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Tau invariants of spatial graphs

by

Katherine Rose Poulsen Vance

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APPROVED, THESIS COMMITTEE:

Shelly Harvey

Shelly Harvey, Chair
Associate Professor of Mathematics

Andy Putman
Associate Professor of Mathematics

Ilinca Stanciulescu
Assistant Professor of Civil Engineering

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Abstract

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In 2003, Ozsváth and Szabó defined the concordance invariant τ for knots in oriented 3-manifolds as part of the Heegaard Floer homology package. In 2011, Sarkar gave a combinatorial definition of τ for knots in S^3 and a combinatorial proof that τ gives a lower bound for the slice genus of a knot. Recently, Harvey and O'Donnol defined a relatively bigraded combinatorial Heegaard Floer homology theory for transverse spatial graphs in S^3 which extends knot Floer homology. We define a $\mathbb{Z}$-filtered chain complex for balanced spatial graphs whose associated graded chain complex has homology determined by Harvey and O'Donnol's graph Floer homology. We use this to show that there is a well-defined τ invariant for balanced spatial graphs generalizing the τ knot concordance invariant. In particular, this defines a τ invariant for links in S^3. Using techniques similar to those of Sarkar, we show that our τ invariant gives an obstruction to a link being slice.

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CHAPTER 1

Introduction

1. Background

A graph is a one-dimensional CW-complex whose edges (one-cells) may be oriented. A spatial graph is a smooth or piecewise linear embedding $f : G \to S^3$, where G is an (oriented) graph. One way to think of spatial graphs is as a generalization of the classical study of knots and links, which are embeddings of one or more ordered S^1 components into S^3. Just as for knots and links, we consider spatial graphs up to ambient isotopy.

The study of knots and links have historically been very important in low-dimensional topology. In the early 1960s, Lickorish [**Lic62**] and Wallace [**Wal60**] proved that every closed orientable 3-manifold is the result of surgery on a link in S^3. Another way to describe 3-manifolds is by Heegaard splittings, which Heegaard defined in his Ph.D. thesis in 1898 [**Hee98**]. A Heegaard splitting is a decomposition of a 3-manifold into two handlebodies glued together by a homeomorphism of their

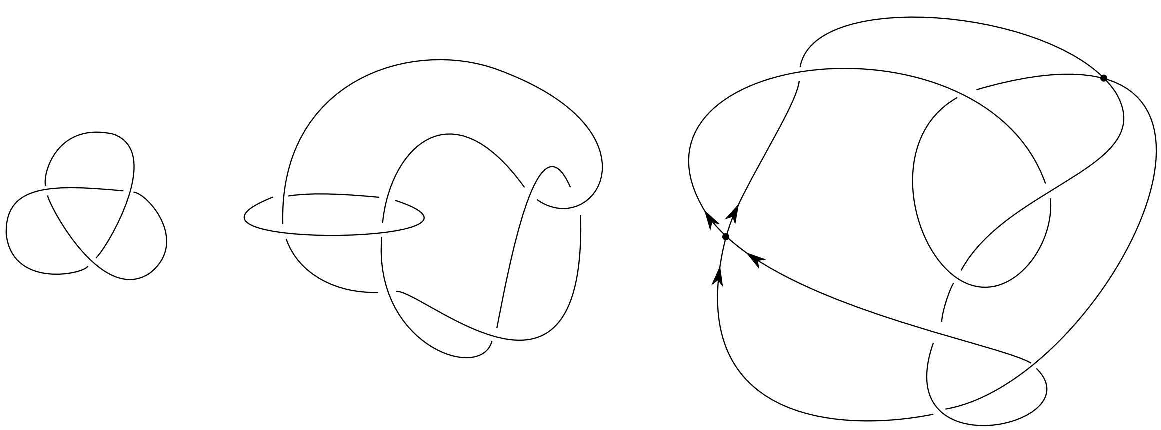

FIGURE 1.1. A knot, a link, and a spatial graph

boundaries. Such a splitting is described by a Heegaard diagram, which consists of the closed Heegaard surface (the boundary of the handlebodies) together with sets of α- and β-curves describing the gluing homeomorphism. In 2001, Ozsváth and Szabó defined Heegaard Floer homology — a package of 3-manifold invariants — as the Lagrangian Floer homology of two Lagrangian submanifolds obtained from the α- and β-curves, respectively, in a symplectic manifold obtained as the symmetric product of a Heegaard surface for the manifold [**OS04c**].

Knot Floer homology was independently defined in 2002 by Ozsváth and Szabó [**OS04b**] and by Rasmussen [**Ras03**]. Knot Floer homology categorifies the Alexander polynomial and has many other nice properties, including detecting the genus of a knot and whether a knot is fibered [**Ni07**]. One invariant to come out of the knot Floer homology package is the τ invariant, which was defined by Ozsváth and Szabó in 2004 [**OS04a**].

One reason the τ invariant is important is its relationship to knot concordance, which is an active area of study in knot theory. A cobordism between knots K and J is a smoothly properly embedded surface in $S^3 \times [0, 1]$ with two boundary components, such that K is the boundary component in $S^3 \times \{0\}$ and J is the boundary component in $S^3 \times \{1\}$. If the genus of the cobordism is zero, we say that K and J are concordant. A knot is slice if it is the boundary of a smoothly properly embedded disk in the four-ball. The slice genus (sometimes called the four-ball genus) of a knot is the minimum genus over all properly embedded smooth surfaces in the four-ball whose boundary is the knot. The τ invariant is a concordance invariant and its absolute value is a lower bound for slice genus [**OS04b**].

The original formulation of knot Floer homology involves pseudo-holomorphic disks in the symmetric product of a Heegaard surface, but in 2006, Manolescu, Ozsváth, and Sarkar [**MOS09**] gave a combinatorial definition using grid diagrams. Then Manolescu, Ozsváth, Szabó, and Thurston proved the invariance of the grid diagram formulation without reference to the original analytic definition [**MOST07**]. In 2011, Sarkar proved the relationship between τ and slice genus combinatorially [**Sar11**]. Recently, Harvey and O'Donnol have defined graph Floer homology for a certain class of spatial graphs in S^3 using a grid diagram construction analogous to that used for knots and links [**HO15**]. However, while knot Floer homology is filtered by the integers, Harvey and O'Donnol's graph Floer homology is not; rather it is relatively graded graded by the first homology group of the spatial graph.

2. Summary of Results

In this thesis, we define a filtered version of graph Floer homology for balanced transverse spatial graphs whose associated graded object is Harvey and O'Donnol's HFG and prove that it is a spatial graph invariant. We prove that the filtered graph Floer chain complex is, up to filtered quasi-isomorphism, an invariant of balanced spatial graphs. Thus we have the following theorem.

THEOREM 5.1. *For grid diagrams g, g' representing $f : G \to S^3$, there exist filtered quasi-isomorphisms $\phi_1 : CF^-(g) \to CF^-(g')$ and $\phi_2 : CF^-(g') \to CF^-(g)$ which preserve the symmetrized filtration $\{\mathcal{F}_s^{-H}\}$.*

This allows us to define a τ invariant for balanced spatial graphs and prove that it is an invariant.

DEFINITION 5.9. For a graph grid diagram g representing a balanced spatial graph $f : G \to S^3$, define the τ invariant of g to be

$$\tau(g) = \min\{m \in \frac{1}{2}\mathbb{Z} | \iota_m \text{ is non-trivial}\}$$

where $\iota_m : H_*(\widehat{\mathcal{F}}_m^H) \to H_*(\widehat{CF}(g))$ is the map induced by inclusion.

COROLLARY 5.1. *If g and $\overline{g}$ are graph grid diagrams representing a balanced spatial graph $f : G \to S^3$, then $\tau(g) = \tau(\overline{g})$.*

Considering links as spatial graphs with one vertex and one edge in each link component, we get a result relating the τ invariant to link cobordisms.

THEOREM 6.1. *If L_1 and L_2 are l- and m-component links, respectively, and F is a connected genus g cobordism from L_1 to L_2, then*

$$1 - g - l \leq \tau(L_1) - \tau(L_2) \leq g + m - 1.$$

As a corollary, we see that the τ invariant can be an obstruction to a link being slice.

COROLLARY 6.1. *If an l-component link L has $\tau(L) > 0$ or $\tau(L) \leq -l$, then L is not slice.*

3. Outline of Thesis

Chapter 2 is about homological algebra. We give several definitions and results about filtered chain complexes that will be necessary in Chapter 4 and Chapter 5.

In Chapter 3, we give an overview of knot concordance, Heegaard Floer homology of 3-manifolds, and knot Floer homology. We discuss the definition of τ for knots and some results relating τ to knot concordance. In Chapter 4 we discuss Harvey and O'Donnol's graph Floer homology.

In Chapter 5 we give the definitions of filtered graph Floer homology and the τ invariant for balanced spatial graphs. We prove that τ is an invariant of balanced spatial graphs.

In Chapter 6 we discuss link cobordisms and "movie moves" on grid diagrams. We prove Theorem 6.1, giving a relationship between the τ invariants of links and link cobordisms between them. We then prove, as a corollary, that the τ invariant can be an obstruction to a link being slice.

CHAPTER 2

Algebraic Background Information

In this chapter, we will give the algebraic background information necessary for the definitions of knot Floer homology and graph Floer homology. We will also prove a lemma about filtered chain complexes that will be used in Chapter 5 in the proof that τ is an invariant of balanced spatial graphs.

Definition 2.1. A chain complex (C, ∂) is a graded group or module, together with a map $\partial : C_n \to C_{n-1}$ satisfying the condition that $\partial \circ \partial = 0$.

$$\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

Definition 2.2. A filtered chain complex is a chain complex (C, ∂) together with a filtration of C

$$0 \subset \ldots \subset \mathcal{F}_{i-1} \subset \mathcal{F}_i \subset \mathcal{F}_{i+1} \subset \ldots \subset C$$

such that $\partial(\mathcal{F}_i) \subset \mathcal{F}_i$ and $\bigcup\limits_i \mathcal{F}_i = C$.

Definition 2.3. A map f between filtered chain complexes (C, ∂) with filtration $\{\mathcal{F}_s\}$ and (C', ∂') with filtration $\{\mathcal{F}'_s\}$ is a filtered chain map if the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] \arrow[d, "f"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "f"] & C_n \arrow[r, "\partial"] \arrow[d, "f"] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r, "\partial'"] & C'_{n+1} \arrow[r, "\partial'"] & C'_n \arrow[r, "\partial'"] & C'_{n-1} \arrow[r, "\partial'"] & \cdots
\end{tikzcd}$$

and if the map respects the filtrations, that is, $f(\mathcal{F}_s) \subseteq \mathcal{F}'_s$.

Definition 2.4. A map ϕ from a chain complex (C, ∂) to a chain complex (C', ∂') is a quasi-isomorphism if it induces an isomorphism on the homology of the chain complexes.

Definition 2.5. A chain map $\Phi_1 : C \to C'$ is a chain homotopy equivalence with chain homotopy inverse $\Phi_2 : C' \to C$ if there exist maps $h_C : C \to C$, $h_{C'} : C' \to C'$ such that

$$\Phi_2 \circ \Phi_1 + \mathbb{I}_C = \partial \circ h_C + h_C \circ \partial$$

and

$$\Phi_1 \circ \Phi_2 + \mathbb{I}_{C'} = \partial' \circ h_{C'} + h_{C'} \circ \partial'.$$

We will need the following two lemmas in Chapter 5, when we prove Theorem 5.1.

Lemma 2.1. *Let (C, ∂) be a filtered chain complex with filtration $\{\mathcal{F}_s\}$ of C such that $H_*(C) \neq 0$ and $\bigcap_s \mathcal{F}_s = 0$. If for each homological grading i, the chain group C_i is finitely generated, then $H_*(\mathcal{F}_s/\mathcal{F}_{s-1}) \neq 0$ for some s.*

Proof. Since $H_*(C) \neq 0$ and $H_*(C) = \oplus H_i(C)$, there exists some i for which $H_i(C) \neq 0$. Therefore there is some non-zero $x \in C_i$ which is homogeneous with respect to the homological grading i, with $\partial x = 0$, and whose homology class is nonzero. We can then choose the minimal filtration level s so that $x \in \mathcal{F}_s$.

Let $\partial_s : \mathcal{F}_s/\mathcal{F}_{s-1} \to \mathcal{F}_s/\mathcal{F}_{s-1}$. Then $\partial_s(x + \mathcal{F}_{s-1}) = \partial x + \mathcal{F}_{s-1} = 0 + \mathcal{F}_{s-1}$. If $x + \mathcal{F}_{s-1}$ is not a boundary in the chain complex $(\mathcal{F}_s/\mathcal{F}_{s-1}, \partial_s)$, then $H_*(\mathcal{F}_s/\mathcal{F}_{s-1}) \neq 0$ and we are done.

If $x + \mathcal{F}_{s-1}$ is a boundary in $(\mathcal{F}_s/\mathcal{F}_{s-1}, \partial_s)$, then there is some $y \in \mathcal{F}_s$ with $x + \mathcal{F}_{s-1} = \partial_s(y + \mathcal{F}_{s-1}) = \partial y + \mathcal{F}_{s-1}$. Set $z = x - \partial y \in \mathcal{F}_{s-1}$. Since x is a cycle, $\partial z = \partial(x - \partial y) = 0$. Therefore $z \in \mathcal{F}_{s-1}$ is a cycle and since x and z differ by a boundary, $[z] = [x] \neq 0$ in $H_i(C)$.

We can repeat this process, choosing the minimal filtration level $r \leq s-1$ so that $z \in \mathcal{F}_r$, yielding a cycle $z_1 \in \mathcal{F}_{r-1}$ with $[z_1] = [z] = [x] \neq 0$ in $H_i(C)$. Iterating this process will produce infinitely many representatives of $[x]$, each in different filtration levels. This contradicts our hypothesis that for each homological grading i, the chain group C_i is finitely generated. $\square$

LEMMA 2.2 ([**McC01**], Theorem 3.2). *If $F : B \to C$ is a filtered chain map which induces an isomorphism on the homology of the associated graded objects of B and C, then F is a filtered quasi-isomorphism.*

CHAPTER 3

Knot Concordance and Heegaard Floer Homology

A knot is an embedding of S^1 into S^3. Two knots $K_0, K_1 : S^1 \hookrightarrow S^3$ are equivalent if there exists an ambient isotopy between them, that is, if there is a homotopy $H : S^3 \times [0,1] \to S^3$ such that for all $t \in [0,1]$, the map $H(\cdot, t)$ is a homeomorphism, $H(\cdot, 0)$ is the identity map, and $H(K_0(\cdot), 1) = K_1(\cdot)$. The set of knots is infinite, and we use knot invariants to distinguish knots and to attempt to understand the structure of the set of knots.

The set of knots does not admit a group structure as is. Although there is a binary operation (connected sum) and an identity element (the unknot), there are no inverses. This fact can be proved using knot genus.

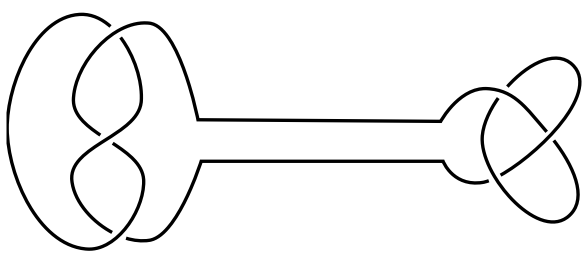

Figure 3.1. The connected sum of two knots

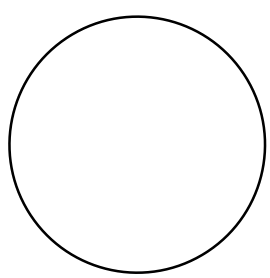

Figure 3.2. The unknot

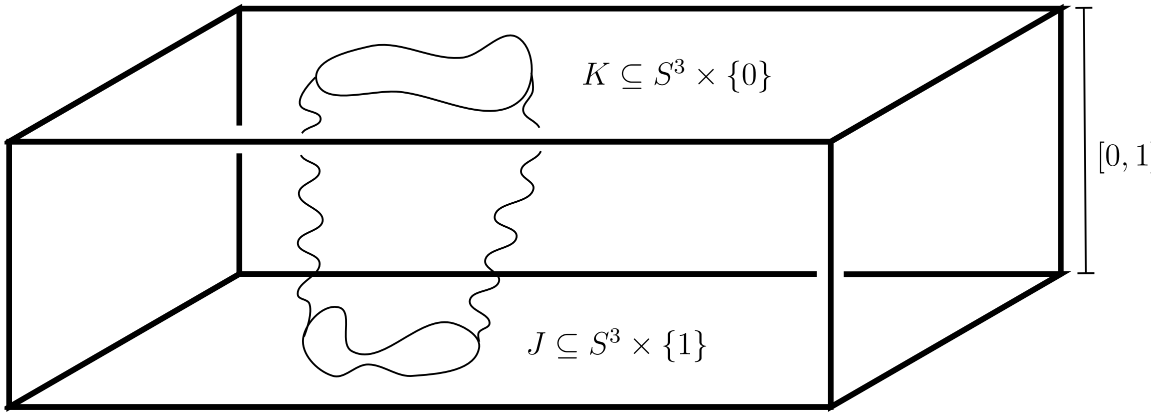

FIGURE 3.3. A concordance between two knots

Given two knots K_1 and K_2, we form the connected sum $K_1 \# K_2$ by taking a point P_i on K_i, and removing a small ball B_i centered at P_i from S^3, with $K_i \cap B_i$ an unknotted arc in B_i. We then glue the two pairs $(S^3 - B_1, K_1 - K_1 \cap B_1)$ and $(S^3 - B_2, K_2 - K_2 \cap B_2)$ together along their boundaries so as to respect the orientations of K_1 and K_2.

A Seifert surface for a knot K is a connected, orientable surface whose boundary is the knot K. Every knot has a Seifert surface [**Sei35**]. The genus of a knot K (sometimes called the Seifert genus) is $g(K) = min\{g(\Sigma) \,|\, \Sigma \text{ is a Seifert surface for } K\}$. The unknot is defined as the only knot of genus zero. Since knot genus is additive under connected sum, and every non-trivial knot has strictly positive genus, we can see that the connected sum of any two non-trivial knots is also non-trivial, and thus there are no inverses in the set of knots.

Although the set of knots is a monoid, not a group, in the 1950s Fox and Milnor introduced the equivalence relation knot concordance. A knot K_0 is concordant to a knot K_1 if there is a smoothly properly embedded genus zero surface in $S^3 \times [0, 1]$ whose boundary components are $K_0 \subset S^3 \times 0$ and $K_1 \subset S^3 \times 1$.

We denote the set of knots modulo concordance by $\mathcal{C}$, and in 1966 Fox and Milnor [**FM66**] proved that this is a group with the operation connected sum. The identity element is the class of slice knots, which are concordant to the unknot, and a knot's inverse is its reverse mirror image.

A knot K is slice if there exists a smoothly embedded disk D in B^4, such that $\partial D = k \subset \partial B^4 = S^3$. It is easy to see that the set of slice knots is exactly the class of knots which are concordant to the unknot. The Alexander polynomial of any slice knot takes the form $\Delta_K(t) = f(t)f(t^{-1})$, where $f(t)$ is a Laurent polynomial. A ribbon knot is one which is the boundary of an immersed disk in S^3 whose self-intersections consist of pairs of arcs, with one in each pair contained in the interior of the disk and the other with endpoints on the boundary of the disk. By pushing a neighborhood of each of these interior arcs into the interior of the four-ball, we can easily visualize a slice disk for any ribbon knot. The slice-ribbon conjecture, which is open, posits that every slice knot is a ribbon knot.

The four-ball genus, sometimes called slice genus, is a related invariant. The four-ball genus of a knot K, denoted $g_4(K)$, is the minimum genus of a connected, orientable surface smoothly properly embedded in B^4 whose boundary is K.

Although in this thesis we will focus on the category of smooth knots, links, and spatial graphs, it is important to note that there also exist concepts of topological concordance, topologically slice knots, and algebraically slice knots. For topological concordance or sliceness, we only require that the embedding of the cylinder or disk be locally flat rather than smooth. Freedman showed that all knots with trivial Alexander polynomial are topologically slice [**Fre82**]. A knot is algebraically slice if it has a Seifert form which is zero on a half-dimensional subspace of the first homology of the knot complement (this is sometimes referred to as "half lives, half dies"). Every smoothly slice knot is topologically slice, and every topologically slice knot is algebraically slice, but the converses are not true.

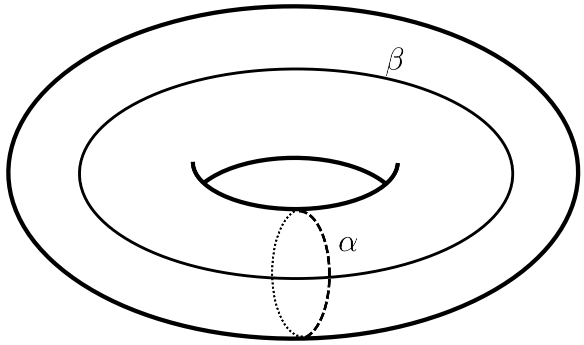

FIGURE 3.4. A Heegaard diagram

1. Heegaard Floer homology

Heegaard Floer homology is a 3-manifold invariant defined by Ozsváth and Szabó [**OS04c**]. The Heegaard Floer homology $\widehat{HF}$ of a 3-manifold Y is an abelian group. It is computed using a Heegaard splitting for Y. Heegaard splittings were introduced by Heegaard in 1898 [**Hee98**], and they decompose 3-manifolds into two solid handlebodies whose intersection is their common boundary, Σ_g. A genus g Heegaard splitting is described by a Heegaard diagram, which consists of the closed genus g surface Σ_g together with two sets α and β, each of which consists of g disjoint simple closed curves in Σ such that $\Sigma - \alpha$ and $\Sigma - \beta$ are connected.

To recover a three-manifold from a Heegaard diagram, we thicken the Heegaard surface and attach g three-dimensional 2-handles to each side of the surface, using the α-curves (on one side of the surface) and β-curves (on the other side) as attaching curves. This yields an manifold with two S^2 boundary components. By capping each of those boundary components with a 3-ball, we obtain the manifold.

There are infinitely many Heegaard diagrams for any given 3-manifold. However, if $(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g)$ and $(\Sigma_{g'}, \alpha'_1, ..., \alpha'_{g'}, \beta'_1, ..., \beta'_{g'})$ are two diagrams representing the same 3-manifold Y, then there is a sequence of isotopies, handle-slides of α or β curves, and stabilizations or destabilizations that begins with $(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g)$ and ends with $(\Sigma_{g'}, \alpha'_1, ..., \alpha'_{g'}, \beta'_1, ..., \beta'_{g'})$.

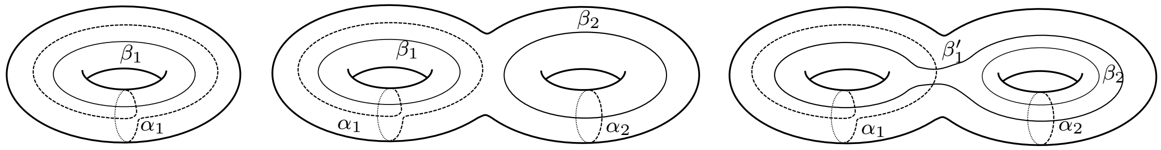

FIGURE 3.5. On the left, a genus one Heegaard diagram, stabilized to the center diagram, then a handleslide of β_1 over β_2 produces the right-hand diagram

An isotopy of the α or β curves is a perturbation of the curves in Σ_g such that each set of curves remains disjoint simple closed curves. A handleslide of α_i over α_j replaces α_i with a curve which is homologous to $\alpha_i + \alpha_j$ in $H_1(\Sigma_g)$. A stabilization increases the genus of the Heegaard surface by one and adds a new pair of curves, α_{g+1} and β_{g+1}, whose intersection is a single point, and such that the resulting diagram satisfies the definition of a Heegaard diagram. A destabilization can be done when there are an α curve and a β curve which intersect each other in one point, and which do not interact with any other α and β curves, and it is the reverse of a stabilization.

Heegaard Floer homology is defined using a pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, z)$, where z is a basepoint in $\Sigma_g - \alpha - \beta$. The generators of the chain complex are intersection points of the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ formed by the α and β curves in the g-fold symmetric product $Sym^g(\Sigma_g)$. The differential map counts pseudo-holomorphic disks properly embedded in $(Sym^g(\Sigma_g), \mathbb{T}_\alpha \cup \mathbb{T}_\beta)$. There are several variations of Heegaard Floer homology: $\widehat{HF}$, $\widetilde{HF}$, HF^-, HF^+, and HF^∞.

2. Knot Floer Homology

Knot Floer homology is a variation of Heegaard Floer homology that was defined in 2002 by Ozsváth and Szabó [**OS04b**] and independently by Rasmussen [**Ras03**]. The knot Floer homology is a bigraded module $\bigoplus_{i,s\in\mathbb{Z}} \widehat{HFK}_i(K,s)$ and it is the categorification of the Alexander polynomial. That is to say, when we take the Euler characteristic of $\widehat{HFK}$, using powers of a variable to to record the s-grading (which is called the Alexander grading), we get the Alexander polynomial of the knot:

$$\Delta_K(t) = \sum_{i,s \in \mathbb{Z}} 9-1)^i t^s rank(\widehat{HFG}_i(K,s)).$$

Like Heegaard Floer homology for 3-manifolds, knot Floer homology comes in several "flavors" - $\widehat{HFK}$, $\widetilde{HFK}$, HFK^-, HFK^+, and HFK^∞. Also like Heegaard Floer homology, its original formulation was defined using the symmetric product of a Heegaard surface.

A knot K in a 3-manifold Y can be represented using a Heegaard diagram for Y with two basepoints, z and w in $\Sigma_g - \alpha - \beta$. To recover the knot, we connect the two basepoints with two unknotted arcs, one in $\Sigma_g - \alpha$ and the other in $\Sigma_g - \beta$. By pushing the interiors of one of these arcs into each handlebody in the Heegaard splitting, we see a bridge decomposition of K with bridge surface Σ_g. Although the original formulation of knot Floer homology is defined for knots in arbitrary closed 3-manifolds, in 2006, Manolescu, Ozsváth, and Sarkar [**MOS09**] gave a combinatorial definition of knot Floer homology for knots in S^3. Since the way graph Floer homology is defined is analogous to the combinatorial definition of knot Floer homology, we well restrict our attention to knots in S^3.

Combinatorial knot Floer homology is defined using grid diagrams, which are essentially multi-pointed genus one Heegaard diagrams for knots in S^3. An index n grid diagram is an $n \times n$ grid, with its top and bottom edges and its left and right edges identified. Its horizontal gridlines are known as α-circles (and are numbered from bottom to top in the planar picture of the grid diagram), and its vertical gridlines are β-circles (which are numbered from left to right). Instead of z and w basepoints, a grid diagram has n O-markings and n X-markings, arranged so that there are exactly one X and one O in each row and column of the grid. To recover the knot or link from a grid diagram, we connect the X to the O in each column and the O to the X in each row, with the vertical strand as the overpass arc at teach crossing. It is easy

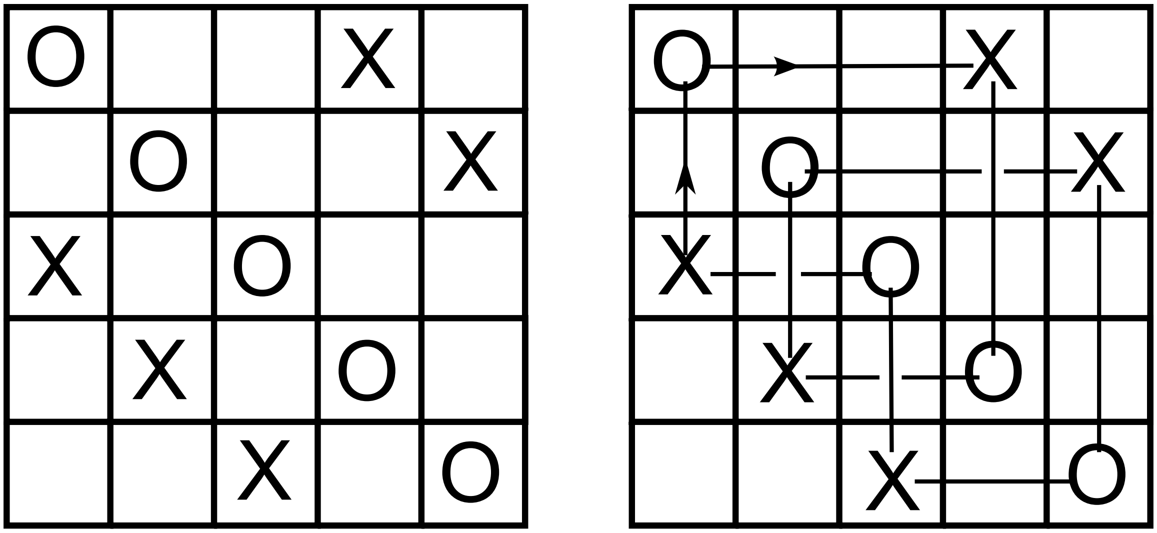

FIGURE 3.6. A grid diagram of the trefoil, and the knot recovered from it

to see that we can create a grid diagram from any knot or link diagram by isotoping the neighborhood of each crossing so that the overpasses all run vertically, "squaring up" the strands of the knot or link, and adjusting so that no two vertical or horizontal segments are at the same level, and then place X's and O's, as appropriate, at the corners. There are infinitely many grid diagrams for any given knot or link, but there is an analog for grid diagrams of Reidemeister's theorem for knot and link diagrams:

THEOREM 3.1 (Cromwell's Theorem, [**Cro95**]). *For any two planar grid diagrams g_1 and g_2 representing a given knot or link, there exists a finite sequence of cyclic permutation, commutation, and stabilization or destabilization moves transforming g_1 into g_2.*

A *cyclic permutation* move takes the column (or row) at one edge of the grid diagram and moves it to the opposite edge. This is equivalent to changing where we cut a toroidal grid diagram to produce a planar grid diagram. A *commutation* move interchanges two adjacent columns (or rows) if the 0-spheres formed by projecting the X- and O-markings in the two columns (rows) onto the grid line between them are not linked. A *stabilization* move at an X-marking adds an additional row and column

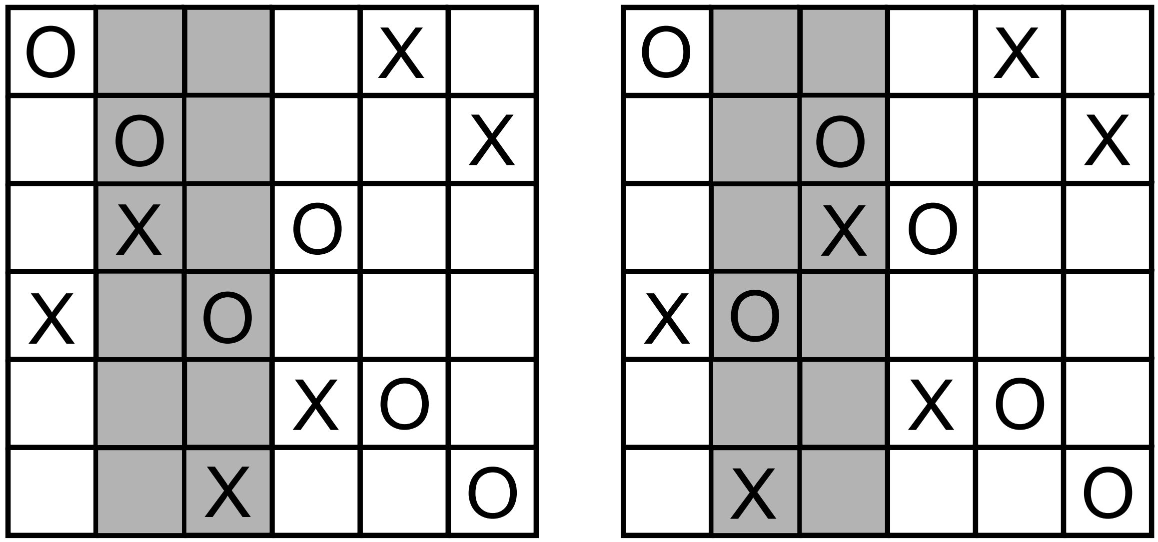

FIGURE 3.7. A commutation move

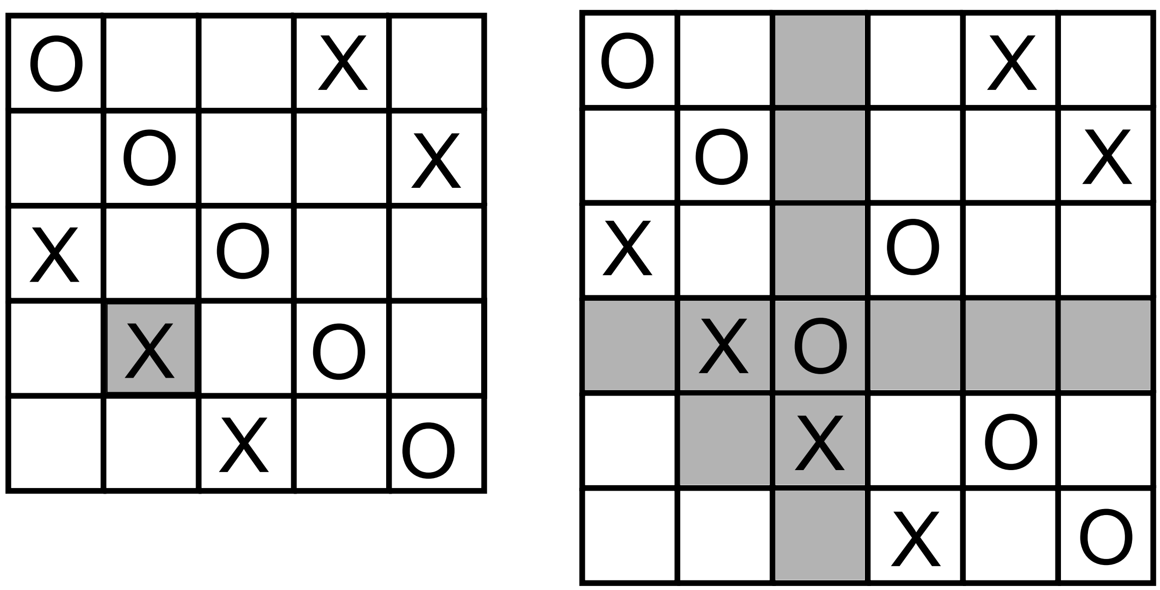

FIGURE 3.8. A stabilization move

adjacent to that X, removes the X, adds an O in the intersection of the new row and new column, and adds X's in the two squares in the intersection of the new column (row) and the row (column) that previously contained the stabilized X. We can also stabilize at an O, by reversing the role of X and O-markings in the description of a stabilization. A destabilization move is the reverse of a stabilization.

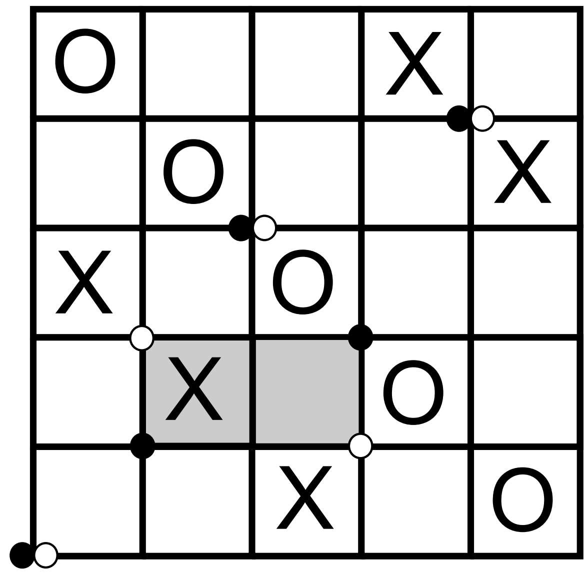

FIGURE 3.9. An empty rectangle from one generator (indicated by solid dots) to another (indicated by empty dots)

We will now give a brief overview of the combinatorial definition of $\widehat{HFK}$. The generating set $\mathcal{G}$ of the knot Floer chain complex $\widehat{CFK}$ is the set of n-tuples of intersection points of the α- and β-circles in an index n toroidal grid diagram representing the knot, with one point on each horizontal and each vertical grid line. Notice that the generating set is in set bijection with the symmetric group on n letters, which means that although $\widehat{HFK}$ is combinatorial, in practice it is very difficult to compute because of the sheer number of generators. The chain complex is a freely generated vector space over $\mathbb{F}_2[U_1, ..., U_{n-1}]$, where $\mathbb{F}_2$ is the field of two elements and the U_i's are formal variables corresponding to O-markings in the grid diagram. One special O-marking, marked with an asterisk, does not have a corresponding U_i.

The differential map $\hat{\partial}$ counts empty rectangles between generators of $\widehat{CFK}$. A rectangle r in the toroidal grid diagram connects a generator $\mathbf{x}$ to another generator $\mathbf{y}$ if $\mathbf{y}$ agrees with $\mathbf{x}$ except in two points, and those two points of $\mathbf{x}$ are the lower left and upper right corners of r. The two points of $\mathbf{y}$ that are not also points of $\mathbf{x}$ are the upper left and lower right corners of r. We say that r is empty if it does not contain any other points of $\mathbf{x}$ or $\mathbf{y}$. The set of empty rectangles connecting $\mathbf{x}$ to $\mathbf{y}$ is denoted $\mathcal{R}^\circ(\mathbf{x},\mathbf{y})$. See Fig. 3.9 for an example of an empty rectangle connecting two generators. The differential map $\hat{\partial}$ is U_i-equivariant and defined as follows on the generators of $\widehat{CFK}$:

$$\hat{\partial}\mathbf{x} = \sum_{\mathbf{y}\in\mathcal{G}} \sum_{\substack{r\in\mathcal{R}^\circ(\mathbf{x},\mathbf{y}) \\ O_*\notin r}} U_1^{n_1(r)} U_2^{n_2(r)} \cdots U_{n-1}^{n_{n-1}(r)} \mathbf{y},$$

where $n_i(r)$ is 1 if $O_i \in r$ and 0 otherwise.

There are two gradings on $\widehat{CFK}$: the Maslov grading, which is the homological grading, and the Alexander grading. Both are defined using the bilinear function $\mathcal{J}$, which for points $a, b \in \mathbb{R}^2$ is defined to be

$$\mathcal{J}(a,b) = \begin{cases} \frac{1}{2} \text{ if } b \text{ is above and to the right or below and to the left of } a \\ 0 \text{ otherwise} \end{cases}$$

and for finite sets A and B of points in the plane, $\mathcal{J}(A,B) = \sum_{\substack{a_i\in A \\ b_j\in B}} \mathcal{J}(a_i,b_j)$. To define the Maslov and Alexander gradings on $\mathcal{G}$, we think of a grid diagram as a subset of the $\mathbb{R}^2$ plane and denote the sets of X- and O-markings by $\mathbb{X}$ and $\mathbb{O}$, respectively. The Maslov grading is integer-valued and is defined as

$$M(\mathbf{x}) = \mathcal{J}(\mathbf{x}-\mathbb{O}, \mathbf{x}-\mathbb{O}) + 1$$

and extended to $\widehat{CFK}$ by $M(0) = M(1) = 0$ and $M(U_i) = -2$ for all i. The Alexander grading is also integer-valued. It is defined as

$$A(\mathbf{x}) = \mathcal{J}\left(\mathbf{x} - \frac{1}{2}(\mathbb{X}+\mathbb{O}), \mathbb{X}-\mathbb{O}\right) - \frac{n-1}{2}$$

and extended to the rest of the chain complex by $A(0) = A(1) = 0$ and $A(U_i) = 0$ for all i. The knot Floer homology of a knot is $\widehat{HFK}(K) = H_*(\widehat{CFK}(K))$.

The Alexander grading of $\widehat{CFK}$ gives rise to a filtration, $\{\widehat{\mathcal{F}}_m\}_{m\in\mathbb{Z}}$. The m^{th} filtration group is made up of those elements of $\widehat{CFK}$ whose Alexander grading is less than or equal to m. The Alexander filtration is used to define the τ invariant:

$$\tau(K) = min\{m \in \mathbb{Z} | \iota_m \text{ is non-trivial}\}$$

where $\iota_m : H_*(\widehat{\mathcal{F}}_m) \to \widehat{HFK}(K)$ is the map induced by inclusion.

The τ invariant is a concordance invariant and it is additive under connected sums. Using the original formulation of knot Floer homology, Ozsváth and Szabó proved that $|\tau(K)| \leq g_4(K)$ [**OS03**]. Sarkar restated this result to say that if there is a genus g cobordism between knots K_1 and K_2, then $|\tau(K_1) - \tau(K_2)| \leq g$, and gave a combinatorial proof of it [**Sar11**].

CHAPTER 4

Graph Floer Homology

In this chapter we give an overview of Harvey and O'Donnol's graph Floer homology, which is defined for transverse spatial graphs. For precise definitions of spatial graphs and transverse spatial graphs, see [**HO15**]. A *spatial* graph is an embedding $f : G \to S^3$ of a 1-dimensional CW-complex G into S^3. An *oriented spatial graph* is a spatial graph with an orientation given for each edge. For each vertex v of an oriented spatial graph, the *incoming edges* of v are the edges incident to v whose orientation points toward v, and the *outgoing edges* of v are the edges incident to v whose orientation points away from v. A *disk graph* is one which has a standard disk $\mathcal{D}$ at each vertex, attached to the graph by identifying the center point of $\mathcal{D}$ with the vertex.

DEFINITION 4.1. A **transverse spatial graph** is an embedding $f : G \to S^3$ of an oriented disk graph G, such that at each vertex the standard disk is embedded in a plane that separates the incoming and outgoing edges, as shown in Fig. 4.1.

In contrast to spatial graph ambient isotopy, in which any combination of edges incident to a vertex can move freely, ambient isotopy of transverse spatial graphs only allows free movement of incoming edges with other incoming edges or outgoing edges with other outgoing edges at each vertex. This is because the edges may not pass through the standard disk at the vertex.

Like the combinatorial definition of knot Floer homology, graph Floer homology is defined using grid diagrams. The definition of spatial graph grid diagrams is very similar to the definition of grid diagrams for knots and links.

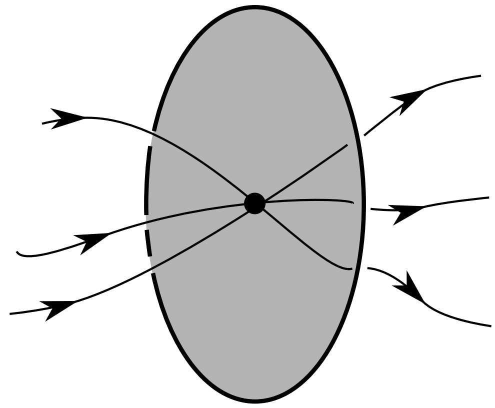

Figure 4.1. The standard disk separating incoming and outgoing edges at a vertex of a transverse spatial graph

Definition 4.2. An index n **grid diagram** for a transverse spatial graph is an n by n grid with some O- and X-markings, such that there is exactly one O in each row and in each column. We make a distinction between standard O-markings, which are those which are in the interior of a graph edge when we recover the spatial graph from the graph grid diagram, and special O-markings, which are vertices of the graph when it is recovered from the graph grid diagram. We mark special O's with an asterisk in the graph grid diagram. Standard O-markings have exactly one X in their row and column, while vertex O's may have any number of X-markings in their row and column. If a transverse spatial graph has more than one connected component, we require that there be at least one special O-marking in each component. A **toroidal graph grid diagram** is one in which we think of the grid as being a torus, with the leftmost and rightmost gridlines identified and the top and bottom gridlines identified.

To recover the spatial graph from a grid diagram, connect the X's to the O's vertically and the O's to the X's horizontally. At each crossing, the vertical strand is the overpass and the horizontal strand is the underpass. At vertex O's (those with more than one X in their row or column) use a straight line to connect the closest X in the row or column to the vertex O and a curved line to connect the more distant X's

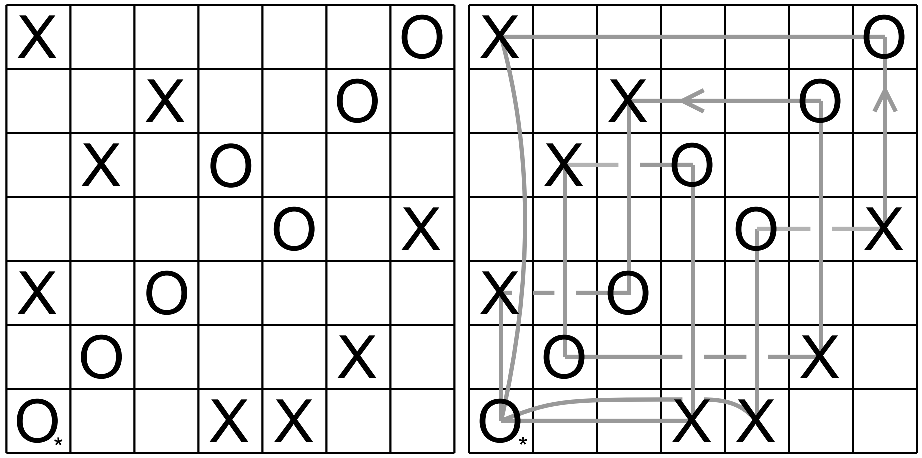

FIGURE 4.2. Graph grid diagram (note the starred vertex O)

to the vertex O, observing the same conventions with regard to the crossings created, so that the line connecting two markings within a column is always the overstrand. See Fig. 4.2. Just as in the case for knots and links, every transverse spatial graph can be represented by a graph grid diagram.

THEOREM 4.1 ([**HO15**]). *Any two planar grid diagrams for a given transverse spatial graph are related by a finite sequence of cyclic permutation, commutation', and (de-)stabilization' moves.*

A **cyclic permutation** moves the top (resp. bottom) row of a grid diagram to the bottom (resp. top) or moves the left (right) column to the far right (left) of the diagram. See the example in Fig. 4.3. Thinking of the grid as a torus, this equates to changing which gridline we "cut" the torus along to get the square diagram.

Two adjacent columns (or rows) may be exchanged using a **commutation'** move if there are vertical (horizontal) line segments LS_1 and LS_2 on the torus such that $LS_1 \cup LS_2$ contain all the X's and O's in the two adjacent columns (rows), the projection of $LS_1 \cup LS_2$ to a single vertical circle β_i (horizontal circle α_i) is β_i (α_i), and the projection of their endpoints, $\partial(LS_1) \cup \partial(LS_2)$, to a single β_i (α_i) is precisely two points. See the example in Fig. 4.4.

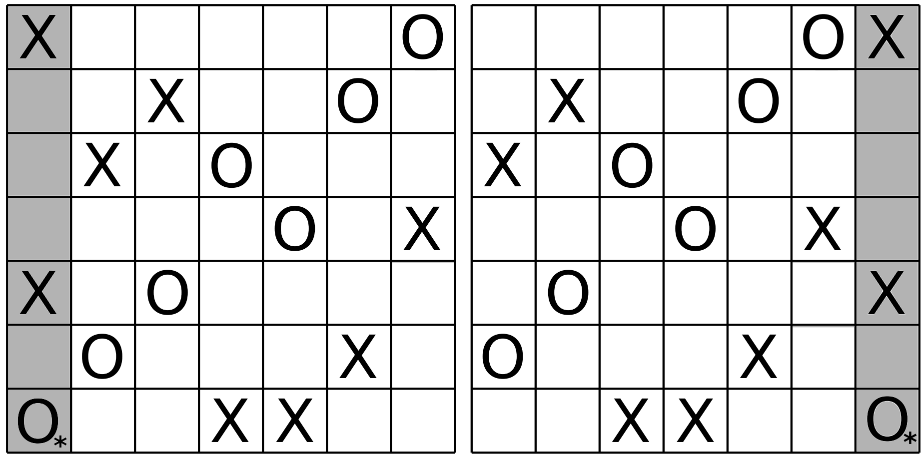

Figure 4.3. A cyclic permutation move

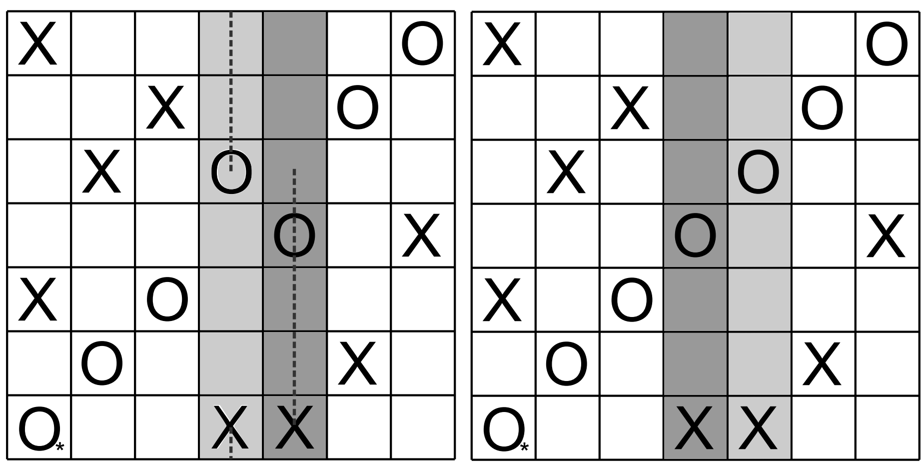

Figure 4.4. A commutation' move: notice the dotted helper arcs LS_1, LS_2 in the left-hand grid

A row (column) **stabilization'** at an X-marking is performed by adding one new row and one new column to the grid next to that X. The X is then moved to the new row (column), remaining in the same column (row), with the O and any other X-markings in which were in the same row (column) as the X being stabilized remaining in the old row (column). A new X-marking is placed in the intersection of the new column (row) and the row (column) previously occupied by the X-marking, and a

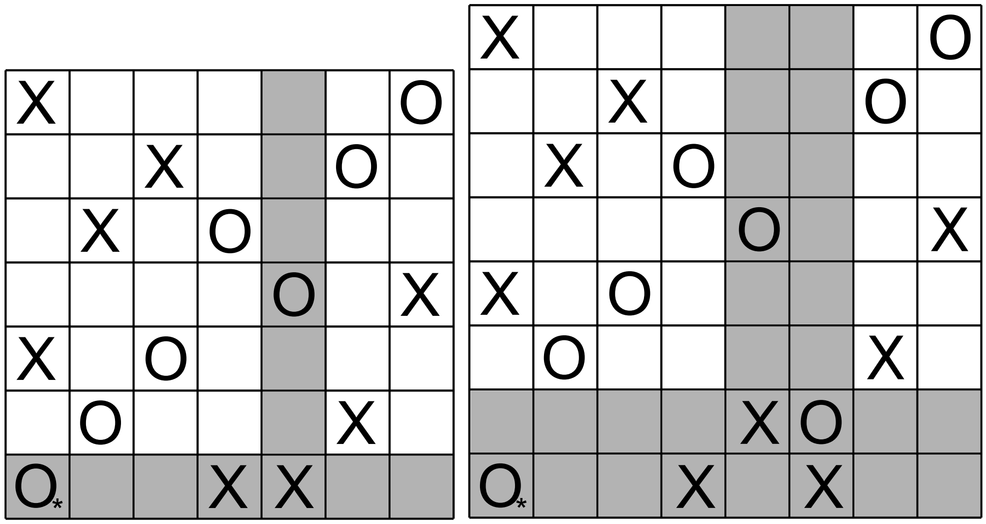

FIGURE 4.5. A row stabilization'

new O is placed in the intersection of the new row and column. See the example in Fig. 5.8. A destabilization' is the opposite of a stabilization'.

Harvey and O'Donnol's graph Floer homology is defined for transverse spatial graphs without sinks or sources. A sink is a vertex with no outgoing edges and a source is a vertex with no incoming edges. In other words, graph Floer homology is defined for spatial graphs whose underlying graph has at least one incoming edge and at least one outgoing edge at every vertex. This corresponds to a requirement that a graph grid diagram representing the spatial graph has at least one X-marking in every row and column.

For a spatial graph $f : G \to S^3$ represented by an $n \times n$ graph grid diagram g, the graph Floer chain complex $(C^-(g), \partial^-)$ is freely generated as a module over $\mathbb{F}_2[U_1, ..., U_n]$, where the U_i's are formal variables corresponding to the O-markings $O_1, ..., O_n$ in the graph grid diagram. The generating set of $C^-(g)$ is

$$\mathbf{S} = \{\mathbf{x} = (x_1, ..., x_n) | x_i = \alpha_i \cap \beta_{\sigma(i)} \text{ for some } \sigma \in S_n\}$$

where S_n is the symmetric group on n letters.

The map $\partial^- : C^-(g) \to C^-(g)$ counts empty rectangles in the toroidal graph grid diagram g. An embedded rectangle r in g connects a generator $\mathbf{x}$ to another generator $\mathbf{y}$ if $x_i = y_i$ for all but two i, if $j < k$ are the two indices for which $\mathbf{x}$ and $\mathbf{y}$ are not equal, and if the corners of r are, clockwise from the bottom left, x_j, y_k, x_k, and y_j. We say that r is empty if the interior of r does not contain any points of $\mathbf{x}$ or $\mathbf{y}$. The set of empty rectangles from $\mathbf{x}$ to $\mathbf{y}$ is denoted $\mathcal{R}^\circ(\mathbf{x}, \mathbf{y})$. The map $\partial^- : C^-(g) \to C^-(g)$ is defined as follows on the generating set $\mathbf{S}$ and then extened to all of $C^-(g)$ as an $\mathbf{F}_2[U_1, ..., U_n]$-module homomorphism:

$$\partial^-(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}} \sum_{\substack{r \in \mathcal{R}^\circ(\mathbf{x}, \mathbf{y}) \\ int(r) \cap \mathbb{X} = \varnothing}} U_1^{O_1(r)} \cdots U_n^{O_n(r)} \mathbf{y}$$

where $O_i(r)$ is zero if O_i is not in r and one if O_i is in r. Note that although this definition is very similar to the definition of $\widehat{\partial}$ given in Chapter 3, it differs in that ∂^- counts rectangles that contain any of the O-markings in g but does not count any rectangles that contain X-markings. This is because knot Floer homology is filtered by $\mathbb{Z} \cong H_1(S^3 - K)$, but since $H_1(S^3 - f(G))$ does not have a natural filtration, Harvey and O'Donnol's graph Floer homology is graded rather than filtered.

Proposition 4.1 ([**HO15**] Proposition 4.9). *For $\partial^- : C^-(g) \to C^-(g)$ as defined above, $\partial^- \circ \partial^- = 0$.*

There are two gradings on $(C^-(g), \partial^-)$. The first is the Maslov grading, which is the homological grading. Its definition is exactly the same as the definition of the Maslov grading in knot Floer homology (see Chapter 3), but we will restate it here. Viewing $\mathbb{X}$ and $\mathbb{O}$ as sets of points in the grid with half-integer coordinates, the Maslov grading of $\mathbf{x} \in \mathbf{S}$ is

$$M(\mathbf{x}) = \mathcal{J}(\mathbf{x} - \mathbf{O}, \mathbf{x} - \mathbf{O}) + 1.$$

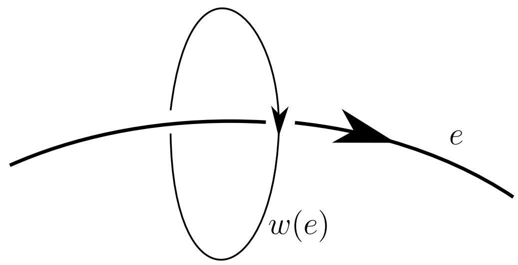

FIGURE 4.6. The weight of an edge

Before we can define the Alexander grading we need to define weights of the edges of G. We define a weight function $w : E(G) \to H_1(S^3 - f(G))$, where $E(G)$ is the set of edges of G, by mapping each edge $e \in E(G)$ to the homology class of the meridian of e, oriented according to the right-hand rule, as shown in Fig. 4.6.

For X-markings and O-markings associated to the interior of an edge e, the weights are $w(X) = w(e)$ or $w(O) = w(e)$. For O-markings associated to a graph vertex v, the weight is $w(O) = \sum_{e \in In(v)} w(e) = \sum_{e \in Out(v)} w(e)$, where $In(v)$ and $Out(v)$ are, respectively, the sets of incoming and outgoing edges of v.

We can now define the Alexander grading on the generating set $\mathbf{S}$:

$$A(\mathbf{x}) = \sum_{p \in \mathbb{X}} \mathcal{J}(\mathbf{x}, p) w(p) - \sum_{p \in \mathbb{O}} \mathcal{J}(\mathbf{x}, p) w(p).$$

This grading is not well-defined on toroidal graph grid diagrams, but Harvey and O'Donnol show that the relative grading $A^{rel}(\mathbf{x}, \mathbf{y}) = A(\mathbf{x}) - A(\mathbf{y})$ is well-defined on toroidal graph grid diagrams ([**HO15**] Corollary 4.12).

The graph Floer chain complex $(C^-(g), \partial^-)$ is bigraded, with an absolute $\mathbb{Z}$-valued grading (the Maslov grading) and a relative $H_1(S^3 - f(G))$-valued grading (the Alexander grading). The graph Floer homology is $HFG^-(f) = H_*(C^-(g), \partial^-)$ for any graph grid diagram g representing f, and it ia also absolutely $\mathbb{Z}$-graded and relatively $H_1(S^3 - f(G))$-graded.

CHAPTER 5

Filtered Graph Floer Homology and the τ Invariant

1. Spatial Graphs and the Chain Complex

In this section, we will define our filtered graph Floer homology chain complex. It is defined for balanced spatial graphs.

Definition 5.1. A transverse spatial graph is **balanced** if there is an equal number of incoming and outgoing edges at each vertex.

For an index n grid diagram g representing a spatial graph $f : G \to S^3$, we choose an ordering for the O-markings of g and denote them $O_1, \ldots, O_n$. Then the chain complex $CF^-(g)$ is freely generated over $\mathbb{F}[U_1, \ldots, U_n]$, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ and each U_i is a formal variable corresponding to O_i. It is generated by the set $\mathbf{S}$ of unordered n-tuples of intersection points in g with one point on each horizontal and vertical gridline. The generating set $\mathbf{S}$ is in bijection with S_n, the set of permutations of n

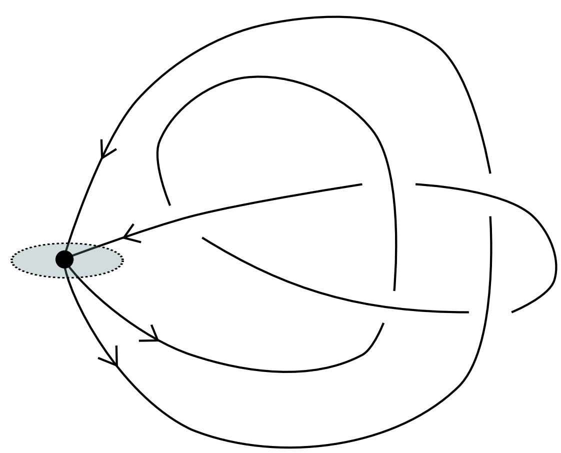

Figure 5.1. A balanced spatial graph

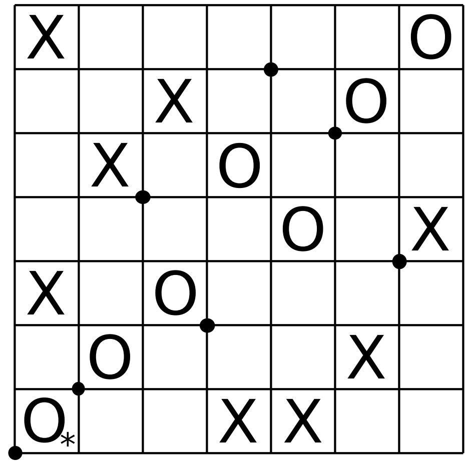

FIGURE 5.2. A generator of $CF^{-}(g)$

elements, so $\mathbf{S} = \{\mathbf{x} = (x_1, \ldots, x_n) | x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some $\sigma \in S_n\}$. See Fig. 5.2 for an example of a generator.

DEFINITION 5.2. A rectangle r in the grid diagram connects a generator $\mathbf{x}$ to another generator $\mathbf{y}$ if its lower left and upper right corners are points in $\mathbf{x}$, its upper left and lower right corners are points in $\mathbf{y}$, and all other points in $\mathbf{x}$ and $\mathbf{y}$ coincide. Such a rectangle is empty if its interior does not contain any points of $\mathbf{x}$ and $\mathbf{y}$. An empty rectangle may contain X- and O-markings. The set of empty rectangles from $\mathbf{x}$ to $\mathbf{y}$ is denoted $\mathcal{R}^{\circ}(\mathbf{x}, \mathbf{y})$.

The boundary map ∂^{-} is defined as follows on the generators and extended linearly to $CF^{-}(g)$:

$$\partial^{-}\mathbf{x} = \sum_{\mathbf{y} \in \mathbf{S}} \mathbf{y} \sum_{r \in \mathcal{R}^{\circ}(\mathbf{x}, \mathbf{y})} U_1^{O_1(r)} \cdots U_n^{O_n(r)}$$

where $O_i(r) = 1$ if O_i is contained in r and 0 otherwise.

If g is a graph grid diagram representing a balanced spatial graph, the chain complex $CF^{-}(g)$ is bigraded over $\mathbb{Z}$. The gradings are defined using the following bilinear map $\mathcal{J}$.

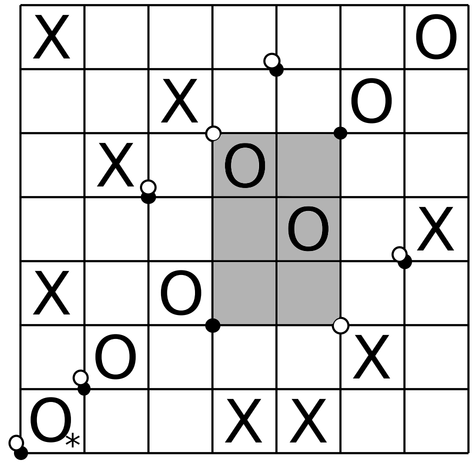

FIGURE 5.3. An empty rectangle connecting the black generator to the white generator

For a point $a = (a_1, a_2)$ and a finite set B of points in the plane, define $\mathcal{J}(a, B)$ to be half of the number of points in B which lie either above and to the right of a or below and to the left of a. That is, $\mathcal{J}(a, B) = \frac{1}{2}(\#\{(b_1, b_2) \in B \mid \text{either } (a_1 < b_1, a_2 < b_2) \text{ or } (a_1 > b_1, a_2 > b_2)\})$. By extending $\mathcal{J}$ bilinearly to formal sums and differences of sets of points in the plane, we can make the following definition, which is the same as the Maslov grading defined in [**MOST07**] and [**HO15**].

DEFINITION 5.3. The Maslov grading, also known as the homological grading, is defined as follows on the generators of the chain complex:

$$M(\mathbf{x}) = \mathcal{J}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) + 1$$

where $\mathbb{O}$ and $\mathbb{X}$ are the sets whose points are the O- and X-markings, respectively. The Maslov grading is extended to the rest of the chain complex by

$$M(U_i) = -2 \text{ for all } i$$

$$M(0) = M(1) = 0.$$

For example, the Maslov grading of the element $U_2 U_3^2 \mathbf{x}$ is $M(U_2 U_3^2 \mathbf{x}) = M(\mathbf{x}) - 6$.

Definition 5.4. The $\mathbb{Z}$-valued Alexander grading can only be defined for grids which represent balanced spatial graphs (for grids representing spatial graphs that are not balanced, an $H_1(S^3 \setminus f(G))$-valued Alexander grading can be defined, as in [**HO15**]).

$$A(\mathbf{x}) = \mathcal{J}(\mathbf{x}, \mathbb{X} - \sum_{O_i \in \mathbb{O}} m_i O_i)$$

where m_i is the weight of O_i: the number of incoming (or equivalently, since we are restricting to balanced graphs, outgoing) edges at O_1. The Alexander grading is extended to the rest of the chain complex by

$$A(U_i) = -m_i \text{ for all } i$$

$$A(0) = A(1) = 0.$$

We can also view the Alexander grading as a relative grading, namely $A(\mathbf{x}) - A(\mathbf{y})$, where $\mathbf{x}, \mathbf{y}$ are elements of the chain complex, computed using rectangles. Any two generators in $\mathbf{S}$ are connected by a sequence of rectangles. This follows from the fact that $\mathbf{S}$ is in bijection with the symmetric group on n letters, S_n. If $\sigma_1, \sigma_2 \in S_n$, there exists a finite sequence of transpositions that will turn σ_1 into σ_2. If $\mathbf{x}_1, \mathbf{x}_2$ are the generators in $\mathbf{S}$ corresponding to σ_1 and σ_2, respectively, then that sequence of transpositions corresponds to a sequence of rectangles connecting $\mathbf{x}_1$ to $\mathbf{x}_2$. The following lemma is very similar to Lemma 4.11 in [**HO15**].

Lemma 5.1. *If $\mathbf{x}, \mathbf{y}$ are generators of the chain complex and r is a rectangle (not necessarily empty) connecting $\mathbf{x}$ to $\mathbf{y}$, then the relative Alexander grading of $\mathbf{x}$ and $\mathbf{y}$ is*

$$A(\mathbf{x}) - A(\mathbf{y}) = |\mathbb{X} \cap r| - \sum_{O_i \in \mathbb{O} \cap r} m_i$$

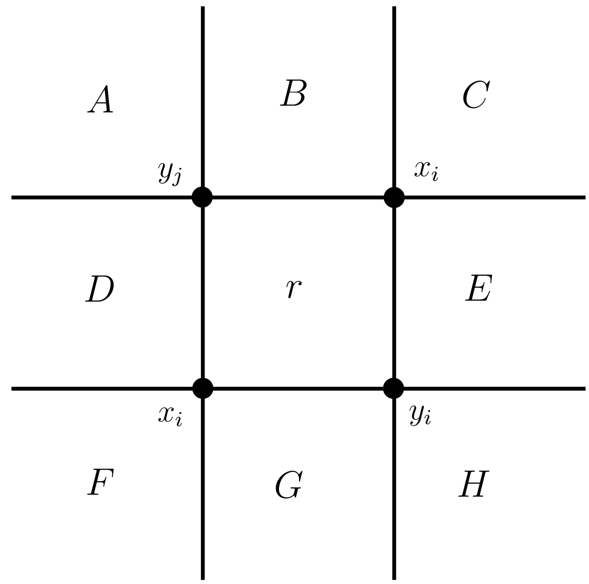

FIGURE 5.4. The regions of the grid referred to in Lemma 5.1

PROOF. By the Definition 5.4,

$$
\begin{aligned}
A(\mathbf{x}) - A(\mathbf{y}) &= \mathcal{J}\left(\mathbf{x}, \mathbb{X} - \sum_{O_i \in \mathbb{O}} m_i O_i\right) - \mathcal{J}\left(\mathbf{y}, \mathbb{X} - \sum_{O_i \in \mathbb{O}} m_i O_i\right) \\
&= \mathcal{J}(\mathbf{x}, \mathbb{X}) - \mathcal{J}(\mathbf{y}, \mathbb{X}) - \left(\sum_{O_i \in \mathbb{O}} m_i \left(\mathcal{J}(\mathbf{x}, O_i) - \mathcal{J}(\mathbf{y}, O_i)\right)\right) \\
&= \frac{1}{2}\left(|\mathbb{X} \cap (C \cup D \cup r \cup F \cup G)| + |\mathbb{X} \cap (B \cup C \cup r \cup E \cup F)|\right) \\
&\qquad - \frac{1}{2}\left(|\mathbb{X} \cap (B \cup C \cup D \cup F)| + |\mathbb{X} \cap (C \cup E \cup F \cup G)|\right) \\
&\qquad - \left(\sum_{O_i \in \mathbb{O} \cap (C \cup D \cup r \cup F \cup G)} \frac{m_i}{2}\right) - \left(\sum_{O_i \in \mathbb{O} \cap (B \cup C \cup r \cup E \cup F)} \frac{m_i}{2}\right) \\
&\qquad + \left(\sum_{O_i \in \mathbb{O} \cap (B \cup C \cup D \cup F)} \frac{m_i}{2}\right) + \left(\sum_{O_i \in \mathbb{O} \cap (C \cup E \cup F \cup G)} \frac{m_i}{2}\right) \\
&= |\mathbb{X} \cap r| - \sum_{O_i \in \mathbb{O} \cap r} m_i
\end{aligned}
$$

Where A, B, C, D, E, F, G, H and r are the regions of the grid indicated in Fig. 5.4.

$\square$

Definition 5.5. The Alexander filtration of $(CF^-(g), \partial^-)$ is $\{\mathcal{F}_m^-\}_{m\in\mathbb{Z}}$, where $\mathcal{F}_m^-$ is generated by those elements of $CF^-(g)$ whose Alexander grading is less than or equal to m.

Proposition 5.1. *$(CF^-(g), \partial^-)$ is a filtered chain complex. That is, $\partial^- \circ \partial^- = 0$, the boundary map decreases by one the Maslov grading of elements which are homogeneous with respect to the Maslov grading, and the boundary map preserves the relative Alexander filtration.*

Proof. That $\partial^- \circ \partial^- = 0$ follows directly from the proof of Proposition 2.10 of [**MOST07**], since graph grid diagrams differ from link grid diagrams only in the X-markings, and the definition of ∂^- does not involve X-markings.

The proof that ∂^- decreases Maslov grading by one is also the same as in [**MOST07**]. By their Lemma 2.5, if r is an empty rectangle from $\mathbf{x}$ to $\mathbf{y}$, then $M(\mathbf{x}) - M(\mathbf{y}) = 1 - 2|\mathbb{O} \cap r|$. Therefore the term in $\partial^-\mathbf{x}$ corresponding to $\mathbf{y}$ will have Maslov grading

$$\begin{aligned} M(U_1^{n_1(r)} \cdots U_n^{n_n(r)}\mathbf{y}) &= M(\mathbf{y}) - \sum_{i=1}^{n} 2n_i(r) \\ &= M(\mathbf{x}) - 1. \end{aligned}$$

To show that ∂^- preserves the relative Alexander filtration, note that if a rectangle r connects $\mathbf{x}$ to $\mathbf{y}$, then $A(\mathbf{y}) = A(\mathbf{x}) - |\mathbb{X} \cap r| + \sum_{O_i \in \mathbb{O}\cap r} m_i$. Therefore the term in $\partial^-\mathbf{x}$ corresponding to $\mathbf{y}$ will have Alexander grading

$$\begin{aligned} A(U_1^{n_1(r)} \cdots U_n^{n_n(r)}\mathbf{y}) &= A(\mathbf{y}) - \sum_{O_i \in \mathbb{O}\cap r} m_i \\ &= A(\mathbf{x}) - |\mathbb{X} \cap r| + \sum_{O_i \in \mathbb{O}\cap r} m_i - \sum_{O_i \in \mathbb{O}\cap r} m_i \\ &= A(\mathbf{x}) - |\mathbb{X} \cap r| \\ &\leq A(\mathbf{x}). \end{aligned}$$

□

Definition 5.6. Suppose the O-markings in g are numbered so that $O_1, \ldots, O_k$ are edge O's and $O_{k+1}, \ldots, O_n$ are vertex O's. Let $\mathcal{U}$ be the minimal subcomplex of $CF^-(g)$ containing $U_{k+1}CF^-(g) \cup \cdots \cup U_nCF^-(g)$. Then $(\widehat{CF}(g), \widehat{\partial})$ is the filtered chain complex obtained from $(CF^-(g), \partial^-)$ by setting $\widehat{CF}(g) = CF^-(g)/\mathcal{U}$ and letting $\widehat{\partial}$ be the map on the quotient induced by ∂^-. We consider $\widehat{CF}(g)$ as a vector space over $\mathbb{F}_2$.

We denote by $\widehat{HFG}(g)$ the homology of the associated graded object of $\widehat{CF}(g)$. It is finitely generated as a vector space over $\mathbb{F}_2$, since all of the U_i's act trivially on it ([**HO15**]).

2. Alexander filtration and the τ invariant

For a knot K, the Alexander filtration of the knot Floer homology chain complex for K is an absolute grading preserved under the maps associated to the commutation and (de)stabilization grid moves. For balanced spatial graphs, as discussed elsewhere in this chapter, only the relative Alexander filtration of the graph Floer homology chain complex is preserved under the maps associated to the commutation' and (de)stabilization' grid moves. Therefore, we need to fix an absolute Alexander grading and filtration of the graph Floer homology chain complex in order to be able to define a τ invariant for balanced spatial graphs.

To be able to fix an absolute Alexander grading, we need to know that the homology of the associated graded complex $\oplus_s \widehat{\mathcal{F}}_s / \widehat{\mathcal{F}}_{s-1}$ is non-trivial. To show this, we appeal to Lemma 2.1. Note that the grid chain complex $(\widehat{CF}(g), \widehat{\partial})$ satisfies the condition in Lemma 2.1 that for each Maslov grading level i, the chain group $\widehat{CF}(g)_i$ is finitely generated. This is because all elements of $\widehat{CF}(g)$ are of the form $U_1^{a_1} \cdots U_k^{a_k}\mathbf{x}$ for some generator $\mathbf{x}$ and with $a_j \geq 0$ for all j, so

$$M(U_1^{a_1} \cdots U_k^{a_k}\mathbf{x}) = M(\mathbf{x}) - 2\left(\sum_{j=1}^{k} a_j\right).$$

Since there are finitely many generators, since $M(\mathbf{x})$ is finite, and since there are only finitely many ways to write a given number i as the sum of finitely many positive integers, the condition is satisfied.

Definition 5.7. For a grid diagram g representing a balanced spatial graph $f : G \to S^3$, define the symmetrized Alexander filtration $\{\widehat{\mathcal{F}}^H_m\}_{m\in\frac{1}{2}\mathbb{Z}}$ to be the absolute Alexander filtration obtained by fixing the relative Alexander grading so that $s_{max}(g) = -s_{min}(g)$, where $s_{max}(g) = max\{s|H_*(\widehat{\mathcal{F}}_s(g)/\widehat{\mathcal{F}}_{s-1}(g)) \neq 0\}$ and $s_{min}(g) = min\{s|H_*(\widehat{\mathcal{F}}_s(g)/\widehat{\mathcal{F}}_{s-1}(g)) \neq 0\}$.

Now that we have symmetrized the Alexander filtration of $\widehat{CF}(g)$, we can lift that filtration to a symmetrized filtration of $CF^-(g)$.

Definition 5.8. Define the symmetrized Alexander filtration of $CF^-(g)$ to be $\{\mathcal{F}^{-H}_m\}_{m\in\frac{1}{2}\mathbb{Z}}$, obtained by fixing the relative Alexander grading of $CF^-(g)$ so that each generator $\mathbf{x} \in \mathbf{S}(g)$ is in the same filtration level of $\{\mathcal{F}^{-H}_m\}_{m\in\frac{1}{2}\mathbb{Z}}$ as it is in $\{\widehat{\mathcal{F}}^H_m\}_{m\in\frac{1}{2}\mathbb{Z}}$.

Remark 5.1. This is not necessarily the only way to symmetrize the Alexander filtration. If we knew that the bigraded Euler characteristic of $\widehat{HFG}(g)$ (which is an Alexander polynomial, see [**HO15**]) were non-zero, then we could fix an absolute Alexander grading so that the maximal and minimal terms with non-zero coefficients in the Alexander polynomial were centered around zero. It would be interesting to answer the question of whether these two ways of fixing the Alexander grading are equivalent.

Definition 5.9. For a graph grid diagram g representing a balanced spatial graph $f : G \to S^3$, define the τ invariant of g to be

$$\tau(g) = \min\{m \in \frac{1}{2}\mathbb{Z}|\iota_m \text{ is non-trivial}\}$$

where $\iota_m : H_*(\widehat{\mathcal{F}}^H_m) \to H_*(\widehat{CF}(g))$ is the map induced by inclusion.

THEOREM 5.1. *For grid diagrams g, g' representing $f : G \to S^3$, there exist filtered quasi-isomorphisms $\phi_1 : CF^-(g) \to CF^-(g')$ and $\phi_2 : CF^-(g') \to CF^-(g)$ which preserve the symmetrized filtration $\{\mathcal{F}_s^{-H}\}$.*

PROOF. For graph grid diagrams g and g' both representing a balanced spatial graph $f : G \to S^3$, we know by Theorem 4.1 [**HO15**] that there is a finite sequence of cyclic permutation, commutation', stabilization', and destabilization' moves which turns g into g'. Thus, once we show that each of these grid moves is associated to a quasi-isomorphism of filtered chain complexes. We can then take the composition of the maps associated to each of the grid moves in the sequence, resulting in a filtered quasi-isomorphism from $CF^-(g)$ to $CF^-(g')$. The proof that each of the grid moves is associated to a quasi-isomorphism of filtered chain complexes consists of three steps:

(1) We need to show that if g and $\overline{g}$ are graph grid diagrams which are related by a cyclic permutation, commutation', stabilization', or destabilization' grid move, there exists a chain map $\Phi : CF^-(g) \to CF^-(\overline{g})$ and an integer m such that for all s, we have $\Phi(\mathcal{F}_s^-(g)) \subset \mathcal{F}_{s+m}^-(\overline{g})$, and such that Φ induces an isomorphism $H_*(\mathcal{F}_s^-(g)/\mathcal{F}_{s-1}^-(g)) \to H_*(\mathcal{F}_{s+m}^-(\overline{g})/\mathcal{F}_{s+m-1}^-(\overline{g}))$. Note that here, we are working with the original Alexander filtration rather than the symmetrized version. This will be proved in Section 3, Section 4, and Section 5 of this chapter.

(2) We need to show that each of the maps from Step (1) induces a quasi-isomorphism on the symmetrized Alexander filtration. That is, we need to show that $\Phi_* : H_*(\mathcal{F}_s^{-H}(g)/\mathcal{F}_{s-1}^{-H}(g)) \to H_*(\mathcal{F}_s^{-H}(\overline{g})/\mathcal{F}_{s-1}^{-H}(\overline{g}))$ is an isomorphism. Since we know from Step (1) that Φ induces an isomorphism on the homology of the associated graded objects, it is sufficient to show that the span $s_{max} - s_{min}$ is the same for both $\widehat{\mathcal{F}}_s(g)$ and $\widehat{\mathcal{F}}_s(\overline{g})$. We will show that $s_{max}(\overline{g}) = s_{max}(g) + m$ and $s_{min}(\overline{g}) = s_{min}(g) + m$.

Assume for the sake of contradiction that $s_{max}(\overline{g}) > s_{max}(g) + m$. Then there exists some $\mathbf{y} \in \widehat{\mathcal{F}}_{s_{max}(\overline{g})}(\overline{g})$ such that $[\mathbf{y}] \in H_*(\widehat{\mathcal{F}}_{s_{max}(\overline{g})}(\overline{g})/\widehat{\mathcal{F}}_{s_{max}(\overline{g})-1}(\overline{g}))$ is non-trivial. Then, since

$$\Phi_* : H_*(\widehat{\mathcal{F}}_{s_{max}(\overline{g})-m}(g)/\widehat{\mathcal{F}}_{s_{max}(\overline{g})-m-1}(g)) \to H_*(\widehat{\mathcal{F}}_{s_{max}(\overline{g})}(\overline{g})/\widehat{\mathcal{F}}_{s_{max}(\overline{g})-1}(\overline{g}))$$

is an isomorphism, there exists some non-trivial $[\mathbf{x}] = \Phi_*^{-1}([\mathbf{y}])$ in $H_*(\widehat{\mathcal{F}}_{s_{max}(\overline{g})-m}(g)/\widehat{\mathcal{F}}_{s_{max}(\overline{g})-m-1}(g))$. This contradicts our assumption that $s_{max}(\overline{g}) > s_{max}(g) + m$, so we have that $s_{max}(\overline{g}) \leq s_{max}(g) + m$. Similar arguments show that $s_{max}(\overline{g}) \geq s_{max}(g) + m$, and that $s_{min}(\overline{g}) = s_{min}(g) + m$. Therefore we have shown that $s_{max}(g) - s_{min}(g) = s_{max}(\overline{g}) - s_{min}(\overline{g})$.

(3) We need to know that the existence of a quasi-isomorphism on the associated graded object of a filtered chain complex implies the existence of a filtered quasi-isomorphism on the filtered chain complex. This is exactly what Lemma 2.2 [**McC01**] says.

□

LEMMA 5.2. *Suppose that there exist filtered quasi-isomorphisms $F : \widehat{CF}(g) \to \widehat{CF}(\overline{g})$ and $F' : \widehat{CF}(\overline{g}) \to \widehat{CF}(g)$. Then $\tau(g) = \tau(\overline{g})$.*

PROOF. Suppose that $\tau(g) = a$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H_*(\widehat{\mathcal{F}}_a(g)) & \xrightarrow{i_*} & H_*(\widehat{CF}(g)) \\ \big\downarrow{\scriptstyle F_*^a} & & \big\downarrow{\scriptstyle F_*} \\ H_*(\widehat{\mathcal{F}}_a(\overline{g})) & \xrightarrow{j_*} & H_*(\widehat{CF}(\overline{g})) \end{array}$$

Thus there is some $x \in H_*(\widehat{\mathcal{F}}_a(g))$ which maps via $F_* \circ i_*$ to a non-zero element of $H_*(\widehat{CF}(\overline{g}))$ to which j_* sends $F_*^a(x)$. Therefore $j_* : H_*(\mathcal{F}_a(\widehat{CF}(\overline{g}))) \to H_*(\widehat{CF}(\overline{g}))$ is non-trivial, so $\tau(\overline{g}) \leq a$.

The same argument using $F' : \widehat{CF}(\overline{g}) \to \widehat{CF}(g)$ says that $\tau(g) \leq \tau(\overline{g})$, so putting the two inequalities together gives the result that $\tau(g) = \tau(\overline{g})$. □

With the previous lemma, we have shown the following corollary to Theorem 5.1.

Corollary 5.1. *If g and $\overline{g}$ are graph grid diagrams representing a balanced spatial graph $f : G \to S^3$, then $\tau(g) = \tau(\overline{g})$.*

Now we have a well-defined τ invariant for balanced spatial graphs.

Definition 5.10. For a balanced spatial graph $f : G \to S^3$, and g is any graph grid diagram representing f, then

$$\tau(f) = \tau(g).$$

3. Cyclic Permutation

Suppose that g and $\overline{g}$ are graph grid diagrams which differ by a cyclic permutation move. Since the chain complex $(CF^-(g), \partial_g^-)$ and $CF^-(\overline{g}), \partial_{\overline{g}}^-)$ are defined from toroidal grid diagrams, the chain map associated to the cyclic permutation grid move is the identity map, so it is a quasi-isomorphism. However, we still need to show that the map preserves the Alexander filtration.

From Lemma 5.1 and Corollary 4.12 in [**HO15**], we know that the relative Alexander grading is well-defined on the toroidal grid diagram. Define new gradings $A_g'(\cdot)$ and $A_{\overline{g}}'(\cdot)$ by shifting the Alexander gradings on $CF^-(g)$ and $CF^-(\overline{g})$, respectively, so that in each one, $\mathbf{x}_{\mathbb{O}}$, the generator whose points are at the lower left corner of each of the grid squares containing and O-marking, has grading zero. Now the identity map preserves this shifted grading. If k and $\overline{k}$ were the shifts from $A_g(\cdot)$ to $A_g'(\cdot)$ and from $A_{\overline{g}}(\cdot)$ to $A_{\overline{g}}'(\cdot)$, respectively, then we see that the identity map sends elements of $CF^-(g)$ with Alexander grading s to elements of $CF^-(\overline{g})$ with Alexander grading

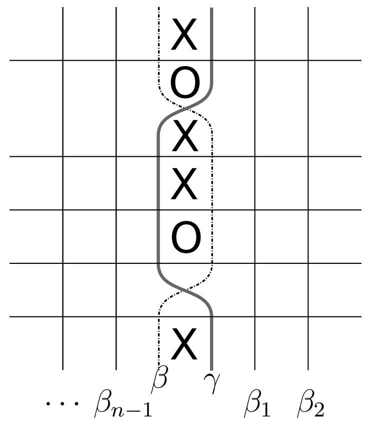

FIGURE 5.5. A grid showing both g and $\overline{g}$. The grid diagram with β but not γ is g, and the diagram with γ but not β is $\overline{g}$.

$s+k-\overline{k}$. Therefore $H_*(\mathcal{F}_s^-(g)/\mathcal{F}_{s-1}^-(g)) \to H_*(\mathcal{F}_{s+k-\overline{k}}^-(\overline{g})/\mathcal{F}_{s+k-\overline{k}-1}^-(\overline{g}))$, the map induced by the identity, is an isomorphism.

4. Commutation'

Let g and $\overline{g}$ be graph grid diagrams which differ by a commutation' move. We can depict both grids in a single diagram, as shown in Fig. 5.5. In this example g is the graph grid diagram obtained from Fig. 5.5 by deleting the line labeled γ, and $\overline{g}$ is the graph grid diagram obtained from it by deleting β. The proof of commutation' invariance closely follows that in [**HO15**].

Recall that the differential map $\partial^- : CF^-(g) \to CF^-(g)$ counts empty rectangles connecting generators in g. In this section, we will consider maps that count empty pentagons and hexagons in the combined grid showing both g and $\overline{g}$. An embedded pentagon p in the combined grid diagram connects $\mathbf{x} \in \mathbf{S}(g)$ to $\mathbf{y} \in \mathbf{S}(\overline{g})$ if $\mathbf{x}$ and $\mathbf{y}$ agree in all but two points, and if the boundary of p is made up of arcs of five grid lines, whose intersection points are, in counterclockwise order, a, x_2, y_2, x_1, y_1, where $a \in \beta \cap \gamma$, $y_1 = \mathbf{y} \cap \gamma$, and $x_1 = \mathbf{x} \cap \beta$. See Fig. 5.6 for an example. Such a pentagon p is *empty* if its interior does not contain any points of $\mathbf{x}$ or $\mathbf{y}$. The set of empty pentagons connecting $\mathbf{x}$ to $\mathbf{y}$ is denoted $Pent^\circ_{\beta\gamma}(\mathbf{x}, \mathbf{y})$.

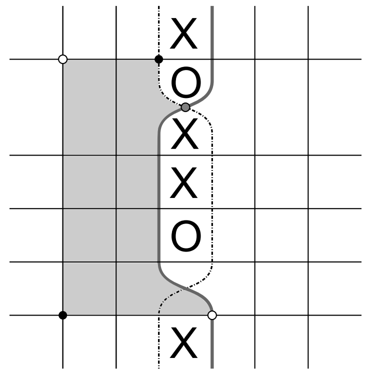

FIGURE 5.6. A pentagon connecting the black generator to the white generator, counted in $\Phi'_{\beta\gamma}$.

DEFINITION 5.11. For $\mathbf{x} \in S(g)$, let

$$\Phi'_{\beta\gamma}(\mathbf{x}) = \sum_{\mathbf{y} \in S(\overline{g})} \left(\sum_{p \in \mathrm{Pent}^{\circ}_{\beta\gamma}} U_1^{O_1(p)} \dots U_n^{O_n(p)} \cdot \mathbf{y} \right)$$

and note that $\Phi'_{\beta\gamma}(\mathbf{x}) \in CF^{-}(\overline{g})$.

LEMMA 5.3. *The map $\Phi'_{\beta\gamma}$ is a chain map which preserves Maslov grading and respects the Alexander filtration, which is to say that $\Phi'_{\beta\gamma}(\mathcal{F}^{-}_{k}(g)) \subset \mathcal{F}^{-}_{k+m}(\overline{g})$ for some $m \in \mathbb{Z}$, where $\{\mathcal{F}^{-}_{k}(g)\}$ is the unsymmetrized Alexander filtration of $CF^{-}(g)$. Moreover, it induces an isomorphism on the homology of the associated graded object, so*

$$(\Phi'_{\beta\gamma})_* : H_*(\mathcal{F}^{-}_{k}(g)/\mathcal{F}^{-}_{k-1}(g)) \to H_*(\mathcal{F}^{-}_{k+m}(\overline{g})/\mathcal{F}^{-}_{k+m-1}(\overline{g}))$$

is an isomorphism for all k.

PROOF. This proof has three parts:

(1) $\Phi'_{\beta\gamma}$ preserves Maslov grading. This follows immediately from Lemma 5.2 in [**HO15**] because the difference between their $\Phi_{\beta\gamma}$ map between associated graded chain complexes and our filtered map $\Phi'_{\beta\gamma}$ between filtered chain complexes is that in the filtered setting pentagons may contain X-markings, but Maslov grading does not involve the X-markings on the grid in any way.

(2) The map $\Phi'_{\beta\gamma}$ preserves the Alexander filtration in the sense given in the statement of the lemma and induces an isomorphism on the homology of the associated graded object. In the proof of Lemma 5.2 in [**HO15**], Harvey and O'Donnol show that their maph $\Phi_{\beta\gamma}$ shifts the Alexander grading by some fixed element $\delta(g, \overline{g}) \in H_1(S^3 \setminus f(G))$. By collapsing their Alexander grading using the obvious map from $H_1(S^3 \setminus f(G))$ to $\mathbb{Z}$, we obtain from their $\Phi_{\beta\gamma}$ the induced map of our $\Phi'_{\beta\gamma}$ on the associated graded objects $\bigoplus_m \mathcal{F}^-_m(g)/\mathcal{F}^-_{m-1}(g) \to \bigoplus_m \mathcal{F}^-_{m+d}(\overline{g})/\mathcal{F}^-_{m+d-1}(\overline{g})$, where $d \in \mathbb{Z}$ corresponds to $\delta(g, \overline{g}) \in H_1(S^3 \setminus f(G))$. Therefore we know that $\Phi'_{\beta\gamma}$ induces an isomorphism on the homology of the associated graded object.

It remains to show that $\Phi'_{\beta\gamma}(\mathcal{F}^-_m(g)) \subset \mathcal{F}^-_{m+d}(\overline{g})$. Notice that $\Phi'_{\beta\gamma}$ can be decomposed into a sum of $\Phi_{\beta\gamma}$ plus terms corresponding to empty pentagons that contain X-markings. We need to show that each term corresponding to an empty pentagon containing at least one X-marking has Alexander grading less than or equal to $m+d$ in $CF^-(\overline{g})$. This is true because for any generator $\mathbf{x} \in \mathbf{S}(g)$ with Alexander grading m, $\Phi'_{\beta\gamma}(\mathbf{x})$ is a sum of terms corresponding to pentagons that do not contain X-markings and terms corresponding to pentagons that do not contain X-markings. As shown above, the terms corresponding to pentagons not containing X-markings are in Alexander grading $m+d$, but the presence of X-markings in the other pentagons reduces the Alexander grading of those terms, so a term in $\Phi'_{\beta\gamma}(\mathbf{x})$ corresponding to a pentagon containing k X-markings is in Alexander grading $m+d-k$.

(3) The map is a chain map, that is $\partial^- \circ \Phi'_{\beta\gamma} + \Phi'_{\beta\gamma} \circ \partial^- = 0$. This follows immediately from the proof of Lemma 3.1 in [**MOST07**].

□

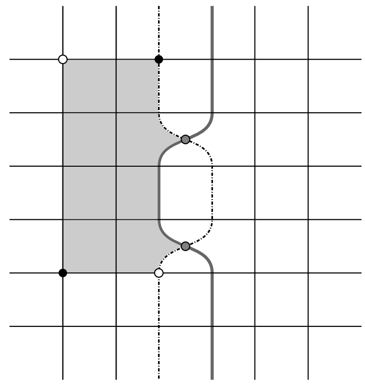

FIGURE 5.7. A hexagon counted in $H_{\beta\gamma\beta}$.

The proof that $\Phi'_{\beta\gamma}$ is a chain homotopy equivalence is the same as the proof in Section 3.1 of [**MOST07**]. An embedded hexagon h in the combined grid showing both g and $\overline{g}$ connects $\mathbf{x} \in \mathbf{S}(g)$ to $\mathbf{y} \in \mathbf{S}(\overline{g})$ if $\mathbf{x}$ and $\mathbf{y}$ agree in all but two points (without loss of generality, say the points where they do not agree are x_1, x_2 and y_1, y_2), and if the boundary of h is made up of arcs of grid lines whose intersection points are, in counterclockwise order, x_1, y_1, a_1, a_2, x_2, and y_2, where $\{a_1, a_2\} = \beta \cap \gamma$, and if the interior angles of h are all less than straight angles. See Fig. 5.7 for an example. A hexagon is empty if its interior does not contain any points of $\mathbf{x}$ or $\mathbf{y}$. The set of empty hexagons connecting $\mathbf{x}$ to $\mathbf{y}$ is denoted $Hex^{\circ}_{\beta\gamma\beta}(\mathbf{x}, \mathbf{y})$. The chain homotopy operator $H_{\beta\gamma\beta} : CF^-(g) \to CF^-(g)$ is defined as follows:

$$H_{\beta\gamma\beta}(\mathbf{x}) = \sum_{\mathbf{y} \in S(g)} \left(\sum_{h \in \mathrm{Hex}^{\circ}_{\beta\gamma\beta}(\mathbf{x},\mathbf{y})} U_1^{O_1(h)} \dots U_n^{O_n(h)} \cdot \mathbf{y} \right).$$

LEMMA 5.4 ([**MOST07**] Proposition 3.2). *The map $\Phi'_{\beta\gamma}$ is a chain homotopy equivalence. That is,*

$$\mathbb{I}_{C(\overline{g})} + \Phi'_{\beta\gamma} \circ \Phi'_{\gamma\beta} + \partial^- \circ H_{\gamma\beta\gamma} + H_{\gamma\beta\gamma} \circ \partial^- = 0$$

and

$$\mathbb{I}_{C(g)} + \Phi'_{\gamma\beta} \circ \Phi'_{\beta\gamma} + \partial^- \circ H_{\beta\gamma\beta} + H_{\beta\gamma\beta} \circ \partial^- = 0.$$

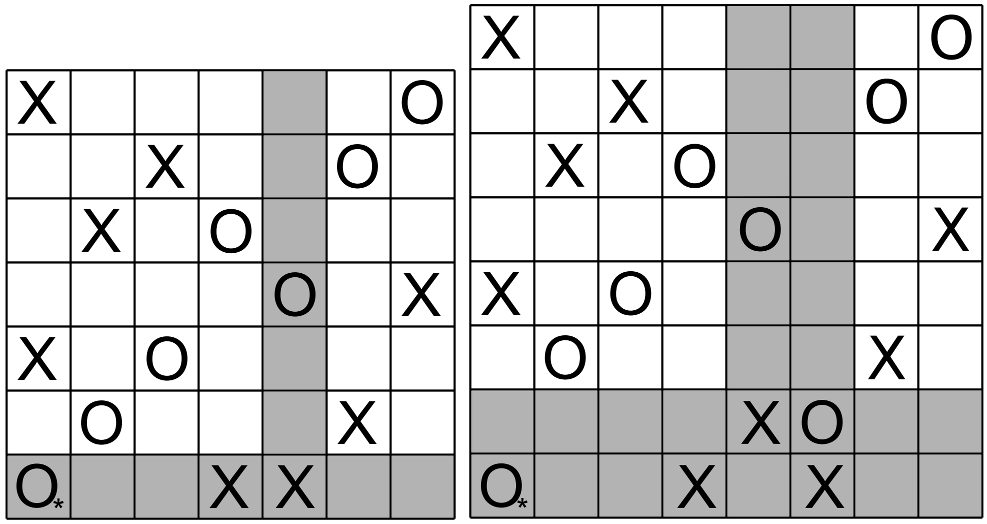

Figure 5.8. A row stabilization'

5. Stabilization'

Let g and g' be two graph grid diagrams such that a stabilization' move on g results in g'. Our proof that the (de)stabilization' move induces filtered quasi-isomorphisms in both directions between $CF^{-}(g)$ and $CF^{-}(g')$ is modeled on Sarkar's proof in [**Sar11**].

Definition 5.12. A row (column) **stabilization'** at an X-marking is performed by adding one new row and one new column to the grid next to that X. The X is then moved to the new row (column), remaining in the same column (row), with the O and any other X-markings in which were in the same row (column) as the X being stabilized remaining in the old row (column). A new X-marking is placed in the intersection of the new column (row) and the row (column) previously occupied by the X-marking, and a new O is placed in the intersection of the new row and column. See the example in Fig. 5.8. A destabilization' is the opposite of a stabilization'.

Sarkar [**Sar11**] distinguishes between two types of (de)stabilizations: those at ordinary O-markings, which he refers to as S^3-grid move (4), and those at special O-markings, which he refers to as S^3-grid move (5) (Special O-markings in the spatial graph case are the vertex O's). The first type can correspond to a (de)stabilization (Link-grid move (3)), which preserves isotopy class, or a birth in the cobordism (Link-grid move (4)), while the second type corresponds to a death in the cobordism (Link-grid move (7)).

Therefore, although both types of stabilization will be needed to prove the link cobordism result in Theorem 6.1, for the purposes of proving the invariance of HFG^- and the τ invariant, we will only need the first type.

Sarkar defines two stabilization maps, s_{11}, s_{22}, and two destabilization maps, d_{11}, d_{22}. The 11 maps correspond to the stabilization in which the new O-marking is placed in the row above the X being stabilized, and the 22 maps correspond to the stabilization in which the new O-marking is placed in the row below the X being stabilized. Because we can use the commutation' move, we only need the graph grid diagram analogs of the 11 maps. The case for which [**Sar11**] uses the s_{22}, d_{22} maps can instead be addressed in the spatial graph case using a commutation' move, then d_{11} or s_{11}, then another commutation' move.

The maps $d_{11} : CF^-(g') \to CF^-(g)$ and $s_{11} : CF^-(g) \to CF^-(g')$ are defined as follows on the generators of the chain complexes:

$$d_{11}(U_0^m \mathbf{x}) = U_j^m \sum_{\mathbf{y}} \mathbf{y} \sum_{D \in \mathcal{S}_1(\mathbf{x}, \mathbf{y} \cup \star, \star)} U_1^{O_1(D)} \cdots U_n^{O_n(D)}$$

$$s_{11}(\mathbf{x}) = \sum_{\mathbf{y}} \mathbf{y} \sum_{D \in \mathcal{S}_3(\mathbf{x} \cup \star, \mathbf{y}, \star)} U_1^{O_1(D)} \cdots U_n^{O_n(D)}$$

Here, O_0 is the the O-marking in g' but not in g, O_j is the O-marking in the row immediately below O_0, and $\mathcal{S}_3(\mathbf{x} \cup \star, \mathbf{y}, \star)$ is the set of snail-like domains illustrated in

Figure 5 of [**Sar11**], which are the same as rotating counterclockwise by 90°the Type R domains described in [**MOST07**], and $\star$ is the intersection point of the α and β curves immediately below and to the left of the new O-marking (see Fig. 5.9).

The map d_{11} is exactly the map F^R defined in [**MOST07**] and used in [**HO15**], considered as a map from C to B, where C is the chain complex associated to the stabilized grid diagram and B is the chain complex associated to the unstabilized grid diagram. Therefore by Lemma 3.5 in [**MOST07**], the map d_{11} is a chain map which preserves the Maslov grading. In Lemma 5.8 in [**HO15**], Harvey and O'Donnol prove that d_{11} induces an isomorphism $\mathcal{F}^-_m(g')/\mathcal{F}^-_{m-1}(g') \to \mathcal{F}^-_{m+a}(g)/\mathcal{F}^-_{m+a-1}(g)$ for all $m \in \mathbb{Z}$. When mapped to the integers, the grading shift is $a = -A^{g'}(\star) - 1$. The proof that d_{11} preserves the Alexander filtration up to a shift by a is similar to the proof in [**HO15**] that it induces an isomorphism on the associated graded object, except that in the filtered case, we allow the domains to contain X-markings, which lowers the Alexander grading of the terms associated to the domains containing X-markings.

Lemma 5.5. *The composition $d_{11} \circ s_{11}$ is the identity map on the associated graded chain complex for the unstabilized grid diagram.*

Proof. In the associated graded chain complexes, the only regions counted in s_{11} and d_{11} are rectangles with the starred grid intersection point as their lower left and upper left corners, respectively. All higher complexity snail-like regions counted in these maps contain the X being stabilized and thus are not counted in the associated graded version. Furthermore, in the associated graded chain complexes the regions counted may not contain any X-markings other than the one in the newly added column.

If D is a rectangle connecting $\mathbf{x} \cup \star$ to $\mathbf{y}$ which is counted in $s_{11}(\mathbf{x})$, then we consider $d_{11}(\mathbf{y})$. If D' is a domain counted in $d_{11}(\mathbf{y})$, then the boundary of $\partial D' \cap \beta_1$ is

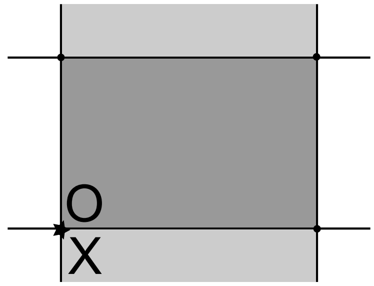

FIGURE 5.9. the rectangles D and D'

$\mathbf{y} - \star$. Therefore the upper boundary of D' is α_1, so the term in $d_{11}(\mathbf{y})$ corresponding to D' is $\mathbf{x}$. See Fig. 5.9.

No U_i's survive in $d_{11} \circ s_{11}(\mathbf{x})$ since the composite map counts domains $D \cup D'$, which as just discussed are the union of entire columns in the grid diagram. Since every column contains at least one X-marking, the only $D \cup D'$ that may be counted is the single column containing the new O-marking. Since the new O-marking is not counted, $d_{11} \circ s_{11}(\mathbf{x}) = \mathbf{x}$ and so the composition $d_{11} \circ s_{11}$ is the identity map. □

LEMMA 5.6. *The map s_{11} is a quasi-isomorphism between the associated graded chain complexes for the unstabilized and stabilized grid diagrams.*

PROOF. We know from the previous lemma that $d_{11} \circ s_{11}$ is the identity map on the associated graded chain complex for the unstabilized grid diagram. The identity map is a quasi-isomorphism, and by Proposition 5.13 in [**HO15**], $d_{11} = F^R$ is a quasi-isomorphism. Then since s_{11} is the one-sided inverse of a quasi-isomorphism, it is also a quasi-isomorphism. □

LEMMA 5.7. *The map s_{11} is a filtered chain map which preserves Maslov grading and respects the Alexander filtration up to a finite shift, so that $s_{11}(\mathcal{F}^-_m(g)) \subset \mathcal{F}^-_{m-a}(g')$ and it induces an isomorphism $\mathcal{F}^-_m(g)/\mathcal{F}^-_{m-1}(g) \to \mathcal{F}^-_{m-a}(g')/\mathcal{F}^-_{m-a-1}(g')$ for all $m \in \mathbb{Z}$ and $a = -A^{g'}(\star) - 1$.*

PROOF. By definition, s_{11} is a module homomorphism. We need to show that it preserves the Maslov grading, it respects the Alexander filtration, and that it is a chain map.

The proof that s_{11} is a chain map is the same as the proof of Lemma 3.5 in [**MOST07**] except that the snail-like domains are rotated 90°counterclockwise.

To show that the Maslov grading is preserved, suppose that there is some snail-like domain D which connects $\mathbf{x}\cup\star$ to $\mathbf{z}$ in the stabilized grid g' which is counted in $s_{11}(\mathbf{x})$. We begin by using the definition of the Maslov grading to compare the grading of $\mathbf{x}\cup\star$ in the stabilized grid g' to the grading of $\mathbf{x}$ in the unstabilized grid g.

$$\begin{aligned}
M(\mathbf{x}\cup\star) &= \mathcal{J}(\mathbf{x}+\star-\mathbb{O}_{g'},\mathbf{x}+\star-\mathbb{O}_{g'})+1\\
&= \mathcal{J}(\mathbf{x},\mathbf{x})+\mathcal{J}(\star,\star)+\mathcal{J}(\mathbb{O}_{g'},\mathbb{O}_{g'})+2\mathcal{J}(\mathbf{x},\star)\\
&\qquad -2\mathcal{J}(\mathbf{x},\mathbb{O}_{g'})-2\mathcal{J}(\star,\mathbb{O}_{g'})+1
\end{aligned}$$

Noting that $\mathbb{O}_{g'}$, the set of O-markings in g', is the same as $\mathbb{O}_g\cup O_{k+1}$, where $\mathbb{O}_g$ is the set of O-markings in g and O_{k+1} is the new O-marking, we can see that

$$\begin{aligned}
M(\mathbf{x}\cup\star)=\mathcal{J}(\mathbf{x},\mathbf{x}) &+ \mathcal{J}(\star,\star)+\mathcal{J}(\mathbb{O}_g,\mathbb{O}_g)+\mathcal{J}(O_{k+1},O_{k+1})+2\mathcal{J}(\mathbb{O}_g,O_{k+1})\\
&+ 2\mathcal{J}(\mathbf{x},\star)-2\mathcal{J}(\mathbf{x},\mathbb{O}_g)-2\mathcal{J}(\mathbf{x},O_{k+1})-2\mathcal{J}(\star,\mathbb{O}_g)\\
&- 2\mathcal{J}(\star,O_{k+1})+1.
\end{aligned}$$

We can use the following observations to simplify the expression:

$$\begin{aligned}
\mathcal{J}(\star,\star) &= \mathcal{J}(O_{k+1},O_{k+1})=0\\
\mathcal{J}(\mathbf{x},\star) &= \mathcal{J}(\mathbf{x},O_{k+1})\\
\mathcal{J}(\mathbb{O}_g,O_{k+1}) &= \mathcal{J}(\star,\mathbb{O}_g)\\
M(\mathbf{x}) &= \mathcal{J}(\mathbf{x},\mathbf{x})+\mathcal{J}(\mathbb{O}_g,\mathbb{O}_g)-2\mathcal{J}(\mathbf{x},\mathbb{O}_g)+1\\
\mathcal{J}(\star,O_{k+1}) &= \frac{1}{2}
\end{aligned}$$

Therefore $M(\mathbf{x}\cup\star) = M(\mathbf{x}) - 1$.

Since D connects $\mathbf{x}\cup\star$ to $\mathbf{z}$, the term corresponding to D in $s_{11}(\mathbf{x})$ is $U_1^{O_1(D)}\cdots U_n^{O_n(D)}\mathbf{z}$. Therefore we need to compare $M(\mathbf{x})$ and $M(U_1^{O_1(D)}\cdots U_n^{O_n(D)}\mathbf{z})$. By the definition of Maslov grading, $M(U_1^{O_1(D)}\cdots U_n^{O_n(D)}\mathbf{z}) = M(\mathbf{z}) - 2\sum_{i=1}^n O_i(D)$, where the summation does not include $i = 0$, which corresponds to the new O-marking. Using Lemma 2.5 in [**MOST07**], we know that $M(\mathbf{x}\cup\star) = M(\mathbf{z}) + 1 + 2m - 2\sum_{i=0}^n O_i(D)$, with $i = 0$ included in the sum and where m is the multiplicity of $\star$ in D. Since the multiplicity of $\star$ in the interior of D is $O_0(D) - 1$, we can put all of this together to see that

$$M(\mathbf{x}) = M(U_1^{O_1(D)}\cdots U_n^{O_n(D)}\mathbf{z}) + 2 - 2O_0(D),$$

so s_{11} preserves the Maslov grading.

For the Alexander filtration, we need to show that $s_{11}(\mathcal{F}_m^-(g)) \subset \mathcal{F}_{m-a}^-(g')$. Suppose that D is a snail-like domain counted in $s_{11}(\mathbf{x})$. Then considered in the stabilized grid g', D is a domain connecting $\mathbf{x}\cup\star$ to some generator $\mathbf{z}$. By Lemma 4.9 in [**HO15**],

$$A^{g'}(\mathbf{x}\cup\star) - A^{g'}(\mathbf{z}) = n_{\mathbb{X}}(D) - \sum_{i=0}^n m_i \cdot O_i(D),$$

where $n_{\mathbb{X}}$ is the number of X-markings contained in D, counted with multiplicity. The term of $s_{11}(\mathbf{x})$ corresponding to the domain D is $U_1^{O_1(D)}\cdots U_n^{O_n(D)}\mathbf{z}$, which has Alexander grading $A^{g'}(\mathbf{z}) - \sum_{i=1}^n O_i(D)$. The shift in the Alexander grading from the s_{11} map is

$$\begin{aligned}
A^{g'}(U_1^{O_1(D)}\cdots U_n^{O_n(D)}\mathbf{z}) &= A^{g'}(\mathbf{z}) - \sum_{i=1}^n m_i O_i(D) - A^g(\mathbf{x}) \\
&= -n_{\mathbb{X}}(D) + O_0(D) + A^{g'}(\mathbf{x}\cup\star) - A^g(\mathbf{x}) \\
&= -n_{\mathbb{X}}(D) + O_0(D) + A^{g'}(\mathbf{x}) + A^{g'}(\star) - A^g(\mathbf{x}) \\
&= -n_{\mathbb{X}}(D) + 1 + A^{g'}(\star).
\end{aligned}$$

Notice that for domains that do not contain any X-markings, which are exactly the domains considered in the associated graded object, the shift in Alexander grading is $1 + A^{g'}(\star)$, which is the negative of the shift for the d_{11} map. For domains that do contain X-markings, the Alexander grading in the terms of $s_{11}(\mathbf{x})$ (for $\mathbf{x}$ in Alexander grading m) corresponding to those domains have Alexander grading less than $m + A^{g'}(\star) + 1$, since the presence of X-markings in the domain reduces their Alexander grading. Therefore $s_{11}(\mathcal{F}_m^-(g)) \subset \mathcal{F}_{m-a}^-(g')$, for $a = -A^{g'}(\star) - 1$, and since $d_{11} \circ s_{11} = id$ induces an isomorphism on $\mathcal{F}_m^-(g)/\mathcal{F}_{m-1}^-(g) \to \mathcal{F}_m^-(g)/\mathcal{F}_{m-1}^-(g)$ for all $m \in \mathbb{Z}$, we know that s_{11} induces an isomorphism $\mathcal{F}_m^-(g)/\mathcal{F}_{m-1}^-(g) \to \mathcal{F}_{m-a}^-(g')/\mathcal{F}_{m-a+1}^-(g')$ for all $m \in \mathbb{Z}$.

□

Using Lemma 2.2 [**McC01**] and the results above that d_{11} and s_{11} are filtered chain maps which are quasi-isomorphisms on the associated graded objects, we see that they are quasi-isomorphisms on the filtered chain complexes on which they are defined.

CHAPTER 6

Link Cobordisms

In this chapter we state the definition of link cobordism and prove an inequality for links analogous to the one proven for knots by Sarkar in [**Sar11**]. This gives an obstruction to sliceness for some links.

Definition 6.1. A cobordism from a link L_0 to another line L_1 is a surface F properly embedded in $S^3 \times [0,1]$, such that $F \cap S^3 \times \{0\} = L_0$ and $F \cap S^3 \times \{1\} = -L_1$. If such a surface exists, we say that L_0 is cobordant to L_1.

If two l-component links L_0 and L_1 are connected by a cobordism consisting of l disjoint annuli, we say that they are concordant, and a link which is concordant to the unlink is slice.

Following [**Sar11**], a cobordism between two links can be represented by a series of link grid moves. These moves are commutations and stabilizations (which correspond to isotopy of links) and births, deaths, X-saddles, and O-saddles.

A grid diagram $\overline{g}$ is obtained from another grid diagram g via a **birth** if adding an additional row and column to g and placing both an O- and an X-marking in the grid square that is the intersection of the new row and column results in $\overline{g}$. See Fig. 6.1 for an example. This move is link-grid move (4) in [**Sar11**].

A grid diagram $\overline{g}$ is obtained from another grid diagram g via a **death** if there are a row and a column in g, each containing exactly one X-marking, whose intersection contains that X- and an O-marking. Then $\overline{g}$ is the result of deleting those markings and deformation retracting the row and column to an α and a β circle. For an example, see Fig. 6.1. This is very similar to link-grid move (7) in [**Sar11**], with the

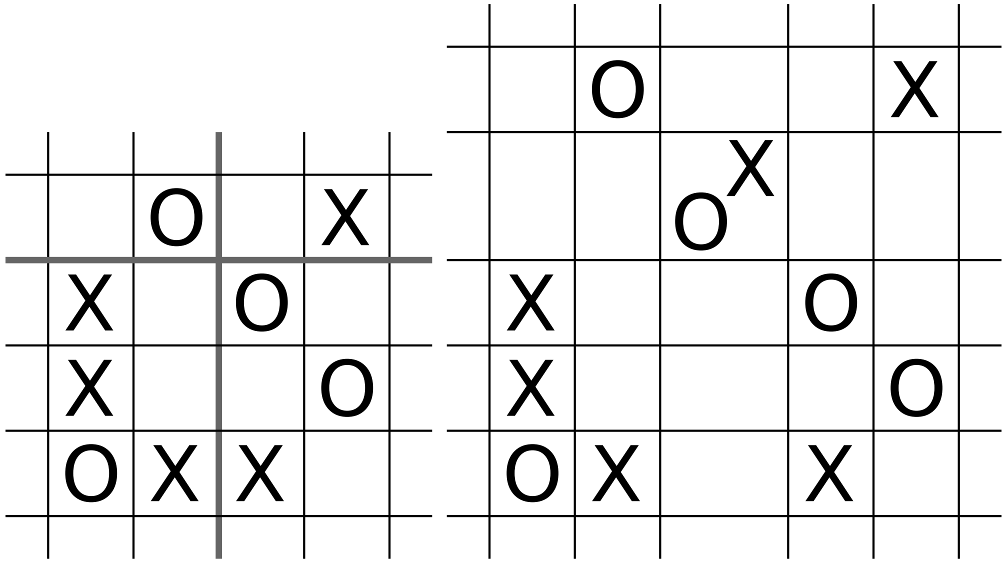

Figure 6.1. A birth in the grid on the left produces the grid on the right; a death in the right-hand grid produces the left-hand grid

difference being that Sarkar required the O-marking in the dying component to be a special O, and here it is a regular O-marking.

There are two grid moves corresponding to saddles in the cobordism. The first, an X-**saddle**, which is link-grid move (5) in [**Sar11**], is used when the saddle merges two components of the graph or when it splits one component into two. If a grid diagram g contains a two-by-two square whose upper left and lower right grid squares contain X-markings and whose upper right and lower left grid squares are unoccupied, then doing this saddle move results in a grid diagram $\overline{g}$. The new grid diagram $\overline{g}$ is exactly the same as g except that in the two-by-two square, the X-markings are placed in the upper right and lower left grid squares, with the upper left and lower right squares unoccupied, as shown in Fig. 6.2.

The second type of saddle grid move, an O-**saddle**, is used only when the saddle in the cobordism splits one component of the graph into two components. This move is link-grid move (6) from [**Sar11**]. It is exactly the same as the first saddle move

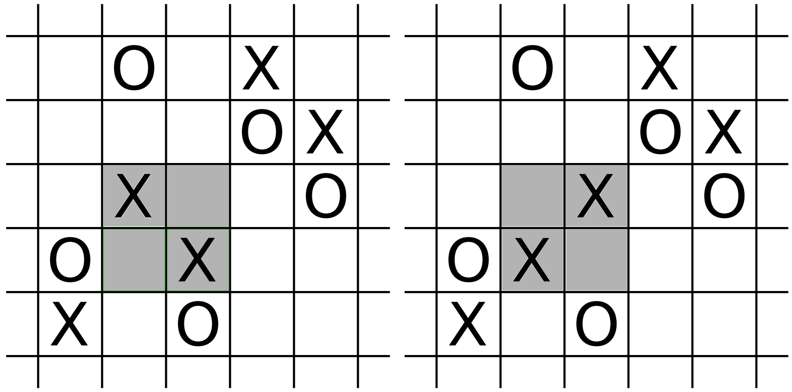

FIGURE 6.2. A saddle move of the first type on the left-hand grid produces the grid on the right

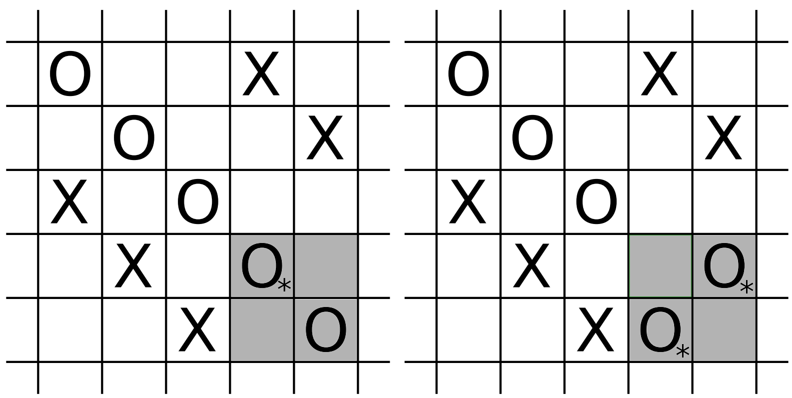

FIGURE 6.3. A saddle move of the second type on the left-hand grid produces the grid on the right

except that the two-by-two square which differs in g and $\overline{g}$ contains a special O-marking in the upper left corner and a regular O in the lower right corner in g, and special O-markings in the uper right and lower left corners in $\overline{g}$. An example is shown in Fig. 6.3.

For the proof of the inequality, we will use the combinatorial definition of Alexander grading from [**Sar11**], which we will denote as A'.

Definition 6.2. For a generator $\mathbf{x}$ in a grid diagram g, the Alexander grading of $\mathbf{x}$ is

$$A'(\mathbf{x}) = \mathcal{J}(\mathbf{x}, \mathbb{X} - \mathbb{O}) - \frac{1}{2}\mathcal{J}(\mathbb{X}, \mathbb{X}) + \frac{1}{2}\mathcal{J}(\mathbb{O}, \mathbb{O}) - \frac{n-1}{2},$$

where n is the grid size of g. For an l-component link, this definition differs slightly from the usual combinatorial definition of Alexander grading $A(\mathbf{x})$ from [**MOST07**], which can be obtained from $A'(\mathbf{x})$ by adding $\frac{l-1}{2}$.

Definition 6.3. For a tight grid diagram g representing an l-component link L in S^3, define

$$\tau'(L) = \min\{m \in \frac{1}{2}\mathbb{Z} | \iota_m \text{ is non-trivial}\}$$

where $\widehat{\mathcal{F}}'_m$ is the Alexander filtration induced by the Alexander grading $A'(\cdot)$ and $\iota_m : H_*(\widehat{\mathcal{F}}'_m) \to H_*(\widehat{CF}(g))$ is the map induced by inclusion.

Lemma 6.1. *The Alexander grading $A(\cdot)$ from* [**MOST07**] *is equal to the Alexander grading I define in my basic definitions document, and for an l-component link the τ^H defined in my basic definitions document is equal to $\tau' + \frac{l-1}{2}$.*

1. Link Cobordisms and the τ Invariant

Theorem 6.1. *If L_1 and L_2 are l- and m-component links, respectively, and F is a connected genus g cobordism from L_1 to L_2, then*

$$1 - g - l \leq \tau(L_1) - \tau(L_2) \leq g + m - 1.$$

Proof. The proof will follow the same basic outline of the proof of the main theorem in [**Sar11**]. Consider the cobordism F as a “movie.” Then there are some number of births, deaths, and saddles in the movie, and the genus $g = \frac{1}{2}(s - b - d) + 1 - \frac{l+m}{2}$, where b, d, and s are the number of births, deaths, and saddles, respectively. We can alter the cobordism slightly so that each of the movie moves happens at a distinct time and so that all of the births take place before any of the saddles, all of the saddles take place before any of the deaths, and the last $m+d-l$ saddles split one link component into two.

Note that $m+d-l$ is always greater than or equal to 0. If $m>l$, then this is obviously true. If $l>m$, then we must have $d \geq l-m$ since both g_1 and g_2 are tight link grid diagrams and deaths are the only move that reduce the number of special O-markings in the grid. Therefore $d \geq l-m>0$, so $m+d-l \geq 0$.

As Sarkar shows in [**Sar11**], the modified cobordism can be represented by a sequence of link grid diagrams, such that the first grid, g_1 is a tight diagram for L_1, the last grid, g_2 is a tight diagram for L_2, and each diagram in the sequence is obtained from the one before it by a commutation, stabilization, destabilization, birth, X-saddle, O-saddle, or death grid move, or by renumbering the ordinary O-markings.

As shown in [**Sar11**], the chain maps associated to renumbering the ordinary O-markings, commutations, stabilizations, and de-stabilizations are quasi-isomorphisms which preserve both the Maslov and A' Alexander gradings. The chain maps associated to births is a quasi-isomorphism which preserves the Maslov grading and shift the Alexander grading A' by $-\frac{1}{2}$. The chain maps associated to X-saddles are the identity maps, and they shift the Alexander grading A' by $+\frac{1}{2}$. The chain maps associated to O-saddles induce injective maps on homology and shift the Alexander grading A' by $-\frac{1}{2}$. The chain maps associated to deaths induce surjective maps on homology and shift the Alexander grading A' by $+\frac{1}{2}$.

Now we will track the overall shift in the Alexander grading A' over the sequence of moves in the (modified) cobordism. Since there are b births, the shift from the births is $-\frac{1}{2}b$. Next we need to figure out how many of the saddles are represented by X-saddle grid moves and how many by O-saddles. Any saddle can be represented by either an X-saddle or an O-saddle grid move, but O-saddles and deaths are the only moves that change the number of special O-markings in the diagrams. Therefore we can choose which saddles will be represented by X-saddles and which by O-saddles so that we will have the correct number of special O-markings at each stage of the cobordism. Since the beginning and ending grid diagrams g_1 and g_2 are tight, we know that g_2 has m special O-markings and g_1 has l special O-markings. Since the death move removes a special O-marking, we need to have $m+d$ special O-markings after all of the saddles have been performed but before the deaths. Therefore we should have $m+d-l$ O-saddles in the cobordism, and these are the last saddles. The fact that we chose that the last $m+d-l$ saddles should be splits ensures that there will not be more than one special O-marking on any one component, so the ending grid diagram g_2 will be tight. The rest of the saddles, which is to say the first $s-m-d+l$ saddles in the cobordism, are X-saddles.

Now we can see that the Alexander grading shift from the X-saddles is $+\frac{1}{2}(s-m-d+l)$ and the shift from the O-saddles is $-\frac{1}{2}(m+d-l)$. Since there are d deaths, the shift from the deaths is $+\frac{1}{2}$. Adding up the grading shifts from all of the cobordism moves, the total shift is $\frac{1}{2}(s-b-d)+l-m$.

Following [**Sar11**], we know that $\tau'(L_1)$ is less than or equal to $\tau'(L_2)$ plus the Alexander grading shift of the cobordism from L_1 to L_2. Therefore

$$\tau'(L_2) \leq \tau'(L_1) + \frac{1}{2}(s-b-d) + l - m,$$

and after some algebraic manipulation, we see that

$$\tau'(L_2) + \frac{m-1}{2} - \tau'(L_1) - \frac{l-1}{2} \leq \left(\frac{s-b-d}{2} + 1 + \frac{m-l}{2}\right) + l - 1.$$

Now we observe that since the genus of F is $g = \frac{s-b-d}{2} + 1 + \frac{m-l}{2}$ and $\tau'(L_1) + \frac{l-1}{2} = \tau(L_1)$, we have

$$\tau(L_2) - \tau(l_1) \leq g + l - 1.$$

To prove the other inequality, we reverse the direction of F and consider it as a cobordism from L_2 to L_1. Following the same proof as for the first inequality, we see that

$$\tau(L_1) - \tau(L_2) \leq g + m - 1.$$

$\square$

2. Application to link sliceness

If an l-component link L is slice, then there is a concordance between L and the l-component unlink. We can modify this concordance by connect-summing the annuli together and capping off all but one of the unlink's components to produce a connected genus zero cobordism from L to the unknot U. Applying Theorem 6.1 to this cobordism, we see that

$$1 - l \leq \tau(L) - \tau(U) \leq 0.$$

Since $\tau(U) = 0$ we have the following corollary:

Corollary 6.1. *If an l-component link L has $\tau(L) > 0$ or $\tau(L) \leq -l$, then L is not slice.*

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