boundary Links

An m-component link $L = K_1 \cup \cdots \cup K_m$ is called a *boundary link* if its components bound disjoint Seifert surfaces. That is, there exists a disjoint collection of compact oriented bicollared surfaces $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_m$ smoothly embedded in S^3 such that $\partial\Sigma_i = K_i$ for each i. In this case Σ is called a *boundary Seifert surface* for L. Equivalently, an m-component link L is boundary if and only if there exists an epimorphism of $\pi_1(S^3 - L)$ onto F_m, the free group on m generators, that takes meridians to generators.

Suppose that $L = K_1 \cup \cdots \cup K_m$ is an m-component boundary link with boundary Seifert surface $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_m$ and $g = g_1 + \cdots + g_m$, where g_i is the genus of Σ_i. Define a pairing $\theta : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ by $\theta(x, y) = \ell k(x, y^+)$, where y^+ is the positive normal push-off of y. We call θ a *boundary Seifert form* for L. Note that since $H_1(\Sigma) = H_1(\Sigma_1) \oplus \cdots \oplus H_1(\Sigma_m)$, we may restrict θ to each $H_1(\Sigma_i) \times H_1(\Sigma_j)$ to obtain the maps $\theta_{ij} : H_1(\Sigma_i) \times H_1(\Sigma_j) \to \mathbb{Z}$. When $i = j$, θ_{ii} is just a Seifert form for K_i. When $i \neq j$, we have

$$\theta_{ij}(x, y) = \ell k(x, y^+) = \ell k(x, y) = \ell k(x^+, y) = \ell k(y, x^+) = \theta_{ji}(y, x),$$

since x and y live on disjoint surfaces.

Choose a basis $\{a_1, \ldots, a_g\}$ for $H_1(\Sigma)$ such that $\{a_1, \ldots, a_{g_1}\}$ is a basis for $H_1(\Sigma_1)$, $\{a_{g_1+1}, \ldots, a_{g_1+g_2}\}$ is a basis for $H_1(\Sigma_2)$, and so on. Then the matrix A representing θ with respect to this basis is a $2g \times 2g$ block matrix composed of m^2 matrices. That is $A = (A_{ij})_{i,j\in\{1,\ldots,m\}}$ where each A_{ij} is a $2g_i \times 2g_j$ matrix. When $i = j$, A_{ii} represents the Seifert form θ_{ii}, and is just a Seifert matrix for K_i, thus we know that $A_{ii} - A_{ii}^T$ is unimodular. Note that this implies that $A - A^T$ is unimodular. When $i \neq j$, A_{ij} represents the pairing θ_{ij}, and since $\theta_{ij}(x, y) = \theta_{ji}(y, x)$, we must have that $A_{ij} = A_{ji}^T$. Such a matrix $A = (A_{ij})_{i,j\in\{1,\ldots,m\}}$ representing a boundary Seifert form θ such that each submatrix A_{ij} represents θ_{ij} is called a *boundary Seifert matrix* [Lia77], [Ko87].

We can construct the universal abelian cover $\widetilde{X}$ of the link exterior $X = S^3 - N(L)$ of an m-component boundary link $L = K_1 \cup \cdots \cup K_m$ with boundary Seifert surface $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_m$ by cutting X along each disjoint surface Σ_i to obtain $Y = S^3 - N(\Sigma \cup L)$, placing a copy of Y at every point in a $\mathbb{Z}^m$ lattice and connecting them in the x_i direction with copies of $N(\Sigma_i)$ (See the figure). The group of deck transformations of $\widetilde{X}$ is $\Gamma = \mathbb{Z}^m$. Then $H_1(\widetilde{X}) = H_1(X; \mathbb{Z}\Gamma)$ as a $\mathbb{Z}\Gamma$-module. We often think of $\mathbb{Z}\Gamma$ as the ring of multivariable Laurent polynomials $\mathbb{Z}\Gamma \cong \mathbb{Z}[t_1, \ldots, t_m]$. Let $g = g_1 + \cdots + g_m$ where g_i is the genus of Σ_i. Let $A = (A_{ij})_{i,j\in\{1,\ldots,m\}}$ be a boundary Seifert matrix for L and Σ. Let τ be the $2g \times 2g$ block diagonal matrix with diagonal blocks $t_1 I_{2g_1}, \ldots, t_m I_{2g_m}$, where I_n is the $n \times n$ identity matrix. It is well-known that $TH_1(X; \mathbb{Z}\Gamma)$ is presented by the matrix $A\tau - A^T$.

We can define a Blanchfield form $\mathcal{B}\ell_{\mathbb{Z}\Gamma}$ on $TH_1(X; \mathbb{Z}\Gamma)$ in the same way as in previous chapters. Recall that for a knot, if A is a Seifert matrix for the knot, then the Blanchfield form is given by $(t - 1)(A - tA^T)$. An analogous result holds for boundary links.

Theorem 6.1 (Hillman, A. Conway [Hil81][Con18]). *The Blanchfield pairing* $\mathcal{B}\ell_{\mathbb{Z}\Gamma}$

on $TH_1(X;\mathbb{Z}\Gamma)$ *for a boundary link* L *with a boundary Seifert matrix* A *of size* $2g\times 2g$ *is represented by the matrix* $\left(A-\tau A^T\right)^{-1}(\tau - I_{2g})$.

Recall that a Seifert surface F for a knot J with corresponding Seifert form θ_F has a metabolizer if there exists a half-rank direct summand $H \subset H_1(F)$ such that $\theta_F(x,y)=0$ for all $x,y\in H$. That is, there exists a Seifert matrix for J and F of the form $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$. Recall that a knot J is called an algebraically slice knot if some Seifert surface for J admits a metabolizer.

Definition 6.2. Let $L = K_1 \cup \cdots \cup K_m$ be an m-component boundary link and let $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_m$ be a boundary Seifert surface for L. We say that the boundary Seifert surface Σ has a *boundary metabolizer* if there exists a half-rank direct summand $H \subset H_1(\Sigma)$ that is self-annihilating with respect to the corresponding boundary Seifert form, that is $\theta(x,y)=0$ for all $x,y\in H_1(\Sigma)$, and such that H decomposes as the direct sum $H = H_1\oplus\cdots\oplus H_m$ where each $H_i \subset H_1(\Sigma_i)$ is a metabolizer for the component Seifert surface Σ_i.

We'll call a boundary link L an *algebraically slice boundary link* if there exists a boundary Seifert surface for L that has a boundary metabolizer.

Every slice knot is algebraically slice, but this result is not known for boundary links. There is a stronger condition for boundary links called boundary slice. A boundary link $L = K_1\cup\cdots\cup K_m$ is called *boundary slice* if it is slice and for some choice of disjoint slice disks $\Delta = \Delta_1\cup\cdots\cup\Delta_m$ there exists an epimorphism $\pi_1(S^3 - L) \twoheadrightarrow F_m$, where F_m is the free group on m generators, that takes meridians to generators and that extends to an epimorphism $\pi_1(B^4-\Delta)\twoheadrightarrow F_m$.

Mimicking the proof that every slice knot is algebraically slice, one can easily prove that every boundary slice boundary link is algebraically slice. But it is unknown whether slice boundary links are algebraically slice. Indeed it is an open question whether slice and boundary slice are equivalent for boundary links.

Recall that a knot if algebraically slice if and only if it is 0.5-solvable. We can prove one direction for boundary links.

Theorem 6.3. *Let $L = K_1 \cup \cdots \cup K_m$ be an m-component boundary link. If L is algebraically slice then L is* 0.5*-solvable.*

Proof. This will be very similar to the proof of Theorems 2.5 and 3.16. Let $L = K_1 \cup \cdots \cup K_m$ be an m-component boundary link. Let $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_m$ be a boundary Seifert surface for L of genus $g = g_1 + \cdots + g_m$ with boundary metabolizer $H = H_1 \oplus \cdots \oplus H_m \subset H_1(\Sigma)$ such that each $H_1 \subset H_1(\Sigma_i)$ is a metabolizer for Σ_i, a genus g_i Seifert surface for the link component K_i. As in the proof of Theorem 2.5, we may isotope each disjoint Seifert surface Σ_i one at a time into disk-band form such that the cores $a_{i,1}, \ldots, a_{i,g_i}$ of the labeled bands in the figure form a basis for the metabolizer H_i. After we do this for each $i = 1, \ldots, m$, we may isotope the entire boundary Seifert surface into the form demonstrated in Figure 6.1. Now the curves

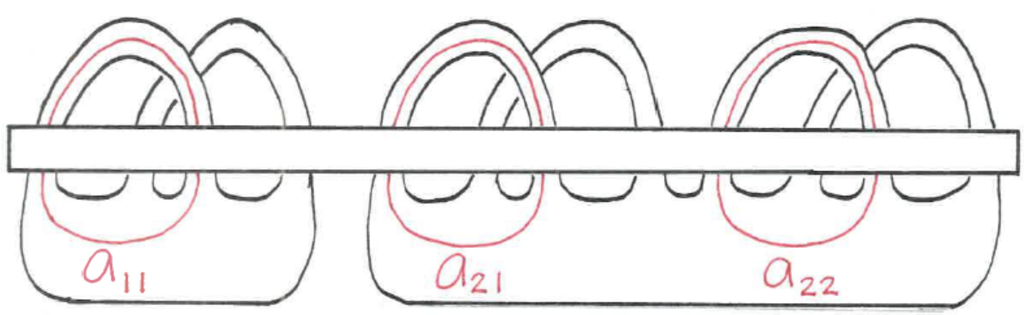

Figure 6.1: A boundary Seifert surface in disk-band form with a boundary metabolizer represented as the cores of half the bands. The box contains a string link on the bands.

$\bigcup_{i=1}^{m} \bigcup_{j=1}^{g_i} a_{ij}$ form a link J, and since they also represent a basis for the boundary metabolizer, J has pairwise linking numbers zero. Thus J is delta equivalent to the unlink. Note that since the Seifert surfaces Σ_i are disjoint, each band involves only one link component. So as we perform delta moves on J to transform it into an unlink

$J = \bigcup_{i=1}^{m} \bigcup_{j=1}^{g_i} a'_{ij}$, we are performing delta moves on the bands and transforming the boundary Seifert surface Σ into some surface $\Sigma' = \Sigma'_1 \sqcup \cdots \sqcup \Sigma'_m$, and we are performing double-delta moves on L and transforming it into some link L'. Note that double delta moves preserve pairwise linking numbers and note that Σ' is a boundary Seifert surface for L', thus L' is still a boundary link. Now that $J' \subset \Sigma'$ is the unlink, we may cut Σ' along the curves a'_{ij} and then cap with $2g$ disks $D_1, \ldots, D_{2g}$ as in the figure to obtain $\Sigma'' = \Sigma''_1 \cup \cdots \cup \Sigma''_m$, an immersed collection of m disks with boundary L'. The only self intersections of Σ'' occur in the disks D_i, and since Σ' was in disk-band form, we can see that the only possible types of self-intersection are ribbon intersections, occurring when a band intersects one of these new disks D_I. Therefore L' is a ribbon link, and is therefore slice. Then L is double-delta equivalent to a slice link, therefore since double-delta equivalence preserves 0.5-solvability, L is 0.5-solvable. □

The converse of this theorem is unknown, and is difficult again due to the possible difference between slice and boundary slice.

Recall that a knot is algebraically slice if and only if its Blanchfield form is hyperbolic. We can prove one direction for boundary links.

Theorem 6.4. *Let $L = K_1 \cup \cdots \cup K_m$ be an m-component boundary link. If L is algebraically slice then the Blanchfield form $\mathcal{B}\ell_{\mathbb{Z}\Gamma}$ on $TH_1(X;\mathbb{Z}\Gamma)$ is hyperbolic.*

Proof. Let $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_m$ be a boundary Seifert surface for L and let $H = H_1 \oplus \cdots \oplus H_m$ be a boundary metabolizer for Σ such that for each i, H_i is a metabolizer for Σ_i. Then there exists a boundary Seifert matrix $A = (A_{ij})_{i,j\in\{1,\ldots,m\}}$ for L and Σ such that each block A_{ij} subdivides as $A_{ij} = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$, where each sub-block is of size $g_i \times g_j$ and * denotes a sub-block with no restrictions on its entries. We will call this "boundary metabolizer form." By Theorem 6.1, the Blanchfield form is represented by the matrix $\mathcal{B}\ell = (A - \tau A^T)^{-1}(\tau - I_{2g})$. First we'll show that the matrix $\mathcal{B}\ell$ has a certain special form.

The transpose of A is given by $A^T = (A^T_{ij}) = \left((A_{ji})^T\right)$, so A^T also has boundary metabolizer form. Multiplying by τ only changes A^T by constants–$\tau A^T = \left(\left(\tau A^T\right)_{ij}\right) = (t_i A^T_{ij}) = (t_i (A_{ji})^T)$. So τA^T also has boundary metabolizer form. Adding A and $-\tau A^T$ preserves the boundary metabolizer form. So $E = A - \tau A^T$ has boundary metabolizer form. Now we invert E. (Note that since $A - A^T$ is unimodular, $\det(A - \tau A^T)$ is a nontrivial Laurent polynomial in m variables, thus $(A - \tau A^T)$ is indeed invertible.)

First apply a change of basis matrix Q. The original boundary Seifert matrix A is with respect to a basis $B = \bigcup_{i=1}^m \{a_{i,1}, \ldots, a_{i,2g_i}\}$, where $\{a_{i,1}, \ldots, a_{i,g_i}\}$ is a basis for the metabolizer $H_i \subset H_1(\Sigma_i)$. We want to rearrange the basis elements so that instead of being ordered by which disjoint Seifert surface they belong to, they are ordered so that the basis elements of the boundary metabolizer come first. Let C be the basis $C = \bigcup_{i=1}^m \{a_{i,1}, \ldots, a_{i,g_i}\} \bigcup_{i=1}^m \{a_{i,g_i+1}, \ldots, a_{i,2g_i}\}$ and let Q be the change of basis matrix, which is just a permutation matrix, so that $E' = Q^T E Q$ is with respect to the basis C. Then E' is of the form $E' = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$.

Recall that $(E')^{-1} = \dfrac{1}{\det(E')}(E')^{\mathrm{adj}}$, where $(E')^{\mathrm{adj}}$ is the adjugate matrix, or classical adjoint matrix, of E, so we need only check that $(E')^{\mathrm{adj}}$ is in boundary metabolizer form. Recall that the ij entry of $(E')^{\mathrm{adj}}$ is given by $(-1)^{i+j}\det(M_{ji})$, where M_{ij} is the $(2g-1) \times (2g-1)$ matrix obtained from E' by deleting the ith row and jth column. Notice that when $i, j > g$, M_{ij} has the form $E' = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$, where the upper left block is a $g \times g$ block of zeros and D is a $(g-1) \times (g-1)$ block.

Now by Lemma 5.8, when $i, j > g$, $\det M_{ij} = 0$. Thus $(E')^{-1}$ has the form

$(E')^{-1} = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$.

Now we return to the original basis. $E^{-1} = \left(QE'Q^T\right)^{-1} = Q(E')^{-1}Q^T$ is now with respect to the basis B, and has the form of a block matrix $E^{-1} = \left((E^{-1})_{ij}\right)$ with blocks $(E^{-1})_{ij}$ of the form $(E^{-1})_{ij} = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$, which we'll call "reverse boundary metabolizer form." Finally, multiplying by $\tau - I_{2g}$ on the right gives us $\mathcal{B}\ell = E^{-1}(\tau - I_{2g}) = \left((E^{-1}(\tau - I_{2g}))_{ij}\right) = \left((E^{-1})_{ij}\,(t_j - 1)\right)$, which is also of reverse boundary metabolizer form.

Now we'll use the fact that $\mathcal{B}\ell$ is in reverse boundary metabolizer form to show that the Blanchfield form is hyperbolic. We have $\mathcal{B}\ell = (\mathcal{B}\ell_{ij})$ is a block matrix where each block $\mathcal{B}\ell_{ij}$ is of size $2g_i \times 2g_j$ and has the form $\mathcal{B}\ell_{ij} = \begin{pmatrix} * & B_{ij} \\ * & 0 \end{pmatrix}$ where each sub-block is of size $g_i \times g_j$. Recall that $TH_1(X; \mathbb{Z}\Gamma) \cong \mathbb{Z}\Gamma^{2g} / \left(A\tau - A^T\right) \mathbb{Z}\Gamma^{2g}$ is generated by the ordered set $B^* = \bigcup_{i=1}^m \{\alpha_{i,1}, \ldots, \alpha_{i,2g_i}\}$. Let P be the submodule of $TH_1(X; \mathbb{Z}\Gamma)$ generated by the ordered set $B^\perp = \bigcup_{i=1}^m \{\alpha_{i,g_i+1}, \ldots, \alpha_{i,2g_i}\}$. Then using the reverse boundary metabolizer form of the matrix $\mathcal{B}\ell$, we see that for any $x, y \in P$, we have $\mathcal{B}\ell_{\mathbb{Z}\Gamma}(x, y) = \vec{x}^T \mathcal{B}\ell \bar{y} = 0$. So $P \subseteq P^\perp$. Recall that $P^\perp = \{x \in TH_1(X; \mathbb{Z}\Gamma) \mid \mathcal{B}\ell_{\mathbb{Z}\Gamma}(x, y) = 0 \text{ for all } y \in P\}$. Let $x \in P^\perp$, and suppose that $x = \sum_{i=1}^m \sum_{j=1}^{2g_i} x_{ij}\alpha_{ij}$ for some $x_ij \in \mathbb{Z}\Gamma$. We know $\mathcal{B}\ell_{\mathbb{Z}\Gamma}(x, \alpha_{ij}) = 0$ for each $g_i + 1 \leq j \leq 2g_i$, since $x \in P^\perp$ and $\alpha_{ij} \in P$ for each $g_i + 1 \leq j \leq 2g_i$. Let F be the $2g \times g$ matrix whose columns are given by the vectors representing the elements of $B^\perp$. Then the product matrix $\mathcal{B}\ell F$ has the form of a block matrix with blocks $(\mathcal{B}\ell F)_{ij} = \begin{pmatrix} B_{ij} \\ 0 \end{pmatrix}$ and $x^T \mathcal{B}\ell F = \vec{0}^T$. Let B be the $g \times g$ block matrix $B = (B_{ij})$. Let $y = \sum_{i=1}^m \sum_{j=1}^{g_i} x_{ij}\alpha_{ij}$. Then $y^T B = \vec{0}^T$. Since $\det \mathcal{B}\ell \neq 0$ and with a change of basis $\mathcal{B}\ell = \begin{pmatrix} * & B \\ * & 0 \end{pmatrix}$, we must have that $\det B \neq 0$. Hence $\vec{0}^T = \vec{0}^T B^{\mathrm{adj}} = y^T B B^{\mathrm{adj}} = y^T \det B$ implies that $y = \vec{0}$. Thus for each $i = 1, \ldots, m$ and each $j = 1, \ldots, g_i$ we have $x_{ij} = 0$. Therefore $x = \sum_{i=1}^{m} \left(0\alpha_{i,1} + \cdots + 0\alpha_{i,g_i} + x_{i,g_i+1}\alpha_{i,g_1+1} + \cdots + x_{i,2g_i}\alpha_{i,2g_i}\right) \in P$. $\square$

The following diagram summarizes what is known for boundary links.

$$\begin{tikzcd}
\text{boundary slice} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{slice} \arrow[r, Rightarrow] & \text{1-solvable} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \text{0.5-solvable} \\
\text{algebraically slice} \arrow[d, Rightarrow] \arrow[urrr, Rightarrow, bend right=10] & & & \\
\mathcal{B}\ell_{\mathbb{Z}\Gamma} \text{ hyperbolic} \arrow[rr, Rightarrow] & & \mathcal{B}\ell_{\mathcal{R}_\varphi} \text{ hyperbolic } \forall\varphi &
\end{tikzcd}$$

The implication on the bottom row follows from the fact that the Blanchfield forms $\mathcal{B}\ell_{\mathcal{R}_\varphi}$ are the localized versions of the Blanchfield form $\mathcal{B}\ell_{\mathbb{Z}\Gamma}$.

References

[CHL08] T. Cochran, S. Harvey, and C. Leidy, *Link concordance and generalized doubling operators*, Algebr. Geom. Topol. **8** (2008), 1593–1646. 1.1, 4.1, 4.7, 4.8, 4.9

[CHL09] ———, *Knot concordance and higher-order Blanchfield duality*, Geom. Topol. **13** (2009), 1419–1482. 1.1, 4.1, 4.7, 4.8, 4.9

[Cim04] David Cimasoni, *Studying the multivariable Alexander polynomial by means of Seifert surfaces*, Bol. Soc. Mat. Mexicana (3) **10** (2004), no. Special Issue, 107–115. MR 2199342 1.1, 3, 3.1, 3.2, 3.1, 3.9

[Coc04] Tim D. Cochran, *Noncommutative knot theory*, Algebr. Geom. Topol. **4** (2004), 347–398. MR 2077670 4, 5, 4.4

[Coh85] P. M. Cohn, *Free rings and their relations*, second ed., London Mathematical Society Monographs, vol. 19, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1985. MR 800091 4

[Con18] Anthony Conway, *An explicit computation of the Blanchfield pairing for arbitrary links*, Canad. J. Math. **70** (2018), no. 5, 983–1007. MR 3831912 6.1

[COT03] T. Cochran, K. Orr, and P. Teichner, *Knot concordance, Whitney towers and L^2 signatures*, Ann. of Math. **157** (2003), no. 2, 433–519. 1, 1.1, 2.1, 2.1, 2.2, 2.2, 4, 4.1, 4.4, 4.1, 4.1, 4.2

[EN85] David Eisenbud and Walter Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematics Studies, vol. 110, Princeton University Press, Princeton, NJ, 1985. MR 817982 1.1, 3, 3.1, 3.2, 3.1

[FM66] R. Fox and J. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math **3** (1966), 257–267. 1

[FP17] S. Friedl and M. Powell, *A calculation of Blanchfield pairings of 3-manifolds and knots*, Moscow Math. J. **17** (2017), 59–77. 1.1, 4, 4.3, 5.1, 5.1, 5.2, 5.2

[Gee08] Carol Gwosdz Gee, *Strong S-equivalence of ordered links*, J. Knot Theory Ramifications **17** (2008), no. 8, 961–981. MR 2439771 3.1

[Har05] Shelly L. Harvey, *Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm*, Topology **44** (2005), no. 5, 895–945. MR 2153977 4, 4.1, 4.2, 5.2

[Hil81] Jonathan A. Hillman, *Alexander ideals of links*, Lecture Notes in Mathematics, vol. 895, Springer-Verlag, Berlin-New York, 1981. MR 653808 6.1

[Kea75] C. Kearton, *Cobordism of knots and Blanchfield duality*, J. London Math. Soc. **10** (1975), 406–408. 1, 2.2, 4

[Kir78] R. Kirby, *A calculus for framed links in S^3*, Invent. Math. **45** (1978), no. 1, 35–56. 1

[Ko87] K. H. Ko, *Seifert matrices and boundary link cobordisms*, Trans. Amer. Math. Soc. **299** (1987), no. 2, 657–681. 6

[Lei] C. Leidy, *Higher-order linking forms for 3-manifolds*, preprint, arXiv:1204.5167. 4, 4.5

[Lei06] Constance Leidy, *Higher-order linking forms for knots*, Comment. Math. Helv. **81** (2006), no. 4, 755–781. MR 2271220 4

[Lia77] C. Liang, *An algebraic classification of some links of codimension two*, Proc. Amer. Math. Soc. **67** (1977), no. 1, 147–151. 6

[Lic62] W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. **76** (1962), no. 3, 531–540. 1

[Lic97] ———, *An introduction to knot theory*, Springer-Verlag New York, 1997. 1, 3.3

[Mar] T. Martin, *Classification of links up to 0-solvability*, preprint, arXiv:1511.00156. 1, 2.7

[Mar13] ———, *Lower order solvability of links*, Ph.D. thesis, Rice University, 2013. 2.3, 2.4, 2.1, 2.7, 2.8

[MN89] H. Murakami and Y. Nakanishi, *On a certain move generating link-homology*, Math. Ann. **284** (1989), no. 1, 75–89. 2.1, 3.3

[MO] T. Martin and C. Otto, *Splittings of link concordance groups*, preprint, arXiv:1606.00481. 1, 2.8

[Pas77] Donald S. Passman, *The algebraic structure of group rings*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977. MR 470211 1

[Rol76] D. Rolfsen, *Knots and links*, Publish or Perish, 1976. 1, 1.1, 5.2

[Ste75] Bo Stenström, *Rings of quotients*, Springer-Verlag, New York-Heidelberg, 1975, Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory. MR 0389953 3, 4, 5, 4.1

[Tro73] H. F. Trotter, *On S-equivalence of Seifert matrices*, Invent. Math. **20** (1973), 173–207. 4

[Wal60] A. Wallace, *Modifications and cobounding manifolds*, Canad. J. Math. **12** (1960), 503–528. 1

[Whi78] George W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York-Berlin, 1978. MR 516508 4.3