Linear Algebra

1. Consider the following three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ drawn in $\mathbb{R}^2$.

   [picture omitted]

   Are they linearly independent or linearly dependent? How do you know?

   Write down the definition of linear dependence for three vectors. How, explicitly, is this definition fulfilled, or not fulfilled, by the above example?

2. Below are six lists of vectors. Without doing any row reduction, you should be able to pick out three lists that are linearly dependent. Which three? For each of those three, explain in a sentence or two how you can tell without doing any calculation.

   (a) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 13 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 15 \\ -2 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 8 \\ 11 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \\ 11 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}$

   (e) $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 12 \\ 0 \\ 23 \\ 35 \\ 0 \end{pmatrix}$

   (f) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$

3. Suppose we wish to prove that the list $\{\mathbf{v_1}; \mathbf{v_2}; \mathbf{v_3}\}$ is linearly independent. Which of the following are reasonable ways to start?

   (a) Start by calculating $0v_1 + 0v_2 + 0v_3 = \mathbf{0}$.

(b) Start by assuming that there are scalars a_1, a_2, a_3 such that $a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} = \mathbf{0}$. Aim to show that they are all zero.

(c) Start by assuming that there are scalars a_1, a_2, a_3 such that $a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} \neq \mathbf{0}$.

(d) Assume that one of the three vectors is in the span of the other two, aiming to derive a contradiction.